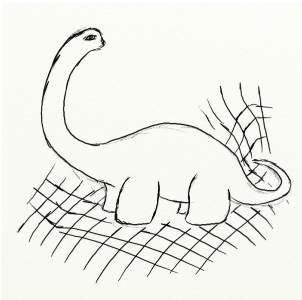

Thick-Brane Worlds

By

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Produced in $\LaTeX\,2_\varepsilon$.

"*If we do not succeed in solving a mathematical problem, the reason frequently consists in our failure to recognize the more general standpoint from which the problem before us appears only as a single link in a chain of related problems. After finding this standpoint, not only is this problem frequently more accessible to our investigation, but at the same time we come into possession of a method which is applicable also to related problems.*"

- David Hilbert, Mathematical Problems (1900)

"*Real life is, to most men, a long second-best, a perpetual compromise between the ideal and the possible; but the world of pure reason knows no compromise, no particular limitations, no barrier to the creative activity embodying in splendid edifices the passionate aspiration after the perfect from which all great work springs.*"

- Bertrand Russell, The Study of Mathematics (1919)

Abstract

We examine 4 + 1-dimensional field theories exhibiting both 3 + 1-dimensional standard model phenomenology and type 2 Randall-Sundrum gravity on a dynamically-generated thick brane. We consider problems with finding and guaranteeing the stability of thick brane solutions, and with the non-renormalizability of Yang-Mills gauge theory in 4 + 1-dimensions. This motivates us to take a more systematic approach to gauge theories and to study a richer variety of field theories beyond standard Yang-Mills gauge theories.

This thesis comes in three parts. In the first part we explicitly construct a model featuring a single infinite extra dimension and an SO(10) grand unified theory, where standard model phenomenology is recovered on a 3 + 1-dimensional domain-wall brane topological defect. A 3 + 1-dimensional graviton and all standard model particles are dynamically localized to the brane. To localize gauge fields we invoke an analogue of the Dvali-Shifman mechanism. Similarly 3 + 1-dimensional left chiral fermions and effective 3 + 1-dimensional general relativity are recovered via the split fermion mechanism, and a Randall-Sundrum type 2 warped metric, respectively. We prove that this model is stable, respectively unstable, for two different regimes of free parameters in the dynamical equations.

In the second part we use Lifshitz anisotropic scaling to fix inherent problems with the non-renormalizability of 4 + 1-dimensional Yang-Mills gauge theories. Here we address problems with the Dvali-Shifman mechanism by paving the way to a 4 + 1-dimensional domain-wall brane model in a Lifshitz field theory with critical exponent $z = 2$. We extend

our analysis to look at specific examples of finite energy, stable, static topological defects in $3 + 1$-dimensional Lifshitz field theories with critical exponent $z = 2$. These defects are forbidden in standard relativistic field theories by Derrick's theorem. As such the observation of cosmic relics would be a hallmark signature of the breakdown of Lorentz invariance at short distances.

In the final part of this thesis we consider grand unified theories (GUTs). These theories are important to a wide range of models in this thesis. We consider the set of vacuum expectation values which break the GUT to differently embedded isomorphic copies of a subgroup H. We define and characterize the relationship between these vacuum expectation values.

I hereby declare that this submission is my own work and to the best of my knowledge it contains no material previously published or written by another person, nor material which to a substantial extent has been accepted for the award of any other degree or diploma at the University of Melbourne or any other educational institution, except where due acknowledgement is made in the thesis. Any contribution made to the research by colleagues, with whom I have worked at the University of Melbourne or elsewhere, during my candidature, is fully acknowledged.

I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in the project's design and conception or in style, presentation and linguistic expression is acknowledged.

Jayne Thompson, Candidate.

Raymond R. Volkas, Principal Advisor.

List of publications

The following publications involve content that is directly relevant to the topic of this PhD thesis.

1. Jayne E. Thompson and Raymond R. Volkas, SO(10) domain-wall brane models. *Phys. Rev D*, **80**: 125016, (2009). [arXiv:0908.4122v2 [hep-ph]]

2. Jayne E. Thompson and Raymond R. Volkas, Domain-wall branes in Lifshitz theories. *Phys. Rev D,* **82**: 116007 (2010). [arXiv:1008.2054v2 [hep-ph]]

3. Jayne Thompson, Bevan Thompson and Michal Fečkan On the stability of a domain-wall brane model. *Nonlinear Analysis,* **74**: 4989-4999 (2011).

4. Archil Kobakhidze, Jayne E. Thompson and Raymond R. Volkas, BPS solitons in Lifshitz field theories. *Phys Rev D,* **83**: 025007 (2011). [arXiv:1010.1068v2 [hep-th]]

5. Damien George, Arun Ram, Jayne E. Thompson, and Raymond R. Volkas, Symmetry breaking, subgroup embeddings and the Weyl group. [arXiv:1203.1048 [hep-th]]

Acknowledgements

Throughout my PhD I have been surrounded by many kind, wonderful and supportive people. I am especially grateful to my family, friends and collaborators who have been helpful, understanding, motivational and at times incredibly patient. I owe a large amount of what I have achieved to their constant efforts and belief in me. I am also extremely lucky to have had an excellent supervisor, Ray. It is hard not to catch his constant enthusiasm for new avenues of research and his openness to fresh ideas. He has taught me to consider all new physics proposals and evaluate each theory critically both in its original context and in its possible strengths and applications to different contexts. This skill and the ability to approach problems critically are invaluable gifts for which I am very grateful to Ray. I am also grateful to all three members of my supervisory panel, Ray Volkas, Nicole Bell and Andy Martin for interesting and engaging conversations about research, for their academic mentoring and advice on applying for postdoctoral positions and finding collaborators over seas.

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Contributions

Statement of contributions to jointly authored works contained in the thesis

Chapter 4 is based on the paper [TV09] by Jayne E. Thompson and Raymond R. Volkas. I am first author on this paper; in addition my contribution to this paper includes all calculations. The idea for the paper was suggested by my supervisor Raymond Volkas, who was interested in extending a model by Davies et al. [DGV08].

Chapter 5 features a stability analysis [TTc11] which was done in collaboration with Bevan Thompson and Michal Fečkan. I am first author on this paper and proposed the original question. Bevan Thompson and Michal Fečkan contributed by giving technical advice and suggesting to apply the Lyapunov-Schmidt method.

Chapter 6 is based on the paper [TV10] by Jayne E. Thompson and Raymond R. Volkas. I am first author on this paper; in addition my contribution involved all the technical calculations for the model. The idea was first proposed by my supervisor Raymond Volkas, and was independently conceived by Damien George.

Chapter 8 is based on the paper [KV11] by Archil Kobakhidze, Jayne E. Thompson and Raymond R. Volkas. I contributed solutions to the differential equations and was involved in the collaborative discussion which formulated the topic and shaped the direction of the paper. The manuscript was jointly written by Archil Kobakhidze and Raymond Volkas.

Chapter 9 is based on a paper by Damien George, Arun Ram, Jayne E. Thompson and Raymond R. Volkas . I wrote the manuscript for this paper supplying all calculations that underpin the results. The introduction section was edited by Raymond Volkas. I owe much of my knowledge in the area and insight into the solution of the problem to Arun Ram.

Statement of contributions by others to the thesis as a whole

Chapter 4 features an idea originally proposed by Raymond Volkas, which generalizes a model proposed by Davies et al. [DGV08].

Chapter 5 features research carried out in collaboration with Bevan Thompson and Michal Fečkan, who contributed technical advice that made the analysis in Chapter 5 possible.

Chapter 6 is based on an idea originally proposed by Damien George and Raymond Volkas. In addition, during the editing process Damien George and Archil Kobakhidze reviewed the presentation of the associated paper and improved it.

Chapter 8 features a model developed in collaboration with Archil Kobakhidze and Raymond Volkas. Many of the technical details in this chapter and the first author role on this paper are shared by Raymond Volkas and Archil Kobakhidze.

Chapter 9 features work done in collaboration with Damien George, Raymond Volkas and Arun Ram. The original question was posed by Damien George. Arun Ram helped me with the analysis.

Statement of parts of the thesis submitted to qualify for the award of another degree

The preliminary analysis of the model in Chapter 4 was submitted as my honours thesis at the University of Melbourne in 2008. The work featured in this thesis is much more comprehensive and radically altered from my original honours thesis. The analysis was done predominantly during the first year of my PhD, 2009, and the paper [TV09] (the foundation for Chapter 4 of this thesis) was written during 2009 and 2010.

List of conferences

Below is a list of conferences where the work in this thesis was presented:

1. SO(10) domain-wall brane models

   AIP 2010, Melbourne.

2. Domain wall branes in 4 + 1-dimensional space time

   2011 National Center for Theoretical Science (South) at National Cheng Kung University, Tainan, Taiwan.

3. Extra dimensions and domain wall branes

   Workshop on Cosmology and Astro Particle Physics 2011, National Taiwan University, Taipei, Taiwan.

4 Domain wall branes in 4 + 1-dimensions

   Kavil Institute for Theoretical Physics China at the Chinese Academy of Sciences 2011, Beijing, China.

Contents

List of Tables

List of Figures

Abbreviations

The following abbreviations are used throughout this thesis.

ED: Extra dimensions

KK: Kaluza-Klein

IR: Infrared

BPS: Bogomolnyi-Prasad-Sommerfield

LBPS: Lifshitz Bogomolnyi-Prasad-Sommerfield

QCD: Quantum chromo dynamics

SM: Standard model

SSB: Spontaneous symmetry breaking

TeV: Tera electron volt

UV: Ultraviolet

VEV: Vacuum expectation value

WKB : WentzelKramersBrillouin approximation

1
Prologue: The path integral towards reality

Particle physics is a reductionist approach to modeling physical reality. We study indivisible particles and their interactions as a way of characterizing the fundamental processes and conservation laws which all physical systems have in common.

It is contentious to say that a complete description of the behaviour of these fundamental components will help us predict physical laws for macroscopic systems. However there are many examples where understanding the behaviour of the macroscopic system has changed our model for microscopic reality, and vice versa. Some examples are the discovery of Nambu-Goldstone bosons in superconducting materials, and the proposal that a colour-flavor locking phase in quark-gluon plasmas causes neutron star glitches. Meanwhile observations of galaxy rotation curves have led us to postulate dark matter halos.

In this thesis we are interested in how changing the dimensions of the space-time manifold affects the dynamics of the subatomic particles and fields. Maintaining a renormalizable Yang-Mills gauge theory in $4 + 1$-dimensions motivates us to rewrite the standard model gauge theory as a Lifshitz field theory. Lifshitz field theories explicitly break Lorentz invariance, causing the speed of light to change with energy scale. This modification of the behavior of subatomic particles becomes phenomenologically detectable in the ultraviolet regime.

At the same time we investigate how Lifshitz field theories may cause stable, long lived, cosmic relics in any $d + 1$-dimensional universes, where $d \in \mathbb{N}$. Detection of long lived cosmic relics would be a macroscopic hallmark signature for Lorentz violation in subatomic field theories.

The dichotomy between large and small is implicit in all three major topics in this thesis. We study mechanisms and phenomenology, related to extra-dimensional domain-wall brane models, which lead us to:

- Construct a domain-wall brane model, with realistic standard model phenomenology on a 3 + 1-dimensional brane.
- Investigate Lifshitz field theories, and use them to create renormalizable 4+1-dimensional Yang-Mills gauge theories.
- Domain-wall brane models feature large grand unified theories (GUT). We use the crystallographic root systems for Lie algebras to understanding all the embeddings of the standard model within these GUTs.

While investigating all three topics, we would like to discover if there is a consistent description of microscopic physics when we alter the macroscopic properties of the universe.

We take a bottom up approach to probing physics beyond the standard model by constructing effective field theory models and investigating their phenomenology. The emphasis is not on constructing a comprehensive fundamental theory, rather we are interested in new approaches to field theory. These approaches arise from solving problems in extra-dimensional field theories and gravity.

Because this thesis encompasses three different topics we briefly review each of them, so that we have placed each in context. Then we spend a paragraph outlining each chapter of this thesis.

1.1 Extra dimensions

Extra dimensional models are based on $4 + n$-dimensional coordinate spaces. They can be broadly classified as either universal extra-dimensional models, which usually feature Planck length sized extra-dimensions, or large extra-dimensional models which incorporate a $3+1$-dimensional brane. In large extra-dimensional models the standard $3+1$-dimensional universe lives on the brane. The complementary space is called the bulk. Typically a mechanism for recovering $3+1$-dimensional Newtonian gravity is introduced and key components for a standard model gauge field theory are localized to the brane. The localization of these fields and the origin of the brane may be assumed a priori as part of the original set up of the model. Alternatively it may be implemented through a dynamical localization scheme, where at low energies the effective dynamics of the $4 + n$-dimensional fields resembles $3 + 1$-dimensional standard model phenomenology on the brane.

There are many motivations for studying extra dimensions. They have been used to create grand-unified theories [Kal21, Kle26] and as a means of cancelling conformal anomalies in string theories.

Recently it has been shown that extra dimensions provide a natural solution to the hierarchy problem. This problem occurs because quantum corrections to the Higgs self-energy diagram predict the Higgs mass to be at the Planck scale. However problems with unitarity in W-boson scattering require the Higgs mass to be below 1 tera electron volt (TeV). This in turn requires the self-energy quantum corrections to be cancelled. Cancelling these quantum corrections involves severe fine tuning, see [RS99b, RS99a].

Extra dimensions are now a leading candidate for extensions to the standard model of particle physics. We will review the development of this field in Chapter 3.

1.2 Lifshitz field theory

In conjunction with extra dimensions we investigate replacing Lorentz invariance in the standard model action with a Lifshitz anisotropic scaling symmetry.

Lifshitz anisotropy is a property of many physical systems. For example, the phase diagram for magnetic materials have a Lifshitz critical point. At this point the material undergoes a transition from a disordered high temperature paramagnetic phase or helicoidal phase, to a low temperature ordered ferromagnetic configuration. In the helicoidal phase the system is isotropic and the magnetic spins are unaligned; in the ferromagnetic phase the system is anisotropic [KS85, Hen02].

More generally consider an anisotropic system in d-dimensions whose free energy is a function of the order parameter $\vec{M}$:

$$F(\vec{M}) = a_2' t M^2 + a_4 M^4 + c_\beta (\nabla_\beta \vec{M})^2 + c_\alpha (\nabla_\alpha^2 \vec{M})^2 + c_\gamma (\nabla_\alpha^2 \vec{M})^2 (\nabla_\beta^2 \vec{M}) \tag{1.1}$$

where $\vec{M}$ is an n component order parameter and t is the reduced temperature. We have introduced a gradient operator ∇_α which acts on the first m coordinates of $\vec{M}$ and a second gradient operator for the remaining $d - m$ coordinates ∇_β. This system exhibits quadratic derivative terms only in the last $d - m$ coordinate directions. Here the Lifshitz point is the critical point where classical behaviour onsets [HLS75].

Recently Hořava attempted to solve the quantum gravity problem [Hor09, HMT10], by replacing Lorentz invariance in the Einstein Hilbert action with a Lifshitz anisotropic scaling symmetry in the ultraviolet (UV) regime. The anisotropy in Hořava's theory refers to invariance of the action under a coordinate rescaling, which is parameterized by the critical exponent z as follows,

$$x \to bx \qquad\qquad t \to b^z t \tag{1.2}$$

In the action, this anisotropic rescaling symmetry results in $2z$ leading order spatial derivative of the spatial-metric tensor and quadratic leading order time derivatives. The theory is power counting renormalizable and Hořava, [Hor09], argues that $z = 1$ is an infrared (IR) fixed point of the model. This causes Lorentz symmetry to manifest at low energies as an "accidental symmetry".

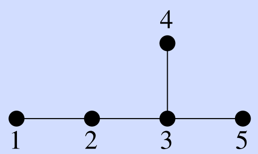
Lifshitz anisotropic scaling invariance (1.2), simultaneously increases the range of power counting renormalization in quantum field theories. The key idea of Lifshitz field theories, in this context, is that by explicitly breaking Lorentz invariance, it is possible to write down an action with higher-order spatial derivatives while maintaining quadratic time derivatives and thus the unitarity of the theory. The UV properties of the radiative corrections in the theory are softened by the higher inverse powers of momentum appearing in the propagator. One interesting application is to field-theoretic models featuring extra dimensions of space. It is well-understood that the UV behaviour degrades as the number of dimensions is increased. Even increasing the spatial dimensions from three to four causes Yang-Mills theory to be nonrenormalizable. Thus, Lorentz-invariant field-theory models involving extra dimensions are inherently UV incomplete, and must be defined with a UV cutoff. But extra-dimensional Lifshitz theories can be UV complete.

Furthermore Lifshitz field theories violate Derrick's theorem [Der64], allowing stable $3+1$-dimensional solitons.

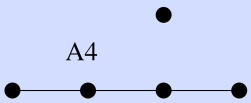
1.3 Spontaneous symmetry breaking, and crystallographic root systems

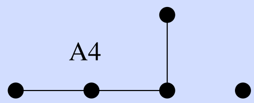
Embeddings of $\mathrm{SU}(5)\times\mathrm{U}(1)$ **inside SO(10)**:

Consider the Dynkin diagrams for the root system D5 for SO(10), where each node has been labelled by one of the 5 simple root $(\xi^{(1)},\ldots,\xi^{(5)})$:



There are two ways to embed the SU(5) root system A4 inside D5. These correspond to the two choices for embedding $\mathrm{SU}(5)\times\mathrm{U}(1)$ inside SO(10). We shall call these two choices standard SU(5), which we will refer to as $\mathrm{SU}(5)_s\times\mathrm{U}(1)_{\mathrm{X}_s}$, and flipped SU(5), correspondingly referred to as $\mathrm{SU}(5)_f\times\mathrm{U}(1)_{\mathrm{X}_f}$. It is conventional to choose flipped $\mathrm{SU}(5)_f$ to be the root system diagram on the left. The diagram on the right is standard $\mathrm{SU}(5)_s$.





In standard $\mathrm{SU}(5)_s$ theories we break $\mathrm{SU}(5)_s\times\mathrm{U}(1)_{\mathrm{X}_s}\to[\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)_{\mathrm{Y}}]\times\mathrm{U}(1)_{\mathrm{X}_s}$. Here the root system for the standard model Lie algebra is generated entirely by the simple roots $(\xi^{(1)},\xi^{(2)},\xi^{(3)},\xi^{(4)})$. This means the standard model subgroup $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)_{\mathrm{Y}}\subset\mathrm{SU}(5)_s$. The remaining Cartan subalgebra generator X_s is a pull back of the simple root $\xi^{(5)}$.

In flipped $\mathrm{SU}(5)_f$ models we break $\mathrm{SU}(5)_f\times\mathrm{U}(1)_{\mathrm{X}_f}\to[\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)_{\mathrm{Y}_f}]\times\mathrm{U}(1)_{\mathrm{X}_f}$. Here $\mathrm{U}(1)_{\mathrm{Y}_f}\not\subset\mathrm{SU}(5)_s$, instead the Cartan subalgebra generator Y_f is now the linear combination of X_s and Y_s, which is linearly independent from X_f (conventionally we choose X_f and Y_f to be mutually orthogonal under the matrix trace).

Finally our work encompasses a study of spontaneous symmetry breaking (SSB). In particular we are interested in generalizations of the flipped SU(5) scenario [Bar82, DKN84].

This scenario arises in SO(10) grand unified theories. Here there are two possible choices for embedding SU(5)×U(1) inside SO(10). One is called standard $SU(5)_s$ the other is called flipped $SU(5)_f$.

We study extensions of this pattern to embeddings of subgroups within extremely large grand unified theories, like embeddings of SU(5) within E_6

$$E_6 \to SO(10) \times U(1)'' \to SU(5) \times U(1)' \times U(1)''. \tag{1.3}$$

There are three possible ways of embedding SU(5) within E_6. We can talk about each embedding in terms of the corresponding candidate weak hypercharge generator. We call three Cartan preserving embeddings standard, flipped, or double-flipped. For convenience we name the hypercharge generators in the same way. Standard hypercharge is a generator of SU(5). The flipped hypercharge is a linear combination of standard hypercharge and the $U(1)'$ generator, while the double-flipped hypercharge also involves an admixture of the generator of $U(1)''$. These different embeddings are related by a Weyl group symmetry.

Understanding the relationship between these different embeddings is of importance to extra-dimensional models because the "clash of symmetries" mechanism (a generalization of the Dvali-Shifman mechanism, which creates a domain-wall brane and confines gauge fields to the brane) requires that at opposite extremes of the extra-dimension a grand unified symmetry, G, must be broken to two differently embedded isomorphic copies of a subgroup H. These issues will be elaborated on in Chapter 9.

1.4 Content of chapters of this thesis

In Chapter 4 of this thesis we construct a domain-wall brane model based on the grand-unification group SO(10). Our model is a classical field theory which physically motivates the origin of the brane and uses well known dynamical localization mechanisms to explain why the field theory appears 3 + 1-dimensional. Our model is an extension of earlier work by Davies et. al. [DGV08], to a larger grand-unification group. We are interested in SO(10) based GUTs because they allow us to incorporate all the fundamental fermions in any generation into a single representation.

The domain-wall brane is provided by a topological defect solitary wave. We invoke the Dvali-Shifman mechanism to dynamically localize gauge bosons on the wall. As part of the implementation of this mechanism an adjoint Higgs field is introduced which spontaneously breaks the SO(10) symmetry inside the wall. We present two scenarios: in the first, the unbroken subgroup inside the wall is SU(5) × U(1), and in the second it is the left-right symmetry group $SU(3) \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$. In both cases we demonstrate that the phenomenologically-correct fermion zero modes can be localized to the wall, and we briefly discuss how the symmetry-breaking dynamics may be extended to induce breaking to the standard model group with subsequent electroweak breaking. Dynamically localized gravity is realized through the type 2 Randall-Sundrum mechanism.

We analyze the stability of this model in Chapter 5. In particular, we consider the field configurations of the topological defect domain-wall brane and the adjoint Higgs field which condenses inside the domain-wall to break SO(10) to SU(5) × U(1). The profiles of these fields are found by solving the dynamical equations of motion. The interaction terms in the dynamical equations contain free parameters, which carry mass dimensions and will therefore run with energy. For the domain-wall brane and adjoint Higgs field profiles given in (4.3) we show there exists an open set in the free parameter space for which the adjoint Higgs field and domain-wall-brane solutions are stable against bifurcation. This is a non-trivial problem: we establish these solutions are not always stable and will bifurcate if the running coupling constants meet a certain hypersurface of the parameters. We use Fredholm theory for compact linear operators combined with the Lyapunov-Schmidt method to prove our results.

In Chapter 6 we examine a fundamental problem with extra-dimensional domain-wall brane models. This problem is the non-renormalizability of Yang-Mills gauge theory in greater than 3 + 1-dimensions. This implies domain-wall brane models can only be effective up to an ultraviolet cutoff, where a putatively UV complete theory must take over. The Dvali-Shifman mechanism for localizing gauge fields to the domain-wall brane requires a 4 + 1-dimensional gauge theory in a confinement phase with a mass gap. With domain-wall brane models based on conventional Yang-Mills gauge theory, we must conjecture that the Dvali-Shifman mechanism is effective and can not rigourously establish the validity of our model.

We directly tackle this problem by analyzing whether or not Lifshitz field theories in 4 + 1-dimensions may provide ultraviolet-complete domain-wall brane models. We first show that Lifshitz scalar field theory can admit topological domain-wall solutions. A Lifshitz fermion field is then added to the toy model, and we demonstrate that 3+1-dimensional Kaluza-Klein zero mode solutions do *not* exist when the four spatial dimensions are treated isotropically. In addition to the anisotropy between space and time, we must postulate breaking full 4-dimensional rotational symmetry down to the subgroup of rotations which mix the usual 3-dimensional spatial directions and fix the extra-dimensional axis. This allows us to recover 3 + 1-dimensional chiral fermions dynamically localized to the domain-wall.

In Chapter 7 we consider the stability of the Lifshitz domain-wall brane (6.5) presented in Chapter 6. Here we establish results for the dynamical evolution of classes of perturbations about (6.5). We do not present a comprehensive analysis along the lines of Chapter 5. Instead we concentrate on reviewing Derrick's theorem and explaining why it does not apply to Lifshitz models. It is a corollary of Derrick's theorem that Lorentz-invariant scalar field theories in 3 + 1-dimensions, and higher dimensions, are unable to support solitons. However Derrick's theorem does not apply to Lifshitz field theories, because they introduce additional derivatives.

Chapter 8 deals with static solitons. In 1 + 1-dimensional Lorentz invariant scalar field theories, static solitons are stable, finite energy, solutions to the $\lambda\phi^4$ second order differential equation. Here we point out and capitalize on one of the key differences between Lifshitz field theories and standard Lorentz invariant ones. Namely, we use the dichotomy from Chapter 7. In 3 + 1-dimensional relativistic field theories, finite energy, static solitons are excluded by Derrick's theorem. However 3 + 1-dimensional Lifshitz field theories with dynamical critical exponent $z = 2$ can support static solitons.

For a 3 + 1-dimensional Lifshitz field theory with dynamical critical exponent $z = 2$, we exhibit three generic types of solitons: non-topological point defects, topological point defects, and topological strings. We focus mainly on Lifshitz theories that are defined through a superpotential and admit BPS solutions. If nature obeys a Lifshitz field theory in the ultraviolet, as would be compatible with the Hořava-Lifshitz quantum point model for gravity [Hor09], then these novel topological defects may exist as relics from the early universe.

Their discovery would prove that relativistically invariant field theories break down at short distance scales.

In Chapter 9 we present a systematic approach to writing adjoint Higgs vacuum expectation values (vevs), which break a symmetry G to differently embedded isomorphic copies of a subgroup belonging to the chain $G \supset H_1 \supset \cdots \supset H_l$, as linear combinations of each other. Given an adjoint Higgs vacuum expectation value h breaking $G \to H$, a full complement of vevs breaking G to different embeddings of the subgroup H can be generated through the Weyl group orbit of h. An explicit formula for recovering each vev is given. We focus on the case when H stabilizes the highest weight of the lowest dimensional fundamental representation, where the formula is exceedingly simple. We also discuss cases when the Higgs field is not in the adjoint representation and apply these techniques to current research problems, especially in domain-wall brane model building.

2

Notation

We discuss our notational conventions, in the following order: conventions relating to space-time symmetries followed by internal symmetries.

We work with d-dimensional space-time manifolds. When we wish to refer to the standard space-time manifold with 3-spatial dimensions and 1-temporal dimension we shall use the term 3 + 1-dimensional. In the case $d > 4$ if we want to emphasis that the space-time manifold has n additional dimensions we will write $4 + n$-dimensional.

The lower case Latin letters will run over spatial coordinates so that $x^i \equiv (x^1, x^2, x^3, x^5, \ldots, x^{4+n})$. Upper case Latin letters will index $4 + n$-dimensional space-time coordinates so that $x^M \equiv (t, x^i)$. In particular we use vector notation for 3-dimensional coordinate vectors, $\vec{x} = (x^1, x^2, x^3)$.

Following standard conventions, the lower case Greek letters μ and ν are reserved for 3 + 1-dimensional 4-vectors so that $x^\mu \equiv (t, x^1, x^2, x^3)$.

The $4 + n$-dimensional metric tensor will be called g^{MN}; however we adopt the conventional notation $g^{\mu\nu}$ for the special 3 + 1-dimensional $n = 0$ case. In both scenarios we adopt the metric signature $(+, -, \ldots, -)$. In many examples we will dimensionally reduce a $4 + n$-dimensional theory to derive the effective 3 + 1-dimensional metric or Einstein Hilbert action. In these cases we will often label the $4 + n$-dimensional metric $\tilde{g}^{MN}$ and contractions of this metric like the Ricci scalar $\tilde{R}$ or metric determinant $\tilde{g}$.

We will occasionally need a metric g_{ij} which can be used to independently raise and lower the spatial coordinates.

The symbol D will stand for a gauge covariant, or gravitationally covariant derivative; this will be clear from context. We will generally use the standard symbol $\Delta \equiv g_{ij}\partial^i\partial^j$.

In 4 + 1-dimensional theories we will reserve several Greek letters to label specific fields. Namely the function $\phi(x^M)$ labels a real scalar while $\Psi(x^M)$ refers to the 4 + 1-dimensional fermion which we will mode expand in a tower of fields $\psi_n(x^\mu)$ corresponding to 3 + 1-dimensional massive Dirac particles. In domain-wall brane models we will always use $\phi(x^M)$ to label the real scalar field which condenses to form the brane. We distinguish the extra-dimensional coordinate with the syntax $x^5 \equiv y$ and by $C(y)$ we mean the space of continuous bounded functions with respect to the y-coordinate space.

In the internal symmetry space, we choose to work exclusively with diagonal Cartan subalgebra generators. This can be done without loss of generality because given an arbitrary Cartan subalgebra it is always possible to simultaneously diagonalize each member using a similarity transformation within the Lie algebra. If the Lie algebra has rank l we choose h^i where $i \in \{1, \ldots, l\}$ to refer to our basis for the Cartan subalgebra.

When working with general results from the theory of Lie algebra, we physically contextualize our analysis using QCD and the weak force as examples. To do this we choose explicit representations. In each case we make use of the adjoint representation and the lowest dimensional fundamental representation, otherwise known as the smallest faithful representation.

In QCD for the 3 representation of SU(3) we use the Gell-Mann matrices $\lambda_1, \ldots, \lambda_8$ as generators. We refer to the gluons as a set of 8 Lorentz 4-vector fields G^i_μ where $i \in \{1, \ldots, 8\}$ distributed over the Gell-Mann matrices; we write $X^\mu_i = G^\mu_i \lambda_i$ where there is no intended sum over i. We also make use of the linear combinations of the off diagonal gluons:

$$Z^1_\mu = \frac{1}{\sqrt{2}}(X^1_\mu + iX^2_\mu), \qquad Z^2_\mu = \frac{1}{\sqrt{2}}(X^4_\mu - iX^5_\mu), \qquad Z^3_\mu = \frac{1}{\sqrt{2}}(X^6_\mu + iX^7_\mu). \tag{2.1}$$

Correspondingly we take linear combinations of the two diagonal gluons, renamed for notational convenience $X^3_\mu = A^1_\mu$ and $X^8_\mu = A^2_\mu$,

$$B^p_\mu = A^i_\mu \alpha^p_i, \tag{2.2}$$

where $p \in \{1, 2, 3\}$ and the α^p_i vectors are the three roots $\alpha^1_i = (1, 0)$, $\alpha^2_i = (1/2, \sqrt{3}/2)$, $\alpha^3_i = (-1/2, \sqrt{3}/2)$. In keeping with this notation we use a relabelling of the diagonal Gell-Mann matrices $\lambda_3 = A^1$ and $\lambda_8 = A^2$ to define the SU(3) Lie algebra generators $\kappa = A^i\alpha^1_i, \rho = A^i\alpha^2_i$ and $\epsilon = A^i\alpha^3_i$ associated respectively with B^1_μ, B^2_μ and B^3_μ. We also give rather unimaginative names to the Lie algebra generators associated with the valence gluons Z^1_μ, Z^2_μ and Z^3_μ:

$$Z^1 = \lambda_1 + i\lambda_2, \qquad Z^{-1} = \lambda_1 - i\lambda_2, \tag{2.3}$$

$$Z^2 = \lambda_4 + i\lambda_5, \qquad Z^{-2} = \lambda_4 - i\lambda_5, \tag{2.4}$$

$$Z^3 = \lambda_6 + i\lambda_7, \qquad Z^{-3} = \lambda_6 - i\lambda_7. \tag{2.5}$$

Notice these are precisely the raising and lowering operators of the SU(3) Lie algebra.

Extending the SU(3) example we will refer to the I-spin, V-spin and U-spin directions in colour space, which describe the three Cartan preserving embeddings of SU(2) inside SU(3). These are the three embeddings which have the Cartan subalgebra generators for SU(2) as a subset of the Cartan generators for SU(3). In terms of the Gell-Mann matrices, the generators of the SU(2) subgroup in each case are

$$\begin{gathered}
\underbrace{\lambda_1, \lambda_2, \kappa}_{\text{2(I-spin)}}, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \\
\lambda_1, \lambda_2, \lambda_4, \lambda_5, \underbrace{\epsilon, \lambda_6, \lambda_7}_{\text{2(V-spin)}}, \epsilon' \\
\lambda_1 \lambda_2, \underbrace{\rho, \lambda_4, \lambda_5}_{\text{2(U-spin)}}, \lambda_6, \lambda_7, \rho'
\end{gathered} \tag{2.6}$$

where we have chosen to introduce complementary matrices to the ρ and ϵ, namely $\rho' = \sqrt{3}/2A^1 + A^2$ and $\epsilon' = -\sqrt{3}/2A^1 + 1/2A^2$ respectively, so that each set of Lie algebra generators contains a diagonal Cartan subalgebra, which is orthogonal under the matrix trace.

In our weak force examples we use the Pauli matrices $\{\tau_1, \tau_2, \tau_3\}$ as a vector space basis for the adjoint representation (note τ_3 is the diagonal generator of the weak isospin gauge group, I_2, in this representation) and the standard notation for the three gauge bosons $W^1_\mu = w^1_\mu \tau^1, W^2_\mu = w^2_\mu \tau^2, W^3_\mu = w^3_\mu \tau^3$.

Analogously to the QCD case we consider linear combinations of the weak force gauge bosons $W^+_\mu = w^+_\mu \tau^+ = W^1_\mu - iW^2_\mu,\ W^-_\mu = w^-_\mu \tau^- = W^1_\mu + iW^2_\mu$ and $W^0_\mu = w^3_\mu \tau^3 = W^3_\mu$, and the corresponding generators $\tau^+ = \tau_1 - i\tau_2,\ \tau^- = \tau_1 + i\tau_2$ and $\tau_0 = \tau_3$. We use this notation because +1, -1 and 0 are the respective $U(1)_Q$ quantum numbers or electric charges of these linear combinations.

The adjoint action of the Lie algebra $\mathrm{ad}_{h^i} \cdot E^\alpha$ on itself is defined by $\mathrm{ad}_{h^i} \cdot E^\alpha = [h^i, E^\alpha]$. In the special cases where the E^α are eigenvectors under the adjoint operation for some h^i we write $[h^i, E^\alpha] = \alpha(h^i)E^\alpha$

We say a linear transformation stabilizes a point if it maps that point back onto itself. For example if $|\lambda\rangle$ is an eigenvector of a Lie algebra generator $t_k \in \mathcal{L}$, so that $t^k \cdot |\lambda\rangle = \lambda(t^k)\,|\lambda\rangle$, then we say $|\lambda\rangle$ is stabilized by t_k.

3
Introduction

We will begin with a broad review of extra-dimensional models. This will be broken down into major topics including a discussion of the progenitive extra-dimensional theory developed by Theadore Kaluza and Oscar Klein. One of the key features of this model is the Kaluza-Klein (KK) mode reduction of a 4 + 1-dimensional field into a tower of 3 + 1-dimensional KK modes or resonances. In section 3.2 we proceed to large extra-dimensional models. Arkani Hamed, Dimopoulos and Dvali (ADD) proposed the foundational model in this area which demonstrated extra dimensions can be used to solve the hierarchy problem. The two seminal models from Randall and Sundrum (RS1 and RS2) are covered in section 3.3. Importantly the RS2 type model is the basis of Chapter 4 of this thesis.

We discuss canonical topological defects in section 3.4. In particular we are interested in domain-walls. A proof of the stability of real scalar field topological defects in 1 + 1-dimensions, respectively instability in higher dimensions, is given through homotopy arguments, respectively Derrick's theorem.

Sections 3.5-3.6 of our review will discuss how particles become localized to the domain-wall brane. We examine Rubakov and Shaposhnikov's fermion zero mode method for localizing fermions to a topological defect [RS83]. This picture is complemented with a discussion of the Dvali-Shifman mechanism for gauge field localization on a topological defect domain-wall brane. We end our discussion of localizing fields to the domain-wall brane by explaining how to extend the Dvali-Shifman mechanism to trap standard model fields using large grand unified groups. In conjunction with this we discuss the potential problems with applying the Dvali-Shifman mechanism to 4 + 1-dimensional field theories.

Section 3.7 and subsection 3.7.4 introduce grand unified theories, and spontaneous symmetry breaking.

To aid with building a renormalizable 4 + 1-dimensions Yang Mills gauge theory section 3.8 introduced Lifshitz anisotropic scaling and field theories. Lifshitz field theories will form the foundation for Chapters 6 and 8.

We conclude the introduction with a discussion of roots and weights, along with the Weyl group. These will be useful tools for studying the grand unified theories we have introduced.

3.1 Kaluza-Klein theory

3.1.1 Kaluza's original idea

Extra-dimensions were first introduced in the 1920s by Oscar Kaluza and Theodore Klein. Kaluza and Klein realized extra-dimensions could be used to unify Maxwell's theory of electromagnetism and Einstein's general relativity. To achieve this Kaluza proposed extending the metric, so that the extended Lorentz indices, M, N, ran over a 4 + 1-dimensional co-ordinate space (x^μ, y), where μ is the normal 3 + 1-dimensional Lorentz index. The extended metric tensor is:

$$\widetilde{g}_{MN} = \begin{pmatrix} g_{\mu\nu} + \kappa^2\phi^2 A_\mu A_\nu & \kappa^2\phi^2 A_\mu \\ \kappa^2\phi^2 A_\nu & \phi^2 \end{pmatrix} \tag{3.1}$$

where $g_{\mu\nu}$ is a rank two tensor under 3 + 1-dimensional spatial isometries, A_μ transforms like a four vector of fields under the 3 + 1-dimensional Lorentz group and ϕ is a 3 + 1-dimensional Lorentz scalar.

Kaluza imposed the "cylindrical condition": None of the metric components depend on the 5-th co-ordinate y. The 4 + 1-dimensional Einstein-Hilbert action for the theory is the same as the 3 + 1-dimensional Einstein-Hilbert action where the Ricci scalar and metric determinant are calculated from (3.1) under the assumption that all the derivatives with respect to y vanish. This gives:

$$S = -\int d^4x\,\sqrt{g}\phi\left(\frac{R}{16\pi G} + \frac{1}{4}\phi^2 F_{\mu\nu}F^{\mu\nu} + \frac{2}{3\kappa^2}\frac{\partial^\mu\phi\partial_\mu\phi}{\phi^2}\right), \tag{3.2}$$

where we have chosen to match the free parameter κ to the gravitational constant $\kappa = 4\sqrt{\pi G}$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the stress energy tensor for A^μ.

There is a limiting case of the action (3.2) where the roles of the various metric components become clearly identifiable. If we choose to suppress the scalar field dynamics by requiring $\phi = 1$, then (3.2) becomes the standard Einstein-Hilbert action for a 3 + 1-dimensional theory of electromagnetism and gravity. Using the principle of stationary action we can derive Einstein's equation for the 3 + 1-dimensional metric tensor with the electromagnetic energy momentum tensor, $T_{\mu\nu} = g_{\mu\nu}F_{\lambda\delta}F^{\delta\lambda}/4 - F_{\mu\lambda}F^\lambda_\nu$, as a source, and a sourceless 3 + 1 dimensional Maxwell equation for the photon [OW97]:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \qquad \nabla^\mu F_{\mu\nu} = 0. \tag{3.3}$$

3.1.2 Compactification

Kaluza's theory introduces new problems, namely how do we hide the extra dimension and why impose such an artificial "cylindrical condition"? Wouldn't it be more natural for the fields to have a nontrivial profile along the extra dimension? This inspired Klein to propose that extra dimensions will be undetectable if they are compactified, or quotiented out by a relation $y \sim y + 2\pi R$ where the compactification radius R is usually taken to be of order the Planck length $l_{Pl} = 10^{-33}$ cm. In this topology each metric component will be periodic with respect to the 5-th co-ordinate y. This means that we can expand each

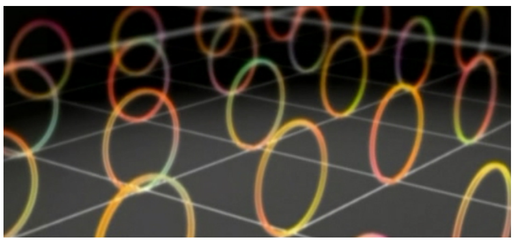


Figure 3.1: You can think about a compactified extra dimension, as attaching a cylinder to each point in space (picture from Brian Green's, The Elegant Universe).

field into a sum over normal modes, or a KK tower:

$$g_{\mu\nu}(x,y) = \sum_{n=-\infty}^{\infty} g_{\mu\nu n}(x) e^{inmy}$$
$$A_{\mu}(x,y) = \sum_{n=-\infty}^{\infty} A_{\mu 5n}(x) e^{inmy}$$
$$\phi(x,y) = \sum_{n=-\infty}^{\infty} \phi_{n}(x) e^{inmy}, \tag{3.4}$$

with $g^{*}_{\mu\nu n}(x) = g_{\mu\nu -n}(x)$ and $m = 2\pi/R$.

This approach has two major advantages. We can start with 4 + 1-dimensional fields and clearly identify the 3 + 1-dimensional particles, which are the coefficients in the expansion (3.4). Furthermore substituting (3.4) directly into the action and dimensionally reducing the theory will allow us to extract the dynamics of the 3 + 1-dimensional particles in the theory. To wit each KK tower contains a zero mode, corresponding to the coefficient of the $n = 0$ term, which does not depend on y. If we ignore the other modes then the Einstein-Hilbert action for this zero mode will be (3.2).

The $n \neq 0$ modes become a distinctive signature for the extra-dimension. These extra modes give rise to massive gravitons $g_{\mu\nu n}(x)$. To see this [Duf94] considers the general co-ordinate parameter, $\widetilde{\xi}^{M}(x,y)$, which parameterizes the symmetry of the 4 + 1-dimensional Einstein-Hilbert action under co-ordinate transformation leading to perturbations in the 4 + 1-dimensional metric

$$\delta\widetilde{g}_{MN} = \partial_{M}\widetilde{\xi}^{L}\widetilde{g}_{LN} + \partial_{N}\widetilde{\xi}^{L}\widetilde{g}_{LM} + \widetilde{\xi}^{L}\partial_{L}\widetilde{g}_{MN}. \tag{3.5}$$

Fourier expanding this general co-ordinate parameter $\widetilde{\xi}^{M}(x,y)$ gives

$$\widetilde{\xi}^{\mu}(x,y) = \sum_{n=-\infty}^{\infty} \xi^{\mu}_{n}(x) e^{inmy}$$
$$\widetilde{\xi}^{4}(x,y) = \sum_{n=-\infty}^{\infty} \xi^{4}_{n}(x) e^{inmy} \tag{3.6}$$

and allowing for $\widetilde{\xi}^{*\mu}_{n} = \widetilde{\xi}^{\mu}_{-n}$, shows that we have an infinite parameter family of local transformations. There is a corresponding infinite parameter global algebra which is the Kac-

Moody-Virasoro generalization of the Poincarè algebra, generated by

$$
\begin{aligned}
P_n^\mu &= e^{inmy}\partial^\mu \\
M_n^{\mu\nu} &= e^{inmy}\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right) \\
Q_n &= ie^{inmy}\partial_{my}.
\end{aligned}
\tag{3.7}
$$

However when we substitute the zero mode expressions into the $4+1$-dimensional Einstein-Hilbert action we find that the vacuum of (3.2) only respects $3+1$-dimensional Poincarè invariance cross producted with a U(1) symmetry arising from the extra-dimension. Consequently when $n \neq 0$ in (3.6) the local gauge parameters ξ_n^μ and ξ_n^4 correspond to spontaneously broken generators. The higher KK modes $A_{\mu n}(x)$ and $\phi_n(x)$ become the Goldstone bosons associated with this broken symmetry, which the spin-2 gravitons $g_{\mu\nu n}(x)$ eat to become massive modes (thereby acquiring 5 degrees of freedom) [Duf94].

3.2 Large extra dimensions and the ADD model

The prevailing wisdom was that extra-dimensions had to be of order the Planck scale. Csáki [Csa04] gives a natural explanation for this. Imagine there are n extra compact dimensions all of the same length $2\pi R$. We call the volume of the extra dimensions $V_n = (2\pi R)^n$. Let the $4+n$-dimensional line element correspond to a product topology of $3+1$-dimensional Minkowski space-time with S^n (solid angle parameters $\Omega_{(n)}$):

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu - R^2 d\Omega_{(n)}^2. \tag{3.8}$$

The metric signature is $+----$ and the $4+n$-dimensional coordinate space is $(x^\mu, R\theta^n)$, where Θ^n is a n-dimensional vector of angles parameterizing the location in the toroidal compactified extra dimensions. Then take the $4+n$-dimensional Einstein-Hilbert action and carefully include a factor of the fundamental gravitational scale M_*, to the power $n+2$ to ensure the action is dimensionless. If we integrate over the extra dimensions we get

$$S^{4+n} = -M_*^{n+2}\int d^{4+n}x\,\sqrt{-g^{(4+n)}}R^{(4+n)} \supset -M_*^{n+2}V_n\int d^4x\,\sqrt{-g}R. \tag{3.9}$$

This last term can be identified with the effective 3 + 1-dimensional Einstein-Hilbert action, leading us to conclude that the fundamental gravitational scale must be related to the Planck scale by

$$M_{Pl} = M_*^n V_n \sim M_*^n R^n. \tag{3.10}$$

Correspondingly in the matter sector of the action consider a 4 + n-dimensional scalar field with action

$$S^{4+n} = \int d^{4+n}x \, \left(\partial_M \phi \partial^M \phi + m^2 \phi^2 - \lambda \phi^4\right). \tag{3.11}$$

If the scalar field is canonically normalized then it has mass dimensions $[\phi] = 1 + \frac{n}{2}$, and therefore the coupling constants m^2 and λ are dimensionful whenever $n \geq 1$. In particular the ϕ^4 interaction term should be suppressed by a mass parameter. The natural choice for this parameter is the cutoff for the theory M_*. It is convenient to rewrite the ϕ^4 interaction term as $\frac{\lambda}{M_*^n}\phi^4$.

However if we substitute a KK tower expansion for ϕ,

$$\phi(x^\mu, y) = \sum_{n=-\infty}^{\infty} \phi_n(x) e^{inmy} \tag{3.12}$$

into (3.11) and dimensionally reduce the action (that is integrate over y to find the effective 3 + 1-dimensional action) then we find that all KK modes $\phi_n(x)$ have mass dimension 1. Therefore in the effective 3 + 1-dimensional field theory there will be an interaction term between KK modes of the form

$$\lambda_{eff} \, \phi_n \phi_k \phi_m \phi_{-n-k-m}, \tag{3.13}$$

where the effective 3 + 1-dimensional coupling constant $\lambda_{eff} = (2\pi R)^n \lambda / M_*^n \sim R^n / M_*^n$. Analogously in the gauge sector where the kinetic term is

$$S^{4+n} = -\int d^{4+n}x \frac{1}{4g^2/M_*^n} F_{MN} F^{MN} \sqrt{g^{(4+n)}}, \tag{3.14}$$

integrating over the extra dimensions gives

$$S = -\int d^4x \frac{V_n}{4g^2/M_*^n} F_{\mu\nu} F^{\mu\nu} \sqrt{g}. \tag{3.15}$$

Hence the 3 + 1-dimensional effective gauge coupling

$$\frac{1}{g_{eff}^2} \sim R^n M_*^n. \tag{3.16}$$

Combining (3.16) and (3.10) we find that the natural size [Csa04] for the extra-dimension is

$$R \sim \frac{1}{M_{Pl}} g_{eff}^{\frac{n+2}{n}}. \tag{3.17}$$

In other words the size of the extra dimension is the same order of magnitude as the Planck length. In this scenario at energies $E < M_{Pl}$, the effective 3+1-dimensional interaction terms in (3.13) are highly suppressed, even amongst 3 + 1-dimensional zero modes. Furthermore there is no hope of probing these microscopic extra dimensions with collider experiments.

This motivated the idea of brane world models with large compact extra dimensions. These are compact extra dimensional models where only gravity is able to propagate in all 4 + n-dimensions while standard model fields are restricted to a 3 + 1-dimensional slice of the extra dimension known as a brane.

Arkani Hamed, Dimopoulos and Dvali proposed using brane world models with large compact extra dimensions to solve the hierarchy problem [AHDD98]. Here we are referring to the discrepancy between the electroweak symmetry breaking scale, M_{EW}, or Higgs mass and the Planck scale. From one loop corrections to the Higgs potential, the Higgs mass receives corrections which are quadratic in the cutoff scale,

$$\delta m_H^2 = \frac{1}{8\pi^2}\left(\lambda_H - \lambda_Y^2\right)\Lambda^2 + (\text{log. div.}) + \text{finite terms}, \tag{3.18}$$

where λ_Y is the Yukawa coupling constant, while λ_H is the Higgs self coupling [PL05]. Precision electroweak experiments put the optimal Higgs mass around 100 GeV. If $\Lambda = M_{Pl}$ is used to set the cutoff scale then it becomes necessary to balance the corrections to an unreasonable degree of precision, both at one loop level and at every subsequent order in perturbation theory.

Arkani Hamed, Dimopoulos and Dvali [AHDD98] suggested that if all standard model fields are trapped on a 3 + 1-dimensional brane then the extra dimensions may be quite large (up to the distance at which Newtonian gravity has been tested by Cavendish type experiments $R \sim 0.2\,\text{mm}$). Therefore because the fundamental gravitational scale is related to the Planck scale by (3.10), if there are two or more compact extra dimensions then the fundamental gravitational scale may be comparable to M_{EW} (around 1 TeV) while maintaining $M_{Pl} \sim 10^{16}$ TeV. (With one extra dimension for $M_* \sim 1$ TeV we need the size of the extra dimension to be comparable to the diameter of our solar system. Needless to say this scenario was observationally ruled out during the 17th century.)

3.3 Randall-Sundrum models

Randall and Sundrum introduced two models called RS1 and RS2. The RS1 paradigm solves the hierarchy problem using a compact extra dimension. The RS2 model recovers Newtonian $1/r^2$ gravity in spite of a single infinite extra dimension. However in doing so, it precludes the solution to the hierarchy problem.

3.3.1 The type 1 Randall-Sundrum paradigm

Randall and Sundrum took the alternate approach to solving the hierarchy problem [RS99a]. Instead of bringing the fundamental gravitational scale down to the electroweak symmetry breaking scale, Randall and Sundrum showed that when you take the back reaction of the brane tension on the 4 + 1-dimensional metric into account then the 4 + 1-dimensional Einstein equations have a warped metric solution. After canonically normalizing the Higgs field in this warped background, it is possible for the electroweak symmetry breaking to occur at 10^{18} TeV and still appear to take place at 1 TeV on a 3 + 1-dimensional hyperplane.

Randall and Sundrum look for solutions to the 4 + 1-dimensional Einstein equations for a 4 + 1-dimensional metric $g^{(5)}_{MN}$, which is described by

$$ds^2 = e^{-\sigma(y)}\eta_{\mu\nu}\,dx^\mu\,dx^\nu - dy^2, \tag{3.19}$$

where the signature of the metric is $+----$, and the 4 + 1-dimensional coordinate space is $x^M = (x^\mu, y)$. The key feature of this metric is that it preserves Lorentz invariance on each 3 + 1-dimensional slice of the extra dimension. However unlike (3.8), it is not factorizable.

By this we mean that the coefficient of the infinitesimal variation in the 3 + 1-dimensional coordinate space dx^μ, is proportional to a function of y, known as the warp factor.

The extra dimension is finite. It runs from $y = -L$ to $y = L$ and there is an orbifold $\mathbb{Z}_2$ symmetry $y \leftrightarrow -y$. There is an excellent reason for the orbifolding condition. Consider perturbations around the 4 + 1-dimensional metric (3.19)

$$ds^2 = e^{-\sigma(y)} \left[\eta_{\mu\nu} + h_{\mu\nu}\right] dx^\mu dx^\nu - A_\mu dx^\mu dy - \phi^2 dy^2, \tag{3.20}$$

where A_μ is a graviphoton, ϕ is the radion and $h_{\mu\nu}$ is the 3 + 1-dimensional graviton. Now if we Kaluza-Klein expand these perturbations in the same KK tower as in (3.4), then because ds^2 has to be symmetric under $y \leftrightarrow -y$, A_μ must be an odd function of y. From the form of the Kaluza-Klein tower (3.4) we can see that $A_n(x) = -A_{-n}(x)$ and $A_0(x) = 0$. There is no zero mode graviphoton, and from the Kaluza-Klein example we anticipate the theory will have a zero mode graviton and radion as well a continuum of massive gravitons.

Randall and Sundrum then introduce a positive tension and a negative tension brane, which they refer to as the hidden brane and the visible brane, respectively. We will see that having both branes is necessary for a consistent solution to the Einstein equations. They place each brane at an orbifolding fixed point. Specifically the positive tension (hidden) brane is located at $y = 0$ while the negative tension (visible) brane is at $y = L$. Again this choice is phenomenologically motivated.

In a similar model Gregory, Rubakov and Sibiryakov [GRS00] create a scenario where gravity exhibits 4 + 1-dimensional characteristics at both UV and IR scales but is localized to a brane at intermediary energies. This model necessitates placing the negative tension brane in between the orbifolding fixed points. The brane is no longer fixed at $y = L$ and can oscillate along the y-direction. The radion kinetic term acquires a negative coefficient which is proportional to the brane tension; oscillations in the position of this brane now lead to negative energy problems [PRZ00].

With these specifications the action for the RS model becomes

$$S = \underbrace{-\int \sqrt{g^{(5)}}\, d^4x\, dy \left[\frac{1}{2M_*^{-3}} R^{(5)} + \Lambda\right]}_{S_{gravity}} \underbrace{-V_{hid} \int \sqrt{g^{hid}}\, d^4x}_{S_{hid}} \underbrace{+V_{vis} \int \sqrt{g^{vis}}\, d^4x}_{S_{vis}}, \quad (3.21)$$

where M_* is the fundamental gravitational scale for the 4 + 1-dimensional theory, V_{hid} and V_{vis} are the brane tension or unit energy per three-volume on the visible and hidden branes respectively, $g^{(5)}$ is the determinant of the 4 + 1-dimensional metric tensor, similarly $R^{(5)}$ is the 4 + 1-dimensional Ricci scalar. We use g_{vis} for the determinant of the 3 + 1-dimensional metric tensor on the visible brane and g_{hid} for the hidden brane case, and we have chosen to incorporate a 4 + 1-dimensional cosmological constant Λ.

The Einstein equations can now be derived and solved. These equations must be solved consistently in the region between the two branes, which contains no sources other than the 4 + 1-dimensional cosmological constant, and at the locations of the two branes. This can be seen directly from the righthand side of Einstein's equations [PL05], which give the source terms to the Einstein equations,

$$G_{MN} = -\frac{1}{M_*^3}\Lambda g_{MN} + \frac{V_{hid}}{M_*^3}\sqrt{\frac{g^{hid}}{g^{(5)}}}\delta^{\mu}_{M}\delta^{\nu}_{N} g^{hid}_{\mu\nu}\delta(y) - \frac{V_{vis}}{M_*^3}\sqrt{\frac{g^{vis}}{g^{(5)}}}\delta^{\mu}_{M}\delta^{\nu}_{N} g^{vis}_{\mu\nu}\delta(y-L). \quad (3.22)$$

This must be matched against the Einstein tensor components for (3.19), the warped metric ansatz:

$$G_{\mu\nu} = -3g^{(5)}_{\mu\nu}\left(-\sigma'' + 2(\sigma')^2\right), \qquad G_{\mu 4} = 0, \qquad G_{44} = -6g^{(5)}_{44}(\sigma')^2. \quad (3.23)$$

From the region between the two branes we obtain our first condition on the metric ansatz (3.19), namely $6(\sigma')^2 = \Lambda/M_*^3$, while at the location of each brane we pick up an Israel junction condition. This tells us the derivative of the exponent of the warp factor, σ, is discontinuous at both branes.

$$3\sigma'' = \frac{V_{vis}}{M_*^3}\delta(y) - \frac{V_{hid}}{M_*^3}\delta(y-L). \quad (3.24)$$

In spite of this discontinuity there is a physical solution to the Einstein equations as the metric will still be smooth. In fact the solution to these equations is

$$\sigma(y) = 2k|y|, \qquad k^2 = \frac{\Lambda}{24M_*^3}, \tag{3.25}$$

provided $V_{vis} = V_{hid}$, and that we fine tune the brane tension against the 4 + 1-dimensional cosmological constant

$$\Lambda = \frac{V_{vis}^2}{24M_*^3}. \tag{3.26}$$

Because of the sign of the action (3.21), we have therefore forced the 4 + 1-dimensional space, bounded by the two branes, to be a slice of Anti-de Sitter space-time. This condition occurs because we are asking for a flat Minkowski metric on the 3 + 1-dimensional space spanned by the brane and for the branes to be static. As such we do not want a 3 + 1-dimensional cosmological constant. Fine tuning is a way of exactly cancelling out the 3 + 1-dimensional cosmological constant.

Now Randall and Sundrum a priori assume that matter is localized to the visible brane. The 3 + 1-dimensional metric components at the location of the hidden brane are $\eta_{\mu\nu}$, however on the visible brane they are $e^{-2kL}\eta_{\mu\nu}$. Therefore, given a Higgs field action is of the form

$$S_{vis} \supset S_{Higgs} = \int \sqrt{g^{vis}} d^4x \{g_{vis}^{\mu\nu}\partial_\mu H^\dagger \partial_\nu H - \lambda(|H|^2 - v_0^2)^2\}, \tag{3.27}$$

where v_0 is the mass parameter, the kinetic term inside the integrand is proportional to e^{-2kL}. To obtain a canonically normalized Higgs field it is therefore necessary to rescale $H \to e^{kL}H$, which modifies the action to

$$S_{Higgs} = \int d^4x \{\eta^{\mu\nu}\partial_\mu H^\dagger \partial_\nu H - \lambda(|H|^2 - e^{-2kL}v_0^2)^2\}. \tag{3.28}$$

Hence we can choose the electroweak symmetry breaking scale $v = e^{-kL}v_0$ to be around 1 TeV. We can simultaneously maintain a fundamental mass scale parameter $v_0 = M_{Pl}$, provided we choose $kL \sim 50$.

By contrast, if we ask how the fundamental gravitational scale M_* is related to M_{Pl}, then by integrating over the y coordinate in $S_{gravity}$ and comparing the result to the 3+1-dimensional

Einstein-Hilbert action, we find that

$$M_{Pl}^2 = \frac{M_*^3}{k}\left[1 - e^{-2kL}\right]; \tag{3.29}$$

here we have used the property of the 4 + 1-dimensional Ricci scalar $R^{(5)} \supset e^{2k|y|}R$, that is, $R^{(5)}$ is directly proportion to the effective 3 + 1-dimensional Ricci scalar R.

For our previous choice of kL, we find the 4 + 1-dimensional fundamental scale is also similar to the Planck scale. This eliminates the hierarchy problem between the Planck and electroweak symmetry breaking scales.

3.3.2 The type 2 Randall-Sundrum paradigm

In RS2, Randall and Sundrum extended the RS1 model to incorporate an infinite extra dimension. The key asset of the RS2 model is that in spite of the presence of an infinite extra dimension, there is a localized zero mode graviton on the domain-wall brane. This leads to effective Newtonian gravity on the brane. Furthermore the RS2 model also recovers normal 3 + 1-dimensional general relativistic phenomenology on the brane.

Randall and Sundrum [RS99b] begin by taking the setup from the RS1 model [RS99a], outlined in the previous section, and analyzing the spectrum of general linearized perturbations about the 3 + 1-dimensional metric

$$ds^2 = \left[e^{-2k|y|}\eta_{\mu\nu} + h_{\mu\nu}\right] dx^\mu dx^\nu - dy^2. \tag{3.30}$$

From invariance of the proper distance under the orbifold symmetry, we can see that the gravitons $h_{\mu\nu}$ must be even functions of y. This gives us a boundary condition on the solutions to the linearized Einstein equation.

In addition to these spin 2 gravitons, there is a radion ϕ which corresponds to fluctuations around the $g_{44}^{(5)}$ component of the metric (3.19).

Because there are no source terms in between the branes, in this region the perturbations can be gauge fixed in a transverse traceless gauge: $\partial_\mu h^\mu_\nu = 0$ and $h^\mu_\mu = 0$. This causes all the different components (indexed by different μ and ν) to be the same. For this reason we suppress these indices. Furthermore we make a Kaluza-Klein decomposition of the graviton and rewrite the 4 + 1-dimensional field as a KK tower of 3 + 1-dimensional modes:

$$h(x, y) = \Sigma_m \hat{\psi}_m(y) H^m(x^\mu). \tag{3.31}$$

It is now possible to use separation of variables in the linearized Einstein equations to obtain the simultaneous equations

$$\left[-\frac{m^2}{2}e^{2k|y|} - \frac{1}{2}\partial_y^2 - 2k\delta(y) + 2k^2\right]\hat{\psi}_m = 0 \qquad \text{and} \qquad \partial_\mu\partial^\mu H^m = -m^2 H^m. \tag{3.32}$$

From the above we conclude the 3 + 1-dimensional gravitons $H^m(x^\mu) = e^{ip^m_\mu x^\mu}$ are freely propagating plane waves in the 3 + 1-dimensional coordinate directions with each satisfying a dispersion relation $(p^m)^2 = m^2$, where the value of m^2 is found by solving the corresponding eigenvalue equation for the profile functions $\hat{\psi}_m(y)$.

If we rescale the wave function $\hat{\psi}_m$, and simultaneously shift to conformal coordinates,

$$z = sgn(y)(e^{k|y|} - 1)/k \qquad \text{and} \qquad \hat{\psi}_m(z) = \hat{\psi}_m(y)e^{k|y|/2}, \tag{3.33}$$

then we can turn the differential equation (3.32) for the graviton profile functions into a Schrodinger equation

$$\left[-\frac{1}{2}\partial_z^2 + V(z)\right]\hat{\psi}_m(z) = m^2\hat{\psi}_m(z), \tag{3.34}$$

where the potential,

$$V(z) = \frac{15k^2}{8(k|z| + 1)^2} - \frac{3k}{2}\delta(z), \tag{3.35}$$

is called a volcano potential because its characteristic form has a deep well centred about $y = 0$; this can be seen from Figure 3.2.

There is a localized zero mode graviton on the $z = 0$ brane (in the original coordinates this is the $y = 0$ brane). This graviton corresponds to the $m = 0$ term in the KK tower. Its 3 + 1-dimensional wave function satisfies a massless dispersion relation $(p^0)^2 = 0$, while the extra dimensional profile function is proportional to the warp factor $\hat{\psi}_0 \propto e^{-2|k|y}$. We say this mode is localized because its profile function is sharply peaked around the location of the $z = 0$ brane. In addition for a quantized spectrum of masses, m, there are corresponding massive modes, whose profile functions can be written as linear combinations of $(|z|+1/k)^{1/2}Y_2(m(|z|+1/k))$ and $(|z|+1/k)^{1/2}J_2(m(|z|+1/k))$, where J_2 and Y_2 are Bessel functions of the first and second kind.

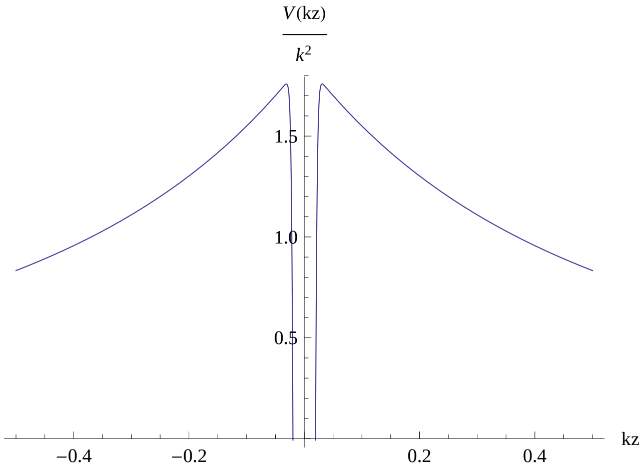


Figure 3.2: An approximation of the Randall Sundrum Volcano potential (where we have replaced the infinitely thin Dirac delta function by a Gaussian wave function $\frac{1}{a\sqrt{\pi}}e^{-z^2/a^2}$, whose limiting value when a tends to zero, is the delta function.) graphed in units of $1/k^2$ with a dimensionless axis variable kz. The deep well at $y=0$ is able to trap a zero mode graviton on the brane, while the warping of the potential $\lim_{kz\to\pm\infty} V(z)/k^2 \rightarrow 0$ gives rise to a continuum of freely propagating massive modes starting from $m=0$.

Remarkably, if we take the negative tension brane to infinity, $L\to\infty$, the localized zero mode graviton remains and we acquire a continuum of massive modes starting from $m=0$. To create an infinite extra dimension we now reflect the space-time about the origin $z=0$; this simply corresponds to removing the boundary condition that all gravitons must be even functions of z.

Randall and Sundrum calculate the Yukawa gravitational attraction potential, $V(r)$, between two objects of mass m_1 and m_2 living on the brane and separated by a distance r:

$$V(r)=G_N\frac{m_1m_2}{r}+\int_0^\infty\frac{dm}{k}G_N\frac{m_1m_2e^{-mr}}{r}\frac{m}{k}. \tag{3.36}$$

The first term is mediated by the zero mode graviton, while the integral is the contribution from the KK modes at $z=0$. The measure dm/k indicates the density of KK modes, while the extra factor m/k is caused by the suppression of the continuum KK modes at $z=0$.

We established in Figure 3.2 that due to the warping of the potential $\lim_{kz\to\infty} V(z)/k^2 \to 0$, these continuum modes are freely propagating as $kz \to \infty$. However to get to the location of the brane, $z = 0$, they must tunnel through a potential barrier caused by the steep slope of the volcano potential surrounding the crater. This leads to a suppression in the coupling of continuum modes to matter localized on the brane.

Evaluating the integral (3.36) gives a modified Newtonian gravitational potential, $V(r)$, which receives only second order corrections in $1/kr$:

$$V(r) = G_N \frac{m_1 m_2}{r} \left(1 + \frac{1}{r^2 k^2} \right). \tag{3.37}$$

At distances $r >> 1/k$ the contribution from the continuum modes to the Newtonian potential will go unnoticed and the extra dimension will be hidden from an observer living on the brane! This has created a dynamic and insightful new field [Vis85, Ant90, AAHDD98].

3.4 Topological defects

In this thesis we are interested in extending the RS2 model. We would like to dynamically generate the brane, rather than introduce it by hand using an explicitly 3 + 1-dimensional term in the action. The RS2 action (3.21) contains the 3 + 1-dimensional term, S_{hid}, which explicitly breaks Poincaré invariance along the extra dimensional coordinate space.

To dynamically generate the brane we introduce a real scalar field topological defect [RS83, Aka82, GW87]. In 4 + 1-dimensional theories this defect is a domain-wall which spontaneously condenses around $y = 0$, thereby spontaneously breaking translational invariance along the extra dimension[1]. In 6-dimensional brane world models, the optimal field theoretical candidate for creating a 3 + 1-dimensional brane is a hedgehog. These defects will

[1]A rigid brane will spontaneously break translational invariance along the extra dimension and 4-dimensional spatial rotational invariance. However if gravity is taken into account and the full complement of perturbations about the brane is considered then there is a massless Goldstone boson associated with the breaking of translational symmetry along the extra dimension. The translational invariance becomes a hidden local symmetry which is not manifestly evident in the action due to the explicit choice of coordinates for embedding the brane. This coordinate choice acts as a gauge fixing condition in the full diffeomorphisms invariant 5-dimensional theory. The Goldstone boson carries the extra degree of freedom. The 4-dimensional spatial rotational invariance remains spontaneously broken, this symmetry does not become a hidden local symmetry because it is associated with the local Lorentz invariance in the tangent space to the 5-dimensional space-time manifold.

also become important in Chapter 8, where we are interested in whether a $3+1$-dimensional universe with Lifshitz anisotropic scaling can contain cosmic relics created by stable, finite energy, static topological defects. For these reasons we spend some time on the topic of topological defects. We will review both domain-wall branes and hedgehogs. In addition we will mention sting defects, which become important in Chapter 8.

3.4.1 Domain-wall topological defects

In $1+1$-dimensional field theories (with 1-spatial dimension, y, and 1 temporal dimension) domain-wall topological defects form static solitary waves with finite total energy. In $4+1$-dimensional field theories they can be used to generate codimension 1 domain-wall branes with finite $3+1$-dimensional energy density.

Canonically they are generated by a ϕ^4 field. This is a real scalar field whose Lagrangian is invariant under a discrete $\mathbb{Z}_2$ reflection symmetry $\phi \to -\phi$. Let

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4}(\phi^2 - v^2)^2. \tag{3.38}$$

The degenerate minima of the potential occur at $\phi = \pm v$; as demonstrated in Figure 3.3.

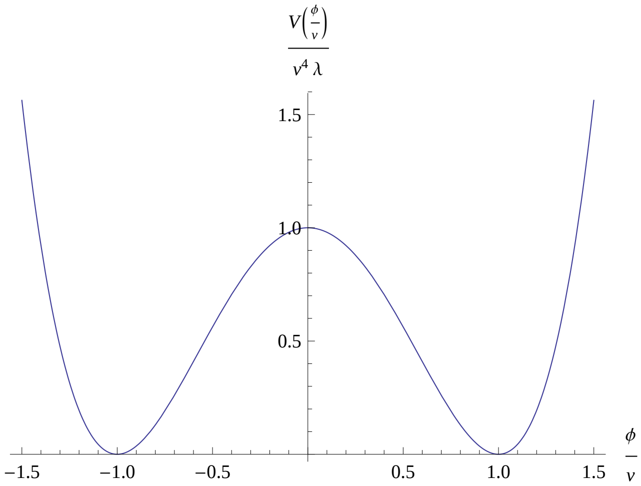


Figure 3.3: The ϕ^4 potential well after rescaling and using a dimensionless coordinate. The invariance of the Lagrangian under $\phi \to -\phi$ creates a disconnected vacuum manifold. This vacuum manifold is generated by $\mathbb{Z}_2$ reflections of the minimum $\phi = v$ (or $\phi = -v$). In the dimensionless coordinate system used to plot the potential well these two points are $\phi/v = 1$ and $\phi/v = -1$.

In addition to the vacuum solutions to the Euler Lagrange equations $\phi = +v$ and $\phi = -v$ there exists a solitary wave solution which satisfies the boundary conditions $\lim_{y\to\pm\infty}\phi(y) = \pm v$. This solution is known as the kink:

$$\phi_{kink}(y) = v\tanh\left(\frac{\sqrt{\lambda}vy}{2}\right). \tag{3.39}$$

Additionally there is a corresponding anti-kink solution which is the negative of (3.39). In 1 + 1-dimensions the total energy of the solitary wave is finite:

$$E = \int dy\left[\partial_\mu\phi_{kink}\partial^\mu\phi_{kink} + \frac{\lambda}{4}(\phi_{kink}^2 - v^2)^2\right] = \frac{4}{3}\sqrt{\lambda}v^3. \tag{3.40}$$

This energy is concentrated around the point of interpolation of the kink, giving rise to a co-dimension 1 domain-wall brane located at $y = 0$. From (3.40) we see that both the kink and anti-kink solutions have higher energy than the vacuum. However neither configuration will relax to the vacuum of the theory. The kink approaches distinct minima which belong to disconnected sectors of the vacuum manifold at opposite extremes of the y-direction. If we were to continuously deform the kink into the flat vacuum configuration $\phi = v$ (or $\phi = -v$), then we would have to pass through intermediary states where $\phi(y)$ asymptotically approaches a nonzero constant $c \neq \pm v$. That is $-v < c < v$. Because c does not belong to the vacuum manifold, these intermediary states have infinite energy.

Another way to see this is to use the conserved current for the Lagrangian [Vac97] which, from Noëther's theorem [aGS93], is

$$j^\mu = \epsilon^{\mu\nu}\partial_\nu\phi, \tag{3.41}$$

where $\mu, \nu \in \{0, 1\}$ and $\epsilon^{\mu\nu}$ is the 2-dimensional Levi-Civita tensor. This leads to a conserved charge in the theory,

$$Q = \int dy\, j^0 = \phi(+\infty) - \phi(-\infty), \tag{3.42}$$

called the topological charge. Clearly the kink possesses a topological charge of $+2v$, while the anti-kink has topological charge $-2v$, and the vacuum has charge 0. We say these configurations belong to different topological sectors. Because none of these field configurations belong to the same topological sector, it is not possible to start out with one solution and spontaneously evolve into another.

We can directly demonstrate that ϕ^4-field theory possesses topologically nontrivial solutions (solutions whose topological charge is different from that of the vacuum and therefore can not relax into the vacuum) using the 0-th homotopy group of the vacuum manifold. We can think of the boundary conditions as forming a map from the boundary of the real line onto the vacuum manifold. Because the vacuum manifold is generated by $\mathbb{Z}_2$ (see Figure 3.3 caption), this mapping is

$$S^0 \to \mathbb{Z}_2. \tag{3.43}$$

In general [Hua92] the n-th homotopy group, $\pi_n(X)$, of a space X, is the group of inequivalent mappings of the n-sphere into X, $S^n \to X$. Inequivalent mappings belong to different topological classes (they are not homotopy equivalent). The zeroth homotopy group, $\pi_0(x)$, simply counts the number of path connected components in X.

$\mathbb{Z}_2$ is not path connected, therefore $\pi_0(\mathbb{Z}_2) \neq 0$. The kink is a solution to the Euler Lagrange equations whose boundary conditions give a mapping, φ_{kink}, from S^0 into $\mathbb{Z}_2$. The flat vacuum $\phi = v$ is also a solution to the Euler Lagrange equations whose boundary conditions give a trivial mapping φ_{vac} from S^0 into $\mathbb{Z}_2$. These mappings φ_{kink} and φ_{vac} are not homotopy equivalent. It is impossible to continuously deform the kink into the vacuum through intermediary states whose boundary conditions also belong to the vacuum manifold. A continuous deformation of the kink into $\phi = v$ would give rise to a homotopy equivalence of φ_{kink} and φ_{vac}.

3.4.2 Higher defects in n-dimensions

We can directly generalize the above to a topological defect in $n+1$-dimensions [DFM10]. First of all start out with a $n+1$-dimensional coordinate space (n-spatial dimensions, 1 temporal dimension) containing a vector of m scalar fields $(\vec{\phi})_a = \phi_a$, where $a = 1, \ldots, m$ and both n, $m > 1$.

Consider a normalized vector of the asymptotic values of these m fields $\lim_{r\to\infty} \frac{1}{|\vec{\phi}|}(\phi_1, \ldots, \phi_m)$. We distinguish between two cases:

1 The fields can take any real value as $r \to \infty$. Therefore this normalized vector belongs

to S^{m-1}.

2 If we were to introduce a Lagrangian, invariant under an internal symmetry G, and insist $\vec{\phi}$ asymptotically approaches a vacuum of the theory which breaks $G \to H$, then the vector $\lim_{r\to\infty} \frac{1}{|\vec{\phi}|}(\phi_1, \ldots, \phi_m)$ will belong to the vacuum manifold G/H.

We follow the [DFM10] example and use case 1 as our default example in this section. In case 1 we do not need to introduce a Lagrangian and talk about the vacuum manifold.

Define a $(n-1)$-sphere, S^{n-1}, which is the boundary of the n-dimensional spatial coordinate space. That is, if we take

$$dl^2 = b(r)(dr^2 + r^2 d\Omega_{n-1}^2) \tag{3.44}$$

as the metric for the n-dimensional spatial coordinate space, then the $(n-1)$-sphere S^{n-1} inherits the metric $b(r)r^2 d\Omega_{n-1}^2$. Let $\vec{n} = \vec{r}/r$ be a point on the $(n-1)$-sphere.

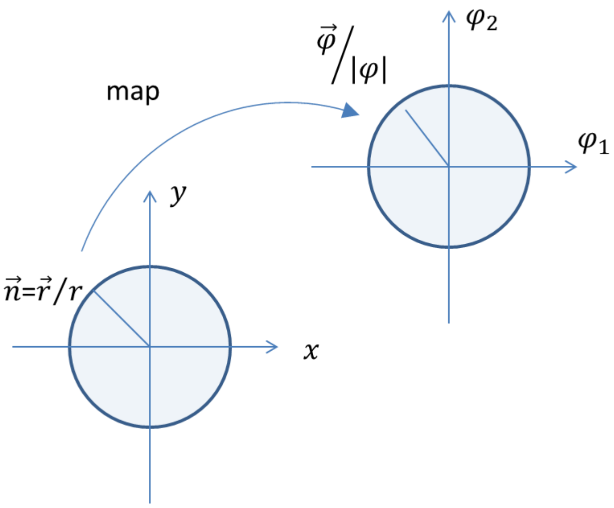


Figure 3.4: An example of a mapping from the boundary of a 2-dimensional spatial coordinate space to the space of asymptotic configurations of a doublet of real scalar fields, $\lim_{r\to\infty} \frac{1}{|\phi|}\vec{\phi} = \lim_{r\to\infty} \frac{1}{|\phi|}(\phi_1, \phi_2)$. This Figure was adapted from [DFM10].

Then there is a mapping from the boundary of n-dimensional spatial coordinate space, S^{n-1}, into the space of asymptotic configurations of the scalar fields

$$\vec{n} = \frac{\vec{r}}{r} \to \lim_{r\to\infty} \frac{\vec{\phi}}{|\vec{\phi}|}. \tag{3.45}$$

See Figure 3.4 for an example of a mapping given by the asymptotic values of a doublet of real scalar fields as $r \to \infty$ in a coordinate space with 2-spatial dimensions.

We now know that the number of topological classes (or inequivalent maps) is

$$\pi_{n-1}(S^{m-1}). \tag{3.46}$$

If this number is non zero then there exists a map which does not belong to the same topological class as the vacuum, and hence the theory possesses soliton solutions which are stable and will not relax to the vacuum.[2] In general

$$\begin{aligned} &\pi_n(S^n) = \mathbb{Z} \\ &\pi_n(S^m) = 0 \qquad \text{if } n < m. \end{aligned} \tag{3.47}$$

The cases $n > m$ can not be succinctly summarized.

3.4.3 Cosmic strings

Box 3.1: *"An example of a vortex is universally encountered by people taking baths or washing dishes. As the water flows down the drain it circulates. We cannot interpolate the circulating velocity field all the way to the centre of the vortex since it would have to become multi-valued. Instead the fluid density in the central region of the vortex vanishes." Tanmay Vachaspati*

The simplest generalization of the domain-wall brane is a vortex defect in $2 + 1$-dimensions (2 spatial and 1 temporal). Here we take a complex scalar field ϕ and modify our Lagrangian (3.38) to be invariant under a U(1) global symmetry $\phi \to e^{i\psi}\phi$,

$$\mathcal{L} = |\partial_\mu \phi||\partial^\mu \phi| - \frac{\lambda}{4}(|\phi|^2 - v^2)^2. \tag{3.48}$$

Of course there is a degenerate vacuum manifold generated by U(1) applied to v. A choice of vacua, $ve^{i\alpha}$, for any $\alpha \in [0, 2\pi]$, spontaneously breaks the symmetry. We demand that the asymptotic configuration of ϕ (as we head to infinity along any axis in the 2-dimensional spatial coordinate space) belongs to the vacuum manifold U(1).

[2]Note there is a subtlety here. If we start talking about the homotopy class $\pi_1(X)$, then we are talking about the number of inequivalent closed paths in X. However these paths are defined with respect to a base point x_0 so really one should be using the the free homotopy group which takes account of this [Vac97].

If we take the 2-spatial dimensions and write them in terms of spherical polar coordinates, say $(x, y) = r(\cos\theta, \sin\theta)$, then the mapping from the boundary of the 2-dimensional spatial coordinate space onto the asymptotic configuration $\lim_{r\to\infty} \phi(x, y)/|\phi| = e^{i\theta}$ is in a distinct topological class from the vacuum and is therefore stable.

The existence of distinct topological classes is guaranteed by the fact that the homotopy group $\pi_1(\mathrm{U}(1)) \neq 0$, since the first homotopy group counts the number of inequivalent paths in the space. Paths in U(1), considered as a manifold, are inequivalent when they have different winding numbers.

If we were to extend this calculation to a $3+1$-dimensional space-time then the vortex would become a string.

3.4.4 Hedgehog solutions in $3 + 1$-dimensional relativistic field theories

In $3 + 1$-dimensional field theories the canonical type of defect is a hedgehog or monopole.

Hedgehog solutions form when we introduce a triplet of real scalar fields $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ and demand that the Lagrangian is invariant under orthogonal transformations $\vec{\phi} \to R\vec{\phi}$ in the (ϕ_1, ϕ_2, ϕ_3) coordinate space, where R is a 3×3 rotations matrix belonging to O(3). Notice that when the scalar fields $\vec{\phi}$ acquire a vacuum expectation value this symmetry is broken down to O(2), the group of orthogonal transformations which fix the vacuum axes in 3-dimensional space. The boundary conditions as $r \to \infty$ for solutions to this theory,

$$\mathcal{L} = \partial_\mu\vec{\phi} \cdot \partial^\mu\vec{\phi} - \frac{\lambda}{4}(\vec{\phi}^2 - v^2)^2, \tag{3.49}$$

must belong to the vacuum manifold O(3)/O(2).

If we adopt normal 3-dimensional spherical polar coordinates and take a mapping from the 2-sphere boundary of the normal 3-spatial coordinate directions (inside a $3 + 1$-dimensional space-time manifold), onto the asymptotic configuration of the scalar field triplet whose asymptotic vacuum configuration is

$$\lim_{r\to\infty} \frac{\vec{\phi}}{\phi} = (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta), \tag{3.50}$$

then we obtain a solution in a different topological class from the vacuum. The topological classes are enumerated by the homotopy group $\pi_2(\mathrm{O}(3)/\mathrm{O}(2))$. Because this homotopy group is nontrivial we can always find a solution to the dynamical equations for (3.49) which is in a different topological class to the homogeneous vacuum.

When evaluated in 3 + 1-dimensions the total energy of this field is infinite. However if we add a gauge symmetry to the hedgehog configuration then the contribution to the energy from the asymptotic configuration of the gauge field cancels against the contribution from the asymptotic behaviour of $\vec{\phi}$ [Vac97]. This gives the stable t'Hooft-Polyakov magnetic monopoles [Pol74].

In the next section we will prove that it is not possible to produce stable, finite energy static solutions to a second order wave equation in relativistic 3 + 1-dimensional field theory (3-spatial directions and 1 temporal).

3.4.5 Instability in higher dimensions

The hedgehog cannot simultaneously be stable and have finite energy density. This is a corollary of Derrick's theorem [Der64]:

Theorem 3.1: Derrick's theorem: *In $d > 2$-spatial dimensions there are no stable, finite energy, localized, static solutions to the second order, $d + 1$-dimensional relativistically covariant nonlinear wave equation derived by the principle of least action applied to*

$$S[\phi] = \int d^{d+1}x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right], \tag{3.51}$$

for any $V(\phi)$.

We prove this statement directly for static solutions to the dynamical equations arising from (3.51) in $d + 1$-dimensional relativistic field theories, for $d > 2$. Assume the energy density is finite and the solution is stable.

The energy density functional for the static scalar field is

$$E[\phi] = \int d^d x \left[\frac{1}{2}\nabla\phi \cdot \nabla\phi + V(\phi)\right] = \underbrace{\int d^d x \frac{1}{2}\nabla\phi \cdot \nabla\phi}_{S_1} + \underbrace{\int d^d x V(\phi)}_{S_3}. \tag{3.52}$$

For a static field, $\phi(\vec{x})$ extremizes (3.51) if and only if $\phi(\vec{x})$ extremizes (3.52).

Let $\phi(\vec{x})$ extremize (3.52). This implies the energy functional is stationary with respect to this field configuration and will vanish, at least to leading order, for all perturbations around $\phi(\vec{x})$. Consider a specific, small perturbation of the form

$$\phi(\vec{x}) \to \phi(k\vec{x}). \tag{3.53}$$

This perturbation will cause the energy density functional to rescale to

$$E_k[\phi] = \int d^d x \left[\frac{1}{2}\nabla\phi(k\vec{x}) \cdot \nabla\phi(k\vec{x}) + V\left(\phi(k\vec{x})\right)\right]. \tag{3.54}$$

When we demand that the corresponding perturbation in the energy density function vanishes to first order in k, around $k = 1$, we are requiring

$$\left.\frac{\partial E_k[\phi]}{\partial k}\right|_{k=1} = 0. \tag{3.55}$$

Now

$$\begin{aligned} E_k[\phi] &= \int d^d\xi \left[k^{2-d}\frac{1}{2}\nabla_{\vec{\xi}}\phi(\vec{\xi}) \cdot \nabla_{\vec{\xi}}\phi(\vec{\xi}) + k^{-d}V\left(\phi(\vec{\xi})\right)\right] \\ &\equiv k^{2-d}S_1 + k^{-d}S_3, \end{aligned} \tag{3.56}$$

where $\vec{\xi} \equiv k\vec{x}$ and $E = S_1 + S_3$. Imposing (3.55) we get the virial relation $(2 - d)S_1 = dS_3$, which implies that $E = 2S_1/d$.

Simultaneously, if the solution is stable, we require this point to be a local minimum in the $E_k[\phi]$ topology (a small variation cannot reduce the energy density). This will correspond to the second order variation with respect to k being positive. However, using the information that $(2-d)S_1 = dS_3$, it follows that $d^2E_k/dk^2|_{k=1} = (2-d)(1-d)S_1+d(d+1)S_3 = (2-d)2S_1$.

Whenever $d > 2$ this is a contradiction since S_1 is positive-definite. Thus one of the original assumptions is violated so either there is no stable solution, or the energy density is not finite. Clearly, after obtaining a candidate soliton, $\phi(\vec{x})$, whose energy density is finite, if we compress this wave, then we will end up with a lower energy configuration.[3]

These results simultaneously apply to cosmic strings and domain-wall branes in $d > 2$ dimensions. These defects have divergent energy functionals.

3.5 Fermion zero modes

In this thesis we are primarily interested in models with a single infinite extra dimension. This scenario leads to simple toy models where extra-dimensional physics exhibits standard model phenomenology with conventional 3 + 1-dimensional gravity on the brane. Therefore we consider thick branes generated by codimension-1 domain-wall topological defects.

We would like some way of localizing fermions to the domain-wall brane. The most promising candidate is the fermion zero mode approach.

To construct a Lorentz invariant 4 + 1-dimensional action for the spinor we need to extend the Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, where the signature on the Minkowski metric is $+, -, -, -$ and $\mu = 0, 1, 2, 3$. We use $\Gamma^M = (\gamma^\mu, -i\gamma^5)$ to introduce a 5-th gamma matrix, Γ^5, which anticommutes with $\gamma^0, \ldots, \gamma^3$ and satisfies $\left(\Gamma^5\right)^2 = \eta_{55} = -1$. However the 4 + 1-dimensional Lorentz transformations are now generated by $\{\Gamma^M, M = 1, \ldots, 5\}$, and therefore the smallest representation for the Lorentz group is a four component spinor. Clearly no subspace of this spinor is invariant under 4 + 1-dimensional Lorentz transformations. This contrasts with the conventional 3 + 1-dimensional relativity where the smallest representation is the 2-component Weyl spinor, while parity transformations are the only symmetries which mix left and right chiral modes. In the 4 + 1-dimensional scenario we have lost these well defined chiral subspaces. Because the weak force breaks parity invariance and only couples

[3]Intuitively Derrick's theorem arises because a hypothetical static, stable, finite energy density solution, $\phi_0(x^i)$, to the Euler Lagrange equations would locally minimize the energy functional. However if we contract the soliton slightly, corresponding to $\phi_0(x^i) \to \phi_0(kx^i)$ where $k \approx 1$, then we find the energy functional scales according to

$$E[\phi_0(kx^i)] = \frac{2k^2 - 1}{k^4} E[\phi_0(x^i)] \tag{3.57}$$

and will decrease as k increases. This contradicts the hypothesis that $E[\phi_0(x^i)]$ is minimal.

to left handed fermions, we will have to recover the notion of chirality in our effective 3 + 1-dimensional theory.

Through the fermion zero mode approach a Lorentz-invariant domain-wall brane can dynamically localize a massless chiral 3 + 1-dimensional fermion.

In Lorentz-invariant 4 + 1-dimensional brane-world models the 4 + 1-dimensional Dirac action, which introduces a Yukawa interaction term between the 4 + 1-dimensional spinor and the domain-wall brane (3.39), is

$$\int d^5x \left[i\overline{\Psi}\Gamma^M \partial_M \Psi - g\phi\overline{\Psi}\Psi \right]. \tag{3.58}$$

This gives rise to a Dirac equation

$$\left[i\Gamma^M \partial_M - g\phi(y) \right] \Psi(x^N) = 0. \tag{3.59}$$

A solution of (3.59) also satisfies the Klein-Gordon equation

$$\left(\partial_t^2 - \vec{\nabla}^2 - \partial_y^2 + g^2\phi(y)^2 + \gamma^5 g \left(\partial_y \phi(y) \right) \right) \Psi(x^M) = 0. \tag{3.60}$$

There exists a complete basis, $\{f_{nL/R}(y)\}$, for the space of bounded continuous functions of the extra-dimensional coordinate, $C(y)$, which can be used to project the solution, $\Psi(x^N)$, to (3.59) and (3.60) onto towers of 3 + 1-dimensional chiral spinors $\{\psi_{nL/R}(x^\mu)\}$:

$$\begin{aligned} \Psi_i(x, y) &= \Psi_{0L}(x^M) + \sum_{n>0} \{\Psi_{nL}(x^M) + \Psi_{nR}(x^M)\}, \\ &= f_{0L}(y)\psi_{0L}(x^\mu) + \sum_{n>0} \{f_{nL}(y)\psi_{nL}(x^\mu) + f_{nR}(y)\psi_{nR}(x^\mu)\}, \end{aligned} \tag{3.61}$$

where we have introduced the notation $\Psi_{nL/R}(x^M) = f_{nL/R}(y)\psi_{nL/R}(x^\mu)$. Moreover, we will write $\Psi_n(x^M) = \Psi_{nL}(x^M) + \Psi_{nR}(x^M)$. There are as many decompositions of this type as there are complete sets of continuous bounded functions of y. But, for two reasons, there is one basis that is special. An expression (3.61) using this basis will be called a Kaluza-Klein decomposition. First, for this basis the $\Psi_n(x^M)$ appearing in (3.61) are independent solutions to the Dirac equation (3.59) satisfying orthonormality conditions in a rigged Hilbert space.

Second, each 3 + 1-dimensional spinor in this tower satisfies the 3 + 1-dimensional Dirac equation for a particle with mass m_n,

$$\begin{aligned} i\gamma^\mu\partial_\mu\psi_{nL}(x^\mu) &= m_n\psi_{nR}(x^\mu), \\ i\gamma^\mu\partial_\mu\psi_{nR}(x^\mu) &= m_n\psi_{nL}(x^\mu). \end{aligned} \tag{3.62}$$

The 3 + 1-dimensional left-chiral zero-mode $\psi_{0L}(x^\mu)$ spinor satisfies (3.62) with $m_0 = 0$. Equation (3.61) does not contain a right-handed massless 3+1-dimensional spinor. Directly calculating the form of $f_{0R}(y)$ when $\psi_{0R}(x^\mu)$ is a massless 3 + 1-dimensional right-chiral fermion and $f_{0R}(y)\psi_{0R}(x^\mu)$ is a solution to (3.59) reveals that $f_{0R}(y)$ does not belong to $C(y)$ and hence cannot form part of a basis for this space.

Explicitly we find that $f_{0L}(y)$ and $f_{0R}(y)$ are given by

$$f_{0L/R}(y) = Ne^{\mp\int_\infty^y dy' g\phi(y')}, \tag{3.63}$$

where N is a normalization factor for the left chiral zero mode. Figure 3.5 shows the integrand for the left chiral zero mode $f_{0L}(y)$. The corresponding plot of the integrand of the right chiral zero mode is the reflection in the x-axis of Figure 3.5.

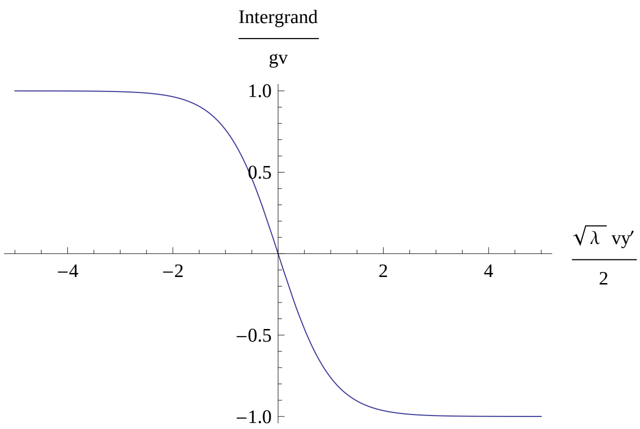


Figure 3.5: Plot of integrand appearing in the exponential of the 3 + 1-dimensional left chiral zero mode fermion profile functions $f_{0L}(y)$. Because the primitive of this function diverges to $-\infty$ in either direction as $y \to \pm\infty$, the profile function is peaked about $y = 0$ and exponentially suppressed as $y \to \pm\infty$. The corresponding plot for the integrand of the $f_{0R}(y)$ function is the negative (or reflection in the x-axis) of this Figure. The integrand of the $f_{0R}(y)$ profile function is the anti-kink. Applying the same logic we find that $f_{0R}(y)$ diverges exponentially as $y \to \infty$.

From (3.62) it follows that each $\psi_{nL}(x^\mu)+\psi_{nR}(x^\mu)$ satisfies a 3+1-dimensional Klein-Gordon equation

$$\begin{aligned}\left(\partial_t^2-\vec{\nabla}^2\right)\psi_{nL}(x^\mu) &= m_n^2\psi_{nL}(x^\mu),\\ \left(\partial_t^2-\vec{\nabla}^2\right)\psi_{nR}(x^\mu) &= m_n^2\psi_{nR}(x^\mu).\end{aligned} \tag{3.64}$$

If we use unbroken 3 + 1-dimensional Poincaré invariance to expand $\psi_{nL}(x^\mu)+\psi_{nR}(x^\mu)$ in terms of plane waves then we find that the dispersion relation describing the propagation of the 3 + 1-dimensional spinor $\psi_{nL}(x^\mu)+\psi_{nR}(x^\mu)$ with energy-momentum 4-vector $p_n = (\omega_n, \vec{p}_n)$ is

$$\omega_n^2-\vec{p}_n\cdot\vec{p}_n = m_n^2. \tag{3.65}$$

From the perspective of a 3 + 1-dimensional observer these particles can now be given the interpretation of propagating free particles with masses m_n which transform according to a spin-1/2 representation of an embedded 3 + 1-dimensional Lorentz space-time symmetry. For each mode, $\Psi_n(x^M)$, which appears in (3.61), a 3 + 1-dimensional observer will see a resonance in the detector at energy m_n caused by what he perceives as a massive fermion $\psi_{nL}(x^\mu)+\psi_{nR}(x^\mu)$. At low energies only the left-chiral zero-mode $\psi_{0L}(x^\mu)$ will be detectable.

We choose to work with this special basis because we have a physical interpretation for the individual modes $\Psi_n(x^M)$. This interpretation allows us to argue that at low energies there is a candidate 3 + 1-dimensional massless chiral fermion and to explain why parity is broken (the kink localizes a zero mode of one chirality only).

We find this special basis and the allowed masses, m_n, of the 3 + 1-dimensional spinors that are present in (3.59) by solving the eigenvalue problem,

$$\left[-\partial_y^2+g^2\phi(y)^2\pm g\partial_y\phi(y)\right]f_{nL/R}(y) = m_n^2 f_{n,L/R}(y). \tag{3.66}$$

Sturm-Liouville theory determines the existence and completeness of eigensystems generated by (3.66).

We can check these conditions are consistent and that the eigenfunctions in (3.66) are the correct 'special basis' functions $\{f_{nL/R}(y)\}$ to use in (3.59). To do this we substitute $\Psi_n(x^M)$

into the 4 + 1-dimensional Dirac equation and use (3.62) to simplify. After isolating the coefficients of the 3 + 1-dimensional spinors, $\psi_{nL}(x^\mu)$ and $\psi_{nR}(x^\mu)$, which correspond to independent degrees of freedom, and setting each coefficient equal to zero independently we arrive at:

$$\begin{aligned} \left[-\partial_y - g\phi(y)\right] f_{nL}(y) &= m_n f_{nR}(y), \\ \left[\partial_y - g\phi(y)\right] f_{nR}(y) &= m_n f_{nL}(y). \end{aligned} \tag{3.67}$$

Uncoupling this first order system automatically generates the two second order differential equations in (3.66). Moreover, by using separation of variables, (3.66) and (3.64), we can show that each mode $\Psi_n(x^M)$ satisfies the Klein-Gordon equation (3.60).

3.5.1 Splitting the fermion profiles

This approach is called split fermions because if we were to introduce another scalar field, $\rho(y)$, into the model which also Yukawa couples to the fermions with coupling strength m_5, then the Dirac equation, (3.59), will be modified to

$$\left[i\Gamma^M \partial_M - g\phi(y) - m_5\rho(y)\right] \Psi(x^N) = 0. \tag{3.68}$$

The profile function of the left chiral zero mode is modified to

$$f_{0L/R}(y) = N e^{\mp \int_\infty^y dy' g\phi(y') + m_5\rho(y')}; \tag{3.69}$$

here again N is a normalization factor for the left chiral zero mode. Assume that either

1 $g\phi(y) + m_5\rho(y)$ has a single x-axis intercept and the same asymptotic behaviour as the kink $\phi(y)$, or

2 $\rho(y) = c$ is a constant satisfying $|m_5 c| < |gv|$, so $g\phi(y)+m_5\rho(y)$ just shifts the kink up or down; we maintain the conditions $\lim_{y\to\infty} g\phi(y) + m_5\rho(y) < 0$ and $\lim_{y\to-\infty} g\phi(y) + m_5\rho(y) > 0$.

Then, by the same argument as is used in the caption of Figure 3.5, the left chiral 3 + 1-dimensional zero mode profile function will belong to $C(y)$ (respectively the 3+1-dimensional right chiral zero mode profile function will not belong to $C(y)$). However the fermion will be localized about the solution of $g\phi(y)+m_5\rho(y) = 0$, instead of about the hyperplane $y = 0$. If we stop and examine scenario 2, then the extra Yukawa coupling term is in fact a 4 + 1-dimensional mass term for the fermion. Let $c = 1$ for convenience. We find that the fermion is now localized about the x-axis intercept of the integrand $g\phi(y) + m_5 = 0$. Therefore, if we introduce two or more fermions with different 4 + 1-dimensional mass terms, then the fermions get confined to parallel hyperplanes split along the bulk direction (y-coordinate axes). Hence the origin of the name "split fermions".

3.6 Dvali-Shifman mechanism

Having successfully localized chiral fermions, we would like to complement this picture by trapping gauge fields on the brane.

The most straightforward approach would be to follow the same method in the gauge sector as with the fermions; that is we could Kaluza-Klein decompose the 4+1-dimensional gauge field, A^N, into a tower of 3+1-dimensional gauge bosons with zero mode term, $h_0(y)a_0^\mu(x^\mu)$, and then try to localize the 3+1-dimensional zero mode to the domain-wall brane. However this approach can only localize *massive* spin-1 fields. This is not what we require, both because photons and gluons are massless, and because gauge coupling constant universality is lost.

To see that gauge coupling constant universality is lost [Rub01, Dav07], consider the 4 + 1-dimensional current term

$$S_{cur} = \int d^5x\,\sqrt{g}V_C^N\overline{\Psi}\gamma^C\Psi A_N \subset \int d^4x\left(\int dy\,e^{-3k|y|}|f_{0L}(y)|^2h^0(y)\right)\overline{\psi}_{0L}\gamma_\mu\psi_{0L}a_0^\mu, \quad (3.70)$$

where V_C^N is the 4 + 1-dimensional vielbein for the RS2 metric. Now (3.70) shows that the effective 3+1-dimensional coupling constant depends on the overlap of the profile functions for the 3 + 1-dimensional gauge boson and left chiral fermion zero modes

$$e_{eff} \propto \int d^4x \left(\int dy e^{-3k|y|} |f_{0L}(y)|^2 h^0(y) \right). \tag{3.71}$$

For the coupling of a fermion to the photon, the 3 + 1-dimensional coupling constants must be the quantized U(1) electromagnetic charges of the fermions. However according to (3.71) this coupling will depend on their 4 + 1-dimensional zero mode profile functions. The only way around this is to make $h^0(y) = c$, a constant, thereby reducing the right hand side of (3.71) to c times the normalization of the left chiral zero mode. However in this scenario the gauge boson has a flat profile function.

An alternate approach which takes advantage of the dynamics of confining gauge theories with a mass gap was proposed by Dvali and Shifman [DS97].

3.6.1 A 2+1-dimensional toy model with a trapped U(1) gauge boson.

Dvali and Shifman consider a 3+1-dimensional toy model based on an SU(2)×$\mathbb{Z}_2$ invariant Lagrangian incorporating SU(2) gauged Yang-Mills theory which breaks to U(1) on a 2+1-dimensional domain wall [DS97]. The Higgs sector in the Lagrangian of this theory,

$$\begin{aligned}\mathcal{L} = & \underbrace{-\frac{1}{4g^2} G^a_{\mu\nu} G^a_{\mu\nu}}_{\text{gauge field kinetic term}} + \underbrace{\bar{\psi}_L \not{D} \psi_L + \bar{\psi}_R \not{D} \psi_R}_{\text{fermion kinetic terms}} - \underbrace{(h\phi\bar{\psi}_L\psi_R + h.c.)}_{\text{fermion coupling to } \phi} + \underbrace{\frac{1}{2}\left(D_\mu \mathcal{X}^a\right)^2 + \frac{1}{2}\left(\partial_\mu \phi\right)^2}_{\text{Higgs kinetic term}} \\ & \underbrace{-\frac{1}{2}\lambda' \left(\mathcal{X}^2 + \kappa^2 - v^2 + \phi^2\right)^2 - \lambda\left(\phi^2 - v^2\right)^2}_{-V(\phi,\mathcal{X})}\end{aligned}, \tag{3.72}$$

contains an SU(2) singlet ϕ which establishes the domain-wall brane as well as an SU(2) adjoint field $\mathcal{X}$. The 2 + 1-dimensional domain-wall brane is located at the point of interpolation of the kink. Provided $\kappa^2 - v^2 < 0$, at this point $\phi = 0$ and the adjoint Higgs field becomes tachyonic and develops a vacuum expectation value which breaks SU(2) → U(1). Asymptotically $\lim_{y\to\pm\infty} \phi = \pm v$, causing the vacuum configuration for $\mathcal{X}$ and its profile to asymptotically approach $\mathcal{X} = 0$, as $y \to \pm\infty$, thereby restoring the full SU(2) symmetry of the theory.

The bulk is presumed to be in confinement phase with a strong coupling regime in the infrared. This creates a mass gap of order the bulk confinement scale $\Lambda_{\text{conf[SU(2)]}}$ in the vector boson spectrum. Away from the wall the gauge fields form bound states with mass greater than $\Lambda_{\text{conf[SU(2)]}}$.

On the domain wall the photon is free, however in order to propagate in the direction transverse to the wall it must become conglomerated into a massive SU(2) glueball. The energy cost associated with moving off the wall is untenable for the massless U(1) gauge boson which effectively becomes trapped on the 2 + 1-dimensional topological defect.

In the 't Hooft-Mandelstam [tH81, Man76] dual superconductivity picture [AHS99], electric field lines from test charges on the brane spread out like normal Coulomb fields along the brane directions. However, when they meet the bulk they get repelled by the dual analogue of the Meissner effect. Thus any two charges located on the brane will interact via a 2 + 1-dimensional Coulomb force at distances much greater than the thickness of the wall.

If the test charge is located in the bulk then the field lines are no longer at liberty to spread out. Instead they form a flux string which tunnels through to the brane and expels the field onto the domain wall. This is the dual superconductor effect corresponding to Abrikosov vortex formation. Hence, regardless of the bulk localisation profiles of the fermions, the effective field on the domain wall will exhibit the same 2 + 1-dimensional Coulomb distribution.

3.6.2 Extension of the Dvali-Shifman mechanism

To create a phenomenologically realistic standard model gauge theory on a 3 + 1-dimension domain-wall brane it will be necessary to extend the Dvali-Shifman mechanism in two respects:

1 Firstly we must reformulate the mechanism so that it traps a full contingent of standard model gauge bosons on the domain-wall brane, instead of a single U(1) gauge field.

2 Secondly we need to extend the space-time dimensions in the theory to incorporate a 3 + 1-dimensional domain-wall brane isometrically embedded in 4 + 1-dimensional space-time.

It is straightforward to extend the Dvali-Shifman argument to deal with a grand unified theory gauge group G breaking to a subgroup H (such as the standard model) on the domain wall. Under these circumstances it is assumed that the confinement scale on the brane, $\Lambda_{\mathrm{conf}[H]}$, is significantly lower than the bulk scale, $\Lambda_{\mathrm{conf}[G]}$.

However, extending the model to cope with a larger number of dimensions is not easy. The problem is that non-Abelian gauge theories are not renormalizable in more than 3 + 1-dimensions.

3 + 1-dimensional non-perturbative lattice simulations provide qualified support for the hypothesis that the Dvali-Shifman mechanism can trap gauge fields on a codimension-1 brane [LMRS04]. But to our knowledge no one has tackled a 4+1-dimensional simulation. At this stage the Dvali-Shifman mechanism remains an intriguing conjecture in 4 + 1-dimensions.

A necessary condition for the Dvali-Shifman mechanism is confinement in the bulk. It is encouraging that lattice gauge theory simulations for SU(2) pure Yang-Mills theory in 4+1-dimensions demonstrate a first order phase transition at finite lattice spacing as a function of the gauge coupling constant: for coupling strengths above a critical value, the theory appears to be confining [Cre79]. This result has been extended to pure SU(5) Yang-Mills gauge theories in [Geo09].

However no conclusion can be drawn about the continuum limit since our non-renormalizable theory has problems with point-like interactions at high energies. But this may not actually be a problem for our application. All field-theoretic brane models are necessarily low-energy effective theories due to their non-renormalisability, so they are implicitly defined with an ultraviolet cutoff. The finite lattice spacing in the simulations is also an ultraviolet cutoff, so it appears sensible to use the lattice gauge results to conclude that confinement in 4 + 1-dimensional effective gauge theories can exist.

While this argumentation provides encouragement to pursue models based on the Dvali-Shifman idea, it does not rigorously establish that any specific model works. For one thing, no such model is a pure Yang-Mills theory. Secondly, for each candidate theory one would need to compute via lattice simulations the critical gauge coupling constant as a function of the cutoff. Since the critical coupling constant in 4+1-dimensions is itself of non zero mass dimension, one would then need to check that this value is compatible with other scales in the problem, such as the inverse width of the domain wall and the effective grand unified symmetry breaking scale inside the wall.

In the next two sections we investigate grand unified gauge groups G which contain the standard model $\mathrm{SU(3)_C \times SU(2)_W \times U(1)_Y}$ as a subgroup. Also we investigate Lorentz violating field theories which exhibit Lifshitz anisotropic scaling. These field theories are able to lift the problems with non-renormalizability of Yang-Mills gauge theory in higher dimensions, thereby addressing issues with building brane world models in 4 + 1-dimensions.

3.7 Spontaneous symmetry breaking and grand unification

Grand unified theories were first introduced in 1974 by Georgi and Glashow. The underlying principle is to take the standard model gauge group $\mathrm{SU(3)_C \times SU(2)_W \times U(1)_Y}$ and embed it within a compact simple Lie group. This approach has the advantage of unifying the fermions, belonging to a given family, in 1- or 2-irreducible representations of the GUT gauge group, a clear improvement over the standard model's piecemeal approach. However grand unified theories must introduce additional Higgs fields which spontaneously break the symmetry down to the standard model, leading to complicated Higgs sectors and new free parameters. They also experience problems with coupling constant unification, degeneracy of mass relations, and proton decay. We choose to cover this topic through the explicit examples of SU(5) and SO(10) grand unification, followed by a discussion of spontaneous symmetry breaking.

In the following, note that there are an infinite number of embeddings of a subgroup $\mathrm{SU(3)_C \times SU(2)_W \times U(1)_Y}$ inside a Lie group G. Selecting an embedding corresponds to an explicit choice of the hypercharge generator Y. Once an embedding has been chosen, conjugating Y by an element of the Lie group $g \in G$ gives a new hypercharge generator $g\mathrm{Y}g^{-1}$. This conjugated $g\mathrm{Y}g^{-1}$ is the hypercharge generator of an isomorphic embedded copy of the standard model gauge group. If the Lie group generator $g \notin \mathrm{SU(3)_C \times SU(2)_W \times U(1)_Y}$, then the copy will be a different embedding. As such, when we talk about a specific embedding we are simply invoking a convention within the literature. In this thesis we will place a special emphasis on embeddings of a subgroup H within a Lie group G, where the Cartan subalgebra for H is a subspace of the Cartan subalgebra for G. Such embeddings are called Cartan preserving.

3.7.1 SU(5) grand unification

For the canonical embedding of $SU(3)_C \times SU(2)_W \times U(1)_Y$ inside SU(5) we describe the transformation properties of the fundamental 5 representation for SU(5) and the antisymmetric 10 representation under the restriction $SU(5) \downarrow SU(3)_C \times SU(2)_W \times U(1)_Y$. The fundamental representation for SU(5) is a column vector T^i, while the 10 representation is a 5×5 antisymmetric matrix T^{ij}. If we choose to refer to the irreducible $SU(3)_C \times SU(2)_W \times U(1)_Y$ representations sitting inside each as A, A_1 etc, then the 5 and 10 representations break down [Cur06] as follows:

$$\begin{array}{cccccccc} 5 & \longrightarrow & (3,1)(-2/3) & + & (1,2)(1) & & \\ {\scriptstyle T^i} & & {\scriptstyle A^\rho = T^\rho} & & {\scriptstyle A_1^p = T^{p+3}} & & \\ \\ 10 & \longrightarrow & (\bar{3},1)(-4/3) & + & (3,2)(1/3) & + & (1,1)(2) \\ {\scriptstyle T^{ij}} & & {\scriptstyle A_p = \epsilon_{pab} T^{ab}} & & {\scriptstyle A_1^{p\nu} = T^{p,\nu+3}} & & {\scriptstyle A_2 = T^{4,5}}. \end{array} \tag{3.73}$$

Here by $(3,2)(1)$ we mean the representation labelled q_L in equation 3.74, is a module of the 3-dimensional representation of the SU(3) colour symmetry. Simultaneously in the other two subspaces of the product space, $(3,2)(1)$ is a module of the 2-dimensional representation of the SU(2) weak isospin gauge group, and $q_L \to e^{i\theta} q_L$ under the U(1) weak hypercharge symmetry. This prompts us to incorporate the first generation of fermions in the standard model into the combination of a 10 and a $\overline{5}$ representation according to

$$\begin{array}{lll} \overline{5} & \to & \underbrace{(\bar{3},1)(2/3)}_{u_R^c} + \underbrace{(1,2)(-1)}_{f_L} \\ 10 & \to & \underbrace{(3,2)(1/3)}_{q_L} + \underbrace{(1,1)(2)}_{e_R^c} + \underbrace{(\bar{3},1)(-4/3)}_{d_R^c} \end{array}, \tag{3.74}$$

where a^c stands for the charge conjugate of the field a, [Sla81]. Simultaneously in the gauge sector we find that the SU(5) adjoint representation contains the 8 gluons arising from an embedded SU(3) adjoint representation, as well as the triplet of weak isospin vector bosons and the hypercharge mediator B^μ,

$$24 \longrightarrow \underbrace{(8,1)(0)}_{G^\mu} + \underbrace{(1,3)(0)}_{W^\mu} + \underbrace{(1,1)(0)}_{B^\mu} + \underbrace{(3,2)\,(-5)}_{X,Y} + \underbrace{(\bar{3},2)\,(5)}_{\overline{X},\overline{Y}}. \tag{3.75}$$

Additionally the 24-dimensional representation furnishes two representations for $SU(3)_C \times SU(2)_W \times U(1)_Y$, the $(3,2)\,(-5)$ and its conjugate representation, which contain gauge bosons carrying nontrivial colour and weak force quantum numbers [Sla81],

$$\begin{pmatrix} X^{-1} \\ Y^{-4} \end{pmatrix} \sim (3,2)\,(-5)\,. \tag{3.76}$$

3.7.2 SO(10) grand unification

We can ask if it is possible to extend the SU(5) scenario and unify the 10 and the $\overline{5}$ into a single representation of a grand unified theory gauge group $G \supset SU(5)$, and thereby incorporate all the fermions in any generation of the standard model in a single irreducible representation of G. The answer is yes, provided we also postulate the existence of a right handed neutrino. This state is currently unobserved, however, due to its lack of electric charge and colour indices, as well as its inability to couple to the weak force vector bosons. This leads us to conclude, through the Gell-Mann-Nishijima formula,

$$Q = I_2 + \frac{1}{2}Y, \tag{3.77}$$

that the right handed neutrino is also uncharged under hypercharge, that it is extremely weakly interacting, and is not phenomenologically ruled out. Under these conditions we choose to work with the higher rank Lie group SO(10) which contains SU(5):

$$SO(10) \supset SU(5) \times U(1) \supset SU(3)_C \times SU(2)_W \times U(1)_Y \times U(1). \tag{3.78}$$

SO(10) has a 16 spinor representation where each state in the representation can be identified with a fermion belonging to the first generation of standard model particles. This follows from the branching rules

$$\begin{array}{rcl} \mathrm{SO}(10) & \longrightarrow & \mathrm{SU}(5)\times\mathrm{U}(1) \\ 16 & \longrightarrow & 10(-1)+\bar{5}(3)+1(-5), \end{array} \tag{3.79}$$

and the identifications in (3.74) [Sla81]. We assign the role $\nu_R^c \equiv 1(-5)$.

The gauge sector has a similar embedding of the 24-dimensional SU(5) adjoint within the SO(10) adjoint:

$$\begin{array}{rcl} \mathrm{SO}(10) & \longrightarrow & \mathrm{SU}(5)\times\mathrm{U}(1) \\ 45 & \longrightarrow & 24(0)+1(0)+10(4)+\overline{10}(-4). \end{array} \tag{3.80}$$

Again from (3.75) we conclude that a correct contingent of standard model gauge bosons will be present in our SO(10) gauged grand unified theory. However there will be more exotic vector bosons like the X and Y which create additional channels for proton decay.

Box 3.2: *The 16 is a spinor representation for SO(10)*: *We explain why SO(10) has a spinor representation and how to explicitly construct this representation.*

An $SO(2n)$ rotation group is defined as the set of transformations of a $2n$-dimensional coordinate vector $\vec{x}$, which leave the bilinear form $B(\vec{x},\vec{x}) = \vec{x}\cdot\vec{x}$ invariant [Li74]. Using a $2n$-dimensional set of anticommuting matrices, $\{\sigma_j \,|\, \{\sigma_j,\sigma_k\} = 2\delta_{j,k} \;\; j,k = 1,\ldots 2n\}$ we can recast the bilinear form as $B(\vec{x},\vec{x}) = \left(x^i\sigma_i\right)^2$. These anticommuting matrices are known as a Clifford algebra.

Now from the transformation properties of the coordinate vector under $SO(2n)$ rotations we conclude that $x^i\sigma_i \to \mathrm{O}^i_j x^j\sigma_i$. Therefore the vector of matrices $\vec{\sigma} = \left(\sigma^1,\ldots,\sigma^{2n}\right)$ furnishes a vector representation for $SO(2n)$.

Because the matrices $\sigma'_j = \mathrm{O}^i_j\sigma_i$ also generate a Clifford algebra we know the two algebras are related by a similarity transformation

$$\mathrm{O}^i_k\sigma_i \to \mathrm{S(O)}\sigma_\mathrm{k}\mathrm{S(O)}^{-1}, \tag{3.81}$$

where S(O) is a spinor representation of the $SO(2n)$ matrix O. From expanding (3.81) to first order in the generators of the representation we can demonstrate that

$$\mathrm{S(O)} = 1 - \frac{1}{8}\left[\sigma_i,\sigma_j\right]\epsilon^{ij} = 1 - \Sigma_{ij}\epsilon^{ij}, \tag{3.82}$$

where ϵ^{ij} is a rank 2 antisymmetric tensor.

The $2n$-dimensional Clifford algebra is formed by taking the direct products of n Pauli matrices;

$$\sigma^{2k-1} = 1\otimes\cdots\otimes 1\otimes\tau_1\otimes\tau_3\otimes\cdots\otimes\tau_3, \tag{3.83}$$

$$\sigma^{2k} = \underbrace{1\otimes\cdots\otimes 1}_{k-1}\otimes\tau_2\otimes\underbrace{\tau_3\otimes\cdots\otimes\tau_3}_{n-k}. \tag{3.84}$$

There is a matrix which anticommutes with all of the $2n$-dimensional Clifford algebra matrices,

$$\sigma_c = (i)^n\sigma_1\ldots\sigma_{2n} = \underbrace{\tau_3\otimes\cdots\otimes\tau_3}_{n}. \tag{3.85}$$

The $2n+1$-dimensional Clifford algebra is constructed from (3.83) and (3.85), and the spinor representations for $SO(2n+1)$ follow from (3.82). Additionally for a $2n$-dimensional Clifford algebra, σ^c commutes with all the generators, Σ_{ij}, of the associated SO(2n) spinor representation. Schur's lemma tells us that SO(2n) spinor representation are reducible. When using (3.83) to construct the SO(10) spinor representation we get 32×32 matrices. The irreducible 16-dimensional representations are then projected out using $(1+\sigma_c)$ and $(1-\sigma_c)$.

3.7.3 Problems with grand unified theories

We have highlighted that grand unified theories introduce fresh problems. We briefly outline some of these problems and will discuss one possible solution to them in Chapter 4.

Coupling constant unification

If the standard model is putatively to become unified in a grand unified theory then, at the scale at which the GUT symmetry breaking occurs, the weak, strong and hypercharge gauge bosons must all become part of a single irreducible adjoint representation of the GUT gauge group. This implies the weak, strong and electromagnetic coupling constants must be the same at this scale. If we presume that it is possible to run the coupling constants up to the symmetry breaking scale using standard model physics (this will be the case if there are no substantial modifications to the standard model between here and the unification scale) then we must be able to identify this GUT symmetry breaking scale as the energy at which the three coupling constants meet. However, using this assumption we can run the coupling constants all the way to M_{Pl} without encountering an energy scale at which all three are the same, within experimental bounds. Therefore there is no feasible grand unified symmetry breaking scale without new physics at an intermediary energy scale.

Proton decay

SU(5) and SO(10) grand unified theories naturally induce proton decay. In SU(5) theories this arises from the presence of the X and Y gauge bosons and the electroweak breaking Higgs field which is introduced in an irreducible SU(5) 5 representation. A pair of fields belonging to this representation play the role of the standard model electroweak doublet, while the remaining 3 components mediate interactions between both leptons and quarks leading to proton decay.

In SO(10) grand unified theories, because of the inclusion of the SU(5) 24 representation within the SO(10) adjoint, we continue to have problems with the X and Y gauge bosons. These problems are now augmented by extra gauge fields sitting in the additional nontrivial SU(5) representations furnished by the 45 the $10(4) + \overline{10}(-4) \supset 45$ under $SO(10) \downarrow SU(5)$. In addition the minimal approach to breaking SO(10) down to the electroweak scale requires an electroweak symmetry breaking Higgs field in the 10 representation of SO(10) to Yukawa couple to the fermions and thereby generate the fermion masses. The standard electroweak doublet is identified with the pair of fields sitting inside this 10 representation which transforms like $(1,2)(-3)(-2)$ under $\mathrm{SO}(10) \downarrow \mathrm{SU}(3)_{\mathrm{C}} \times \mathrm{SU}(2)_{\mathrm{W}} \times \mathrm{U}(1)_{\mathrm{Y}} \times \mathrm{U}(1)$, and the other components of this 10 representation can mediate proton decay.

Gauge boson mediated interactions, which cause proton decay, can be suppressed by setting the grand unified symmetry breaking scale to be much higher than the electroweak symmetry breaking scale. The gauge bosons mediating proton decay are associated with broken generators of the Lie group. When the Higgs field develops a vacuum expectation value (vev) which breaks the grand unified theory, they do not annihilate this vev. These gauge bosons pick up masses proportional to the grand unified symmetry breaking scale, leading to a very strong suppression of proton decay in the electroweak breaking energy regime.

However to fix problems with the Higgs mediated proton decay we have to arrange for the electroweak symmetry breaking components of the Higgs multiplet to acquire a vev at M_{EW}, while keeping the other proton decay mediating component at a considerably higher mass scale. From the perspective of the grand unified theory one set of interactions are unnaturally suppressed. This situation is highly contrived and is known as the “doublet-triplet splitting problem”.

Fermion mass degeneracy

In the standard model the fermion masses are determined by the product of the electroweak symmetry breaking Higgs vev and the Yukawa coupling λ. This follows from the form of the Higgs field’s interaction with the fermions $\mathcal{L}_{yuk} = \lambda\overline{\psi}\phi\psi$. Thus each fermion in a different representation of the weak isospin gauge group will have a different Yukawa coupling constant and the relationship between their masses will be completely unconstrained.

One of the great advantages of grand unified theories is that, by putting all the fermions in one or two representation, they cut down the number of free parameters and therefore make the theory more predictive. However this applies to the Yukawa coupling terms where the GUT makes predictions about the relationship between the masses of different fundamental particles at the GUT energy scale. These predictions carry implications for degeneracies at the electroweak breaking scale, such as $m_s/m_d = m_\mu/m_e$, which are completely incongruent with the experimentally measured standard model values.

3.7.4 Spontaneous symmetry breaking

Clearly the major problem with introducing grand unified theories is caused through the process of breaking the GUT symmetry. This makes spontaneous symmetry breaking an extremely pertinent topic in these theories.

Spontaneous symmetry breaking occurs when the action has a symmetry G, but the lowest energy configuration of the potential (the vacuum) is stabilized by a subgroup H, called the little group of G. Because the potential is invariant under G, the orbit of H in G, $\mathcal{M} = G/H$, forms a manifold of degenerate minima of the potential known as the vacuum manifold. There exists a point, P, in the vacuum manifold associated with the coset $1H$. All the elements t_l, belonging to the Lie subalgebra of H, $\mathcal{L}_H$, generate infinitesimal diffeomorphisms which fix P, while elements, m, belonging to the complement of $\mathcal{L}_H$ in the Lie algebra $\mathcal{L}$ induce parallel translation along geodesics at point P in $\mathcal{M}$.

A parallel translation (induced by m) from P to a neighbouring point in the vacuum manifold, Q, is accompanied by a continuous change in the space of diffeomorphisms which fix P to the space of diffeomorphisms which fix Q. The latter is given by gHg^{-1}, where g is the Lie group element generated by m. Physically, members of the complement of $\mathcal{L}_H$ in $\mathcal{L}$ generate symmetries of the action which nevertheless shift the vacuum of the theory associated with point P. If the vacuum at point P is designated $|0\rangle$ and has the property $H|0\rangle = |0\rangle$ then the vacuum at Q is $g|0\rangle$ and is consequently fixed by $gHg^{-1}\,g|0\rangle = g|0\rangle$.

The principle of spontaneous symmetry breaking is completely general; once illustrated in a particular context it is easily transferable to other contexts. The natural context would be electroweak symmetry breaking. However in this thesis we are not directly concerned with the phenomenological implications of electroweak symmetry breaking. Rather we spend a significant amount of time investigating the specific cases of breaking SO(10) grand unified symmetries and the more general case of breaking an internal symmetry G down to a subgroup H, especially for the case when the Higgs field is in the adjoint representation. Therefore we choose to follow Li's approach [Li74] and highlight the simplest case of breaking SU(2) ≡ O(3) to U(1) by the real triplet of Higgs fields $\vec{\phi}$. By doing this we address the theory of symmetry breaking while forgoing a parallel commentary on the standard model phenomenological context which gives significance to electroweak symmetry breaking.

Further we mention the conditions under which an adjoint Higgs field, in either an SU(5) or an SO(10) grand unified theory, condenses. We examine extra dimensional models with SU(5) or SO(10) bulk gauge symmetry. In these models an adjoint Higgs field can be used to break the grand unified theory to a subgroup on the domain-wall brane. If the Dvali-Shifman mechanism is effective this will trap the massless gauge fields belonging to this subgroup on the brane.

A toy O(3) symmetry breaking example

Consider a Lagrangian based on the O(3) gauge symmetry [Li74],

$$\begin{aligned} \vec{A}_\mu &\rightarrow \vec{A}_\mu + \vec{\theta}\times\vec{A}_\mu + \frac{1}{g}\partial_\mu\vec{\epsilon} \\ \vec{\phi} &\rightarrow \phi + \vec{\theta}\times\vec{\phi}. \end{aligned} \tag{3.86}$$

This Lagrangian, $\mathcal{L}$, can be written in terms of the field strength tensor

$$\vec{F}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\nu + g\vec{A}_\mu\times\vec{A}_\nu \tag{3.87}$$

and the generators of the adjoint representation. These generators form a triplet of 3×3 matrices with components $(T^i)_{jk} = i\epsilon_{ijk}$. In particular,

$$\mathcal{L} = -\frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} + \frac{1}{2}\left[\left(\partial_\mu - g\vec{T}\cdot\vec{A}_\mu\right)\vec{\phi}\right]^2 + \frac{1}{2}\mu^2\vec{\phi}^2 - \frac{1}{4}\lambda\left(\vec{\phi}^2\right)^2, \tag{3.88}$$

where $\vec{\phi}^2 = \vec{\phi}\cdot\vec{\phi}$.

The minimum for the potential $V = -\mu^2/2\,\vec{\phi}^2 + \lambda/4\left(\vec{\phi}^2\right)^2$ occurs at a non zero value of $\vec{\phi}^2$. Under these circumstances the third component of the Higgs triplet can develop a non zero vacuum expectation value

$$\langle\phi_i\rangle = \delta_{i3}v. \tag{3.89}$$

The dynamical field will be fluctuations about this vacuum, so it is appropriate to redefine the fields to have a vanishing vacuum expectation value $\phi_i' = \phi_i - \langle\phi_i\rangle = \phi_i - \delta_{i3} v$. Expressing the Lagrangian in terms of ϕ_i' gives

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{int}, \text{ where} \\
\mathcal{L}_0 &= -\frac{1}{4}\left(\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu\right)^2 + \frac{1}{2} g v^2 \left(A_{1\mu}^2 + A_{2\mu}^2\right) + \frac{1}{2}\left(\partial_\mu \vec{\phi}'\right)^2 + \\
&\left[\frac{1}{2}\left(\mu^2 - \lambda v^2\right)\vec{\phi}_3'^2 - \lambda v^2 \phi_2'^2\right] + \left(\mu^2 - \lambda v^2\right) v \phi_3'; \qquad (3.90)
\end{aligned}$$

here $\mathcal{L}_0$ contains all the terms that are at most quadratic in ϕ_i', while $\mathcal{L}_{int}$ contains third and fourth order terms in the perturbative fields ϕ_i' which define the interactions.

If we choose $v^2 = \mu^2/\lambda$, which corresponds to choosing to keep $\langle\phi_i'\rangle = 0$ at tree level, it becomes clear that the gauge fields $A_{1\mu}$ and $A_{2\mu}$ have gained a mass gv^2 by eating the degrees of freedom of the massless Goldstone bosons ϕ_1' and ϕ_2'. The symmetry of the theory has been broken down to a U(1) gauge symmetry of the remaining massless gauge boson $A_{\mu 3}$.

Adjoint symmetry breaking and Dvali-Shifman

Consider a Lagrangian which possesses an internal symmetry G, and a Higgs field ϕ transforming under a non-trivial representation for G. The symmetry will break whenever the minimum of the Higgs field potential is situated at $\langle\phi\rangle \neq 0$. Under these circumstances the vacuum expectation value of ϕ is not fixed by the full symmetry group. That is $\{\langle\phi\rangle\} \not\supset G \cdot \langle\phi\rangle$.

In the case of SU(n) and SO(n) adjoint representations we take the most general quartic potential,

$$V_X = -\frac{1}{2}\mu^2 \mathrm{Tr}\,X^2 + \frac{1}{3}\lambda_1 \mathrm{Tr}\,X^3 + \frac{1}{4}\lambda_2 \left(\mathrm{Tr}\,X^2\right)^2 + \frac{1}{4}\lambda_3 \mathrm{Tr}\,X^4, \qquad (3.91)$$

which is invariant under the internal symmetry. This quartic potential is a linear combination of the Casimirs invariants.

We can express the Casimir invariants as traces over products of the adjoint Higgs field. For SO(n) the condition $X^i_j = -X^j_i$ implies the third order Casimir, $\mathrm{Tr}\,X^3$, will vanish as it is a contraction over an odd number of X fields, with a permutation symmetry on the labels. It is possible to further simplify this formula for the potential by noting that there is a transformation simultaneously mapping every element in the Lie algebra onto the Cartan subalgebra. In each case we seek to minimize this potential subject to the constraint that $\mathcal{X}$ belongs to the Lie algebra of SU(n) and SO(n), respectively. This adds a Lagrange multiplier condition implying we should consider extrema of the potential which occur within the set of traceless Cartan subalgebra generators, that is, the minima of $V_{\mathcal{X}} - g\mathrm{Tr}\,\mathcal{X}$.

Now let us examine the application of this to the Dvali-Shifman example for an $\mathrm{SU}(5) \times \mathbb{Z}_2$ invariant action with a Higgs sector composed of the domain-wall brane ϕ and an SU(5) adjoint Higgs $\mathcal{X}$. The $\mathbb{Z}_2$ symmetry must be imposed to create a domain-wall brane topological defect. Under this symmetry we impose $\phi \to -\phi$ and $\mathcal{X} \to -\mathcal{X}$. The most general Higgs sector potential under these conditions is

$$V_{\mathcal{X},\phi} = \left(c\phi^2 - \mu_{\mathcal{X}}^2\right) \mathrm{Tr}\,\mathcal{X}^2 + \lambda_1 \phi \mathrm{Tr}\,\mathcal{X}^3 + \lambda_2 \left(\mathrm{Tr}\,\mathcal{X}\right)^2 + \lambda_3 \mathrm{Tr}\,\mathcal{X}^4 + \lambda \left(\phi^2 - v^2\right)^2 . \tag{3.92}$$

When we choose $cv^2 - \mu_{\mathcal{X}}^2 > 0$ and $\lim_{y \to \pm\infty} \phi = \pm v$, then we implicity choose the boundary condition $\lim_{y \to \pm\infty} \mathcal{X} = 0$, that is $\mathcal{X}$ approaches a vacuum as $y \to \pm\infty$ preserving the full SU(5) symmetry.

The domain-wall is located at $y = 0$. At $y = 0$ the kink, $\phi(0) = 0$. The potential at this point $V_{\mathcal{X},\phi=0}$ is a special case of $V_{\mathcal{X}}$ for $\lambda_1 = 0$. From solving the Lagrange multiplier equations, if $\lambda_3 > 0$ at this point[4] the vacuum for the theory causes

$$\mathrm{SU}(n) \to \mathrm{SU}(n_1) \times \mathrm{SU}(n - n_1) \times \mathrm{U}(1), \tag{3.93}$$

where $n_1 = \lceil 1/2\, n \rceil$ is the ceiling of $1/2\, n$, while if $\lambda_2 < 0$, it causes

$$\mathrm{SU}(n) \to \mathrm{SU}(n-1) \times \mathrm{U}(1). \tag{3.94}$$

From the above we conclude that $\mathcal{X} = 0$ is not the vacuum configuration on the domain-wall brane.

As $\mathcal{X} = 0$ is no longer the vacuum inside the domain-wall, by invoking Dvali and Shifman's argument, it follows that the Higgs field will condense breaking SU(5). Provided the bulk is

[4] A solution to this problem is readily available in [Li74].

in a confining regime with a mass gap the massless gauge bosons, belonging to the unbroken subgroup within the domain-wall, become trapped.

Later we will consider the stability of the domain-wall brane and adjoint Higgs field profiles. This will be done by considering small perturbations around the domain-wall and adjoint Higgs field. We say the configuration is stable, respectively unstable, if these perturbations remain small under dynamical evolution, respectively diverge. The dynamical evolution is found by solving the coupled systems for the linearized perturbation equations. In general this is a nontrivial problem.

However if we choose $\lambda_3 > 0$ and choose the SU(5) adjoint Higgs field to condense to break $SU(5) \to SU(3) \times SU(2) \times U(1)$, on the domain-wall brane, then inside the domain wall the perturbations will not become tachyonic, leading to a better chance of long term stability.

3.8 Lifshitz anisotropic scaling

Lifshitz anisotropic scaling can be used to pave the way to UV complete 4 + 1-dimensional domain-wall brane theories. Improving the renormalization properties of field theories with $(d > 3)$-spatial dimensions avoids potential problems with the convergence of lattice gauge theory stimulations of the Dvali-Shifman mechanism, for localizing gauge fields on the domain-wall brane, in the limit of infinitesimal lattice spacing.

In section 1.2 we touched on the main thrust of Lifshitz field theories. Namely a Lifshitz field theory is formulated so that the marginal operators in the action are invariant under an anisotropic rescaling of the spatial coordinates as defined in (1.2). For a critical exponent $z = 1$, the theory exhibits normal Lorentz invariance. We are primarily interested in the case $z > 1$. For this case we introduce the concept of weighted scaling dimensions, and then argue that Lifshitz scalar field theories, as well as gauge theories, are power counting renormalizable with respect to the weighted scaling dimensions. We establish that these weighted scaling dimensions are indeed the correct indicators of the renormalization properties of the theory. We do this by calculating the superficial degree of divergence of a $d+1$-dimensional Lifshitz theory, and showing that the theory is renormalizable for the case $z \geq d$, when all coupling constants have nonnegative Lifshitz weighted scaling mass dimensions.

We mention research into the key problem in this area [IRS09], namely Lorentz invariance must be recovered in the IR regime . However in this theory the maximum attainable speed of each particle has positive weighted scaling dimensions and therefore runs with energy scale. Thus as the energy rises, the particle's maximum accessible speed also increases. Furthermore the coupling constant for each individual field runs differently from the others. This is in direct conflict with special relativity, so the model must be fine tuned to guarantee all particles recover the correct speed of light at experimentally accessible energies. It is not enough to set the maximum attainable speed of two particles to be equal at the Lorentz symmetry violating scale, they must somehow converge to be equal at the electroweak scale. An interesting question is whether this pair of coupling constants both converge to the speed of light sufficiently fast in the IR to avoid detection by current experiments.

3.8.1 Weighted scaling dimensions

In a $d+1$-dimensional Lifshitz type theory the space-time manifold is foliated into a product $\mathbb{R} \times \mathbb{R}^d$ with coordinates

$$(t, x^i), \tag{3.95}$$

where $i = 1, \ldots, d$. The action is invariant under d-dimensional spatial rotations and translations but not under Lorentz boosts. The Lifshitz anisotropic scaling, or the degree of anisotropy in the space-time manifold, is characterized by the value of the critical exponent z. The critical exponent automatically sets the highest power of $\Delta^z = \left(\partial_i \partial^i\right)^z$ appearing in the action for a scalar field, ϕ. Thus if we discard all relevant operators, then the kinetic terms in the action are

$$S_{lif} = \int d^{d+1}x \left(\dot{\phi}^2 - \phi\,(-\Delta)^z\,\phi\right), \tag{3.96}$$

where we have used the breaking of the Lorentz symmetry to simultaneously absorb the coefficient of the $\dot{\phi}^2$ term into a rescaling of the temporal coordinate, and the coefficient of the spatial derivative term into a rescaling of the spatial coordinate space. This action possess a rescaling symmetry,

$$t \longrightarrow \lambda^z t, \qquad x^i \longrightarrow \lambda x^i, \qquad \phi \to \lambda^{\frac{z-d}{2}} \phi. \tag{3.97}$$

Because the structure of (3.96) automatically adapts the loop propagators, the correct dimensions to use when evaluating whether the theory is power counting renormalizable are the weighted scaling dimensions, where

$$[t]_s = -z, \qquad \left[x^i\right]_s = -1, \qquad [\phi]_s = \frac{d-z}{2}. \tag{3.98}$$

Hereinafter by $[\tau]_s$ we mean the weighted scaling dimensions of τ which are not to be confused with the usual mass dimensions. Lorentz invariance corresponds to $z = 1$.

To make expressions more compact we shall write $d + 1_z$-dimensions for a foliated space-time manifold with weighted scaling characterized by (3.98) and we drop the '1' subscript in the isotropic space-time case.

3.8.2 Power counting renormalizability of Lifshitz scalar field theories

Adding relevant operators which softly break the Lifshitz anisotropic scaling symmetry does not change the renormalization properties of the theory. Hence consider the full action [Vis09] with all the relevant operators for Lifshitz scalar fields,

$$\begin{aligned} S_{\phi-lif} &= S_{kin} + S_{int} \\ &= \int d^{d+1}x \left\{\dot{\phi}^2 - \phi\left[-c^2\Delta + \ldots + (-\Delta)^z\right]\phi\right\} - \int d^{d+1}x \sum_{n=1}^{N} g_n.\phi^n. \end{aligned} \tag{3.99}$$

Evaluating the mass dimensions of each of the coupling constant above, we find

$$[c]_s = [x]_s \,/\, [t]_s = z - 1 \qquad \text{and} \qquad [g_n]_s = d + z + (d-z)n/2. \tag{3.100}$$

Whenever Lorentz invariance is broken and $z \neq 1$, the speed of light is a dimensionful quantity which cannot be absorbed into a coordinate rescaling, so we are unable to set $c = 1$ and we expect this parameter will run with the energy scale.

Further the theory is weighted power counting renormalizable if all the coupling constants have positive weighted scaling dimensions. If $z \geq d$, then $[g_n]_s > 0$ and the normal ordered theory is weighted power counting renormalizable for all N. However if $z < d$, then we require $N \leq 2(d+z)/(d-z)$. For example, a $4 + 1_2$-dimensional theory with 6-th order interaction terms is renormalizable.

3.8.3 Power counting renormalizability of a toy Lifshitz gauge theory

Because our primary interest in Lifshitz scaling is in the application to renormalizable $4+1$-dimensional Yang-Mills field theories, we repeat this analysis in the gauge sector, for an arbitrary gauge group G.

A standard Lorentz invariant Yang-Mills action is built from contractions of the gauge invariant field strength tensor $F_{\mu\nu}$. However since we wish to introduce an extra $2z$-th order spatial differential operator acting on the gauge fields we need to separate out the electric field strength tensor and the magnetic field strength tensors

$$\begin{aligned} E_i &= \partial_t A_i - \partial_i A_0 - i\,[A_i, A_0] \\ F_{ij} &= \partial_i A_j - \partial_j A_i - i\left[A_i, A_j\right], \end{aligned} \tag{3.101}$$

where we have used the index i to denote the spatial dimensions, and the gauge fields belong to the adjoint representation of the Lie algebra for G. That is $A_i = A_i^a \mathrm{T}_a$, where $(\mathrm{T}_a)_{bc} = i f_a^{bc}$ is defined in terms of the structure constants for the Lie algebra,

$$[\mathrm{T}_a, \mathrm{T}_b] = i f_{ab}^{c} \mathrm{T}_c, \tag{3.102}$$

and $\mathrm{Tr}\left(\mathrm{T}^a \mathrm{T}^b\right) = 1/2\,\delta_{ab}$. The weighted scaling dimensions of the fields become

$$[A_0]_s = z \qquad \text{and} \qquad [A_i]_s = 1. \tag{3.103}$$

Following [CH10], we abandon the canonical Lorentz invariant action for the gauge fields and simply write down the most general action possible with two caveats. We maintain the d-dimensional spatial rotational invariance of the theory; we maintain the unitarity of the theory. Preserving unitarity in the theory requires the temporal derivative terms are no higher than second order. Thus

$$S_{A_\mu - lif} = \frac{1}{2}\int d^{d+1}x \left[\frac{1}{g_E^2}\mathrm{Tr}\ (E_i E_i) + \sum_{J\geq 2}\frac{1}{g_E^J}\sum_{n=0}^{N_J}\lambda_{J,n}D^{2n}F^J\right], \tag{3.104}$$

where we have suppressed the indices of the gauge covariant derivative D_k and of the magnetic field strength tensor F_{ij}, defined in (3.101). In the above sum the rule for enumerating these indices is that all spatial indices must be contracted, and that we include one term for each inequivalent contraction. For example, when $J = 2$ and $n = 2$ there are two inequivalent terms, $\left(D_i D_j F^{ij}\right)^2$ and $D_k D^k F^{ij} F_{ij}$. The sum over even powers of D_k is to ensure that there are no terms with an odd number of spatial indices which would break spatial rotational invariance. The weighted scaling dimension [CH10] of the electric field strength gauge coupling constant is

$$[g_E]_s = \frac{z-d}{2} + 1. \tag{3.105}$$

Hence to keep the mass dimensions of this coupling constant positive we need

$$z \geq d - 2. \tag{3.106}$$

We can find the highest order, N_J, of derivatives of a contraction of J copies of the magnetic field strength tensor by canonically normalizing the gauge field, $A_i \to \tilde{A}_i = A_i/g_E$, and then demanding that the weighted scaling dimensions of the coupling constants satisfy $[\lambda_{J,n}]_s \geq 0$ [CH10]. The result is

$$N_J = \left\lfloor \frac{z+d}{2} + \frac{z-d-2}{4}J \right\rfloor. \tag{3.107}$$

We can check these results for the Lorentz invariant case $z = 1$, where (3.105) predicts the theory will be power counting generalizable only for $d \leq 3$-spatial dimensions. However, when the critical exponent $z = 2$ it is clear that we can construct a $4 + 1_2$-dimensional gauge field action which is weighted power counting renormalizable.

3.8.4 Superficial degree of divergence

We corroborate our results from subsection 3.8.3 by showing that, when a Lifshitz scalar field theory is weighted power counting renormalizable for $N = \infty$ in (3.99), the superficial degree of divergence of the theory is always negative. We follow [Vis09] and invoke the standard result that, when the superficial degree of a Feynman diagram and of all subgraphs is negative, then the diagram is convergent. Our aim is to establish that the weighted scaling dimensions are the correct weighted power counting dimensions for this theory.

Define conjugate variables to position and time, k and ω, which carry units of momentum and energy, respectively:

$$[k]_s = 1/\,[dx]_s = 1 \text{ and } [\omega]_s = 1/\,[dt]_s = z. \tag{3.108}$$

In these units we consider the contribution to a Feynman matrix element from an integral over the internal momentum in a loop. From the integral we pick up

$$\int d\omega_l\, d^d k_l \ldots \tag{3.109}$$

which makes a contribution of $\left[\omega k^d\right]_s = z + d$ to the superficial degree of divergence of the diagram. The scalar field propagator inside the loop integral is

$$\frac{1}{(\omega_l - \omega_e)^2 - \left\{m^2 + c^2\left(\vec{k}_l - \vec{k}_e\right)^2 + \cdots + \left[\left(\vec{k}_l - \vec{k}_e\right)^2\right]^z\right\}}, \tag{3.110}$$

where $\left(\omega_e, \vec{k}_e\right)$ is some linear combination of the external momenta, and $\left(\omega_l, \vec{k}_l\right)$ are the loop energy and momenta. Each propagator makes a contribution of $-2z$ to the superficial degree of divergence. Hence if there are L loops and I propagators in the Feynman matrix element the superficial degree of divergence [Vis09] is

$$\delta = (d + z)L - 2Iz = (d - z)L - 2(I - L)z. \tag{3.111}$$

Since there is at least one propagator sitting inside every loop integral, the second term is always negative and the upper bound on this expression is $(d - z)L$. Hence in theories where

$z > d$, the superficial degree of divergence is always negative and all Feynman diagrams converge. If the theory is normal ordered, then the case $z = d$ will also be renormalizable. An alternative way to state this result is when all coupling constants have positive weighted scaling dimensions the theory is renormalizable, even in the presence of an arbitrarily large number of interaction terms $N = \infty$ in (3.99).

3.8.5 Problems with coupling constant running

Ongoing research is critically examining whether coupling constant running can plausibly cause Lorentz symmetry to emerge as an accidental symmetry in the infrared limit of $z \neq 1$, Lifshitz models [IRS09, IS10, Ans09a, Ans09b, Ans09c, CH10]. The recent insights of Hořava and Melby Thompson [HMT10] now make it more plausible that the infrared limit of a suitably-defined version of Hořava-Lifshitz gravity can closely resemble pure general relativity. Thus we can hope that some similar progress will be made for emergent relativity in general.

In the context of $4 + 1_2$-dimensional Lifshitz scalar field theory, in an example with a 6-th order potential, Iengo et. al. [IRS09] examined the coupling constant running in this theory and found that c, defined in (3.99), exhibits logarithmic running

$$c^2(E) = c^2(\Lambda)\left[1 + f\log\left(\frac{E}{\Lambda}\right)\right]^n, \tag{3.112}$$

where Λ is the energy scale at which the theory exhibits Lifshitz anisotropic scaling, while f governs the reorganization group flow, and n is a first order parameter which depends on the field ϕ. Repeating this analysis for two species of particles, the difference between the maximal attainable velocities, $\delta c^2 = c^2_{\phi_1} - c^2_{\phi_2}$, also runs logarithmicly. Iengo et. al. [IRS09] explicitly demonstrate that in general, even when the maximal attainable velocities are equal at the Lorentz violating scale, they will not be equal in the infrared regime; arranging for $\delta c^2 = 0$ in the IR requires fine tuning of the model.

3.9 Crystallographic root systems and the Weyl group

Throughout this thesis we will cultivate a strong interest in roots and weights. Here we review the concepts of roots and weights. These provide an extremely compact formalism for dealing with the structure of large grand unified groups as well as insights into lower rank groups like SU(3).

While confining gauge fields on a domain-wall brane in extra-dimensional models, through either the Dvali-Shifman mechanism or an alternative, closely related, method known as the clash of symmetries mechanism, it is necessary to understand embeddings of all sub-groups, H ⊂ G, inside a given Lie group G. Roots and weights provide a systematic way of approaching this problem and will be a vital background concept in Chapter 9.

3.9.1 Crystallographic root systems

Consider the closed reflection groups which act on a Euclidean vector space E. Let (a, b) be a bilinear form on the Euclidean space. Each reflection fixes a hyperplane in E and flips the vector orthogonal to this hyperplane. We label the hyperplanes $H^{\alpha^\vee}$, while the respective orthogonal vector, α, is called a root.

Together the roots form a *root system* Δ [Kan01]. This means that the collection of roots Δ satisfy the conditions:

Definition 3.1: *Δ is a root system if for all $\alpha, \gamma, \beta \in \Delta$:*

1. *If $\alpha \in \Delta$, then $\chi\,\alpha \in \Delta$ if and only if $\chi = \pm 1$;*
2. *The reflection of β in the hyperplane perpendicular to γ: $s^\gamma \cdot \beta = \beta - (\beta, \gamma^\vee)\gamma$. also belongs to Δ.*

We call the reflection group generated by involutions in the hyperplanes orthogonal to the roots in Δ the *Weyl group*,

$$W = \{s^\gamma |\, \gamma \in \Delta\}, \tag{3.113}$$

where the action of s^γ on Δ follows from condition (2) of the root system Definition 3.1.

By construction, the Weyl group permutes the roots in Δ. We will assume that the Weyl group is irreducible. Under these circumstances the root system has no invariant subspaces.

If we construct the Euclidean space so that $E = \operatorname{span}\Delta$, then the root system is called essential.

We say that Δ is crystallographic if, in addition to the above, for all roots $\alpha, \gamma \in \Delta$, the inner product of α with $\gamma^\vee = 2\gamma/(\gamma,\gamma)$ is an integer, that is,

$$(\beta, \gamma^\vee) \in \mathbb{Z}, \tag{3.114}$$

$\gamma^\vee$ is called a co-root. From now on we assume that we are dealing with crystallographic, essential root systems.

Every Lie algebra is associated with a crystallographic root system. Hence when looking for the embeddings of a Lie subalgebra or calculating vertex factors and interaction terms between particles belonging to nontrivial representations of the Lie group it is possible to do the calculation either by choosing an explicit representation or by understanding the relationship between the roots and the weights. Our aim is to understand all the Cartan preserving embeddings of a subgroup H inside a Lie group G. This will allow us to extend the ideas presented in section 1.3, and to index all the embeddings of the standard model gauge group inside a grand unified group. We introduce some of the theory necessary to understand the root system picture, including how to link it to matrix representations of the Lie algebra and how to generate explicit representations like the adjoint.

Simple roots

It follows from the definition of Δ, Definition 3.1, that for each root α, there exists $-\alpha$. This leads us to partition the root system into two disjoint sets: the positive and the negative roots. We elect to call a root, α, whose first non zero component is positive, a “positive root”. The corresponding negated positive root, $-\alpha$, is termed a “negative root”. Not all of these roots are linearly independent. In fact a rank l Lie algebra has l independent Cartan subalgebra generators and therefore inherits a set of l linearly independent roots $\{\zeta^{(1)}, \ldots, \zeta^{(l)}\}$, called simple roots.

The simple roots are conventionally chosen to be the l-dimensional subset of the positive roots, with the property that every positive root can be written as a non-negative linear combination of $\{\zeta^{(1)}, \ldots, \zeta^{(l)}\}$. It follows that every negative root can be written as a non-positive linear combination of the simple roots. For each collection of positive roots there exists a unique choice of simple roots satisfying this condition.

Coxeter presentations

The simple roots generate the root system in the sense that:

1 The simple roots generate the Weyl group, W. Any element of W can be expressed as a sequence of refections in the hyperplanes orthogonal to the simple roots. This gives rise to a presentation of the Weyl group, called the Coxeter presentation, generated by reflections in the hyperplanes orthogonal to the simple roots, ζ^i. If we refer to the angle between any two simple roots ζ^i and ζ^j as π/m_{ij}, then the *Coxeter presentation* is

$$W = \left\{ s^{\zeta^i} | \left(s^{\zeta^i} s^{\zeta^j} \right)^{m_{ij}} = 1, \left(s^{\zeta^i} \right)^2 = 1 \right\}. \tag{3.115}$$

The Coxeter presentation expression for each element, $s^\gamma \in W$, is not unique. However, if we define the length of an expression to be the number of reflections, s^{ζ^i}, it contains, then the relations can be used to reduce all Coxeter presentations for s^γ to a fixed minimum length. This fixed length is a property of γ relative to the choice of $\{\zeta^1, \ldots, \zeta^l\}$.

To understand the relations in (3.115), let $\mathcal{H}^{\zeta^{i\vee}}$ be the (l -1)-dimensional hyperplane orthogonal to ζ^i. Because ζ^1 and ζ^2 are linearly independent, the intersection $\mathcal{H}^{\zeta^{1\vee}} \cap \mathcal{H}^{\zeta^{2\vee}}$ is an $(l-2)$-dimensional space. The complementary space is the plane spanned by ζ^1 and ζ^2. A reflection in $\mathcal{H}^{\zeta^{1\vee}}$ followed by a reflection in $\mathcal{H}^{\zeta^{2\vee}}$, $s^{\zeta^1} s^{\zeta^2}$, is the same as a rotation by twice the angle between $\mathcal{H}^{\zeta^{1\vee}}$ and $\mathcal{H}^{\zeta^{2\vee}}$ (that is a rotation by $2\pi/m_{12}$) in the (ζ^1, ζ^2) plane. Therefore the relation $(s^{\zeta^1} s^{\zeta^2})^{m_{12}} = 1$ is equivalent to the statement that m_{12} concatenations of a rotation by an angle $2\pi/m_{12}$ is the identity transformation.

2 The Weyl group action on the root system is regular, that is, for all $\alpha, \beta \in \Delta$, there exists precisely one $s^\gamma \in W$ such that $\beta = s^\gamma \cdot \alpha$. Hence given the simple roots and the

Coxeter presentation for W, we can generate Δ.

Highest root

It follows that all roots in an irreducible crystallographic root system must belong to a lattice

$$Q = \mathbb{Z}\zeta^{(1)} + \cdots + \mathbb{Z}\zeta^{(l)}. \tag{3.116}$$

Furthermore the difference between any two roots will also belong to Q. From these two conditions it is possible to define a partial ordering on a crystallographic root system with respect to the choice of simple roots. We can rank any pair of roots $\alpha, \beta \in \Delta$, according to $\alpha > \beta$ if $\alpha - \beta \in Q^+$, where $Q^+ \subset Q$ is the subspace

$$Q^+ = \{x_1\zeta^{(1)} + \cdots + x_l\zeta^{(l)} | \, x_i \geq 0 \text{ for all } i\}, \tag{3.117}$$

of the lattice Q with strictly non-negative coefficients. For each choice of simple roots there is a unique $\zeta^0 \in \Delta$ such that $\zeta^0 > \alpha$ for all $\alpha \in \Delta$ with $\alpha \neq \zeta^0$. In appendix A.2 we show how to write the highest root of each crystallographic root system, as a linear combination of the simple roots.

Closed subroot systems

We are interested in all subspaces $\Delta_H \subset \Delta$ which are also crystallographic root systems. Once we have associated a Lie algebra with Δ, these will characterize all Lie subalgebras. From Definition 3.1 of a root system , we can see a subspace $\Delta_H \subset \Delta$ is a subroot system whenever it is closed. A closed subroot system is defined as follows.

Definition 3.2: *A closed subroot system is a root system, $\Delta_H \subset \Delta$, such that for all $\alpha, \beta \in \Delta_H$ if $\alpha + \beta \in \Delta$ then $\alpha + \beta \in \Delta_H$.*

Furthermore if Δ is crystallographic then any subroot system Δ_H is also crystallographic.

The Weyl group of the subroot system Δ_H, $W_H = \{s^\alpha | \alpha \in \Delta_H\}$, is the subgroup of W which permutes the subset of the roots belonging to Δ_H .

For each subroot system,[5] Δ_H, there is a systematic way of choosing a basis of simple roots consisting of a proper subset $I_H \subset \{\zeta^1, \ldots, \zeta^i\} \cup \{-\zeta^0\}$, the union of the simple roots for Δ and the negated highest root. The method is outlined in Box 3.3 below and follows from a straightforward application of the Borel-de-Siebenthal theorem.

The Coxeter presentation for W_H is generated by reflections in the hyperplanes $H^{\zeta^{j\vee}}$, $\zeta^j \in I_H$ orthogonal to the simple roots of Δ_H.

Box 3.3: The Borel-de-Siebenthal theorem gives a method for finding the maximal closed subroot systems, that is, of finding the subroot systems $\Delta_H \subset \Delta$ such that for all other subroot systems $\Delta_k \subset \Delta$ we have $\Delta_H \not\subset \Delta_K$. Non maximal closed subroot systems can be found by iteratively applying the Borel-de-Siebenthal theorem.

Theorem 3.2: (Borel-de-Siebenthal): *Let Δ be an irreducible crystallographic root system. Let $\{\zeta^1, \ldots, \zeta^l\}$ be the simple roots for Δ. Let ζ^0 be the highest root of Δ with respect to $\{\zeta^1, \ldots, \zeta^l\}$ so that*

$$\zeta^0 = \Sigma_i c_i \zeta^i. \tag{3.118}$$

Then the maximal closed subroot systems of Δ, up to Weyl group reflections, have simple roots

- $\{\zeta^1, \zeta^2, \ldots, \hat{\zeta}^i, \ldots, \zeta^l\}$ *where $c_i = 1$;*
- $\{-\zeta^0, \zeta^1, \ldots, \hat{\zeta}^i, \ldots, \zeta^l\}$ *where $c_i = p$ (prime);*

where "$\hat{\zeta}^i$" denotes the elimination of ζ^i.

Extended Dynkin diagram

The simple roots and Coxeter relations encode all the information necessary to generate the entire crystallographic root system. When we add the highest root, we know all the closed crystallographic subroot systems. We would like to summarize all this information in a compact and easily accessible form known as the extended Dynkin diagram.

The extended Dynkin diagram incorporates both the simple roots and the highest root along with the relationship between the simple roots in the Coxeter presentation (3.115). The caption of Figure 3.6 provides a key for reading these relations. The extended Dynkin diagrams

[5] or one of its Weyl group conjugates $s^\gamma \cdot \Delta_H$, where $s^\gamma \in W$

for all the different possible crystallographic root systems are exhibited in Figure 3.6. The restricted range of possible Dynkin diagrams allows us to classify the different crystallographic root systems.

The crystallographic root systems are broadly classified into the A_n, B_n, C_n and D_n series as well as the exceptional cases E_6, E_7, E_8, F_4 and G_2.

Consider an extended Dynkin diagram associated with a crystallographic root system Δ and the expression for the highest root $\zeta^0 \in \Delta$. If we want to find a maximal crystallographic subroot systems Δ_H then we use the Borel-de-Siebenthal theorem to choose a set of simple roots for Δ_H. Next isolate a node in the Dynkin diagram for Δ. If the associated simple root $\zeta^{(k)}$ (or highest root ζ^0) belongs to Δ_H, then we leave the node. If it does not belong to Δ_H, then we delete the node along with any adjacent edges. Do this for each node in the extended Dynkin diagram for Δ.

The end product is the Dynkin diagram for Δ_H. Using Figure 3.6 we can identify the closed crystallographic subroot system Δ_H.

To find all possible closed subroot systems of Δ this process must be repeated iteratively to obtain the non-maximal closed subroot systems.

3.9.2 Root systems

We have claimed that the simple roots and the Coxeter presentation contain all the information about the root system, and that this information can be summarized in the extended Dynkin diagram. Therefore we must be able to take the extended Dynkin diagram and reproduce the entire crystallographic root system.

To do this we take the highest root and use the Coxeter presentation of the Weyl group to produce the Weyl group orbit of ζ^0. As the Weyl group is regular this orbit generates all of Δ. For the example of the crystallographic root system E_8 we reconstruct all 248 roots from the extended Dynkin diagram. In Figures 3.8 and 3.9 we have replaced each of the 248 roots in the E_8 root system by a small node. A coloured line linking any two nodes indicates that the two roots are related by reflecting in the hyperplane orthogonal to a simple root. The extended E_8 Dynkin diagram, Figure 3.7, has been colour coordinated

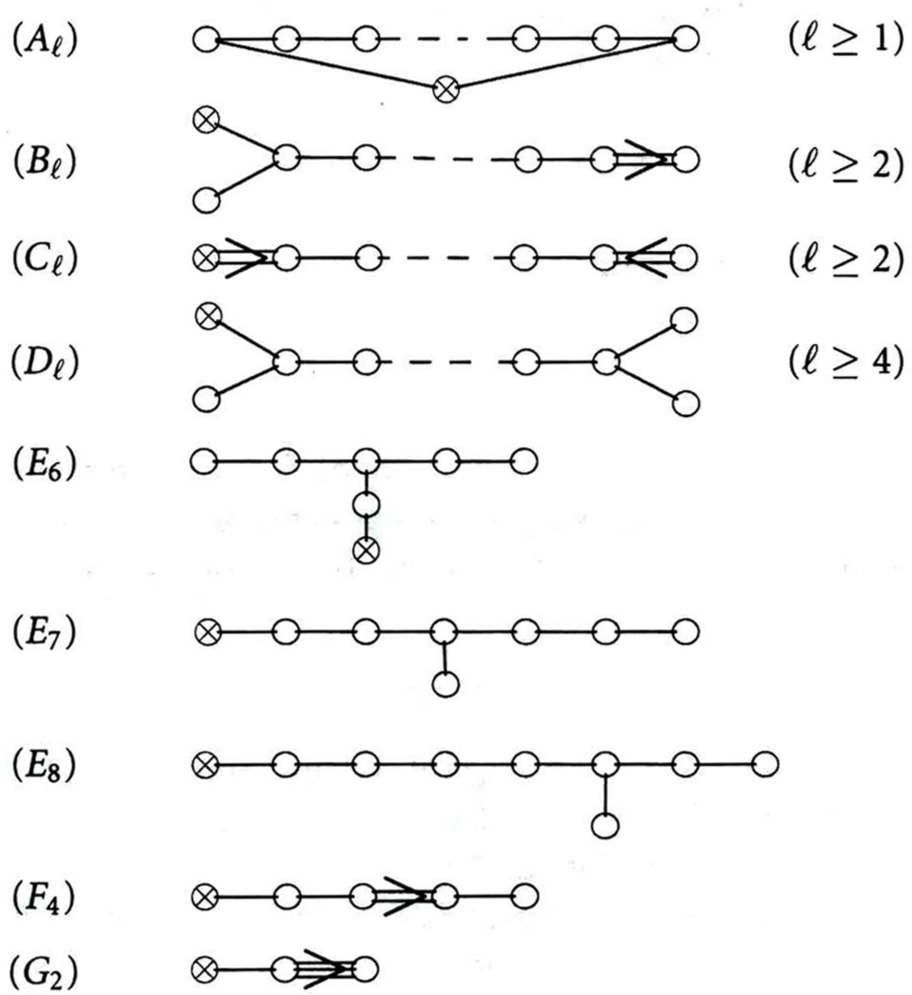


Figure 3.6: The extended dynkin diagram of an irreducible crystallographic root system has a node for each simple root. In addition there is one extra node, which is marked with a cross. This node corresponds to the highest root ζ^0. If two simple roots, $\zeta^{(i)}$ and $\zeta^{(j)}$, are related by $m_{ij} = 3$ in (3.115), then the respective nodes are joined by a single edge, while if $m_{ij} = 4$ the edge is doubled, and when $m_{ij} = 5$ the edge is tripled. These are the only possibilities for closed crystallographic root systems. Notice that some edges are directed. When the Dynkin diagram has a directed edge, some of the simple roots are shorter than others. Conventionaly the arrow points in the direction of the shorter simple roots. For example in the B_n diagram one of the simple roots, $\zeta^{(k)}$, is shorter than the others, that is, $||\zeta^{(k)}|| < ||\zeta^{(i)}||$ for all $i \neq k$. Picture from R. Kane *Reflection Groups and Invariant Theory* Springer 2001 [Kan01].

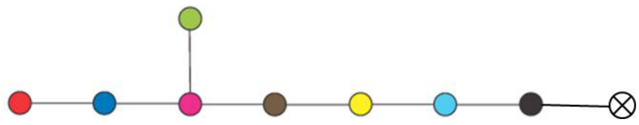


Figure 3.7: A colour coordinated extended Dynkin diagram for E_8. Picture from AIM (American Institute of Mathematics).

with Figures 3.8 and 3.9, so that if a line in Figure 3.8 or Figure 3.9 is colour X then the line represents reflection in the hyperplane orthogonal to the simple root associated with the X-colored Dynkin diagram node.

In this picture the highest root is at the top. All other roots have been obtained from the highest root by a series of reflections in the simple roots.

See appendix A.2 for a more detailed version of Figures 3.8 and 3.9 which include an explicit expression for each root belonging to a Euclidean subspace of $\mathbb{R}^8$.

3.10 Weights

We have defined our crystallographic root system with respect to a bilinear form $B(a, b) = (a, b)$ on the Euclidean space. There is a dual space to the roots under this bilinear form. The vectors in this dual space are called the weights.

The action of the Weyl group on the root system can be extended to an action on the weight space. Weyl group reflection of a weight ν in the hyperplane orthogonal to root κ is

$$s^{\kappa} \cdot \nu = \nu - (\nu, \kappa^{\vee})\kappa. \tag{3.119}$$

It is possible to define a basis for the weight space $\{\omega^1, \dots, \omega^l\}$, which is dual to the simple roots, that is

$$\omega^i \zeta^{j\vee} = \delta^{ij}. \tag{3.120}$$

We call $\{\omega^1, \dots, \omega^l\}$ fundamental weights. A linear combination of $\{\omega^1, \dots, \omega^l\}$ with non-negative coefficients is called a dominant weight.

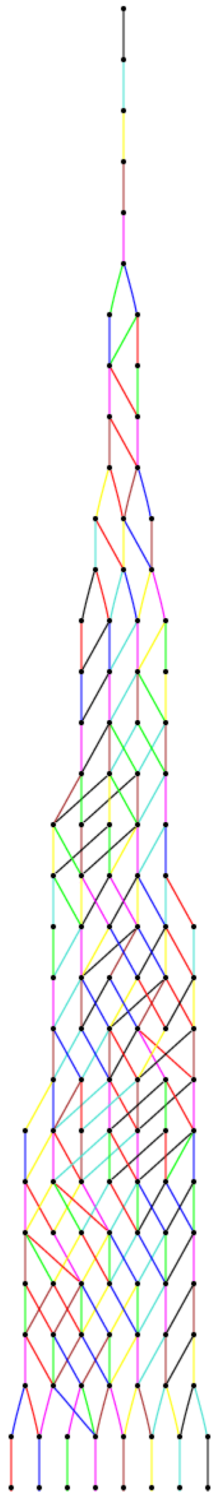


Figure 3.8: The positive roots for E_8 and the eight simple roots are represented as nodes. Edges are reflections in the simple roots, which can also be thought of as lowering operators $\tilde{f}_i$ on the root system. They are colour coordinated with the extended Dynkin diagram nodes so that if the extended Dynkin diagram node of a simple root $\zeta^{(i)}$ is colour X, then the reflection in the hyperplane orthogonal to the simple root $\zeta^{(i)}$ is colour X. The highest root is at the top of the diagram while the eight simple roots are at the bottom. Picture from AIM (American Institute of Mathematics).

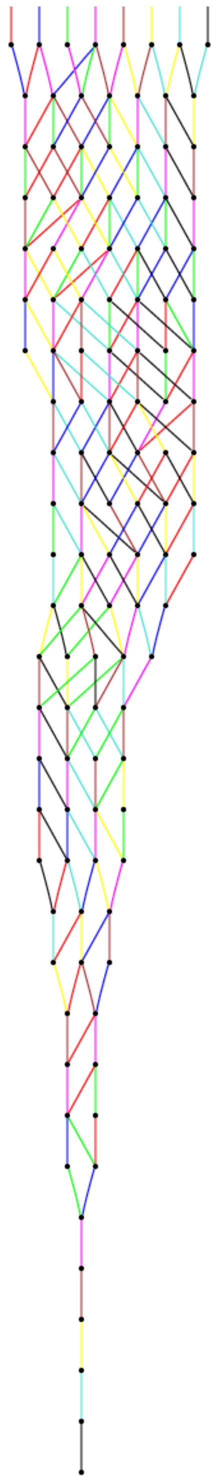


Figure 3.9: The corresponding negative roots for E_8. Picture from AIM.

3.11 Lie algebras

Crystallographic root systems are in one-to-one correspondence with semi-simple complex Lie algebras. This correspondence comes from a direct identification of the roots with elements of the adjoint representation for the Lie group.

3.11.1 An SU(2) example

Let us start with an example of this correspondence. We explicitly identify the roots for the SU(2) weak-isospin gauge group.

Consider the self-interacting gauge bosons for the weak force,

$$W_\mu^+ = \begin{pmatrix} 0 & w_\mu^+ \\ 0 & 0 \end{pmatrix}, \qquad W_\mu^- = \begin{pmatrix} 0 & 0 \\ w_\mu^- & 0 \end{pmatrix}. \tag{3.121}$$

The gauge bosons are associated with the SU(2) raising operator τ^+, and the SU(2) lowering operator τ^-, respectively. These are eigenstates of the adjoint action of the weak-isospin generator $I_2 = \operatorname{diag}(1, -1)$, that is, $\mathrm{ad}_{I_2} \cdot \tau^\pm = [I_2, \tau^\pm] = \alpha_\pm(I_2)\,\tau^\pm$. This sets $\alpha_\pm(I_2) = \pm 2$.

We say that SU(2) contains two roots α_+ and $\alpha_- = -\alpha_+$ (both of which are single component vectors). They are related by reflection in the hyperplane orthogonal to α_+, that is, $s^{\alpha_+} \cdot \alpha_+ = -\alpha_+ = \alpha_-$. If we allow the root system to inherit an inner product $(\alpha^+, \alpha^-) = B(\tau^+, \tau^-)$ from the Cartan Killing form on the Lie algebra, then the root system is crystallographic; the Cartan Killing form for the Lie algebra[6] is defined for all $T^i, T^j \in \mathcal{L}$ through

$$B(T^i, T^j) = \mathrm{trace}(\mathrm{ad}_{T^i} \cdot \mathrm{ad}_{T^j}). \tag{3.122}$$

In this context the roots are the isospin charges of W^+ and W^-:

$$\left[I_2, W_\mu^+\right] = w_\mu^+ \,[I_2, \tau^+] = 2W_\mu^+ \qquad \left[I_2, W_\mu^-\right] = w_\mu^- \,[I_2, \tau^-] = -2W_\mu^-. \tag{3.123}$$

[6]Note the trace here is over the Lie algebra, that is, the Cartan Killing form is directly proportional to the contraction $\sum_{mk} f_k^{mi} f_m^{kj}$ of the structure constants for the Lie algebra.

3.11.2 The correspondence between Lie algebras and root systems

In general, it is possible to represent a semi-simple rank l Lie Algebra, $\mathcal{L}_{\mathrm{G}}$, for a Lie group G using two types of generators:

- a set of l mutually commuting diagonalisable generators, $h^1, \ldots, h^l$, which together with the linear combinations $\Sigma_i a^i h^i$, form a Cartan subalgebra, C_G and,
- a collection of simultaneous eigenstates E^α of the adjoint action of every Cartan subalgebra generator.

By definition these generators satisfy the commutation relations:

$$
\begin{aligned}
&[h^i, h^j] = 0 \text{ and} \\
&[h^i, E^\alpha] = \mathrm{ad}_{h_i} \cdot E^\alpha = \alpha(h^i) E^\alpha.
\end{aligned}
\tag{3.124}
$$

We shall set $\alpha^i = \alpha(h^i)$, for convenience. Each eigenstate, E^α, can be labelled by an l-dimensional vector $\alpha = (\alpha^1, \ldots, \alpha^l)$ called a *root*. The root is a list of the l eigenvalues (structure constants) for the commutator, $[h^i, E^\alpha]$, of E^α with each $h^i \in C_G$. Dynkin proved that the collection of roots Δ labelling the Lie algebra generators form a crystallographic root system Δ.

The length of the roots depends on choosing a consistent normalization scheme for the generators. We fix the normalization of our Lie algebra generators by choosing $\mathrm{Tr}\,(E^\alpha E^{-\alpha}) = 2/(\alpha, \alpha)$, where (a, b) is an invariant inner product. For example, if one used an invariant inner product on the Lie algebra generators, such as the Cartan-Killing form or the regular trace and restricted this inner product to the Cartan generators, then because the root space is dual to the Cartan subalgebra this induces an invariant inner product on the root space. This is a condition known as the Chevalley-Serre basis. It guarantees the components of the roots are integers.

There are two more relations which characterize the Lie algebra,

$$[E^{\alpha}, E^{-\alpha}] = h^{\alpha} \text{ and}$$
$$[E^{\alpha}, E^{\beta}] = N_{\alpha,\beta} E^{\alpha+\beta} \text{ if } \alpha \neq -\beta, \tag{3.125}$$

where h^{α} is a linear combination of the h^i and if $\left[E^{\alpha}, E^{\beta}\right] \notin \mathcal{L}_{\mathrm{G}}$, then $N_{\alpha,\beta} = 0$.

These conditions are equivalent to the Jacobi relation. All Lie algebras must satisfy the Jacobi relation, which states that for any $T^i, T^j, T^k \in \mathcal{L}$

$$\left[T^i\left[T^j, T^k\right]\right] + \left[T^k\left[T^i, T^j\right]\right] + \left[T^j\left[T^k, T^i\right]\right] = 0. \tag{3.126}$$

It follows from (3.124) that for each root $\alpha \in \Delta$, labelling a generator $E^{\alpha} \in \mathcal{L}$, we automatically use the root $-\alpha \in \Delta$, to label the hermitian conjugate generator $E^{-\alpha} = E^{\alpha\dagger} \in \mathcal{L}$. We refer to E^{α} as a raising operator, and $E^{-\alpha}$ as a lowering operator. We use our definition from subsubsection 3.9.1.3, to classify the roots as either positive or negative, and choose a set of simple roots $\{\zeta^{(1)}, \ldots, \zeta^{(l)}\}$. A rank l Lie algebra has l independent Cartan subalgebra generators and therefore l linearly independent simple roots.

It is clear from (3.124) that each root α is the pullback of a member of the Cartan subalgebra,

$$h^{\alpha} = [E^{\alpha}, E^{-\alpha}]. \tag{3.127}$$

Multiplying this expression on the left by $h^j \in \{h^1, \ldots, h^l\}$ and taking the matrix trace we see that

$$h^{\alpha} = \alpha^{\vee}_j h^j, \tag{3.128}$$

where we have assumed an implicit sum over $j \in \{1, \ldots, l\}$ and $\alpha^{\vee}_j = g_{ij} 2\alpha^i/(\alpha, \alpha)$; again we sum over $i \in \{1, \ldots, l\}$, while $g_{ij} = \left[\mathrm{Tr}\,(h^i h^j)\right]^{-1}$ is the inverse of the $l \times l$ matrix whose ij-th element is $g^{ij} = \left[\mathrm{Tr}\,(h^i h^j)\right]$.

Linearity of the commutator bracket now allows us to extend our definition of the adjoint action to any h^β acting on the Lie algebra according to

$$\mathrm{ad}_{h^\beta} \cdot E^\alpha = \alpha(h^\beta)E^\alpha = (\alpha,\beta^\vee)E^\alpha. \tag{3.129}$$

3.11.3 The Weyl group representation in the Lie algebra space

The Weyl group has a natural analogue in the matrix picture [Hel78]. Here conjugation by the operator

$$w^\gamma = \exp(E^\gamma)\exp(E^{-\gamma})\exp(E^\gamma), \tag{3.130}$$

acts on the Cartan subalgebra according to

$$w^\gamma \cdot h^\beta = w^\gamma h^\beta w^{-\gamma} = (s^\gamma \cdot \beta^\vee)_i h^i = (s^\gamma \cdot \beta)^\vee_i h^i, \tag{3.131}$$

where $w^{-\gamma} = (w^\gamma)^{-1}$. We can check that (3.130) is a matrix representation for the Weyl group, acting as a module on the Cartan subalgebra C_G, by checking that $w^\gamma \cdot h^\beta = h^{s^\gamma \cdot \beta}$. This follows directly from the action of $w^\gamma \cdot h^\beta$ on E^α:

$$\begin{aligned} [w^\gamma \cdot h^\beta, E^\alpha] &= (s^\gamma \cdot \beta)^\vee_i [h^i, E^\alpha] = \alpha^i (s^\gamma \cdot \beta)^\vee_i E^\alpha \\ &= (\alpha, (s^\gamma \cdot \beta)^\vee)E^\alpha = [h^{s^\gamma \cdot \beta}, E^\alpha]. \end{aligned} \tag{3.132}$$

Conversely, conjugating (3.132) by $w^{-\gamma}$ gives

$$\begin{aligned} [h^\beta, w^{-\gamma}E^\alpha w^\gamma] &= (s^\gamma \cdot \beta, \alpha^\vee)w^{-\gamma}E^\alpha w^\gamma \\ &= (\beta, (s^\gamma \cdot \alpha)^\vee)w^{-\gamma}E^\alpha w^\gamma = [h^\beta, E^{s^\gamma \cdot \alpha}]. \end{aligned} \tag{3.133}$$

This leads us to conclude that $w^{-\gamma} \cdot E^\alpha = E^{s^\gamma \cdot \alpha}$. Therefore (3.130) also furnishes a matrix representation for the Weyl group acting as a module on the space of generators $\{E^\alpha |\ \alpha \in \Delta\}$.

In the matrix picture the Cartan subalgebra is an invariant subspace for the Weyl group and the Weyl group permutes the raising and lowering operators $E^{\pm\alpha}$. The Weyl group orbit of the generator, E^α, can be calculated from the Weyl group orbit of its root α.

3.11.4 Generic representations of the Lie algebra

Lie algebras also have a close connection to the concept of weights.

More generally, the physical significance of being able to simultaneously diagonalize the Cartan subalgebra is that, for any representation space of the Lie group, there exists a basis, $\mathcal{B}$, of simultaneous eigenvectors, $|\nu\rangle$, of the entire Cartan subalgebra. Each eigenvector $|\nu\rangle \in \mathcal{B}$ can be labelled by the l-dimensional vector, $\nu = (\nu^1, \dots, \nu^l) = (\nu(h^1), \dots, \nu(h^l))$, formed by listing its eigenvalues, $h^i |\nu\rangle = \nu(h^i) |\nu\rangle$, for $h^i = h^1, \dots, h^l$. These l-dimensional vectors are the weights.

Box 3.4 (An SU(2) weak force example)**:** *Consider the* SU(2)*-weak lepton doublet,*

$$l_L = \begin{pmatrix} \nu_{e_L} \\ e_L \end{pmatrix} \sim (1, 2)(-1). \tag{3.134}$$

The SU(2) *weights of the two states in this representation are the isospin quantum numbers of the fermions. The electron neutrino, ν_e, has isospin quantum number $+1/2$. This is the highest weight of the representation. The electron, e, has isospin quantum number $-1/2$. This is the lowest weight of the representation.*

In the adjoint representation the weights are the root vectors. If the Lie group representation acts as a module over a vector space of n-dimensional column vectors (analogously to the SU(2)-weak lepton doublet), then the weights are the eigenvalues under left matrix multiplication by the Cartan subalgebra generators. The eigenvector labelled by the highest weight is annihilated by all raising operators.

We defined the Weyl group action on the weight space in (3.119). We check that under the matrix representation of the Weyl group (3.130), our eigenbasis $\mathcal{B}$ transforms in the same way as (3.119).

This follows because an arbitrary representation space for the Lie group furnishes a representation space for the Weyl group. Directly computing the action of (3.130) on $|\nu\rangle \in \mathcal{B}$ shows that

$$w^{-\kappa} \cdot |\nu\rangle = \left| s^{\kappa} \cdot \nu \right\rangle . \tag{3.135}$$

The action of $h^i \in C_G$ on $w^{-\kappa}|\nu\rangle$ is given by

$$\begin{aligned}
h^i w^{-\kappa}|\nu\rangle &= w^{-\kappa}\left(w^{\kappa}h^i w^{-\kappa}\right)|\nu\rangle \\
&= w^{-\kappa}\left(h^i - \kappa^i h^{\kappa}\right)|\nu\rangle \\
&= w^{-\kappa}\left(h^i - \kappa^i \kappa_j^{\vee} h^j\right)|\nu\rangle \\
&= \left(\nu^i - (\nu, \kappa^{\vee})\kappa^i\right) w^{-\kappa}|\nu\rangle \\
&= [s^{\kappa}\cdot\nu]^i\, w^{-\kappa}|\nu\rangle \\
&= h^i\left|s^{\kappa}\cdot\nu\right\rangle, \qquad (3.136)
\end{aligned}$$

where the second equality follows by using $h^j = \Sigma_i \delta_i^j h^i$ in (3.131).

We introduce some terminology to facilitate our discussion on weights. The weights belonging to the Weyl group orbit of the highest weight are called extremal weights.

Consider a representation which has highest weight ν, and let $E^{\delta} \in \mathcal{L}$ be a generic raising operator for this representation. Then it is easy to see that each extremal weight $\mu = s^{\kappa}\cdot\nu$, where $\kappa \in \Delta$, is also the highest weight with respect to a different choice of positive roots, because $|\mu\rangle$ is eliminated by an equivalent set of raising operators $w^{\kappa}\cdot E^{\delta} \in \mathcal{L}$. However the Weyl group permutes the set of Lie algebra generators, so both representations have the same generators. We would like a way of distinguishing between these representations and others which have qualitatively different sets of generators. Id est representation with qualitatively different sets of generators are not related to each other by a similarity transformation and are also generally of different dimension.

It is necessary to work with the basis of fundamental weights $\{\omega^1, \ldots, \omega^l\}$ defined in section 3.10, and satisfying (3.120). Under these circumstances every dominant weight is the highest weight of a representation, and up to conjugation by the Weyl group every highest weight is dominant.

3.11.5 Lie subalgebras and embeddings

Lie subalgebras $\mathcal{L}_H \subset \mathcal{L}$ have generators labeled by closed subroot systems $\Delta_H \subset \Delta$; see Definition 3.2 for a definition of closed subroot systems.

When Δ_H is a closed subroot system, the Lie subalgebra $\mathcal{L}_H$ is closed, that is, for all $E^\alpha, E^\beta \in \mathcal{L}_H$ such that $N_{\alpha,\beta} \neq 0$, we have $[E^\alpha, E^\beta] \in \mathcal{L}_H$.

Therefore if Δ_H is a closed subroot system then $\mathcal{L}_H$ satisfies the Lie algebra commutation relations (3.124) and (3.125).

We can now use the Borel-de-Siebenthal theorem from subsubsection 3.9.1.4 and the matrix representation for the Weyl group (3.130) to construct a representation of the Weyl group for the subalgebra. We consider different embeddings of the subalgebra $\mathcal{L}_H$ within $\mathcal{L}$ which are related by Weyl group conjugation.

The orbit $W \cdot \Delta_H$ represents all the embeddings of Δ_H inside Δ. However we know that W_{Δ_H} maps Δ_H back onto itself, so each element in the orbit $W_{\Delta_H} \cdot \Delta_H = \Delta_H$ gives rise to the same embedding of $\mathcal{L}_H$ inside $\mathcal{L}$. Therefore the set $W/W_{\Delta_H} \cdot \Delta_H$ represents all the "qualitatively different" embeddings of Δ_H and $\mathcal{L}_H$ inside Δ and $\mathcal{L}$, respectively. By "qualitatively different" we mean the raising and lowering operators belonging to $\mathcal{L}_H$ and $w^\kappa \mathcal{L}_H w^{-\kappa}$ are distinct subsets of the full complement of raising and lowering operators belonging to $\mathcal{L}$.

One outcome is that the number of embeddings of $\mathcal{L}_H$ inside $\mathcal{L}$ is $|W/W_{\Delta_H}|$. Because the Weyl group is finite we can simplify this expression[7] to $|W|/|W_{\Delta_H}|$.

3.12 An explicit representation

We use the above to explicitly write down the matrices belonging to the adjoint representation for the Lie algebra.

We have been working with an explicit choice of bases for the Lie algebra defined through the commutation relations (3.124) and (3.125). These are a special way of writing the most general commutation relations for the Lie algebra

$$\left[T^l, T^j\right] = i f_{lj}^k T_k, \tag{3.137}$$

where f_{lj}^k are the structure constants for the Lie algebra. We highlight the parallels between

[7]This follows directly from the orbit stabilizer theorem: Suppose that a linear algebraic group G acts on the set X. If G is finite then $|G| = |G \cdot x| \cdot |\text{Stabilizer}(x)|$, that is, the order of the orbit of x, $|G \cdot x|$, divides $|G|$.

the structure constants f_{lj}^{k} in (3.137) and the structure constants $N_{\alpha\beta}$ in our special basis (3.124) and (3.125). By choosing $T^{l} = E^{\alpha}$ and $T^{j} = E^{\beta}$ in (3.137) we find that $N_{\alpha\beta} = if_{\alpha\beta}^{(\alpha+\beta)}$.

In general the adjoint representation is given by

$$i\left(T_j\right)_k^m = f_{jk}^m. \tag{3.138}$$

We know the adjoint representation satisfies the Lie algebra commutation relations (3.137), because it follows from the Jacobi identity (3.126) that

$$f_{jk}^m f_{lm}^n + f_{lj}^m f_{km}^n + f_{kl}^m f_{jm}^n = 0. \tag{3.139}$$

This implies the adjoint representation can be calculated directly from the structure constants.

For the simple roots $\zeta^{(i)}$ and $\zeta^{(j)}$, it is very easy to determine the $N_{\zeta^{(i)}\zeta^{(j)}}$. We will write $N_{\zeta^{(i)}\zeta^{(j)}} \equiv N_{ij}$. This implies we can easily find the adjoint Lie algebra elements $E^{\zeta^{(i)}}$. Using $E^{\zeta^{(i)}}$ we can build a full Lie algebra adjoint representation satisfying the commutation relations (3.124) and (3.125).

Firstly we outline how to get the N_{ij} and then we outline how to generate all $E^{\alpha} \in \mathcal{L}$ from the $E^{\zeta^{(i)}}$.

For a crystallographic root system Δ we have the following result.

Lemma 3.1: *Let $\alpha, \beta \in \Delta$ satisfy $\alpha + \beta \in \Delta$. Let q be the largest integer j such that $\beta - j\alpha \in \Delta$. Then*

$$N_{\alpha\beta} = \pm(q+1). \tag{3.140}$$

Now draw the root diagram for the crystallographic root system (see Figure 3.8 for an example) and find all edges in the diagram which correspond to Weyl group reflections in a simple root $\zeta^{(i)}$. If two roots in the diagram are related by $\alpha = s^{\zeta^{(i)}} \cdot \beta$ then $\alpha = \beta - (\beta, \zeta^{(i)\vee})\zeta^{(i)}$. The root diagram catches all of these relations and therefore we can find the length of each of these $\beta, \zeta^{(i)}$ chains. From here we find all $N_{\zeta^{(i)}\beta}$ and hence we can construct $E^{\zeta^{(i)}}$.

To find a matrix E^{α} for some $\alpha \in \Delta$ we first identify α with a member of the root lattice (3.116) by writing α as a linear combination of the simple roots. If for example $\alpha = \zeta^{(1)}+\zeta^{(2)}$ then from (3.124) we know that $E^{\alpha} \propto \left[E^{\zeta^{(1)}}, E^{\zeta^{(2)}}\right]$. If α has a more complicated expression it will be necessary to use nested commutators. In every case once we know the root diagram we can write down an explicit expression for each root as a linear combination of the simple roots and find a series of nested commutators which tell you how to construct the equivalent generator.[8]

This formalism will be used extensively in Chapter 9 where we are interested in finding all the embeddings of a subgroup H inside a Lie group G, be examining the relationship between the crystallographic subroot systems $\Delta_{\mathrm{H}} \subset \Delta$

[8] Informally for the E_8 example the method would be to start at the bottom of Figure 3.8 and top of Figure 3.9 with the 8 simple roots and then look at the row directly below, call this row the first row. Every root in the first row is related by a Weyl group reflection to a simple root. Choose a specific root in this first row α, such that $\alpha = s^{\zeta^{(i)}} \cdot \zeta^{(j)}$. Then we can choose $\alpha = \zeta^{(j)} - (\zeta^{(j)}, \zeta^{(i)\vee})\zeta^{(i)}$. Now we can construct E^{α} from $E^{\zeta^{(i)}}$ and $E^{\zeta^{(j)}\dagger}$. Move down the diagram another row and use the same argument: every root in the second row is related by a Weyl group reflection to a root in the first row. This will allow us to construct the Lie algebra element E^{α} for each of the roots in Figure 3.9. To complete the Lie algebra for each E^{α} we need to add the conjugate transpose matrix $E^{-\alpha}$.

4

SO(10) domain-wall brane models

As indicated in the introduction, over the last decade extra spatial dimensions have become an eligible and phenomenologically interesting extension to the standard model.

RS2 [RS99b] is of considerable interest because the extra dimension is treated analogously to the usual three dimensions of space, unlike the compact extra dimension paradigm. However, pure RS2 leaves open the question of what the brane is (D-brane of string theory?) and why non-gravitational fields are localized to it. Also, the extra dimension is not exactly on the same footing as the other dimensions, because the placement of the fundamental brane explicitly breaks translational invariance along the extra dimension.

We would like to bring together the RS2-type localized 3 + 1-dimensional gravity on a 3-brane [RS99b] with the field theoretical mechanisms for creating a brane and localizing standard model fields. By doing this we can create a model for a $3+1$-dimensional universe embedded as a hyperplane in 4 + 1-dimensional space-time. Such a universe is brane-like, but it has finite thickness and the origin of the brane is specified. Brane formation is now an instance of the spontaneous rather than the explicit breakdown of translational invariance. Furthermore, one can envisage that *all* fields are localized for dynamical reasons to the brane, not just the graviton as in pure RS2. Thereby we can reproduce realistic 3 + 1-dimensional gravity and standard model phenomenology on the brane.

Our model is meant to be a very simple illustrative example. We are not trying to introduce a complete theory and we have neglected certain standard model processes. For example, in this chapter we introduce only the first generation of fermions and, of course, thereby neglect neutrino mass mixing. The theory can indeed accommodate these phenomena, through an extension of [CV11]. However to incorporate these features correctly we would need to do a full phenomenological parameter fitting. We feel it is premature, and the concepts in this model are easier to understand without the fiddly technical details.

We acknowledge that Yang-Mills gauge theory is non-renormalizable in 4 + 1-dimensions. In this chapter we assume that our model is effective up to some ultraviolet cutoff scale. We adopt the position that there exists a putative ultraviolet complete theory, which will take over above this cutoff scale. In chapter 6 we will examine a candidate ultraviolet complete theory.

Furthermore our model cannot directly address the hierarchy problem and will not lead to the unification of gravity and field theory (the two original reasons for introducing extra-dimensions). However it is interesting that we have all the ingredients necessary to create this model.

We are partially motivated by a recent model (to be called the DGV model hereinafter) that may realise the dynamical localization of an $SU(3)_C \times SU(2)_W \times U(1)_Y$ gauge theory plus gravity to a solitonic domain-wall brane with one warped extra dimension [DGV08]. The DGV paper speculates that Dvali-Shifman gauge field localization is implemented by confining SU(5) bulk gauge dynamics.

The purpose of this chapter is to extend this model by enlarging the bulk gauge group to SO(10). This allows the domain-wall localized gauge theory to be either $SU(5) \times U(1)_X$ or the left-right symmetric model, depending on the choice of Higgs potential.

In the next section, we discuss the qualitative features of this kind of domain-wall brane model, as a warm up for the detailed constructions presented in sections 4.2 and 4.3. We wrap up this chapter with a discussion of salient points for our model.

4.1 The Dvali-Shifman domain-wall brane setup

The most basic structure in domain-wall brane models is a kink configuration for a real scalar field. We assume the Higgs sector of our model contains the $\mathbb{Z}_2$ invariant ϕ^4 Lagrangian, (3.38), for a real scalar field ϕ. Taking the coordinate y as the extra dimensional axis, we choose the scalar field to asymptotically approach the vacuum configurations $\lim_{y\to\pm\infty}\phi(y) = \pm v$. This selects the kink solution (3.39) to the dynamical equations for this theory, where v is the $\mathbb{Z}_2$-breaking vacuum expectation value, while $\frac{\sqrt{\lambda}v}{2}$ is the inverse width of the kink. From subsection 3.4.1 we know that the energy density profile for this solution is localized around the zero of the kink ($y = 0$); the real scalar field has condensed to form a domain wall about $y = 0$. The region away from the 3 + 1-dimensional domain wall is called the bulk.

The prototype domain wall must be embedded in a considerably richer theoretical structure in order to produce a model that might have realistic phenomenology. In particular, the requirement that massless gauge bosons be dynamically localized to the wall is most plausibly met by invoking the Dvali-Shifman mechanism [DS97]. In subsection 3.6.2 we explained that it is possible to extend the Dvali-Shifman mechanism to trap a non-Abelian, H-gauged, pure Yang-Mills gauge theory on the wall. This required the bulk gauge group G ⊃ H. Additionally the bulk gauge theory must be in a confinement phase.

The DGV model uses G = SU(5) and H = $SU(3)_C \times SU(2)_W \times U(1)_Y$, thus potentially realising a domain-wall localized standard model. It is also a novel reinterpretation of SU(5) grand unification. As with standard SU(5) theories, the extension to SO(10) immediately suggests itself because all the standard model fermions of a given family may be assembled into a single irreducible representation, the 16. Thus, in this chapter we start with G = SO(10). Also we need a discrete symmetry outside of SO(10) for topological-stability reasons; we choose $\mathbb{Z}_2$ for simplicity and economy.[1]

Therefore we consider an SO(10) × $\mathbb{Z}_2$ gauge-invariant Lagrangian, eventually to be coupled to a Randall-Sundrum warped 4 + 1-dimensional metric in order to dynamically localize the graviton. Group-theoretically extrapolating the original argument presented by Dvali and Shifman, an SO(10) singlet scalar field in conjunction with an SO(10) adjoint Higgs is used to dynamically generate the required domain wall. The singlet Higgs takes on a kink configuration, while the adjoint Higgs configuration is non-zero inside the wall,

[1] An earlier attempt at SO(10) domain-wall brane models can be found in [SV04]; it is based on the clash of symmetries idea [DTVW02]. However, these schemes are unable to produce phenomenologically acceptable fermion localization [Cur06].

spontaneously breaking SO(10), and asymptotes to zero as $|y| \to \infty$, thus restoring SO(10) in the bulk. For quartic potentials, we shall show that the unbroken group inside the wall is $SU(5) \times U(1)_X$, while the extension to sixth order permits the further breaking to the left-right group $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$. If the Dvali-Shifman mechanism is operative, then the gauge bosons of these subgroups are dynamically localized.

Chiral 3 + 1-dimensional fermions are localized through the split fermion mechanism from section 3.5. These fermions interact with both the domain-wall brane and adjoint Higgs field. In this scenario the adjoint Higgs field plays the role of the $\rho(y)$ field from subsection 3.5.1. This causes each fermion to become confined around a 3 + 1-dimensional hypersurface parameterised by certain values of the bulk coordinate y. These values differ according to fermion species, because the adjoint Higgs configuration "splits" their localization points [RS83].

For completeness we discuss the subsequent breaking of $SU(5) \times U(1)_X$, respectively, of $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ to the standard model on the domain-wall brane. This will require additional Higgs fields which must condense inside the domain wall as well as some fine tuning conditions to induce the required gauge hierarchies.

An important advantage of this type of 4 + 1-dimensional grand-unified theory is that the usual tree level mass relations between the quark and lepton masses, outlined in subsubsection 3.7.3.3, no longer appear. This is precisely because the fermions are split along the extra dimension. Each fermion's 3 + 1-dimensional mass scale depends on overlap integrals of the fermion's bulk profile with the extra-dimensional profile functions of the additional Higgs fields which we introduced to break the symmetry down to the standard model gauge group. For the $SU(5) \times U(1)_X$ case, the bulk profiles for the fermions depend on their distinct $U(1)_X$ charges, thus these overlap integrals contribute factors to the 3 + 1-dimensional mass parameters which are different for fermions in different $SU(5) \times U(1)_X$ representations. Hence the tree level mass relations are reduced to the $SU(5) \times U(1)_X$ subset of the normal SO(10) mass relations. A similar effect occurs for the left-right symmetric alternative model.

4.2 The SO(10) → SU(5) × U(1)$_X$ model

Our goal is an SU(5)×U(1)$_X$ gauge theory on the 3+1-dimensional brane with a full SO(10) unified theory in the bulk. For simplicity we shall start with a Minkowski flat action. Then we compile our theory using the following algorithm:

- Construct a domain wall using a scalar SO(10) singlet.
- Use an adjoint Higgs to break SO(10) to SU(5)×U(1)$_X$ on the domain wall and invoke Dvali-Shifman gauge field localization.
- Confine zero mode chiral fermions.
- Add a Randall-Sundrum warped metric.
- Discuss breaking SU(5) × U(1)$_X$ to the standard model on the domain wall as well as electroweak symmetry breaking.

4.2.1 Domain wall construction and gauge field localization

We consider a Higgs sector constituted by an SO(10) Higgs singlet $\phi \sim 1$ and an adjoint Higgs $\mathcal{X} \sim 45$. We impose the discrete Z_2 symmetry, $y \to -y$, $\phi \to -\phi$, $\mathcal{X} \to -\mathcal{X}$. The Higgs potential is

$$V_{\mathcal{X},\phi} = -\frac{\mu^2}{2}\mathrm{Tr}\mathcal{X}^2 + \frac{\lambda_1}{4}(\mathrm{Tr}\mathcal{X}^2)^2 + \frac{\lambda_2}{4}\mathrm{Tr}\mathcal{X}^4 + \frac{\kappa}{2}\phi^2\mathrm{Tr}\mathcal{X}^2 + \frac{\lambda}{4}(\phi^2 - v^2)^2, \tag{4.1}$$

where we have truncated our potential at fourth order. We argue that the general expression for $V_{\mathcal{X},\phi}$ can be expanded as a polynomial in the fields $\mathcal{X}$ and ϕ where higher order terms are suppressed in an effective low energy theory by their dimensionful coupling constants. We include 4th order terms because orthogonality of the SO(10) generators remonstrates that truncating our potential at second order would have introduced an accidental symmetry whereby each of the 45 SO(10) adjoint-Higgs field components could be transformed independently under SO(10). Since our theory is non-renormalizable in 4 + 1-dimensions and is therefore considered to be effective only up to an ultraviolet cutoff we do not expect pathologies from the dimensionful coupling constants. The adjoint Higgs multiplet is represented by the 10×10 antisymmetric matrix, $\chi = \sum_i T_i \chi_i$, where the T_i are the generators of the fundamental representation normalized so that $\mathrm{Tr}\left(T_i T_j\right) = \frac{1}{2}\delta_{ij}$. In a certain regime of parameter space the vacuum manifold is $(\chi = 0, \phi = \pm v)$. From section 3.4, the solitonic solutions to the Euler-Lagrange equations must obey Dirichlet boundary conditions, where we require both Higgs fields to asymptotically approach vacuum configurations. We choose the boundary condition to be $(0, -v)$ at $y = -\infty$ and $(0, v)$ at $y = +\infty$.

Since the vacuum manifold is not connected, the relevant homotopy group which describes the different maps from the boundary of the real line onto the vacuum manifold $\pi_0(\mathrm{SO}(10) \times \mathbb{Z}_2/\mathrm{SO}(10))$ is non-trivial. Our choice of boundary conditions ensures the kink belongs to a different topological class than the homogeneous vacua $\phi = +v$ and $\phi = -v$. Any solution to the Euler-Lagrange equations satisfying the boundary conditions $\lim_{y\to\pm\infty}(\chi(y), \phi(y)) = (0, \pm v)$ cannot spontaneously evolve into a solution from another topological sector, irrespective of relative energy densities. Id est, any map such that the image includes both disconnected pieces cannot be deformed continuously to give the trivial map from S^0 to $\mathrm{SO}(10) \times \mathbb{Z}_2/\mathrm{SO}(10)$.

Under these conditions, section 3.4 explains that the stable solution to the Euler-Lagrange equations is the minimum energy density solution belonging to this topological sector. Like Dvali and Shifman, we identified stable solutions by checking the dynamical evolution of perturbative linear modes. Also we checked our results numerically to account for higher order effects. The energy density of each solution is dependent on the coupling constants, hence different solutions are stable in different coupling constant regimes.

We find that $\lambda_2 = 0$ is a bifurcation point for the manifold of solutions to the Euler-Lagrange equations. When $\lambda_2 > 0$ the SO(10) adjoint field will condense about $y = 0$, so that the non-zero components of χ arrange for $\mathrm{SO}(10) \to \mathrm{SU}(5) \times \mathrm{U}(1)_X$ on the domain wall. When $\lambda_2 < 0$ our analysis indicates that to first order the χ field components will try to adopt a configuration which breaks $\mathrm{SO}(10) \to \mathrm{U}(1)^5$ on the domain wall. A more detailed discussion is given in chapter 5.

Let us take a closer look at the $\lambda_2 > 0$ scenario. Solutions to the Euler-Lagrange equations persist for a wide range of parameter values. Purely for the sake of convenience, however, we choose to focus on a concrete analytic solution that exists provided the parameters obey

$$(4\mu^2 - 2\kappa v^2 + \lambda v^2)(20\lambda_1 + 2\lambda_2) - 5\kappa(4\mu^2 - \kappa v^2) = 0. \tag{4.2}$$

We emphasize that this is not a fine-tuning condition. Rather, it defines a special slice through parameter space that happens to admit an analytical solution.

Under these circumstances the Euler-Lagrange equations have a stable solution of the form,

$$\phi(y) = v \tanh(my), \qquad \mathcal{X}_1 = A\mathrm{sech}(my), \tag{4.3}$$

when $A^2 = \frac{40\mu^2 - 10\kappa v^2}{10\lambda_1 + \lambda_2}$, $m^2 = -2\mu^2 + \kappa v^2$ and $\mathcal{X}_1$ is associated with the U(1)$_X$ generator which we embed inside SO(10) according to $T_1 = \frac{1}{\sqrt{20}}\mathrm{diag}(\tau_2, \tau_2, \tau_2, \tau_2, \tau_2)$, where τ_2 is the second Pauli matrix. Aside from having a closed form, this solution is convenient since the dynamical equations for the first order terms in a perturbative expansion about this solution can be transformed into hypergeometric differential equations. In chapter 5 we solve these equations analytically and show that all perturbative modes are oscillatory. Thus, the solution is stable against further condensation of the remaining 44 $\mathcal{X}$ field components.

As promised, this solution spontaneously breaks the SO(10) gauge symmetry down to SU(5) × U(1)$_X$ on the domain wall. So if we assume the Dvali-Shifman mechanism to be effective, that is assume the 4 + 1-dimensional bulk SO(10) gauge theory is in a confining phase which is valid up to some ultraviolet cutoff Λ_{UV}, then SU(5) × U(1)$_X$ gauge field localization will follow as a consequence of the background Higgs field configuration.

Having established the existence of a stable Dvali-Shifman domain-wall solution, we now turn our attention to the localization of fermions.

4.2.2 Localizing Fermions

We use the split fermion mechanism developed in section 3.5 to localize the first generation of standard model fermions plus the right handed neutrino.

Throughout this chapter we continue to use the notation $\Gamma^M = (\gamma^\mu, -i\gamma^5)$, to represent the extended Dirac algebra. These matrices satisfy $\{\Gamma^N, \Gamma^M\} = 2\eta^{MN}$. The fermions are contained in a $\Psi \sim 16$ representation of SO(10), and have the Lagrangian

$$\mathcal{L}_{\text{Yukawa}} = i\bar{\Psi}\Gamma^M \partial_M \Psi - h_\chi \bar{\Psi}\sigma^a \sigma^b \chi_{ab}\Psi - h_\phi \phi \bar{\Psi}\Psi, \tag{4.4}$$

where χ_{jk} is the j,k-th entry of the 10×10 antisymmetric matrix used to represent χ and σ^j is a member of the 10 dimensional Clifford algebra [Li74].

Box 3.2, in subsection 3.7.2, explains how to contract a general Clifford algebra. In (4.4) we need the 10-dimensional Clifford algebra:

$$\sigma^{2k-1} = 1 \otimes \cdots \otimes 1 \otimes \tau_1 \otimes \tau_3 \otimes \cdots \otimes \tau_3, \tag{4.5}$$

$$\sigma^{2k} = \underbrace{1 \otimes \cdots \otimes 1}_{k-1} \otimes \tau_2 \otimes \underbrace{\tau_3 \otimes \cdots \otimes \tau_3}_{5-k}, \tag{4.6}$$

where it is understood that the index k runs over $\{1, \ldots, 5\}$.

To maintain the discrete reflection symmetry we now simultaneously transform $\Psi \to i\Gamma^5\Psi$ in conjunction with $y \to -y$, $\phi \to -\phi$ and $\chi \to -\chi$.

Each spinor $\Psi_i(x, y)$, where the index i denotes the different irreducible $\text{SU}(5) \times \text{U}(1)_X$ components of the 16, can be decomposed in terms of a set of simultaneous eigenfunctions $\psi_{nLi}(x)$ and $\psi_{nRi}(x)$ for the 3 + 1-dimensional Dirac Hamiltonian and (to remove the degeneracy) the 3 + 1-dimensional chirality operator γ^5. The eigenfunctions, which appear as non-zero coefficients in our expansion, will be the 3 + 1-dimensional chiral zero mode, ψ_{0L} or ψ_{0R} depending on the case, plus a finite number of discrete massive modes, as well as a continuum of massive modes, as per

$$\Psi_i(x, y) = \sum\!\!\!\!\!\!\int_n \{f_{nLi}(y)\psi_{nLi}(x) + f_{nRi}(y)\psi_{nRi}(x)\}, \tag{4.7}$$

where the sum is understood to include an integration over the continuum. This is a dimensional reduction or generalised Kaluza-Klein procedure whereby the dependence on the y coordinate of $\Psi_i(x, y)$ is subsumed into a complete set of mode functions or profiles $f_{n,L/R,i}$, and the 4 + 1-dimensional field is redescribed as an infinite tower of 3 + 1-dimensional fields. The discrete mode functions are square-integrable, while the modes from the continuum are delta-function normalizable.

We want the 4 + 1-dimensional Dirac fermions in the 16 representation of SO(10) to supply 16 left-handed 3 + 1-dimensional zero modes to be identified with a family of quarks and leptons. To display these modes, it suffices to truncate the full mode decomposition to just the chiral zero mode term $f_{0Li}(y)\psi_{0Li}(x)$. Now this ansatz is substituted into the 4 + 1-dimensional Dirac equation with the background domain-wall configuration playing the role of the mass term:

$$0 = i\Gamma^M \partial_M \Psi(x,y) - h_X \sigma^a \sigma^b X_{ab}(y)\Psi(x,y) - h_\phi \phi(y)\Psi(x,y). \tag{4.8}$$

The separation of variables allows us to isolate the dependence on the y-coordinate, after requiring that each of the ψ_{0Li} be left chiral and obey the usual massless 3 + 1-dimensional Dirac equation. Fortunately, for $X = A\mathrm{sech}(my)T_1$ the matrix $\sigma^a\sigma^b X_{ab}(y)$ is diagonal. We label the diagonal entry belonging to row i and column i by $(\sigma^a\sigma^b X_{ab}(y))_{ii}$, where there is no intended sum over the index i. The differential equations decouple and the bulk localization profiles for the fermions are then easily found to be

$$f_{0Li}(y) = N_i e^{-\int_{y_0}^{y} dy'\, h_X(\sigma^a\sigma^b X_{ab}(y'))_{ii} + h_\phi \phi(y')}, \tag{4.9}$$

where N_i is a normalization constant. We choose the signs of the coupling constants, h_X and h_ϕ, so that the integrand cuts the axis with a positive slope. The asymptotic behaviour of the kink implements the localization of each fermion to a 3+1-dimensional hyperplane coplanar with the zero of its respective integrand, which occurs at $y = y_0$. Due to coupling to the SO(10) adjoint Higgs, fermions belonging to different SU(5) representations are localized around different parallel hyperplanes in the bulk, see Figure 4.1.

These 3 + 1-dimensional localized massless fermions are our SU(5) brane world matter. In flat space-time there is a mass gap between the zero mode and the lowest allowed massive 3 + 1-dimensional mode. Hence, at low energies, interactions between 3 + 1-dimensional zero mass modes can only produce other zero mass fermions, [Rub01]. This, in conjunction with the Dvali-Shifman gauge boson localization conjecture, produces a candidate low-energy theory with dimensional reduction down to 3 + 1-dimensions.

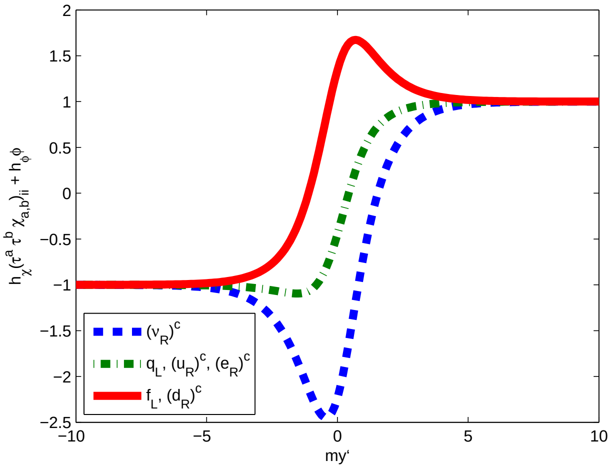


Figure 4.1: The graph displays the integrand in the exponent of (4.9) which gives the extra dimensional profiles of the left handed zero modes associated with first generation fermions in the SU(5)×U(1)$_X$ brane world model. The graph shown here corresponds to the parameter choices $h_\phi = h_\chi = 1$ and the flat space background solution to the Euler-Lagrange equations $\chi_1 = A\text{sech}(my)$, $\phi = v\tanh(my)$ with $A = v = 1$. $3+1$-dimensional left chiral fermions will be confined to the hyperplane corresponding to the zero of their integrand.

We need to explain the physical interpretation we give to individual terms in our mode decomposition. We argued in section 3.5 that equation (4.7) is simply a convenient way of projecting the $4+1$-dimensional field onto a collection of $3+1$-dimensional coefficients $\psi_{nL\backslash Ri}(x)$ by resolving the y-dependence of the field in terms of a complete set of orthonormal functions $f_{nL\backslash Ri}(y)$ which span the 1-dimensional rigged Hilbert space. We did this because it allows us to see how $3+1$-dimensional states with definite mass feel the classical background. It is necessary to treat the different $3+1$-dimensional eigenfunctions of γ^5 independently since we have associated Γ^5 with ∂_y. However, mathematically (4.7) is simply a convenient way of writing the field.

Since the physics is basis independent, we can choose to expand a general field, $\Xi(x,y) = \Sigma_n g_n(y)\xi_n(x)$ $(x \equiv x^\mu)$, in terms of any complete set of states $\{g_n(y)\}$. For example, if the $\Xi(x,y)$ is an SO(10) gauge singlet then, without loss of generality, we may conveniently choose the $\{g_n(y)\}$ so that the coefficients $\{\xi_n(x)\}$ are solutions to the massive $3+1$-dimensional Klein-Gordon equation. In this case, the profile functions, $g_n(y)$, appearing in the above expansion, correspond to physical states of the effective $3+1$-dimensional theory which propagate as free particles in the confining bulk SO(10) gauge theory. In our case the $\psi_{nL\backslash Ri}(x)$ are charged under U(1)$_X$ and will be forced, by a confining bulk, to propagate as constituent particles of a bound SO(10) gauge singlet. This is conceptually akin to writing the QCD Lagrangian in terms of quarks and gluons with the implicit understanding that the propagating states are hadrons. Since SO(10) is broken inside the wall, the 3+1-dimensional fields behave in a manner consistent with the standard model gauge interactions.

We argue that, because our zero mode profile functions are sharply peaked around the domain wall and the SO(10) confinement dynamics are suppressed here, our classical localization profiles give us a reasonable first order approximation for the landscape of these low energy particles. See [DGV08, Geo09] for more details.

4.2.3 Adding warped gravity

The last vital component of our model is warped gravity. We search for solutions to the Einstein-Klein-Gordon equations for the action

$$S = \int d^5x\,\sqrt{G}\{-2M^3R - \Lambda + \mathcal{T} - V_{\chi\phi} + \mathcal{L}_{\text{Yukawa}}\}, \tag{4.10}$$

where G is the determinant of the five-dimensional metric tensor, M is the 5-dimensional fundamental gravitational mass scale, R is the Ricci scalar, Λ is the bulk cosmological constant, and $\mathcal{T}$ is simply the gauge field and Higgs boson kinetic terms. We would like a solution with 3 + 1-dimensional Minkowski space and zero mode gravitons localized on our field theoretical brane. To achieve a Minkowski metric on the brane we impose a fine tuning condition on the bulk cosmological constant. The necessity of balancing the bulk cosmological constant against the brane tension is a feature of Randall-Sundrum like models. At the same time we need the qualitative forms of both χ and ϕ to be similar to the flat-space case. Substituting a Randall-Sundrum warped metric ansatz,

$$ds^2 = e^{-p(y)/6M^3}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2, \tag{4.11}$$

into the Einstein-Klein-Gordon equations produces four coupled second order differential equations which exhibit appropriate solutions for a wide range of parameter values. By choosing, as before purely for convenience, our parameters to lie on the manifold,

$$\begin{aligned} 6m^2M^3 + v^2m^2 - 3M^3\lambda v^2 + 3M^3\kappa v^2 &= 0, \\ -3m^2M^3 - v^2m^2 - 6M^3\mu^2 + 3M^3\kappa v^2 &= 0, \\ 20\lambda_1 + 2\lambda_2 - 10\kappa + 5\lambda &= 0, \end{aligned} \tag{4.12}$$

we find an analytic solution of the form:

$$p(y) = v^2 \ln(\cosh[my]), \qquad \phi(y) = v \tanh[my], \qquad \mathcal{X}_1 = 2v\text{sech}[my]. \tag{4.13}$$

This solution is consistent with all the localization properties mentioned above. It provides the same Higgs configuration which we postulate induces Dvali-Shifman SU(5) × U(1)$_X$ gauge field localization on the brane.

Box 4.1: We also know the solution (4.13) has a localized zero mode graviton. In fact we can use Peano's theorem to find the zero mode perturbation around the solution to Einstein equations. Note that when we are dealing with second order differential equations we replace y by a vector of fields $\vec{y} = (y_1, y_2)$ and then replace (4.14) with the initial value problem for the coupled system of differential equations $y_2 = y_1'$ and $y_2' = f(t, \vec{y}, z)$, with $\vec{y}(t_0) = (y_{1_0}, y_{2_0})$.

Theorem 4.1: (Peano's Theorem from [Har64]) Let $f(t, y, z)$ be continuous on an open (t, y, z)-set E and possess continuous first order partials $\frac{\partial f}{\partial y^k}$, $\frac{\partial f}{\partial z^j}$ with respect to the components of y and z:

- Then the unique solution $y = \eta(t, t_0, y_0, z)$ of the initial value problem

$$y' = f(t, y, z) \qquad y(t_0) = y_0 \tag{4.14}$$

  is of class C^1 on an open domain $\omega_- < t < \omega_+$, $(t_0, y_0, z) \in E$, where $\omega_\pm$ are functions of the initial conditions, and of z, which you can think about as a free parameter in your equation, like a coupling constant. That is $\omega_\pm = \omega_\pm(t_0, y_0, z)$.

- Furthermore, if $J(t) = J(t, t_0, y_0, z)$ is the Jacobian matrix $\left(\frac{\partial f}{\partial y}\right)$ of $f(t, y, z)$ with respect to y at $y = \eta(t, t_0, y_0, z)$,

$$J(t) = J(t, t_0, y_0, z) = \left(\frac{\partial f}{\partial y}\right) \qquad \text{at } y = \eta(t, t_0, y_0, z), \tag{4.15}$$

then $x = \frac{\partial \eta(t, t_0, y_0, z)}{\partial {y_0}^k}$ is the solution of the initial value problem,

$$x' = J(t)x, \qquad x(t_0) = e_k, \tag{4.16}$$

where $e_k = (e_k^1, \ldots, e_k^d)$ with $e_k^j = 0$ if $j \neq k$ and $e_k^k = 1$; $x = \frac{\partial \eta(t, t_0, y_0, z)}{\partial z^j}$ is the solution of

$$x' = J(t)x + g_j(t), \qquad x(t_0) = 0 \tag{4.17}$$

where $g_j(t) = g_j(t, t_0, y_0, z)$ is the vector $\frac{\partial f(t, y, z)}{\partial z^j}$ at $y = \eta(t, t_0, y_0, z)$ and $\frac{\partial \eta(t, t_0, y_0, z)}{\partial t_0}$ is given by

$$\frac{\partial \eta}{\partial t_0} = -\Sigma_{k=1}^d \frac{\partial \eta}{\partial y_0^k} f^k(t_0, y_0, z). \tag{4.18}$$

In changing from flat space to a warped metric we must include vielbeins in the fermion action so that fields written at each point in terms of a local Lorentz coordinate system in the tangent space transform correctly under general coordinate transformations [Wei72]. Also we have included the spin connections to ensure the derivative is covariant under Lorentz transformations. With a metric-geometry described by (4.11) the fermionic terms in the action must be rewritten in terms of the vielbeins and spin connections

$$V^{\mu}_{A} = \delta^{\mu}_{A} e^{p(y)/12M^3}, \qquad \omega_{\mu} = \frac{i}{24M^3} p'(y) e^{-p(y)/6M^3} \gamma_{\mu}\gamma^5, \tag{4.19}$$

$$V^{5}_{A} = \delta^{5}_{A}, \qquad \omega_5 = 0, \tag{4.20}$$

where capital letters from the start of the Latin alphabet have been used to label the vielbeins' Lorentz indices and the Greek alphabet characters μ, ν are coordinate indices. Explicitly these become incorporated into the spin-covariant derivative $D_N = \partial_N + \omega_N$ and gamma matrices $\Gamma^N = V^N_A \Gamma^A$. With this simplified notation the fermions contribute to the warped space 5-dimensional action according to:

$$S_{\psi} = \int d^5x \sqrt{G}\{i\bar{\Psi}\Gamma^N D_N \Psi - h_{\chi}\bar{\Psi}\sigma^a\sigma^b \chi_{ab}\Psi - h_{\phi}\phi\bar{\Psi}\Psi\}. \tag{4.21}$$

We use an analysis presented in [DG07] to qualitatively demonstrate that, if we decompose $\Psi(x, y)$ according to (4.7), then, in warped space time, there is no mass gap between the zero mode left handed fermion and the continuum of 3 + 1-dimensional massive left and right chiral modes. To make this statement transparent we use separation of variables in the Dirac equation, to obtain eigenvalue equations for the extra dimensional profiles. However we now consider the extra dimensional profiles of the massive modes, f_{nLi}, as well as the zero mode. These satisfy the equations:

$$\begin{aligned} 0 &= 6M^3 f'_{0Li} - p' f_{0Li} + 6M^3 g_j\phi_j f_{0Li} \quad n = 0 \\ 24M^6 m_n^2 e^{p(y)/6M^3} f_{nLi} &= -24M^6 f''_{nLi} + 10M^3 p' f'_{nLi} \\ &\quad +[4M^3 p'' - p'^2 + 24M^6 W] f_{nLi} \quad n > 0 \end{aligned} \tag{4.22}$$

where the generic scalar field coupling term $g_j\phi_j = h_{\chi}(\tau^a\tau^b\chi_{ab})_{ii} + h_{\phi}\phi(y)$ has been used to simplify the expression for $W = (g_j\phi_j)^2 - g_j\phi'_j + g_j\phi_j\frac{p'}{12M^3}$ and m_n is the mass of the n-th 3 + 1-dimensional chiral mode. That is:

$$i\gamma^{\mu}\partial_{\mu}\psi_{nLi} = m_n\psi_{nRi} \qquad \text{and} \qquad i\gamma^{\mu}\partial_{\mu}\psi_{nRi} = m_n\psi_{nLi}. \tag{4.23}$$

We can solve for the zero mode extra dimensional profile immediately,

$$f_{0Li}(y) = N_0 e^{-\int_{y_0}^{y} dy' \; g_j\phi_j - p'/6M^3}. \tag{4.24}$$

For the solution presented in (4.13) there is a normalized $3+1$-dimensional left handed zero mode confined to the plane parameterized by

$$my = \ln\left[\mathrm{root}\left(15h_\phi M^3 t^2 + \left(6\sqrt{5}h_X \Sigma_{a,b}\left(\sigma^a\sigma^b\right)_{ii}(i\delta_{a,b+1} - i\delta_{a,b-1})\,M^3 - 5vm\right)t - 15h_\phi M^3\right)\right] \tag{4.25}$$

provided $6M^3 h_\phi - vm > 0$.

For an arbitrary point in the parameter space which will give rise to a different solution to the Einstein-Klein-Gordon equations we can still say something about the existence of a confined zero mode fermion. This is because square integrability of (4.24) depends on the asymptotic properties of p and ϕ_j. We know that in the case of a scalar field Lagrangian, solving the Einstein-Klein-Gordon equations in a vacuum yields a warp factor $p \propto |y|$, so we expect p to approach $|y|$ as $y \to \infty$. Also we impose the boundary conditions $lim_{y\to\pm\infty}\mathcal{X} = 0$ and $lim_{y\to\pm\infty}\phi = \pm v$, so that any solution to the Einstein-Klein-Gordon equations will have the same asymptotic form as (4.13).

For a continuous, bounded warp factor and Higgs field expressions with the same asymptotic form as (4.13) there is a delta function normalizable zero mode, confined to the y-plane parameterized by the zero of the integrand in (4.24), provided $6M^3 h_\phi - vm > 0$. So far our results are consistent with the flat space case. However we must now attend to the massive chiral 3 + 1-dimensional modes.

Previously we argued that the presence of a mass gap between the zero mode and the lowest discrete mode explained why electroweak scale experiments could not detect the tower and continuum modes. It has been shown by [DG07] that in Randall-Sundrum warped space the continuum modes start from zero mass. To see this we make the substitution $\tilde{f}_{nLi} = e^{-p(y)/6M^3} f_{nLi}$ in (4.22) and change variables to conformal coordinates, $\frac{dz}{dy} = e^{p(y)/12M^3}$, to obtain

$$\left[-\frac{d^2}{dz^2} + e^{-p(y(z))/6M^3} W\right] \tilde{f}_{nLi} \quad = \quad m_n^2 \tilde{f}_{nLi}. \tag{4.26}$$

The change to conformal coordinates, $f : y \to z$, is a diffeomorphism from $\mathbb{R}$ into a connected segment of the real line so we can analyze the potential $e^{-p(y(z))/6M^3} W$ as a function of y and interpret the results as corresponding to the z-coordinate space factored through the mapping $f^{-1} : z \to y$. This is well defined since f^{-1} is bijective.

Using our Einstein-Klein-Gordon solutions (4.13) we can easily see that $e^{-p(y)/6M^3} W$ asymptotes to zero. Thus we have a continuum of delta function normalized modes $\tilde{f}_{nLi}(z)$ starting at zero $3+1$-dimensional mass, m_n, which approach plane waves asymptotically. This translates into a continuum of eigenfunctions, $f_{nLi}(y)$, $\forall m_n > 0$, which are delta function normalized with respect to the weight function $e^{-p(y)/4M^3}$. Davies and George [DG07] argue that these are precisely the normalization conditions required to reduce the kinetic term in the $4+1$-dimensional action (4.21) to its regular $3+1$-dimensional counterpart. We take this as the condition for proper normalization of the profile functions. Hence the continuum modes $f_{nLi}(y)\psi_{nLi}(x)$ and their counterparts $f_{nRi}(y)\psi_{nRi}(x)$, which have analogous conditions omitted here for simplicity, constitute properly normalized solutions to the Dirac equation.

Thus there is a continuum of massive chiral $3+1$-dimensional modes starting from zero mass. It is argued that provided $0 < \frac{\partial p}{\partial y} << 1$ the potential, $e^{-p(y)/6M^3} W$, will decay slowly to 0. This provides a wide barrier for the low energy, asymptotically free, continuum modes to tunnel through. Hence the corresponding wave functions will be heavily suppressed at the position of the brane: $y = 0$. The same behavior is exhibited by the spectrum of Kaluza-Klein modes for the general linearlized fluctuations around the metric in the Randall-Sundrum delta function brane case [RS99b]. We argue that it is possible for the cross section of any process involving interaction between the zero mode and light continuum modes to be imperceptibly low.

Because the $\psi_{nL\backslash Ri}$ are not SO(10) gauge singlets, individual modes will propagate as constituent particles of gauge singlet states in the confining bulk. However as argued in subsection 4.2.2 this does not compromise the integrity of our choice of mode decomposition for $\Psi(x, y)$. Furthermore bound states comprised of low energy, massive, 3 + 1-dimensional chiral, continuum modes will propagate far from the brane, while the zero mode fermion will be trapped around the 3 + 1-dimensional topological defect. Hence the cross section for processes involving interactions between the zero mode and the low energy continuum modes will still be extremely low.

4.2.4 Additional symmetry breaking

The final components of our model are the SU(5)-breaking and electroweak breaking Higgs fields, $\zeta(x^\mu, y)$ and $\eta(x^\mu, y)$, respectively. We introduce these fields now. There are many well documented ways of breaking SU(5) × U(1)$_X$ down to the electroweak gauge group and beyond. We choose a specific scenario to illustrate the general process.

The symmetry breaking pattern:

$$\begin{aligned} \mathrm{SO}(10) \supset\ & \mathrm{SU}(5)\times\mathrm{U}(1)_X \\ & \longrightarrow \mathrm{SU}(3)_C\times\mathrm{SU}_W(2)\times\mathrm{U}_Y(1) \\ & \longrightarrow \mathrm{SU}(3)_C\times\mathrm{U}(1)_Q \end{aligned} \tag{4.27}$$

can be achieved easily by using a pair of 16 dimensional representations for SO(10). The branching rules for this representation are [Sla81]:

$$\begin{aligned} \mathrm{SO}(10) &\supset \mathrm{SU}(5)\times\mathrm{U}(1)_X \\ 16 &\rightarrow 10(-1) + 5^*(3) + 1(-5) \\ \mathrm{SO}(10) &\supset \mathrm{SU}(3)_C\times\mathrm{SU}(2)_W\times\mathrm{U}(1)_Y\times\mathrm{U}(1)_X \\ 16 &\rightarrow (3,2)(1)(-1) + (3^*,1)\,(-4)\,(-1) + (1,1)(6)(-1) + \\ & \quad (3^*,1)(2)(3) + (1,2)(-3)(3) + (1,1)(0)(-5). \end{aligned} \tag{4.28}$$

We would like the component of $\zeta(x^\mu, y)$ which transforms like $(1,1)(0)(-5)$ under SU(3)$_c$× SU(2)$_W$ × U(1)$_Y$ × U(1)$_X$ to condense inside the domain wall. The other 15 components of $\zeta(x^\mu, y)$ should not condense. This will ensure that the gauge group is broken to the standard model on the brane.

Similarly for the field $\eta(x^\mu, y)$, we would like the field component which is uncharged under the embedded $U(1)_Q$ belonging to the doublet $(1,2)(-3)(3)$ to condense inside the domain wall and all other components not to condense. This will implement electroweak symmetry breaking on the brane and circumvent the doublet triplet splitting problem from subsubsection 3.7.3.2. The most general 4th order SO(10) invariant potential felt by these Higgs fields is

$$V_{\zeta,\eta} = \Sigma_{i,j}\{\lambda_{H1,h_i,h_j} h_i^\dagger h_j + \lambda_{H2,h_i,h_j}\left(h_i^\dagger h_j\right)^2 + \lambda_{H3}\left(h_i^\dagger h_i\right)\left(h_j^\dagger h_j\right) + \lambda_{H4,h_i,h_j} h_i^\dagger h_j \phi^2$$
$$+\lambda_{H5,h_i,h_j} h_i^\dagger h_j \mathrm{Tr}\chi^2 + \lambda_{H6,h_i,h_j} h_i^\dagger (\tau^a\tau^b\chi_{ab})^2 h_j + \lambda_{H7,h_i,h_j} h_i^\dagger \tau^a\tau^b\chi_{ab}\phi h_j\}, \quad (4.29)$$

where the indices i, j run over the set $\{1,2\}$ with $h_1 = \zeta(x^\mu, y)$ and $h_2 = \eta(x^\mu, y)$.

We factor the $k-$ th entry in the 16-dimensional column vector for each Higgs field into a sum of a complete set of extra dimensional profile functions $g_{h_i,k,n}(y)$ with coefficients given by solutions of the 3 + 1-dimensional massive Klein-Gordon equation, $\theta_{h_i,k,n}(x^\mu)$. That is,

$$[h_i]_k = \sum\!\!\!\!\!\!\int_n g_{h_i,k,n}(y)\theta_{h_i,k,n}(x^\mu), \quad (4.30)$$

where

$$\partial_\mu\partial^\mu\theta_{h_i,k,n} + m^2_{n_{h_i,k}}\theta_{h_i,k,n} = 0. \quad (4.31)$$

Thus if we let $\frac{\partial V_{\zeta,\eta}}{\partial [h_i^\dagger]_k} = U_{h_i,k,h_j,p}[h_j]_p + O(h_i^2)$, where the indices k and p are being used to label the 16 components of each Higgs field, while i and j still run over the set $\{1,2\}$ with the same definition as before for $h_1(x^\mu, y)$ and $h_2(x^\mu, y)$, then we must find the solutions to the Euler-Lagrange equations:

$$- \quad \frac{\partial^2 g_{h_i,k,n}(y)}{\partial y^2}\theta_{h_i,k,n}(x) + \frac{p'}{3M^3} g_{h_i,k,n}(y)\theta_{h_i,k,n}(x) + U_{h_i,k,h_j,p} g_{h_j,p,n}(y)\theta_{h_j,p,n}(x)$$
$$= \quad m^2_{n_{h_i,k}} e^{p(y)/6M^3} g_{h_i,k,n}(y)\theta_{h_i,k,n}(x). \quad (4.32)$$

To get a qualitative idea of the behavior of solutions we consider the simplest case of (4.32). That is we choose the coupling constants in (4.29) so that $U_{h_i,k,h_j,p} = 0$ when $k \neq p$ or $h_i \neq h_j$. We would like the appropriate symmetry breaking components of $\eta(x^\mu, y)$ and $\zeta(x^\mu, y)$ to have a Kaluza-Klein mode with tachyonic mass. This indicates an instability in the background solution $h_i = 0$. There is sufficient parameter freedom for this to be possible while keeping $\frac{p'}{3M^3} + U_{h_i,k,h_i,k} - m^2_{n_{h_i},k} e^{p(y)/6M^3}$ as a potential well centered about the coordinate of the domain-wall brane. This is necessary to ensure that the condensed component of each Higgs field is localized to the domain wall. For the remaining 15 components of each Higgs field it is only necessary to ensure that one of these two conditions is voided so that the field does not condense inside the domain wall. A thorough analysis would then require us to go back and solve for both the flat space domain-wall configuration and the Einstein-Klein-Gordon equations consistently with this new background to ensure the back reaction of having additional Higgs fields with tachyonic components does not destabilize the brane. This presents a considerable computational task and we do not attempt it here.

The zero mode fermions can now acquire masses through coupling to the SU(5) and electroweak breaking Higgs fields. Coupling constants in the effective $3+1$-dimensional theory arise from the putative 5-dimensional coupling constant multiplied by the overlap integral of the bulk profile functions. This follows from integrating out the y-dependence of the terms in the 5-dimensional action. Hence fermions with different bulk profile functions will have different tree level masses. Since the fermion profile functions are split along the bulk according to their $U(1)_X$ charge the normal SO(10) tree level mass relations are reduced to the less phenomenologically infringing SU(5) mass relations.

4.3 The SO(10) → $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ model

We would like to engineer a Higgs potential which breaks SO(10) → $SU(3)_C \times SU(2)_R \times SU(2)_L \times U(1)_{B-L}$ directly with the background SO(10) adjoint Higgs. Unfortunately, for our fourth order potential, any Higgs solution respecting this symmetry on the domain wall is perturbatively unstable against further condensation. This point is considered in more detail in chapter 5. However it is possible to break SO(10) down to the left right symmetric model using a sixth-order potential,

$$\begin{aligned} V_{X,\phi} = &-\frac{\mu^2}{2}\mathrm{Tr}X^2 + \frac{\lambda_1}{4}(\mathrm{Tr}X^2)^2 + \frac{\lambda_2}{4}\mathrm{Tr}X^4 + \frac{\lambda_3}{4}\mathrm{Tr}X^6 + \frac{\lambda_4}{4}\mathrm{Tr}X^2\mathrm{Tr}X^4 + \frac{\lambda_5}{4}(\mathrm{Tr}X^2)^3 \\ &+\frac{\lambda}{4}(\phi^2 - v^2)^2(\phi^2 - \xi^2) + \frac{\kappa}{4}\phi^2\mathrm{Tr}X^2 + \frac{\lambda_6}{4}(\mathrm{Tr}X^2)^2\phi^2 + \frac{\lambda_7}{4}\mathrm{Tr}X^4\phi^2 + \frac{\beta}{4}\phi^4\mathrm{Tr}X^2. \quad (4.33)\end{aligned}$$

This introduces a plethora of dimensionful coupling constants. However, since we are unable to renormalize $4+1$-dimensional Yang-Mills gauge theories, we have assumed there is an inherent cutoff scale beyond which our model is no longer effective, so we are at liberty to add sixth-order terms.

The Euler-Lagrange equations still exhibit flat space solutions of the form

$$\phi(y) = v\tanh[my], \qquad \mathcal{X}_1 = A\text{sech}[my], \tag{4.34}$$

with $m^2 = -2\mu^2+\kappa v^2+\beta v^4$ and $A^2 = \frac{8(2\mu^2-\kappa v^2-\beta v^4)-4\lambda v^2\xi^2+4\lambda v^4}{\kappa+2v^2\beta}$. We have relabeled the SO(10) generators so that T_1 now refers to $T_1 = \frac{1}{\sqrt{12}}\text{diag}(\tau_2,\tau_2,\tau_2,0,0)$. However the conditions are now a lot more convoluted with the solution contingent on the relations

$$\begin{aligned} (\lambda_3+2\lambda_4+36\lambda_5)A^4-(48\lambda_6 v^2+8\lambda_7 v^2)A^2+48\beta v^4 &= 0, \\ 12m^2+(6\lambda_1+\lambda_2)A^2-6\kappa v^2+6\lambda_6 v^2A^2+\lambda_7 v^2 A^2-12\beta v^4 &= 0, \\ 72\lambda v^4+(6\lambda_6+\lambda_7)A^4-24v^2\beta A^2 &= 0. \end{aligned} \tag{4.35}$$

A general perturbative linear analysis must now be done numerically, as is explained in chapter 5 and appendix B.1. However we note that for the parameter regime

$$\begin{aligned} 2\lambda_1 A^2+\lambda_2 A^2-2\kappa v^2+2\lambda_6 A^2 v^2+\lambda_7 A^2 v^2-4\beta v^4 &> 0, \\ (5\lambda_3 A^4+14\lambda_4 A^4+36\lambda_5 A^4-48\lambda_6 v^2 A^2-24\lambda_7 v^2 A^2+48\beta v^4) &> 0, \end{aligned} \tag{4.36}$$

it is possible to guarantee that all perturbations about (4.34) are oscillatory and that SO(10) will break stably to $\text{SU}(3)_C\times\text{SU}(2)_L\times\text{SU}(2)_R\times\text{U}(1)_{B-L}$. We cannot predict the behaviour of the perturbations outside this parameter regime. A general method for identifying the bifurcation points has yet to be determined. The fermions are localized by exactly the same techniques as in the $\text{SU}(5)\times\text{U}(1)_X$ case.

The zero mode fermion profiles are now localized about the x-axis intercepts of Figure 4.2.

We can also solve the Einstein-Klein-Gordon equations under these circumstances. We find that our previous ansatz, (4.13), will satisfy the Einstein-Klein-Gordon equations for the sixth order potential provided we impose the same bulk cosmological constant fine tuning

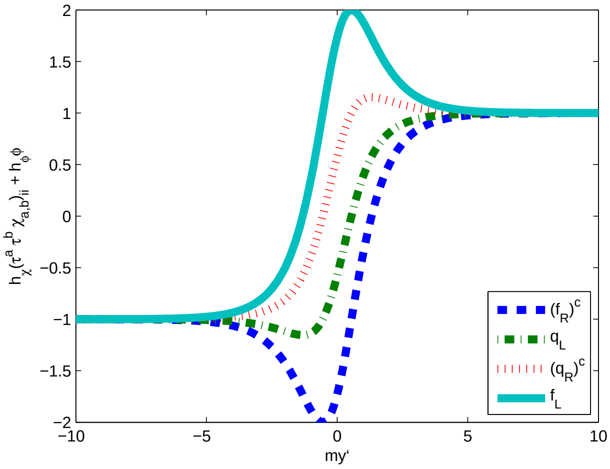


Figure 4.2: The graph displays the integrands in the exponent of (4.9) associated with extra dimensional profile functions for the left right symmetric model bulk fermions. The graph shown here corresponds to the parameter choices $h_\phi = h_\chi = 1$ and to the flat space solution to the Euler-Lagrange equations $\mathcal{X}_1 = A\text{sech}(my)$, $\phi = v\tanh(my)$ with $A = v = 1$. 3 + 1-dimensional left chiral fermions will be confined to the hyperplane associated with the zero of their integrand.

and we are in the slice through the parameter regime:

$$\begin{aligned}
-3\lambda\xi^2 - 12\beta v^2 + 3\lambda v^2 - 6\kappa + 12v^2\lambda_6 + 2\lambda_7 v^2 + 2\lambda_2 + 12\lambda_1 &= 0, \\
3M^3m^2 + v^2m^2 + 6M^3\mu^2 - 3M^3\kappa v^2 - 3M^3\beta v^4 &= 0, \\
\lambda_3 + 6\lambda_4 + 36\lambda_5 - 12\lambda_6 - 2\lambda_7 + 3\beta &= 0, \\
6M^3m^2 + v^2m^2 - 3M^3\lambda v^4 + 3M^3\lambda v^2\xi^2 + 3M^3\kappa v^2 + 6M^3\beta v^4 &= 0, \\
9\lambda + 12\lambda_6 + 2\lambda_7 - 12\beta &= 0.
\end{aligned} \tag{4.37}$$

This symmetry breaking Higgs pattern selects a left right symmetric model on the brane [SM75]. Subsequently it is possible to break our left right symmetric model down to the standard model by a technique similar to the one outlined in subsection 4.2.4.

4.4 Further remarks

There are a couple of salient points for 4+1-dimensional domain-wall brane models, that are based on the Dvali-Shifman mechanism, which we have not yet considered. Firstly, since we are constructing an SO(10) grand unified theory we need the standard model coupling constants to unify at some high scale. Secondly, it was mentioned in subsection 3.6 that we need a hierarchical ordering of the inherent physical scales for our model to work. Both these points were discussed in the paper, [DGV08], on the DVG model. However, in going from SU(5) to SO(10) gauge invariance in the bulk we have introduced a new intermediate symmetry breaking scale. For this reason we explicitly review both ideas in the context of our model.

The spectrum of Kaluza-Klein modes will affect standard model gauge coupling constant running. In particular Kaluza-Klein modes for 4 + 1-dimensional fields carrying different $U(1)_X$ charges which belong to the same SO(10) multiplet in the unified SO(10) gauge theory generally have different masses. That is, the Kaluza-Klein modes originate in the SO(10) gauge theory as split SO(10) multiplets. Because the Kaluza-Klein modes form split SO(10) multiplets, and at specific energy scales only certain $U(1)_X$ components of the split multiplet will be able to contribute to the beta functions, the comparative running of the standard model coupling constants will change. Although coupling constant unification is ruled out for $3+1$-dimensional non-supersymmetric SO(10) GUTs, it is still possible for our model. The full calculation would require us to do a phenomenological parameter fitting: this will fix the higher mass Kaluza-Klein modes and therefore determine the coupling constant running. We have not undertaken this particular task, however we find it tantalizing that the coupling constant may unify at some high scale [Dav07].

To maintain the internal consistency of our model, we require a specific ranking of the magnitudes of the inherent physical scales. For simplicity we will look specifically at the $SO(10) \to SU(5) \times U(1)_X$ model. The concepts do not change for the $SO(10) \to SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ model, hence everything we say here is relevant to this model as well.

We need to look carefully at 5 specific physical scales: the ultraviolet cutoff Λ_{UV}; the SO(10) and SU(5) breaking scales on the brane $\Lambda_{SO(10)} \sim [\mathcal{X}(my = 0)]^{2/3}$ and $\Lambda_{SU(5)} \sim \langle \zeta_{(1,1)(0)(-5)}(my = 0) \rangle^{2/3}$, respectively; the SO(10) confinement scale in the bulk Λ_{conf}; the inverse wall width $\Lambda_{\mathrm{DW}} \equiv m$. These scales must be ranked according to:

$$\Lambda_{\mathrm{DW}} < \Lambda_{\mathrm{conf}} < \Lambda_{\mathrm{SU(5)}} < \Lambda_{\mathrm{SO(10)}} < \Lambda_{\mathrm{UV}}. \tag{4.38}$$

This hierarchy is necessary for the integrity of the Dvali-Shifman mechanism. Lattice gauge theory simulations [LMRS04] tell us that the width of the domain wall must be greater than the SO(10) glueball radius for the Dvali-Shifman mechanism to work. This translates into the relation: the inverse wall width is less than the bulk confinement scale.

The confinement scale in the bulk must be lower than the SO(10) and SU(5) symmetry breaking scales on the brane. Otherwise the non-perturbative bulk SO(10) confinement dynamics would dominate on the brane and the behaviour of the ζ and $\mathcal{X}$ fields, which set these scales, would be dictated by their strong SO(10) interactions. In this case the classical dynamical equations we have solved for our background ζ and $\mathcal{X}$ configurations would not be appropriate, and our theory would have an internal inconsistency.

Finally, our model is an effective field theory which is assumed to be valid only up to the ultraviolet cutoff. This establishes Λ_{UV} as the highest energy scale in the theory.

Amongst the above scales, Λ_{DW}, $\Lambda_{\mathrm{SU(5)}}$, $\Lambda_{\mathrm{S0(10)}}$, and Λ_{UV} are all functions of the free parameters in the Lagrangian. We argue that there is sufficient parameter freedom to rank them according to (4.38). The bulk SO(10) confinement scale is putatively to be calculated from the value of the dimensionful gauge coupling constant g for the bulk SO(10) gauge theory. Because this gauge theory is not renormalizable in 4 + 1-dimensions and because we have introduced a UV cutoff, Λ_{conf} will depend on both Λ_{UV} and g. If we follow the approach adopted in section 3.6 and assume that our model has a transition to a confining regime for values of the dimensionful gauge coupling constant g, greater than a critical value $g_c(\Lambda_{\mathrm{UV}})$, then the bulk confinement scale will be set by Λ_{UV} and a $g > g_c(\Lambda_{\mathrm{UV}})$. This calculation is beyond the scope of the present work.

4.5 Conclusions

In this chapter we identified all the ingredients needed for a 4 + 1-dimensional field theoretical model to produce an effective theory of 3 + 1-dimensional fields localized to a domain-wall brane with SO(10) gauge invariance in the bulk. We complemented the effective 3 + 1-dimensional field theory with Randall-Sundrum type-2 [RS99b] localized gravity.

We proposed a field theoretical mechanism for generating the brane. In our model the brane is created by a real scalar field which condenses over a topological defect. This type of brane has a finite width along the extra-dimension and different phenomenological signatures from the delta function thin brane.

In contrast to the Randall-Sundrum type-2 scenario this approach has the advantage that all four spatial dimensions are treated equally in the fundamental action. Furthermore the breakdown of translational invariance along the extra dimension now occurs spontaneously due to the condensation of the real scalar field domain-wall brane.

In addition we were able to dynamically localize gauge fields and fermions to the domain wall. We assumed that gauge field localization takes place via a 4 + 1-dimensional extension of the Dvali-Shifman mechanism. Further investigation into the implementation of Dvali-Shifman gauge field localization arguments in 4 + 1-dimensions will be necessary to establish a rigorous foundation for future models. We will return to this problem in chapter 6.

The Dvali-Shifman mechanism directly motivated an extension to a grand unified theory in the bulk, SO(10) grand unification is an ideal candidate. We examined two scenarios where the localized Yang-Mills gauge theory on the brane is either a $SU(5) \times U(1)_X$ unified model or a left right symmetric model.

In both these scenarios we took the first generation of fermions and discussed how to localize a 3 + 1-dimensional massless left chiral zero mode for each particle. We localized each 3 + 1-dimensional fermion by applying the split fermion mechanism. In this mechanism we use the fermion's coupling to the domain-wall brane and the SO(10) adjoint Higgs field. The SO(10) adjoint Higgs field couples more strongly to some fermions than to others. This causes the fermions to become localized to a series of parallel hyperplanes split along the bulk direction.

In both scenarios we discussed how to introduce a pair of Higgs fields which condense inside the brane. The Higgs fields break the gauge symmetry on the brane to the standard model and subsequently break the electroweak gauge group. We argued it is feasible to implement this symmetry breaking without localizing additional components of the SO(10) Higgs multiplet which mediate proton decay.

The massless 3 + 1-dimensional chiral zero mode fermions can now gain mass via their interaction with the electroweak symmetry breaking Higgs field. The 3 + 1-dimensional mass term depends on the overlap of their profile functions with the electroweak Higgs field profile function. Hence the degeneracy of SO(10) mass relations is slightly ameliorated.

In the gravity sector, we reproduced Randall and Sundrum's localization of the zero mode linearized fluctuation about the metric. This gives rise to localized 3+1-dimensional gravity.

We reiterate that the signature points of our model are:

- All 4 spatial dimensions are infinite.
- The brane is supplied by a dynamically generated domain wall.
- Standard model fields are localized by their interactions with the domain-wall-engineering Higgs fields.
- All standard model fermions get unified in a single SO(10) representation in the bulk. However SO(10) mass relations are conspicuously absent from our model.

5

On the stability of a domain-wall brane model

In chapter 4 we presented a model based on a domain-wall brane solution and an adjoint Higgs solution to the coupled dynamical equations

$$
\begin{aligned}
0 &= \Box\phi + \frac{\kappa}{2}\mathcal{X}_{ij}\mathcal{X}_{ji}\phi + \lambda\phi\left(\phi^2 - v^2\right), \\
0 &= \Box\mathcal{X}_{ab} - 2\mu^2\mathcal{X}_{ab} + 2\lambda_1\mathcal{X}_{ab}\left(\mathcal{X}_{ij}\mathcal{X}_{ji}\right) + 2\lambda_2\left(\mathcal{X}^3\right)_{ab} + \kappa\phi^2\mathcal{X}_{ab}.
\end{aligned} \tag{5.1}
$$

We imposed the topological boundary conditions $\lim_{y\to\pm\infty}(\mathcal{X}, \phi) = (0, \pm v)$. We call this system of equations the DWB equations. These boundary conditions have topological charge $2v$. We claimed that the stable solution to the dynamical equations is the lowest energy solution belonging to this topological sector.

The homogeneous vacuum solutions $(\mathcal{X}, \phi) = (0, \pm v)$ both have topological charge 0 and are therefore not accessible.

We presented an analytic solution to the DWB equations of the form

$$
\phi_0(x^\mu, y) = v\tanh(my), \qquad \mathcal{X}_0(x^\mu, y)_{i,j} = A\,\text{sech}(my)T_{i,j}, \tag{5.2}
$$

where $A^2 = \frac{40\mu^2 - 10\kappa v^2}{10\lambda_1 + \lambda_2}$, while $m^2 = -2\mu^2 + \kappa v^2$ and $T = \frac{1}{\sqrt{20}}\text{diag}(\sigma_2, \sigma_2, \sigma_2, \sigma_2, \sigma_2)$ is written in terms of the second Pauli matrix

$$
\sigma_2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}.
$$

The solution (5.2) establishes the domain-wall brane and ensures the adjoint Higgs field condenses inside the brane to break SO(10) $\rightarrow$ SU(5) $\times$ U(1)$_X$. The extra-dimensional profile of (5.2) leads to an SU(5) grand unified gauge theory on the domain-wall brane and traps fermions on parallel hyperplanes split along the bulk direction. If $(\mathcal{X}_0, \phi_0)$ is not the lowest energy solution with topological charge $2v$ then the system will be unstable. A small perturbation, $(\delta\bar{\mathcal{X}}(x, y, t), \delta\phi(x, y, t))$, about $(\mathcal{X}_0, \phi_0)$ will grow with time. This perturbation can break the SU(5) grand unified theory on the brane, making it impossible to recover 3+1-dimensional standard model phenomenology.

In this chapter we argue that if a perturbation is not allowed because of the symmetry of the background domain-wall brane configuration or if a perturbation has oscillatory time dependence so that $\frac{||\delta\bar{\mathcal{X}}(x,y,t)||}{||\mathcal{X}_0(y)||}, \frac{||\delta\phi(y,t)||}{||\phi_0(y)||} << 1,\ \forall t > 0$, then the domain-wall brane formed by $(\mathcal{X}_0(y), \phi_0(y))$ is stable.

The coupling constants $\mu^2, \lambda_1, \lambda_2, v, \lambda$, and κ are all free parameters. However, when the theory is quantized they will be replaced by solutions to the renormalization group equations which run with the energy scale. Thus a physically interesting solution to the DWB equations must be stable for an open set in the free parameter space.

While the analytic form of (5.2) exists only when the free parameters satisfy constraints given in (4.2), we have numerically solved (5.1) for a wide region of parameter space surrounding this slice. Our numerical results show that, on a ball in the free parameters surrounding this slice, the solutions to (5.1) are of the form (5.2).

Therefore in this chapter we prove that a domain-wall brane solution (5.2) is stable on an open region in parameter space encompassing (4.2).

5.1 Notation

This chapter is more in depth and will require some technical notation. We have included it at the beginning of the chapter for the convenience of the reader.

5.1.1 Banach spaces

Let O denote the order symbol so that $l = O(\varepsilon)$ means there exists a constant $c > 0$ such that $|l| \leq c|\varepsilon|$ as $\varepsilon \to 0$.

Let $C(\mathbb{R})$, $C_0(\mathbb{R})$, and $C^2(\mathbb{R})$ denote, respectively, the spaces of bounded continuous functions, of continuous functions that converge to 0 at $\pm\infty$, and of twice continuously differentiable real valued functions on the line $\mathbb{R}$. Thus $C_0(\mathbb{R}) \subset C(\mathbb{R})$ are Banach spaces with the supremum norm

$$||y|| = \sup\{|y(z)| \, : \, z \in \mathbb{R}\}.$$

We will abbreviate these to C, C_0 and C^2, respectively.

Throughout the chapter, Z and Y denote real Banach spaces. If $\mathcal{A} : Z \to Y$ then $\ker \mathcal{A}$ and $\mathrm{Ra}\,\mathcal{A}$ denote the kernel and range of $\mathcal{A}$ respectively. Let $\dim V$ denote the dimension of a subspace V.

Let L^1 and L^2 denote, respectively, the spaces of Lebesgue integrable and of Lebesgue square integrable functions on $\mathbb{R}$. If $v \in L^1$ and $h \in C_0$ or $v, h \in L^2$, let

$$\langle v, h \rangle = \int_{-\infty}^{\infty} vh \, dz.$$

5.1.2 Coordinate change

We change variables from y to $z = my$ so that we are working with a dimensionless coordinate system. In this coordinate system we are able to rewrite (5.1) entirely in terms of a set of parameters with zero mass dimensions

$$\{\lambda_1, \lambda_2, \lambda, \kappa, v^2, A^2, \omega^2\} \to \{\lambda_1 m, \lambda_2 m, \lambda m, \kappa m, v^2/m^3, A^2/m^3, \omega^2/m^2\}. \tag{5.3}$$

We note that because m has to be real valued to ensure a solitonic background solution for $\phi(y)$, our condition for perturbative stability, in terms of the new parameter ω^2/m^2, is $\omega^2/m^2 \leq 0$. To make it easier to cross reference the results of this stability analysis with subsection 4.2.1 we choose not to relabel these new dimensionless parameters. Instead of relabeling, we point out that stability conditions written in terms of these new parameters, will look exactly the same as conditions written in terms of the old dimensionful parameters if we set $m = 1$. We hope it is clear that this is not a fine tuning condition; it is a shift of notation to a convention where all parameters and variables have zero mass dimension.

The perturbation $\delta\bar{X}(z,t)$ is allowed to vary independently in the direction of each of the 45 components of the adjoint representation for SO(10). We use the notation $\delta\bar{X}(x,z,t) = \delta X(x,z,t) + N(x,z,t)$, where $\delta X(x,z,t)$ is a matrix containing the perturbations along the directions of the Cartan subalgebra generators for SO(10), and $N(x,z,t)$ contains all the perturbations in the subspace of the adjoint representation that is orthogonal to the Cartan subalgebra under the trace operator. We choose to work with the explicit choice of generators $C_1 = \frac{1}{2}\text{diag}(\sigma_2, 0, 0, 0, 0), \ldots, C_5 = \frac{1}{2}\text{diag}(0, 0, 0, 0, \sigma_2)$, for the Cartan subalgebra of the 10-dimensional SO(10) representation. Thus $N(x,z,t)$ is a matrix with 2×2 blocks of zero matrices along the diagonal.

Outline of Proof

We will break this argument down by ruling out classes of perturbations. We will start by arguing that perturbations which break the 3-dimensional rotational invariance respected by the solution $(X_0(z), \phi_0(z))$ are not allowed. Hence we will suppress the coordinate label x and write $(X(z,t), \phi(z,t)) = (X_0(z,t) + \delta X(z,t) + N(z,t), \phi_0(z,t) + \delta\phi(z,t))$. Then we successively rule out all perturbations to $X_0(z)$ of the form $N(z,t)$. Finally we will show that all perturbations of the form $(\delta X(z,t), \delta\phi(z,t))$ are oscillatory in time and hence the background solution $(X_0(z), \phi_0(z))$ is stable under dynamical evolution.

Perturbations breaking 3-dimensional rotational invariance

Since our solution $(X_0(z), \phi_0(z))$ is invariant under 3-dimensional spatial rotations there is no preferred direction in the hyperplane orthogonal to the z-coordinate. We argue that because the background solution treats all spatial directions orthogonal to the bulk coordinate, z, the same, it does not make sense to say a perturbation has formed along the direction of a specific vector in this orthogonal space. This is because all directions are relative to an arbitrary choice of reference coordinate system [PV00]. Hence we consider only perturbations of the form $(\delta X(z,t) + N(z,t), \delta\phi(z,t))$.

Perturbations along non-Cartan directions

It is argued in [PV01] that if a perturbation does not change the energy density of our solution $(\mathcal{X}_0(z), \phi_0(z))$ by introducing a term which is linear in the perturbative fields then we can have a stable solution where these perturbations are set to zero in the expansion for $(\mathcal{X}(z,t), \phi(z,t))$.

A term which is linear in the perturbative fields would be introduced through an expansion of the SO(10) Casimir invariants appearing in the potential. Hence we consider the form of $\mathrm{Tr}\mathcal{X}^2$ and $\mathrm{Tr}\mathcal{X}^4$ for $\mathcal{X}(z,t) = \mathcal{X}_0(z) + \delta\bar{\mathcal{X}}(z,t) = \{\mathcal{X}_0(z) + \delta\mathcal{X}(z,t)\} + N(z,t)$ which we will abbreviate to $\mathcal{X} = \{\mathcal{X}_0 + \delta\mathcal{X}\} + N$. For $\mathrm{Tr}\mathcal{X}^2$ the expression is

$$\mathrm{Tr}\left[\{\mathcal{X}_0 + \delta\mathcal{X}\} + N\right]^2 = \mathrm{Tr}\left[\{\mathcal{X}_0^2 + 2\mathcal{X}_0\delta\mathcal{X} + (\delta\mathcal{X})^2\} + 2\{\mathcal{X}_0 + \delta\mathcal{X}\}N + N^2\right]. \tag{5.4}$$

Clearly this Casimir invariant has introduced a term which is linear in the perturbations along the 5-independent Cartan subalgebra direction $C_1, \ldots, C_5$. Hence we must examine perturbations along the Cartan subalgebra directions in more detail to determine their stability. However for the perturbations along the direction of the other 40 adjoint Higgs field generators, orthogonality of our specific choice of generators for the representation of SO(10) under the matrix bilinear form, trace, means that the linear term disappears. In the above equation this is $\mathrm{Tr}\left[\mathcal{X}_0 N\right] = 0$. A similar observation can be made for all the Casimir invariants for SO(10) because the only possible terms in a perturbative expansion which are first order in the perturbations, will be of the form $\mathrm{Tr}\mathcal{X}^{2k} \supset \mathrm{Tr}\left[\mathcal{X}_0^{2k-1}(\delta\mathcal{X} + N)\right]$, where $k \in \mathbb{Z}^+$. The exponent, n, for a general Casimir invariant can always be written as $n = 2k$ because the antisymmetry of the SO(10) generators implies $\mathrm{Tr}\mathcal{X}^{2k+1} = 0$. It is easy to check that for any solution $\mathcal{X}_0$ written as fields distributed over $C_1, \ldots, C_5$, $\mathcal{X}_0^{2k-1}$ will also contain non-zero components only in the direction of the SO(10) Cartan subalgebra generators, $C_1, \ldots, C_5$, and hence $\mathrm{Tr}\left[\mathcal{X}_0^{2k-1}N\right] = 0$ [PV01]. This means that the $\mathrm{Tr}\mathcal{X}^4$ does not contribute any terms to the energy density that are linear in the perturbations $N(z,t)$. So a stable solution $(\mathcal{X}(z,t), \phi(z,t))$ can be constructed using only the Cartan subalgebra generators and we do not need to worry about linear perturbations along the directions of SO(10) generators which are orthogonal to the Cartan subalgebra $C_1, \ldots, C_5$.

Perturbations along Cartan directions

It is left to establish that the perturbations $\delta\mathcal{X}(z,t)$, along the Cartan subalgebra directions $C_1,\ldots,C_5$, have oscilatory time dependence under dynamical evolution. For definiteness we choose $\delta\mathcal{X}(z,t)$ to be the 10×10 matrix:

$$\delta\mathcal{X}(y,t) = \delta\mathcal{X}_{12}(z,t)C_1 + \delta\mathcal{X}_{34}(z,t)C_2 + \delta\mathcal{X}_{56}(z,t)C_3 + \delta\mathcal{X}_{78}(z,t)C_4 \ + \delta\mathcal{X}_{910}(z,t)C_5. \quad (5.5)$$

Thus in shorthand our perturbed solution is:

$$\mathcal{X}(z,t) = \mathcal{X}_0(z) + \delta\mathcal{X}(z,t), \qquad \phi(z,t) = \phi_0(z) + \delta\phi(z,t). \quad (5.6)$$

We substitute this expansion into (5.1) and argue that initially time evolution will be dictated by terms of first order in the small quantities $\delta\phi(z,t)$ and $\delta\mathcal{X}(z,t)$. Discarding all higher order terms we are left with coupled linear homogeneous equations, for $\delta\phi(z,t)$ and the *a,b* th entry of $\delta\mathcal{X}(z,t)$, of the form:

$$\begin{aligned} 0 &= \Box\delta\phi + \kappa\phi_0\,(\mathcal{X}_0)_{ij}\,(\delta\mathcal{X})_{ji} + \left(\frac{\kappa}{2}\,(\mathcal{X}_0)_{ij}\,(\mathcal{X}_0)_{ji} + \lambda\left(3\phi_0^2 - v^2\right)\right)\delta\phi, \\ 0 &= \Box\delta\mathcal{X}_{ab} + 4\lambda_1\,(\mathcal{X}_0)_{ab}\,(\mathcal{X}_0)_{ij}\,(\delta\mathcal{X})_{ji} + \left(-2\mu^2 + 2\lambda_1\,(\mathcal{X}_0)_{ij}\,(\mathcal{X}_0)_{ji} + \kappa\phi_0^2\right)\delta\mathcal{X}_{ab} + \\ &\quad 2\lambda_2\,(\delta\mathcal{X})_{ai}\,(\mathcal{X}_0)_{ij}\,(\mathcal{X}_0)_{jb} + 2\lambda_2\,(\mathcal{X}_0)_{ai}\,(\delta\mathcal{X})_{ij}\,(\mathcal{X}_0)_{jb} \\ &\quad +2\lambda_2\,(\mathcal{X}_0)_{ai}\,(\mathcal{X}_0)_{ij}\,(\delta\mathcal{X})_{jb} + (2\kappa\phi_0\,(\mathcal{X}_0)_{ab})\,\delta\phi. \end{aligned} \quad (5.7)$$

These equations can be reduced to ordinary differential equations by using Fourier decomposition to factor out the time dependence of $\delta\phi(z,t)$ and $\delta\mathcal{X}(z,t)$. We consider the evolution of a specific normal mode of the coupled system $(e^{\omega t}\delta\mathcal{X}(z), e^{\omega t}\delta\phi(z))$.

Each mode must independently satisfy the boundary conditions: $\delta\mathcal{X}(z), \delta\phi(z) \to 0$ as $z \to \pm\infty$ since a global change to the profile of either $\mathcal{X}(z)$ or $\phi(z)$ would require an infinite 3+1-dimensional energy density and could not be accomplished by perturbative effects alone.

If $\omega^2 \le 0$ for all coupled solutions $(e^{\omega t}\delta\mathcal{X}(y), e^{\omega t}\delta\phi(z))$ of the above equations, then the perturbation will be oscillatory and the background solution $(\mathcal{X}_0, \phi_0)$ will be perturbatively stable over short periods of dynamical evolution. Of course the solutions to (5.7) depend on the parameters $\mu^2, \lambda_1, \lambda_2, \lambda, \kappa$. Thus we are looking for relations between the parameters which guarantee $\omega^2 \leq 0$ for all normal modes.

We will assess when our analytic solution (4.3) is stable. We provide an example where SO(10) breaks to SU(5) × U(1)$_X$ and stops there. We also establish scenarios where (4.3) is unstable. Hence we perturb about $(\mathcal{X}_0(z), \phi_0(z)) = (A\text{sech}(z), v\tanh(z))$ and divide our problem into two scenarios: either there exists two components of the superdiagonal of $\delta\mathcal{X}(z)$, say $-i\delta\mathcal{X}_{k,k+1}(z)$ and $-i\delta\mathcal{X}_{n,n+1}(z)$, that are distinct or all non-zero components of the superdiagonal of $\delta\mathcal{X}(z)$ are equal.

Two Cartan perturbations diverge relative to each other

In the first scenario: there exist $k, n \in \{1, ...9\}$ such that $-i\delta\mathcal{X}_{k,k+1}(z)$ and $-i\delta\mathcal{X}_{n,n+1}(z)$ are distinct. Therefore we have a non-trivial differential equation for $\delta\mathcal{X}_\epsilon = \delta\mathcal{X}_{k,k+1} - \delta\mathcal{X}_{n,n+1}$. We seek conditions which guarantee $\omega^2 \leq 0$ for all solutions to

$$-\frac{\partial^2 \delta\mathcal{X}_\epsilon}{\partial z^2} - \left(2 - \frac{\lambda_2 A^2}{5}\right)\text{sech}^2 z\, \delta\mathcal{X}_\epsilon = \left(-\omega^2 - 1\right)\delta\mathcal{X}_\epsilon. \tag{5.8}$$

Let $U = \left(2 - \frac{\lambda_2 A^2}{5}\right)$. Equation (5.8) is just a 1-dimensional time independent Schrödinger equation and if we impose the condition $U > 0$, then negative eigenvalues $-\omega^2 < 0$ will correspond to bound states of the potential $-U\text{sech}^2(z)+1$. So in what follows we are simply determining the conditions necessary for this potential well to be sufficiently shallow so that it does not admit bound states.

We use the following lemma from [Yag99], to conclude that if $\lambda_2 > 0$ and $\lambda_2 A^2 < 10$, then there are no non-trivial solutions to (5.8) when $\omega^2 > 0$.

Lemma 5.1: [Yag99] *Let $\kappa > 0$, $\lambda > 0$. The equation*

$$\ddot{v} + (-\lambda + \kappa\,\text{sech}^2 t)v = 0 \tag{5.9}$$

has a bounded solution if and only if there exists an integer M such that

$$\begin{aligned} \lambda &= \tfrac{1}{4}(\sqrt{4\kappa+1} - 4M - 1)^2 \quad \textit{for} \quad 0 \leq M < \tfrac{1}{4}(\sqrt{4\kappa+1} - 1) \\ \textit{or} \quad \lambda &= \tfrac{1}{4}(\sqrt{4\kappa+1} - 4M - 3)^2 \quad \textit{for} \quad 0 \leq M < \tfrac{1}{4}(\sqrt{4\kappa+1} - 3). \end{aligned} \tag{5.10}$$

If $\lambda_2 < 0$, then $\omega^2 > 0$ and there will be a perturbative mode which grows with time. This indicates that our analytical solution (4.3) will be unstable under dynamical evolution and it is not possible to stably break SO(10) to SU(5) × $U(1)_X$ with this solution. This appears fortuitous as we would expect a cascade effect where the solution for $\mathcal{X}(y)$ evolved spontaneously into a form which would break SO(10) to a subgroup of SU(5) × $U(1)_X$ on the domain wall. However $\delta\mathcal{X}_\epsilon$ is given by the difference of any two arbitrary superdiagonal entries in $\delta\mathcal{X}(y)$. Hence if $\lambda_2 < 0$, then each non-zero component of $\delta\mathcal{X}$ will diverge relative to all the others. Thus what we will be left with is an SO(10) adjoint Higgs profile that breaks SO(10) down to $U(1)^5$ on the domain-wall brane.

All Cartan perturbations diverge uniformly

In the case where the perturbations along all the Cartan directions are the same we will use

$$\delta\mathcal{X}(z,t) = \chi(z)e^{\omega t}\left(C_1 + \cdots + C_5\right), \qquad \delta\phi(z,t) = \varphi(z)e^{\omega t}, \tag{5.11}$$

in the linearized equations (5.7). We retain the boundary conditions $\chi(z), \varphi(z) \to 0$ as $z \to \pm\infty$. This produces the coupled equations

$$\begin{aligned}\varphi'' &= \left[\omega^2 + \left(\frac{\kappa A^2}{4}\,\mathrm{sech}^2 z + 3\lambda v^2 \tanh^2 z - \lambda v^2\right)\right]\varphi + \frac{5A\kappa v}{\sqrt{20}}\,\mathrm{sech}\, z \tanh z\, \chi \\ \chi'' &= \left[\omega^2 + 1 + \left(3A^2\gamma_1 - \kappa v^2\right)\mathrm{sech}^2 z\right]\chi + \frac{4A\kappa v}{\sqrt{20}}\,\mathrm{sech}\, z \tanh z\, \varphi\end{aligned} \tag{5.12}$$

subject to the boundary conditions

$$\lim_{|z|\to\infty} \varphi(z) = 0 = \lim_{|z|\to\infty} \chi(z). \tag{5.13}$$

Here $\gamma_1 = \lambda_1 + \frac{\lambda_2}{10}$ while $A, \gamma_1, \kappa, \mu^2, \lambda$, and v are parameters which satisfy the constraints

$$2 + \frac{\kappa A^2}{4} - \lambda v^2 = 0, \quad -1 - 2\mu^2 + \kappa v^2 = 0, \quad 2 + \gamma_1 A^2 - \kappa v^2 = 0. \tag{5.14}$$

Note (5.14) is equivalent to

$$\mu^2 = \frac{1+\gamma_1 A^2}{2}, \quad \nu^2 = \frac{2+\gamma_1 A^2}{\kappa}, \quad \lambda = \frac{8+\kappa A^2}{8+4\gamma_1 A^2}\kappa. \tag{5.15}$$

Hence the $\Theta = (A, \gamma_1, \kappa)$ may be regarded as free parameters satisfying $\gamma_1 A^2 \geq -1$, $\kappa > 0$ (note these inequalities are necessary and sufficient), while the remaining parameters are given by (5.15).

The objective is to establish an open region in the free parameter space for which $\omega^2 \leq 0$, that is, ω is an imaginary complex number, for all solutions $\chi(z), \varphi(z)$ to the linearized equations (5.12).

By a solution we mean a twice continuously differentiable vector valued function, (φ, χ), satisfying (5.12) and (5.13).

Since φ and χ are continuous it follows from (5.13) that they are bounded while it follows from (5.12) that they are analytic.

We assume throughout that $\omega \neq 0$. First we make some observations about the system (5.12) and use (5.15) to eliminate μ^2, ν, and λ.

Since the system depends on ω^2 it suffices to consider $\omega > 0$. Multiplying the first equation in (5.12) by $4\nu/\sqrt{20}$ and setting $\Phi = 4\nu\varphi/\sqrt{20}$, system (5.12) becomes

$$\begin{aligned} \Phi'' &= \left[\omega^2 + \left(\frac{\kappa A^2}{4}\operatorname{sech}^2 z + 3\lambda\nu^2\tanh^2 z - \lambda\nu^2\right)\right]\Phi \; + \nu^2 A\kappa \operatorname{sech} z \tanh z\, \chi \\ \chi'' &= \left[\omega^2 + 1 + \left(3A^2\gamma_1 - \kappa\nu^2\right)\operatorname{sech}^2 z\right]\chi \qquad\qquad + \; A\kappa \operatorname{sech} z \tanh z\, \Phi \end{aligned} \tag{5.16}$$

which depends on ν^2 rather than ν and so solutions (Φ, χ) depend on ν^2. In particular if (φ, χ) is a solution of (5.12) with $\nu = C < 0$ then $(-\varphi, \chi)$ is a solution of (5.12) with $\nu = -C > 0$. Thus we need only consider the case $\nu \geq 0$. By a similar argument applied to A in place of ν and using (5.14) we need only consider the case $A \geq 0$. Now $\kappa > 0$, $\lambda > 0$ and $\nu \neq 0$ by (5.14). Moreover in our main result we will assume that $\gamma_1 > 0$.

Using (5.14) and replacing Φ by φ and using (5.15) to eliminate $\kappa\nu^2$ it follows that (5.16) has the form

$$\begin{aligned}\varphi'' &= \omega^2\varphi + \left(4 - 6\,\mathrm{sech}^2 z + \frac{\kappa A^2}{2}\tanh^2 z\right)\varphi + A(2+\gamma_1 A^2)\,\mathrm{sech}\, z\tanh z\,\chi\\ \chi'' &= \omega^2\chi + \left(1 - 2\,\mathrm{sech}^2 z + 2\gamma_1 A^2\ \mathrm{sech}^2 z\right)\chi + A\kappa\,\mathrm{sech}\, z\tanh z\,\varphi.\end{aligned} \tag{5.17}$$

It suffices to establish non-existence and existence for system (5.17) and (5.13) with the parameters given in Θ.

To facilitate the presentation of our results we introduce the following notation. Let

$$\Theta_1 = (\omega_0, A_0, \gamma_1, \kappa)\,, \quad \Theta_2 = (\omega_0, A_0, \kappa)\,,$$

and let $\bar\Theta$ and $\bar\Theta_i$ be defined naturally so, for example, $\bar\Theta_1 = \left(\bar\omega_0, \bar A_0, \bar\gamma_1, \bar\kappa\right)$. Let

$$g(\omega, A, \gamma_1, \kappa) = 2\omega^2 + \frac{A^2}{6}(16\gamma_1 - \kappa). \tag{5.18}$$

For $u \in \mathbb{R}^n$ let $B^n_r(u)$ be the open ball of radius r and centre u in $\mathbb{R}^n$ and $S^1 = \{v \in \mathbb{R}^2 : v_1^2 + v_2^2 = 1\}$.

For $d,\ \eta,\ k > 0$ let

$$\begin{aligned}\mathfrak{V}_{d,k} &= (-d,d)\times(0,k)^2\\ \mathfrak{V}_{\omega,d,\eta,k} &= \{\Theta\in\mathfrak{V}_{d,k} :\ g(\omega,A,\gamma_1,\kappa) > \eta(\omega^2 + A^2)\}.\end{aligned}$$

Let $S^1_d(u) = S^1 \cap B^2_d(u)$ for $u \in S^1$. For $d,\ l > 0,\ v,\ w \in \mathbb{R}$, and $u \in S^1$ let

$$\begin{aligned}U_d(0,u,v) &= (-d,d)\times S^1_d(u)\times B^1_d(v)\,,\\ U_{d,l}(0,u,v,w) &= (-d,d)\times S^1_d(u)\times B^1_l(v)\times B^1_d(w)\,.\end{aligned}$$

For open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}$, let $C^1(U;V)$ denote the space of continuously differentiable functions from U to V.

We establish the following non-existence (stability) result for the domain-wall brane

Theorem 5.1: *If $k > 4$ and $\eta \in (0,2)$ there exists $d(\eta,k) > 0$ such that $(\varphi,\chi) = (0,0)$ is the only bounded solution to* (5.12) *and* (5.13) *when $\omega \neq 0$ and $\Theta \in \mathfrak{V}_{\omega,d,\eta,k}$.*

Remark 5.1: *1. We chose $\eta \in (0,2)$ to ensure that $\mathfrak{B}_{\omega,d,\eta,k} \neq \emptyset$, since $\hat{\Theta} = (0, k/2, k/2) \in \mathfrak{B}_{\omega,d,\eta,k}$.*

2. Note $\mathfrak{B}_{\omega,d(\eta,k),\eta,k}$ is a non-empty open set of parameter space.

3. We will see that if $\omega \neq 0$, $A = 0$, $\Theta \in \mathbb{R}^3$, then there are no non-trivial solutions (φ, χ) of (5.12) *and* (5.13).

4. If $(16\gamma_1 - \kappa)/6 > \eta$ for some $\eta \in (0,2)$, then $g(\omega, A, \gamma_1, \kappa) > \eta(\omega^2 + A^2) > 0$ for all $(\omega, A) \neq (0,0)$. Then clearly $\mathfrak{B}_{\omega,d,\eta,k} = \mathfrak{B}_{d,k}$.

Moreover we prove the following existence result.

Theorem 5.2: *Let $(\bar{\gamma}_1, \bar{\kappa}) \in \mathbb{R}^2$, $\bar{\omega}_0 \in \mathbb{R}$, $\bar{\omega}_0 \bar{A}_0 \neq 0$, $(\bar{\omega}_0, \bar{A}_0) \in S^1$ and $g(\bar{\omega}_0, \bar{A}_0, \bar{\gamma}_1, \bar{\kappa}) = 0$. There exist d, $l > 0$ and $\gamma_1 \in C^1(U_d(0, \bar{\Theta}_2); B_l^1(\bar{\gamma}_1))$ such that*

(i) $\bar{\gamma}_1 = \gamma_1(0, \bar{\Theta}_2)$*, and*

(ii) (5.17) *and* (5.13) *has a bounded solution with $\omega = \varepsilon\omega_0 \neq 0$ and $A = \varepsilon A_0$ satisfying $(\varepsilon, \Theta_1) \in U_{d,l}(0, \bar{\Theta}_1)$ iff $\gamma_1 = \gamma_1(\varepsilon, \Theta_2)$.*

The plan for the remainder of this chapter is as follows.

In section 5.2 we introduce the background results that we use.

In section 5.3 we prove our main result Theorem 5.1. There are four steps in the proof.

In the first step we show that for $k > 4$ and $\omega \geq 4$ there is $a = a(k) > 0$ such that $(\varphi, \chi) = (0,0)$ is the only solution of (5.17) when $\Theta \in \mathfrak{B}_{a,k}$.

In the second step we observe that the system decouples to

$$\begin{aligned} \varphi'' &= \left[\omega^2 + 4 - 6\,\mathrm{sech}^2 z\right]\varphi \\ \chi'' &= \left[\omega^2 + 1 - 2\,\mathrm{sech}^2 z\right]\chi, \end{aligned} \tag{5.19}$$

when $A = 0$. In this case we use Lemma 5.1 to conclude that $(\varphi, \chi) = (0,0)$ is the only solution of (5.19) when $\omega \geq \varepsilon > 0$, that is, ω is positive and bounded from below.

In the third step we use a limiting argument to show that for $4 > \omega \geq \varepsilon > 0$ and $k > 4$ there exists $a = a(\varepsilon, k) > 0$ such that $(\varphi, \chi) = (0, 0)$ is the only solution of (5.12) when $\Theta \in \mathfrak{V}_{a,k}$.

In the final step of the proof we apply Fredholm theory for compact linear operators combined with the Lyapunov-Schmidt method to prove for $k > 4$ and $\eta \in (0, 2)$ there is $\varepsilon_0 = \varepsilon_0(\eta, k) > 0$ sufficiently small that if $(\omega_0, A_0, \gamma_1, \kappa) \in S^1 \times (-k, k)^2$ and $g(\omega_0, A_0, \gamma_1, \kappa) > \eta > 0$ then $(\varphi, \chi) = (0, 0)$ is the only solution of (5.17) and (5.13) when $\omega = \varepsilon\omega_0 \neq 0$, $A = \varepsilon A_0$, and $0 < \varepsilon < \varepsilon_0$. Thus there is $\varepsilon_0 > 0$ sufficiently small that $(\varphi, \chi) = (0, 0)$ is the only solution of (5.17) and (5.13) when $k > 4$, $\omega \neq 0$, $16\gamma_1 > \kappa$, and $\Theta \in \mathfrak{V}_{\varepsilon_0,k}$.

In section 5.4 we assume that $\bar{\omega}_0 \bar{A}_0 \neq 0$, $(\bar{\omega}_0, \bar{A}_0) \in S^1$, $(\bar{\gamma}_1, \bar{\kappa}) \in (-k, k)^2$, and that $g(\bar{\omega}_0, \bar{A}_0, \bar{\gamma}_1, \bar{\kappa}) = 0$. We use the implicit function theorem to prove that there are non-trivial solutions (φ, χ) of (5.17) and (5.13) when ε is small, $\omega = \varepsilon\omega_0$, $A = \varepsilon A_0$, $(\omega_0, A_0) \in S^1$ is close to $(\bar{\omega}_0, \bar{A}_0)$, and (γ_1, κ) is close to $(\bar{\gamma}_1, \bar{\kappa})$ iff $\gamma_1 = \gamma_1(\varepsilon, \omega_0, A_0, \kappa)$ for a smooth function γ_1. Thus there is a hypersurface of parameters for which there are solutions in a small sector of (ω, A) space containing $\varepsilon(\bar{\omega}_0, \bar{A}_0)$ for ε small. Moreover this hypersurface is contained in the boundary of the region where there are no solutions.

Finally we note that the study of existence of bounded solutions to linear ordinary differential equations like (5.12) is closely related to the bifurcation of transversal homoclinic solutions in dynamical systems and so to the existence of deterministic chaos (for more see details [Bat90, FG04, FG08, Gru92, Li96, Pal84, Yag99]). Consequently our approach can be applied also to these kinds of dynamical problems.

5.2 Background results

For the convenience of the reader we recall some basic facts about Fredholm theory in Banach space.

Let $c > 0$, $h \in C_0$ and

$$g_c(z, s) = \begin{cases} -\frac{1}{2c} \mathrm{e}^{-c(z-s)}, & \text{for } s \leq z \\ -\frac{1}{2c} \mathrm{e}^{c(z-s)}, & \text{for } z \leq s. \end{cases} \tag{5.20}$$

It follows that the boundary value problem

$$y'' = c^2 y + h(z), \text{ for } z \in \mathbb{R} \tag{5.21}$$

$$0 = \lim_{|z|\to\infty} y(z) \tag{5.22}$$

has a unique bounded solution $y \in C_0 \cap C^2$ iff $y \in C_0$ satisfies the integral equation

$$y(z) = -\frac{1}{2c}\int_{-\infty}^{z} \mathrm{e}^{-c(z-s)} h(s)\,ds - \frac{1}{2c}\int_{z}^{\infty} \mathrm{e}^{c(z-s)} h(s)\,ds \tag{5.23}$$

$$= \int_{-\infty}^{\infty} g_c(z,s)h(s)\,ds =: \mathcal{G}_c h(z). \tag{5.24}$$

Indeed, if $y \in C_0$ satisfies (5.24) it is easy check that $|y(z)| \leq \|h\|/c^2$,

$$y'(z) = \int_{-\infty}^{\infty} \frac{\partial}{\partial z} g_c(z,s)h(s)\,ds \tag{5.25}$$

and that (5.21) and (5.22) are satisfied. Thus $\mathcal{G}_c : C_0 \to C_0$ is bounded linear. Moreover it follows from (5.25) that $|y'(z)| \leq \|h\|/c$.

For $(z,s) \in \mathbb{R}^2$, let

$$q_c(z,s) = g_c(z,s)\operatorname{sech} s, \text{ and}$$

$$k_c(z,s) = g_c(z,s)\operatorname{sech}^2 s.$$

For $h \in C_0$, let

$$\mathcal{Q}_c h(z) = \int_{-\infty}^{\infty} q_c(z,s)h(s)\,ds, \text{ and}$$

$$\mathcal{K}_c h(z) = \int_{-\infty}^{\infty} k_c(z,s)h(s)\,ds.$$

Since $|\operatorname{sech} s| \leq 2e^{-|s|}$ it is easy to show that for $c > 1$

$$|\mathcal{Q}_c(h)(z)|, |\mathcal{K}_c(h)(z)| \leq \frac{2}{c(c-1)}\|h\|\,\mathrm{e}^{-|z|}, \text{ for } z \in \mathbb{R}. \tag{5.26}$$

Moreover, by a similar argument to that above

$$|\mathcal{Q}_c(h)'(z)|, |\mathcal{K}_c(h)'(z)| \leq \frac{1}{c}\|h\|, \text{ for } z \in \mathbb{R}, \tag{5.27}$$

so that $\mathcal{Q}_c$, $\mathcal{K}_c : C_0 \to C_0$ are completely continuous.

Define $\mathcal{K}_{c,b}$, $\mathcal{A}_{c,b} : C_0 \to C_0$ for $y \in C_0$ by

$$\begin{aligned} \mathcal{K}_{c,b}y &= -b\mathcal{K}_c y \text{ and} \\ \mathcal{A}_{c,b} &= I - \mathcal{K}_{c,b}, \end{aligned}$$

where I is the identity on C_0. For $c > 1$ and $h \in C_0$ it is easy to see that $y \in C_0 \cap C^2$ is a solution to

$$y'' = (c^2 - b\,\mathrm{sech}^2 z)y + h(z), \text{ for } z \in \mathbb{R} \tag{5.28}$$

satisfying (5.22) iff $y \in C_0$ is a solution to

$$\mathcal{A}_{c,b}y = \mathcal{G}_c h. \tag{5.29}$$

Note by the above arguments that $y', y'' \in C(\mathbb{R})$. Now $\mathcal{K}_{c,b}$ is completely continuous on C_0 so it follows by Fredholm theory that:

(i) $Z_{0,c,b} = \ker \mathcal{A}_{c,b} \subset C_0$ is finite dimensional and we may write C_0 as a direct sum $C_0 = Z_{0,c,b} \bigoplus Z_{c,b}$ where $Z_{c,b} \subset C_0$ is a closed subspace.

(ii) $Y_{c,b} = \mathrm{Ra}\,\mathcal{A}_{c,b} \subset C_0$ is a closed subspace and we may write C_0 as a direct sum $C_0 = Y_{0,c,b} \bigoplus Y_{c,b}$ where $Y_{0,c,b} \subset C_0$ is a closed subspace with $\dim Y_{0,c,b} = \dim Z_{0,c,b}$.

(iii) $\mathcal{A}_{c,b} : Z_{c,b} \to Y_{c,b}$ is one-one and onto with bounded inverse so that for $l \in Y_{a,b}$ there is a unique $y \in Z_{c,b}$ such that

$$\mathcal{A}_{c,b}\, y = l$$

and there is $C > 0$ such that

$$\frac{1}{C}||l|| \leq ||y|| \leq C||l||. \tag{5.30}$$

For more information on Fredholm theory see [Zei88, sections 8.4 and 8.5].

When $\omega = 0 = A$ the system (5.12) decouples to

$$\begin{aligned} \mathcal{L}_1\varphi &:= \varphi'' - \left[4 - 6\,\mathrm{sech}^2 z\right]\varphi = 0 \\ \mathcal{L}_2\chi &:= \chi'' - \left[1 - 2\,\mathrm{sech}^2 z\right]\chi = 0. \end{aligned} \tag{5.31}$$

The first equation in (5.31) corresponds to (5.28) where $h = 0$, $c = 2$ and $b = 6$ and the kernel is spanned by φ_0 where $\varphi_0(z) = \mathrm{sech}^2 z$.

Similarly the second equation in (5.31) corresponds (5.28) where $h = 0$, $c = 1$ and $b = 2$ and the kernel is spanned by χ_0 where $\chi_0(z) = \mathrm{sech}\, z$.

With $\varphi_0(z) = \mathrm{sech}^2 z$ and $\chi_0(z) = \mathrm{sech}\, z$ we associate projections $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, defined by

$$\begin{aligned} \mathcal{P}_1 h &= h - \varphi_0\langle h, \varphi_0\rangle/\|\varphi_0\|^2 \\ \mathcal{P}_2 h &= h - \chi_0\langle h, \chi_0\rangle/\|\chi_0\|^2, \end{aligned} \tag{5.32}$$

for $h \in C_0$.

If $c = 2$ and $b = 6$ and (5.28) has a solution, then it is easy to check that $\langle \varphi_0, h\rangle = 0$. It follows from (i) and (ii) that $Y_{2,6} = \mathrm{Ra}\,\mathcal{A}_{2,6} = \mathcal{P}_1\, C_0 = Z_{2,6}$. Moreover if $c = 1$ and $b = 2$ and (5.28) has a solution it is easy to check that $\langle \chi_0, h\rangle = 0$. It follows from (i) and (ii) that $Y_{1,2} = \mathrm{Ra}\,\mathcal{A}_{1,2} = \mathcal{P}_2\, C_0 = Z_{1,2}$.

For the case $A = 0$ we use Lemma 5.1 in part four of the proof of Theorem 5.1.

The idea of the proof of Lemma 5.1 is to write the solution as a product of a power of sech z and a hypergeometric function with argument $-\sinh^2 z$; see the appendix of [Yag99] for details. See [FG08] for a discussion of this result and associated references.

5.3 The main result

Proof of Theorem 5.1. First we show that for $k > 4$ and $\omega \geq 4$ there is $a = a(k) > 0$ sufficiently small that $(\varphi, \chi) = (0, 0)$ is the only bounded solution of (5.17) when $\Theta \in \mathfrak{B}_{a,k}$.

Setting $c = \omega$ in (5.21) and noting that $A \geq 0$, $\kappa > 0$, and $\gamma_1 > 0$ it follows from (5.17) that

$$\begin{aligned}
\|\varphi\| + \|\chi\| &\leq \frac{1}{\omega^2}\left(\frac{\kappa A^2}{2} + 4\right)\|\varphi\| + \frac{A(2+\gamma_1 A^2)}{\omega^2}\|\chi\| \\
&\quad + \frac{1}{\omega^2}\left(2\gamma_1 A^2 + 1\right)\|\chi\| + \frac{A\kappa}{\omega^2}\|\varphi\| \\
&\leq \frac{1}{\omega^2}\left(\frac{\kappa A^2}{2} + 4 + 2\gamma_1 A^2 + A(\kappa + 2 + \gamma_1 A^2)\right)(\|\varphi\| + \|\chi\|).
\end{aligned} \tag{5.33}$$

If

$$\frac{\kappa A^2}{2} + 4 + 2\gamma_1 A^2 + A(\kappa + 2 + \gamma_1 A^2) < \omega^2 \tag{5.34}$$

it follows that $\varphi = \chi = 0$. Further, if $A = 0$ then (5.34) becomes

$$4 < \omega^2. \tag{5.35}$$

Thus if $\omega \geq 4$ and $k > 4$ there is $a = a(k) > 0$ sufficiently small, that $(\varphi, \chi) = (0, 0)$ is the only bounded solution of (5.17) when $\Theta \in \mathfrak{B}_{a,k}$. As a matter of fact, we can take $a(k)$ as the only positive root of the cubic equation

$$4 - \omega^2 + (2 + k)a + \frac{5}{2}ka^2 + ka^3 = 0$$

whose existence is guaranteed by (5.35). However this formula is awkward so for the case $ak \leq 1$, $\omega \geq 4$, and $k > 4$ we proceed as follows: since

$$4 - \omega^2 + (2 + k)a + \frac{5}{2}ka^2 + ka^3 \leq -11 + \frac{9}{2k} + \frac{1}{k^2} < -\frac{157}{16},$$

we may take $a(k) = \frac{1}{k}$.

Now suppose that $0 < \omega < 4$ and $A = 0$. Substituting $A = 0$ into (5.14) and (5.17) we obtain (5.19).

Applying Lemma 5.1 we see that $(\varphi, \chi) = (0, 0)$ is the only bounded solution of (5.19) and (5.13) when $A = 0$ and $\omega > 0$. Using this fact, we have the following result.

If $k > 4$ and $\varepsilon > 0$ we claim that there is $a = a(\varepsilon, k) > 0$ sufficiently small that $(\varphi, \chi) = (0, 0)$ is the only bounded solution of (5.17) and (5.13) when $\omega \geq \varepsilon$ and $\Theta \in \mathfrak{B}_{a,k}$.

Suppose the result is false. Then there are $(\omega, \Theta) = (\omega_j, \Theta_j)$ and corresponding solutions $(\varphi, \chi) = (\varphi_j, \chi_j)$ of (5.17) and (5.13) such that $A_j \to 0^+$. By the first step of the proof if $k > 4$ and $\omega \geq 4$ there is a so we assume that $k > 4$ and $4 > \omega \geq \varepsilon$. Thus (ω_j, Θ_j) is bounded and after rescaling we may assume that $\max\{\|\varphi_j\|, \|\chi_j\|\} = 1$. But then by the above arguments, we also obtain $\max\{\|\varphi_j'\|, \|\chi_j'\|, \|\varphi_j''\|, \|\chi_j''\|\} \leq \Upsilon$ for a constant Υ depending on a, k, ε. Choosing a subsequence and relabeling we may assume that (ω_j, Θ_j) converges to a limit which, by abuse of notation, we again denote by (ω, Θ). Moreover, by the Arzela-Ascoli theorem, we may assume that (φ_j, χ_j) converges uniformly on any compact interval of $\mathbb{R}$ together with their 1 st and 2 nd derivatives to (φ, χ). Thus $4 \geq \omega \geq \varepsilon$, $\max\{\|\varphi\|, \|\chi\|\} = 1$ and φ, χ satisfy (5.19). But then (5.13) is satisfied as well [Bat90]. On the other hand, we already know from Lemma 5.1 that $(\varphi, \chi) = (0, 0)$ which is a contradiction since $\max\{\|\varphi\|, \|\chi\|\} = 1$. The result follows.

For the final step we study (5.17) and (5.13) as both ω and A approach zero.

Setting $\omega = \omega_0\varepsilon$ and $A = A_0\varepsilon$ for $A_0^2 + \omega_0^2 = 1$, in (5.17) we obtain

$$\begin{aligned} \mathcal{L}_1\varphi &= \varepsilon\left[\varepsilon O_1\varphi + O_2\chi\right] \\ \mathcal{L}_2\chi &= \varepsilon\left[O_3\varphi + \varepsilon O_4\chi\right] \end{aligned} \tag{5.36}$$

where

$$\begin{aligned} O_1 &= \omega_0^2 + \frac{\kappa A_0^2}{2}\tanh^2 z, \qquad O_{2a} = 2A_0 \operatorname{sech} z \tanh z,\ O_{2b} = \gamma_1 A_0^3 \operatorname{sech} z \tanh z, \\ O_4 &= \omega_0^2 + 2\gamma_1 A_0^2 \operatorname{sech}^2 z, \qquad O_3 = A_0\kappa \operatorname{sech} z \tanh z,\ O_2 = O_{2a} + \varepsilon^2 O_{2b}. \end{aligned} \tag{5.37}$$

We apply the Lyapunov-Schmidt method to (5.36). Assuming that there is a non-trivial solution for some $\varepsilon > 0$ there are φ_1 and χ_1 such that $\varphi = \varepsilon\varphi_1 + c_1\varphi_0$ and $\chi = \varepsilon\chi_1 + c_2\chi_0$, where

$$\langle\varphi_1, \varphi_0\rangle = 0, \quad \langle\chi_1, \chi_0\rangle = 0 \tag{5.38}$$

and $c_1^2 + c_2^2 = 1$. Substituting $\varphi = \varepsilon\varphi_1 + c_1\varphi_0$ and $\chi = \varepsilon\chi_1 + c_2\chi_0$ in (5.36) and using the definition of $\mathcal{P}_1$ and $\mathcal{P}_2$ we obtain

$$\mathcal{L}_1\varphi_1 = \mathcal{P}_1\left[\varepsilon O_1(\varepsilon\varphi_1 + c_1\varphi_0) + O_2(\varepsilon\chi_1 + c_2\chi_0)\right] \tag{5.39}$$

$$\mathcal{L}_2\chi_1 = \mathcal{P}_2\left[O_3(\varepsilon\varphi_1 + c_1\varphi_0) + \varepsilon O_4(\varepsilon\chi_1 + c_2\chi_0)\right] \tag{5.40}$$

and

$$\begin{aligned}&\langle \varepsilon O_1(\varepsilon\varphi_1 + c_1\varphi_0) + O_2(\varepsilon\chi_1 + c_2\chi_0), \varphi_0\rangle = 0\\ &\langle \varepsilon O_4(\varepsilon\chi_1 + c_2\chi_0) + O_3(\varepsilon\varphi_1 + c_1\varphi_0), \chi_0\rangle = 0.\end{aligned} \tag{5.41}$$

Since $\langle\varphi_1, \varphi_0\rangle = 0$, $\langle\mathcal{P}_1\left[\varepsilon O_1(\varepsilon\varphi_1 + c_1\varphi_0) + O_2(\varepsilon\chi_1 + c_2\chi_0)\right], \varphi_0\rangle = 0$, $\langle\chi_1, \chi_0\rangle = 0$ and $\langle\mathcal{P}_2\left[O_3(\varphi_1 + c_1\varphi_0) + \varepsilon O_4(\chi_1 + c_2\chi_0)\right], \chi_0\rangle = 0$ it follows from the Fredholm alternative and (5.30) that we can uniquely solve (5.39) for φ_1 and (5.40) for χ_1 to obtain

$$||\varphi_1|| \leq C||\mathcal{P}_1\left[\varepsilon O_1(\varepsilon\varphi_1 + c_1\varphi_0) + O_2(\varepsilon\chi_1 + c_2\chi_0)\right]|| \tag{5.42}$$

$$||\chi_1|| \leq C||\mathcal{P}_2\left[O_3(\varepsilon\varphi_1 + c_1\varphi_0) + \varepsilon O_4(\varepsilon\chi_1 + c_2\chi_0)\right]||. \tag{5.43}$$

Adding these inequalities and moving terms involving $||\varphi_1||$ and $||\chi_1||$ to the left we see that there is a constant $C > 0$ such that for $\varepsilon > 0$ small

$$||\varphi_1|| + ||\chi_1|| \leq C(||\varphi_0|| + ||\chi_0||),$$

so $||\varphi_1||$ and $||\chi_1||$ are bounded; here we have used $c_1^2 + c_2^2 = 1$. Now

$$\langle O_2\chi_0, \varphi_0\rangle = (2A_0 + \gamma_1 A_0^3\varepsilon^2)\int_{-\infty}^{\infty} \operatorname{sech}^4 z \tanh z\, dz = 0, \text{ and} \tag{5.44}$$

$$\langle O_3\varphi_0, \chi_0\rangle = A_0\kappa\int_{-\infty}^{\infty} \operatorname{sech}^4 z \tanh z\, dz = 0. \tag{5.45}$$

Using (5.41), (5.44) and (5.45) we obtain

$$F = \begin{pmatrix}\langle O_1(\varepsilon\varphi_1 + c_1\varphi_0) + O_2\chi_1, \varphi_0\rangle\\ \langle O_4(\varepsilon\chi_1 + c_2\chi_0) + O_3\varphi_1, \chi_0\rangle\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}. \tag{5.46}$$

Setting $F = (F_1, F_2)$ we show below that the F_i can be expressed as power series in ε^2

computing the first term in the series for F_1 and F_2. Moreover we show by induction that $c_1\omega_0^2$ is a factor of F_1. Combining this with the first term in the series for F_1 we will see that

$$F_1 = c_1\omega_0^2\left(\frac{4}{3} + O(\varepsilon^2)\right).$$

Since $\omega = \omega_0\varepsilon \neq 0$ it follows that if ε is sufficiently small then $F_1 = 0$ only if $c_1 = 0$.

We use this to show that there is $\varepsilon_0 > 0$ such that $F = (F_1, F_2) = (0,0)$ implies $c_1^2 + c_2^2 < 1$, when $0 < \varepsilon \leq \varepsilon_0$, $g(\omega_0, A_0, \gamma_1, \kappa) > \eta > 0$, $\omega = \varepsilon\omega_0$, $A = \varepsilon A_0$, $\omega_0^2 + A_0^2 = 1$, a contradiction. It follows that $(\varphi, \chi) = (0,0)$ is the only solution of (5.39), (5.40), (5.46), and (5.13) when $0 < \varepsilon \leq \varepsilon_0$, $g(\omega_0, A_0, \gamma_1, \kappa) > \eta > 0$, and $\omega^2 + A^2 = \varepsilon^2$, as required.

To see that F_i can be expressed as power series in ε we note from section 5.2 that for $f \in C_0$ and $(c,b) \in \{(2,6), (1,2)\}$

$$y'' - \left[c^2 - b\,\mathrm{sech}^2 z\right] y = \mathcal{P}_{c,b} f \tag{5.47}$$

has a unique solution

$$y = \mathcal{A}_{c,b}^{-1}(\mathcal{P}_{c,b} f) \in Z_{c,b}$$

where

$$\mathcal{A}_{c,b}^{-1} : Z_{c,b} \to Z_{c,b}$$

is bounded while $\mathcal{P}_{2,6} = \mathcal{P}_1$ and $\mathcal{P}_{1,2} = \mathcal{P}_2$.

Thus (5.39) and (5.40) can be rewritten as

$$\left(I - \varepsilon^2\mathcal{A}_{2,6}^{-1}\mathcal{P}_1 O_1\right)\varphi_1 - \varepsilon\mathcal{A}_{2,6}^{-1}\mathcal{P}_1 O_2\chi_1 = \mathcal{A}_{2,6}^{-1}\mathcal{P}_1\left[\varepsilon c_1 O_1\varphi_0 + c_2 O_2\chi_0\right] \tag{5.48}$$

$$\left(I - \varepsilon^2\mathcal{A}_{1,2}^{-1}\mathcal{P}_2 O_4\right)\chi_1 - \varepsilon\mathcal{A}_{1,2}^{-1}\mathcal{P}_2 O_3\varphi_1 = \mathcal{A}_{1,2}^{-1}\mathcal{P}_2\left[\varepsilon c_2 O_4\chi_0 + c_2 O_3\varphi_0\right]. \tag{5.49}$$

It follows from elementary operator theory that

$$\varphi_1 = \sum_{i=0}^{\infty}\varphi_{1,i}\varepsilon^i \tag{5.50}$$

$$\chi_1 = \sum_{i=0}^{\infty}\chi_{1,i}\varepsilon^i \tag{5.51}$$

where $\varphi_{1,i} \in Z_{2,6}$ and $\chi_{1,i} \in Z_{1,2}$ and the series converge in C_0 for ε small. Substituting in (5.48) and (5.49) and equating coefficients of powers of ε it is not difficult to see that $\varphi_{1,i}$, $\chi_{1,i} \in C^2 \cap C_0$ satisfy

$$\mathcal{L}_1\varphi_{1,0} = c_2 O_{2a}\chi_0 \tag{5.52}$$

$$\mathcal{L}_1\varphi_{1,1} = \mathcal{P}_1 [O_{2a}\chi_{1,0} + c_1 O_1\varphi_0] \tag{5.53}$$

$$\mathcal{L}_1\varphi_{1,2} = \mathcal{P}_1 [O_{2a}\chi_{1,1} + O_{2b}c_2\chi_0 + O_1\varphi_{1,0}] \tag{5.54}$$

$$\mathcal{L}_1\varphi_{1,i} = \mathcal{P}_1 [O_{2a}\chi_{1,i-1} + O_{2b}\chi_{1,i-3} + O_1\varphi_{1,i-2}] \text{ for } i \geq 3 \tag{5.55}$$

$$\mathcal{L}_2\chi_{1,0} = c_1 O_3\varphi_0 \tag{5.56}$$

$$\mathcal{L}_2\chi_{1,1} = \mathcal{P}_2 [O_3\varphi_{1,0} + c_2 O_4\chi_0] \tag{5.57}$$

$$\mathcal{L}_2\chi_{1,i} = \mathcal{P}_2 [O_4\chi_{1,i-2} + O_3\varphi_{1,i-1}] \text{ for } i \geq 2 \tag{5.58}$$

where $\varphi_{1,i}(z)$, $\chi_{1,i}(z) \to 0$ as $|z| \to \infty$. Here we used $\mathcal{P}_1 O_2\chi_0 = O_2\chi_0$ and $\mathcal{P}_2 O_3\varphi_0 = O_3\varphi_0$. Indeed if $x \in C_0$ is odd then $\mathcal{P}_1 x = x = \mathcal{P}_2 x$.

Substituting the expressions for φ_1 and χ_1 given in (5.50) and (5.51) into (5.46) we obtain

$$\begin{aligned} F_1 &= \langle O_1 c_1\varphi_0 + O_{2a}\chi_{1,0}, \varphi_0\rangle + \varepsilon\langle O_1\varphi_{1,0} + O_{2a}\chi_{1,1}, \varphi_0\rangle \\ &\quad + \sum_{n=2}^{\infty} \varepsilon^n \langle O_1\varphi_{1,n-1} + O_{2a}\chi_{1,n} + O_{2b}\chi_{1,n-2}, \varphi_0\rangle \end{aligned} \tag{5.59}$$

$$F_2 = \langle O_4 c_2\chi_0 + O_3\varphi_{1,0}, \chi_0\rangle + \sum_{n=1}^{\infty} \varepsilon^n \langle O_3\varphi_{1,n} + O_4\chi_{1,n-1}, \chi_0\rangle. \tag{5.60}$$

To show the coefficients of odd powers of ε are all 0 we need the following lemma.

Lemma 5.2: *If*

$$y'' - (c^2 - b\,\text{sech}^2 z)y = f(z)$$

satisfies $y(z) \to 0$ *as* $|z| \to \infty$ *where* $(c, b) \in \{(2, 6), (1, 2)\}$ *and* y, $f \in Z_{c,b}$ *and* $f \in C_0$ *then*

(a) y is even if f is even, and

(b) y is odd if f is odd.

Proof. We consider the case $(c, b) = (2, 6)$; the case $(c, b) = (1, 2)$ follows by a similar argument. If f is even then $x(z) = (y(z) - y(-z))/2 \in Z_{2,6}$ and $\mathcal{L}_1 x = 0$ so that $x = 0$ and y is even. If f is odd then $x(z) = (y(z) + y(-z))/2 \in Z_{2,6}$ and $\mathcal{L}_1 x = 0$ so that $x = 0$ and y is odd. □

Corollary 5.1: *Let $\varphi_{1,i}$ and $\chi_{1,i}$ satisfy* (5.52) *to* (5.58). *Then*

(a) $\varphi_{1,i}$, $\chi_{1,i}$ *is odd if i is even, and*

(b) $\varphi_{1,i}$, $\chi_{1,i}$ *is even if i is odd.*

Proof. It follows from (5.52), (5.56) and Lemma 5.2 that $\varphi_{1,0}$ and $\chi_{1,0}$ are odd. The result follows by induction using this, (5.52) to (5.58) and Lemma 5.2. □

Since φ_0 and χ_0 are even it follows from Corollary 5.1, (5.59) and (5.60) that the coefficients of ε^{2m+1} in (5.59) and (5.60) are 0.

Thus

$$F_1 = \sum_{m=0}^{\infty} a_{2m}\varepsilon^{2m} \text{ and} \tag{5.61}$$

$$F_2 = \sum_{m=0}^{\infty} b_{2m}\varepsilon^{2m} \tag{5.62}$$

where

$$a_0 = \langle O_1 c_1 \phi_0 + O_{2a}\chi_{1,0}, \varphi_0\rangle \tag{5.63}$$

$$b_0 = \langle O_4 c_2 \chi_0 + O_3\varphi_{1,0}, \chi_0\rangle. \tag{5.64}$$

It is easy to check that the solutions of (5.52) and (5.56) are

$$\varphi_{1,0}(x) = -c_2\frac{A_0}{2} z\,\mathrm{sech}^2 z \tag{5.65}$$

$$\chi_{1,0}(z) = -c_1\frac{A_0\kappa}{4}\,\mathrm{sech}\, z \tanh z \tag{5.66}$$

so that

$$a_0 = c_1\omega_0^2\frac{4}{3} \quad (5.67)$$

$$b_0 = c_2\left(2\omega_0^2 + \frac{A_0^2}{6}(16\gamma_1 - \kappa)\right). \quad (5.68)$$

It is easy to see that

$$a_{2m} = \langle O_1\varphi_{1,2m-1} + O_{2a}\chi_{1,2m} + O_{2b}\chi_{1,2m-2}, \varphi_0\rangle. \quad (5.69)$$

We show that a_{2m} has $c_1\omega_0^2$ as a factor, for all $m \geq 1$. For $m \geq 1$ define ξ_m and η_m by

$$\xi_m = 2A_0\chi_{1,2m} + \gamma_1 A_0^3\chi_{1,2m-2} \quad (5.70)$$

$$\eta_m = \varphi_{1,2m-1} \quad (5.71)$$

so that $O_{2a}\psi_{1,2m} + O_{2b}\psi_{1,2m-2} = \xi_m \operatorname{sech} z \tanh z$. Thus it suffices to show that ξ_m and η_m have $c_1\omega_0^2$ as a factor, for all $m \geq 1$.

First we show that $\xi_1 = 2A_0\chi_{1,2} + \gamma_1 A_0^3\chi_{1,0}$ and $\eta_1 = \varphi_{1,1}$ have $c_1\omega_0^2$ as a factor. Then we show by induction that both ξ_m and η_m have $c_1\omega_0^2$ as a factor, for all $m \geq 1$.

Thus we need to solve (5.53) and (5.58) with $i = 2$ to find $\varphi_{1,1}$ and $\chi_{1,2}$.

First we solve (5.53) for $\varphi_{1,1}$.

Now

$$\begin{aligned}\mathcal{L}_1\varphi_{1,1} &= \mathcal{P}_1\left(O_{2a}\chi_{1,0} + c_1 O_1\varphi_0\right)\\ &= c_1\mathcal{P}_1\left(-\frac{A_0^2\kappa}{2}\operatorname{sech}^2 z\tanh^2 z + (\omega_0^2 + \frac{\kappa A_0^2}{2}\tanh^2 z)\operatorname{sech}^2 z\right) = 0\end{aligned}$$

since $\mathcal{P}_1 \operatorname{sech}^2 = 0$. Since $\mathcal{L}_1$ is linear it follows that $\varphi_{1,1} = 0$ and, in particular, $\varphi_{1,1}$ has a factor of $c_1\omega_0^2$.

Now we solve (5.58) with $i = 2$ for $\chi_{1,2}$, and substitute in (5.69).

Since $O_4\chi_{1,0} + O_3\varphi_{1,1}$ is odd

$$\begin{aligned}\mathcal{L}_2\chi_{1,2} &= \mathcal{P}_2\left(O_4\chi_{1,0} + O_3\varphi_{1,1}\right) = O_4\chi_{1,0} + O_3\varphi_{1,1} \\ &= -c_1\frac{A_0\kappa}{4}\left(\omega_0^2 \operatorname{sech} z \tanh z + 2\gamma_1 A_0^2 \operatorname{sech}^3 z \tanh z\right).\end{aligned}$$

Thus using additivity of solutions it is easy to check that

$$\chi_{1,2} = c_1\frac{A_0\kappa}{8}\left(\gamma_1 A_0^2 \operatorname{sech} z \tanh z + \omega_0^2 z \operatorname{sech} z\right).$$

Thus

$$\xi_1 = 2A_0\chi_{1,2} + \gamma_1 A_0^3\chi_{1,0} = c_1\omega_0^2\frac{A_0^2\kappa}{4} z \operatorname{sech} z$$

so ξ_1 has $c_1\omega_0^2$ as a factor.

Thus ξ_m and η_m have $c_1\omega_0^2$ as a factor for $m = 1$.

Assume that ξ_m and η_m have $c_1\omega_0^2$ as a factor for $1 \leq m \leq k$. It follows from (5.55) and (5.58) that

$$\begin{aligned}\mathcal{L}_1\eta_{k+1} &= \mathcal{L}_1\varphi_{1,2k+1} = \mathcal{P}_1\left(O_1\varphi_{1,2k-1} + O_{2a}\chi_{1,2k} + O_{2b}\chi_{1,2k-2}\right) \\ \mathcal{L}_1\eta_{k+1} &= \mathcal{P}_1\left(O_1\eta_k + \xi_k \operatorname{sech} z \tanh z\right) && (5.72) \\ \mathcal{L}_2\xi_{k+1} &= \mathcal{L}_2\left(2A_0\chi_{1,2k+2} + \gamma_1 A_0^3\chi_{1,2k}\right) \\ &= \mathcal{P}_2\left(O_4\left(2A_0\chi_{1,2k} + \gamma_1 A_0^3\chi_{1,2k-2}\right) + O_3\left(2A_0\varphi_{2k+1} + \gamma_1 A_0^3\varphi_{2k-1}\right)\right) \\ \mathcal{L}_2\xi_{k+1} &= \mathcal{P}_2\left(O_4\xi_k + O_3(2A_0\eta_{k+1} + \gamma_1 A_0^3\eta_k)\right). && (5.73)\end{aligned}$$

It follows from (5.72) and the induction hypothesis that η_{k+1} has $c_1\omega_0^2$ as a factor. It then follows from (5.73) and the induction hypothesis that ξ_{k+1} has $c_1\omega_0^2$ as a factor, as required.

So thus,

$$\begin{aligned}F_1 &= c_1\omega_0^2\left(\frac{4}{3} + O(\varepsilon^2)\right), && (5.74) \\ F_2 &= c_2 g(\omega_0, A_0, \gamma_1, \kappa) + O(\varepsilon^2). && (5.75)\end{aligned}$$

Since $\omega = \omega_0\varepsilon \neq 0$, it follows from (5.74) that $F_1 = 0$ for ε small only if $c_1 = 0$. Assume that $g(\omega_0, A_0, \gamma_1, \kappa) = 2\omega_0^2 + \frac{A_0^2}{6}(16\gamma_1 - \kappa) > \eta > 0$. Thus $F_2 = 0$ only if $c_2 = O(\varepsilon^2)$. Thus $F = (F_1, F_2) = (0, 0)$ and (5.17) and (5.13) has a solution for ε small only if $c_1^2 + c_2^2 = O(\varepsilon^2)$, a contradiction since $c_1^2 + c_2^2 = 1$.

Since $\omega_0^2 + A_0^2 = 1$ there is $\varepsilon_0 = \varepsilon_0(\eta, k) > 0$ such that $(\varphi, \chi) = (0, 0)$ is the only solution of (5.17) and (5.13) when $0 < |\varepsilon| \leq \varepsilon_0$, $g(\omega, A, \gamma_1, \kappa) > \eta\varepsilon^2 > 0$ and $\omega^2 + A^2 = \varepsilon^2$. Now if $0 < \omega < \varepsilon_0/\sqrt{2}$ and $0 < A < \varepsilon_0/\sqrt{2}$ then $\omega^2 + A^2 < \varepsilon_0^2$. Thus we may summarize our results as follows: If $k > 4$, $\eta \in (0, 2)$ and $d = \min\{\varepsilon_0/\sqrt{2}, a(k), a(\varepsilon_0/\sqrt{2}, k)\} > 0$, then $(\varphi, \chi) = (0, 0)$ is the only solution of (5.17) and (5.13) when $\omega \neq 0$ and $\Theta \in \mathfrak{B}_{\omega,d,\eta,k}$.

Hence our analytic solutions (4.3) to the Euler-Lagrange equations (5.1) will maintain a form which breaks SO(10) $\to$ SU(5) $\times$ U(1)$_X$ under short term dynamical evolution for an open ball, $\Theta \in \mathfrak{B}_{\omega,d,\eta,k}$, in the free parameter space.

□

5.4 Existence of non-trivial solutions

We now prove Theorem 5.2. Recall that $\Theta_1 = (\omega_0, A_0, \gamma_1, \kappa)$ and $\Theta_2 = (\omega_0, A_0, \kappa)$.

Proof. We assume that $\Theta_1 \in S^1 \times (-k, k)^2$ where $\omega_0 \neq 0$ and $k > 4$. In the third part of the proof of Theorem 5.1 we showed there is $\varepsilon_0 > 0$ such that (5.17) and (5.13) has a solution when $A = \varepsilon A_0$, $\omega = \varepsilon\omega_0 \neq 0$, and $0 < |\varepsilon| \leq \varepsilon_0$ iff $F = (F_1, F_2) = (0, 0)$, where F_1 is given by (5.74) and F_2 is given by (5.75). Note $F_i = F_i(\varepsilon, \Theta_1)$, $i = 1, 2$. Moreover we showed that $\varepsilon_0 > 0$ may be chosen so that $F_i \in C^1((-\varepsilon_0, \varepsilon_0) \times S^1 \times (-k, k)^2; \mathbb{R})$, $F_2 = c_2 g(\omega_0, A_0, \gamma_1, \kappa) + O(\varepsilon^2)$ and $F_1 = 0$, iff $c_1 = 0$. Now $c_2^2 + c_1^2 = 1$ for non-trivial solutions so assume that $c_2 = 1$.

Let $\bar{\Theta}_1 \in S^1 \times (-k, k)^2$ satisfy $\bar{\omega}_0 \bar{A}_0 \neq 0$ and $g(\bar{\Theta}_1) = 0$. Now $F_2(0, \Theta_1) = g(\Theta_1) = g(\omega_0, A_0, \gamma_1, \kappa)$ so $F_2(0, \bar{\Theta}_1) = 0$ and

$$\frac{\partial F_2}{\partial \gamma_1}(0, \bar{\Theta}_1) = \frac{8}{3}\bar{A}_0^2 \neq 0.$$

By the implicit function theorem there are neighbourhoods, $U_d(0,\bar{\Theta}_2)$ of $(0,\bar{\Theta}_2)$ given by

$$U_d(0,\bar{\Theta}_2) = (-d,d) \times S^1_d(\bar{\omega}_0,\bar{A}_0) \times B^1_d(\bar{\kappa})$$

and $U_{d,l}(0,\bar{\Theta}_1)$ of $(0,\bar{\Theta}_1)$ given by

$$U_{d,l}(0,\bar{\Theta}_1) = (-d,d) \times S^1_d(\bar{\omega}_0,\bar{A}_0) \times B^1_l(\bar{\gamma}_1) \times B^1_d(\bar{\kappa})$$

as well as $\gamma_1 \in C^1\left(U_d(0,\bar{\Theta}_2); B^1_l(\bar{\gamma}_1)\right)$ such that $\bar{\gamma}_1 = \gamma_1(0,\bar{\Theta}_2)$ and $F_2 = 0$ in $U_{d,l}(0,\bar{\Theta}_1)$ iff $\gamma_1 = \gamma_1(\varepsilon,\Theta_2)$. □

5.5 Long term evolution

Ultimately, mode mixing and other non-linear effects arising from terms of higher order in δX and $\delta\phi$ will influence the long term evolution of the system. Assessing these effect is beyond the scope of our quantitative analysis. We fall back on our claim that numerical solutions to (5.1) for X and ϕ indicate that the analytic solutions (4.3) are stable under long term dynamical evolution for an open ball in the free parameter space.

Please see the appendix B.1 for a discussion of the stability of the left right symmetric domain-wall-brane model.

5.6 Conclusions

In this chapter we considered the stability of the SO(10) domain-wall brane solutions, (ϕ_0, X_0) in (5.2), to the DWB equations (5.1). We did this by examining the dynamical evolution of linear perturbations $\delta\phi$, respectively δX, about ϕ_0, respectively X_0. The dynamical evolution of these perturbations is governed by equation (5.7). We used a Fourier transformation to expand $\delta\phi$ and δX in terms of normal modes. We examined a specific mode with time dependence $e^{\omega t}$. The solution (ϕ_0, X_0) in (5.2) is called **stable if there are *no* nontrivial solutions**, $(\delta\phi, \delta X)$, to the linear perturbation equations (5.7) when $\omega^2 > 0$. The solution (ϕ_0, X_0) in (5.2) is called **unstable if there exist nontrivial solutions** $(\delta\phi, \delta X)$ to (5.7) when $\omega^2 > 0$. Because the coupling constants run with energy when the theory is quantized we insisted that solutions of the form (5.2) were stable for a range of free parameters in the Euler Lagrange equations (5.1).

Because an SO(10) adjoint Higgs field perturbation, $\delta\mathcal{X}$, has independent components along 45 different directions, we broke the problem down by considering subclasses of the perturbations. The nontrivial case was where the perturbations along all Cartan directions are the same[1]. We referred to the specific subclass of the perturbations $(\delta\phi, \delta\mathcal{X})$, where the perturbations along all Cartan directions are the same as (φ, χ). We defined the free parameters in the model to be $\Theta = (A, \gamma_1, \kappa)$. Our free parameters were chosen to be the coupling constants in the DWB equations (5.1) satisfying the conditions (5.15) for our original solution (5.2). We used Fredholm theory for compact linear operators and the LyapunovSchmidt method to establish that there exists an open ball, $\mathfrak{B}_{\omega,d,\eta,k}$, in the free parameter space such that when $\Theta \in \mathfrak{B}_{\omega,d,\eta,k}$, the only solution to the linearized perturbation equations about (5.2), for any $\omega^2 > 0$, is $(\varphi, \chi) = 0$. In section 5.4 we established this result is nontrivial. Namely we established there exists a certain alternate hypersurface of the free parameters on which nontrivial solutions (φ, χ) to the linearized perturbation equations exist with nonzero $A = \epsilon A_0$ and nonzero $\omega = \epsilon\omega_0$ such that $\omega^2 > 0$. This is the first time the LyapunovSchmidt method has been applied to a dynamical systems problem. This new approach is relevant to other areas involving the study of dynamical systems and the study of the existence of deterministic chaos.

[1] At the same time we are requiring all other perturbations in the remaining 40, SO(10) adjoint Higgs field directions to be zero.

6

Domain-wall branes in Lifshitz theories

6.1 Introduction

We lack a renormalizable 4 + 1-dimensional Yang-Mills gauge theory. If we can introduce an ultraviolet complete 4 + 1-dimensional Yang-Mills gauge theory then chapter 4 will provide a feasible structure for a complete brane-world model. This model will exhibit 3+1-dimensional field theory and localized gravity. Furthermore the model should reproduce standard model phenomenology, subject to a phenomenological parameter fitting, akin to [CV11]. Therefore in the context of domain-wall brane models, we are extremely interested in finding a logically consistent regulator for 4 + 1-dimensional Yang-Mills gauge theory.

We found the inspiration for improving the ultraviolet properties of 4+1-dimensional Yang-Mills gauge theory in Hořava-Lifshitz gravity models.

As outlined in section 1.2, Hořava [Hor09, HMT10] used the Lifshitz anisotropic scaling properties of magnetic materials as a regulator for quantum gravity. This involves modifying the Einstein-Hilbert action by introducing higher spatial derivative curvature terms, which break Lorentz invariance but ameliorate graviton-loop renormalization effects.

Hořava's success has sparked enquiries into whether the good ultraviolet (UV) behaviour of Hořava-Lifshitz gravity may be transferable to certain non-renormalizable Yang-Mills gauge theories [IRS09, IS10]. Models which include higher spatial derivative extensions à la Hořava are collectively referred to as Lifshitz field theories.

As demonstrated in subsections 3.8.3-3.8.4 a $4 + 1$-dimensional Lifshitz scalar field theory with critical exponent $z < 4$ is power counting renormalizable whenever the potential has $2(d + z)/(d - z)$ th leading order interaction terms, and is superrenormalizable when $z > 4$. Furthermore provided $z \geq 2$ any $4+1$-dimensional Lifshitz gauge theory will satisfy (3.106). Hence we can write a power counting renormalizable action for the gauge fields. We are hopeful that we can provide the machinery necessary to build a $4 + 1$-dimensional Lifshitz Dvali-Shifman analogue which generalizes section 3.6. The trapping of gauge fields on the wall due to non-perturbative dynamics in the $4 + 1$-dimensional bulk can be checked now by lattice gauge simulations in the limit of infinitesimally small lattice spacing.

The purpose of this chapter is to take the standard building blocks for a $4 + 1$-dimensional Lorentz invariant brane-world model, collated in [DGV08, TV09, RS83], and see if they can be extended to construct a Lifshitz power counting renormalizable brane-world model with quartic leading order spatial derivatives. We are principally interested in whether there is a Lifshitz analogue to the domain-wall brane and if we can dynamically localize fermions. These two elements (together with the Dvali-Shifman mechanism for dynamical gauge boson localization [DS97]) form the backbone for a dynamically localized domain-wall brane standard model. Being an introduction to the subject, we restrict our analysis here to a toy model featuring just a scalar and a fermion field. Of course a full theory would require many more ingredients including gravity and gauge fields, but that is too ambitious a construction to attempt in one step.

We stress that a Lifshitz domain-wall brane model would be markedly different to previous work on extra-dimensional field theories and would be valuable because preceding work in this field has been entirely devoted to effective field theories and, by construction, is only predictive up to a UV cutoff scale [DGV08, TV09].

To this end we take the most general power-counting-renormalizable action for a real Lifshitz scalar field ϕ living in $4 + 1$-dimensions which is consistent with a discrete reflection symmetry $\phi \to -\phi$, where the Z_2 symmetry is necessary for generating topological boundary conditions. For this model we demonstrate that a kink or domain-wall type of topological defect in the Lifshitz scalar field can condense to form a brane. However when we extend the action to incorporate fermions we find that the standard interpretation of the $3+1$-

dimensional left-chiral fermion, as the dynamically-localized zero mode in a Kaluza-Klein tower, fails.

In section 3.5 we argued that for isotropic 4 + 1-dimensional models, at low energies 4 + 1-dimensional spinors behave like 3 + 1-dimensional left-chiral fermions localized on the brane. For a solution, $\Psi(x^M)$, to the isotropic 4 + 1-dimensional Dirac equation, there exists a basis for the space of continuous bounded functions of a single coordinate, $x_5 \equiv y$, which can be used to project the field $\Psi(x^M)$ onto a 'Kaluza-Klein' tower. This Kaluza-Klein tower contains a normalizable massless 3 + 1-dimensional left-chiral zero-mode fermion plus massive 3 + 1-dimensional fermions. At low energies only the dynamically-localized massless chiral zero mode in this tower is kinematically allowed and the effective low-energy dynamics carry the right phenomenological signature needed to model the quarks and leptons of the standard model.

In this chapter we take a solution to the Lifshitz 4 + 1-dimensional Dirac equation with quartic leading order spatial derivatives and with Yukawa interactions terms coupling the fermions to the Lifshitz scalar.[1] We write this solution as a Kaluza-Klein tower of 3 + 1-dimensional spinors and we show that this tower does not contain a massless 3 + 1-dimensional left-chiral zero mode fermion. We find that spatially isotropic Lifshitz Dirac equations do not have zero mode solutions as part of their Kaluza-Klein towers. To overcome this difficulty we consider other models which do not treat all four spatial dimensions symmetrically. We present the zero mode solution for a model where 4 + 1-dimensional Lorentz invariance is broken explicitly to SO(3) spatial rotational invariance.

We refer the reader to chapter 2 for a guide to the notation used in this chapter and section 3.8.1 of the Introduction for a discussion of weighted scaling dimensions.

In section 7.1 of this chapter we shall explicitly demonstrate that, for the minimal case of a 4 + 1-dimensional model with quartic spatial derivatives, the Euler-Lagrange equations

[1] We choose to stop at quartic order purely for simplicity. A complete theory including renormalizable gravity in 4 + 1-dimensions will require an action containing at least order-eight derivatives. In the same spirit we consider only Yukawa interactions. This approach makes the analysis easier and allows for a direct comparison to the analogous 4 + 1-dimensional Lorentz invariant scenario. For completeness we present an action containing the full complement of fermion and scalar interaction terms invariant under 4-dimensional spatial-rotations and generalize our Yukawa theory conclusion to cover this action by arguing that interaction terms do not alter the no-go result.

for a general power counting renormalizable Lifshitz scalar field action have a domain-wall brane solution. In section 6.3 we explain why it is impossible to isolate and dynamically localize a 3 + 1-dimensional left chiral zero mode fermion in Lifshitz spatially isotropic 4+1-dimensional domain-wall brane models. In section 6.4 we discuss alternative models featuring compact extra-dimensions and smaller unbroken spatial symmetries. Our final section is a conclusion.

We will apply standard terminology directly to our model without qualifying that we are talking about a 4 + 1-dimensional, quartic leading order spatial derivatives extension to the standard theory before each statement. We will use these descriptors only when our intention is not clear from the context.

We remind the reader of our convention from section 3.8.1. We use $d + 1_z$-dimensions for a foliated space-time manifold with weighted scaling characterized by (3.98), and drop the subscript "1" when talking about the Lorentz invariant limiting case $z = 1$. For a given $d + 1_z$-dimensions Lifshitz model we continue to use $[\tau]_s$ to refer to the weighted scaling dimensions of τ.

6.2 Domain-wall brane

6.2.1 The existence of a domain-wall brane

We start by looking for the analogue of the domain-wall brane in $4+1_z$-dimensional Lifshitz scalar field theories. In section 3.4.1, in the context of isotropic 4 + 1-dimensional models, we introduced the canonical ϕ^4 Lagrangian and kink-like domain-wall brane. In $4 + 1_z$-dimensional Lifshitz field theories the real scalar field Lagrangian and topological defect solutions are modified by the presence of higher spatial derivatives.

In this section we isolate the real scalar field terms in the action.[2] We work under the assumption that there is an equivalence relation on the space of all actions such that actions S_1 and S_2 are identified if they contain the same terms up to global boundary terms. By con-

[2]It is physically justifiable to solve the scalar field dynamical equations, and then work out the motion of fermions propagating in the scalar field background because, in the framework of domain-wall brane models, the fermions are incorporated as a perturbative mode expansion about an empty vacuum state.

sistently following this principle we will discard all terms in the equations of motion which arise from boundary effects. Therefore from an empirical perspective, actions belonging to the same equivalence class will be phenomenologically indistinguishable.

From subsection 3.4.1 we know that, to create a domain-wall brane our action must exhibit the discrete reflection symmetry $\phi(x^M) \to -\phi(x^M)$. To simplify the model we choose to work in flat space-time.[3] We consider the most general action for $\mathbb{R} \times \mathbb{R}^4$ with critical exponent $z = 2$. This is consistent with the above and in the methodology of weighted scaling dimensions is renormalizable with a full complement of relevant and marginal operators:

$$\begin{aligned} S_\phi &= S_{\phi \mathrm{free}} + S_{\phi \mathrm{int}}, \\ &= \int d^5x \left[\left(\frac{1}{2}(\partial_t\phi)^2 - \frac{a^2}{2\Lambda^2}(\Delta\phi)^2 - \frac{c^2}{2}(\partial_i\phi)^2 \right) + \right. \\ & \left. \left(-\frac{g_2}{2}\phi^2 - \frac{g_4}{4!\Lambda}\phi^4 - \frac{b}{4\Lambda^3}(\phi\partial_i\phi)^2 - \frac{g_6}{6!\Lambda^4}\phi^6 \right) \right], \end{aligned} \tag{6.1}$$

where Λ is being used to keep track of the natural mass dimensions in isotropic space-time. We have absorbed the coupling constant in front of $(\partial_t\phi)^2$ into a rescaling of the field $\phi(x^M)$. In the weighted scaling dimensions of our theory we have

$$[t]_s = -z = -2, \qquad \left[x^i\right]_s = -1, \qquad [\phi]_s = \frac{d-z}{2} = 1. \tag{6.2}$$

[3]Incorporating gravitation will be far from trivial, but not hopeless. First, one would have to face the difficulties [CNPS09, LP09, BPS09, BPS10, Kob10] that have been identified in the original versions of 3 + 1-dimensional Hořava gravity [Hor09]. Reference [HMT10] provides a way forward. Then one would need to develop a 4 + 1-dimensional extension, which will require going to a $z = 4$ theory, and then would need to establish that the brane tension causes the domain-wall to be a preferred observation hyperplane from which gravity appears to be 3 + 1-dimensional, and that the effective gravity theory on the wall is sufficiently close to general relativity. For domain-wall branes with 4 + 1-dimensional general relativity, the answer is provided by the Randall-Sundrum [RS99a, RS99b] warped metric solution which involves pasting together two semi-infinite regions of anti-de Sitter space-time with a matching junction condition at the location of the brane. This configuration can be used to dynamically localize a 3 + 1-dimensional Kaluza-Klein zero mode graviton. Something resembling this solution would presumably also have to exist within the putative 4 + 1-dimensional Hořava gravity theory. The most straight-forward approach would be to verify the anti-de Sitter space-time solutions in [HMT10] could be extended to the putative 4 + 1-dimensional Hořava gravity theory before looking for an analogue to the Randall-Sundrum warped metric ansatz. These challenging problems are well beyond the scope of this chapter.

If we set $[\Lambda]_s = 0$ then the coupling constants have weighted scaling dimensions:

$$\left[a^2\right]_s = 2z - 4 = 0, \qquad \left[c^2\right]_s = 2z - 2 = 2,$$
$$[b]_s = 3z - d - 2 = 0, \qquad [g_n]_s = d + z - \frac{n(d-z)}{2} = 6 - n. \tag{6.3}$$

From the above weighted scaling dimensions, we confirm that $4 + 1_2$-dimensional Lifshitz scalar field theory with a ϕ^6 leading order potential is power counting renormalizable. Note that the parameter c which plays the role of the maximum obtainable velocity in the free particle IR dispersion relation, becomes a running parameter in the quantized theory. It would thus be misleading to absorb the energy-scale-dependent c^2 by a relative rescaling of spatial and temporal coordinates.

The Klein-Gordon equation obtained from this action using the principle of stationary action is

$$\partial_t^2\phi + \frac{a^2}{\Lambda^2}\Delta^2\phi - c^2\Delta\phi - \frac{b}{2\Lambda^3}\phi^2\Delta\phi - \frac{b}{2\Lambda^3}\phi\partial_i\phi\partial^i\phi + \frac{g_6}{5!\Lambda^4}\phi^5 + \frac{g_4}{3!\Lambda}\phi^3 + g_2\phi = 0. \tag{6.4}$$

The domain-wall is the static solitary wave solution to the Klein-Gordon equation which is isotropic in three spatial directions and interpolates between distinct minima of the potential as a function of y.

Numerical domain-wall brane solutions to the Klein-Gordon equation have been found for a wide range of parameters. In Figure 6.1 we give an example of a numerical domain-wall brane configuration. In addition we can give an explicit example of an analytic domain-wall brane,

$$\phi = v\tanh(uy), \tag{6.5}$$

which is a solution to the Klein-Gordon equation for:

$$v = \sqrt{-\frac{5g_4\Lambda^3}{g_6} + \frac{\sqrt{5}\sqrt{-6g_6g_2 + 5g_4^2\Lambda^6}}{g_6}}, \qquad u = \sqrt{\frac{bv^2}{32a^2\Lambda} - \frac{\sqrt{45b^2v^4\Lambda^2 - 16a^2g_6^4\Lambda^2}}{96\sqrt{5}a^2\Lambda^2}},$$
$$\text{and} \quad g_6v^4 + 10\Lambda(-6bu^2v^2 + \Lambda(48a^2u^4 + \Lambda(v^2g_4 - 12c^2u^2\Lambda))) = 0. \tag{6.6}$$

We clarify that this is not a fine-tuning condition, it is merely a prototypical example of

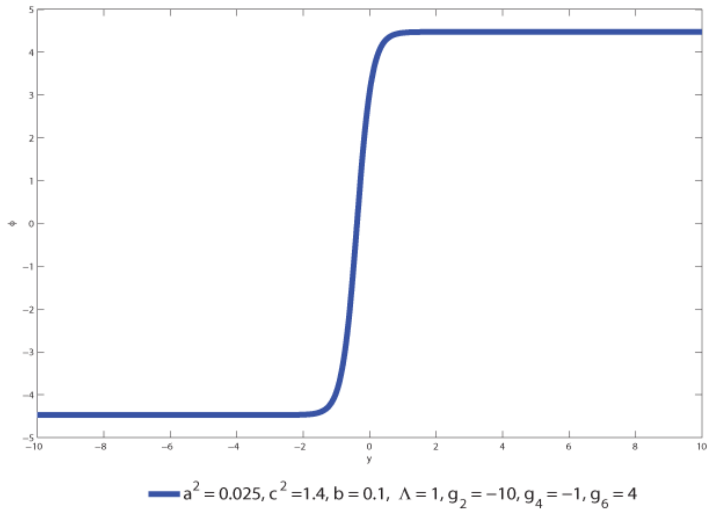


Figure 6.1: The graph displays a numerical domain-wall brane solution to the Lifshitz Klein-Gordon equation (6.4) for a choice of parameters which do not belong to the slice defined in (6.6).

the kink solutions that exist for a large region of parameter space. The numerical solution depicted in Figure 6.1 corresponds to a point which is not on this slice. Because we want the solutions described by (6.5) and (6.6) to be real valued and because v must be a minimum of the potential for (6.5) to be a solitary wave, we can bound the allowed region of parameter space by the inequalities $\Lambda, g_6, -g_2, -g_4 > 0$. Indeed none of the results presented in this chapter are contingent on providentially choosing parameters which satisfy (6.6).

This establishes the existence of a domain-wall brane for $R\times R^4$ space-time with anisotropic scaling characterized by critical exponent $z = 2$.

In the next chapter we will discuss the stability of the Lifshitz domain-wall brane (6.5). In this case we are less interested in a tour de force analysis of the stability. Instead we are intrigued that Lifshitz scalar field theories can evade Derrick's theorem (see section 3.4.5), thereby permitting long lived stable 3 + 1-dimensional topological defects.

As part of examining the stability of our Lifshitz domain-wall brane model, in chapter 7 we will prove Derrick's theorem does not apply to Lifshitz scalar field theories. We do this as preparation for chapter 8, where we focus on topological defects in $3 + 1_2$-dimensional Lifshitz scalar field theories.

6.3 Fermions

In section 3.5 we explained why Lorentz-invariant domain-wall branes dynamically localize a massless chiral 3 + 1-dimensional fermion.

We need to look at how the situation changes in the $4 + 1_2$-dimensional case.

First we must clarify what we mean by a fermion in $4 + 1_2$-dimensional space-time. We consider $\Psi(x^M)$ to be a spinorial wave function on $\mathbb{R} \times \mathbb{R}^4$ which transforms under SO(4) spatial rotations $x^j \to x'^j = O^j_k x^k \approx (\delta^j_k - \epsilon^j_{ki}\theta^i)x^k$ according to

$$\Psi(x^M) \longrightarrow S(O)\Psi(x^M), \qquad S(O) = e^{\omega^{jk}\Sigma_{jk}}, \tag{6.7}$$

where for convenience we set $\omega^{jk} = \epsilon^{jki}\theta_i$ and $\Sigma_{jk} = -\frac{i}{4}[\Gamma_j, \Gamma_k]$. As we are talking about 4-dimensional spatial rotations here the $\Gamma^j \in \{\gamma^1, \gamma^2, \gamma^3, \gamma^5\}$.

In the IR limit we are assuming the theory exhibits an accidental Lorentz symmetry. This Lorentz group incorporates a 3 + 1-dimensional subgroup given by the subset of SO(3) spatial rotations augmented by boosts in the 3 + 1-dimensional coordinate space which are a symmetry of the low-energy Lagrangian. Chirality will be restored in this low-energy Lorentz-invariant theory provided we can localize a Kaluza-Klein zero-mode fermion and provided this zero-mode fermion is an eigenstate of the γ^5 operator which commutes with the 3 + 1-dimensional Lorentz subgroup.

We consider the most general action for the $4 + 1_2$-dimensional Dirac field $\Psi(x^M)$ invariant under 4-dimensional spatial rotations. We are guided in our construction by the following considerations. The $4 + 1_2$-dimensional Dirac action will putatively flow towards (3.58) in the low energy limit. Therefore the $4 + 1_2$-dimensional Dirac action must contain all the operators present in (3.58). To suppress perturbative fermion loop diagram contributions to transition amplitudes we must incorporate second order spatial derivatives. We choose to do this in such a way that the propagation of the free $4 + 1_2$- dimensional scalar boson and free $4 + 1_2$-dimensional fermion will be described by the same dispersion relation. We will absorb the coefficient in front of the temporal derivative into a rescaling of the field $\Psi(x^M)$.

The most general action consistent with these principles is[4]

$$S = \int d^5x \Big[i\overline{\Psi}\Gamma^0\partial_t\Psi + i\beta\overline{\Psi}\Gamma^i\partial_i\Psi - \alpha\overline{\Psi}\Delta\Psi - g\phi\overline{\Psi}(1+\delta_1\Gamma^0)\Psi$$
$$-h\phi^2\overline{\Psi}(1+\delta_2\Gamma^0)\Psi + k\partial_i\phi\overline{\Psi}\Gamma^i(1+\delta_3\Gamma^0)\Psi \Big]. \quad (6.8)$$

We choose to work with a cut down version of (6.8). Our principal motivation is that the calculations are simpler, however we are also keen to present the result in a manner whereby it is easy to directly compare each step with the isotropic 4 + 1-dimensional case. We will argue that the no-go result holds independently of which interaction terms we include. The action we analyze is

$$S_\Psi = \int d^5x \Big[i\overline{\Psi}\Gamma^0\partial_t\Psi + i\beta\overline{\Psi}\Gamma^i\partial_i\Psi - \alpha\overline{\Psi}\Delta\Psi - g\phi\overline{\Psi}\Psi \Big]. \quad (6.9)$$

Therefore our simple Dirac equation is

$$\Big[i\Gamma^0\partial_t + i\beta\Gamma^i\partial_i - \alpha\Delta - g\phi(y) \Big] \Psi(x^M) = 0, \quad (6.10)$$

and the associated Klein-Gordon equation is

$$0 = \Big[\partial_t^2 - \beta^2\Delta + (\alpha\Delta + g\phi(y))^2 - g\beta\gamma^5\left(\partial_y\phi(y)\right) \Big] \Psi(x^M). \quad (6.11)$$

Choosing the dispersion relations for the free $4 + 1_2$-dimensional fermion and scalar to be the same, means a solution to the free field $4+1_2$-dimensional Dirac equation also satisfies a Klein-Gordon equation which contains the same differential operators as the Klein-Gordon equation for the free scalar field. Comparing the above equation to (6.4) will confirm that the free field kinetic operators are the same.

We will argue that for our $z = 2$ model, the Kaluza-Klein 'zero mode' $f_{0L}(y)\psi_{0L}(x^\mu)$ is not a solution to the $4 + 1_2$-dimensional Dirac equation. Therefore it is not present in the Kaluza-

[4]We have implicitly included terms like $\phi\overline{\Psi}\Gamma^i\partial_i\Psi$ which arise from integration by parts of the interaction term $\partial_i\phi\overline{\Psi}\Gamma^i\Psi$.

Klein tower expression (4.7) and cannot provide a candidate $3 + 1_2$-dimensional left chiral fermion.

We follow the logic: If $f_{0L}(y)\psi_{0L}(x^\mu)$ is a solution to the $4 + 1_2$-dimensional Dirac equation then it is also a solution of the Klein-Gordon equation (6.11). Thus, if $f_{0L}(y)\psi_{0L}(x^\mu)$ is not a solution to (6.11) then it is not a solution to the $4 + 1_2$-dimensional Dirac equation. Thus it suffices to show $f_{0L}(y)\psi_{0L}(x^\mu)$ is not a solution to (6.11).

We do this because the operator in the Klein-Gordon equation (6.11) is diagonal when acting on a Kaluza-Klein tower (4.7) and for the left-chiral zero mode this operator can be simplified to an identity matrix acting on the spinor space multiplied by a differential operator. For the left chiral zero mode, (6.11) is

$$0 = \left[\partial_t^2 - \beta^2\Delta + (\alpha\Delta + g\phi(y))^2 + g\beta\left(\partial_y\phi(y)\right)\right] f_{0L}(y)\psi_{0L}(x^\mu). \tag{6.12}$$

It should become clear immediately that the kinetic operator in (6.12) contains terms like $2\alpha\partial_y^2\vec{\nabla}^2$ which act on both the $f_{0L}(y)$ component of this solution and on the $\psi_{0L}(x^\mu)$ component of the solution. This is different from the isotropic $4 + 1$-dimensional Klein-Gordon equation (3.60). It follows that using separation of variables in (6.12) will no longer work.

For any solution $f_{0L}(y)\psi_{0L}(x^\mu)$ to (6.12), we can use unbroken $3 + 1_2$-dimensional Poincaré invariance to expand $\psi_{0L}(x^\mu)$ in terms of plane waves

$$\psi_{0L}(x^\mu) = \sum\!\!\!\!\!\!\int d^4p_0\, e^{-i(\omega_0 t - \vec{p_0}\cdot\vec{x})}\psi_{0L}(p_0^\mu), \tag{6.13}$$

where the coefficient

$$\psi_{0L}(p_0^\mu) = \frac{1}{(2\pi)^4}\int d^4x\, e^{i(\omega_0 t - i\vec{p_0}\cdot\vec{x})}\psi_{0L}(x^\mu) \tag{6.14}$$

and the allowed values of p_0^μ are determined by substituting this expansion into (6.12). At this stage, before finding the allowed values of p_0^μ, we must be clear that p_0^μ cannot be a function of y.

First, if p_0^μ depends on y then (6.13) will imply that $\psi_{0L}(x^\mu)$ depends on y and our interpretation of the Kaluza-Klein tower (4.7), as a projection of the solution $\Psi(x^M)$ to the $4 + 1_2$-

dimensional Dirac equation down onto a space of $3 + 1_2$-dimensional spinors $\psi_{nL/R}(x^\mu)$ using a complete basis for $C(y)$, is incorrect. This will mean there is a logical inconsistency in our theory.

Second, there will be phenomenological problems for our domain-wall brane model because when the brane is created by a soliton it has finite width. For the analytic solution given in (6.5) this width is characterized by $1/u$. If p_0^μ depends on y then a particle which starts on the $y < 0$ side of the brane and propagates minutely in the y-direction to the $y > 0$ side will experience a sudden change in energy or momentum $\vec{p}_0$. This is particularly true if p_0^μ depends in any way on the derivative $\partial_y \phi(y)$ of the topological defect which is large around $y = 0$ because $\phi(y)$ is varying rapidly at the location of the wall. Shrinking the width of the wall will make the gradient of $\phi(y)$ larger around $y = 0$ and make any dependence of p_0^μ on precise y-coordinate location more pronounced. With this assumption we derive the dispersion relations:

$$\begin{aligned} 0 = \left[\omega_0^2 - \alpha^2 \left(\vec{p}_0^{\,2}\right)^2\right] f_{0L}(y) + \vec{p}_0^{\,2} \underbrace{\left[2\alpha^2\partial_y^2 + 2\alpha g\phi(y) - \beta^2\right]}_{o1} f_{0L}(y) \\ - \underbrace{\left[\left(\alpha\partial_y^2 + g\phi(y)\right)^2 - \beta^2\partial_y^2 + g\beta\left(\partial_y\phi(y)\right)\right]}_{o2} f_{0L}(y). \end{aligned} \tag{6.15}$$

For this dispersion relation to be y-independent $f_{0L}(y)$ will have to be an eigenfunction of both $o1$ and $o2$. Although we are free to choose any complete basis we like, there is no function $f_{0L}(y)$ which is a simultaneous eigenfunction of both the operators $o1$ and $o2$. Thus we have a contradiction. There is no 'zero-mode' solution to the $4 + 1_2$-dimensional Dirac equation and hence no candidate $3 + 1_2$-dimensional left-chiral fermion at any energy regime.[5]

We follow the source of this problem back to the higher spatial derivative appearing in the kinetic terms in the Klein-Gordon equation. These higher spatial derivative operators include terms like $2\alpha^2\partial_y^2\vec{\nabla}^2$ which acted on both $f_{0L}(y)$ and $\psi_{0L}(x^\mu)$ and, ultimately, caused the operator $o1$ to appear in (6.15). Since the problem is due to the kinetic operator in the

[5]It is also possible to establish that there is no 'zero-mode' solution to the $4+1_2$-dimensional Dirac equation by direct computation. This is most easily accomplished in the Weyl basis. When the Kaluza-Klein 'zero-mode' ansatz, $\Psi_{0L}(x^M) = f_{0L}(y)\psi_{0L}(x^\mu)$, is substituted into the Dirac equation (6.10) inconsistent equations are generated.

Klein-Gordon equation (6.11) it cannot be fixed by adding additional interaction terms to (6.10). These higher spatial derivative terms will be present in any $4+1_z$-dimensional SO(4) invariant model with $z \geq 2$. They are uniquely fixed by SO(4) spatial rotational invariance, so the culprit in this theory is spatial dimensional democracy.

6.4 Breaking spatial isotropy

If we intend to project out effective $3 + 1_2$-dimensional left-chiral fermions via Kaluza-Klein decomposition then we cannot treat all four spatial dimensions symmetrically. This immediately prompts us to consider the alternative compact extra-dimensions paradigm.

Consider a model where space-time is a direct-product of a 3 + 1-dimensional manifold, $\mathcal{M}^4$, and a compact extra-dimension with an orbifolding symmetry (the extra dimension is identified into cosets S^1/Z_2). Equip the space $\mathcal{M}^4 \times S_1/Z_2$ with a preferred foliation into constant time sheets and impose anisotropic scaling characterized by critical exponent $z = 2$.

As before the standard tactic for recovering 3 + 1-dimensional chiral fermions is to Kaluza-Klein mode expand $\Psi(x^\mu, y)$ using a complete set $\{f_{nL/R}(y)\}$ of eigenfunctions of the y-dependent component of the Dirac equation. The crucial difference is that the boundary conditions are fixed by periodicity with respect to y of the wave-function and the transformation of the spinor under the orbifolding symmetry. In the isotropic space-time case these boundary conditions eliminate the right-handed zero mode profile function and 3 + 1-dimensional chirality discriminating physics is again an emergent low energy phenomenon. Unfortunately the conceptual differences between compact and infinite extra-dimensions change the boundary conditions for the Euler-Lagrange equations rather than the form of the kinetic operator. Because compact extra-dimension models still rely on the existence of a separable solution $f_{0L}(y)\psi_{0L}(x^\mu)$ to the Klein-Gordon equation (6.11) they will encounter similar problems.

The only option left is to strongly break 4 + 1-dimensional Lorentz invariance to SO(3) spatial rotational invariance. If we keep our anisotropic scaling with $z = 2$, we can write down a Dirac action with a separable free particle dispersion relation. The most general action consistent with this condition is,

$$S = \int d^5x \Big[\overline{\Psi} i\Gamma^0\partial_t\Psi + i\beta\overline{\Psi}\Gamma^i\partial_i\Psi + \alpha\overline{\Psi}\Gamma^0\vec{\nabla}^2\Psi + \lambda\overline{\Psi}\partial_y^2\Psi - g\phi\overline{\Psi}(1+\delta_1\Gamma^0+\epsilon_1\Gamma^5)\Psi \\ -h\phi^2\overline{\Psi}(1+\delta_2\Gamma^0+\epsilon_2\Gamma^5)\Psi + k\partial_i\phi\overline{\Psi}(1+\delta_3\Gamma^0+\epsilon_3\Gamma^5)\Gamma^i\Psi\Big]. \quad (6.16)$$

Again to make the analysis simpler and to set the problem up so that we can directly compare our results to previous cases given in this chapter, we look at an action with the simple Yukawa couplings [6]

$$S_\Psi = \int d^5x \Big[\overline{\Psi} i\Gamma^0\partial_t\Psi + i\beta\overline{\Psi}\Gamma^i\partial_i\Psi + \alpha\overline{\Psi}\Gamma^0\vec{\nabla}^2\Psi + \lambda\overline{\Psi}\partial_y^2\Psi - g\phi\overline{\Psi}\Psi\Big]. \quad (6.17)$$

The derived Dirac equation is:

$$\Big[i\Gamma^0\partial_t + i\beta\Gamma^i\partial_i + \alpha\Gamma^0\vec{\nabla}^2 + \lambda\partial_y^2 - g\phi(y)\Big]\Psi(x^M) = 0. \quad (6.18)$$

In constructing (6.17) we have deliberately multiplied the differential operator $\vec{\nabla}^2$ by a matrix which anti-commutes with Γ^5. This matrix cannot be Γ^1, Γ^2 or Γ^3 because we are demanding SO(3) rotational invariance, therefore it must be Γ^0. We need to introduce this matrix operator for two reasons. First, because it engineers a Klein-Gordon equation which has a separable solution of the form $\Psi_{0L}(x^M) = \psi_{0L}(x^\mu)f_{0L}(y)$. Explicitly the form of the Klein-Gordon equation is

$$\Big[-\left(i\partial_t + \alpha\vec{\nabla}^2\right)^2 - \beta^2\vec{\nabla}^2 + \left(\lambda\partial_y^2 - g\phi(y)\right)^2 - \beta^2\partial_y^2 - \beta\gamma^5\left(\partial_y g\phi(y)\right)\Big]\Psi(x^M) = 0. \quad (6.19)$$

There are no operators in (6.19) which operate on both the x^μ and y coordinate spaces simultaneously.

Second, to enable the effective $3 + 1_2$-dimensional theory to be chiral, we need the left- and right-chiral components of the zero mode to be independent solutions of (6.18). We

[6] Also we are mindful that the Klein-Gordon equation is derived from the action (6.16) where all interaction terms contains operators like $2g\delta_1\phi(y)\vec{\nabla}^2$. As a consequence of these cross terms which mix y-dependent quantities with 3-dimensional spatial derivatives, the vector $\vec{p}_0$ describing the zero-mode particle's momentum in the 3-dimensional spatial subspace will depend on the particle's location in the extra dimension. This will create a logical inconsistency in our Kaluza-Klein analysis entirely analogously to the spatially isotropic $4+1_2$-dimensional scenario.

can then arrange for the right-chiral profile function $f_{0R}(y)$ not to belong to $C(y)$ so that the right chiral 'zero mode' is excluded from the Kaluza-Klein tower. We collect the x^μ-coordinate space differential operators in (6.18) and interpret the $3 + 1_2$-dimensional zero-mode massless spinor to be the solution of

$$\left[i\gamma^0\partial_t + i\beta\left(\gamma^1\partial_1 + \gamma^2\partial_2 + \gamma^3\partial_3\right) + \alpha\gamma^0\vec{\nabla}^2\right]\psi_{0L/R}(x^\mu) = 0. \tag{6.20}$$

It is easy to check that $\psi_{0L}(x^\mu)$ and $\psi_{0R}(x^\mu)$ will be independent solutions to (6.20) only when the $\vec{\nabla}^2$ differential operator term in (6.20) is multiplied by γ^0.

After using (6.20) to simplify the algebra we find that $\Psi_{0L/R}(x^M) = f_{0L/R}(y)\psi_{0L/R}(x^\mu)$ will be a solution to (6.18) if

$$\begin{aligned} 0 &= \lambda\partial_y^2 f_{0L}(y) - \beta\partial_y f_{0L}(y) - g\phi(y)f_{0L}(y), \\ 0 &= \lambda\partial_y^2 f_{0R}(y) + \beta\partial_y f_{0R}(y) - g\phi(y)f_{0R}(y). \end{aligned} \tag{6.21}$$

If we make the substitution $f_{0L/R}(y) = e^{\pm\beta y/2\lambda}F_{0L/R}(y)$ then these equations can be converted into a time independent Schrödinger equation for $F_{0L/R}(y)$:

$$\left[-\partial_y^2 + \frac{g}{\lambda}\phi(y) + \frac{\beta^2}{4\lambda^2}\right]F_{0L/R}(y) = 0. \tag{6.22}$$

We observe that for the kink form of $\phi(y)$ given in (6.5) and parameters belonging to the open set $4gv\lambda > \beta^2$, the solution $F_{0L/R}(y)$ to (6.22) describes a freely propagating particle in the region $y < u\,\text{arctanh}(-\beta^2/4gv\lambda)$ which is incident on a potential barrier for $y > u\,\text{arctanh}(-\beta^2/4gv\lambda)$. Because all numerical solutions have the same kink like behaviour we can extend this analysis to cover all domain-wall branes. We treat the analytic case here because it has a closed form expression for the location of the potential barrier in (6.22). We can use the WKB method to find an approximate solution to (6.22) and write the zero-mode profile functions in terms of our analytic solution (6.5) for $\phi(y)$, as

$$f_{0L/R}(y) \approx \tag{6.23}$$

$$e^{\pm\beta y/2\lambda}\begin{cases} \sqrt[4]{\frac{4\lambda^2}{4\lambda g\phi(y)+\beta^2}}\,e^{-\int_{u\,\text{arctanh}(-\beta^2/4gv\lambda)}^{y} dy'\,\sqrt{g\phi(y')/\lambda+\beta^2/4\lambda^2}} & : y > u\,\text{arctanh}(-\beta^2/4gv\lambda) \\ \sqrt[4]{\frac{4\lambda^2}{-\beta^2-4\lambda g\phi(y)}}\,e^{i\int_{u\,\text{arctanh}(-\beta^2/4gv\lambda)}^{y} dy'\,\sqrt{-\beta^2/4\lambda^2-g\phi(y')/\lambda}} & : y < u\,\text{arctanh}(-\beta^2/4gv\lambda). \end{cases}$$

This approximation breaks down at the cusp of (6.23), $y = u\,\mathrm{arctanh}(-\beta^2/4gv\lambda)$. Provided $\beta > 0$, the profile function $f_{0R}(y)$ grows exponentially as $y \to -\infty$ and the right handed 'zero mode' will not belong to our Kaluza-Klein tower. In addition $f_{0L}(y)$ satisfies $\lim_{y\to\pm\infty} f_{0L}(y) = 0$. This follows from substituting the asymptotic form $\lim_{y\to\pm\infty}\phi(y) \to \pm gv$ for the analytic kink $\phi(y)$ given in (6.5) into the expression for $f_{0L}(y)$ in (6.23) and taking the appropriate limits. The left chiral profile function $f_{0L}(y)$ decays exponentially in both directions and therefore given $\epsilon > 0$, $\exists y(\epsilon)$ such that $|f_{0l}(y)| < \epsilon$, $\forall |y| \geq y(\epsilon)$ and on the compact region $|y| < y(\epsilon)$ we can argue that $f_{0L}(y)$ is continuous and hence attains a finite maximum. Thus $f_{0L}(y)$ is bounded.

Now we can solve (6.20) to obtain the form of the $3 + 1_2$-dimensional spinor $\psi_{0L}(x^\mu)$. We choose to write the full result for the left-chiral zero mode in terms of the solution for $f_{0L}(y)$ given in (6.23) as:

$$\Psi_{0L}(x^M) = N f_{0L}(y)\begin{pmatrix}\chi_+\\0\end{pmatrix} e^{-i(\omega_0 t - \vec{p}_0\cdot\vec{x})}, \tag{6.24}$$

where N is a normalization constant, the two component column vector χ_+ is the eigenstate of the helicity operator[7] corresponding to $\vec{\sigma}\cdot\vec{p}_0\chi_+ = |\vec{p}_0|\chi_+$, the lower two components of the four component spinor are zero and the energy and momentum of the plane wave $p_0^\mu = (\omega_0, \vec{p}_0)$ is fixed by the dispersion relation,

$$\left(\omega_0 - \alpha\vec{p}_0^{\,2}\right)^2 - \beta^2\vec{p}_0^{\,2} = 0, \tag{6.25}$$

which is derived by substituting this solution into the Klein-Gordon equation (6.19).

This energy-momentum relation for the $3 + 1_2$-dimensional subspace no longer depends on y.

However, if we derive the free scalar field Klein-Gordon equation directly from a $z = 2$, SO(3) invariant action for $\phi(x^M)$, then we will arrive at

$$\left(\partial_t^2 - \frac{a_1^2}{\Lambda^2}\left(\vec{\nabla}^2\right)^2 - \frac{a_2^2}{\Lambda^2}\partial_y^4 - \frac{a_3^2}{\Lambda^2}\partial_y^2\vec{\nabla}^2 - c_1^2\vec{\nabla}^2 - c_2^2\partial_y^2\right)\phi(x^M) = 0. \tag{6.26}$$

[7] The helicity operator gives the component of the spin of $\psi_{0L}(x^\mu)$ in the direction of propagation in the 3-dimensional $\vec{x}$ coordinate space. It is defined in terms of a vector of Pauli matrices $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and the particle's 3-momentum, $\vec{p}_0$, as $\vec{\sigma}\cdot\vec{p}_0/|\vec{p}_0|$.

The domain-wall brane (6.5) will still be a solution to (6.26). In fact in the spatially anisotropic scenario where Lorentz invariance is broken directly to SO(3) spatial rotational invariance, and the real scalar field Klein-Gordon equation (involving self interaction terms) has the form,

$$\partial_t^2\phi - \frac{a_1^2}{\Lambda^2}\left(\vec{\nabla}^2\right)^2\phi - \frac{a_2^2}{\Lambda^2}\partial_y^4\phi - \frac{a_3^2}{\Lambda^2}\partial_y^2\vec{\nabla}^2\phi - c_1^2\vec{\nabla}^2\phi - c_2^2\partial_y^2\phi - \frac{b_1}{2\Lambda^3}\phi^2\vec{\nabla}^2\phi$$
$$-\frac{b_1}{2\Lambda^3}\phi\vec{\nabla}\phi\cdot\vec{\nabla}\phi - \frac{b_2}{2\Lambda^3}\phi^2\partial_y^2\phi - \frac{b_2}{2\Lambda^3}\phi\partial_y\phi\partial_y\phi - \frac{g_6}{5!\Lambda^4}\phi^5 - \frac{g_4}{3!\Lambda}\phi^3 - g_2\phi = 0, \quad (6.27)$$

our solitary wave domain-wall brane (6.5) will be a solution, provided the free parameters satisfy (6.6) with the identifications $a \equiv a_2$, $c \equiv c_2$, and $b \equiv b_2$.

From a comparison of (6.26) and (6.19) we infer that the fermions and bosons will now propagate according to different dispersion relations. It is easy to check that if we insist on writing down a Dirac equation for which the solutions $\Psi(x^M)$ are also solutions to (6.26) for the case $a_3 = 0$ (this will circumvent our previous problem of mixed partial derivative operators) then we will have to use a 7-dimensional Clifford algebra. This forces the smallest representation for our spinor $\Psi(x^M)$ to consist of 8-component column matrices and, ultimately, once this Dirac equation has been solved we find that the 'zero mode' has too many degrees of freedom to be given the interpretation of a massless chiral fermion.

6.5 Conclusions

In this chapter we have examined a $4 + 1_z$-dimensional Lifshitz scalar field theory, with critical exponent $z = 2$.

We were motivated by the observation that $4 + 1_2$-dimensional Lifshitz Yang-Mills gauge theories can be power counting renormalizable, see section 3.8.3. We have laid the foundations for ultraviolet complete domain-wall brane models and for further investigation into trapping gauge bosons on a $4 + 1_2$-dimensional Lifshitz brane through an analogue of the Dvali-Shifman mechanism.

We recollect from section 3.6 that the Dvali-Shifman mechanism deals with trapping a U(1) gauge boson on a 2 + 1-dimensional domain-wall brane. The mechanism is operative for a 3 + 1-dimensional Lorentz invariant SU(2) Yang-Mills gauge theory. The U(1) gauge field becomes trapped as a result of non-perturbative dynamics in the bulk, when the bulk is in a confinement phase of SU(2) with a mass gap. Lattice gauge simulations support the hypothesis that the Dvali-Shifman mechanism can trap a U(1) gauge boson on a domain-wall brane [LMRS04].

Lattice gauge stimulations for a Lorentz invariant 4 + 1-dimensional analogue of the Dvali-Shifman mechanism are fundamentally limited by the fact that Lorentz invariant 4 + 1-dimensional gauge theories are non-renormalizable.

We recognize Lifshitz anisotropic scaling can act as a regulator for $4 + 1_2$-dimensional Yang-Mills gauge theories. We were interested in this model primarily because it provides an ultraviolet complete $4 + 1_2$-dimensional Yang-Mills gauge theory which could be used in conjunction with localization mechanisms in chapters 4 and 6 to provide a feasible, complete brane-world model. In this scenario lattice gauge simulations can be used to test gauge field localization in the limit of infinitesimal lattice spacing. However a proper treatment of gauge field localization with the accompanying simulations is beyond the scope of this chapter.

We have demonstrated that a topological defect in the scalar field can spontaneously condense to form a domain-wall brane.

We considered the dynamics of a Lifshitz fermion in this background and showed that a $3 + 1_2$-dimensional left chiral Kaluza-Klein zero-mode fermion will become trapped on the wall only when four fold spatial rotational invariance is strongly broken to three-fold rotational invariance. So it is not possible to keep 4-dimensional spatial isotropy and localize a Kaluza-Klein zero-mode candidate chiral fermions in $4+1_2$-dimensional Lifshitz domain-wall brane models. While we worked with $z = 2$ theory for simplicity, we do not expect the results to be qualitatively different for the more realistic case of $z = 4$, necessary to have a renormalizable quantum theory of gravity in 4 + 1-dimensions.

Many additional challenges remain before a realistic UV-complete domain-wall brane model could be contemplated. These include the dynamical localization of gravity in some type of Hořava-Lifshitz theory, and the incorporation of dynamically-localized gauge fields through the Dvali-Shifman mechanism, and supporting lattice stimulations.

7

Stability of the Lifshitz domain-wall brane

7.1 Introduction: the question of stability

The Lifshitz domain wall is topologically stable for the same reason the usual domain wall is stable: the enforced discrete symmetry, when spontaneously broken, produces disconnected vacua which serve as the boundary conditions for the domain-wall solution. The kink is a mapping from the boundary of the real line, parameterizing the extra dimension, onto the disconnected manifold $\{-v, v\}$. This mapping does not belong to the same topological class as the spatially homogeneous vacua $\phi = \pm v$. Thus, the domain wall is prevented from decaying to the lower-energy spatially-homogeneous vacua $\phi(x^M) = \pm v$.

We have not analytically shown that our kink domain-wall brane is the lowest energy solution to the Euler-Lagrange equations satisfying these boundary conditions.

In the following discussion we guarantee the kink solution has finite $3+1$-dimensional energy density (energy per unit of 3-dimensional volume). Furthermore we show that the kink is stable under perturbations corresponding to a contraction or dilation of the transverse wall width. This suggests the kink is stable because perturbations which alter the profile in any of the $\vec{x}$-coordinate directions are forbidden by SO(3)-rotational invariance. Our background kink solution is invariant under 3-dimensional spatial rotations and there is no preferred direction in the hyperplane orthogonal to the y-coordinate. Thus it does not make sense to say a perturbation has formed along a specific direction in the $\vec{x}$-coordinate space because all directions are relative to an arbitrary choice of reference axis [PV00].

We are left with conjecturing about perturbations which do not deform the kink in any direction perpendicular to the bulk coordinate axis and do not correspond to rescaling the width of the domain wall. This means that we can only have perturbations which locally deform the kink according to some nonlinear dependence on y. These are allowed because translational invariance is spontaneously broken by the condensation of the kink.

We do not attempt a full perturbative stability analysis along the lines of chapter 5. We therefore cannot address the most general case of a perturbation which depends nonlinearly on the extra-dimensional coordinate y.

Instead, we concentrate on showing that Lifshitz topological defects in $4 + 1_2$-dimensions and $3 + 1_2$-dimensions evade Derrick's theorem.

Our aim is to highlight, in situ, information needed for the general discussion of stable Lifshitz topological defects presented in chapter 8.

Our analytic kink (6.5) has finite 3 + 1-dimensional energy density given by

$$\sigma = \frac{v^2(-23g_6v^4 + 120\Lambda^2(24a^2u^4 + \Lambda(v^2(3bu^2 - 5g_4) - 45g_2\Lambda + 30c^2u^2\Lambda)))}{5400u\Lambda^4} < \infty. \quad (7.1)$$

Naturally if we integrate this quantity with respect to the 3-dimensional volume element to obtain the total energy the answer will be infinite. This is the statement that domain-wall branes are technically only solitons in 1+1-dimensions.

7.2 Derrick's theorem for $4+1_2$-dimensional Lifshitz scalar field

In fact, in isotropic 4 + 1-dimensional theories Derrick's theorem [Der64] implies that all static non-homogeneous[1] solutions to the Klein-Gordon equation have infinite total energy, independent of the precise set of self interaction terms in the action.

In Lifshitz theories the higher spatial derivatives change the scaling properties of the energy functional. Effectively the higher spatial derivatives stabilize these perturbations and Derrick's theorem can be evaded. This means we cannot rule out finite total energy solutions to

[1] We note that the qualification that flat vacuum solutions $\phi = \pm v$ which have trivial x^i-dependence evade Derrick's theorem.

the Klein-Gordon equation in Lifshitz theories. However a finite total energy solution cannot be homogeneous on 3 + 1-dimensional hyperplanes. Thus, if we wish to avoid breaking 3-dimensional spatial rotational invariance, then it will have to be radially symmetric and will therefore correspond to a point-like topological defect. This configuration cannot provide a 3 + 1-dimensional brane. We find the prospect fascinating for other reasons but have delayed a full investigation to chapter 8.

The higher spatial derivatives in Lifshitz theories also modify the virial theorem. For a 1+1-dimensional, solitary wave solution $\phi_0(y)$ to the Klein-Gordon equation (6.4), there is a (virial) relation between the contribution to the energy density from the gradient of the field, and from the potential energy density. This can be derived from the total 3 + 1-dimensional energy density. If we assume that all boundary terms which arise from integration by parts are zero, then the 3 + 1-dimensional energy density is given by

$$\sigma[\phi_0(y)] = \underbrace{\int dy\, \frac{a^2}{2\Lambda^2}(\partial_y^2\phi_0)^2}_{\tilde{S}_1} + \underbrace{\int dy\, \tilde{K}_0\partial_y^2\phi_0}_{\tilde{S}_2} + \underbrace{\int dy\, \tilde{V}_0}_{\tilde{S}_3} = \tilde{S}_1 + \tilde{S}_2 + \tilde{S}_3, \tag{7.2}$$

where

$$\begin{aligned} \tilde{K}_0 &= \frac{b}{12\Lambda^3}\phi_0^3 + \frac{c^2}{4}\phi_0 \\ \tilde{V}_0 &= \left[\frac{g_2}{2}\phi_0^2 + \frac{g_4}{4!\Lambda}\phi_0^4 + \frac{g_6}{6!\Lambda^4}\phi_0^6\right]. \end{aligned} \tag{7.3}$$

The virial relation is derived by rescaling the solution $\phi_0(y) \to \phi_0(ky)$ and computing $\frac{\partial\,\sigma[\phi_0(ky)]}{\partial k}\Big|_{k=1} = 0$. This result is a direct consequence of requiring that any static solitary solution, $\phi_0(y)$, extremizes the energy density. Here the virial relation is

$$3\tilde{S}_1 + \tilde{S}_2 = \tilde{S}_3. \tag{7.4}$$

Using this information in the second derivative $\frac{\partial^2\,\sigma[\phi_0(ky)]}{\partial k^2}\Big|_{k=1}$ we find that

$$\left.\frac{\partial^2\,\sigma[\phi_0(ky)]}{\partial k^2}\right|_{k=1} = 18\tilde{S}_1 + 2\tilde{S}_2. \tag{7.5}$$

Provided $b > 0$ the integrands of $\tilde{S}_1, \tilde{S}_2$ can be written as the sum of perfect squares[2], so that the second derivative must be strictly positive. As a corollary a 1+1-dimensional, solitary wave solution $\phi_0(y)$ to the Klein-Gordon equation (6.4) is stable under perturbations corresponding to a contraction or dilation of the transverse wall width.

7.3 Derrick's theorem for a $3 + 1_2$-dimensional real scalar field

We can repeat this exercise for a $3 + 1_2$-dimensional Lifshitz scalar field theory. In 3 + 1-dimensions with dynamical critical exponent $z = 2$ a weighted power-counting renormalizable action includes the marginal and relevant operators

$$\dot{\phi}^2 \tag{7.6}$$

$$(\vec{\nabla}^2\phi)^2 \tag{7.7}$$

$$\phi^n\nabla^2\phi, \quad \text{with} \quad 0 \leq n \leq 5 \tag{7.8}$$

$$\phi^m, \quad \text{with} \quad 0 \leq m \leq 10. \tag{7.9}$$

The relevant subleading operators explicitly, but softly, break the anisotropic scale invariance. We include them because they do not destroy the "good renormalization" properties of the Lifshitz scalar field theory.

Assuming we can discard all total derivative terms, the most general action may be compactly written as

$$S = \int d^3x\, dt \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla^2\phi)^2 - K\nabla^2\phi - V\right], \tag{7.10}$$

where $K\nabla^2\phi$ is taken to be a general linear combination of the terms (7.8), and the potential $V(\phi)$ is a general linear combination of (7.9). Here we choose to keep K and V completely general. We can recover the relativistic speed of light parameter, c, as the coefficient of the $n = 1$ term inside K.

If we apply the principle of stationary action, then we arrive at the $3 + 1_2$-dimensional

[2]This is most easily seen by undoing the integration by parts to get $\tilde{S}_1 + \tilde{S}_2 = \int dy\, \frac{a^2}{2\Lambda^2}(\partial_y^2\phi_0)^2 + \int dy\, \frac{c^2}{2}(\partial_y\phi_0)^2 + \frac{b}{4\Lambda^3}(\phi_0\partial_y\phi_0)^2$.

dynamical equations

$$\ddot{\phi} + \nabla^2(\nabla^2\phi) - K_{\phi\phi}\nabla\phi \cdot \nabla\phi - 2K_\phi\nabla^2\phi + V_\phi = 0. \tag{7.11}$$

For certain cases, like $K(\phi) = \frac{2}{R^2 v^2}(4\phi - 3v)(\phi - v)^2$ and $V = K^2/2$, this equation possesses a static solution ϕ_0. We will demonstrate a numerical solution to (7.11), under these circumstances, in section 8.2.1.

The energy functional for this static solitary wave is

$$E[\phi] = \int d^3x \left[\frac{1}{2}(\nabla^2\phi_0)^2 + K_0\nabla^2\phi_0 + V_0\right] = S_1 + S_2 + S_3. \tag{7.12}$$

Applying the variation $\phi_0(\vec{x}) \to \phi_0(k\vec{x}) = \phi(\xi)$ to (7.12) we obtain

$$E_k[\phi] = kS_1 + k^{-1}S_2 + k^{-3}S_3, \tag{7.13}$$

where $S_1 \equiv \int d^3x\,dt\,(\nabla^2\phi_0)^2/2$ and $S_2 \equiv \int d^3x\,dt\,K_0\nabla^2\phi_0$. This implies the virial relation

$$S_1 - S_2 - 3S_3 = 0. \tag{7.14}$$

The second-order variation is now equal to $2S_2 + 12S_3 = 4S_1 - 2S_2$ by the virial relation. Because this quantity can be either positive or negative (depending on the precise forms of both K and V) Derrick's theorem does not apply to the $z = 2$ Lifshitz case. A good place to start looking for stable Lifshitz defects is $K = \phi^5$, which sees S_2 turn into an integral of the negative-definite function $-\phi^4\nabla\phi \cdot \nabla\phi$ after integration by parts. Alternatively the simple case $K = 0$ has a positive second-order variation because $S_2 = 0$. The higher-derivative terms tend to stabilise scalar-field solitons. This result is also encountered in the (relativistic) Skyrme model [Sky62].

7.4 Conclusions

In this chapter we developed a partial formalism for examining the stability of a Lifshitz domain-wall brane in $4 + 1_2$-dimensions. We proved that the $4 + 1_2$-dimensional domain-wall brane is stable against perturbations which break 3-dimensional spatial rotational invariance, and against contractions which shrink the transverse wall width. However we did not rule out perturbations which are a general nonlinear function of the fifth coordinate $y \equiv x_5$; this is beyond the scope of our current analysis.

Derrick's theorem shows relativistic field theories in $d > 2$ spatial dimensions do not support finite energy, stable, static defects. During our stability analysis we discovered that $3 + 1$ and $4 + 1$-dimensional Lifshitz field theories with critical exponent $z = 2$, are exempt from Derrick's theorem. This opens up the intriguing possibility that the breakdown of Lorentz invariance in the ultraviolet may leave hallmark signatures like cosmic relics which are readily observable at any energy scale.

In the next chapter we present an in depth perspective on this issue.

8 BPS solitons in Lifshitz field theories

8.1 Introduction

The concept of stable static 3+1-dimensional Lifshitz topological defects is intriguing. If Hořava's Lifshitz anisotropic scaling is the natural regulator for quantum gravity, then it is reasonable to assume quantum field theories may also exhibit anisotropic scaling in the deep ultraviolet (UV) regime.

In chapter 7 we showed that Lifshitz scalar field theories in $3 + 1$-dimensions with critical exponent $z = 2$ evade Derrick's theorem and therefore are not fundamentally prohibited from having stable, static, finite-energy topological defects. However, evasion of Derrick's theorem does not prove that solitons actually exist. We argued in chapter 7 that applying a nonlinear perturbation (outside of the field variations considered in Derrick's theorem) may still cause the candidate soliton to spontaneously relax into a lower energy configuration. Therefore we turn to explicit computations.

In this chapter, we explicitly show examples of stable, finite energy, static Lifshitz solitons. We specifically choose to work with a $3 + 1_2$-dimensional Lifshitz scalar field theory. Here by $3 + 1_2$-dimensional field theories we mean a $3 + 1$-dimensional Lifshitz field theory with critical exponent $z = 2$.

We study $z = 2$ models because they lead to the simplest differential equations. It would be more realistic to work with critical exponent $z = 3$, which is featured in Hořava's quantum gravity model. However the former choice $z = 2$ introduces fourth-order spatial derivatives, while the latter choice $z = 3$ leads to sixth order differential equations. It is sufficient to demonstrate that Derrick's theorem is violated in $z = 2$ scenarios, and that $3 + 1_2$-dimensional field theories support finite energy, stable, static topological defects. This result will presumably generalize to an arbitrary $z > 1$ scenario.

In general, Lifshitz theories are much more difficult to analyze than the usual second-order relativistic theories. However we can simplify the calculations by using the field theory analogue of Hořava's "detailed balance" condition [Hor10]. This is the Lifshitz version of the Bogomolnyi-Prasad-Sommerfield (BPS) [Bog76, Pra75] superpotential scenario. We choose to work with Lifshitz BPS (BPS) models because it allows us to reduce the dynamical equations back to second-order. This approach will greatly simplify the analysis in this chapter. The paper [KV11] further motivates the Lifshitz BPS approach by highlighting the connection between BPS systems and stochastic quantisation. We also note it creates parallels between our work and supersymmetry.

Here we warm up with the canonical example of a non-topological Lifshitz point-like defect. We then derive both "hedgehog" defects and string-like solitons, which we can call "Lifshitz poles (L-poles)" and "Lifshitz strings (L-strings)", respectively.

Derrick's theorem [Der64] tells us that there is no analogue for these solitons in standard second-order Lorentz-invariant $3 + 1$-dimensional field theories.

As such the discovery of cosmic relics such as L-poles and/or L-strings would be a strong fundamental argument for Lifshitz anisotropic scaling replacing standard Lorentz invariance at short distances.

There are many potential observational signatures for Hořava Lifshitz quantum point gravity in the UV regime, for instance delays in the arrival time for very high energy cosmic ray events. However no strong consensus has been reached as to whether Lifshitz scaling or Lorentz invariance accurately describes the UV regime.

Observational evidence of cosmological relics which are inexplicable by standard Lorentz invariant field theories would be a strong signature of new physics.

In the next section we derive the Lifshitz BPS equation. Subsequently we use special cases of this equation to find Lifshitz BPS topological defects. In section 8.3 we clearly show the BPS condition is a convenience not a necessary condition for Lifshitz solitons. We do this by numerically solving a general non-BPS fourth-order system. Our finite-energy solution to this system constitutes an existence proof. The final section provides a conclusion.

8.2 The Lifshitz BPS soliton solutions

We need to develop the Lifshitz BPS description. We devote this section to writing down an explicit Lifshitz BPS model. This will allow us to clarify our terminology.

Box 8.1: *We first examine standard relativistic* $3+1$*-dimensional BPS scenarios with action*

$$S[\phi] = \int dx\,dt \left[\frac{1}{2}\left(\partial_\mu \phi\right)^2 + V(\phi)\right], \tag{8.1}$$

In the special case where the potential, $V(\phi)$*, is a function of the superpotential* W

$$V(\phi) = \frac{1}{2}\left(\frac{dW(\phi)}{d\phi}\right)^2. \tag{8.2}$$

Here the integrand of the energy density,

$$E[\phi] = \int dx \left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + V(\phi)\right], \tag{8.3}$$

can be factorized according to

$$E[\phi] = \mp\left[W\left(\phi(x=+\infty)\right) - W\left(\phi(x=-\infty)\right)\right] + \int_{-\infty}^{+\infty} dx \left(\frac{d\phi}{dx} \pm \frac{dW}{d\phi}\right)^2. \tag{8.4}$$

Because the integrand of (8.4) is positive definite, for given boundary conditions it is easy to see that the minimum energy static solution to the Euler-Lagrange equations, arising from applying Hamilton's principle to the action (8.1) satisfies

$$\frac{d\phi}{dx} \pm \frac{dW}{d\phi} = 0. \tag{8.5}$$

These solutions are stable BPS solitons.

Lifshitz field theories can emulate the above whenever we impose the condition,

$$V(\phi) = \frac{1}{2}K(\phi)^2, \tag{8.6}$$

in the Lifshitz $3 + 1_2$-dimensional action given in (7.10).

Under these circumstances the fundamental theorem of calculus tells us that the superpotential W is the indefinite integral of K with respect to ϕ. It is conventional to negate the above relation so that $K = -W_\phi$.

The energy functional is a linear combination of perfect squares, with positive coefficients,

$$E[\phi] = \int d^3x \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\vec{\nabla}^2\phi + K(\phi)\right)^2\right]. \tag{8.7}$$

Therefore static solutions, ϕ_0, to the dynamical equations (7.11) which minimize the energy-*density* satisfy

$$\dot{\phi}_0 = 0, \qquad \vec{\nabla}^2\phi_0 + K_0 = 0. \tag{8.8}$$

In fact solutions ϕ_0 of (8.8) have zero energy-*density*.

We briefly characterize the vacuum of the Lifshitz BPS system. Using the condition (8.6) in (7.11) we find that

$$\ddot{\phi} + \nabla^2(\nabla^2\phi) - K_{\phi\phi}\nabla\phi\cdot\nabla\phi - 2K_\phi\nabla^2\phi + K_\phi K = 0. \tag{8.9}$$

Clearly the static homogeneous solutions of (8.9) are $\phi_0 = v$ such that $K_\phi K|_{\phi_0=v}= 0$. These solutions minimize the potential. This requires either $K_\phi|_{\phi_0=v}= 0$ and/or $K|_{\phi_0=v}= 0$.

Now if $K|_{\phi_0=v}\neq 0$ then we cannot have $\vec{\nabla}^2\phi_0 + K|_{\phi_0}= 0$. However the homogeneous vacuum always minimizes the energy-*density* and therefore must solve (8.8). Therefore the vacuum $\phi_0 = v$ occurs if and only if $K|_{\phi_0=v} = 0$. Lifshitz BPS solutions for the allowed models are degenerate with the vacua, and hence at least perturbatively stable.

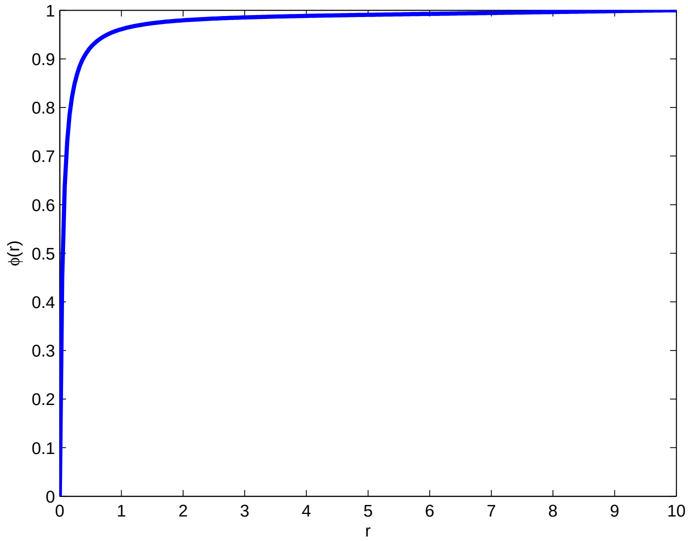


Figure 8.1: Numerical solution for the non-topological BPS L-pole corresponding to the choice of K given in (8.13) with $\lambda = v = 1$.

8.2.1 Non-topological BPS L-pole

Let us examine some specific examples of Lifshitz BPS solitons. We begin with an Lifshitz BPS point-like defect. This is the easiest possible example: a non-topological defect which introduces a single radially symmetric field $\phi = \phi_0(r)$.

In spherical polar coordinates the dynamical equation for the Lifshitz BPS non-topological point-like soliton arise from the specific case of (8.8), where:

$$\left(\frac{2}{r}\frac{d}{dr} + \frac{d^2}{dr^2}\right)\phi_0(r) = -K\left(\phi_0(r)\right). \tag{8.10}$$

We are looking for a solution ϕ_0 which asymptotically approaches a finite value v as $r \to \infty$.

For the special choice of free parameters in the Lifshitz potential (8.6) corresponding to

$$K(\phi) = \frac{2}{R^2 v^2}\left(4\phi - 3v\right)\left(\phi - v\right)^2, \tag{8.11}$$

equation (8.10) has an analytic solution

$$\phi_0(r) = \frac{v r^2}{R^2 + r^2}, \tag{8.12}$$

where R is a parameter that sets the characteristic size of the defect. Notice we have allowed the Lifshitz potential (8.11) to be fine tuned, thereby arranging for an analytic solution (8.12). This is natural because for any differential equation an analytic solution only exists on a thin slice of the parameter space. The dimensions of this slice are set by the number of unfixed parameters in the analytic solution. In (8.12) the free parameters are R and v, making the slice 2-dimensional.

In general, other forms of the potential (8.6) also can lead to soliton solutions to the Lifshitz BPS equation (8.10). However more general scenarios will necessitate numerical solutions.

In Figure 8.1 we give an example of a non-topological BPS L-pole found by numerically solving (8.10) when

$$K(\phi) = \lambda\phi(\phi^2 - v^2)^2. \tag{8.13}$$

For this choice of K we are not aware of any analytic solution.

8.2.2 Topological BPS L-pole or hedgehog

Our second example is an Lifshitz BPS hedgehog. Recollect from subsection 3.4.4 that a hedgehog solution requires us to introduce a triplet of scalar fields $\vec{\phi} = (\phi_i),\ i = 1, 2, 3$. This triplet transforms as a vector under a 3-dimensional representation of O(3). We work with an O(3) invariant action such that the corresponding energy functional for the static configuration is

$$E[\vec{\phi}] = \int d^3x \left[\vec{\nabla}^2\phi_i + \phi_i F(\vec{\phi}\cdot\vec{\phi})\right]\left[\vec{\nabla}^2\phi^i + \phi^i F(\vec{\phi}\cdot\vec{\phi})\right], \tag{8.14}$$

where we have summed over contracted indices and

$$F = \lambda(\vec{\phi}\cdot\vec{\phi} - v^2)^2. \tag{8.15}$$

Notice that for the special choice, (8.15), of free the parameters for our potential $V = \vec{\phi}\cdot\vec{\phi}F^2$, the scalar fields $\vec{\phi}$ acquire a vacuum expectation value which breaks O(3) $\rightarrow$ O(2). Again we have a Lifshitz generalization of the conditions needed to recreate the hedgehog solution in subsection 3.4.4.

The hedgehog ansatz in spherical polar coordinates is

$$\vec{\phi}(r,\theta,\psi) = h(r)\,(\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta). \tag{8.16}$$

The Lifshitz BPS equations $\vec{\nabla}^2\phi_i + \phi_i F = 0$ reduce to

$$\frac{d^2h}{dr^2} + \frac{2}{r}\frac{dh}{dr} - 2\frac{h}{r} + \lambda h(h^2 - v^2)^2 = 0. \tag{8.17}$$

This is actually a straightforward generalization of Polyakov's original monopole [Pol74] to include fifth-order non-derivative terms. These appear in Lifshitz BPS hedgehog equations, because the Lifshitz weighted power counting conventions extend the normal range of power counting renormalization to 10th order interaction terms.

In Figure 8.2 we demonstrate a numerical solution which is regular at the origin and asymptotes to the vacuum expectation value $\vec{\phi}_{vac} = v\,(\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)$ breaking O(3) to O(2).

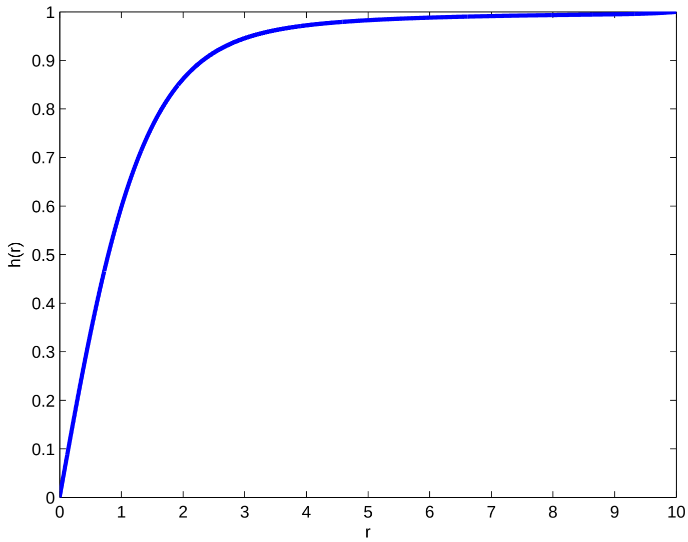


Figure 8.2: Numerical solution to (8.17) with $\lambda = v = 1$.

8.2.3 Topological BPS L-string

We also explicitly demonstrate the existence of a BPS L-string. These defects arise from a Lifshitz $3 + 1_2$-dimensional complex scalar field Φ, whose action and energy functional

$$E[\vec{\Phi}] = \int d^3x \left[\vec{\nabla}^2\Phi^* + \lambda\Phi^*(\Phi^*\Phi - v^2)^2\right]\left[\vec{\nabla}^2\Phi + \lambda\Phi(\Phi^*\Phi - v^2)^2\right], \tag{8.18}$$

are invariant under a global phase symmetry $\Phi \to e^{i\alpha}\Phi$.

Notice that the vacuum, which can be any state in the vacuum manifold $\{e^{i\alpha}v|\alpha \in (0, 2\pi)\}$, breaks the U(1) phase symmetry.

In cylindrical coordinates (ρ, θ, z), the topological global string ansatz is

$$\Phi(\rho, \theta) = e^{i\theta} f(\rho), \tag{8.19}$$

where the string defines the z-axis.

Substituting this ansatz into the Lifshitz BPS string dynamical equations $\vec{\nabla}^2\Phi + \lambda\Phi(\Phi^*\Phi - v^2)^2 = 0$ leads to

$$\frac{d^2 f}{d\rho^2} + \frac{1}{\rho}\frac{df}{d\rho} - \frac{f}{\rho^2} + \lambda f(f^2 - v^2)^2 = 0. \tag{8.20}$$

In Figure 8.3 we demonstrate a numerical string solution to (8.20) which asymptotes to the vacuum $\Phi_{vac} = e^{i\theta v}$ and is regular at the origin.

8.3 General Lifshitz systems

Up to this point we have considered only Lifshitz BPS solitons. We have chosen to investigate BPS systems because the differential equations are easier and solutions are degenerate in energy with the vacuum. This means a full stability analysis is not necessary. In particular to examine the stability of the non-topological Lifshitz BPS point-like defect we would need to develop a fourth-order linear stability formalism.

In this section we provide a numerical solution for a non-BPS Lifshitz hedgehog. We include this solution because it shows that the BPS condition was not essential for creating

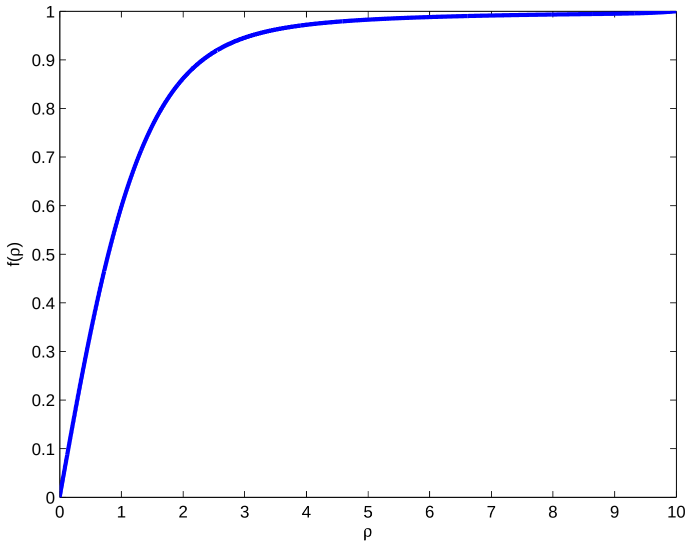


Figure 8.3: Numerical solution to (8.20) with $\lambda = \nu = 1$.

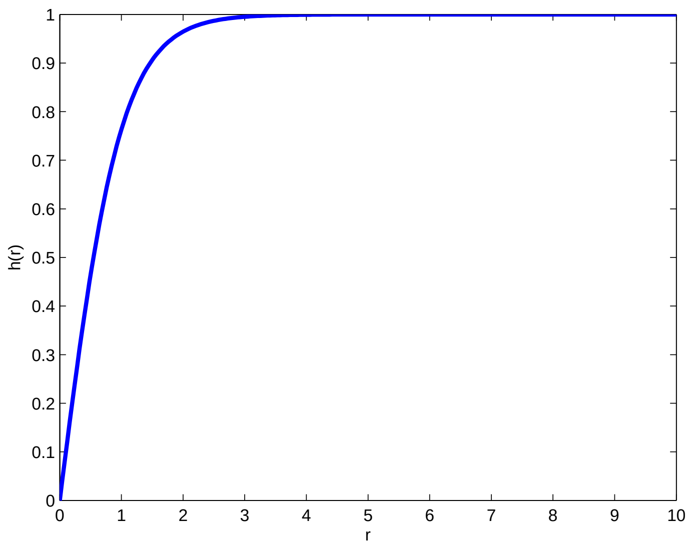


Figure 8.4: Numerical solution to (8.22) with $\lambda = \nu = 1$.

Lifshitz topological defects.

In particular we choose the explicitly non-BPS combination $K = 0$ and

$$V = \frac{\lambda}{10}\left(\vec{\phi}\cdot\vec{\phi} - v^2\right)^5 . \tag{8.21}$$

The dynamical equations, using the ansatz (8.16), are

$$\frac{d^4h}{dr^4} + \frac{4}{r}\frac{d^3h}{dr^3} - \frac{4}{r^2}\frac{d^2h}{dr^2} + \lambda h(h^2 - v^2)^4 = 0. \tag{8.22}$$

In Figure. 8.4 we give a numerical solution for $h(r)$ which is regular at the origin and asymptotes to v . This configuration is topologically stable.

8.4 Conclusions

In this chapter we have demonstrated finite-energy, static soliton solutions to 3+1-dimensional Lifshitz dynamical field equations with critical exponent $z = 2$. These solitons are not possible in standard $3 + 1$-dimensional relativistic field theories.

We worked primarily with the Bogomolnyi-Prasad-Sommerfield conditions which allowed us to simplify the 4th order $3 + 1_2$-dimensional Lifshitz dynamical equations. Solutions to the Lifshitz BPS equations have the dual advantages:

- A $3 + 1_2$-dimensional Lifshitz BPS solution can be found by solving second order differential equations.
- Lifshitz BPS solutions are degenerate in energy with the vacuum and therefore perturbatively stable.

Explicitly we showed instances of a Lifshitz BPS non-topological point-like defect as well as an Lifshitz BPS hedgehog and an Lifshitz BPS string defect. These solutions constitute an existence proof that finite energy (actually zero energy-density), stable, static Lifshitz defects exist in $3 + 1$-dimensions when the Lifshitz critical exponent $z = 2$.

Furthermore we demonstrated a numerical solution for a Lifshitz hedgehog soliton in a non-

BPS scenario. We clarify that the BPS condition was chosen for convenience rather than necessity. The condition was also chosen because there is a very nice connection with the bosonic sectors of supersymmetric theories that are themselves related to a stochastic field theory in one higher dimension. This connection is described in [KV11].

The observation of cosmic relics in the form of Lifshitz solitons would be a clear sign that the standard Lorentz invariant description of quantum field theory breaks down in the ultraviolet regime. Our motivation for studying Lifshitz topological defects was that, if Hořava's proposal of a Lifshitz quantum point gravity is correct, then it is natural to assume that quantum field theory may also exhibit Lifshitz anisotropic scaling in the ultraviolet regime. In this scenario these finite-energy defects would be a powerful diagnostic tool for detecting the ultraviolet behavour of both quantum gravity and field theory.

9

Symmetry breaking, subgroup embeddings and the Weyl group

9.1 Introduction

The final result we wish to present in this thesis is a description of the symmetry breaking patterns for the adjoint representation of a Lie group G. Specifically we describe how (and when) the vacuum expectation values (vevs) of a collection of symmetry breaking fields, which cause an internal symmetry, G, to break along a chain of subgroups $G \supset H_1 \supset \cdots \supset H_l$, can be written as a linear combination of an equivalent set of vevs which break G to an isomorphic but differently embedded subgroup chain $G \supset gH_1g^{-1} \supset \cdots \supset gH_lg^{-1}$, for some $g \in G$. For the special case of the subgroup, H, which stabilizes the highest weight of the lowest dimensional fundamental representation, we highlight a simple method for constructing linear combinations relating vevs which pick out differently embedded isomorphic copies of H.

More generally we consider an arbitrary representation. For an explicit choice of Cartan subalgebra $h^1, \ldots h^l$ we construct the subgroup chain $G \supset H_1 \supset \cdots \supset H_l$ such that each $H_i = H_i' \times \mathrm{U}(1)_{H_1} \times \cdots \times \mathrm{U}(1)_{H_i}$ includes $\mathrm{U}(1)_{H_1,\ldots,H_i}$ factors generated by $h^1, \ldots, h^i$, respectively, and a semi-simple factor H_i'. Under conjugations of the Lie algebra $\mathcal{L}$ by $g \in G$ we give results for how (and when) the Cartan subalgebra $gh^1g^{-1}, \ldots, gh^lg^{-1}$ can be written as a linear combination of $h^1, \ldots, h^l$. We cover the special case of linear combinations relating the generator h of the $\mathrm{U}(1)_H$ factor in the subgroup, $H = H' \times \mathrm{U}(1)_H$, which stabilizes the highest weight of the representation. We also cover how to write the weights of vevs, which break the symmetry to these differently embedded subgroups, as linear combinations of each other.

On the level of the Lie algebra $\mathcal{L}_G$ which generates G. We are only interested in embeddings of Lie subalgebras $\mathcal{L}_H$, $\mathcal{L}_{gHg^{-1}} \subset \mathcal{L}_G$ which are Cartan preserving. That is, given a Cartan subalgebra C_G for $\mathcal{L}_G$, the Cartan subalgebras for $\mathcal{L}_H$ and for $\mathcal{L}_{gHg^{-1}}$ are subspaces of C_G. In this context, we showed in section 3.11.5 that, the differently embedded isomorphic subalgebras are related via Weyl group conjugation, that is, $g \in W$.

We contextualize our presentation by discussing current fields where these results are useful. Symmetry breaking is a crucial aspect of modern particle physics. This makes our results applicable to a diverse array of model building scenarios. These include, but are not limited to, extra-dimensional models [SV04, DGK+08, DTVW02]. In particular:

- Our main motivation for studying this problem arises from a generalization of the Dvali-Shifman mechanism [DS97], known as the Clash of Symmetries mechanism [SV04, DGK+08, DTVW02].
- The "flipped" grand unified theories, explained in section 1.3, arise because of an automorphism of the Dynkin diagram (implicitly an automorphism of the Dynkin diagram gives an automorphism of the root system and therefore of the Lie algebra) [Bar82, DKN84].
- The low energy limit of Yang-Mills gauge theory predicts the breakdown of Weyl group symmetry in the effective QCD action [GN11, CKP06].
- Our results are also relevant to special cases of the "vacuum alignment" scenario. Here multiple Higgs fields, $\Phi_1, \Phi_2, \ldots$ are introduced in the same representation. These can acquire vevs $\langle\Phi_1\rangle, \langle\Phi_2\rangle, \ldots$ which break the gauge symmetry down to isomorphic but differently embedded copies of a subgroup .

We review the clash of symmetries mechanism in subsection 9.2.1. For a balanced perspective, we briefly address the low energy limit of Yang-Mills theories in subsection 9.2.2 as well as the vacuum alignment scenarios in subsection 9.2.3.

In section 9.3 we will state clearly the formula for recovering the adjoint Higgs vevs which break G to different embeddings of a subgroup H as linear combinations of vevs breaking G along the chain $G \supset H_1 \supset H_2 \supset \cdots \supset H_l$. We also treat the relationship between the weights of vevs causing G to break to different embeddings of a subgroup H, for a non-adjoint Higgs field. The remaining sections contain case studies which physically contextualize the root systems discussion in this paper and explicitly apply the formulas derived in section 9.3, as well as the conclusion.

We will rely heavily on sections 3.9-3.11.5 from the Introduction. These contain technical details necessary to understand the proof in section 9.3. In addition we adopt the notational conventions outlined in chapter 2.

9.2 Motivation

We preface our analysis with a few short comments on its potential applications to model building.

9.2.1 Domain-wall brane models

This work was directly motivated by a study of domain wall topological defects created by an adjoint scalar field, $\mathcal{X}$. In particular we study the case where the Lagrangian is invariant under a discrete symmetry, $\mathbb{Z}$, and a continuous internal symmetry G, but along two distinct antipodal directions the asymptotic configuration of the scalar field breaks $\mathbb{Z} \times G$ down to differently embedded isomorphic copies of $H \subset G$. This construction has a natural manifestation in grand-unified models with gauge group G and a single infinite extra dimension. Here the adjoint scalar field interpolates between two vacuum configurations preserving subgroups H and $zgHg^{-1}$ (for some $z \in \mathbb{Z}$ and $g \in G$) as a function of the extra dimensional co-ordinate, y. If the gauge coupling constants g_H and $g_{gHg^{-1}}$ are assumed to be below the confinement scale in each region, then the model will putatively trap gauge bosons belonging to non-Abelian factors in $H \cap zgHg^{-1}$ on a codimension-1 domain wall brane located at $y = 0$.

In this model the space-time on either side of the domain-wall brane exhibits dual superconducting behaviour. The idea is that the gauge theories in both the semi infinite regions $y > 0$ and $y < 0$ are in a confining regime with a mass gap. This causes the massless gauge bosons, belonging to the non-abelian factors in the intersection $H \cap zgHg^{-1}$, to become dynamically localized to the domain wall. This is, of course, an emergent effect, construed from low energy effective dynamics which must be complemented by an assumed UV-complete theory. The UV-completion is tasked with making Yang-Mills gauge theory renormalizable in 4+1-dimensions. When used effectively it is coined the "clash of symmetries mechanism" [DTVW02, SV04, DGK+08].

The clash of symmetries mechanism reduces to the Dvali-Shifman mechanism whenever $g = 1$. To illustrate the main difference between the Dvali-Shifman topological domain-wall brane scenario and the clash of symmetries we have included Figure 9.1 from [DGK+08]. Figure 9.1 is a comprehensive graph of the different types of domain walls, namely the non-topological, topological and clash of symmetries varieties.

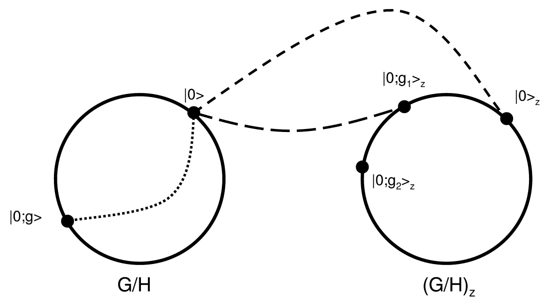


Figure 9.1: This is a graph of the vacuum manifold $G/H \cup (G/H)_z$. This vacuum manifold arises when $G \times \mathbb{Z} \to H$, where $\mathbb{Z}$ is a discrete reflection symmetry and $H \subset G$. The two circles represent the connected components of the vacuum manifold G/H and $(G/H)_z$. Each point on the first circle represents a vacuum $|0; g\rangle$, respectively each point on the second circle represents the $(G/H)_z$ equivalent vacuum. The end points of the three dotted lines represent asymptotic boundary conditions for three different types of domain walls. The dotted line gives an example of a non-topological domain-wall brane while the short dashed line indicates boundary conditions for a topological domain-wall brane. The end points of the long dashed lines are one possible example of the boundary conditions for a clash of symmetries domain-wall brane. This figure is taken from [DGK+08].

To implement the clash of symmetries mechanism we must solve the Euler Lagrange equations for $\mathcal{X}$ for boundary conditions as $y \to \pm\infty$ breaking $G \times Z$ to H and $zgHg^{-1}$, respectively. Therefore it is necessary to understand how the boundary conditions breaking G to gHg^{-1} can be written as a linear combination of the adjoint scalar field vevs breaking G

along the $H_{1,\dots l}$ branching direction in the Cartan subalgebra.

Solutions to the Euler Lagrange equations satisfying different boundary conditions have different energies. Furthermore a boundary condition preserving a symmetry H can be continuously transformed into a boundary condition preserving any other isomorphic coset $g_1 H g_1^{-1}$ inside G. The phenomenology of each domain-wall solution is different because each non-isomorphic intersection $H \cap g_1^{-1} g H g^{-1} g_1$ will give rise to a different gauge theory on the domain wall. Hence an exhaustive search for the lowest energy stable domain-wall configuration must be executed. This search must range through all solutions to the Euler Lagrange equations with different boundary conditions. In this case a systematic method for finding all the different possible configurations must be established. To trap a copy of the standard model gauge group on the domain wall, the grand unified gauge group must have a comparatively high rank, for example E_6, as in [DGK+08]. For high rank groups a method for writing one set of boundary conditions in terms of another becomes critical.

To find the vev for the adjoint $\mathcal{X}$ breaking G to a subgroup gHg^{-1} as a linear combination of vevs along the $H_{1,2,3,\dots,l}$ branching direction in the Cartan subalgebra, the authors of [DGK+08] wrote down the Casimir operators (invariants) for a general linear combination of the Cartan subalgebra, $h^1, \dots, h^l$. The co-ordinates in the Cartan subalgebra space which extremized the Casimir operators correspond to linear combinations which break G to gHg^{-1}. The physical motivation for this is: invariance of the action under the internal symmetry forces the potential to be a polynomial in the Casimir invariants. Therefore extrema of the Casimir operators correspond to degenerate minima in the vacuum manifold associated with spontaneous breaking of the internal symmetry G down to differently embedded isomorphic copies of a subgroup $H = H' \times \mathrm{U}(1)_H$. Hence the coefficients in the linear combination which extremize the Casimir invariants are precisely the components of the adjoint Higgs field in the original Cartan subalgebra basis which combine to give the ghg^{-1} generator which spontaneously condenses to break $G \to gHg^{-1}$. This approach is labor intensive.

9.2.2 Low-energy limit of Yang-Mills theory

We have found a natural motivation for our work in domain-wall formation due to the breaking of a global symmetry on cosmological scales. At the other end of the spectrum, in low energy effective models for SU(3) (and SU(2)) pure Yang-Mills gauge theories, domain walls form due to a breakdown of Weyl group symmetry caused by gluon condensation. This traps gauge fields on the domain wall. Galilo and Nedelko [GN11] work with an effective potential generated by loop order corrections in a low energy effective field theory approach to QCD:

$$U_{\text{eff}} = \frac{1}{12}\text{Tr}\left(C_1\hat{F}^2 + \frac{4}{3}C_2\hat{F}^4 - \frac{16}{9}C_3\hat{F}^6\right). \tag{9.1}$$

Here the potential is confining provided, $C_1 > 0,\ C_2 > 0,\ C_3 > 0$ and the non-Abelian gauge field strength tensor, $\hat{F}_{\mu\nu}$, can be written in terms of the SU(3) Lie algebra structure constants f^{abc},

$$F^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - if^{abc}G^b_\mu G^c_\nu, \qquad \left(\hat{F}_{\mu\nu}\right)_{bc} = F^a_{\mu\nu}T^a{}_{bc}, \quad T^a_{bc} = -if^{abc}. \tag{9.2}$$

The second order Casimir invariant $\text{Tr}(\hat{F})^2 = -3F^a_{\mu\nu}F^a_{\mu\nu} \leq 0$, causes the minimum of the effective potential to occur at a non-zero gluon field strength.

$$F^a_{\mu\nu}F^a_{\mu\nu} = \frac{4}{9C_3^2}\left(\sqrt{C_2^2 + 3C_1C_3} - C_2\right)^2\Lambda^4 > 0, \tag{9.3}$$

where Λ is the QCD confinement scale.

Galilo and Nedelko [GN11] look at the effective potential for $\hat{F}_{\mu\nu} = h^\chi B^\chi_{\mu\nu}$, which involves restricting the full SU(3) gauge theory to the U(1) × U(1) Abelian subspace, where the generators are given as linear combinations of the diagonal Gell-Mann matrices,

$$h^\chi = \chi^1\lambda_3 + \chi^2\lambda_8, \tag{9.4}$$

and the associated field strength $B^\chi_{\mu,\nu}$ can be found by using the Abelian subalgebra version of (9.2) on $B^{\chi}_{\mu} = \chi^1 G^3_{\mu} + \chi^2 G^8_{\mu}$. The minima of the effective potential are located at

$$\chi = (\cos\frac{(2n+1)\pi}{6}, \sin\frac{(2n+1)\pi}{6}) \text{ for } n \in \{0, \ldots, 5\}. \tag{9.5}$$

They are related by a discrete Weyl group symmetry. The requirement that QCD remains unbroken despite a non-zero background field strength means the background field must be the average of an ensemble of gauge field configurations with a high degree of disorder and spatial variation of the direction χ in colour-space. This causes different vacua to be selected in different spatial regions. Galilo and Nedelko [GN11] explain that domain-wall configurations are formed by gauge fields interpolating between these vacua. Collectively the h^{χ} describe the vevs of an adjoint Higgs field which break SU(3) to U(1) × U(1). Here they again form the boundary conditions for the domain wall.

In the pure SU(2) Yang-Mills theory, domain walls form between vacua preserving different embeddings of a $U(1)_{\alpha}$ symmetry associated with magnetic charge [Kob11].

In both the above models there is an opportunity to trap gauge fields on the domain wall. This analysis can be generalized to SU(n) pure Yang-Mills theory where the rank of the algebra will again necessitate a systematic way of identifying all the boundary conditions for the domain walls.

9.2.3 Vacuum alignment

In general there are a wide variety of extensions to the standard model which introduce a gauge symmetry G and multiple Higgs fields $\Phi_1,\ \Phi_2,\ \ldots$ such that all the Higgs fields are in identical representations for G. Under these circumstances, when the Higgs fields acquire vacuum expectations values $\langle\Phi_1\rangle,\ \langle\Phi_2\rangle,\ \ldots$ any pair of vevs which are not aligned will be capable of breaking G to different subgroups. In the scenario where all these subgroups are isomorphic, differently embedded copies of $H \subset G$, the analysis presented in section 9.3 explains how to obtain one vev from the other. This analysis can also be used to find the unbroken symmetry.

9.3 Statement of Proof

We are looking for a complete set of embeddings of the subgroup chain $G \supset H_1 \supset H_2 \supset \cdots \supset H_l$, where the vevs of the adjoint Higgs fields which break G down to these subgroups and define a basis for the Cartan subalgebra $h^1, \ldots, h^l$, can be written as linear combinations of each other.

In subsection 3.11.4 we established that the embeddings of H_1 within G arise from conjugation of the Lie algebra $\mathcal{L}_{H_1}$ for H_1 by the Weyl group $W/W_{\Delta_{H_1}}$, where $W_{\Delta_{H_1}}$ is the Weyl group of the maximal subgroup H_1. Moreover, we know conjugation by any Weyl group element, $w^\kappa \in W/W_{\Delta_{H_1}}$, acts on the Cartan subalgebra or vevs $h^1, \ldots, h^l$ according to

$$w^\kappa \cdot h^j = \Sigma_i(\delta_{ij} - \Sigma_n \kappa^n \delta_{nj} \kappa_i^\vee) h^i = h^j - \kappa^j h^\kappa. \tag{9.6}$$

So after identifying the generators (roots) excluded from the embedding of $H_1 \subset G$ ($\Delta_{H_1} \subset \Delta$) we have a general formula for writing the vevs of the adjoint Higgs field, $w^\kappa \cdot h^1, \ldots, w^\kappa \cdot h^l$, causing the breaking of $G \supset w^\kappa H_1 w^{-\kappa} \supset w^\kappa H_2 w^{-\kappa} \supset \cdots \supset w^\kappa H_l w^{-\kappa}$, as linear combination of $h^1, \ldots, h^l$. If, after choosing an embedding of H_1 within G, identified with $\mathcal{L}_{H_1} \subset \mathcal{L}$, we wish to find all the different embeddings of H_2 within H_1, which have $\mathcal{L}_{H_2} \subset \mathcal{L}_{H_1}$, then we simply repeat this procedure for $W_{\Delta_{H_1}}/W_{\Delta_{H_2}}$.

In the case where we are looking for the adjoint Higgs vevs, $w^\kappa \cdot h$, breaking G to different embeddings of the subgroup $H = H' \times \mathrm{U}(1)_H$ which stabilizes (the representation space state labeled by) the highest weight of the lowest dimensional fundamental representation, $|\lambda\rangle$, these linear combination have a remarkably simple formula. We show that the linear combination giving each vev is $\Sigma_i \mu(h^i) h^i$ for an extremal weight μ of the fundamental representation.

We first prove the adjoint Higgs vev, h, which breaks G to H is given by the linear combination $h = \Sigma_i \lambda(h^i) h^i$, where the coefficients are the coordinates of highest weight of the fundamental representation. Then we explain why other generators breaking G to different embeddings $w^\kappa \cdot H$, $w^\kappa \cdot h = \Sigma_i \mu(h^i) h^i$, are the linear combinations of $h^1, \ldots, h^l$ which have the co-ordinates of the extremal weights, $\mu(h^i)$ as coefficients.

If $\Sigma_i \lambda(h^i)h^i$ is the adjoint Higgs vev which breaks G to H, then it is the generator of the $U(1)_H$ factor in $H = H' \times U(1)_H$. Therefore $\Sigma_i \lambda(h^i)h^i$ must stabilize $|\lambda\rangle$ (be a generator of H) and it must commute with each generator, $E^\alpha \in \mathcal{L}_G$, if and only if $E^\alpha \in \mathcal{L}_H$.

It is clear that $\Sigma_i \lambda(h^i)h^i$ is a generator of H because $\Sigma_i \lambda(h^i)h^i\,|\lambda\rangle = \left(\Sigma_i \lambda(h^i)^2\right)|\lambda\rangle$.

Furthermore, let $E^\alpha \in \mathcal{L}_H$. Then E^α is a raising or lowering operator and E^α stabilizes $|\lambda\rangle$, therefore we must have $E^\alpha\,|\lambda\rangle = 0$. If $\alpha \in \Delta_H$ then $-\alpha \in \Delta_H$, and by the same logic $E^{-\alpha}\,|\lambda\rangle = 0$. Consider the commutator

$$\begin{aligned}
[E^\alpha, \Sigma_i \lambda(h^i)h^i] &= \Sigma_i \lambda(h^i)[E^\alpha, h^i] \\
&= \Sigma_i \lambda(h^i)\alpha^i E^\alpha \\
&= \lambda(\Sigma_i \alpha^i h^i)E^\alpha \\
&= \lambda(h^\alpha)E^\alpha \\
&= \lambda([E^\alpha, E^{-\alpha}])E^\alpha \\
&= 0.
\end{aligned} \tag{9.7}$$

Therefore $\Sigma_i \lambda(h^i)h^i$ commutes with all the elements of $\mathcal{L}_H$.

Assume $\Sigma_i \lambda(h^i)h^i$ commutes with a generator $E^\kappa \notin \mathcal{L}_H$ which does not belong to the Lie algebra of H (that is E^κ does not stabilize $|\lambda\rangle$). Then we have

$$\begin{aligned}
\Sigma_i \lambda(h^i)h^i &= w^\kappa \Sigma_i \lambda(h^i)h^i w^{-\kappa} \\
&= \Sigma_i \lambda(h^i)h^i - \Sigma_i \lambda(h^i)\kappa(h^i)h^\kappa \\
&= \Sigma_i \lambda(h^i)h^i - \Sigma_{ij} \lambda(h^i)\kappa(h^i)\kappa^\vee(h^j)h^j \\
&= \Sigma_i \lambda(h^i)h^i - \Sigma_j \frac{2(\lambda,\kappa)}{(\kappa,\kappa)}\kappa(h^j)h^j \\
&= \Sigma_i \left(\lambda(h^i) - (\lambda,\kappa^\vee)\kappa(h^i)\right)h^i \\
&= \Sigma_i \left[s^\kappa \cdot \lambda\right](h^i)h^i.
\end{aligned} \tag{9.8}$$

This creates a contradiction because we are insisting E^κ does not stabilize $|\lambda\rangle$, so $w^\kappa\,|\lambda\rangle = |s^\kappa \cdot \lambda\rangle \neq |\lambda\rangle$ and the two sets of coefficients (of the linearly independent Cartan subalgebra generators $h^1, \ldots, h^l$) in the above sum must be different. We have proved $\Sigma_i \lambda(h^i)h^i$ is the adjoint Higgs vev, h, which breaks G to H.

Now each embedding $w^\kappa \cdot H = w^\kappa H w^{-\kappa}$ will stabilize a state in the representation labeled by an extremal weight $w^\kappa \cdot |\lambda\rangle = |\mu\rangle$. By the above argument, the center of the subgroup $w^\kappa \cdot H$ which stabilizes $|\mu\rangle$ is generated by $\Sigma_i \mu(h^i) h^i$. We have a remarkably easy formula for reproducing the vevs which break G to all the different embeddings of the subgroup which stabilizes the highest weight of the lowest dimensional fundamental representation, H, as a linear combination of the Cartan subalgebra $h^1, \ldots, h^l$. Notice that w^κ must belong to a non-trivial coset in W/W_{Δ_H}, because conjugation by w^κ only takes us from one embedding to another when s^κ does not fix the highest weight.

We present a systematic method for determining the subgroup H directly from the extended Dynkin diagram for the Lie group G. Each unmarked node in the extended Dynkin diagrams is labeled by a simple root. The node with a cross in the center is ζ^0. To find the Dynkin diagram for H we determine which of the simple roots in Δ are also in Δ_H. We also need to work out if the highest root ζ^0 is in Δ_H. The subset of $\{\zeta^1, \ldots, \zeta^l\} \cup \{-\zeta^0\}$ belonging to Δ_H, will be the simple roots for Δ_H.

First we determine which subset of the simple roots $\{\zeta^1, \ldots, \zeta^l\}$ belong to Δ_H. Take the highest weight, λ, and write it as a linear combination of the fundamental weights.

$$\lambda = a_1 \omega_1 + \cdots + a_l \omega_l. \tag{9.9}$$

We assume this highest weight is dominant, that is, $a_1, \ldots, a_l \geq 0$. If it is not then it is always possible to replace λ by one of the extremal weights which is dominant. Construct a set $S_\lambda = \{j | a_j = 0\}$. For all $j \in S_\lambda$ we have $(\lambda, \zeta^{j\vee}) = 0$. We claim that $\zeta^j \in \Delta_H$, that is $E^{\pm\zeta^j} |\lambda\rangle = 0$, for all $j \in S_\lambda$. Otherwise if $E^{\pm\zeta^j} |\lambda\rangle \neq 0$ consider the norm $N_{\lambda\pm\zeta^j} = \langle\lambda| E^{\pm\zeta^j \dagger} E^{\pm\zeta^j} |\lambda\rangle$. Because λ is the highest weight of the representation $N_{\lambda+\zeta^j} = 0$ while $N_{\lambda-\zeta^j} = \langle\lambda| [E^{\zeta^j}, E^{-\zeta^j}] |\lambda\rangle = \langle\lambda|\lambda\rangle(\lambda, \zeta^{j\vee}) = 0$. For the remaining simple roots labeled by $k \notin S_\lambda$, we have $s^{\zeta^k} \cdot \lambda \neq \lambda$, therefore $w^{\zeta^k} |\lambda\rangle \neq |\lambda\rangle$ and, from (3.130), we know that one of $E^{\pm\zeta^k}$ does not stabilize λ.

The highest root (negated highest root) $\pm\zeta^0$ does not belong to Δ_H. This follows from the fact that ζ^0 is some linear combination of all the simple roots (with positive coefficients), therefore if the set S_λ is non-empty, then $(\lambda, \zeta^0) > 0$.

So the Dynkin diagram for H can be reconstructed from the connected components of the Dynkin diagram for G labeled by simple roots $\{\zeta^j | j \in S_\lambda\}$. This uniquely defines the non-Abelian factor H' of H. The full subgroup H which stabilizes the highest weight is a product of H' with one Abelian factor U(1) for each $k \notin S_\lambda$. These extra U(1) factors are generated by the Cartan subalgebra generators h^{ζ^k}, $k \notin S_\lambda$, which (by definition) stabilize λ, even when the associated raising/lowering operators E^{ζ^k} do not.

If the Higgs field does not belong to the adjoint representation then the above analysis generalizes. The Weyl group reflections still give the different embeddings of the subgroup chain $G \supset H_1 \supset \cdots \supset H_l$. Consider a Cartan subalgebra $h^1, \ldots, h^l$ which is defined as the generators of $\mathrm{U}(1)_{H_i}$ factors appearing in the subgroup chain through $H_i = H_i' \times \mathrm{U}(1)_{H_1} \times \cdots \times \mathrm{U}(1)_{H_i}$, where H_i' is some product of non-Abelian Lie groups. Equation (9.6) gives the linear combinations for the equivalent Cartan subalgebra generator for the $\mathrm{U}(1)_{w^\kappa \cdot H^i}$ factors belonging to the differently embedded subgroup chain $G \supset w^\kappa H_1 w^{-\kappa} \supset \cdots \supset w^\kappa H_l w^{-\kappa}$, where $w^\kappa \in W/W_{\Delta_H}$.

If a subgroup $H \subset G$ annihilates a column vector $|\nu\rangle$, labeled by a weight ν, then the differently embedded subgroup $w^\kappa H w^{-\kappa}$ annihilates the column vector $w^\kappa |\nu\rangle$. Hence if $|\nu\rangle$ breaks G to H, then $|s^\kappa \cdot \nu\rangle$ breaks $G \supset w^\kappa H w^{-\kappa}$ and it follows directly from (3.136) that (3.119) gives the coordinates of the new weights as a linear combination of ν (and κ).

9.4 Insight

We wish to firmly ground the above discussion by applying these concepts to physical systems. We physically contextualize the key concepts in sections 3.9 and 3.11.5 via the smallest effective example: embeddings of U-spin, I-spin and V-spin within the SU(3) QCD gauge group. We also tackle the non-trivial problem of finding a full complement of domain-wall boundary conditions for an adjoint Higgs field which break E_6 to different embeddings of SO(10) × U(1) in order to demonstrate the effectiveness of the techniques developed in section 9.3.

9.4.1 A Quantum chromodynamics example

Consider the Weyl group conjugations giving rise to differently embedded copies of the subgroups SU(2)×U(1) inside SU(3). Following [CKP06] we rewrite the SU(3) pure Yang-Mills quantum chromodynamics Lagrangian in terms of the off diagonal gluons $Z_\mu^p,\ p \in \{1,2,3\}$ and the dual potentials to the roots $B_\mu^p,\ p \in \{1,2,3\}$ defined in chapter 2:

$$\mathcal{L} = -\frac{1}{4}\mathcal{G}_{\mu\nu}\mathcal{G}^{\mu\nu} = \Sigma_p \left\{ -\frac{1}{6}\left(F_{\mu\nu}^p\right)^2 + \frac{1}{2}\left|D_{p\mu}Z_\nu^p - D_{p\nu}Z_\mu^p\right|^2 - igF_{\mu\nu}^p Z_p^{\mu *} Z_p^\nu \right.$$
$$\left. -\frac{1}{2}g^2\left[\left(Z_\mu^{p*}Z_\nu^p\right)^2 + \left(Z_\mu^{p*}\right)^2\left(Z_\nu^p\right)^2\right]\right\}, \qquad (9.10)$$

where

$$F_{\mu\nu}^p = \partial_\mu B_\nu^p - \partial_\nu B_\mu^p, \qquad D_{p\mu}W_\nu^p = \left(\partial_\mu - igB_\mu^p\right)W_\nu^p. \qquad (9.11)$$

The Weyl group permutes the roots $\{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3\}$ of the SU(3) Lie algebra. Hence the Weyl group action on the above Lagrangian will cause a permutation of the dual potentials $B_\mu^p,\ p \in \{1,2,3\}$ spanning the Cartan subalgebra and also will cause a permutation of the raising and lowering operators $Z_\mu^1 \in span\{Z^1, Z^{-1}\}$, $Z_\mu^2 \in span\{Z^2, Z^{-2}\}$, $Z_\mu^3 \in span\{Z^3, Z^{-3}\}$ which is labeled by the roots. The permutation is concordant with the geometric picture of the Weyl group reflections of their root labels. Therefore the invariance of the above Lagrangian under Weyl group reflections is encapsulated in the sum over the index p.

The direct interpretation for the Weyl group role in (9.10) arises because the associated generators ϵ, ρ and $Z^{\pm p}$ where $p \in \{1,2,3\}$ (it is not necessary to include κ in this list because SU(3) has rank 2, however we can substitute it for either ϵ or ρ if we wish), form a useful computational basis for the Lie algebra: the Chevalley basis. Here each of the three subset $\{\kappa, Z^{\pm 1}\}$, $\{\rho, Z^{\pm 2}\}$ and $\{\epsilon, Z^{\pm 3}\}$ defines an embedding of SU(2) inside SU(3). These correspond to the closed crystallographic root systems $\{\pm\alpha^1\}$ whose Weyl group fixes a point on the hyperplane orthogonal to α^1, and to the closed crystallographic root systems $\{\pm\alpha^2\}$

and $\{\pm\alpha^3\}$, whose Weyl groups fix analogous points. Cross checking this with chapter 2 we see these are precisely the I-spin, V-spin and U-spin embeddings. Each embedding commutes with one of the Abelian subgroup generators $\lambda_8(=\kappa')$, ρ' or ϵ', which now we can rewrite as $\Sigma_i \mu(h^i)h^i$ for any diagonal Cartan subalgebra $\{h^1, h^2\}$ for SU(3), (in chapter 2 our Cartan subalgebra was chosen to be λ_3 and λ_8) and the three extremal weights of the lowest dimensional fundamental representation for SU(3).

9.4.2 DWB adjoint Higgs field breaking $E_6 \to SO(10) \times U(1)$

We use the method developed in the previous section to find all the adjoint Higgs vevs which break E_6 to all the different embeddings of SO(10)×U(1); this example is directly motivated by an extra-dimensional "clash of symmetries" domain-wall brane model [DGK+08]. Our choice of Cartan subalgebra for E_6 is explained in Table 9.1. The entries of this table follow directly from the branching rules [Sla81]:

$$\begin{aligned} E_6 &\supset SO(10) \times U(1)_{h^1} \supset SU(5) \times U(1)_{h^1} \times U(1)_{h^2} \\ &\supset SU(3) \times SU(2) \times U(1)_{h^1} \times U(1)_{h^2} \times U(1)_{h^3} \\ &\supset SU(3) \times U(1)_{h^1} \times U(1)_{h^2} \times U(1)_{h^3} \times U(1)_{h^4} \\ &\supset SU(2) \times U(1)_{h^1} \times U(1)_{h^2} \times U(1)_{h^3} \times U(1)_{h^4} \times U(1)_{h^5} \\ &\supset U(1)_{h^1} \times U(1)_{h^2} \times U(1)_{h^3} \times U(1)_{h^4} \times U(1)_{h^5} \times U(1)_{h^6}. \end{aligned} \tag{9.12}$$

As mentioned in section 9.2, our primary motivation for studying this problem arose from a co-dimension-1 clash-of-symmetries domain-wall brane. The brane originates from an E_6 adjoint Higgs field $\mathcal{X}$ which condenses spontaneously to break translational invariance along the extra dimension of a 4+1-dimensional space-time manifold.

The Lagrangian for this theory is invariant under a $\mathbb{Z}_2 \times E_6$ internal symmetry. It is a linear combination of the invariant kinetic term $\mathrm{Tr}\left[D^\mu \mathcal{X} D_\mu \mathcal{X}\right]$, a potential formed from the E_6 Casimir invariants $I_2 = \mathrm{Tr}\mathcal{X}^2$ and $I_6 = \mathrm{Tr}\mathcal{X}^6$, and the powers I_2^2 and I_2^3. Casimir invariants corresponding to odd powers of $\mathcal{X}$, must be omitted due to the imposed Z_2, $\mathcal{X} \to -\mathcal{X}$, symmetry. The potential is truncated at 6 th order because the coupling constants of higher order invariants have negative mass dimensions and therefore are suppressed by powers of the pu-

	$60h^1$	$60h^2$	$60h^3$	$60h^4$	$60h^5$	$60h^6$
1	20	0	0	0	0	0
2	-10	$2\sqrt{15}$	$3\sqrt{10}$	$-5\sqrt{6}$	0	0
3	-10	$2\sqrt{15}$	$3\sqrt{10}$	$5\sqrt{6}$	0	0
4	-10	$2\sqrt{15}$	$-2\sqrt{10}$	0	$5\sqrt{2}$	$5\sqrt{6}$
5	-10	$2\sqrt{15}$	$-2\sqrt{10}$	0	$5\sqrt{2}$	$-5\sqrt{6}$
6	-10	$2\sqrt{15}$	$-2\sqrt{10}$	0	$-10\sqrt{2}$	0
7	-10	$-2\sqrt{15}$	$-3\sqrt{10}$	$5\sqrt{6}$	0	0
8	-10	$-2\sqrt{15}$	$-3\sqrt{10}$	$-5\sqrt{6}$	0	0
9	-10	$-2\sqrt{15}$	$2\sqrt{10}$	0	$-5\sqrt{2}$	$-5\sqrt{6}$
10	-10	$-2\sqrt{15}$	$2\sqrt{10}$	0	$-5\sqrt{2}$	$5\sqrt{6}$
11	-10	$-2\sqrt{15}$	$2\sqrt{10}$	0	$10\sqrt{2}$	0
12	5	$-5\sqrt{15}$	0	0	0	0
13	5	$3\sqrt{15}$	$-3\sqrt{10}$	$5\sqrt{6}$	0	0
14	5	$3\sqrt{15}$	$-3\sqrt{10}$	$-5\sqrt{6}$	0	0
15	5	$3\sqrt{15}$	$2\sqrt{10}$	0	$-5\sqrt{2}$	$-5\sqrt{6}$
16	5	$3\sqrt{15}$	$2\sqrt{10}$	0	$-5\sqrt{2}$	$5\sqrt{6}$
17	5	$3\sqrt{15}$	$2\sqrt{10}$	0	$10\sqrt{2}$	0
18	5	$-\sqrt{15}$	$\sqrt{10}$	$-5\sqrt{6}$	$5\sqrt{2}$	$5\sqrt{6}$
19	5	$-\sqrt{15}$	$\sqrt{10}$	$-5\sqrt{6}$	$5\sqrt{2}$	$-5\sqrt{6}$
20	5	$-\sqrt{15}$	$\sqrt{10}$	$-5\sqrt{6}$	$-10\sqrt{2}$	0
21	5	$-\sqrt{15}$	$\sqrt{10}$	$5\sqrt{6}$	$5\sqrt{2}$	$5\sqrt{6}$
22	5	$-\sqrt{15}$	$\sqrt{10}$	$5\sqrt{6}$	$5\sqrt{2}$	$-5\sqrt{6}$
23	5	$-\sqrt{15}$	$\sqrt{10}$	$5\sqrt{6}$	$-10\sqrt{2}$	0
24	5	$-\sqrt{15}$	$-4\sqrt{10}$	0	$-5\sqrt{2}$	$-5\sqrt{6}$
25	5	$-\sqrt{15}$	$-4\sqrt{10}$	0	$-5\sqrt{2}$	$5\sqrt{6}$
26	5	$-\sqrt{15}$	$-4\sqrt{10}$	0	$10\sqrt{2}$	0
27	5	$-\sqrt{15}$	$6\sqrt{10}$	0	0	0

Table 9.1: The six diagonal generators h^{1-6} of E_6. The diagonal elements of the generator h^n are found by taking the n^{th} column and multiplying it by $1/60$. Also the rows give the coefficients f_{1-6} of these generators that yield a linear combination that breaks $E_6 \to SO(10) \times U(1)$. This table is reproduced from [Geo09].

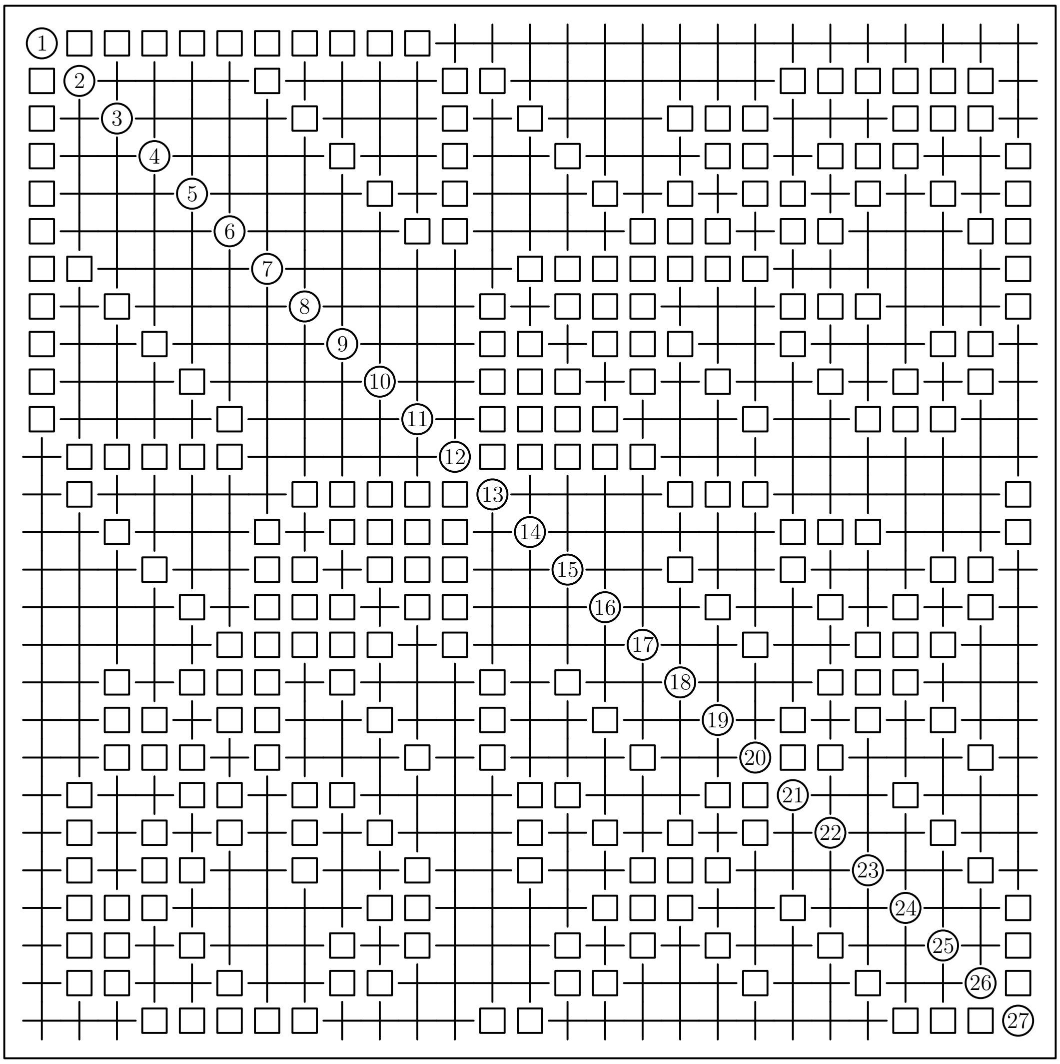


Figure 9.2: A pictorial representation of the twenty seven rearrangements of the diagonal generator h^1 of E_6. Each rearrangement can be reconstructed from one of the twenty seven rows (or columns) of symbols in this picture. To find the diagonal entries of the n−th rearrangement, read along the n−th row and translate the symbols according to: circles ○ correspond to the single $1/3$ entry, squares □ to $-1/6$, and crosses + to $1/12$ (note that adjacent crosses are touching). The number in the centre of each circle tells its row and column number (being the same). Row n of this picture corresponds precisely to row n of Table 9.1 in the sense that the linear combination $\sum_{a=1}^{6} f_a h^a$, where the f_{1-6} are chosen from row n of Table 9.1, yields the rearranged version of the generator h^1, represented by the symbols of row n in this picture. This figure comes from [Geo09].

tative ultraviolet completion scale (see section 9.2). Yet the 4-th order invariants exhibit an accidental O(78) symmetry, so we must include a $\mathrm{Tr}X^6$ term. A subset of the local minima of the Casimir invariants occur at adjoint Higgs vevs which break $\mathbb{Z}_2 \times E_6 \to SO(10) \times U(1)$. If the solution X to the associated Euler-Lagrange equations interpolates between vacuum expectation values which break $\mathbb{Z}_2 \times E_6$ to a specific pair of differently embedded copies of $SO(10) \times U(1)$ then [DGK+08] postulates a copy of the standard model particles can be trapped on the 3 + 1-dimensional domain-wall brane. To find X it is necessary to write the boundary conditions at the two antipodal extremes of the extra dimension as a linear combination of the adjoint Higgs vevs $h^1, \ldots, h^6$, the generators of the Abelian subgroup factors given in (9.12).

Because $SO(10) \times U(1)$ stabilizes the highest weight of the lowest dimensional fundamental representation for E_6 this is now a trivial problem. Each of the possible boundary conditions which break $E_6 \to SO(10) \times U(1)$ can be written as a linear combination of $h^1, \ldots, h^6$ using $\Sigma_i \mu(h^i) h^i$ where μ is one of the 27 extremal weights of the lowest dimensional fundamental representation for E_6. Explicitly the 27 different vevs breaking $E_6 \to SO(10) \times U(1)$ are

$$\langle X \rangle \propto \Sigma_{a=1}^{6} f_a h^a, \tag{9.13}$$

where the sextuplet $f_{1,\ldots,6}$ takes values from one of the rows of the Table 9.1. Figure 9.2 graphically identifies the diagonal entries of each of these 27 vevs breaking $E_6 \to SO(10) \times U(1)$. The Figure 9.2 comes from [Geo09].

9.5 Conclusions

Symmetry breaking is an ubiquitous idea in particle physics. This is due to the wide range of applications, and its capacity to act as a powerful tool. Electroweak symmetry breaking explains the origin of mass in the standard model, while the Frogatt-Nielson mechanism can explain why some interactions are unnaturally heavily suppressed. A large class of models such as little Higgs scenarios and left right symmetric models use additional symmetry breaking fields to extend the standard model and to suggest possible solutions to current open questions.

The diverse array of models presented in this chapter, ranging from the clash of symmetries mechanism to the effects of glueball formation in low energy Yang-Mills theories, demonstrate how important it is to attain a better understanding of symmetry breaking.

We find it fascinating that there is an alternative representation for the Lie algebra, in terms of crystallographic root systems. A more general formulation of Yang-Mills gauge theory, starting from this perspective would be invaluable. In this chapter we have tried to address one small aspect of breaking Yang-Mills gauge theories. Given a collection of adjoint Higgs vevs which break $G \supset H_1 \supset H_2 \supset \cdots \supset H_l$ we have described how to choose general linear combinations which break G to differently embedded isomorphic copies of $\{H_1, \ldots H_l\}$. These copies will belong to a chain $G \supset gH_1g^{-1} \supset gH_2g^{-1} \supset \cdots \supset gH_lg^{-1}$, for some $g \in G$. In section 9.3 we have highlighted the simple case when the subgroup we are breaking to stabilizes the highest weight of the lowest dimensional fundamental representation for G.

We also covered the more general case when the Higgs field is not in the adjoint representation. Here we discussed the relationship between the weights of vevs breaking G to differently embedded copies of a particular subgroups. In addition we canvassed the relationship between the Cartan subalgebra generators $h^1, \ldots, h^l$ which generate the Abelian subgroups in a chain $G \supset H_1 \supset H_2 \supset \cdots \supset H_l$ where each $H_i = H_i' \times \mathrm{U}(1)_{H_1} \times \cdots \times \mathrm{U}(1)_{H_i}$ and the Cartan subalgebra generators from the conjugated chain $G \supset gH_1g^{-1} \supset gH_2g^{-1} \supset \cdots \supset gH_lg^{-1}$.

We took this formalism and applied it to rewriting the QCD Lagrangian. Here we explained how Weyl group invariance arises. For the adjoint representation, we considered the eigenbasis of the Cartan subalgebra action. The Weyl group action permutes the members of this basis. We found that the QCD Lagrangian possesses a Weyl group symmetry because it is a permutationally invariant polynomial in this eigenbasis. The three SU(2) embeddings, I-spin, U-spin and V-spin are related by Weyl group conjugation.

We took the techniques developed in section 9.3 an applied them to an E_6 grand unified clash of symmetries model. Here we gave a comprehensive list of the vevs breaking E_6 to Cartan preserving embeddings of $\mathrm{SO}(10) \times \mathrm{U}(1)$. These vevs furnish a comprehensive set of clash of symmetries domain-wall boundary conditions. Id est for the $\mathrm{E}_6 \times \mathbb{Z}_2 \to \mathrm{SO}(10) \times \mathrm{U}(1)$ example they generate the first circle in Figure 9.1. In [DGK+08] the discrete symmetry is $\mathbb{Z}_2$ and therefore the second vacuum manifold, $(G/H)_z$ in Figure 9.1, is the negative of the first. We provided a remarkably simple formula for recovering all members of the $E_6/SO(10)$, vacuum manifold.

10

Conclusions

This thesis explored new techniques and approaches to field theory. These approaches were motivated by model building in extra dimensions.

Extra-dimensional models are a leading candidate for "beyond the standard model" physics. Several of the constructions presented during this thesis demonstrate that extra-dimensional models can exhibit realistic standard model phenomenology, at low energies. Therefore we consider that extra-dimensions provide a viable description of the universe, and are a worthwhile option to investigate. Ultimately new experimental or observational data will be needed to confirm or exclude their existence. Alternatively new inspiration may be found in crossover fields like condensed matter physics, or from a more systematic treatment of current field theories and gravity, starting from a general analysis of Yang-Mills gauge theories and then determining the acceptable gauge groups, interactions and interpretations.

We are less concerned with explicit constructions and more interested in the new ideas and techniques which arise during our analysis. Because these models are field theory constructions, we find the concept of adaptations of Yang-Mills gauge theory to Lifshitz scaling scenarios interesting. Also we are motivated to study new approaches to understanding the internal symmetries and gauge groups, including their breaking.

We summarize the three main topics addressed in this thesis and our contribution to each.

10.1 Thick-brane extra-dimensional models

Extra-dimensional models can be classified into two general varieties: infinite extra dimensions and compact extra-dimensions. In this thesis we adopted the stance that infinite extra-dimensions are more appealing because they are more "dimensionally democratic" from the field theory perspective. By dimensionally democratic we mean that the spatial dimensions are indistinguishable in the fundamental action. In this scenario we investigated explicit models and mechanisms for recovering standard model physics on a 3 + 1-dimensional brane.

In chapter 4 we considered a 4 + 1-dimensional model for the universe with a single infinite extra dimension. Here the most likely field theoretical candidate for generating a 3 + 1-dimensional brane is a domain-wall topological defect, called the kink. Again we prefer to introduce a domain-wall brane (thick brane) rather than a fundamental brane because the former leads to a more dimensionally democratic model. In thick-brane models Poincaré symmetry is spontaneously broken due to the condensation of the real scalar field topological defect, and the origin of the brane is explained. In the fundamental brane scenario, the presence of a Dirac delta function in the fundamental action, explicitly breaks translational invariance along the extra dimension as well as dimensional democracy. Furthermore a fundamental brane may have string theoretical origins, however we would prefer not to assume an additional new formalism like string theory.

We systematically described the field theoretical mechanisms for confining the standard model gauge fields and the first generation of fermions on the brane.

To trap gauge fields we employed an SO(10) grand unified theory and a 4 + 1-dimensional generalization of the Dvali-Shifman mechanism. This approach has the advantage of preserving gauge coupling constant universality. To implement the Dvali-Shifman mechanism we introduced an SO(10) adjoint Higgs field with the most general adjoint Higgs field-kink interaction terms. In addition, we assumed that the bulk gauge theory is in a confining phase of SO(10), with a mass gap. After solving the equations of motion for the kink and the adjoint Higgs field simultaneously, the resulting extra-dimensional profiles effectively trap a SU(5) pure Yang-Mills gauge theory on the domain-wall brane.

To trap fermions we used the split fermion mechanism to localize a full complement of massless left chiral 3 + 1-dimensional fermions belonging to the first generation of the standard model. As in standard SO(10) models we introduced a right handed neutrino. The left chiral 3 + 1-dimensional fermions are projected out of the 4 + 1-dimensional theory using a Kaluza-Klein decomposition of the 4 + 1-dimensional Lorentz spinor.

We recovered the conventional 3 + 1-dimensional relativistic formulation of gravity and Newtonian $1/r^2$ gravity by exhibiting a type 2 Randall-Sundrum warped metric solution.

The final component of our model is a pair of Higgs fields $\zeta(x^\mu, y)$ and $\eta(x^\mu, y)$, which break the SU(5) symmetry on the domain-wall down to the standard model, and subsequently implement electroweak symmetry breaking. We argued that there is sufficient parameter freedom for the appropriate components of both Higgs fields to condense inside the wall.

The massless 3 + 1-dimensional fermions can now acquire masses by coupling to the electroweak symmetry breaking field. This has strong implications for the observed SU(3) × SU(2) × U(1) gauge theory on the domain-wall brane. The mass degeneracies, such as $m_s/m_d = m_\mu/m_e$, in standard grand unified theories are absent from this model. Proton decay, mediated by the extra components of the irreducible SO(10) representation for the electroweak symmetry breaking Higgs field, is suppressed.

Additional Kaluza-Klein modes will modify the coupling constant running. This implies that for 4 + 1-dimensional SO(10) grand unified theories the coupling constants may meet and thereby establish a GUT symmetry breaking scale. However calculating the coupling constant running will require a full phenomenological parameter fitting. Since our model still requires additional generations of fermions and a realistic mechanism for generating neutrino mixing angles, a detailed calculation of the coupling constant running is premature.

There are many open questions for this model. The most pressing is a strong appraisal of the effectiveness of the Dvali-Shifman mechanism in 4 + 1-dimensions. A necessary condition for the Dvali-Shifman mechanism to be operative is confinement in the bulk. Lattice simulations of 4 + 1-dimensional SU(2) and SU(5) Yang-Mills gauge theories demonstrate a first order phase transition at finite lattice spacing as a function of the gauge coupling constant: for coupling strengths above a critical value, the theory appears to be confining

[Cre79, Geo09]. However the result can not be extended to the continuum limit because $4+1$-dimensional Yang-Mills is nonrenormalizable. A further open question is the stability of the model.

In chapter 5 we analyzed the stability of the domain-wall brane model. We addressed the question of stability by considering the dynamical evolution of classes of perturbations about the solutions (5.2) to the domain-wall brane equations (5.1). Because the coupling constants in (5.1) run with energy we considered these perturbations for an open set in the free parameters space of (5.1). We say the domain-wall brane is unstable, respectively stable, when there exists a perturbation which grows exponentially with time, respectively all perturbations are oscillatory with time and remain small.

We established stability and non-stability results for a domain-wall brane model. In particular, we showed the nonexistence of perturbations which diverge exponentially with time, for an open set of the free parameter space. Moreover we established the existence of nontrivial perturbations which diverge exponentially with time, for a hypersurface of the parameters. We used Fredholm theory for compact linear operators combined with the Lyapunov-Schmidt method to prove our results. This establishes the stability of the domain-wall brane, respectively instability, in two distinct free parameter regimes. That is: there exists a bifurcation point in the phase diagram of (5.1), in one region of the free parameter space (4.3) is a stable solution.

10.2 Lifshitz field theories

We investigated features of $4+1$-dimensional and $3+1$-dimensional Lifshitz field theories with critical exponent $z = 2$. We were motivated by recent results that $4+1$-dimensional Lifshitz scalar field theories and Yang-Mills gauge theories can be power counting renormalizable. Moreover, we were intrigued by the prospect that $3+1$-dimensional Lifshitz field theories with critical exponent, $z = 2$, support novel topological defects which are forbidden in relativistic field theories.

Our primary objective was to construct a $4+1$-dimensional, $z = 2$, Lifshitz, domain-wall brane model. In chapters 6 and 7 we established the existence of a topologically stable domain-wall brane. We considered the dynamics of a $4 + 1$-dimensional Lifshitz fermion propagating in this background. We found that $3 + 1$-dimensional Kaluza-Klein zero mode solutions do *not* exist when the four spatial dimensions are treated isotropically. In $z > 1$, Lifshitz domain-wall brane models, to recover a candidate $3 + 1$-dimensional fermion, it is necessary to break the 4-dimensional spatial rotational symmetry down to the subgroup of rotations which fix the extra-dimensional axis.

Again there are open questions for this model. For example we still need to dynamically localize a $3+1$-dimensional graviton. This will first require us to promote the $z = 2$ Lifshitz critical exponent to a more realistic $z = 4$ scenario. In the $z = 4$ formalism we need a Hořava Lifshitz generalization of the Randall-Sundrum type 2 warped metric scenario.

An interesting exercise would be to incorporate a renormalizable SU(2) Yang-Mills gauge theory in the bulk and an adjoint Higgs field which breaks SU(2) $\rightarrow$ U(1) on the domain-wall brane. This will hopefully clear the way for a simulation of the Dvali-Shifman mechanism in the limit of infinitesimal lattice spacing.

In chapters 7 and 8 we capitalized on one of the key differences between Lifshitz and relativistic field theories. Derrick's theorem forbids finite energy, stable, static solitons in $3 + 1$-dimensional, and higher dimensional, relativistic field theories. In Lifshitz theories the presence of higher spatial derivatives in the action, circumvents Derrick's theorem.

For $3 + 1$-dimensional Lifshitz scalar field theories with critical exponent $z = 2$, we constructed canonical examples of finite energy, stable, static Lifshitz defects. These included a non topological Lifshitz point-like defect, a Lifshitz hedgehog defect and a string-like soliton.

We worked primarily with Bogomolnyi-Prasad-Sommerfield (BPS) superpotential scenarios. These scenarios are the simplest from the point of view of the differential equations. However we established that our results hold independently of the BPS condition by exhibiting a non-BPS Lifshitz hedgehog.

If Hořava Lifshitz quantum point gravity is the ultraviolet regulator for quantum gravity, then it is natural to assume that the UV limit of quantum field theory also obeys Lifshitz scaling conventions. Under these circumstances the observation of cosmic relics which cannot arise in standard relativistic field theories, would strongly indicate Lorentz invariance breaks down at short distances.

10.3 Symmetry breaking and Weyl group invariance

The final topic in this thesis is a systematic approach to finding different embeddings of a subgroup H within a Lie group G. We approached this problem from the perspective of spontaneous symmetry breaking. Given a vacuum expectation value, in a particular representation for G, we developed a formalism which identifies a basis of weights for that representation, and then described all the states in the vacuum manifold G/H belonging to the linear vector space spanned by that basis. In the adjoint representation the weights are the roots and the basis elements are the simple roots which can be pulled back to define the Cartan subalgebra $C_G = \text{span}\{h_1, \ldots, h_l\}$. Our formalism identifies the points in the vacuum manifold G/H which belong to the vector space, C_G, spanned by the Cartan subalgebra. These points are related by a Weyl group symmetry. Given an adjoint Higgs vacuum expectation value, h, breaking $G \to H$, a full complement of vevs breaking G to different Cartan preserving embeddings of the subgroup H can be obtained through this method. We gave an explicit formula for recovering each vev.

We presented a more detailed treatment of the case where H stabilizes the highest weight of the lowest dimensional fundamental representation. This case admits the simplest formula for recovering all vevs breaking $G \to H$, when the Cartan subalgebra for H is a subspace of the Cartan subalgebra for G.

Furthermore we discussed more general cases when the Higgs field is not in the adjoint representation. These techniques offer an insight into current research questions such as model building in extra dimensions and low energy effective models for quantum chromodynamics.

10.4 Prospectus

There is a rich collection of "beyond the standard model" theories; field theories are perhaps too adaptable. However they are also remarkably successful at describing the known physics of this universe. Therefore they are a good place to start looking for new physics. We hope that, regardless of future developments, the tools and perspectives offered in this thesis will be valuable.

A

Root Systems Appendix

In this appendix we give information to complement the root systems picture in sections 3.9-3.11.

A.1 The highest root

Figure A.1 depicts the Dynkin diagrams for each of the crystallographic root systems.

For each Dynkin diagram the highest root in the corresponding crystallographic root system is the linear combination of the simple roots

$$\zeta^0 = \Sigma_i h_i \zeta^{(i)}, \tag{A.1}$$

where the coefficients of the simple root $\zeta^{(i)}$ is the label of the extended dynkin diagram node corresponding to $\zeta^{(i)}$.

A.2 E_8 root system

We include a more detailed version of Figures 3.8 and 3.9. The figures included in this appendix have explicit expressions for each root; this detail is difficult to find online and can be invaluable when doing explicit calculations. The roots are difficult to read in the printed version, however all details are clearly visible in the electronic submission. Again the lowering operators f_i corresponding to a Weyl group reflection in the simple root $\zeta^{(i)}$ are

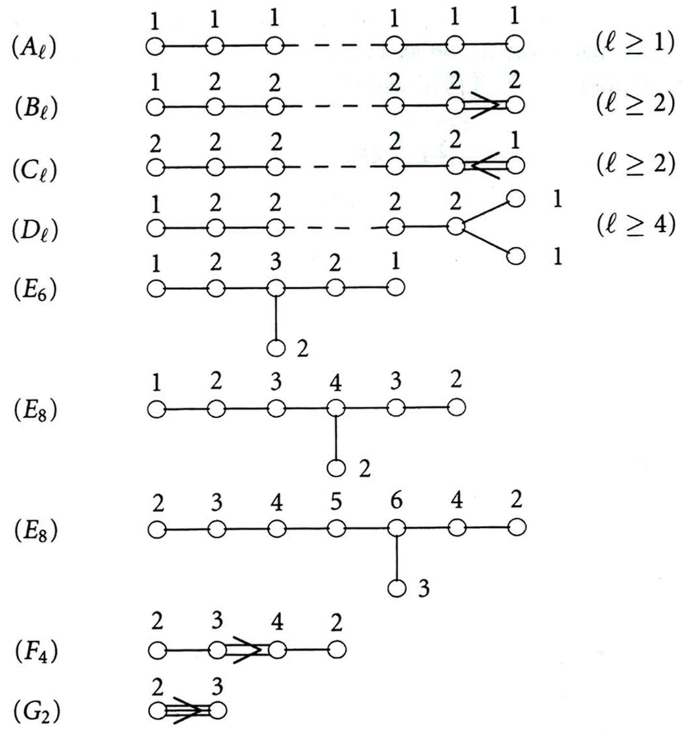


Figure A.1: This figure includes the Dynkin diagram for every viable crystallographic roots systems. For each diagram, the nodes correspond to the simple roots $\zeta^{(i)}$, of the corresponding crystallographic roots system. Each node is labeled by the coefficient, of the associated simple root, which appears in an expression for the highest root.

color coded. The key is included in Figure 3.9.

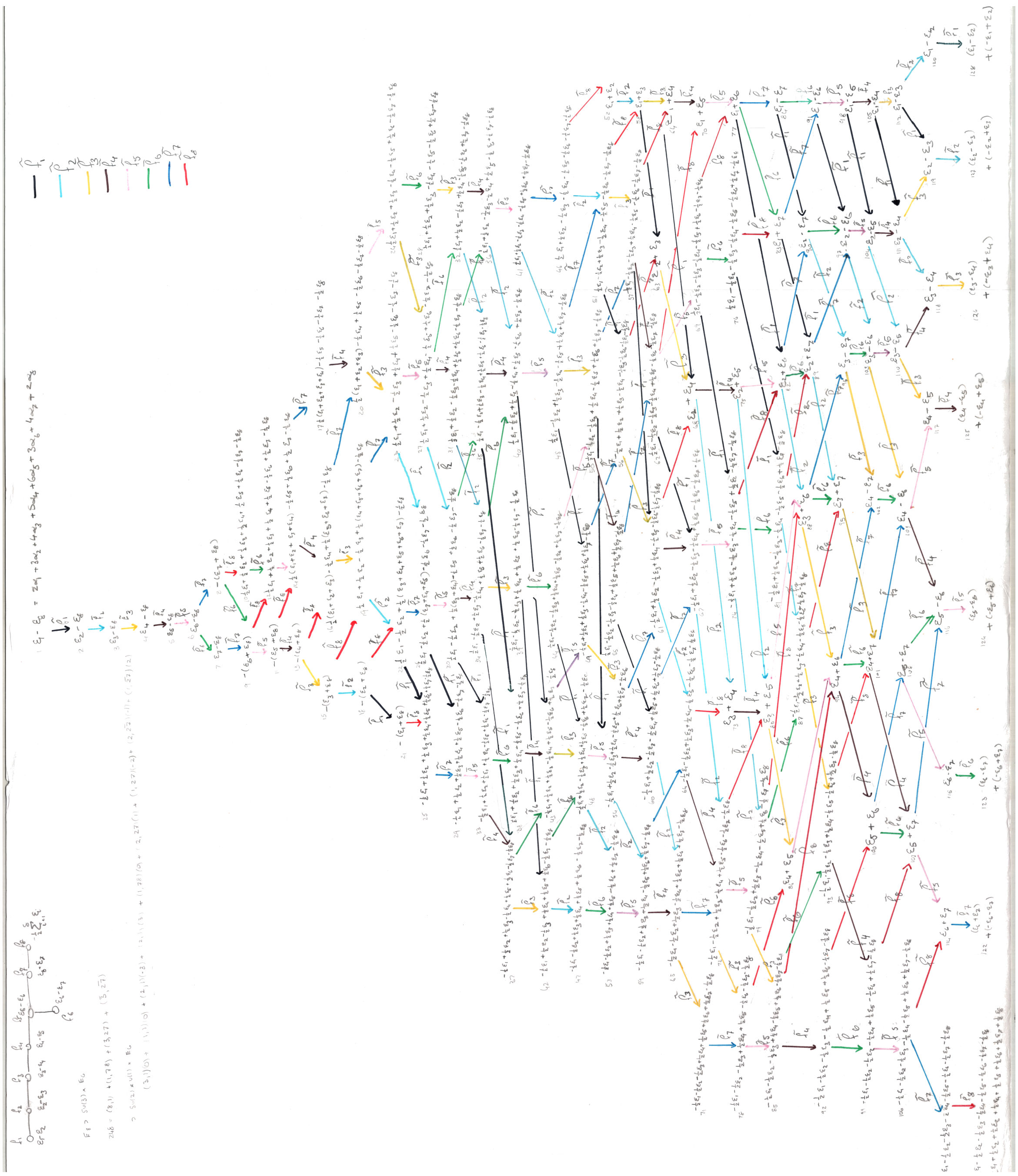


Figure A.2: The positive roots for E_8. This is a detailed description of Figure 3.8, including an explicit expression for each of the E_8 roots, belonging to the root system, $\Delta \subset \mathbb{R}^8$. Please see the electronic version of this thesis for a clearer image of the positive root diagram.

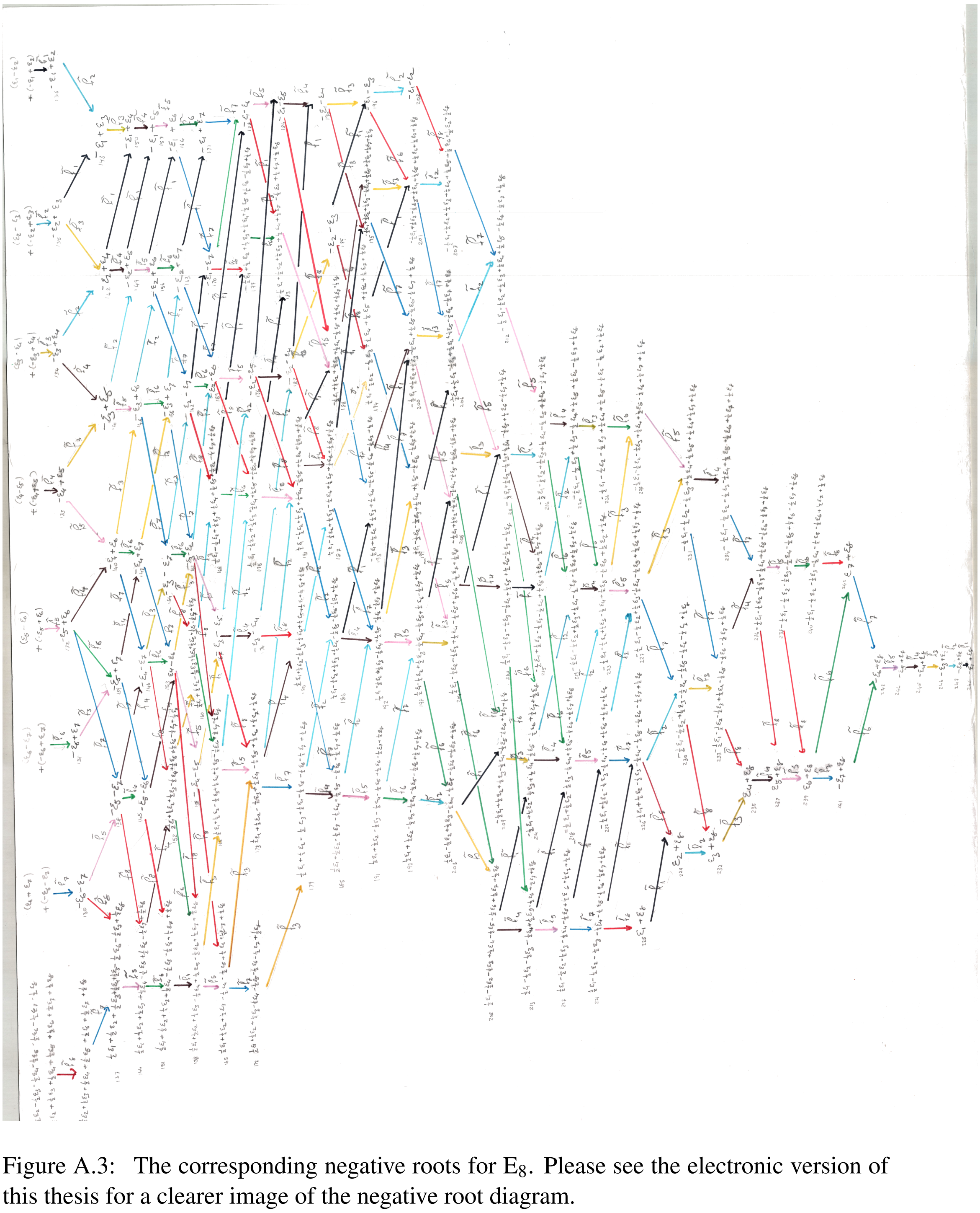


Figure A.3: The corresponding negative roots for E_8. Please see the electronic version of this thesis for a clearer image of the negative root diagram.

$\mathcal{B}$

Stability analysis Appendix

In this Appendix we give information to complement the stability analysis of the SO(10) $\to$ $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ domain-wall brane (4.34) from section 4.3.

B.1 Stability of the $SO(10) \to SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ model

Section 4.3 of chapter 4 deals with a sixth order potential for $\mathcal{X}$ and ϕ, (4.33), where the solution to the Euler-Lagrange equations creates a symmetry breaking pattern SO(10) $\to$ $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$. In this appendix we would like to use the techniques presented in chapter 5 to analyze the analytic solution, (4.34), for the Euler-Lagrange equations in section 4.3. Thereby we will have demonstrated a slice of the parameter space where SO(10) breaks to $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ stably. While most of the results for perturbations of the form $\left(\delta\bar{\mathcal{X}}, \delta\phi\right) = (\delta\mathcal{X} + N, \delta\phi)$ discussed in the context of the 4th order case can be easily generalized to the 6th order potential, there will be a slight twist because we can no longer use lemma 5.1.

One way of looking at lemma 5.1 is to make a co-ordinate substitution $s = \tanh z$ in equation (5.9). This will convert the equation (5.9) into a Riemann equation with regular singular points at $s = \pm 1$. The equation is an example of a hypergeometric differential equation for which the eigenspectrum is well known [NU88]. The associated eigenvectors are orthogonal polynomials where the inner product is given by

$$\langle u, l\rangle = \int_{-1}^{1} u(s)l(s)ds = \int_{-\infty}^{\infty} u(z)l(z)\,\mathrm{sech}^2 z dz. \qquad \text{(B.1)}$$

In the case of the 6th order equation if we make the same co-ordinate substitution $s = \tanh z$ in equation (B.2) we find that $s = \pm 1$ are no longer regular singular points and we are no longer able to easily construct a power series solution about these points. Hence we do not have polynomial solutions associated with eigenvalues which cause the power series to truncate at a finite degree and the previous ideas will not work under these circumstances.[1]

We fix this hole in our analysis. Let $\delta\mathcal{X}_\epsilon = \delta\mathcal{X}_{j,j+1} - \delta\mathcal{X}_{n,n+1}$ be nontrivial then for $m = 1$ we have:

$$\begin{aligned}((-\omega^2) - 1)\delta\mathcal{X}_\epsilon &= -\frac{\partial^2\delta\mathcal{X}_\epsilon}{\partial z^2} + \left(\lambda_1 A^2 + \frac{\lambda_2}{2}A^2 - \kappa v^2 + \lambda_6 A^2 v^2 + \frac{\lambda_7}{2}A^2 v^2 - 2\beta v^4\right)\mathrm{sech}^2 z\delta\mathcal{X}_\epsilon + \\ &\left(\frac{5\lambda_3}{48}A^4 + \frac{7\lambda_4}{24}A^4 + \frac{3\lambda_5}{4}A^4 - \lambda_6 v^2 A^2 - \frac{\lambda_7}{2}v^2 A^2 + \beta v^4\right)\mathrm{sech}^4 z\delta\mathcal{X}_\epsilon. \quad \text{(B.2)}\end{aligned}$$

This is just a 1-dimensional time independent Schrödinger equation with potential

$$\begin{aligned}U &= 1 + \Big(\lambda_1 A^2 + \frac{\lambda_2}{2}A^2 - \kappa v^2 + \lambda_6 A^2 v^2 + \frac{\lambda_7}{2}A^2 v^2 - 2\beta v^4)\mathrm{sech}^2 z + \\ &(\frac{5\lambda_3}{48}A^4 + \frac{7\lambda_4}{24}A^4 + \frac{3\lambda_5}{4}A^4 - \lambda_6 v^2 A^2 - \frac{\lambda_7}{2}v^2 A^2 + \beta v^4\Big)\mathrm{sech}^4 z. \qquad \text{(B.3)}\end{aligned}$$

If we impose the conditions featured in (4.36) then the potential will be positive definite and the only admissible solutions will be bound states with positive energy, hence $\omega^2 \leq 0$ for all modes, under these conditions.

[1] For further insight into lemma 5.1 see section 5.2 and in particular the references to the appendix of [Yag99] and [FG08]

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