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On Einstein–Born–Infeld conformally invariant theory

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Abstract A new Weyl Born Infeld model is presented. It takes as basis the formalism for the mathematical description of conformal gravity based on the local twistor geometry from Merkulov (Class Quantum Gravity 1:349–354, 1984). As in the pure Maxwell case, the electromagnetic field can be naturally incorporated into this scheme by a modification of the local twistor parallel transport law. The dynamical equations of the linear case, namely the Bach equations for gravity and the Maxwell equations for the electromagnetic field, can not be obtained by varying a single quantity-the modified twistor connection, unless a geometric condition on a particular function of the invariants is imposed. When that condition is met, the Weyl–Maxwel gravity system are contained in the obtained set. The invariance under standard dualities and other generalized maps are briefly discussed.

Contents

1 Introduction

We have known for some time that the model par excellence, a central element of theoretical physics, is the Born Infeld model [8]. It has properties that make it unique among the entire zoo of nonlinear models in the literature. Recently conformally invariant models of nonlinear electrodynamics have emerged such as Mod Max [9]. Many classical points and also considering higher spin approach have been analyzed and developed in some sense with such models [10].

But what is the connection of these models with a true Born Infeld conformal model? We know well that there are certain conditions that have to be met to consider this particular type of non-linear models, as we and other authors have remarked in many references over the last few years [4]. Consequently one of the main subjects of this work is to find the specific form of the nonlinear Lagrangian and to see the pro and cons in comparison with other similar proposals in the actual research. By the other hand, due that we use as a geometrical basis of our construction the local twistor connection from [6], we will test the equations of motion with respect to the Weyl-Maxwell case given in detail also in [6]. But why use the local twistor connection instead to define the functional action more directly as in the standard Born–Infeld case? The reason lies mainly in the difficulty of defining the determinant of the sum of tensors of different rank, such as the electromagnetic field and the Weyl tensor. Consequently due the spinorial language and the fact to have both ingredients electromagnetic field and the Weyl tensor in a simple object: the twistor local connection, seems to be natural to use the determinant of its strength field in the proposed Lagrangian.

The organization of the paper is as follows: in Sect. 2 details of the construction of a new conformal and determinant action model of the Born–Infeld type are presented starting from the structure of a twistor local connection based on the Ref. [6]. In Sect. 3 the dynamic equations are derived by means of the variation with respect to the twistor connection splitting them into a Yang Mills type equation plus a constraint involving geometric invariants in a non-linear way:

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the equations of Maxwell–Weyl linear model [6] are contained in the obtained set. In Sect. 4 the results from Sect. 2 discussed from the point of view of the dualities as a generalized mapping. Finally Sect. 5 is devoted to the concluding remarks and outlook, comparing our proposal with respect to other models in particular with the called Mod Max, where we will elucidate clearly the origin of such action.

2 Conformal Born Infeld from twistor formulation

As we briefly commented in the introduction and with more detail in a future work [7], the difficulty to define the action as in the standard Born–Infeld case lies not only in the different ranks of the tensors involved, but also in the fact that if we use for example the (trace free rank two) Bach tensor as the symmetrical part of the determinant, although it has all the required properties, it leads to a problem of interpretation of the equations of motion and higher derivatives. One possibility is based on the observation that local twistor invariant curvature H_{mn}, as discussed in [6] depends on the Weyl tensor C_{abcd} as we will see below.

By definition the modified twistor local connection is [6]

$$A_m \rightarrow \begin{pmatrix} 0 & \delta^A_M \delta^{B'}_{M'} \\ -i\, Q^{A'\,B}_{MM'} & 0 \end{pmatrix},$$

where the tensor $Q_{MM'\,A'\,B}$ is splitted in the same [6] notation as $Q_{(mn)} = P_{mn}$ ($P_{MM'A'\,B} \rightarrow \frac{1}{2}R_{mn} - \frac{1}{12}g_{mn}R$: Schouten tensor [11]) and $Q_{[mn]} = \kappa F_{mn}$, κ : parameter, constant in principle, that can be related with the absolute field b. It was shown in [6] that the local twistor curvature

$$H_{mn} = \nabla_m A_n \nabla_n A_m - [A_m, A_n] + S_{mn}$$

transforms under conformal rescalings as

$$\widetilde{H}_{mn} = G H_{mn} G^{-1}.$$

S_{mn} appears due the noncommutativity of the covariant derivative operators, namely:

$$S_{mn} = \begin{pmatrix} S^A_{mn\,B} & 0 \\ 0 & -\overline{S}^{B'}_{mnA'} \end{pmatrix},$$

where $S^A_{mn\,B} = \varepsilon_{M'N'}\psi^A_{BMN} - \delta^A_M P_{NBN'M'} - \delta^A_N P_{MBM'N'}$. Consequently as its is easily seen, introducing S_{mn}, A_m in H_{mn} (the modified strength of A_m) we obtain explicitly

$$H_{mn} = \begin{pmatrix} H^A_{mn\,B} & 0 \\ H_{mnA'B} & -\overline{H}^{B'}_{mnA'} \end{pmatrix},$$

where

$$\begin{aligned} H^A_{mn\,B} &= \varepsilon_{M'N'}\psi^A_{BMN} + \kappa\left(\delta^A_M F_{NN'M'B} - \delta^A_N F_{MM'N'B}\right), \\ H_{mnA'\,B} &= -i\left[\varepsilon_{M'N'}\nabla^D_{A'}\psi_{DMNB} + \varepsilon_{MN}\nabla^{D'}_B \overline{\psi}_{D'M'N'A'} - \kappa\right. \\ &\quad \left.(\nabla_{MM'}F_{NN'A'B} - \nabla_{NN''}F_{MM'A'B})\right]. \end{aligned}$$

Note that the twistor local curvature depends on the Weyl tensor $C_{abcd} \rightarrow \varepsilon_{AB}\varepsilon_{CD}\overline{\psi}_{A'B'C'D'} + \psi_{ABCD}\varepsilon'_{A'B'}\varepsilon_{C'D'}$.

2.1 Twistor-Born Infeld conformal model

Since the strength of the twistor connection H_{mn} contains the combination of C_{abcd} and F_{ab}, the direct possibility to obtain a consistent conformal version of the Born–Infeld action is

$$S = \varrho \int \sqrt{\left|\det H^{(-)}_{mn}\right|}\, d^4x,$$

whereas

$$H^{(-)}_{mn} = \begin{pmatrix} H^{(-)A}_{mnB} & 0 \\ H^{(-)}_{mnA'\,B} & -\overline{H}^{(-)B'}_{mnA'} \end{pmatrix},$$

and where the electromagnetic part is redefined as

$$\begin{aligned} H^{(-)A}_{mnB} &= \varepsilon_{M'N'}\psi^A_{BMN} + \kappa\left(\delta^A_M F^{(-)}_{NN'M'B} - \delta^A_N F^{(-)}_{MM'N'B}\right), \\ H^{(-)}_{mnA'\,B} &= -i\left[\varepsilon_{M'N'}\nabla^D_{A'}\psi_{DMNB} + \varepsilon_{MN}\nabla^{D'}_B \overline{\psi}_{D'M'N'A'} - \kappa\right. \\ &\quad \left.\left(\nabla_{MM'}F^{(-)}_{NN'A'B} - \nabla_{NN''}F^{(-)}_{MM'A'B}\right)\right]. \end{aligned}$$

with $F^{(-)}_{NN'M'B}$ the anti self-dual part of the electromagnetic field [18,19] also defined in that follows. The explicit calculation of the determinant is carried out respecting the dimensionality of space-time. Due the trace free properties of tensors C and F, we will have $\langle H^2\rangle^2$ and $\langle H^4\rangle$ in the explicit computation of the determinant [1–4].

$$\begin{aligned} \det|H_{mn}| &= \frac{g}{4!}\left(3\langle H^2\rangle^2 - 6\langle H^4\rangle\right), \\ \langle H^2\rangle &= \frac{1}{2}C_{abcd}C^{abcd} + 2\kappa^2 F^{(-)}_{ab}F^{(-)ab}, \\ \langle H^4\rangle &= \frac{1}{4}C_{abcd}C^{cdef}C_{efgh}C^{ghab} + \kappa^2 C_{abcd}C^{abgh}F^{(-)}_{gh}F^{(-)cd} \\ &\quad + 4\kappa^4 F^{(-)}_{ab}F^{(-)bc}F^{(-)}_{cd}F^{(-)da} \end{aligned}$$

$$\begin{aligned} S &= \frac{\varrho}{4}\int\sqrt{g}\sqrt{2\,\mathrm{Re}\left(\langle H^2\rangle^2 - 2\langle H^4\rangle\right)}\,d^4x \\ &= \frac{\varrho}{2}\int\sqrt{g}\left[\mathrm{Re}\left(\frac{1}{8}\left(\left(C_{abcd}C^{abcd}\right)^2\right.\right.\right. \\ &\quad \left. -2C_{abcd}C^{cdef}C_{efgh}C^{ghab}\right) \\ &\quad +\frac{\kappa^2}{2}\left(C_{abcd}C^{abcd}F^{(-)}_{ab}F^{(-)ab} - 2C_{abcd}C^{abgh}F^{(-)}_{gh}F^{(-)cd}\right) \\ &\quad \left.\left.-\kappa^4\left(F^{(-)*}_{ab}F^{(-)ab}\right)^2\right)\right]^{1/2} d^4x, \end{aligned} \quad (1)$$

where we can appreciate that the first order in expansion of the square root is the action studied by Bach [5] and $F^{(\pm)}_{ab} \equiv \frac{F_{ab} \pm i^* F_{ab}}{\sqrt{2}}$ (notation,[Israel,Penrose]): therefore in (1) we take the real part.

It is very important to remark at this stage that the choice $F^{(-)}_{ab}$ instead F_{ab} lies that the action in the limit of curvature goes to zero, it goes to the determinant of F: e.g. topological invariant or surface term. Consequently the existence in the flat limit of a consistent action implies in a simple way the introduction of a complex field of the type $F^{(-)}_{ab}$. In this way only the determinant action becomes in the planar limit conformal invariant and invariant under the Galliard and Zumino duality (in addition to the true MTW duality rotations) as we will see later.

$$S_{flat} \to -\frac{1}{4}\int \sqrt{\left|\left(\left(F^{ij}F_{ij}\right)^2 + \left({}^*F^{ij}F_{ij}\right)^2\right)\right|}d^4x,$$

then it is justified to replace $F^{(-)}_{ab}$ in H_{mn} standard from Ref. [6]. Having clarified that important point, the action

$$\begin{aligned} S &= \frac{\varrho}{4}\int \sqrt{g}\sqrt{2\,\mathrm{Re}\left(\langle H^2\rangle^2 - 2\langle H^4\rangle\right)}d^4x \\ &= \frac{\varrho}{2}\int \sqrt{g}\left[\frac{1}{8}\left(\left(C_{abcd}C^{abcd}\right)^2 - 2C_{abcd}C^{cdef}C_{efgh}C^{ghab}\right)\right. \\ &\quad + \frac{\kappa^2}{2}\left(C_{abcd}C^{abcd}F_{ab}F^{ab} - 2C_{abcd}C^{abgh}F_{gh}F^{cd}\right) - \kappa^4 \\ &\quad \left.\left(\left(F^{ij}F_{ij}\right)^2 + \left({}^*F^{ij}F_{ij}\right)^2\right)\right]^{1/2} d^4x, \end{aligned} \tag{2}$$

is the action of Born Infeld type conformally invariant required.

3 The dynamical equations

In Ref. [6] it was concluded that the metric variation and the variation with respect to the twistor connection (that leads to Yang–Mills type eqs,) coincide. However in our case that it is a determinant of a non linear conformal action, both variational procedures don't coincide unless a particular function of two invariants is covariantly constant. In this case, as we will explain in detail below, it is possible to arrive at the same conclusions as in [6] for Weyl–Maxwell gravity.

3.1 Field equations

From our action $S = \frac{\varrho}{4}\int \sqrt{g}\underbrace{\sqrt{2\,\mathrm{Re}\left(\langle H^2\rangle^2 - 2\langle H^4\rangle\right)}d^4x}_{L_{TBI}}$ where we define $\mathbb{H}^{mn} \equiv \frac{\partial L_{TBI}}{\partial H_{mn}}$ after variation with respect to the twistor connection A_m the field equations are obtained:

$$\nabla_m \mathbb{H}^{mn} + \left[A_m, \mathbb{H}^{mn}\right] = 0 \tag{3}$$

We need to note that if the Lagrangian contains a current term A_m−dependent, a $\frac{\partial L_{TBI}}{\partial A_m}$ part must to be included into the field equations (3). Also it is interesting to see that explicitly that $\mathbb{H}^{mn} = \frac{4(H_{ab}H^{ab} + H^*_{ab}H^{ab})H^{mn}}{\sqrt{2\,\mathrm{Re}\left(\langle H^2\rangle^2 - 2\langle H^4\rangle\right)}} \equiv f(I_1, I_2)\,H^{mn}$(with ${}^*H^{ab} = \frac{1}{2}\varepsilon^{abcd}H_{cd}$ and: $I_1 = H_{ab}H^{ab}$, $I_2 = H^*_{ab}H^{ab}$) it is factorizable due conformal properties (which does not happen in Born Infeld standard model), being able to write (3) as

$$\nabla_m\left(f(I_1, I_2)\,H^{mn}\right) + \left[A_m, f(I_1, I_2)\,H^{mn}\right] = 0 \tag{4}$$

giving two solutions: one trivial with $f(I_1, I_2) = 0$ and evidently one fulfilling

$$H^{mn}\nabla_m f(I_1, I_2) = 0 \tag{5}$$

plus a Yang–Mills type

$$\nabla^m H_{mn} + \left[A^m, H_{mn}\right] = i\begin{pmatrix} 0 & 0 \\ U_{MM'A'B} + \overline{U}_{M'MBA'} & 0 \end{pmatrix} = 0 \tag{6}$$

$$\begin{aligned} U_{AB'A'B} &= \nabla^M_{A'}\nabla^N_{B'}\psi_{MNAB} - P^{MN}_{A'B'}\psi_{MNAB} \\ &+ \frac{\kappa^2}{2}\left(F_{AA'D'D}F_{BB'C'C}\varepsilon^{CD}\varepsilon^{C'D'}\right. \\ &\left.-\frac{1}{4}F_{MNM'N'}F^{MNM'N'}\varepsilon_{AB}\varepsilon_{A'B'}\right) \\ &- \kappa\left[\left(\nabla^R_{S'}\nabla^{S'}_R + \frac{R}{3}\right)F_{AB'A'B} - \psi_{ACDB}F_{EB'A'F}\varepsilon^{CE}\varepsilon^{DF}\right], \end{aligned} \tag{7}$$

consequently from (6) we have $U_{MM'A'B} + \overline{U}_{M'MBA'} = 0$ that implies precisely the gravity Weyl-Maxwell system namely

$$B_{ab} = 8\pi\kappa^2 T_{ab}. \tag{8}$$

We must to note that from (7) the terms corresponding to the Bach tensor [5] and the energy-momentum tensor of the electromagnetic field, namely $T_{ab} \to -\frac{1}{2\pi}\left(F_{ad}F^d_b - \frac{g_{ab}}{4}F_{cd}F^{cd}\right)$ are clearly identified where κ^2 identified with the Einstein constant. And the third line from (7) we have

$$\nabla^m\nabla_m F_{ab} = C_{abcd}F^{cd} - \frac{1}{3}RF_{ab} \tag{9}$$

supplemented with $\nabla^a F_{ab} = 0$ and $\nabla_{[c}F_{ab]}$ forming the complete set of field equations: (5), (8) and (9).

As we clearly see from (2), taking this Lagrangian directly and carrying out the usual metric variation procedure (alternative to the variation with respect to the connection), the Maxwell–Weyl equations are not obtained, unless the trivial condition $f(I_1, I_2) = 0$ or condition (5) is fulfilled.

4 Generalized duality and conformal properties

Now let's analyze the result of the previous section from the fundamental mappings or dualities. These mappings or generalized dualities were studied and developed in some of our previous works, in particular [15]. They are based on the observation that there is always a mapping function (tensor in general) characteristic of each model or theory in particular, which connects geometrically defined fields e.g. $N_{\alpha\beta\ldots}$ with the conjugates e.g. $\mathbb{N} \to \frac{\partial L}{\partial N_{\alpha\beta\ldots}}$. These generalized tensor mappings are performed by means of a matrix that in the Maxwellian case, as in the Born Infeld conformal case of our proposal, is proportional to the identity in sharp contrast with the standard Born–Infeld (BI) model, as we will see in detail below.

The true duality is the one that relates the dynamic fields that are defined from the functional action, and the geometric fields related to intrinsic properties of the space-time as fundamental differentiable manifold. In previous references we have seen that, in general, this symmetry is expressed as

$$\underbrace{\begin{pmatrix} \frac{\partial L}{\partial F} \equiv \mathbb{F} \\ * \frac{\partial L}{\partial F} \equiv^{*} \mathbb{F} \end{pmatrix}}_{dynamic\ fields} = \mathbb{M}\left(S, P\right) \quad \underbrace{\begin{pmatrix} F \\ {}^{*}F \end{pmatrix}}_{geometric\ fields}$$

where $\mathbb{M}\left(S, P\right)$ will be an operator with matrix representation that in Maxwell's case is proportional to the identity. As an example, the Born–Infeld action, and the model in general, plays with 2 types of fields

$$\underbrace{\begin{pmatrix} \mathbb{F} \\ {}^{*}\mathbb{F} \end{pmatrix}}_{dynamic\ fields} = \frac{1}{R}\left(\sigma_0 + i P \sigma_2\right) \quad \underbrace{\begin{pmatrix} F \\ {}^{*}F \end{pmatrix}}_{geometric\ fields}$$

with $\sigma_0 = I_2$ and σ_2 the corresponding Pauli matrix (precisely $i\sigma_2$ is the charge conjugator). Finally, it should be noted that more general admissible transformations consider as a typical example an action $Sp(2R)$ to the general electromagnetic field corresponding schematically as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} F \\ \widetilde{F} \end{pmatrix} \to \begin{pmatrix} aF + b\widetilde{F} \\ cF + d\widetilde{F} \end{pmatrix}$ for example (this type of transformations were studied in the context of symmetries of the BI model in [15] and applying it to dilaton and axion in [16].

Duality rotation type transformations have been known for some time, considering electrodynamic models in the context of general relativity taking into account Rainich conditions [12]. These operate as a rotation in the space of the fields by means of an operator $e^{\alpha *} = \cos\alpha + \sin\alpha\, *$. Below is a table describing the invariance

Quantity	General proper Lorentz transf.	Duality rotation
$F^{ij}F_{ij}$ and $\left({}^{*}F^{ij}F_{ij}\right)^2$	Unchanged	Transformed
Components of T_{ij}^{EM}	Transformed	Unchanged
Combination $\left(F^{ij}F_{ij}\right)^2 + \left({}^{*}F^{ij}F_{ij}\right)^2$	Unchanged	Unchanged

Consequently the proposed modmax action would not respect duality invariance over canonical fields, because it contains a term $\sim F^{ij}F_{ij}$ to give the Maxwellian limit. However, if one considers the Gaillard–Zumino dualities [13,14] (which are mixed) it would fulfill them.

5 Concluding remarks

In this article we have presented a new non-linear conformal action, constructed in the same way as in the standard Born Infeld model, which is defined by the determinant of the field strength of the local twistor connection, of Ref. [6]. This new action meets similar requirements as in the case of Born–Infeld, however due that the local twistor connection contains fundamentally the Weyl and electromagnetic tensors in a simple way, the square root of the determinant of the strength of the twistor connection bring us a good description of a nonlinear gravitational conformal model coupled with nonlinear electrodynamics.

As a second important point, also motivated by the result of Ref. [6], we test the variational method through the analysis and explicit calculation of the field equations from our Weyl–Born–Infeld model. As a conclusion we were able to demonstrate that the results of the linear case are evidently contained in the solutions of our nonlinear case providing that a function of two invariants of the field strength of the local twistor connection (of a tensor character in general), either null or covariantly constant.

Regarding dualities: our model is invariant to all theoretically known dualities [12] plus those defined in contexts of string or membrane theory also [13,14] (which mix or combine some fields in a non-trivial way). With respect to the generalized mappings of the section, the representative matrix of the mapping coming from the field equations of our model is proportional to the identity, as in the case of gravitational Maxwell–Weyl model given in [6], as expected since topologically both cases would be functionally connected.

As next steps of the treatment of conformal symmetries in this theoretical context, problems of continuum mechanics in the cosmological context [17] (caustics, singularities and the structure of the universe on a large scale) and the possibility of the extension of this model in SUGRA will be developed.

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Declarations

Conflict of interest There are not competing interests.

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