Symmetries in $\mathcal{N} = 4$ Supergravities

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Chapter 1

Introduction

To describe the world around us at a fundamental level it is necessary to know the constituents of nature and the rules of the game the constituents play. The length scales at which nowadays experiments can be performed are around 10^{-18} meters, which requires energies around 1 tera electron Volt (TeV), corresponding to the mass of about 1000 protons. The discipline within physics that tries to describe phenomena, both the constituents and the rules of the game, at these length scales at a fundamental level is therefore called high-energy physics.

One of the main achievements of high-energy physics is the Standard Model, which was developed by Salam, Glashow and Weinberg between 1970 and 1973, for which they were rewarded with a Noble prize. The Standard Model describes the elementary particles and the three forces they can exert on each other; the weak force, the electromagnetic force and the strong force. An elementary particle feels the force from a second elementary particle by the interchange of a third particle; the forces are carried by the 'force mediators'. The agreement of the predictions of the Standard Model with experiments has encouraged the physics community to think that the Standard Model is a major step in the right direction towards a theory of everything. However, there are fundamental objections to the standard model.

As one of the important objections, we mention that the most familiar force is absent in the Standard Model; the force that keeps us with both feet on the Earth, gravity, is not incorporated in the Standard Model. It has been thought for a long time that gravity can be incorporated in the Standard Model, but that we just do not know how. Nowadays a large number of physicists has the opinion that new ideas are needed and that a theory that incorporates both the physics of the Standard Model and gravity does not resemble the Standard Model at all.

One such a new idea is superstring theory (the word 'new' is not quite correct since the birth of superstring theory dates back to the 1970's), which states that the constituents of nature can be described by one-dimensional objects, called strings. One can debate whether superstring theory is a step in the right direction towards a theory of everything but this debate we will avoid. In the opinion of the author, superstring theory already taught us about many interesting aspects of theories in general and if in the end string theory is wrong, we are at least a lot wiser.

Superstring theory only admits a consistent quantum mechanical formulation in ten space-time dimensions and it turns out that five different consistent formulations exist. All five superstring theories predict a whole tower of states, where the lowest mass states correspond to massless particles and the first states of nonzero mass have masses around the Planck mass $\sim 10^{19}\text{GeV} \sim 20\mu\text{g}$. Due to the huge mass of the massive states, the massive states are certainly beyond the reach of present-day accelerators and thus one is mainly interested in the physics of the massless states. The massless states can be described at a classical level by a supergravity theory.

A supergravity theory is a classical field theory that has a symmetry called supersymmetry and incorporates gravity. Supersymmetry is a symmetry that relates the bosonic degrees of freedom to the fermionic degrees of freedom. Supersymmetric theories have certain appealing properties and are used in various places. A disadvantage of supersymmetric theories is that nature does not seem to be supersymmetric. However, a theory can still be supersymmetric, although the solutions do not seem to be supersymmetric.

Supergravity theories can be formulated in dimensions less than or equal to eleven. Many supergravities that live in space-time dimensions less than ten can be obtained from a ten-dimensional supergravity by writing ten-dimensional space-time as a direct-product space $X \times Y$ where X is four-dimensional and Y is six-dimensional, compact and of small size[1]. The procedure to obtain lower-dimensional theories from higher-dimensional theories is called dimensional reduction. If string theory is to describe the world around us, a thorough understanding of dimensional reductions is indispensable, since the observable world surely is not ten-dimensional.

Supergravity theories are often given a number $\mathcal{N}$, as in the title of this thesis. The number $\mathcal{N}$ denotes the amount of supersymmetry and it is a nonnegative integer. In four space-time dimensions $\mathcal{N}$ can take values from 1 to 8 and in ten space-time dimensions it is 1 or 2. This thesis is about four-dimensional $\mathcal{N} = 4$ supergravities, of which some can be obtained from a dimensional reduction of a ten-dimensional $\mathcal{N} = 1$ supergravity. However, not all $\mathcal{N} = 4$ supergravities can be obtained from a dimensional reduction; in the construction in four dimensions of $\mathcal{N} = 4$ supergravity it is possible to include so-called $SU(1,1)$-angles. A higher-dimensional origin of these $SU(1,1)$-angles has up to now not been given.

There is no clear motivation why one should study $\mathcal{N} = 4$ supergravities in particular. This thesis one should see more as a small piece of a huge jigsaw puzzle than

[1] Since experiments have not seen the extra dimensions, their size needs to be smaller than 10^{-18} meters. String theory predicts even sizes of around 10^{-35} meters.

as a work on one puzzle on its own. Either string theory is wrong or it is right, but we cannot choose between wrong and right if we have not studied the properties of string theory in great detail. The investigation of supergravities is just one part of finding the solution of the puzzle and within that part the $\mathcal{N} = 4$ supergravities are a small piece. However, this small piece is an interesting piece. The $\mathcal{N} = 8$ supergravities do not allow for much freedom and the theory is so to say fixed; there is no multiplet of fields that can be coupled to the fields of $\mathcal{N} = 8$ supergravity. In $\mathcal{N} = 1, 2$ supergravities there is on the contrary much freedom in coupling various fields to the supergravity fields and the most general version of the theory involves a lot of unknown functions. The $\mathcal{N} = 4$ supergravity can be coupled to sets of other fields, but there exists only one such set and the way different copies of this set can be coupled to the supergravity fields is restricted. Hence $\mathcal{N} = 4$ supergravity contains some freedom, which a theoretical physicist can play with, but not too much freedom, in which a theoretical physicist would drown.

To investigate a theory one can try to solve the equations of motion and investigate the solutions. This programme is not feasible in most cases. Hence we have to resort to other methods, some of which use the concept of symmetry. One can gather much information about the solutions of a theory by investigating the symmetries of the equations of motion and of the Lagrangian. Supersymmetric field theories have a large group of symmetries; not only are there the symmetries that relate fermions and bosons, but also symmetries rotating the fermions into fermions and bosons into bosons.

But not only in supergravities symmetry is an important tool; already in elementary physics lectures symmetry arguments are used to simplify solving problems. More importantly, the interactions between particles in the Standard Model are manifestations of symmetries[2]. A wave function of an electron can be given an arbitrary global phase and hence a quantum theory of electrons has a global $U(1)$-symmetry. Promoting the symmetry to a local symmetry requires the the incorporation of a massless vector field to which the electron couples and electromagnetic interactions are born.

Promoting a global symmetry to a local symmetry is called gauging and gauged supergravities play an important role in understanding supergravities and string theory. In the process of gauging a supergravity supersymmetry is broken; to restore supersymmetry one has to modify the Lagrangian and the supersymmetry transformation rules for the fields. The modified Lagrangian contains a scalar potential in general. The gauging of $\mathcal{N} = 4$ supergravities is not trivial; the symmetry groups that can be gauged are not known explicitly and the scalar potential that arises due to the gauging is complicated and finding stationary points is a difficult task.

One of the main ideas of this thesis is that many aspects of $\mathcal{N} = 4$ supergravity can be understood from symmetry principles. Therefore the discussion is presented

[2] In this thesis we mean by a symmetry always a continuous symmetry, unless otherwise stated.

with an emphasis on symmetries and the use of the mathematics of symmetries called Lie group theory. For the reader not familiar with Lie groups and Lie algebras this thesis contains a detailed appendix on Lie groups and Lie algebras, which - we hope - contains all definitions and theorems that are needed to understand the text.

A part of the philosophy behind this thesis is to present the details and the precise formulas, or to refer to the literature in which the precise details can be found. This makes the discussion sometimes lengthy and boring for the reader who wants a fancy story. But the author experienced that fancy stories bring little details, which are so crucial for starting Ph.D.-students. This motivated to write a text in which precise definitions and formulas are given and that can be a guide for starting Ph.D.-students. Therefore we apologize in advance for referring in a few cases to overview- and review-papers instead of to the original literature.

Before we finish the introduction and present the plan of the thesis, we want to pose a few questions, which we hope to answer at the end of the thesis. The questions are not the research questions but serve as 'things to keep in mind' while reading.

-1- What is the role played by the $SU(1,1)$-angles and is there a higher-dimensional origin?

-2- What symmetry groups can be gauged in $\mathcal{N}=4$ supergravity?

-3- Why do the scalars in supergravity theories often parameterize a coset G/K with K the maximal compact subgroup of G?

-4- What is the relation between the isometries of the compact manifold over which a higher-dimensional supergravity is reduced and the symmetries of the lower-dimensional supergravity?

-5- What is the fate of string theories and supergravities?

In the last chapter we come back to these questions and try to formulate an answer.

The plan of the thesis is as follows. In chapter 2 we introduce indispensable tools to understand gravity, supersymmetry and supergravity at a level that is used in the thesis. In chapter 3 we develop the general theory of Kaluza–Klein reductions. Kaluza–Klein reductions are a particular kind of dimensional reductions and are useful in understanding the relation between ten-dimensional $\mathcal{N}=1$ and four-dimensional $\mathcal{N}=4$ supergravities. In chapter 4 we present the $\mathcal{N}=4$ supergravity theory and discuss the role of the $SU(1,1)$-angles and the potential of the gauged theory. In chapter 5 we conclude and try to answer the five questions posed above. In appendix A we explain our conventions and give a few useful formulas, in appendix B we present a detailed account on Lie group and Lie algebra theory and in appendix C we explain the concept of spinors and fermions and present our conventions on spinors. The work of the author is mainly discussed in chapters 3 and 4. In appendix D we give the published papers.

Chapter 2

Gravity and Supergravity

Gravity is a force that is familiar to everyone and it was also the first force that was described in a quantitatively accurate way by Newton in the end of the seventeenth century. From the Newton era until the beginning of the twentieth century Newton's theory was accepted as the theory that describes gravity.

It was Einstein who came up with a totally new view on space and time, a view that resulted in a completely new explanation for gravity. According to Einstein's theory of General Relativity space-time is a dynamical object and is curved in presence of matter. Space-time in turn influences through its curvature the trajectory that matter follows. Experiments have verified that General Relativity describes gravitational interactions more accurately than Newton's theory and nowadays General Relativity has taken the prominent place of Newton's theory to be the theory of gravity.

In contrast to gravity supersymmetry is less familiar and has not even been observed by an experiment. But supersymmetry is attractive from a theoretical point of view; it can solve some problems encountered in the Standard Model. Supersymmetry has also been an indispensable tool in constructing consistent string theories, which are viable candidates for the so-called theories of everything but are little supported by experiments[1]. In combining supersymmetry and General Relativity one obtains supergravity. It turns out that the supergravity theories are classical field theories that describe the low-energy limit of string theory.

In section 2.1 some elementary differential geometry and General Relativity are discussed. This section is more a kind of summary of concepts and formulas that are used later than a textbook-like introduction. In section 2.2 we discuss the concept of supersymmetry. In section 2.3 the chapter is concluded with a discussion on super-

[1]Some readers might prefer to say that there is no experimental evidence for string theory, but the author shares the opinion of E. Witten, who remarked that string theory is the first theory to predict the existence of gravity [1], which has been confirmed to exist experimentally.

gravities and their properties. The reader who is not familiar with basic elements of General Relativity or elementary concepts of differential geometry is referred to [2–9] for good introductions, reviews and complete lists of references.

2.1 Gravity

2.1.1 Structures on Manifolds

One of the starting points of General Relativity is the equivalence principle, stating that [4]: "There is no local experiment that can distinguish between a nonrotating free fall in a gravitational field and a uniform motion in space in the absence of gravity". It follows that physics can locally be described in a frame that is not influenced by gravity. Since the trajectory of matter is influenced by gravity through the curving of space-time, the absence of gravity means that space-time is not curved. The equivalence principle can thus be translated to the statement that locally space-time is flat, that is, it has no curvature. To describe curvature we will introduce the Riemann tensor in equation 2.1.9. The Riemann tensor is one of the main ingredients for the Einstein equation 2.1.13, which describes the interaction between space-time and matter. The Einstein equation and the action from which the Einstein equation can be derived are of major importance for both gravity and supergravity.

At every point p on an n-dimensional manifold[2] $\mathcal{M}$ there is a vector space, called the tangent space at p and denoted $T\mathcal{M}_p$. The tangent space at p describes the tangent vectors to all curves through p. All tangent spaces together are called the tangent bundle $T\mathcal{M}$. An element of $T\mathcal{M}$ assigns to each $p \in \mathcal{M}$ an element of $T\mathcal{M}_p$ and is called a vector field.

At every point $p \in \mathcal{M}$ there exist basis vectors $e^1_\mu, \ldots, e^n_\mu$, where $n = \dim \mathcal{M}$, that span the tangent space $T\mathcal{M}_p$ at p. If the basis vectors e^i_μ are smoothly varying, one speaks of a frame bundle[3]; locally the tangent bundle $T\mathcal{M}$ looks like $\mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$. In a patch $U \subset \mathcal{M}$ we can use local coordinates x^μ and for vector fields the standard basis ∂_μ. The set $\{\partial_1|_p, \ldots, \partial_n|_p\}$ is called a coordinate frame at $p \in \mathcal{M}$. With $T\mathcal{M}_p$ we associate the dual vector space $T\mathcal{M}^*_p$.

When space-time is curved, the metric g, seen as a map $g : T\mathcal{M} \to T\mathcal{M}^*$, is in general not diagonal, that is, in general we have $g(\partial_\mu, \partial_\nu) \neq \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric (see Appendix A). However, locally there exist vector fields $t_a = t^\mu_a(x)\partial_\mu$ satisfying $g(t_a, t_b) = \eta_{ab}$ and the set of vectors t_a is called a (local) orthonormal frame. Since both $\{t_a\}$ and $\{\partial_\mu\}$ are frames we have $t^\mu_a(x) \in GL(n; \mathbb{R})$ for all x.

We introduce the dual of the orthonormal frame $\{t_a\}$, consisting of 1-forms $\sigma^a = \sigma^a_\mu(x)\mathrm{d}x^\mu$ satisfying $\sigma^a(t_b) = \delta^a_b$. Now we define a covariant derivative, or connection,

[2] We assume the manifold admits a metric with Lorentzian signature -++...+.

[3] In this context a frame means a set of basis vectors attached to a point p on $\mathcal{M}$.

∇, which maps a vector field $v = v^a(x)t_a(x) \in T\mathcal{M}$ to a vector-valued one-form - that is, an element of $T\mathcal{M} \otimes T\mathcal{M}^*$ - in such a way that ∇ satisfies the Leibniz rule and that ∇ is linear: if f is a function on $\mathcal{M}$ and $v, w \in T\mathcal{M}$ then

$$\nabla(fv) = df \otimes v + f\nabla v\,, \quad \nabla(v+w) = \nabla v + \nabla w\,. \tag{2.1.1}$$

Hence ∇ is fully determined by its action on the basis vectors t_a. We define the connection coefficients ${\omega_a}^c{}_b$ by

$$\nabla t_a = {\omega_b}^c{}_a t_c \otimes \sigma^b \;\Rightarrow\; \nabla t_a(t_b) = {\omega_b}^c{}_a t_c\,, \tag{2.1.2}$$

where the Einstein convention for repeated indices is used (see appendix A). For convenience one defines the connection 1-forms by ${\omega^a}_b = {\omega_c}^a{}_b \sigma^c$; the 1-forms ${\omega^a}_b$ are called spin connections. Similarly the Christoffel connections[4] $\Gamma_{\mu\nu}^{\ \lambda}$ are defined by

$$\nabla(\partial_\mu) = \Gamma_{\mu\nu}^{\ \lambda}\partial_\lambda \otimes \mathrm{d}x^\nu\,. \tag{2.1.3}$$

The Lie bracket $[X, Y]$ of two vector fields X, Y is defined by

$$[X,Y] = Z = Z^\mu\partial_\mu, \quad Z^\mu = X^\nu\partial_\nu Y^\mu - Y^\nu\partial_\nu X^\mu\,. \tag{2.1.4}$$

Combining the equations 2.1.2- 2.1.4 gives

$$\nabla t_b(t_a) - \nabla t_a(t_b) = [t_a, t_b] + t_b^\mu t_a^\nu(\Gamma_{\mu\nu}^{\ \rho} - \Gamma_{\nu\mu}^{\ \rho})\partial_\rho\,, \tag{2.1.5}$$

where the last term gives rise to the torsion tensor τ;

$$\tau(\partial_\mu, \partial_\nu) = (\Gamma_{\mu\nu}^{\ \rho} - \Gamma_{\nu\mu}^{\ \rho})\partial_\rho\,, \tag{2.1.6}$$

which is an example of a $(2,1)$-tensor (see appendix A). By using the product rule for differentiation one can show that

$$\mathrm{d}\sigma^a(t_b, t_c) = -\sigma^a([t_b, t_c]) \Rightarrow \mathrm{d}\sigma^a = -{\omega_b}^a{}_c\sigma^b \wedge \sigma^c + \tau^a\,, \tag{2.1.7}$$

where τ^a is a component of the torsion tensor with respect to the vector basis $\{t_a\}$.

The extension of the spin connection to vector valued forms $\alpha = t_a \otimes \alpha^a$ (see appendix A) is given by

$$\nabla\alpha = \nabla t_a \hat{\otimes} \alpha^a + t_a \otimes \mathrm{d}\alpha^a\,, \tag{2.1.8}$$

where the symbol $\hat{\otimes}$ is defined as follows: for a vector valued p-form $\beta = \beta_a \otimes \omega^a$, where ω^a is a p-form, and a q-form γ we define $\beta\hat{\otimes}\gamma = \beta^a \otimes (\omega^a \wedge \gamma)$. From this one can show that $\nabla\nabla$ is a tensor:

$$\nabla\nabla(t_a) = t_c {R^c}_a, \quad {R^c}_a = \mathrm{d}{\omega^c}_a + {\omega^c}_b \wedge {\omega^b}_a\,, \tag{2.1.9}$$

[4] One can argue whether the name Christoffel connections is appropriate here; we take a pragmatic approach and simply take this as the definition of the Christoffel connections and use it throughout.

and the tensor $R^a{}_b$ is called the Riemann tensor.

One can show that there exists just one spin connection that is torsionless and metric compatible[5]. Metric compatibility means that the equation

$$\mathrm{d}\eta_{ab}(t_c) = g(\nabla t_a, t_b)(t_c) + g(t_a, \nabla t_b)(t_c) = 0\,, \tag{2.1.10}$$

is satisfied. The requirement 2.1.10 results in $\omega_{ab} = -\omega_{ba}$ where $\omega_{ab} = \eta_{ac}\omega^c{}_b$. The connection 1-forms are thus elements of the Lie algebra $\mathfrak{so}(1, n-1)$. But two different frames at a point $p \in \mathcal{M}$ are related by an $\mathfrak{so}(1, n-1)$ transformation, i.e. a local Lorentz transformation, and so the result might not be totally unexpected.

In General Relativity one uses the word *vielbeins* for the dual 1-forms: $e^a_\mu \mathrm{d}x^\mu = \sigma^a$. Since both the set $\{\partial_\mu\}$ and $\{t_a\}$ are frames, the map between the two is invertible and hence the vielbein e^a_μ is a nonsingular 'matrix' with inverse E^μ_a, satisfying $E^\mu_a e^b_\mu = \delta^b_a$ and $E^\mu_a e^a_\nu = \delta^\nu_\mu$. One sees that the inverse vielbeins E^μ_a are the components of the vectors t_a.

For most applications in General Relativity one sets the torsion to zero, which is consistent with local Lorentz symmetry - no local Lorentz transformation can be used to map $\tau = 0$ to a nonzero τ'. Hence we obtain

$$\mathrm{d}e^a = -\omega^a{}_b e^b\,, \tag{2.1.11}$$

which can be solved by introducing the objects of anholonomicity $\Omega^a{}_{bc}$ through $\mathrm{d}e^a = \frac{1}{2}\Omega^a{}_{bc} e^b \wedge e^c$. One obtains

$$\omega_{abc} = \tfrac{1}{2}\left(-\Omega_{abc} + \Omega_{cab} + \Omega_{bca}\right)\,, \tag{2.1.12}$$

where the objects with indices down are obtained by contracting with η_{ab}. From the spin connections the Riemann tensor can be calculated with the use of equation 2.1.9.

The Riemann tensor is the input for the Einstein equations that govern the interaction between space-time and matter. The Einstein equations are given by

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}\,, \tag{2.1.13}$$

where $R_{\mu\nu} = R_{a\mu}{}^a{}_\nu$ is the Ricci tensor, $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar, κ is the gravitational coupling constant and $T_{\mu\nu}$ is the energy-momentum tensor of the matter interacting with gravity. The left-hand side of equation 2.1.13 is purely determined by the geometry of space-time and the right-hand side is determined by the matter distribution. The Einstein equations thus tell us that the way matter is distributed in space-time determines the structure of space-time.

[5]We show this below by giving an expression for a torsionless and metric compatible spin connection.

2.1.2 Gravity as a Gauge Theory

There is another sometimes more convenient point of view on the role of the connection 1-forms $\omega^a{}_b$ and the Christoffel connections $\Gamma^{\rho}_{\mu\nu}$; one can view gravity as a kind of gauge theory. The symmetry that is uplifted to a local symmetry is Lorentz symmetry. In 1961 Kibble [10] presented a nice derivation of the Einstein equations starting from local Lorentz invariance.

In the gauge theory language the spin connection is the gauge field and the Riemann tensor is the field strength. The similarity between gauge theories and General Relativity ceases to hold if equations of motion come into play; the Einstein equations 2.1.13 are not of the form $\partial_\mu F^{\mu\nu} = j^\nu$ for some field strength $F^{\mu\nu}$ and some current j^ν.

Fields in physics are tensors and spinors. The tensors have components with respect to the coordinate frame ∂_μ and the orthonormal frame t_a, and their duals $\mathrm{d}x^\mu$, σ^a respectively. The components with respect to the orthonormal frame are inert under a change of coordinate frame, i.e. a general coordinate transformation, but rotated into each other by local Lorentz transformations. The converse holds for the components with respect to the coordinate frame.

Both general coordinate transformations and local Lorentz transformations are groups, denoted $GCT(\mathcal{M})$ and $SO(1, n-1)$ respectively. The vielbeins and the inverse vielbeins are the transformation between a representation of the one group into a representation of the other.

For a field ϕ in a representation ρ of the Lorentz group, an infinitesimal[6] local Lorentz transformation acts on ϕ as

$$\delta\phi(x) = \tfrac{1}{2}\epsilon^{ab}(x)\rho(\Sigma_{ab})\phi(x)\,, \tag{2.1.14}$$

where $\Sigma_{ab} = -\Sigma_{ba}$ are a basis of the Lie algebra $\mathfrak{so}(1, d-1)$ and satisfy

$$[\Sigma_{ab}, \Sigma_{cd}] = \eta_{bc}\Sigma_{ad} + \eta_{ad}\Sigma_{bc} - \eta_{ac}\Sigma_{bd} - \eta_{bd}\Sigma_{ac}\,. \tag{2.1.15}$$

For the spin 1/2 representation we can take the usual $2^{[n/2]}$-dimensional representation of the Clifford algebra generated by the Γ_a, $1 \le a \le n$, with the relation

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}\,. \tag{2.1.16}$$

Then the spin 1/2 representation of the Lie algebra $\mathfrak{so}(1, n-1)$ is given by

$$\rho(\Sigma_{ab}) = \tfrac{1}{2}\Gamma_{ab} \equiv \tfrac{1}{2}\Gamma_{[a}\Gamma_{b]} = \tfrac{1}{4}(\Gamma_a\Gamma_b - \Gamma_b\Gamma_a)\,. \tag{2.1.17}$$

For the spin 1 representation one checks that the map

$$\rho^{(1)} : \Sigma_{cd} \mapsto \rho(\Sigma_{cd})^a{}_b = (\delta^a_c\eta_{db} - \delta^a_d\eta_{bc}) \tag{2.1.18}$$

[6] We immediately leave the group and study the Lie algebra.

indeed defines a representation. The matrices Γ_a are Lorentz-invariant, since a Γ-matrix is both an endomorphism in the spin 1/2 representation and an element of a spin 1 representation (see appendix C).

A covariant derivative transforms as the fields on which it acts, hence without derivatives on the parameters of the transformation. We take the spin connection as the gauge field for the Lorentz group. A covariant derivative of a field in a representation ρ of the Lorentz group is defined as

$$\mathrm{D}_\mu(\omega)\phi = \left(\partial_\mu + \tfrac{1}{2}\omega_\mu{}^{ab}\rho(\Sigma_{ab})\right)\phi\,. \tag{2.1.19}$$

Note that in this definition the spin connection is already antisymmetric; $\omega_\mu{}^{ab} = -\omega_\mu{}^{ba}$ and thus metric compatibility and the requirement that $\mathrm{D}_\mu(\omega)$ is an $\mathfrak{so}(1,n{-}1)$-covariant derivative are the same. The transformation of the spin connection is

$$\delta\omega_\mu{}^{ab} = -\mathrm{D}_\mu(\omega)\epsilon^{ab}\,. \tag{2.1.20}$$

The spin connection is not covariant as is to be expected from a connection, but the difference of two connections is covariant.

The scalar φ, the vector X^μ and the co-vector Y_μ are representations of $GCT(\mathcal{M})$ of which all other representations can be built by taking tensor products. For spinors one needs to take a little care; we take φ, X^μ and Y_μ to be nonspinorial. A spinor of spin 1/2 behaves under general coordinate transformations as a scalar. Any other spinor can be obtained by taking tensor products between nonspinorial representations and a spin 1/2 representation. The map from $\mathfrak{so}(1,n{-}1)$ representations to $GCT(\mathcal{M})$[7] representations is thus 2 : 1 in a sense. The infinitesimal action of an infinitesimal general coordinate transformation $\delta x^\mu = \xi^\mu(x)$ on φ, X^μ and Y_μ is given by

$$\delta\varphi = -\xi^\nu\partial_\nu\varphi\,, \tag{2.1.21a}$$
$$\delta X^\mu = \partial_\nu\xi^\mu X^\nu - \xi^\nu\partial_\nu X^\mu\,, \tag{2.1.21b}$$
$$\delta Y_\mu = -\partial_\mu\xi^\nu Y_\nu - \xi^\nu\partial_\nu Y_\mu\,. \tag{2.1.21c}$$

The equations 2.1.21a-2.1.21c describe a Lie algebra representation, but the Lie algebra is not finite-dimensional. From 2.1.21a-2.1.21c we see

$$[\delta_{\xi_1},\delta_{\xi_2}] = \delta_{\xi_3}\,,\quad \xi_3^\mu = \xi_2^\lambda\partial_\lambda\xi_1^\mu - \xi_1^\lambda\partial_\lambda\xi_2^\mu = \mathcal{L}_{\xi_1}(\xi_2)\,, \tag{2.1.22}$$

and hence the Lie algebra structure is given by the Lie derivative.

If we now pursue the approach taken and define an independent connection for the group $GCT(\mathcal{M})$ we do not obtain gravity. What does give gravity is identifying the two connections and define the covariant derivative ∇_μ with respect to general

[7] For the general coordinate transformations, both the group and the algebra are denoted $GCT(\mathcal{M})$.

coordinate transformations by using the Lorentz transformation covariant derivative. This can be done as follows (see also [3]): (1) the field, which is in some $GCT(\mathcal{M})$-representation, is transformed to a Lorentz representation by contracting with vielbeins (and/or inverses), (2) the covariant derivative $\mathrm{D}_\mu(\omega)$ acts on that field and (3) the result is transformed back using vielbeins and inverse vielbeins. One obtains

$$\nabla_\mu \varphi = \partial_\mu \varphi \,, \tag{2.1.23a}$$
$$\nabla_\mu X_\nu = \partial_\mu X_\nu - \Gamma_{\mu\nu}^{\ \ \rho} X_\rho \,, \tag{2.1.23b}$$
$$\nabla_\mu Y^\nu = \partial_\mu Y^\nu + \Gamma_{\mu\rho}^{\ \ \nu} Y^\rho \,, \tag{2.1.23c}$$
$$\Gamma_{\mu\nu}^{\ \ \rho} = -e_\nu^a \mathrm{D}_\mu(\omega) E_a^\rho = E_a^\rho \mathrm{D}_\mu(\omega) e_\nu^a \,, \tag{2.1.23d}$$

where the $\Gamma_{\mu\nu}^{\ \ \nu}$ are identified with the Christoffel connection. For a spin 1/2 spinor λ we get $\nabla_\mu \lambda = \mathrm{D}_\mu(\omega)\lambda$, since $\partial_\mu \lambda$ is covariant under general coordinate transformations but not covariant under local Lorentz transformations.

One can see that equation 2.1.23d implies that

$$\mathrm{D}_\mu(\omega) e_\nu^a = \Gamma_{\mu\nu}^{\ \ \rho} e_\rho^a, \quad \mathrm{D}_\mu(\omega) E_a^\rho = -\Gamma_{\mu\nu}^{\ \ \rho} E_a^\nu \,, \tag{2.1.24}$$

which is equivalent to the vielbein postulate. The vielbein postulate states that the vielbein is covariantly constant:

$$\nabla_\lambda e_\mu^a = \mathrm{D}_\lambda(\omega) e_\mu^a - \Gamma_{\lambda\mu}^\rho e_\rho^a = \partial_\lambda e_\mu^a + \omega_\lambda{}^a{}_b e_\mu^b - \Gamma_{\lambda\mu}^\rho e_\rho^a = 0 \,. \tag{2.1.25}$$

For a field ϕ in a representation ρ of the Lorentz Lie algebra we have

$$[\mathrm{D}_\mu(\omega), \mathrm{D}_\nu(\omega)]\phi = (\partial_{[\mu}\omega_{\nu]}{}^{ab} + \omega_{[\mu}{}^a{}_{|c|}\omega_{\nu]}{}^{cb})\rho(\Sigma_{ab})\phi \equiv \tfrac{1}{2} R_{\mu\nu}{}^{ab}(\omega)\rho(\Sigma_{ab})\phi \,, \tag{2.1.26a}$$
$$[\mathrm{D}_a(\omega), \mathrm{D}_b(\omega)]\phi = \tfrac{1}{2} R_{ab}{}^{cd}(\omega)\rho(\Sigma_{cd})\phi - \tau_{ab}{}^c \mathrm{D}_c(\omega)\phi \,, \tag{2.1.26b}$$

with

$$R_{ab}{}^{cd}(\omega) = E_a^\mu E_b^\nu R_{\mu\nu}{}^{cd}(\omega) \,, \tag{2.1.27a}$$
$$\tau_{ab}{}^c(P) = E_a^\mu E_b^\nu (\mathrm{D}_\mu(\omega) e_\nu{}^c - \mathrm{D}_\nu(\omega) e_\mu{}^c) = 2 E_a^\mu E_b^\nu \Gamma_{[\mu\nu]}^{\ \ \ \rho} e_\rho^c \,. \tag{2.1.27b}$$

We see that the Riemann tensor $R_{\mu\nu}{}^{ab} = R_{\nu\mu}{}^{ba}$ plays the role of a field strength.

Since the torsion is the antisymmetric part of the Christoffel connection, we can write

$$\Gamma_{\mu\nu}^{\ \ \rho} = \left\{ {\rho \atop \mu\nu} \right\} + \tfrac{1}{2} \tau_{\mu\nu}{}^\rho \,, \tag{2.1.28}$$

where $\left\{ {\rho \atop \mu\nu} \right\}$ denotes the symmetric part of the Christoffel connection. We have

$$\nabla_\mu g_{\nu\rho} = 0 \ \Rightarrow \ \left\{ {\rho \atop \mu\nu} \right\} = \tfrac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right) \,. \tag{2.1.29}$$

For the antisymmetric part we can write

$$\tfrac{1}{2}\tau_{bc}{}^{a} e^{b} \wedge e^{c} = ((de)_{cb} + \omega_{c}{}^{a}{}_{b})\, e^{c} \wedge e^{b} \,. \tag{2.1.30}$$

The easiest way to solve for the spin connections is to write $\omega = \omega_0 + \omega_1$, where ω_0 is a solution of $de^a + \omega^a_{0\,b} \wedge e^b = 0$ and ω_1 is a solution of $\omega^a_{1\,b} \wedge e^b = \frac{1}{2}\tau^a \equiv \frac{1}{2}\tau_{bc}{}^a e^b \wedge e^c$. The solution for ω_0 is given by 2.1.12. For ω_1 we have similarly

$$\begin{aligned} &\tau_{bc}{}^{a} e^{b} \wedge e^{c} = \omega_{1b}{}^{a}{}_{c} e^{b} \wedge e^{c} \;\Rightarrow \\ &\omega_{1abc} = \tfrac{1}{2}(\tau_{bca} + \tau_{acb} - \tau_{abc}) \,. \end{aligned} \tag{2.1.31}$$

Though general relativity with torsion is not really understood - we could ask ourselves what space-time with nonzero torsion would look like -, for supergravity we will see it is inevitable to take torsion into account. If the torsion is zero, the connection is determined uniquely by metric compatibility; when torsion is introduced the uniqueness is lost.

2.1.3 An Action for Gravity

The Einstein equation 2.1.13 can be deduced from the action

$$S[e^a_\mu, \Phi] = \tfrac{1}{2\kappa^2} \int_{\mathcal{M}} \mathrm{d}^n x\, eR(e^a_\mu, \omega) + \int_{\mathcal{M}} \mathrm{d}^n x\, e\mathcal{L}(\Phi, \partial\Phi, e^a_\mu) \,, \tag{2.1.32}$$

where Φ stands for some fields collectively and $R(e, \omega) = E^\mu_a E^\nu_b R_{\mu\nu}{}^{ab}(\omega)$ is the Ricci scalar and $e = \det e^a_\mu$. From the action 2.1.32 one sees that the similarity between gauge theories and General Relativity ceases to hold when the Einstein equations 2.1.13 are introduced; the action 2.1.32 is not the action for a gauge theory with field strength $R_{\mu\nu}{}^{ab}$.

In a general situation there might be torsion but we have yet no way to find the torsion. To find the torsion we treat the vielbein and the spin connection as independent; this formalism is called the first-order formalism [10–12]. The action of uncoupled gravity

$$S = \int_{\mathcal{M}} \mathrm{d}^n x\, \mathcal{L}_{EH} = \tfrac{1}{2\kappa^2} \int_{\mathcal{M}} \mathrm{d}^n x\, eR(e^a_\mu, \omega) \,, \tag{2.1.33}$$

is called the Einstein–Hilbert action. Variation with respect to the spin connection and inverse vielbein independently gives (see appendix A for some convenient formulas)

$$\delta\mathcal{L}_{EH} = -\tfrac{1}{\kappa^2} \mathrm{D}_\mu \left(eE^\mu_a E^\nu_b\right) \delta\omega_\nu{}^{ab} + \tfrac{1}{\kappa^2} \left(eE^\nu_b R_{\mu\nu}{}^{ab}(\omega) - \tfrac{1}{2} eRe^a_\mu\right) \delta E^\mu_a \,. \tag{2.1.34}$$

The equation of motion for the spin connection puts the torsion to zero, while the equation of motion of the vielbein gives the Einstein equation in vacuum:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 0 \,, \tag{2.1.35}$$

which equals 2.1.13 for $T_{\mu\nu} = 0$.

2.2 Supersymmetry

In the early days of quantum field theories the study of quantum field theories describing particles and their interactions seems to be governed by regularization and renormalization. The calculations within these theories were plagued with so-called infinities. The infinities are generated by loop-diagrams and it turns out that the contributions from a fermionic loop-diagram and a bosonic loop-diagram have opposite signs. Therefore one can suggest that invoking a symmetry between bosons and fermions can help in cancelling infinities, and in a few cases it turns out to be so. The symmetry between fermions and bosons is called supersymmetry.

Another motivation for studying supersymmetry is found in reference [13] and the argument goes as follows: gravity is always attractive and therefore gravity should be mediated by even integer spin quanta[8], whereas the well-known weak and electromagnetic forces are mediated by spin 1 quanta. Some no-go theorems (such as that of [14]) forbid symmetry transformations between fields of different integer spin. Hence if we want to unify gravity with the other forces we are faced with a problem, which can be circumvented by introducing symmetry transformations between fields differing half integer in spin. Hence supersymmetry is a necessary ingredient of any theory unifying gravity with the other forces into a single quantum field theory.

Supersymmetry is also of major importance in the development of superstring theory; without supersymmetry superstring theory is inconsistent [15–17].

2.2.1 Super Lie Algebras

A symmetry between fermions and bosons in a field theory in d dimensions means that the particle states form representations of an algebra that contains (1) the algebra $\mathfrak{M}_d$ of translations and Lorentz transformations called the Poincaré algebra of d-dimensional space-time and (2) generators Q_α that map a bosonic state $|boson\rangle$ to a fermionic state $|fermion'\rangle$ and vice versa;

$$Q_\alpha \begin{pmatrix} |boson\rangle \\ |fermion\rangle \end{pmatrix} \sim \begin{pmatrix} |fermion'\rangle \\ |boson'\rangle \end{pmatrix} . \qquad (2.2.1)$$

So we seek a way to uplift the Poincaré algebra with these necessarily fermionic generators Q_α to a 'super Poincaré algebra'. The way to do this is by using super Lie algebras.

A super Lie algebra $\mathfrak{S}$ consists of a set theoretical disjoint sum of two vector spaces $\mathfrak{B}$ and $\mathfrak{F}$ with a $\mathbb{Z}_2$-grading. This means that there is a map $|.| : \mathfrak{S} \to \{0,1\}$, with $|B| = 0, \forall B \in \mathfrak{B}$ and $|F| = 1, \forall F \in \mathfrak{F}$. To get a well defined operation $|.|$ the elements of $\mathfrak{S}$ are either linear combinations of elements of $\mathfrak{F}$, or linear combinations of elements

[8]An assumption made is that a quantum field theory of gravity does not deviate too much from the standard quantum field theories.

of $\mathfrak{B}$ but not a combination of both; this is what is meant by a set theoretical disjoint sum[9].

As a Lie algebra $\mathfrak{S}$ is also equipped with a bilinear product $[.,.] : \mathfrak{S} \times \mathfrak{S} \to \mathfrak{S}$, which has the following graded symmetry and gradation property:

$$[A, B] = -(-1)^{|AB|}[B, A] \,, \quad |[A, B]| = (|A| + |B|)(\mathrm{mod}2) \,. \tag{2.2.2}$$

We define the adjoint action of an element A on an element B by $\mathrm{ad}A(B) = [A, B]$. For fixed $A \in \mathfrak{S}$ the map $\mathrm{ad}A$ is a linear map from $\mathfrak{S}$ to $\mathfrak{S}$. The map ad can be seen as a map from $\mathfrak{S}$ to the set of linear maps from $\mathfrak{S}$ to $\mathfrak{S}$. In order to promote $\mathfrak{S}$ to a super Lie algebra we impose the so-called graded Jacobi identities:

$$\mathrm{ad}F_1 \circ \mathrm{ad}F_2 + \mathrm{ad}F_2 \circ \mathrm{ad}F_1 = \mathrm{ad}[F_1, F_2] \,, \tag{2.2.3a}$$
$$\mathrm{ad}B_1 \circ \mathrm{ad}F_2 - \mathrm{ad}F_2 \circ \mathrm{ad}B_1 = \mathrm{ad}[B_1, F_2] \,, \tag{2.2.3b}$$
$$\mathrm{ad}B_1 \circ \mathrm{ad}B_2 - \mathrm{ad}B_2 \circ \mathrm{ad}B_1 = \mathrm{ad}[B_1, B_2] \,, \tag{2.2.3c}$$

where the F_i, B_i denote generic elements of $\mathfrak{F}$ resp. $\mathfrak{B}$. Any $\mathfrak{S} = \mathfrak{F} \cup \mathfrak{B}$ with the properties described above is a super Lie algebra [13, 18].

A representation of a Lie super algebra is a linear map ρ and a vector space V such that $\rho : \mathfrak{S} \to \mathrm{End}(V)$ and

$$\rho\left([A, B]\right) = \rho(A)\rho(B) - (-1)^{|AB|}\rho(B)\rho(A) \,. \tag{2.2.4}$$

The adjoint action $\mathrm{ad}A : B \mapsto [A, B]$ defines a representation by virtue of the graded Jacobi identities 2.2.3a-2.2.3c.

We sometimes write $[F_1, F_2] = \{F_1, F_2\}$. From the Jacobi identities equations 2.2.3a-2.2.3c follows that $\mathfrak{B}$ is an ordinary Lie algebra and that the fermionic generators fit into a representation of the bosonic Lie algebra $\mathfrak{B}$.

To get a super Poincaré algebra, we take for $\mathfrak{B}$ a direct sum of some Lie algebra $\mathfrak{U}$ (which corresponds to an internal symmetry) and the Poincaré algebra $\mathfrak{M}_d$, under which all fermionic generators transform in the spin 1/2 representation; fermionic generators of spin higher than 1/2 are excluded [19]. The Poincaré algebra $\mathfrak{M}_d$ is spanned by the $d(d-1)/2$ spin generators $\Sigma_{\mu\nu}$ (as defined in section 2.1) and the d momentum operators P_μ, which form a commutative subalgebra and have the following commutation relation with the spin generators $\Sigma_{\mu\nu}$:

$$[\Sigma_{\mu\nu}, P_\lambda] = -\eta_{\mu\lambda}P_\nu + \eta_{\nu\lambda}P_\mu \,. \tag{2.2.5}$$

The total number of fermionic generators is an integer multiple of the number of components of the irreducible spinor in d dimensions. This integer we denote $\mathcal{N}$

[9] Sometimes it is convenient to extend the definition to include linear combination of both $\mathfrak{B}$ and $\mathfrak{F}$. This alteration does not influence the following discussion.

and in high-energy physics literature a super Poincaré algebra is often denoted by the numbers $\mathcal{N}$ and d. One should keep in mind that this notation does not fix the bosonic Lie algebra $\mathfrak{U}$. Therefore one also writes a super Poincaré algebra as a triple $(d, \mathcal{N}, \mathfrak{U})$ to prevent confusion.

To illustrate the idea we work out an important example; the $d = 4$ and $\mathcal{N} = 4$ super Poincaré algebra. For the bosonic part of the super Lie algebra we take $\mathfrak{B} = \mathfrak{su}(4) \oplus \mathfrak{M}_d$. We take a basis $\{\mathfrak{t}_m\}$ of $\mathfrak{su}(4)$ such that the structure constants are given by: $[\mathfrak{t}_m, \mathfrak{t}_n] = f_{mn}{}^p \mathfrak{t}_p$.

As mentioned before a supersymmetry generator $Q \in \mathfrak{F}$ has spin $\frac{1}{2}$, hence we add spin $\frac{1}{2}$ generators, which belong to a $\mathfrak{su}(4)$ representation, say the vector representation $\mathbf{4}$ together with the complex conjugate in the $\bar{\mathbf{4}}$ representation of $\mathfrak{su}(4)$. So, we have left-handed four-component spinors Q^i and right-handed $Q_i = (Q^i)^*$. For the Clifford algebra representation we will take the charge conjugation matrix $\mathcal{C}$ to be antisymmetric and the Γ-matrices are real[10]. By working out the graded Jacobi identities one finds:

$$
\begin{aligned}
&[\mathfrak{t}_m, P_\mu] = [\mathfrak{t}_m, \Sigma_{\mu\nu}] = [P_\mu, Q^i] = [P_\mu, Q_i] = 0 \\
&[P_\mu, P_\nu] = \{Q^i, Q^j\} = \{Q_i, Q_j\} = 0\,, \\
&\{Q^i, Q_j\} = \delta^i_j (\mathbb{1} + \Gamma_5) \Gamma^\mu \mathcal{C}^{-1} P_\mu\,, \quad [\Sigma_{\mu\nu}, Q^i] = \tfrac{1}{2}\Gamma_{\mu\nu} Q^i\,, \quad [\Sigma_{\mu\nu}, Q_i] = \tfrac{1}{2}\Gamma_{\mu\nu} Q_i\,, \\
&[\mathfrak{t}_m, \mathfrak{t}_n] = f_{mn}{}^p \mathfrak{t}_p\,, \qquad [\mathfrak{t}_m, Q^i] = (t_m)^i{}_j Q^j\,, \qquad [\mathfrak{t}_m, Q_i] = (t_m)_i{}^j Q_j\,, \\
&[\Sigma_{\mu\nu}, \Sigma_{\lambda\rho}] = \eta_{\nu\lambda}\Sigma_{\mu\rho} - \eta_{\mu\lambda}\Sigma_{\nu\rho} - \eta_{\nu\rho}\Sigma_{\mu\lambda} + \eta_{\mu\rho}\Sigma_{\nu\lambda}\,, \\
&[\Sigma_{\mu\nu}, P_\lambda] = -\eta_{\mu\lambda} P_\nu + \eta_{\nu\lambda} P_\mu\,,
\end{aligned}
\tag{2.2.6}
$$

where we denote $(t_m)^i{}_j = \left((t_m)_i{}^j\right)^*$. Since the $\mathbf{4}$-representation consists of anti-Hermitian matrices leaving the norm $(v, w) = \sum_i (v_i)^* w^i$ invariant we have the constraint

$$
(t_m)^i{}_j + (t_m)_j{}^i = (t_m + t_m^\dagger)^i{}_j = 0\,, \tag{2.2.7}
$$

which is a necessary condition for the graded Jacobi identities to hold.

Because the commutator of two supersymmetry charges is a translation, making supersymmetry local in a theory means that there is a local translation symmetry, which is diffeomorphism invariance. Hence local supersymmetry implies gravity and leads to supergravity. Supergravity is actually nothing more than a field theory with local supersymmetry. But before we introduce supergravity, we discuss some other important general features of the super Poincaré algebras.

The first feature is the possibility of introducing central charges. The anticommutator between two Q's can be modified to include an element of the center of the super Lie algebra (thus by definition commuting with the whole algebra) on the right

[10] For conventions and technical details on spinor representations and Clifford algebras see appendix C.

hand side. In the example 2.2.6 we can try to put:

$$\{Q_i, Q_j\} = (\mathbb{1} - \Gamma_5)\mathcal{C}Z_{ij}\,. \tag{2.2.8}$$

According to the Haag–Łopuszanski–Sohnius theorem [19] the right hand side of 2.2.8 should be Lorentz invariant, which singles out $(\mathbb{1} - \Gamma_5)\mathcal{C}$.

The matrix $\mathcal{C}$ can be symmetric or antisymmetric, which forces Z_{ij}, the so-called central charge, to be either a symmetric or an antisymmetric two-tensor of the Lie algebra $\mathfrak{U}$. But since Z_{ij} commutes with everything, it has to be an invariant tensor, which restricts the Lie algebra $\mathfrak{U}$ since it should admit invariant (anti-)symmetric two-tensors. If $\mathcal{C} = -\mathcal{C}^T$ then depending on whether the Q^i are Majorana or Weyl spinors, one is forced to take $\mathfrak{U} \subset \mathfrak{sp}([\mathcal{N}/2]\,, \mathbb{R})$ or $\mathfrak{U} \subset \mathfrak{sp}([\mathcal{N}/2]\,, \mathbb{C})$ respectively[11]. For the above example of a super Lie algebra 2.2.6 it is not possible to include central charges Z_{ij} since $\mathfrak{su}(4)$ does not admit an antisymmetric two-tensor.

There is the possibility to neglect the Lorentz invariance and include a whole set of central charges [20];

$$\{Q_i, Q_j\} = 2\sum_p \Gamma^{\mu_1\cdots\mu_p}\mathcal{C}Z_{\mu_1\ldots\mu_p;ij}\,. \tag{2.2.9}$$

The central charges $Z_{\mu_1\ldots\mu_p;ij}$ are associated with charges of p-extended objects of string theory or solitons of a supersymmetric field theory (see e.g. [21–24] for a more complete discussion and more details). The presence of these objects breaks Lorentz symmetry.

The second feature to discuss is R-symmetry. As often in high-energy physics the concept is lacking a unique widespread definition. What we mean with R-symmetry can also be found in references [13, 25]. A super Lie algebra admits automorphisms like ordinary Lie algebras. Thus one can find a grading preserving map $\psi : \mathfrak{S} \to \mathfrak{S}$ satisfying $\psi([A, B]) = [\psi(A), \psi(B)]$, $\forall A, B \in \mathfrak{S}$. The automorphisms of a super Lie algebra $\mathfrak{S}$ form a group $\mathrm{Aut}(\mathfrak{S})$ with Lie algebra $\mathrm{Der}(\mathfrak{S})$. An element $D \in \mathrm{Der}(\mathfrak{S})$ is called a derivation and D is a grading preserving map satisfying $D([A, B]) = [D(A), B] + [A, D(B)]$, $\forall A, B \in \mathfrak{S}$. The adjoint action of elements of $\mathfrak{B}$ define a subalgebra of $\mathrm{Der}(\mathfrak{S})$. Hence exponentiation of the adjoint action of $\mathfrak{B}$ defines a nontrivial subgroup of $\mathrm{Aut}(\mathfrak{S})$ thereby proving that super Lie algebras admit automorphisms. The R-symmetry group H_R is by definition the group of automorphisms $\mathrm{Aut}(\mathfrak{S})$ that act trivially on the Poincaré subalgebra; the restriction of H_R to $\mathfrak{M}_d$ gives the identity on $\mathfrak{M}_d$.

The central charges restrict the R-symmetry group as we have seen, but from 2.2.6 we see that the anticommutator between Q^i and Q_j also restricts the R-symmetry group. For the $\mathcal{N} = 4$ $d = 4$ super Lie algebra we see that $H_R \subset \mathrm{Aut}(\mathfrak{S}) \subset U(4)$.

[11] For our convention on the notation of the groups and algebras, see appendix A. For conventions and notation of spinors, see appendix C.

A third and final subject concerns representations. There exist so-called state representations, which are finite-dimensional, and there exist field representations, which are infinite-dimensional. Since this thesis mainly deals with field representations we only sketch how to obtain a state representation.

For a state representation one chooses a rest-frame and fixes the momentum P_μ; one therefore should decide whether one wishes to work out a massive or a massless multiplet. Then the anticommutation relation $\left\{Q_i, Q^j\right\} = \delta_i^j(\mathbb{1}+\Gamma_5)\Gamma^\mu \mathcal{C}^{-1} P_\mu$ can be written as a set of anticommutation relations of creation and annihilation operators, i.e. one obtains a set of equations of the form $\left\{b_{i\alpha}, b^\dagger_{j\beta}\right\} = \delta_{\alpha\beta}\delta_{ij}$, where $1 \leq i, j \leq \mathcal{N}$. Then one chooses a vacuum state $|P_\mu, 0\rangle$ and acts on it with the creation operators $b^\dagger_{j\beta}$ to obtain all states in the multiplet. For a nice discussion on $\mathcal{N} = 1$ multiplets in any dimensions see [25].

An important feature of field representations is that the momentum is not something chosen but is represented by the operator ∂_μ. The following section 2.2.2 is about supersymmetric field theories, which are built with field representations.

2.2.2 Supersymmetric Field Theories

To incorporate the ideas of section 2.2.1 into a field theory we take a set of fields that form a representation of a super Lie algebra. The fermionic generators $Q \in \mathfrak{F}$ change a boson into a fermion and vice versa. Also, acting twice with a supersymmetry generator we find a translation. Hence it is necessary that the number of fermions N_f equals the number of bosons N_b. But at this point there is a choice: either we impose the equality $N_b = N_f$ before imposing the equations of motion or after. Since the number of degrees of freedom on-shell and off-shell are different, this does make a difference. One therefore speaks of off-shell supersymmetry multiplets and of on-shell supersymmetry multiplets. In this thesis we mostly work with on-shell multiplets; for examples on off-shell multiplets and more details, see [13, 26].

For on-shell multiplets in theories with supersymmetry it is a general feature that the algebra closes, i.e. the transformation rules of the fields do define a representation, only on-shell, hence up to equations of motion and symmetries of the latter.

To find an on-shell multiplet one starts with a state multiplet. From the state multiplet one finds the Lorentz representations present in the multiplet and the number of on-shell degrees of freedom in each Lorentz representation. Then for each Lorentz representation one finds the correct number of fields with the appropriate spin such that the number of degrees of freedom match.

Therefore it is necessary to know the degrees of freedom of some particles. In d dimensions the graviton, the gravitino, the massless vector, the massless fermion and the massless scalar contain the degrees of freedom (denoted d.o.f.) as denoted in table 2.2.1 (see also [27]). The values of the function $c(d)$ are given in appendix C.

particle	d.o.f.
graviton	$\frac{1}{2}(d-1)(d-2)-1$
gravitino	$c(d)(d-3)$
vector	$d-2$
fermion	$c(d)$
scalar	1

Table 2.2.1: Degrees of freedom. $c(d)$ denotes the degrees of freedom of a spin 1/2 particle in d dimensions, which is explained in appendix C.

Having found such a multiplet of fields one constructs a Lagrangian and transformation rules and it is always possible to find a free theory, i.e. without interactions. It is more difficult to find the transformation rules and the Lagrangian for a theory with interactions. Below we present some techniques to find a Lagrangian and supersymmetry transformation rules for a theory with interactions, but we try by no means to be complete[12].

The conceptually simplest strategy is to take a multiplet of fields with equal bosonic and fermionic degrees of freedom and to write down the most general Ansatz with unknown coefficients for the supersymmetry transformations and for the Lagrangian. Then one tries to fit the coefficients by demanding that the algebra closes on-shell. The strategy of trial and error works in many cases but can be cumbersome and does not provide much insight.

There exists a procedure to start with the free theory and to incorporate interactions order by order in a coupling constant g. One adjusts the Lagrangian and transformation rules order by order in g. This procedure is called the Noether procedure and involves no highbrow mathematics or insightful physics, but it is powerful. See [11, 29–33] for more explanation on this procedure and examples.

A more sophisticated and evenly powerful method is developed by Salam and Strathdee [34] and uses superspace; one enlarges space-time to a manifold with fermionic degrees of freedom. In this thesis we do not use the methods of superspace. Therefore we do not go into more detail but refer to the vast amount of literature on this subject, see e.g. [12] and references therein.

For local supersymmetry there exists a nice insightful method developed by West and Chamseddine [35] and the idea is as follows. With every generator T_α of the super Poincaré algebra we associate a gauge field V_μ^α and a gauge parameter ϵ^α. We build a compound gauge field $V_\mu = V_\mu^\alpha T_\alpha$ and a compound gauge parameter $\Lambda^\alpha = T_\alpha \epsilon^\alpha$.

[12]There exists a vast amount of literature on how to construct supersymmetric Lagrangians. We refer the reader to the literature on the subject for a more complete discussion; see e.g. [12, 13, 28] and references therein.

Since a gauge field always transforms as a covariant derivative of the gauge parameter, we have:

$$\delta V_\mu = \mathrm{D}_\mu \Lambda = \partial_\mu \Lambda + [V_\mu, \Lambda] = \delta V_\mu^\alpha T_\alpha \,, \tag{2.2.10}$$

from which we read off the coefficients in front of each generator. This method thus boils down to gauging the super Poincaré algebra. In finding an invariant action one has to be careful not to simply take $\mathcal{L} \sim \mathrm{Tr}F^2$ - where F^α is calculated from the commutator of two covariant derivatives D_μ - since the Einstein–Hilbert Lagrangian is linear in the Ricci scalar. For more explanation on this method, see e.g. [12,35,36].

There is a clear distinction between theories with global supersymmetry and local supersymmetry. Those with local supersymmetry necessarily include gravity, i.e. the supersymmetry multiplet contains a symmetric rank two tensor $g_{\mu\nu}$, the graviton. Since the graviton has spin two, supersymmetry dictates us that there should be a spin 5/2 or spin 3/2 particle. There is a problem however in coupling spin 5/2 particles consistently to other particles [37], and hence a theory with local supersymmetry is constructed with a metric tensor of spin 2 and a gravitino of spin 3/2, see e.g. [11, 29, 38, 39] and many more.

2.3 Simple Supergravities

Having a way of obtaining supersymmetric field theories, we can incorporate gravity into these theories and try to build an invariant Lagrangian for a supermultiplet containing the graviton $g_{\mu\nu}$. An $\mathcal{N} = 1$ supermultiplet that contains the metric is an $\mathcal{N} = 1$ supergravity multiplet and the corresponding field theory describing the dynamics is called an $\mathcal{N} = 1$ supergravity, or a simple supergravity. The Lie algebra $\mathfrak{U}$ is irrelevant since the fields make up a one-dimensional representation.

2.3.1 $\mathcal{N} = 1$ Supergravity in $d = 4$

From table 2.2.1 and appendix C we conclude that in four dimensions a gravitino contains 2 on-shell degrees of freedom. Since the graviton in four dimensions also has 2 degrees of freedom, a supersymmetry multiplet consisting of a gravitino ψ_μ and a graviton $g_{\mu\nu}$ might exist in $d = 4$. And indeed one finds that the variations

$$\delta e_\mu^a = \tfrac{\kappa}{2}\bar{\epsilon}\Gamma^a \psi_\mu \,, \quad \delta\psi_\mu = \tfrac{1}{\kappa}\mathrm{D}_\mu(\omega)\epsilon \,, \tag{2.3.1}$$

with ϵ a spin 1/2 Majorana spinor, leave the following action invariant,

$$\mathcal{L}_{d=4,\mathcal{N}=1} = \tfrac{1}{2\kappa^2} eR(e,\omega) - \tfrac{1}{2} e\bar{\psi}_\mu \Gamma^{\mu\nu\rho} \left(\partial_\nu + \tfrac{1}{4}\omega_\nu{}^{ab}\Gamma_{ab}\right)\psi_\rho \,, \tag{2.3.2}$$

and define a supersymmetry algebra [11,38]. The variation of the action with respect to the spin connection gives the torsion

$$\tau_{\mu\nu}{}^a = \tfrac{\kappa^2}{2}\bar{\psi}_\mu \Gamma^a \psi_\nu \,. \tag{2.3.3}$$

The spin connection is determined from the torsion equation 2.1.27b;

$$\mathrm{D}_\mu(\omega)e_\nu^a - \mathrm{D}_\nu(\omega)e_\mu^a = \tau_{\mu\nu}{}^a \,. \tag{2.3.4}$$

Choosing the spin connection to satisfy its equation of motion is useful in checking the supersymmetry invariance of the action; the coefficient in front of $\delta\omega$ vanishes on-shell. Performing the supersymmetry variations is still nontrivial and details can be found in [3].

A few general remarks can be made by looking at the above example. The first concerns the variation of the gravitino; a general gravitino variation looks like

$$\delta\psi_\mu = \tfrac{1}{\kappa}\partial_\mu\epsilon + \dots\,, \tag{2.3.5}$$

and hence the gravitino behaves like a gauge field of supersymmetry.

The second remark is about the 'covariant derivative' of the gravitino, which one often defines as

$$\hat{\mathrm{D}}_\mu(\omega)\psi_\nu = \left(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab}\right)\psi_\nu\,, \tag{2.3.6}$$

which is clearly not covariant, since there is no Christoffel connection. Therefore the coupling of the gravitino to gravity is called nonminimal [29, 38]. However, it is not the covariant derivative that enters the Lagrangian, but the 'covariant field strength' $\hat{\mathrm{D}}_\mu(\omega)\psi_\nu - \hat{\mathrm{D}}_\nu(\omega)\psi_\mu \equiv \psi_{\mu\nu}$, which is covariant under both $GCT(\mathcal{M})$ and $\mathfrak{so}(1,3)$.

The final remark is on the calculations involved; already the simple supergravity examples as the one above give rise to tedious calculations and as a sign of that we remark that for the first papers on supergravity and even for the one where the above example was constructed [29], a computer was used to complete the calculations.

The action 2.3.2 is finite at one loop [40], which was already hinted at in the beginning of section 2.2. A similar supergravity theory, where an $\mathcal{N}=1$ vector multiplet is added to the supergravity multiplet, also renders finite one-loop corrections as was shown in the same paper. The four-dimensional Einstein–Hilbert action 2.1.33 is also finite at one-loop but the coupling of gravity to other fields without supersymmetry destroys the one-loop finiteness [41]. These observations made supergravity theories interesting from a quantum gravity point of view. However, it is generally believed that higher-loop calculations will show that supergravity theories are not finite to all orders.

2.3.2 $\mathcal{N}=1$ Supergravity in $d=11$

An important development in supergravity theory was the construction of eleven-dimensional supergravity [42]. Eleven is the highest dimension where a supersymmetry multiplet exists that does not contain a spin 5/2 particle; necessarily it is an $\mathcal{N}=1$ multiplet.

The metric contains 44 degrees of freedom, whereas the gravitino contains 128 degrees of freedom, a mismatch that can be resolved by adding a three-form $A_{\mu\nu\rho}$ gauge field to the multiplet.

Two different spin connections are used; one that is on-shell and another that is 'supercovariant'. The latter is denoted $\hat{\omega}_\mu{}^{ab}$ and satisfies the torsion-like equation

$$\mathrm{D}_\mu(\hat{\omega})e^a_\nu - \mathrm{D}_\nu(\hat{\omega})e^a_\mu = K_{\mu\nu}{}^a \equiv \tfrac{\kappa^2}{2}\bar{\psi}_\mu\Gamma^a\psi_\nu\,. \tag{2.3.7}$$

The tensor $K_{\mu\nu}{}^a$ is not the torsion but differs from the torsion by a term proportional to $\bar{\psi}_\alpha\Gamma_{\mu\nu}{}^{a\alpha\beta}\psi_\beta$; this term vanishes in four dimensions but enters in the equation of motion for the spin connection in eleven dimensions. The supercovariant spin-connection transforms without derivatives of the supersymmetry parameter ϵ, hence the name. The supercovariant spin connection does not satisfy the equation of motion of the spin connection, but the on-shell spin connection $\bar{\omega}_\mu{}^{ab}$ does:

$$\bar{\omega}_{\mu ab} = \hat{\omega}_{\mu ab} + \frac{\kappa^2}{8}\bar{\psi}_\nu\Gamma_\mu{}^{\nu\rho}{}_{ab}\psi_\rho\,. \tag{2.3.8}$$

The reason to introduce two different connections is to simplify some of the tedious calculations.

For the three-form one defines a supercovariant field strength:

$$\hat{F}_{\mu\nu\lambda\rho} = 4\partial_{[\mu}A_{\nu\lambda\rho]} + \tfrac{3\kappa}{\sqrt{2}}\bar{\psi}_{[\mu}\Gamma_{\nu\lambda}\psi_{\rho]}\,. \tag{2.3.9}$$

The supercovariance of $\hat{F}$ and $\hat{\omega}$ is easily checked for the following supersymmetry variations:

$$\delta e^a_\mu = \tfrac{\kappa}{\sqrt{2}}\bar{\epsilon}\Gamma^a\psi_\mu\,, \tag{2.3.10a}$$

$$\delta\psi_\mu = \tfrac{\sqrt{2}}{\kappa}\mathrm{D}_\mu(\hat{\omega})\epsilon + \tfrac{1}{144}\left(\Gamma_\mu{}^{\nu_1\nu_2\nu_3\nu_4} - \delta^{\nu_1}_\mu\Gamma^{\nu_2\nu_3\nu_4}\right)\hat{F}_{\nu_1\nu_2\nu_3\nu_4}\epsilon\,, \tag{2.3.10b}$$

$$\delta A_{\mu\nu\lambda} = -\tfrac{3}{2}\bar{\epsilon}\Gamma_{[\mu\nu}\psi_{\lambda]}\,. \tag{2.3.10c}$$

To find the correct transformation rules for the fields it is important to note that $\delta\psi_\mu$ is supercovariant. The gravitino is identified with the connection for local supersymmetry and hence is not covariant, but the difference of two connections is covariant. Hence if one varies $\delta\psi_\mu$ with respect to a second supersymmetry parameter $\tilde{\epsilon}$, the result should contain no derivatives of $\tilde{\epsilon}$, hence $\delta\psi_\mu \sim \frac{1}{\kappa}\mathrm{D}_\mu(\hat{\omega})\epsilon + \ldots$ where the ellipsis contains the supercovariant field strength $\hat{F}$.

The invariant action is:

$$\begin{aligned}\mathcal{L} =& \tfrac{e}{2\kappa^2}R(e,\bar{\omega}) - \tfrac{e}{48}F_{\mu\nu\lambda\rho}F^{\mu\nu\lambda\rho} - \tfrac{e}{2}\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\mathrm{D}_\nu(\tfrac{\bar{\omega}+\hat{\omega}}{2})\psi_\rho\\ &- \tfrac{1}{192\sqrt{2}}e\kappa\left(\bar{\psi}_\sigma\Gamma^{\sigma\tau\mu\nu\lambda\rho}\psi_\tau + 12\bar{\psi}^{[\mu}\Gamma^{\nu\lambda}\psi^{\rho]}\right)\left(F_{\mu\nu\lambda\rho} + \hat{F}_{\mu\nu\lambda\rho}\right)\\ &- \tfrac{\sqrt{2}\kappa}{(144)^2}\varepsilon^{\mu_1\ldots\mu_{11}}F_{\mu_1\ldots\mu_4}F_{\mu_5\ldots\mu_8}A_{\mu_9\mu_{10}\mu_{11}}\,,\end{aligned}$$

where some four fermion terms are captured in $D_\mu(\frac{\bar{\omega}+\hat{\omega}}{2})$.

An important feature of eleven-dimensional supergravity is that many lower-dimensional supergravity theories can be deduced from it by dimensional reduction. This simplifies the construction of supergravity theories, since the methods for constructing supergravity theories such as the Noether method, can be quite elaborate, whereas dimensional reductions are in many cases easier. Hence eleven-dimensional supergravity is an (almost) all encompassing supergravity from which other lower-dimensional supergravities can be obtained by a relatively straightforward procedure.

Eleven-dimensional supergravity was constructed before the first string theory revolution, which was in 1984. Since superstring theory lives in ten dimensions eleven-dimensional supergravity was kicked off its pedestal. But due to the second string revolution in 1996, when an eleven-dimensional M-theory was conjectured [43], it was quickly rehabilitated.

In the rest of the thesis we mainly work with the bosonic sector of supergravity theories and for eleven-dimensional supergravity the bosonic action reads[13]:

$$S_{d=11} = \int \tfrac{1}{2\kappa^2} \star \mathbb{1} R - \tfrac{1}{2} \star F_{(4)} \wedge F_{(4)} + \tfrac{\sqrt{2}\kappa}{6} F_{(4)} \wedge F_{(4)} \wedge A_{(3)} \,, \tag{2.3.11}$$

where $F_{(4)} = dA_{(3)}$.

2.3.3 $\mathcal{N} = 1$ Supergravities in $d = 10$

Of special interest to us are the simple ten-dimensional supergravities since upon dimensional reduction these give four-dimensional $\mathcal{N} = 4$ supergravities when no supersymmetry is broken by the dimensional reduction.

In ten dimensions the irreducible spin 1/2 representation of the Lorentz group is 16-dimensional and on-shell a spinor thus has 8 degrees of freedom, which equals the degrees of freedom of a massless vector. Thus a Yang–Mills vector A_μ and a spinor ψ can build up a supermultiplet; this multiplet is called the $\mathcal{N} = 1$ Super Yang–Mills multiplet. The supersymmetry transformations and the (flat space-time) action are given in section 4.2.1, where the $\mathcal{N} = 1$ Super Yang–Mills theory is treated in more detail. The Super Yang–Mills multiplet can be a nonabelian Super Yang–Mills multiplet where A_μ and ψ take values in a faithful representation of a Lie algebra.

The $\mathcal{N} = 1$ Super Yang–Mills multiplet can be coupled to the ten-dimensional $\mathcal{N} = 1$ supergravity multiplet consisting of a vielbein e_μ^a, a two-form $B_{\mu\nu}$, a dilaton ϕ, a Majorana–Weyl gravitino ψ_μ and a Majorana–Weyl spin 1/2 spinor λ of opposite chirality, sometimes called dilatino. It turns out that there are two different anomaly-free $\mathcal{N} = 1$ supergravity theories in ten dimensions, namely the Type I, corresponding to the low-energy effective action of Type I string theory and the Heterotic super-

[13]For conventions on forms and Hodge duals, see appendix A.

gravity, corresponding in turn to the low-energy effective action of Heterotic string theory. The field contents are the same, but the actions are different.

The bosonic Lagrangian of heterotic supergravity is[14]

$$\begin{aligned} \mathcal{L}_{Het} &= \mathrm{e}^{-2\phi}\left(\star\mathbb{1}R + 4\star\mathrm{d}\phi\wedge\mathrm{d}\phi - \tfrac{1}{2}\star H\wedge H - \tfrac{1}{2}\mathrm{Tr}\star F\wedge F\right)\,, \\ H &= \mathrm{d}B - \tfrac{1}{2}\mathrm{Tr}(A\wedge F + \tfrac{2}{3}g^2 A\wedge A\wedge A)\,, \\ F &= \mathrm{d}A + gA\wedge A\,, \end{aligned} \tag{2.3.12}$$

where g is the Yang–Mills coupling constant and the gauge group has to be $SO(32)$ or $E_8\times E_8$ to cancel the anomalies. Some terms that are needed to cancel the anomalies are not presented in 2.3.12, for the full result see e.g. [16, 17].

The bosonic Lagrangian of type I supergravity is

$$\mathcal{L}_I = \mathrm{e}^{-2\phi}\left(\star\mathbb{1}R + 4\star\mathrm{d}\phi\wedge\mathrm{d}\phi\right) - \tfrac{1}{2}\star H\wedge H - \tfrac{1}{2}\mathrm{e}^{-\phi}\mathrm{Tr}\star F\wedge F\,, \tag{2.3.13}$$

with identical definitions for H and F and with the same omissions as in 2.3.12 but for type I the gauge group has to be $SO(32)$.

The Lagrangians 2.3.13 and 2.3.12 are mapped to each other by a Weyl rescaling $g_{\mu\nu}\mapsto \mathrm{e}^{\phi}g_{\mu\nu}$ of the metric together with the mapping $\phi\mapsto-\phi$.

Besides the two[15] simple supergravities, there are two extended supergravities living in ten dimensions, called type IIA and type IIB.

2.4 Extended Supergravities

In the preceding sections we looked at $\mathcal{N}=1$ supergravities where the representations of the Lie algebra $\mathfrak{U}$ are trivial. Taking the supersymmetry generators Q in a nontrivial $\mathcal{N}$-dimensional representation of the Lie algebra $\mathfrak{U}$ gives rise to extended supergravity. To not get spin 5/2 particles, the total supersymmetry generator components $\mathcal{N}c(d)$ should not exceed 32.

Extended supergravities have some features that are not found in simple gravities. The most important is that all fields are in a representation of the Lie algebra $\mathfrak{U}$. In most applications the supersymmetry generators Q are assembled in a vector representation and hence one often writes Q^i where the index i is a $\mathfrak{U}$ vector index. If there is one metric $g_{\mu\nu}$, the gravitino ψ_μ is in the vector representation and hence one writes ψ^i_μ for the $\mathfrak{U}$-vector components of ψ_μ.

If the number of supersymmetry generators in a supergravity theory is the maximal number such that spin 5/2 or higher-spin fields are not present, one speaks of maximal

[14] The gravitational has been chosen such that the Einstein–Hilbert term has coefficient 1. For similar presentations of the supergravities in ten dimensions see [44, 45].

[15] One often speaks of five different ten-dimensional supergravities; one then counts Heterotic $SO(32)$ and Heterotic $E_8\times E_8$ as two different theories. Since the gauge group in our case is not important we count them as one.

supergravity; eleven-dimensional $\mathcal{N} = 1$ supergravity is the only simple maximal supergravity.

Below we give some examples of extended supergravities. We postpone a more complete discussion of $\mathcal{N} = 4$ supergravity in four dimensions to chapter 4. The examples are examples of "ungauged" supergravities; no global symmetry has been gauged. The gauging of global symmetries is discussed in section 2.4.4.

2.4.1 Type IIA and IIB Supergravity

In ten dimensions it is possible to formulate $\mathcal{N} = 2$ supergravity, where the supersymmetry algebra contains two fermionic supersymmetry generators Q^i, $i = 1, 2$ both with 8 components. Since the irreducible spinor in ten dimensions is Majorana–Weyl, there are two possibilities; either the supersymmetry generators of the supergravity theory have opposite chirality, say $\Gamma^{11} Q^1 = +Q^1$ and $\Gamma^{11} Q^2 = -Q^2$, or they have the same chirality, say $\Gamma^{11} Q^i = Q^i$. The first possibility is called type IIA supergravity and the second possibility type IIB supergravity.

The bosonic action of type IIA is

$$\begin{aligned}\mathcal{L}_{IIA} =& \mathrm{e}^{-2\phi}\left(\star\mathbb{1}R + 4\star\mathrm{d}\phi\wedge\mathrm{d}\phi - \tfrac{1}{2}\star H_{(3)}\wedge H_{(3)}\right) - \tfrac{1}{2}\star G_{(2)}\wedge G_{(2)} \\ & - \tfrac{1}{2}\star G_{(4)}\wedge G_{(4)} - \tfrac{1}{2}B_{(2)}\wedge \mathrm{d}C_{(3)}\wedge \mathrm{d}C_{(3)}\,,\end{aligned} \tag{2.4.1}$$

where the index between parenthesis indicates the rank of the form and $H_{(3)} = \mathrm{d}B_{(2)}$, $G_{(2)} = \mathrm{d}C_{(1)}$ and $G_{(4)} = \mathrm{d}C_{(3)} - H_{(3)} \wedge C_{(1)}$.

There is a common sector with both type I and Heterotic supergravities and this sector is called the NS-NS sector (for an excellent explanation on the nomenclature see [15,16,45]) and contains the metric $g_{\mu\nu}$, the Kalb–Ramond two-form $B_{(2)}$ and the dilaton ϕ, also called NS-NS fields.

The sector different from type I and Heterotic supergravity contains the so-called type IIA R-R fields $C_{(1)}$ and $C_{(3)}$. Type IIB supergravity has the same NS-NS sector but the R-R sector is different from that of type IIA; the type IIB R-R fields are the gauge fields $C_{(0)}$, called the axion, $C_{(2)}$ and $C_{(4)}$.

The action of type IIB is

$$\begin{aligned}\mathcal{L}_{IIB} =& \mathrm{e}^{-2\phi}\left(\star\mathbb{1}R + 4\star\mathrm{d}\phi\wedge\mathrm{d}\phi - \tfrac{1}{2}\star H_{(3)}\wedge H_{(3)}\right) - \tfrac{1}{2}\star G_{(1)}\wedge G_{(1)} \\ & - \tfrac{1}{2}\star G_{(5)}\wedge G_{(5)} + \tfrac{1}{2}C_{(4)}\wedge \mathrm{d}C_{(2)}\wedge H_{(3)}\,,\end{aligned} \tag{2.4.2}$$

where $H_{(3)} = \mathrm{d}B_{(2)}$, $G_{(1)} = \mathrm{d}C_{(0)}$, $G_{(3)} = \mathrm{d}C_{(2)} - H_{(3)} \wedge C_{(0)}$ and:

$$G_{(5)} = \mathrm{d}C_{(4)} - \tfrac{1}{2}C_{(2)}\wedge H_{(3)} + \tfrac{1}{2}B_{(2)}\wedge \mathrm{d}C_{(2)}\,. \tag{2.4.3}$$

The field strength $G_{(5)}$ is self-dual; $G_{(5)} = \star G_{(5)}$. The self-duality condition should be imposed at the level of the equations of motion [46]; this condition does not follow from the action and imposing it in the action 2.4.2 makes the term $\star G_{(5)} \wedge G_{(5)}$ vanish. There exists a Lorentz invariant action that does not require the self-duality condition, but its construction needs the introduction of auxiliary fields [47].

There is another type IIA supergravity called the Romans massive type IIA supergravity [48](for a particular nice and introductory discussion see reference [49]). This theory contains a mass parameter that deforms the theory and in the limit where the mass parameter goes to zero one obtains the ordinary massless type IIA supergravity. The massive theory does not admit a Minkowski vacuum and has been of great importance in the theory of D8-branes [50–52]. The massless type IIA supergravity theory can be obtained from eleven-dimensional supergravity by dimensional reduction, whereas for the massive type IIA and the type IIB no higher-dimensional origin is known.

2.4.2 $\mathcal{N} = 4$ Supergravity in $d = 4$

There are two main ingredients for four-dimensional $\mathcal{N} = 4$ supergravity; the $\mathcal{N} = 4$ Super Yang–Mills multiplet and the $\mathcal{N} = 4$ supergravity multiplet. In this section we briefly comment on these ingredients. In chapter 4 we treat $\mathcal{N} = 4$ supergravity and the coupling to the $\mathcal{N} = 4$ Super Yang–Mills multiplet in more detail.

The $\mathcal{N} = 4$ Super Yang–Mills multiplet contains a vector A_μ, four Weyl fermions λ^i, which are in the vector representation of $\mathfrak{su}(4)$, and six scalars $\phi^{ij} = -\phi^{ji}$ in the $\mathbf{6}^-$ of $\mathfrak{su}(4)$ [16]. All fields are in a representation of a Lie algebra $\mathfrak{g}_{YM}$, which commutes with the super Poincaré algebra. The action and supersymmetry transformation rules are given in section 4.2.1.

The $\mathcal{N} = 4$ Super Yang–Mills theory in flat space is not only renormalizable, but it is even finite; all contributions in perturbation theory that could give rise to infinities precisely cancel [53–57]. This makes $\mathcal{N} = 4$ Super Yang–Mills interesting from a purely theoretical point of view, but the spectrum does not agree with observations. There are for example no chiral fermions in the theory.

To obtain some concordance with observed nature one might hope to have a dynamical way of breaking supersymmetry at some high energy E_h. To solve the hierarchy problem there should be an intermediate energy $E_m < E_h$ where not all supersymmetry is broken and still $\mathcal{N} = 1$ supersymmetry exists. To get chiral fermions there should be a lower energy $E_l < E_m$ where the $\mathcal{N} = 1$ supersymmetry is broken and no supersymmetry is left. From a phenomenological point of view it is thus desirable to break supersymmetry in steps. However, this does not seem to be possible, unless we include gravity (see e.g. [13]).

The $\mathcal{N} = 4$ supergravity multiplet contains a vielbein e_μ^a, four gravitini ψ_μ^i, six vectors A_μ^r, four fermions χ^i and two scalars ϕ^α. The scalars ϕ^α parameterize an $SU(1,1)/U(1)$ coset; we discuss this and other cosets in section 2.4.5 and in chapter

[16] For explanation on $SU(4)$ representations see appendix B.

3. The action and transformation rules of the fields of the ungauged theory was constructed using different formalisms around 1977 [58, 59].

The coupling of $\mathcal{N} = 4$ Super Yang–Mills multiplets to the $\mathcal{N} = 4$ supergravity theory resulted in the so-called matter coupled $\mathcal{N} = 4$ supergravity and was done in the beginning of the 80's by De Roo [60]. The action is invariant under global $SO(6,n)$-rotations, where n is the number of added Super Yang–Mills multiplets. The $6n$ scalars from the Super Yang–Mills multiplets parameterize an $SO(6,n)/SO(6) \times SO(n)$ coset. The $\mathcal{N} = 4$ supergravity theory thus has a global symmetry group $G = SO(6,n) \times SU(1,1)$ and the scalar parameterize the coset G/K where $K = SO(6) \times SO(n) \times U(1)$, which is the maximal compact subgroup of G.

In reference [61] the ten-dimensional type I supergravity with the $SO(32)$-vector fields truncated away was dimensionally reduced. The result is matter coupled $\mathcal{N} = 4$ supergravity where the number n of added Super Yang–Mills multiplets is six[17].

The group $SU(1,1)$ is a global symmetry of the equations of motion but not a symmetry of the action; it rotates the Bianchi identities and the equations of motion and thus the electric and magnetic field strengths into each other. There exist two formulations of $d = 4$ $\mathcal{N} = 4$ supergravity where the $SU(1,1)$-transformations leave the action invariant, but either one has to give up that $SO(6,6)$ is a symmetry of the action [62] or one has to give up general coordinate invariance [63].

2.4.3 Dualities

The highest dimension where a supergravity can be formulated is eleven. The lowest dimension is three; in two dimensions gravity is trivial since then the Ricci scalar is a topological number. Hence for $3 \leq d \leq 11$ one can formulate supergravity theories.

All maximal supergravities in $d \leq 9$ are unique[18] and dimensional reduction over a torus of eleven-dimensional supergravity gives rise to a maximal supergravity; hence all maximal supergravities are related to eleven-dimensional supergravity by a dimensional reduction over a torus.

In d dimensions the Lorentz group is $SO(1, d-1)$ and on-shell massless fields are representations of the little group $SO(d-2)$. Hence the field content of maximal supergravities can be found by decomposing the $SO(9)$ representations of the eleven-dimensional fields $\left\{e_\mu^a, \phi_\mu, A_{\mu\nu\rho}\right\}$ into $SO(d-2)$ representations. Thus dimensional reduction is one way to relate different supergravity theories. But there is another more intriguing way to relate supergravity theories; there exist dualities between some theories.

[17]The value six is rather special; the global symmetry group is then maximally noncompact and later we will see that this has important consequences.

[18]It should be understood that this holds for ungauged maximal supergravities.

S-Duality

S-duality is a generalization of the electromagnetic duality in four dimensions; it relates a theory at strong coupling constant to a theory at weak coupling constant.

The string coupling constant in ten dimensions g_s is related to the expectation value of the dilaton: $g_s \sim \mathrm{e}^{<\phi>}$. Hence in ten dimensions S-duality relates Heterotic $SO(32)$ supergravity to Type I supergravity.

Type IIB is S self-dual since the theory has a global $SL(2;\mathbb{R})$-symmetry and the S-duality transformation is contained in the $SL(2;\mathbb{R})$-transformations. This can be seen by transforming the action 2.4.2 to the Einstein frame, where the Ricci scalar does not have a dilatonic prefactor. The scalars ϕ and $C_{(0)}$ parameterize an $SL(2;\mathbb{R})/SO(2)$ coset and can be combined in a complex scalar $\tau = C_{(0)} + i\mathrm{e}^{-\phi}$. The $SL(2;\mathbb{R})$ symmetry acts on the complex scalar τ as

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{R}\,, \; ad - bc = 1\,. \tag{2.4.4}$$

The other fields are in linear representations of $SL(2;\mathbb{R})$ (see e.g. [16, 49, 64, 65] for more details).

The $SL(2;\mathbb{R})$ transformations contain the transformation $\tau \mapsto -1/\tau$ acting on the imaginary part as $\mathrm{e}^{-\phi} \mapsto \mathrm{e}^{\phi}$ if $C_{(0)} = 0$, i.e. in a vanishing axion background. Hence S-duality transformations form a discrete subgroup of $SL(2;\mathbb{R})$ and type IIB supergravity is S self-dual.

The situation for four-dimensional $\mathcal{N} = 4$ supergravity theory is similar to the IIB-case. The scalars ϕ^α, as mentioned in section 2.4.2, parameterize the coset $SU(1,1)/U(1)$ and the group $SU(1,1) \cong SL(2;\mathbb{R})$ is a global symmetry rotating electric and magnetic field strengths into each other. Hence $d = 4$ $\mathcal{N} = 4$ supergravity is S self-dual.

T-Duality

T-duality is a string theory inspired duality; if d-dimensional bosonic string theory is put on a geometry $S^1 \times M^{d-1}$, where M^{d-1} is $(d-1)$-dimensional Minkowski and S^1 is a one-dimensional compact dimension with length $2\pi R$, the spectrum is determined by two quantum numbers, called winding numbers (n) and momentum numbers (m) [66,67]. The spectrum is invariant under interchanging n and m together with inverting the radius of S^1: $R \leftrightarrow \alpha'/R$. For superstrings the story is even nicer; the spectra of type IIA and type IIB superstring theory are interchanged under $R \leftrightarrow \alpha' R$ and $n \leftrightarrow m$. This kind of duality is called T-duality.

T-duality manifests itself also in the supergravity theories; if the IIA and IIB theories are dimensionally reduced to $d = 9$ the actions can be mapped to each other[19].

[19] This is not trivial; one has to be careful how the treat the self-dual five-form field strength.

The two Heterotic string theories are also T-dual to each other [68]. From the supergravity point of view this might come as a little surprise since the gauge groups are different. But the gauge groups break down to their maximal abelian subalgebras as explained in reference [69, 70]. Hence upon reduction both Heterotic supergravity theories have a $U(1)^{16}$ Yang–Mills sector and they are trivially the same. Hence a dimensional reduction of any ten-dimensional supergravity theory with 16 supercharges gives the same result; changing the sign of the dilaton is in a supergravity theory nothing more than a field redefinition.

When a supergravity theory is reduced over an n-torus, there are more radii to invert and thus T-duality becomes a group of symmetries. Though this group of symmetries is discrete, in the supergravity it is a continuous group. Since type IIA and type IIB supergravity are T-dual, for $d \leq 9$ T-duality manifests itself as a global symmetry of a maximal supergravity (it does not follow from this discussion that it cannot be local, but at least it s a global symmetry). Similarly, since all half-maximal supergravities with 16 supercharges get mapped to the same supergravity theory by a dimensional reduction over a torus, T-duality manifests itself as a global symmetry group in supergravity theories with a total of 16 supercharges.

2.4.4 Gauged Supergravity

In sections 2.4.3 and 2.4.2 we came across the concept of a global symmetry group, which is a symmetry of the equations of motion. Below ten dimensions S- and T-duality manifest themselves as a global symmetry, and the combination of these gives rise to a so-called U-duality group. But also R-symmetry gives rise to a global symmetry group. We write G for the total global symmetry group containing the R-symmetry and T- and S-dualities. For example, $d = 4$ $\mathcal{N} = 4$ matter coupled supergravity has $G = SO(6, n) \times SU(1, 1)$.

Except for four-dimensional $\mathcal{N} = 1, 2$ and five-dimensional $\mathcal{N} = 2$ supergravity, all supergravities share the property that the scalars parameterize a coset G/K where K is the maximal compact subgroup of G [71, 72]. The subgroup K must be a local symmetry since the scalars form a coset G/K (see section 2.4.5). The R-symmetry group H_R has a maximal compact subgroup denoted K_R and some authors prefer to reserve the name 'R-symmetry group' for the compact group K_R. For maximal supergravities it turns out that $K = H_R = K_R$.

Only a subgroup G' of G is a global symmetry of the action; in the example of $d = 4$ $\mathcal{N} = 4$ matter coupled supergravity we have $G' = SO(6, n)$. The vector fields present in the theory transform in a linear representation of G', but the field strengths and their duals make up a representation of G. Elements of G that are not in G' thus rotate electric and magnetic field strengths into each other. In $d = 4$ $\mathcal{N} = 4$ matter coupled supergravity the elements in G that are not in G' form the group $SU(1, 1)$ and as mentioned before this is not a symmetry of the action but rotates the electric

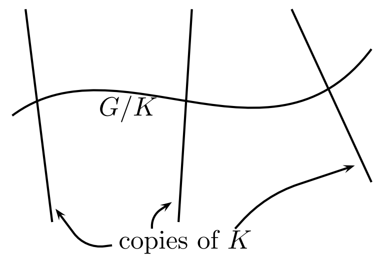


Figure 2.4.1: A graphical representation of the group G as the manifold G/K with at every point $p \in G/K$ a copy of K attached.

and magnetic field strengths into each other.

Suppose there are m vector fields transforming in a linear representation of G', then we can try to gauge a subgroup $F \subset G'$ of dimension m. The gauging can be done by introducing minimal couplings with a coupling constant g and where the gauge fields are in the adjoint of F. To retain supersymmetry the fermion supersymmetry transformation rules must include extra terms linear in g and the action obtains a potential proportional to g^2. The resulting supergravity is called a gauged supergravity and the symmetry group is now the product of F and K and this is a local symmetry.

If G is compact, the whole construction becomes almost trivial since the scalars then parameterize the trivial coset 1, i.e. there are no scalars and no scalar potential can arise.

2.4.5 Coset Scalars

As mentioned before, the scalars in supergravity theories often parameterize cosets. In this section we give some details about cosets and the nonlinear sigma models, i.e. models where the scalars parameterize a coset. Most of the material can also be found in reviews and textbooks, such as [25, 73, 74].

Suppose G is a group and K is a subgroup of K. We can define an equivalence relation in G by calling two elements g and g' of G equivalent if there is an element $k \in K$ such that $g = g'k$. The equivalence classes $[g]$ form a coset. Since every element of G belongs to an equivalence class one can define a projection $\pi : G \to G/K$; the projection π is called the canonical projection. If $[g] = [g']$, then $g = g'k$ for some $k \in K$, and $\pi(g) = \pi(g')$ and hence $\pi(K) = e$ with e the identity element of G. The inverse map $\pi^{-1}[g] = \{gk | k \in K\}$ defines a copy of K at g. This gives an intuitive picture of cosets; the group G can be seen as the coset G/K where at each point of G/K is a copy of K attached.

For every equivalence class one can choose a representative g, which defines a map from G/K to G. If G is thought of as G/K with at every point of G/K a copy of K attached, then the representative can be thought of as a slice through G that intersects every copy of K precisely once.

If G is a Lie group and K is any Lie subgroup of G, the (left) coset G/K is a manifold and can thus be described locally by coordinates y^i. Since any two points p and p' on G/K can by construction be connected by an action of G, the manifold is a homogeneous space.

Suppose the Lie algebras of G and K are $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Then the Lie algebra can be decomposed as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is the complement of $\mathfrak{k}$ in $\mathfrak{g}$. If the decomposition can be done such that the commutation relations are schematically as

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}\,, \;\; [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}\,, \;\; [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}\,, \tag{2.4.5}$$

then the homogeneous space is called symmetric. As an example we mention the n-sphere S^n: $S^n \cong SO(n+1)/SO(n)$.

If ρ is a faithful representation of $\mathfrak{g}$ then by exponentiating $\rho(\mathfrak{p})$ we obtain a parametrization of G/K. More explicitly, if p_i is a basis for $\mathfrak{p}$ the manifold G/K can be parameterized by $\exp y^i \rho(p_i)$. Another and equivalent parametrization of G/K is obtained by right-multiplying this parametrization by an (y-dependent) element of K. The parametrization of G/K by $\exp y^i \rho(p_i)$ is in most cases not useful for practical calculations.

A representative $L(y)$ of G/K is an element of G such that if $y \neq y'$ there is no element k of K such that $L(y)k = L(y')$. Left-multiplication by an element of $g \in G$ takes one coset into another, hence we have

$$\rho(g)L(y) = L(y')k\,, \quad k \in K\,, \tag{2.4.6}$$

where the element $k \in K$ depends on both y and g. Right-multiplication by an element of K leaves the point fixed, but gives another representative; in terms of the picture 2.4.1, every representative is shifted along the 'fibre' K.

The parametrization of G/K by exponentiating $\rho(\mathfrak{p})$ is thus an example of constructing representatives of G/K in an explicit representation ρ of G. For a representative in general we write $L(y)$ but if the coordinate is of no importance we just write L. It is convenient to think of $L(y)$ as being in an explicit representation of G.

The form $L(y)^{-1}\mathrm{d}L(y)$ is a one-form on G/K that takes values in the Lie algebra $\mathfrak{g}$ and can be decomposed as follows

$$L^{-1}\mathrm{d}L = \omega + e\,, \;\; \omega \in \mathfrak{k}\,, \, e \in \mathfrak{p}\,. \tag{2.4.7}$$

We call e the G/K-vielbein and ω the G/K-connection. The notation and nomenclature is on purpose suggestive; ω can be thought of as a kind of spin connection and

e as a kind of vielbein. In fact, space-time can be thought of as a coset G/K where $G = GCT(\mathcal{M}_d)$ and $K = SO(1, d-1)$.

Suppose we act on L from the left by a y-independent element U of G, then $L^{-1}\mathrm{d}L$ is invariant. If we however act from the right with a local (i.e. y-dependent) element O of K we have

$$\omega \mapsto O^{-1}\omega O + O^{-1}\mathrm{d}O, \;\; e \mapsto O^{-1}eO\,. \tag{2.4.8}$$

Thus ω transforms as a connection while e is covariant. Since $\mathrm{d}^2 = 0$ we obtain

$$\mathrm{d}(L^{-1}\mathrm{d}L) = -L^{-1}\mathrm{d}L \wedge L^{-1}\mathrm{d}L = -(\omega + e) \wedge (\omega + e)\,, \tag{2.4.9}$$

which can again be decomposed in a $\mathfrak{k}$-part and a $\mathfrak{p}$-part. Thus we have

$$\begin{aligned} \mathcal{F}(\omega) &\equiv \mathrm{d}\omega + \omega \wedge \omega = -e \wedge e\,, \\ \mathcal{T} &\equiv \mathrm{d}e + \omega \wedge e + e \wedge \omega = 0\,, \end{aligned} \tag{2.4.10}$$

where $\mathcal{F}$ can be compared with the Riemann tensor and $\mathcal{T}$ with the torsion. The torsion $\mathcal{T}$ vanishes since G/K is a symmetric space.

A nonlinear sigma model describes scalars that parameterize a coset. The scalars can be seen as a map from the space-time manifold $\mathcal{M}_d$ to the coset G/K. Hence we can use the pull-back to obtain one-forms on $\mathcal{M}_d$: $\mathrm{d}\phi^i = \partial_\mu \phi^i \mathrm{d}x^\mu$.

Suppose $\mathcal{V}(\phi^i)$ defines a representative of G/K and takes values in a representation ρ of G, we define $\mathcal{V}(x) \equiv \mathcal{V}(\phi^i(x))$ and

$$\Omega_\mu \mathrm{d}x^\mu \equiv \mathcal{V}^{-1}\mathrm{d}\mathcal{V} = \mathcal{Q}_\mu \mathrm{d}x^\mu + \mathcal{P}_\mu \mathrm{d}x^\mu\,, \tag{2.4.11}$$

where $\mathcal{Q}_\mu$ is the projection of Ω_μ onto $\mathfrak{k}$ and $\mathcal{P}_\mu$ is the projection of Ω_μ onto $\mathfrak{p}$.

Under a local K transformation $\mathcal{V} \mapsto \mathcal{V}k$ we have

$$\mathcal{Q}_\mu \mapsto k^{-1}\mathcal{Q}_\mu k + k^{-1}\partial_\mu k\,, \;\; \mathcal{P}_\mu \mapsto k^{-1}\mathcal{P}_\mu k\,, \tag{2.4.12}$$

and hence a K-covariant derivative is given by

$$\mathrm{D}_\mu \mathcal{V} = \partial_\mu \mathcal{V} - \mathcal{V}\mathcal{Q}_\mu = \mathcal{V}\mathcal{P}_\mu\,. \tag{2.4.13}$$

A Lagrangian that is invariant under both global G-transformation and local K-transformations is given by

$$\mathcal{L} = \tfrac{1}{2}\mathrm{Tr}_\rho\left(\mathrm{D}_\mu \mathcal{V}^{-1}\mathrm{D}^\mu \mathcal{V}\right) = -\tfrac{1}{2}\mathrm{Tr}_\rho\left(\mathcal{P}_\mu \mathcal{P}^\mu\right)\,. \tag{2.4.14}$$

We introduce the projection operator $\Pi : \mathfrak{g} \mapsto \mathfrak{p}$ that projects $\mathfrak{g}$ onto the subspace $\mathfrak{p}$. Using 2.4.5 we find

$$\mathrm{Tr}_{\mathrm{ad}}\left(\mathrm{ad}x\mathrm{ad}(\Pi y)\right) = 0, \quad \forall x \in \mathfrak{k}\,, \forall y \in \mathfrak{g}\,. \tag{2.4.15}$$

For semisimple Lie algebras the trace in any representation can be chosen to be proportional to the trace in the adjoint representation (see appendix B). Therefore if G is semisimple the Lagrangian 2.4.14 can be written as:

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{Tr}_\rho \left(\mathcal{V}^{-1}\partial_\mu \mathcal{V}\Pi(\mathcal{V}^{-1}\partial^\mu \mathcal{V})\right) . \tag{2.4.16}$$

There is another approach to nonlinear sigma models [75]. We start with the representative $\mathcal{V}$ of G/K in an explicit representation ρ and construct a Lagrangian with global G invariance and local K invariance as follows. We introduce a connection A_μ taking values in $\mathfrak{k}$, construct a K-covariant derivative $\mathrm{D}_\mu(A)\mathcal{V} = \partial_\mu \mathcal{V} - \mathcal{V}A_\mu$ and demand that A_μ transforms as a connection under local K-transformations and is invariant under global G-transformation. For this to work we need that if $\mathcal{V}$ is in the representation ρ of G, then A_μ is in the derived representation $\mathrm{d}\rho$ of $\mathfrak{k} \subset \mathfrak{g}$ (see Appendix B).

A Lagrangian that is invariant under local K-transformations and global G-transformations is given by

$$\mathcal{L}' = -\tfrac{1}{2}\mathrm{Tr}_\rho \left(\mathcal{V}^{-1}\mathrm{D}_\mu(A)\mathcal{V}\mathcal{V}^{-1}\mathrm{D}^\mu(A)\mathcal{V}\right) . \tag{2.4.17}$$

The connection A_μ appears only algebraically and thus we can solve its equation of motion. Varying the Lagrangian $\mathcal{L}'$ with respect to A_μ we obtain

$$\mathrm{Tr}_\rho \left(\mathrm{D}_\mu(A)\mathcal{V}\mathcal{V}^{-1}\delta A^\mu\right) = 0\,, \tag{2.4.18}$$

and hence we see that $\mathrm{D}_\mu(A)\mathcal{V}\mathcal{V}^{-1}$ must be perpendicular to $\mathfrak{k}$ since δA_μ is an arbitrary element of $\mathfrak{k}$. For semisimple G this means $\mathrm{D}_\mu(A)\mathcal{V}\mathcal{V}^{-1} \in \mathfrak{p}$, i.e. the gauge field A_μ projects out the $\mathfrak{k}$-part of $\mathrm{D}_\mu(A)\mathcal{V}\mathcal{V}^{-1}$. Thus if we use the equation of motion of A_μ we obtain the Lagrangian for semisimple G:

$$\mathcal{L}' = -\tfrac{1}{2}\mathrm{Tr}_\rho \left(\mathrm{D}_\mu(A)\mathcal{V}\mathcal{V}^{-1}\Pi(\mathrm{D}^\mu(A)\mathcal{V})\right) . \tag{2.4.19}$$

The Lagrangian 2.4.19 is identical to the Lagrangian 2.4.16. Therefore we can conclude that the two methods to arrive at a nonlinear sigma model are equivalent.

The action 2.4.19 still contains local K-symmetry. Therefore we can use K-symmetry to bring $\mathcal{V}$ into a 'nice form', which has a physical interpretation as choosing a gauge. If the subgroup K is the maximal compact subgroup of G the 'nice form' can be made more explicit. According to the Iwasawa decomposition the Lie algebra $\mathfrak{g}$ can be decomposed as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ where $\mathfrak{k}$ is the maximal compact subalgebra, which is the Lie algebra of K, and $\mathfrak{s}$ is a solvable Lie algebra. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ is orthogonal with respect to the Cartan–Killing form. It is a theorem of Lie (see e.g. [76]) that the matrices representing the solvable subalgebra $\mathfrak{s}$ can be brought into upper-triangular form. The upper-triangular form is what is meant by a 'nice form'; the coset G/K can be parameterized by $\exp \mathfrak{s}$ and hence we can use a gauge such that $\mathcal{V} = \exp \rho(S)$ for some $S \in \mathfrak{s}$.

If K is the maximal compact subgroup of G, the matrix $\mathcal{V}$ has an interpretation of a vielbein [75]. The associated 'metric' is given by:

$$\mathcal{M} = \mathcal{V}\mathcal{V}^{\#} \,, \tag{2.4.20}$$

where # stands for generalized transpose (see appendix B). The metric $\mathcal{M}$ is invariant under local K-transformations. The nonlinear sigma model 2.4.16, or equivalently 2.4.19, can be written compactly as:

$$\mathcal{L} = \tfrac{1}{8}\mathrm{Tr}\left(\partial_\mu \mathcal{M}^{-1} \partial^\mu \mathcal{M}\right) \,. \tag{2.4.21}$$

The 'vielbein' $\mathcal{V}$ should not be confused with the object e of equation 2.4.7, which can also be defined as a vielbein, but then as a vielbein on G/K. The 'vielbein' interpretation $\mathcal{V}$ is only restricted to equation 2.4.20.

2.4.6 Relation with String Theory

Supergravity is a low-energy approximation of string theory. In this section we explain briefly what is meant by this.

String theory starts with the assumption that the fundamental building blocks of nature are one-dimensional objects, called strings. A string sweeps out a two-dimensional surface in space-time. This surface is called the worldsheet. The dynamics of a free string can be determined by a field theory living on the two-dimensional worldsheet. Interactions are described by attaching so-called vertex operators to the surface; the vertex operators describe the merging or splitting of strings and are linear in the string coupling constant. By means of the vertex operators one can calculate scattering amplitudes in any order of the string coupling constant. Hence string theory is intrinsically perturbative and therefore one often speaks of string perturbation theory instead of string theory in the hope that we will once find a nonperturbative description.

In ordinary quantum field theory the interactions are described by interaction terms in the Lagrangian. However for string theory there is yet not a (classical) action known that describes the dynamics and interaction of strings. This has serious consequences for obtaining a low-energy approximation of string theory.

The spectrum of string theory consists of a finite number of massless fields, which we collectively denote by ϕ_0, and an infinite tower of massive fields, which we collectively denote by ϕ_H. All masses are integer multiples of $m_s \sim 10^{19}$ GeV in order that the graviton, which is one of the massless fields, interacts with such a strength that Newtonian gravity is reproduced. The massive states are so massive that they cannot be seen by any experiment and thus it is not necessary to take them into account for low-energy physics, i.e. physics at an energy lower than 10^{19} GeV. It is therefore desirable to have a low-energy approximation to string theory.

In ordinary quantum field theory a low-energy approximation is obtained by integrating out the massive modes. In this way an effective low-energy action can be obtained. Suppose string theory were described by an action $S(\phi_0, \phi_H)$, then the effective low-energy action $S_{eff}(\phi_0)$ would be given by

$$\mathrm{e}^{iS_{eff}(\phi_0)} \sim \int D\phi_H \mathrm{e}^{iS(\phi_0,\phi_H)} \,. \tag{2.4.22}$$

The path integral includes loops of massive fields. Therefore the effective action $S_{eff}(\phi_0)$ can be seen as a power series expansion in the Planck constant h.

However, there is no action for interacting strings and hence the procedure of integrating out the massive modes is not at our disposal. One can proceed by studying scattering amplitudes and trying to construct a classical action that reproduces these amplitudes. The leading terms (h^0) in the effective action are the classical limits of the low-energy approximations. Since string theory contains both gravity and supersymmetry, this classical limit of the low-energy action describes a supergravity theory.

Though string theory was thought of as a candidate of a unifying theory, there are five different consistent string theories. All five string theories live in ten-dimensional space-time. Hence we expect at least five ten-dimensional supergravities serving as classical low-energy approximations to the five string theories. It is quite special that the five supergravities exhaust all possible ten-dimensional massless supergravities. In ten dimensions there is a one-to-one correspondence between string theories and supergravity theories.

One might then wonder what the role is of eleven-dimensional supergravity. In a seminal paper [43] Witten showed that the dynamics of strongly coupled Type IIA strings can at low energies be described by an eleven-dimensional supergravity. This result might at first look weird but an elegant intuitive picture to keep in mind is that the eleventh dimension is already present in ten-dimensional Type IIA string theory but has size zero [77]. By increasing the string coupling constant the size of the eleventh dimension grows and at the same time the strings grow in this dimension and become two-dimensional objects, membranes. And indeed, eleven-dimensional supergravity contains a three-form $A_{\mu\nu\rho}$ that can couple naturally to a three-dimensional world volume and thus the three-form couples to membranes.

The theory describing the full dynamics, also including massive fields, of the eleven-dimensional theory is not known yet but already a name is given: M-theory [43,78,79]. It is conjectured that taking limits of different parameters in M-theory one ends up with a superstring/supergravity theory; for example, if we take a classical low-energy limit we obtain eleven-dimensional supergravity. The dualities between different string theories nicely fit into this picture. The dualities correspond to going from one limit to another.

Since string theory is only known in the regime where the string coupling constant is small, the features of the nonperturbative M-theory lie outside the scope of string theory. The nonperturbative regime can however be explored by supergravity theories by means of so-called BPS-configurations. Therefore supergravity is an important tool to get some grip of the until now mysterious M-theory.

Even from another totally different point of view supergravities are interesting. A supergravity theory is a classical field theory and this makes supergravities more accessible for explicit calculations. It is a drawback of superstring theory that its calculations are seldom simple.

Chapter 3

Dimensional Reduction

In the previous chapter we saw that string theory has as a low-energy limit a supergravity theory. String theory is only consistent in ten dimensions making ten-dimensional supergravity theories interesting. In light of the conjectured M-theory [43, 78, 79], eleven-dimensional supergravity [42] becomes evenly interesting. However, nature seems to be four-dimensional. Fortunately almost all supergravities in nine or less dimensions can be obtained from a ten- or eleven-dimensional supergravity by a procedure called dimensional reduction.

The procedure of dimensional reduction relies on a basic idea; imagine a theory living in $D+1$ dimensions and suppose that one dimension is compact and circle-like. If one shrinks the radius of the circle to a size that we cannot see by experiment, we don't see it but it is there.

Getting sensible four-dimensional supergravity theories with a stringy origin is not the only way dimensional reductions are used. Finding solutions in supergravities corresponding to solitonic objects (p-branes), of which it is conjectured that they play the role of a fundamental particle in M-theory, is facilitated by using dimensional reductions. As a first step one obtains from a ten- or eleven-dimensional supergravity a lower-dimensional supergravity by dimensional reduction. Finding a solution in the lower-dimensional supergravitiy and uplifting the solution back to ten (or eleven) dimensions gives rise to a solution in ten (or eleven) dimension, often corresponding to a configuration of p-branes. Trying to find p-brane solutions directly in ten (or eleven) dimensions turns out to be more difficult than using a dimensional reduction (see for example [80]).

This chapter starts with an easy example from which some general features can be understood. Then follows a general discussion on Kaluza–Klein dimensional reduction and some of its applications in studying supergravity theories after which we discuss briefly other kinds of dimensional reduction.

3.1 Example: Fourier on a Circle

Before going into technical details, let us treat a pedagogical example. From this easy example we can already see some important issues and where problems might pop up, if any. See [81] for a similar discussion.

Since in ungauged supergravities most fields are massless, it is not a great loss of generalization if we study massless particles. We take a free massless scalar φ living in $D+1$ dimensions and we curl up one dimension, say the z-direction, to a compact interval of length $2\pi R$ with the endpoints identified. Then the scalar admits a discrete Fourier expansion;

$$\varphi(x^\mu, z) = \sum_{n\in\mathbb{Z}} \varphi_n(x^\mu) \mathrm{e}^{in\frac{z}{R}} \,. \tag{3.1.1}$$

The equation of motion of the scalar φ in $D+1$ dimensions becomes

$$\Box_{D+1}\varphi(x^\mu, z) = 0 \;\Rightarrow\; (\Box_D - \left(\tfrac{n}{R}\right)^2)\varphi_n(x^\mu) = 0 \,, \; \forall n \in \mathbb{Z} \,, \tag{3.1.2}$$

where $\Box_m$ denotes the d'Alembertian in m dimensions. We see that from a D-dimensional point of view the scalar φ has decomposed in a tower of scalars φ_n with masses $|\frac{n}{R}|$. If the radius R is shrunk to a size such that the characteristic mass $M_R = 1/R$ becomes unreachable for detectors, only the massless scalar is an observable particle. In the limit $R \to 0$ we get an infinite mass gap and the massive states decouple.

Keeping all the fields φ_n is not useful and a truncation to a finite set is desirable. From the masses of the fields φ_n we can argue that a low-energy approximation only retains the field φ_0 and sets all the fields φ_n for $n \neq 0$ to zero. With interactions to gravity the concepts of mass and energy become troubled and in general finding a consistent truncation to a finite set of fields corresponding to a low-energy approximation becomes more difficult. Consistency in this context means that the fields that are not put to zero do not give rise to nonzero source terms in the original equations of motion for the fields that are put to zero. In other words, the fields that are truncated away should not reappear through the dynamics of the higher-dimensional theory. For this example the issue of consistency is dealt with easily since there is no interaction between the scalar modes φ_n. In the more general case there are interactions and the issue of consistency becomes more involved. However there exists an argument based on symmetries for dealing with the issue.

We now present a symmetry argument to find a consistent truncation. The group of diffeomorphisms of the circle is generated by an infinite-dimensional Virasoro algebra [82]. This group has a finite one-dimensional subgroup $U(1)$ generated by constants shift along the z-direction. The $U(1)$-symmetry is also present in the lower-dimensional theory. The tower of scalars φ_n comprises all irreducible representations of $U(1)$, which are labelled by integers. Under the action of a $U(1)$ element

$\mathrm{e}^{iz} \mapsto e^{iz+ic}$, corresponding to shifting z to $z+c$, we see that

$$\varphi_n(x) \mapsto \mathrm{e}^{inc/R}\varphi_n(x) \tag{3.1.3}$$

and the scalar φ_0 is the only singlet.

The truncation to a singlet of a symmetry group G is always consistent, as we now argue. The equations of motion of a field $\phi^{(\alpha)}$, where the superscript (α) denotes the representation, are schematically

$$\mathbb{D}\phi^{(\alpha)} = J^{(\alpha)} \tag{3.1.4}$$

where $\mathbb{D}$ denotes some differential operator and $J^{(\alpha)}$ is a function of fields, acting as a source term, which has to be in the same representation of G as $\phi^{(\alpha)}$. If all fields that are not singlets under G are put to zero, the source $J^{(\alpha)}$ must be zero, since out of singlets alone, one cannot build a nontrivial representation. Hence, truncating a theory to the singlets of a symmetry (sub-)group is always a consistent truncation.

3.2 Kaluza–Klein Reductions

The name Kaluza–Klein is from the physicist Klein and the mathematician Kaluza who studied general relativity in 5 dimensions [83, 84]. The reason they did this was that reducing the theory over a circle gave Einstein's theory of General Relativity in four dimensions coupled to Maxwell's theory of electromagnetism. The work of Kaluza and Klein was one of the earlier attempts to unify gravity with other forces.

Formally, a Kaluza–Klein dimensional reduction can be broken down into four steps (the definition that we use for a Kaluza–Klein dimensional reduction follows below in section 3.2.1). These four steps we first explain shortly and in the sections 3.2.1, 3.2.2, 3.2.3 and 3.2.4 we go into more detail for all four steps. See for a similar discussion [85].

The first step is finding a (stable) vacuum in the higher-dimensional theory with a geometry $Y \times X$, where Y is a maximally symmetric space and X is a compact homogeneous symmetric space; $X \cong G/K$ for a group G with subgroup K and if $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K respectively then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.

The second step is analyzing the fluctuations of the fields around their vacuum values. The fluctuations are the lower-dimensional fields. The analysis of the fluctuations is similar to the analysis of the Fourier expansion of the example in section 3.1.

The third step is a truncation to a finite set of fields. The truncation needs to be consistent if we want to use the dimensional reduction procedure as a solution generating technique. At the end of this chapter we shortly discuss some aspects of nonconsistent reductions. The final step consists of finding a lower-dimensional Lagrangian that describes the dynamics of the lower-dimensional fields.

Often these four steps are less distinguished in the literature; one directly writes down a (hopefully) consistent Ansatz for the fields and their fluctuations using a finite set of fields. For clearness we stick as much as possible to the four steps mentioned above.

3.2.1 Different Vacua

A solution that can be found in many supergravities is the solution where the space-time geometry is maximally symmetric and all the fields, except the metric, vanish. The Riemann tensor in a maximally symmetric space is of the special form

$$R_{\mu\nu\lambda\rho} = \tfrac{1}{3}\Lambda(g_{\mu\lambda}g_{\nu\rho} - g_{\nu\lambda}g_{\mu\rho})\,. \tag{3.2.1}$$

The sign of Λ gives three different maximally symmetric spaces. (1) $\Lambda = 0$: D-dimensional Minkowski space-time M^D with zero curvature, (2)$\Lambda < 0$: D-dimensional anti-de Sitter space-time AdS_D with negative curvature and (3)$\Lambda > 0$: D-dimensional de Sitter space-time dS_D with positive curvature.

For a maximally symmetric space, the number of Killing vectors takes the maximal value $\frac{1}{2}D(D+1)$. A Killing vector is a vector field of which the flow consists of lines on which the metric is constant. Hence if X is a Killing vector, then the Lie derivative of the metric with respect to X vanishes: $(\mathcal{L}_X g) = 0$. It follows that a Killing vector satisfies

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0\,. \tag{3.2.2}$$

Another set of vacua is given by spaces of the form $Y^d \times X^p$, $p + d = D$, where Y^d is a d-dimensional maximally symmetric space and X^p is a p-dimensional symmetric compact space. In general the number of symmetries will have decreased. So a physical process $Y^D \to Y^d \times X^p$, which corresponds physically to a spontaneous compactification of space-time, involves symmetry breaking; the higher-dimensional equations of motion exhibit a covariance under general coordinate transformation whereas a solution of the form $Y^d \times X^p$ breaks this covariance. Since we are working with theories involving gravity, we cannot simply calculate the energy difference between two vacua and decide which is the true vacuum. Even if the space-times are of the form $M^d \times T^p$, where T^p denotes a p-torus, which has energy zero we cannot say that these vacua have the same energy as we can not compare "zero apples to zero oranges" [86].

The direct-product structure of the vacuum means that the metric decomposes into a block-diagonal structure

$$\hat{g}_{\hat{\mu}\hat{\nu}}(x,y) = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{\alpha\beta}(y) \end{pmatrix}\,. \tag{3.2.3}$$

The hat indicates the higher-dimensional fields and indices (for conventions on splitting up coordinates and indices, see appendix A). The requirement of maximally symmetric Y is maintained if we take a warped-product metric where $g_{\mu\nu}$ is multiplied by a y-dependent function: $\hat{g}_{\mu\nu}(x,y) = f(y)g_{\mu\nu}(x)$. The geometry of a warped-product space differs significantly from that of a direct-product space (see [85]).

The definition of a Kaluza–Klein dimensional reduction that we use lies in the choice of the compact manifold X. A Kaluza–Klein dimensional reduction is a dimensional reduction where the compact manifold X^p is isomorphic to a symmetric space G/K. The cosets G/K fall into three classes:

- $G/K \cong U(1)^p$. The compact manifold is flat and isomorphic to a torus. Sometimes people refer to these dimensional reductions as the Kaluza–Klein reductions. We have chosen to take a wider definition of Kaluza–Klein reductions following [87].

- $G/K \cong P \subset G$. The compact manifold is a (nonabelian) Lie group P. These reductions are called group manifold reductions and DeWitt seems to be the first to consider them [88].

- G/K a true coset, G nonabelian. For example: the n-sphere[1] $S^n = SO(n+1)/SO(n)$ if $n \neq 3$ and the complex projective spaces $CP^n = SU(n+1)/SU(n)$. It seems Pauli invented these dimensional reductions first in 1953 [89, 90].

We assume G is compact implying that G is a direct product of a semisimple Lie group G_0 and a torus: $G = G_0 \otimes U(1)^m$. We give a proof of this in section 3.4.2.

The Vielbein on G/K

Let us discuss the vacuum configuration of the vielbein on G/K in more detail. The analysis becomes trivial for a torus and hence we assume G to be semisimple and compact. The vielbein vacuum configuration is in a similar fashion as the metric in equation 3.2.3 given by

$$\hat{e}_{\hat{\mu}}^{\hat{a}} = \begin{pmatrix} e_\mu^a & 0 \\ 0 & e_\alpha^m \end{pmatrix}, \tag{3.2.4}$$

where e_μ^a is a vielbein of a maximally symmetric space. We now claim that for e_α^m we can take the components of the G/K-vielbein $e = e_\alpha \mathrm{d}y^\alpha$ as defined by equation 2.4.7, where $e_\alpha \in \mathfrak{p}$. A torus is flat and hence the vielbein can be taken constant and diagonal: $e_\alpha^m = \delta_\alpha^m$. We prove the claim by showing that this choice gives rise to a homogeneous metric, i.e. a metric invariant under the action of G.

[1] Of the spheres S^n only one is a group manifold: $S^3 = SO(4)/SO(3) \cong SO(3) \times SO(3)/SO(3) \cong SO(3)$.

The homogeneous metric is given by the line element:

$$ds^2_{G/K} = -\mathrm{tr}_{\mathrm{ad}}(e \otimes e) = -\mathrm{tr}_{\mathrm{ad}}(e_\alpha e_\beta)\mathrm{d}y^\alpha \otimes \mathrm{d}y^\beta. \tag{3.2.5}$$

Since e is G-invariant, so is $\mathrm{tr}_{\mathrm{ad}}(e_\alpha e_\beta)$. The metric $\langle , \rangle$ defined by 3.2.5 is completely determined by its values at the origin since G-invariance requires $\langle X, Y\rangle_{g^{-1}K} = \langle g_* X, g_* Y\rangle_K$ where g_* is the push-forward of the map on G/K induced by $L(y) \mapsto gL(y)$. We now use the symbol g for the action of $g \in G$ on G/K. For the metric to be well-defined it has to be K-invariant since if $g^{-1}g' \in K$ for $g, g' \in G$, then we need $\langle X, Y\rangle_{gK} = \langle X, Y\rangle_{g'K}$, which is true if $\langle , \rangle$ is K-invariant. But since $\langle , \rangle$ is G-invariant, it is also K-invariant. The choice of parametrization of G/K should have no influence on the metric, hence under $L(y) \mapsto L(y)k(y)^{-1}$, $k(y) \in K$, the metric should be invariant. Under $L(y) \mapsto L(y)k(y)^{-1}$, $k(y) \in K$ we have $e \mapsto kek^{-1}$. Hence a reparametrization acts as a rotation the vector space $T(G/K)_K \cong \mathfrak{p}$ by means of the Adjoint action of K on $\mathfrak{p}$. The metric $\langle , \rangle$ is clearly invariant under the Adjoint action of K on $\mathfrak{p}$ since the trace in the adjoint representation is Ad_G-invariant. This proves the claim (**QED**).

The claim just proved has far reaching consequences. We introduce the matrices $D(k) = \mathrm{Ad}_G(k)|_{\mathfrak{p}}$, $k \in K$ [2], and $|_{\mathfrak{p}}$ denotes the restriction to $\mathfrak{p}$. The group G is semisimple and compact and hence $\mathfrak{g}$ is a compact Lie algebra, implying that the metric defined by 3.2.5 is positive definite. The $\mathrm{Ad}_G(K)$-invariance of the metric implies that the matrices $D(k)$ should be matrices in the vector representation of $SO(p)$. In other words, the representation of K defined by $D(k)$ should be embedded in the vector representation of $SO(p)$, with $p = \dim G/K = \dim \mathfrak{p}$. For the Lie algebra this implies that the adjoint representation of $\mathfrak{k}$ restricted to its action on $\mathfrak{p}$ should consist of antisymmetric matrices, which form the vector representation of $\mathfrak{so}(p)$.

We can in get explicit formulas for the embedding. The basis for the vector representation of $\mathfrak{so}(p)$ is given by the matrices $(\Sigma_{mn})_{pq} = \delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np}$. We introduce a set of basis elements $k_{\bar{i}}$ for $\mathfrak{k}$; we use the basis elements t_m for $\mathfrak{p}$. Barred indices correspond to components with respect to the basis $k_{\bar{i}}$ of $\mathfrak{k}$ while unbarred indices correspond to components with respect to the basis t_m of $\mathfrak{p}$.

From the embedding we see that there should be numbers $c_{\bar{i}np} = -c_{\bar{i}pn}$ such that

$$D(k_{\bar{i}}) = \sum_{p,q} c_{\bar{i}pq}\Sigma_{pq}\,. \tag{3.2.6}$$

From the structure constants $f^p{}_{\bar{m}n}$ defined by $[k_{\bar{m}}, t_n] = t_p f^p{}_{\bar{m}n}$ it follows that the matrices $D(k_{\bar{m}})$ satisfy $D(k_{\bar{m}})_{pq} = -D(k_{\bar{m}})_{qp} = f^p{}_{\bar{m}q} = -f^q{}_{\bar{m}p}$. Using this and taking traces of equation 3.2.6 with Σ_{rs} we find

$$c_{\bar{m}rs} = \tfrac{1}{4}(f^s{}_{\bar{m}r} - f^r{}_{\bar{m}s}) = -\tfrac{1}{2}f^r{}_{\bar{m}s} \Rightarrow D(k_{\bar{m}}) = -\tfrac{1}{2}\sum_{p,q} f^p{}_{\bar{m}q}\Sigma_{pq}\,. \tag{3.2.7}$$

[2] In mathematics, the representation of the group K defined by the matrices $D(k)$ is called the isotropy-representation of K w.r.t. G/K, see e.g. [91]

This result was also obtained by [87].

We can summarize the analysis of the vacuum configuration of the vielbein as follows: the vielbein on G/K, with G semisimple, in the vacuum can be read off from the projection of $L^{-1}\mathrm{d}L$ to $\mathfrak{p}$ and the adjoint action of K on $\mathfrak{p}$ should be embedded in the vector representation of the rotation group $SO(p)$. This last sentence is sometimes paraphrased as that the p-vector of $SO(p)$ should contain the K-content of G/K.

Supersymmetry; Killing Spinors

The vacuum configuration can break supersymmetry. Since in the vacuum configuration the fermions are zero, supersymmetry acts trivially on the bosons. The question whether supersymmetry is broken or not, is whether the supersymmetry variation of the fermions vanishes or not. If there is no flux the Einstein equation is $\hat{R}_{\hat{\mu}\hat{\nu}} = 0$ and hence the internal manifold is Ricci flat and the lower-dimensional space-time is Minkowski. The variation of the gravitino takes the form $\nabla_{\hat{\mu}}\epsilon = 0$; this is the Killing spinor equation. The integrability condition for the Killing spinor equation is

$$[\nabla_{\hat{\mu}}, \nabla_{\hat{\nu}}]\epsilon = 0 \;\Rightarrow\; R_{\hat{\mu}\hat{\nu}}{}^{\hat{a}\hat{b}}\Gamma_{\hat{a}\hat{b}}\epsilon = 0. \tag{3.2.8}$$

Hence the integrability equation implies that vacuum configurations of the form Minkowski $\times$ torus do not break supersymmetry.

We define the subgroup H of the $Spin(1, D-1)$ group[3] to be generated by the elements $\hat{R}_{\hat{\mu}\hat{\nu}}{}^{\hat{a}\hat{b}}\Gamma_{\hat{a}\hat{b}}$. The group H is called the restricted holonomy group. The integrability equation implies that a covariant spinor exists that is a singlet under the restricted holonomy group. It follows that the decomposition of the spin 1/2 representation of the group $Spin(1, D-1)$ under the subgroup H should contain at least a singlet if we want an unbroken supersymmetry. This puts restrictions on the group H and thus on the geometry. For some examples and a deeper discussion on the relation between Killing spinors see the lecture notes of Gauntlett [92].

If fluxes are present we obtain a modified Killing spinor equation, which schematically takes on the form:

$$\mathrm{D}_\mu\epsilon + X_\mu\epsilon = 0\,. \tag{3.2.9}$$

The integrability condition now implies that an unbroken supersymmetry exists only if space-time has a nontrivial curvature. This of course is not so strange since a flux contributes to the energy momentum tensor, and hence the Einstein equation guarantees that the Ricci tensor does not vanish.

[3]The $Spin(1, d-1)$ is the group generated by the Γ_{ab} and $Spin(1, d-1)$ turns out to be the double cover of $SO(1, d-1)$; also see appendix C.

3.2.2 Expansions of the Fluctuations

The fluctuations of the fields around their vacuum value are the lower-dimensional fields. Nonzero vacuum values of the higher-dimensional fields correspond to parameters in the lower-dimensional theory. Before we truncate, the fluctuations are arbitrary and therefore the expansion of the fluctuations involves the harmonic analysis on symmetric spaces. There is little known for a general symmetric space, due to the lack of an invariant measure. However, the symmetric spaces that we are interested in are of the form G/K with G compact. In these cases an invariant measure exists [93].

Harmonic Expansion on a Group Manifold

If K is trivial, the analysis becomes easier. Let us write $\hat{G}$ for the set of all inequivalent unitary irreducible representations of G. One can prove that all unitary irreducible representations of a compact group are finite-dimensional [93, 94]. Let us label the representations $\rho^s \in \hat{G}$ by the upper index s: $\hat{G} = \{\rho^s | s = 1, 2, \ldots\}$. For a representation $\rho^s \in \hat{G}$ we write the (ij)-matrix element of the unitary matrix representing the group element $g \in G$ as $D^{(s)}(g)_{ij}$ and the dimension of the representation ρ^s we denote by d_s. The matrix-elements are functions on the group G and satisfy the Schur orthogonality relation

$$\int_G D^{(s)}(g)_{ij} D^{(s')}(g^{-1})_{kl} d\mu(g) = \frac{\delta_{il}\delta_{jk}\delta^{ss'}}{d_s}\,, \tag{3.2.10}$$

where μ is the unique invariant Haar measure on G normalized such that $\mu(G) = 1$. Hence the functions $\sqrt{d_s} D^{(s)}(g)_{ij}$ form an orthonormal base for a subset of $L^2(\mu, G)$, the set of all square integrable functions on G. It is the famous Peter–Weyl theorem [95] that states that the matrix-elements are a dense subset in $L^2(\mu, G)$.

The Peter–Weyl theorem implies that any function $\Psi \in L^2(\mu, G)$ can be expanded as

$$\Psi(g) = \sum_s \sum_{1 \le i,j \le d_s} \sqrt{d_s} D^{(s)}(g)_{ij} \hat{\psi}^s_{ji}\,, \tag{3.2.11}$$

where

$$\hat{\psi}^s_{ij} = \sqrt{d_s} \int_G D^{(s)}(g^{-1})_{ij} \Psi(g) d\mu(g)\,. \tag{3.2.12}$$

The equations 3.2.11-3.2.12 determine the harmonic expansion on G.

If $G = U(1)^n$ the irreducible representations are given by the one by one matrices $D^{\vec{n}}(\vec{\theta}) = \mathrm{e}^{\vec{\theta}\cdot\vec{n}}$, where $\vec{n} \in \mathbb{Z}^n$ is a vector with n integers and $\vec{\theta} = (\theta_1, \ldots, \theta_n)$. Therefore on the torus the standard Fourier analysis is obtained.

Tensorial fields on G are fields that carry a representation of the group that rotates the tangent space. The largest possible group that rotates the tangent space isometrically is $SO(\mathrm{d})$, with $\mathrm{d} = \dim G$. On a group manifold all fields carry in a natural way representations of G; therefore every $SO(\mathrm{d})$-representation is decomposed in irreducible G-representations. A field on G that takes values in a G-representation can be expanded in a similar way as 3.2.11; the columns of a matrix in a representation of G are basis elements of this representation. Therefore a field $\psi_i(g)$ in the irreducible representation ρ can be decomposed as

$$\psi_i(g) = \sum_j \sqrt{d_\rho} D^{(\rho)}(g)_{ij} \hat{\psi}_j \,. \tag{3.2.13}$$

The inversion formula to obtain $\hat{\psi}_j$ is obtained using the orthogonality relation 3.2.10 of the matrix elements $\sqrt{d_\rho} D^{(\rho)}(g)_{ij}$.

Spinors can be defined if the group manifold admits a spin structure. We do not go into details on the existence of spin structures.

Harmonic Expansion on a Symmetric Space

The harmonic expansion on coset manifolds was outlined in [87]. We show how the procedure works and give an easy example. For more details we refer to [85,87,96,97].

From picture 2.4.1 we can see the group manifold G can be seen as $G \cong (G/K) \ltimes K$. This suggests that we can still use the matrix-elements of the group G to expand the fields on G/K. The fields carry representations of the tangent space group, which is $SO(n)$, where $n = \dim G/K$. Since the compact group K is a subgroup of $SO(n)$ we can decompose all tangent space representations into K-representations, see also the discussion about the vielbein on G/K in section 3.2.1.

We give two examples. On the seven sphere $S^7 \cong SO(8)/SO(7)$ the tangent space group is $SO(7)$, which coincides with K and hence all decompositions are trivial; any irreducible $SO(7)$-representation is an irreducible K-representation. The manifold $SU(3)/SU(2) \times U(1)$ has dimension 4 and therefore the tangent space group is $SO(4) \cong SU(2) \times SU(2)$. The vector representation of $SO(4)$ is thus the $(2,2)$ representation, i.e. the tensor product of the vector representations of both $SU(2)$ factors. The decomposition of this vector representation with respect to the subgroup is [87]:

$$(\mathbf{2},\mathbf{2}) \rightarrow \mathbf{2}_{-1} \oplus \mathbf{2}_{+1} \,, \tag{3.2.14}$$

where the symbol $\mathbf{2}_{-1}$ means the $\mathbf{2}$ representation of $SU(2)$ where all vectors in this representation carry $U(1)$-weight -1.

We take a field $\phi_i(g)$ where the index i indicates the components in a representation μ of K. If g_1 and g_2 are in the same coset, there is an element $k \in K$ such that $g_1 = g_2 k$. In terms of the picture 2.4.1 this corresponds to moving along a fibre. Therefore for the field ϕ_i we require

$$\phi_i(gk) = \sum_j D^{\mu}_{ji}(k) \phi_j(g) \,, \tag{3.2.15}$$

where $D^\mu(k)$ is a matrix in the representation μ of K. We would like to relate this to the expansion 3.2.11. This can only be done if the sum includes only those representations of G that contain the representation μ of K. For every representation λ of G we have to do a decomposition with respect to K and see whether the decomposition contains μ. The decomposition does depend on the embedding of K in G; for example there is a family of manifolds corresponding to the cosets $SO(5)/SO(3)$ differing in how the $SO(3)$ subgroup is embedded into $SO(5)$ and the members of the family differ in topology and in how the $SO(5)$ irreducible representations are decomposed into $SO(3)$ irreducible representations [85].

If the decomposition of an irreducible G-representation λ into irreducible K-representations reads $\lambda \to \mu_1 \oplus \ldots \oplus \mu_r$ then any matrix representing an element $k \in K$ in the λ representation of G can be written as

$$D^\lambda_{ij}(k) = \begin{pmatrix} D^{\mu_1}(k) & 0 & \ldots \\ 0 & D^{\mu_2}(k) & \ldots \\ \vdots & \ddots & \ldots \\ 0 & \ldots & D^{\mu_r}(k) \end{pmatrix}_{ij} . \tag{3.2.16}$$

Since a K-representation μ can occur multiply in the decomposition of λ, we introduce an extra label ζ that labels the μ representations in λ:

$$\lambda \to \bigoplus_{\zeta=1}^{m(\lambda,\mu)} \mu^\zeta \oplus \ldots , \tag{3.2.17}$$

where the ellipsis contains representations not equivalent to μ and $m(\mu,\lambda)$ is the number how many times μ occurs in the decomposition of λ.

Since the columns of a matrix in a representation of any group form a basis for the module associated with that representation, one sees that the appropriate expansion of the field ϕ_i in the μ-representation of K is

$$\phi_i(g) = \sum_{\lambda \in \hat{G}} \sum_{\zeta=1}^{m(\mu,\lambda)} \sum_m D^\lambda_{m,(i\zeta)}(g) \phi_{\lambda\zeta m} , \tag{3.2.18}$$

where the index $(i\zeta)$ is a compound index corresponding to the block decomposition 3.2.16, i.e. ζ labels the blocks corresponding to the μ representation and within a block the rows and columns are labelled by the index i.

We can resolve an element $g \in G$ into a product $L(y)k$ and obtain the inversion formula to 3.2.18:

$$\phi_{\lambda\zeta m} = \frac{V_K d_\lambda}{d_\mu} \int_{G/K} \mathrm{d}\mu(L(y)) \phi_i(L(y)) D^\lambda_{(i\zeta),m}(L^{-1}(y)) , \tag{3.2.19}$$

where d_μ denotes the dimension of the irreducible representation λ, V_K denotes the volume of K and $\mathrm{d}\mu(L(y))$ denotes the invariant Haar measure on G evaluated at $L(y)$.

Example: Harmonic Expansion on S^2

The two-sphere S^2 is the symmetric space $SO(3)/SO(2)$. The irreducible representations of $SO(2)$ are labelled by integers n and a vector v in the one-dimensional irreducible representation of $SO(2)$ associated with the integer n transforms as $v \mapsto \mathrm{e}^{in\theta}v$, where θ parameterizes $SO(2)$: $0 \leq \theta < 2\pi$. The irreducible nonspinorial representations of $SO(3)$ are labelled by nonnegative integers j and have dimensions $2j+1$. The decomposition of the j-representation of $SO(3)$ with respect to $SO(2)$ is:

$$j \to 1^{-j} \oplus 1^{-j+1} \oplus \dots 1^{j-1} \oplus 1^{j}\,, \tag{3.2.20}$$

where the upper index denotes the integer corresponding to the $SO(2)$ irreducible representation. The decomposition 3.2.20 follows from the fact that any element of $SO(3)$ can be taken to generate the Cartan subalgebra, which is isomorphic to $SO(2)$. The n-representation of $SO(2)$ is thus contained in all $SO(3)$-representations with $j \geq |n|$.

Let us denote the three independent generators in $SO(3)$ by T_1,T_2 and T_3 with commutation relations $[T_i, T_j] = \sum_k \epsilon_{ijk} T_k$. The $SO(2)$-subgroup is generated by T_3. The two-sphere can be parameterized by $L(\theta,\phi) = \exp(-\theta T_1 - \phi T_2)$, which is inconvenient for practical calculations. A parametrization more suited for practical calculations is:

$$L(y) = \exp(-\phi T_3)\exp(-\theta T_2)\,. \tag{3.2.21}$$

The appearance of the T_3 generator and no T_1 generator might worry the reader, but using the Baker–Campbell–Hausdorff formula the parametrization 3.2.21 can be related to a parametrization involving T_1 in the exponent. Calculating $L^{-1}\mathrm{d}L$ one finds the vielbein e and the K-connection ω:

$$e = -T_2\mathrm{d}\theta + T_1 \sin\theta \mathrm{d}\phi, \quad \omega = -\cos\theta T_3 \mathrm{d}\phi\,. \tag{3.2.22}$$

Therefore the invariant metric is given by the line element

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\,. \tag{3.2.23}$$

The decomposition of irreducible $SO(3)$-representations, which are labelled by the nonnegative integer j, into $SO(2)$ irreducible representations corresponds to taking T_3 diagonal with eigenvalues $-\sqrt{-1}j, \dots, +\sqrt{-1}j$. Since the tangent space is $SO(2)$ there is no need to decompose the tangent space group with respect to the subgroup $K = SO(2)$. The label ζ running from 1 to $m(n,j)$ is now unnecessary since the

$SO(2)$ representations occur at most once in every $SO(3)$ representation. Any field ψ_n in the n-representation of $SO(2)$ can now according to 3.2.18 be decomposed as

$$\psi_n(\theta,\phi) = \sum_{j=|n|}^{\infty} \sum_{m=-j}^{j} \sqrt{2j+1} D^j_{mn}(\theta,\phi)\,. \tag{3.2.24}$$

The so-called harmonics $D^j_{mn}(\theta,\phi)$ satisfy

$$\begin{aligned} D^j_{mn}(\theta,\phi) &= D^j_{mn}(\mathrm{e}^{-\phi T_3}\mathrm{e}^{-\theta T_2}) = \sum_{k=-j}^{j} D^j_{mk}(\mathrm{e}^{-\phi T_3}) D^j_{kn}(\mathrm{e}^{-\theta T_2}) \\ &= \mathrm{e}^{-im\phi} D^j_{mn}(\mathrm{e}^{-\theta T_2}) = \mathrm{e}^{-im\phi} D^j_{mn}(0,\theta)\,, \end{aligned} \tag{3.2.25}$$

since T_3 is diagonal. For $n = 0$ the familiar expansion of a function in harmonics on the sphere as treated in most quantum mechanics textbooks is recovered:

$$\psi(\theta,\phi) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \sqrt{2j+1}\mathrm{e}^{-im\phi} P^j_m(\theta)\,, \tag{3.2.26}$$

where $P^j_m(\theta) = D^j_{m,0}(0,\theta)$.

The harmonics on the sphere are usually found by solving the Laplace equation on the sphere. This holds in general: the harmonics are found by analyzing the spectrum of a G-invariant second or first order differential operator (the mass operator), such as the Laplacian, Dirac and Lichnierowicz operator, see e.g. [85, 93, 97]. The eigenfunctions of these operators form a dense subset of $L^2(\mu, G/K)$.

The Full Expansion of the Fluctuations

For the fluctuations of the fields around their vacuum we can thus write an expansion like 3.2.18 but where the coefficients $\hat{\psi}_{\lambda m}$ are functions of x, the lower-dimensional space-time coordinate. This is in principle how it works, but as we shall see, we don't have to follow this tedious programme; things can be simplified significantly.

Without proof we quote a result of [87, 98] that states that the spectrum of fluctuations of the metric always contains a graviton and Yang–Mills fields. In a few cases it is explicitly proved that the Yang-Mills fields are related to the Killing vectors of G/K. It is generally believed that the off-diagonal metric fluctuations always (and not just in those cases where it was shown to be so) contain the terms [99]

$$g(x,y)_{\mu\alpha} = \sum_I A^I_\mu(x) K^I_\alpha(y) + \dots\,, \tag{3.2.27}$$

where $K^I_\alpha = g_{\alpha\beta} K^{I\beta}$ and $K^{I\alpha}\partial_\alpha$ is a Killing vector of the vacuum metric $g_{\alpha\beta}$ on G/K. The field A^I_μ is called the Kaluza–Klein vector and it is a gauge boson in the lower-dimensional theory. This can be seen as follows. The Killing vectors generate a Lie subgroup of G;

$$[K^I, K^J] = \mathcal{L}_{K^I}(K^J) = f_L^{IJ} K^L \,. \tag{3.2.28}$$

For a general coordinate transformation with parameter $\xi^\alpha = \epsilon^I(x) K^{I\alpha}(y)$ we have

$$\nabla_{(\alpha}\xi_{\beta)} = 0 \,. \tag{3.2.29}$$

From this it follows that from a lower-dimensional point of view, the field A_μ^I has changed according to $\delta A_\mu^I = \mathrm{D}_\mu \epsilon^I$, where D_μ is the covariant derivative of the gauge group generated by the Killing vectors. Hence the field A_μ^I behaves like a Yang–Mills gauge fields.

One might expect that the Killing vectors generate the full isometry group G, but when fluxes are turned on, the vacuum can transform nontrivially under the isometry group and the symmetry group of the vacuum can be broken down to a subgroup of the isometry group G [100–102].

3.2.3 Consistent Truncation

Keeping all fluctuations corresponds to keeping the full theory, but written in an inelegant form. A crucial ingredient in doing a consistent dimensional reduction is finding a good truncation. A consistent dimensional reduction is by definition a dimensional reduction of which the solutions of the lower-dimensional theory can be uplifted to a solution of the higher-dimensional theory. A necessary condition for consistency is that the fields that are truncated away do not reappear through their higher-dimensional equations of motion.

In the example of section 3.1 we truncated the massive fields away, corresponding to a low-energy approximation. For a general vacuum we do not know what is meant by mass or energy. But if there is a gauge symmetry, we have a sense of massless, since a gauge field is massless.

There is no general recipe for finding a suitable truncation. For the toroidal and group reductions we can find general truncations, which we explain in the section where these reductions are discussed. For (truly) coset reductions the matter is tedious and only for a few reductions consistency has been established. Luckily these are precisely the interesting cases from a string/M-theory point of view; for example eleven-dimensional supergravity on $AdS_4 \times S^7$ is known to admit a consistent Kaluza–Klein reduction [103].

Sometimes it is possible to break supersymmetry by the truncation. The fluctuations in general make up supersymmetry multiplets although the vacuum might not be supersymmetric. If the fluctuations can be truncated to a subsector that does not itself form a supersymmetry multiplet with respect to the original supersymmetry algebra, the lower-dimensional theory has less supersymmetry.

3.2.4 The Lower-Dimensional Lagrangian

To obtain the lower-dimensional Lagrangian one takes often the Kaluza–Klein Ansatz and plugs this into the higher-dimensional Lagrangian and then expressing everything in lower-dimensional quantities and integrating over G/K. Hence we need to know what the Kaluza–Klein Ansatz is and how to find it.

The word Kaluza–Klein Ansatz is a misnomer, it is not an Ansatz. After the truncation we are left with a finite number of lower-dimensional fields. The fluctuations of the higher-dimensional fields can be expanded in the lower-dimensional fields. This expansion is called the Kaluza–Klein Ansatz [100].

In a toroidal reduction the truncation is clear and the Kaluza–Klein Ansatz follows straightforward. For the toroidal reduction and group manifold reduction we explicitly discuss the Kaluza–Klein Ansätze in the corresponding sections. For the (truly) coset reduction we refer to the existing literature, e.g. [96, 104–111] and references therein. Note that a Kaluza–Klein Ansatz is not unique; two different Kaluza–Klein Ansätze are related by a lower-dimensional field redefinition.

Typical for Kaluza–Klein reductions is that the expansion around the vacuum is not written explicitly, but implicitly. Let us give an example, where we do a dimensional reduction from five to four dimensions. We choose as vacuum Minkowski in five dimensions - we suppose this is a solution. The vacuum metric is thus

$$\hat{g}_{\hat{\mu}\hat{\nu}}(x,y) = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix} . \tag{3.2.30}$$

The most general five-dimensional metric can be written as

$$\hat{g}_{\hat{\mu}\hat{\nu}}(x,y) = \begin{pmatrix} g_{\mu\nu}(x,y) + \phi^2(x,y)A_\mu(x,y)A_\nu(x,y) & \phi(x,y)A_\mu(x,y) \\ \phi(x,y)A_\nu(x,y) & \phi(x,y) \end{pmatrix} . \tag{3.2.31}$$

Equation 3.2.31 describes the full metric and can be used to obtain the expansions of the fluctuations. On a circle this means we Fourier expand the fields $g_{\mu\nu}$, A_μ and ϕ. The truncation is then to the y-independent sector, which is consistent by the symmetry argument of section 3.1. The truncated metric is given by

$$\hat{g}_{\hat{\mu}\hat{\nu}}(x) = \begin{pmatrix} g_{\mu\nu}(x) + \phi^2(x)A_\mu(x)A_\nu(x) & \phi(x)A_\mu(x) \\ \phi(x)A_\nu(x) & \phi(x) \end{pmatrix} . \tag{3.2.32}$$

In this form the vacuum metric and truncated expansion are put in one single metric. Equation 3.2.32 is called *the* Kaluza–Klein Ansatz for the metric. Since this nomenclature is so often used we stick to the tradition and (ab)use it as well.

For every isometry of the higher-dimensional vacuum configuration there is a massless gauge boson. In this case, massless is defined by having a gauge transformation. The higher-dimensional theory has in many cases symmetries; the symmetries that are not broken or truncated away in the dimensional reduction are present in the lower-dimensional theory. The massless gauge bosons of the lower-dimensional theory are thus in general associated with a subgroup of the isometry group of the vacuum and remnants of higher-dimensional gauge symmetries.

3.3 Toroidal Reductions

A torus is flat and has vanishing Ricci tensor. Hence if there are no fluxes the maximally symmetric space-time Y^d is Minkowski. Thus in the absence of fluxes there appear no new parameters in the lower-dimensional theory. Hence starting with a ten-dimensional supergravity (not Romans type IIA) we will end up in a lower-dimensional ungauged supergravity.

In this section we carry out the dimensional reduction over a torus without fluxes. We first discuss the issue of the expansion and truncation and then motivate the Kaluza–Klein Ansatz. The global symmetry group of the ungauged supergravity and how to find it is discussed in section 3.3.2.

The Field Expansions and Truncations

For the torus T^p the vacuum configuration is almost trivial; the torus is flat and therefore the vielbein on the torus can be taken to be diagonal. Expanding the fields around their vacuum values can be done by standard Fourier analysis and the truncation is to the $U(1)^p$ singlets; i.e. all retained fluctuations are independent of the coordinates on the torus.

The action of the $U(1)^p$ of the torus commutes with the supersymmetry algebra. Hence the zero modes also form a supersymmetry multiplet of a supersymmetry algebra with the same number of supercharges as in the original theory. Therefore the truncation to the zero modes does not break supersymmetry.

Unbroken supersymmetry implies that we already know what the result of the dimensional reduction is. Suppose we start with Type I supergravity, which has 16 supercharges, then in four dimensions we end up with an $\mathcal{N} = 4$ ungauged supergravity since the irreducible spinor in four dimensions has four real components. The number of vector multiplets that is coupled to the pure supergravity multiplet is easily calculated; the two-form gives six vector as well as the metric. Hence we end up with a total of 12 vectors in four dimensions. Pure $\mathcal{N} = 4$ supergravity in four dimensions contains 6 vectors, so we have $\mathcal{N} = 4$ supergravity coupled to 6 vector multiplets. This is also what Chamseddine found [61].

The Kaluza–Klein Ansatz

The Kaluza–Klein Ansatz is a generalization of 3.2.32. The correct generalization describing the lower-dimensional fields is most conveniently found by expanding the fields in the tangent space as we explain below.

The lower-dimensional fields transform under lower-dimensional general coordinate transformations and under internal symmetries that are remnants of the higher-dimensional general coordinate transformations. We want the fields to transform in 'the right way' under the transformations induced by the higher-dimensional general coordinate transformations. The only general coordinate transformations that are consistent with the truncation to the y-independent sector are

$$\text{lower-dimensional g.c.t.} : x^\mu \mapsto x^\mu + \xi^\mu(x) \tag{3.3.1a}$$

$$\text{internal symmetries} : y^\alpha \mapsto \xi^\alpha(x) + \Lambda^\alpha{}_\beta y^\beta \,. \tag{3.3.1b}$$

To find the fields that transform correctly under these symmetries it is convenient to write the Kaluza–Klein Ansatz in the tangent space, since the fields with tangent space indices are inert under general coordinate transformations. This statement also holds for group manifold reductions.

To illustrate the ideas we work out the toroidal reduction of the bosonic sector of ten-dimensional Heterotic supergravity. We illustrate how global symmetries arise and give a general method for finding the global symmetry group from the reduction. We do not write all calculations down in every detail; the details can be found in for example [112].

3.3.1 Heterotic Supergravity on a Torus

We reduce the bosonic sector of ten-dimensional Heterotic supergravity to D dimensions over a d-torus in the absence of fluxes. The ten-dimensional action of the bosonic sector of Heterotic supergravity is given by (see 2.3.12)

$$\mathcal{L}_{Het} = \mathrm{e}^{-\phi}\left(\star\mathbb{1}R + \star\mathrm{d}\phi\wedge\mathrm{d}\phi - \tfrac{1}{2}\star H\wedge H - \tfrac{1}{2}\mathrm{Tr}\star F\wedge F\right) \,. \tag{3.3.2}$$

where we rescaled the ten-dimensional dilaton ϕ. The Kaluza–Klein Ansatz for the metric is given by the line element

$$\mathrm{d}s^2_{10} = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + G_{\alpha\beta}(\mathrm{d}y^\alpha - V^\alpha)(\mathrm{d}y^\beta - V^\beta)\,. \tag{3.3.3}$$

In this line element $V^\alpha = V^\alpha_\mu \mathrm{d}x^\mu$ is the Kaluza–Klein vector. We can equally well specify the vielbein by

$$\hat{e}^a = e^a, \quad \hat{e}^m = \Phi^m_\alpha(\mathrm{d}y^\alpha - V^\alpha) \equiv \Phi^m_\alpha f^\alpha\,. \tag{3.3.4}$$

The object Φ^m_α is to be identified with the y-independent term in the expansion of the fluctuations of the vielbein on the torus and it satisfies $\Phi^m_\alpha \delta_{mn} \Phi^n_\beta = G_{\alpha\beta}$.

As mentioned before the Kaluza–Klein Ansatz for the fields is found by expanding the higher-dimensional fields - from now on indicated with a hat - in lower-dimensional fields in the tangent space; we expand the fields with respect to the basis e^a and f^α. Instead of working with e^m we work with f^α; this has the advantage that $\mathrm{d}f^\alpha = -\mathrm{d}V^\alpha \equiv -F^\alpha$. The Kaluza–Klein Ansätze are:

$$\begin{aligned} \hat{\phi} &= \phi + \tfrac{1}{2}\ln\det G_{\alpha\beta}\,, \\ \hat{B} &= B^{(2)} + B^{(1)}_\alpha \wedge f^\alpha + \tfrac{1}{2} B_{\alpha\beta} f^\alpha \wedge f^\beta\,, \\ \hat{A}^I &= A^{I(1)} + A^I_\alpha f^\alpha\,, \end{aligned} \tag{3.3.5}$$

where $B^{(2)} = \frac{1}{2} B_{ab} e^a \wedge e^b = \frac{1}{2} B_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ and similarly: $B^{(1)}_\alpha = B_{\mu\alpha}\mathrm{d}x^\mu$ and $A^{I(1)} = A^I_\mu \mathrm{d}x^\mu$. The index I on A^I is a Yang–Mills index.

In the absence of fluxes the vacuum configuration for the nonmetric fields is trivial; all fields vanish in the chosen vacuum. Hence equations 3.3.5 represent the y-independent terms in the fluctuations of the fields around the vacuum value 0.

Suppose we want nontrivial vacuum values. One then adds a term to the Kaluza–Klein Ansatz of the fields; this term gives rise to a flux-term in the field strength. For example, we could have taken $\hat{A}^I = A^{I(1)} + A^I_\alpha f^\alpha + \alpha^I$, where α^I is a one-form on the torus that is only defined locally. Acting on it with the differential operator gives the flux-term $M^I = d\alpha^I$; this equation means that locally the flux-term M^I can be written as the d of a one-form. If it is possible to write M^I globally on the torus as $M^I = \mathrm{d}\alpha^I$ the flux, which is defined by the integral of M^I over a two-cycle on the torus, vanishes. For the moment we do not use fluxes.

We restrict ourselves to the maximal abelian subalgebra of the Yang–Mills algebra (see also section 2.4.3). This restriction also simplifies the analysis. The Cartan subalgebra of both $SO(32)$ and $E_8 \times E_8$ has dimension 16, but we take this number to be N. This makes contact with four-dimensional $\mathcal{N} = 4$ supergravity, which admits the possibility to couple any number of vector multiplets to the supergravity multiplet.

It is straightforward to calculate from equations 3.3.5 the field strengths. The field strengths can be reassembled in the Lagrangian, which can be integrated over the torus. Setting $\int_{T^\mathrm{d}} d^\mathrm{d}y = 1$, one obtains the D-dimensional Lagrangian

$$\begin{aligned} \mathcal{L} =& \mathrm{e}^{-\phi}\Big(\star\mathbb{1}R + \star\mathrm{d}\phi \wedge \mathrm{d}\phi - \tfrac{1}{2}\sum_I \star F^I \wedge F^I - \tfrac{1}{2}\sum_I G^{\alpha\beta} F^I_\alpha F^I_\beta \\ & - \tfrac{1}{2}\star \mathrm{d}G^{\alpha\beta} \wedge \mathrm{d}G_{\alpha\beta} - \tfrac{1}{2} G_{\alpha\beta} \star F^\alpha \wedge F^\beta - \tfrac{1}{2}\star H^{(3)} \wedge H^{(3)} \\ & - \tfrac{1}{2} G^{\alpha\beta} \star H^{(2)}_\alpha \wedge H^{(2)}_\beta - \tfrac{1}{4} G^{\alpha\beta} G^{\gamma\delta} \star H^{(1)}_{\alpha\gamma} \wedge H^{(1)}_{\beta\delta}\Big)\,. \end{aligned} \tag{3.3.6}$$

The field strengths are given by[4]

$$\begin{aligned} F^I &= \mathrm{d}A^I - A^I_\alpha F^\alpha\,, \qquad F^I_\alpha = \mathrm{d}A^I_\alpha\,,\\ H &= \mathrm{d}B + B_\alpha \wedge F^\alpha - \tfrac{1}{2}A^I \wedge F^I\,,\\ H_\alpha &= \mathrm{d}B_\alpha + B_{\alpha\beta}F^\beta - \tfrac{1}{2}A^I \wedge \mathrm{d}A^I_\alpha\,,\\ H_{\alpha\beta} &= \mathrm{d}B_{\alpha\beta} + \tfrac{1}{2}\left(A^I_\alpha \mathrm{d}A^I_\beta - A^I_\beta \mathrm{d}A^I_\alpha\right)\,. \end{aligned} \tag{3.3.7}$$

To extract physical information from a supergravity action it is useful to 'go the Einstein frame'. By this we mean that a Weyl rescaling $g_{\mu\nu} \mapsto \mathrm{e}^{a\phi} g_{\mu\nu}$ is performed such that the resulting action contains the Ricci scalar without a dilatonic prefactor. If the Weyl rescaling has been performed the kinetic term for the dilaton acquires the canonical form $-\frac{1}{2} \star \mathrm{d}\phi \wedge \mathrm{d}\phi$.

As mentioned before, the action 3.3.6 describes ungauged $\mathcal{N} = 4$ supergravity coupled to 6 abelian Yang–Mills multiplets. In accordance with this, there appears no potential in the action 3.3.6.

The scalars $G_{\alpha\beta}$, $B_{\alpha\beta}$ and A^I_α describe an $SO(\mathrm{d}, \mathrm{d}+N)/SO(\mathrm{d}) \times SO(\mathrm{d}+N)$ coset [75]. In reference [75] it was shown that the scalars can be rearranged into the $(2\mathrm{d}+N) \times (2\mathrm{d}+N)$-matrix

$$M = \begin{pmatrix} G^{-1} & -G^{-1}C & -G^{-1}a^T \\ -C^T G^{-1} & G + C^T G^{-1} C + a^T a & C^T G^{-1} a^T + a^T \\ -aG^{-1} & aG^{-1}C + a & 1 + aG^{-1}a^T \end{pmatrix}\,, \tag{3.3.8}$$

where $a = (A^I_\alpha)$, $B = (B_{\alpha\beta})$ and $C = \frac{1}{2}a^T a + B$. The matrix M is a symmetric $SO(\mathrm{d}, \mathrm{d}+N)$-matrix; M satisfies $M^T \eta M = \eta$ where

$$\eta = \begin{pmatrix} 0 & \mathbb{1} & 0 \\ \mathbb{1} & 0 & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix}\,. \tag{3.3.9}$$

The kinetic terms of the scalars appearing in the action 3.3.6 can now be written as $\frac{1}{8}\mathrm{tr}(\partial_\mu M^{-1} \partial^\mu M)$. The reader who wants to check this is warned; the calculation is tedious. For more details on this construction see [75]. In section 3.3.2 we develop a method how to recognize the coset that is parameterized by the scalars.

Circle-by-Circle Approach

The n-torus is topologically equivalent to the direct product of n circles. Hence we can do a torus reduction step by step, where at every step we do a circle reduction. Though this procedure might look a little clumsy, as we will see in the next section, one can obtain useful information in a circle-by-circle approach.

[4] Any repeated Yang–Mills index $I, J, \ldots$ is summed over.

The Kaluza–Klein Ansatz for the metric for reducing over a circle is:

$$\mathrm{d}s^2_{D+1} = \mathrm{e}^{2\alpha\varphi}\mathrm{d}s^2_D + \mathrm{e}^{-2\alpha(D-2)\varphi}(\mathrm{d}y + V_\mu \mathrm{d}x^\mu)^2\,, \tag{3.3.10}$$

where the number α is given by

$$\alpha = \frac{1}{\sqrt{2(D-1)(D-2)}} \equiv \tfrac{1}{2}s\,. \tag{3.3.11}$$

The value of α is chosen such that the action obtained from the reduction maintains in Einstein frame if the action was originally in Einstein frame. By performing a circle-by-circle reduction the internal metric is upper triangular. At every reduction step we 'produce' a dilaton φ_i and they can be assembled in a vector $\vec{\varphi}$. Every Kaluza–Klein vector V^i_μ gives rise to scalars A_{ij}, which is only nonzero if $i < j$.

All steps can be rearranged in one step, where the line element of the Kaluza–Klein Ansatz is given by

$$\mathrm{d}s^2_{10} = \mathrm{e}^{\frac{1}{2}\vec{g}\cdot\vec{\varphi}}\mathrm{d}s^2_D + \sum_{i=1}^{\mathrm{d}} \mathrm{e}^{2\vec{\gamma}_i\cdot\vec{\varphi}}(h^i)^2\,, \quad \mathrm{d} + D = 10\,, \tag{3.3.12}$$

where

$$\vec{\varphi} = (\varphi_1, \ldots, \varphi_{\mathrm{d}})\,, \quad \vec{g} = 2(s_1, s_2, \ldots, s_{\mathrm{d}})\,, \tag{3.3.13a}$$

$$\vec{\gamma}_i = \frac{1}{4}\vec{g} - \frac{1}{2}\vec{f}_i\,, \quad \vec{f}_i = (\underbrace{0, \ldots, 0}_{i-1}, (9-i)s_i, s_{i+1}, \ldots, s_{\mathrm{d}})\,, \tag{3.3.13b}$$

$$h^i = \mathrm{d}y^i + V^i_\mu \mathrm{d}x^\mu + \sum_{i<j\leq n} A_{ij}\mathrm{d}y_j\,. \tag{3.3.13c}$$

From 3.3.13c one sees that the internal metric is upper triangular. A disadvantage of performing a dimensional reduction following the procedure sketched above is that the distinction between tangent space indices and coordinate indices is less clear though this distinction can be restored [D].

3.3.2 Global Symmetries Analysis

In all but three ungauged supergravity theories the scalars parameterize a coset G/K, where G is the global symmetry group and K is the maximal compact subgroup of G. The scalar manifolds of this kind that appear in supergravities are maximally noncompact irreducible Riemannian symmetric spaces.

An irreducible Riemannian symmetric space is a manifold G/K associated to a triple $(\mathfrak{g}, \mathfrak{k}, \theta)$, where $\mathfrak{g}$ is the real simple Lie algebra of G, $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$ and the Lie algebra of K and θ is an involutive automorphism of which $\mathfrak{k}$ is the eigenspace with eigenvalue +1. The triples $(\mathfrak{g}, \mathfrak{k}, \theta)$ with the just mentioned properties are classified [94, 113].

We are interested in the cases where the manifold G/K is such that $\mathfrak{k}$ is the maximal compact subalgebra of $\mathfrak{g}$; then the involutive automorphism θ becomes the Cartan involution and G/K is the maximally noncompact irreducible Riemannian symmetric space. In appendix B we give a classification of the maximally noncompact irreducible Riemannian symmetric spaces.

The real Lie algebra $\mathfrak{g}$ can be decomposed as:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \,, \tag{3.3.14}$$

where $\mathfrak{k}$ is as before, $\mathfrak{a}$ is the noncompact Cartan subalgebra and $\mathfrak{n}$ is the nilpotent subalgebra spanned by the positive restricted root generators. The decomposition 3.3.14 is called the Iwasawa decomposition of $\mathfrak{g}$. It follows from the Iwasawa decomposition that $G/K \cong \exp(\mathfrak{a} \oplus \mathfrak{n})$. The elements of $\mathfrak{a}$ are called noncompact Cartan generators. In the remainder we try to minimize the Lie algebra technicalities, which we review in appendix B.

The classification of the noncompact cosets G/K gives rise to a simple technique for finding the global symmetry group G from the toroidal dimensional reduction without having to do the sophisticated guess as in equation 3.3.8. For maximal supergravities this ‘trick’ was already found by [114, 115]. With a little group theory the analysis can be extended to nonmaximal supergravities as we show in this section. The discussion is based on [D] and details can be found in this reference.

The analysis of the global symmetries starts with a circle-by-circle reduction of an action in Einstein frame. At every step the Kaluza–Klein Ansatz for the metric is given by equations 3.3.10 and 3.3.11. At the ith step we write α_i for the corresponding α and φ_i for the corresponding φ.

The idea behind the analysis is the following. The scalars that emerge from the dimensional reduction appear in two disguises; either they appear only with derivatives or they also appear in exponential couplings to other fields. The scalars of the first kind are called axions and the scalars of the second kind are called dilatons. The dilatons φ_i can be arranged in a vector $\vec{\varphi}$ and the coupling of the dilatons to other fields is of the form

$$\exp(\vec{\varphi} \cdot \vec{c}) \star F_{(p+1)} \wedge F_{(p+1)} \,, \tag{3.3.15}$$

where $F_{(p+1)}$ is the field strength of a p-form. The vectors $\vec{c}$ are called dilaton coupling vectors. The crucial point is that the dilaton coupling vectors are identified with the positive restricted roots of a real form of a semisimple Lie algebra $\mathfrak{g}$, which has a subalgebra $\mathfrak{k}$ that is compactly embedded in $\mathfrak{g}$. Accordingly with every dilaton φ_i we associate a noncompact Cartan generator H_i and every axion is identified with an element of the positive restricted root subalgebra $\mathfrak{n}$. From the positive restricted roots one can find $\mathfrak{g}$ and $\mathfrak{k}$ and hence G and K.

To explain the technique of finding the global symmetry group G we use as example the dimensional reduction of Heterotic supergravity to $D > 2$ dimensions. The Yang–Mills sector is restricted to the Cartan subalgebra and the dimension 16 of the Cartan subalgebra is generalized to N. The action in Einstein frame reads:

$$S = \int_{\mathcal{M}_{10}} d^{10}\, x\, e\Big(R - \frac{1}{2}(\partial\Phi_0)^2 - \frac{1}{12}\mathrm{e}^{-\Phi_0}H^2 - \frac{1}{4}\mathrm{e}^{-\frac{1}{2}\Phi_0}\sum_{I=1}^{N} F^I_{\mu\nu}F^{I\mu\nu}\Big)\,, \tag{3.3.16}$$

where $e = \det e_\mu^a$, Φ_0 is the ten-dimensional dilaton and $F^I = \mathrm{d}A^I$. The field strength H contains the Yang–Mills Chern–Simons term: $H = \mathrm{d}B - \sum_{I=1}^{N} \frac{1}{2} A^I \wedge F^I$.

The ten-dimensional metric reduces to a metric $g_{\mu\nu}$, Kaluza–Klein vectors V_μ^i and axions A_{ij}, which follow from the reduction of the Kaluza–Klein vectors and are only nonzero if $i < j$. The gauge potential $\hat{A}^I$ reduces to a gauge potential A^I and scalars A^I_j. The result of the reduction of the Kalb–Ramond potential $\hat{B}^{(2)}$ is a two-form $B_{\mu\nu}$, vectors $B_{\mu i}$ and scalars B_{ij} and the dimensional reduction of the dilaton results in a dilaton.

The Kaluza–Klein Ansatz for the Kalb–Ramond field $\hat{B}_{\hat{\mu}\hat{\nu}}$ is given by

$$\hat{B}^{(2)} = \tfrac{1}{2}B_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + B^{(1)}_{\mu i}\mathrm{d}x^\mu \wedge h^i + \tfrac{1}{2}B^{(0)}_{ij}h^i \wedge h^j\,, \tag{3.3.17}$$

and for the other fields similar, see [D]. To obtain the lower-dimensional Lagrangian, the precise form of the field strengths is unimportant. At the step of the reduction going from $D+1$ to D dimensions one uses that for any $(n+1)$-form field strength $F_{(n+1)}$ the kinetic term reduces as

$$\tfrac{1}{(n+1)!}\hat{e}\hat{F}^2_{(n+1)} \to \tfrac{1}{(n+1)!}e\mathrm{e}^{-2n\alpha\varphi}F^2_{(n+1)} + \tfrac{1}{n!}e\mathrm{e}^{2\alpha(D-n-1)\varphi}F^2_{(n)}\,, \tag{3.3.18}$$

where $e = \det e_\mu^a$. The Ricci scalar reduces as

$$\hat{e}\hat{R} \to eR - \tfrac{1}{2}e(\partial\phi^i)^2 - \tfrac{1}{4}e\mathrm{e}^{-2\alpha(D-1)\varphi}(F^i)^2\,, \tag{3.3.19}$$

where F^i is the Kaluza–Klein field strength of V^i, which can contain axionic terms.

In D dimensions we obtain the Lagrangian $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$, with

$$\begin{aligned}
e^{-1}\mathcal{L}_1 &= R - \tfrac{1}{2}\partial_\mu\vec{\Phi}\cdot\partial^\mu\vec{\Phi} - \tfrac{1}{2}\sum_{1\le i<j\le \mathrm{d}}(F_{ij})^2\mathrm{e}^{\vec{B}_{ij}\cdot\vec{\Phi}} - \tfrac{1}{4}\sum_{i=1}^{\mathrm{d}}(F_i)^2\mathrm{e}^{\vec{B}_i\cdot\vec{\Phi}}\,,\\
e^{-1}\mathcal{L}_2 &= -\tfrac{1}{12}\mathrm{e}^{\vec{A}\cdot\vec{\Phi}}H^2 - \tfrac{1}{4}\sum_{i=1}^{\mathrm{d}}\mathrm{e}^{\vec{A}_i\cdot\vec{\Phi}}(H^i)^2 - \tfrac{1}{2}\sum_{1\le i<j\le \mathrm{d}}\mathrm{e}^{\vec{A}_{ij}\cdot\vec{\Phi}}(H_{ij})^2\,,\\
e^{-1}\mathcal{L}_3 &= -\tfrac{1}{4}\mathrm{e}^{-\frac{1}{2}\vec{G}\cdot\vec{\Phi}}\sum_{I=1}^{N}(F^I)^2 - \tfrac{1}{2}\sum_{i=1}^{\mathrm{d}}\sum_{I=1}^{N}\mathrm{e}^{\vec{C}_i\cdot\vec{\Phi}}(F_i^I)^2\,,\,,
\end{aligned} \tag{3.3.20}$$

where[5]

$$\begin{aligned}
H &= \mathrm{d}B - (\mathrm{d}B_i)\Gamma_{ij}V^j + \tfrac{1}{2}(\mathrm{d}B_{ij})\Gamma_{im}\Gamma_{jn}V^mV^n - \tfrac{1}{2}(A^I - A^I_i\Gamma_{ij}V^j)F^I\,, \\
H_n &= (\mathrm{d}B_i)\Gamma_{in} - (\mathrm{d}B_{ij})\Gamma_{im}\Gamma_{jn}V^m - \tfrac{1}{2}(A^I - A^I_i\Gamma_{ij}V^j)F^I_n - \tfrac{1}{2}A^I_nF^I\,, \\
H_{mn} &= (\mathrm{d}B_{ij})\Gamma_{im}\Gamma_{jn} + \tfrac{1}{2}\left(A^I_p\Gamma_{pm}F^I_n - A^I_p\Gamma_{pn}F^I_m\right)\,, \\
F_{ij} &= (\mathrm{d}A_{im})\Gamma_{mj}\,, \\
F^i &= \mathrm{d}V^i - (\mathrm{d}A_{im})\Gamma_{mn}V^n = \mathrm{d}V^i - F_{ij}V^j\,,
\end{aligned} \tag{3.3.21}$$

and

$$\begin{aligned}
&\vec{\Phi} = (\Phi_0, \varphi_1, \ldots, \varphi_{\mathrm{d}}) = (\Phi_0, \vec{\varphi})\,, && \vec{F}_i = (0, \vec{f}_i)\,, \\
&\vec{B}_{ij} = -\vec{F}_i + \vec{F}_j\,, && \vec{G} = (1, \vec{g})\,, \\
&\vec{A}_i = \vec{F}_i - \vec{G}\,, && \vec{B}_i = -\vec{F}_i\,, \\
&\vec{A}_{ij} = \vec{F}_i + \vec{F}_j - \vec{G}\,, && \vec{A} = -\vec{G}\,, \\
&C_i = \vec{F}_i - \tfrac{1}{2}\vec{G}\,,
\end{aligned} \tag{3.3.22}$$

where the vectors $\vec{\varphi}$, $\vec{f}_i$ and $\vec{g}$ are as in 3.3.13c. The vectors from equation 3.3.22 contain the information about the coset structure.

Recognizing the Scalar Coset

As explained above the dilaton coupling vectors can be identified with the positive restricted roots of a real semisimple Lie algebra $\mathfrak{g}$. In particular we assume that the inner product between the dilaton coupling vectors is proportional to the inner product between the corresponding restricted roots. At the end of this section we show that the assumption is true.

The vectors 3.3.22 satisfy the following relations

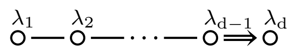
$$\begin{aligned}
\vec{A}_{ij} + \vec{B}_{ik} &= \vec{A}_{kj}\,, & \vec{A}_{ij} + \vec{B}_{jk} &= \vec{A}_{ik}\,, \\
\vec{C}_i + \vec{C}_j &= \vec{A}_{ij}\,, & \vec{C}_j &= \vec{B}_{ij} + \vec{C}_i\,,\ i<j\,, \\
\vec{G}\cdot\vec{G} &= \tfrac{8}{D-2}\,, & \vec{F}_i\cdot\vec{F}_j &= 2\delta_{ij} + \tfrac{2}{D-2}\,, \\
\vec{F}_i\cdot\vec{G} &= \tfrac{4}{D-2}\,, & \vec{A}_i\cdot\vec{G} &= -\tfrac{4}{D-2}\,, \\
\vec{A}_{ij}\cdot\vec{G} &= 0\,, & \vec{B}_{ij}\cdot\vec{B}_{kl} &= 2\delta_{ik} - 2\delta_{il} - 2\delta_{jk} + 2\delta_{jl}\,, \\
\vec{C}_i\cdot\vec{C}_j &= 2\delta_{ij}\,, & \vec{A}_{ij}\cdot\vec{B}_{kl} &= -2\delta_{ik} + 2\delta_{il} - 2\delta_{jk} + 2\delta_{jl}\,.
\end{aligned} \tag{3.3.23}$$

All dilaton coupling vectors can be written as a sum of the vectors $\vec{C}_1$ and $\vec{B}_{i,i+1}$ with integer coefficients. Hence the restricted root vectors $\lambda_{\mathrm{d}-i}$ and λ_{d} associated to the vectors $\vec{B}_{i,i+1}$ and $\vec{C}_1$ respectively generate the whole lattice of positive restricted

[5] Every repeated index is summed over.

root vectors; they are the simple restricted root vectors. The simple restricted root vectors make up a root system just like ordinary simple root vectors of semisimple Lie algebras with the minor differences that the root system need not be reduced and that the root vectors can have multiplicity greater than one; the dimension of the subspace corresponding to a particular restricted root can be greater than one.

We can draw a Dynkin diagram for the restricted root system:



The multiplicities are found from the dimensional reduction; the multiplicity of a restricted root vector is how often it appears in the Lagrangian. Looking at the action 3.3.20 one sees that the multiplicity of λ_i is 1 for $1 \leq i < d$ and N for $i = d$, and the multiplicities of $2\lambda_i$ are zero.

If we know the multiplicities of the simple restricted root vectors λ_i and of twice the simple restricted root vectors $2\lambda_i$, together with the Dynkin diagram of the λ_i, then the Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$ are known and can be read off from tables B.4.1 and B.4.2 in appendix B. The lower-dimensional Lagrangian as obtained by a circle-by-circle reduction contains all information to read off the coset G/K. For Heterotic supergravity the coset in $D > 4$ dimensions is $SO(\mathrm{d}, \mathrm{d}+N)/SO(\mathrm{d}) \times SO(\mathrm{d}+N)$.

There are subtleties if $D = 4$ and $D = 3$. In four dimensions the two-form $B_{\mu\nu}$ can be dualized. More generally, in D dimensions one should always dualize all $(D-2)$-forms to obtain the so-called maximal scalar manifold [115]. Having dualized the two-form in four dimensions one can again read off the restricted root vectors and count multiplicities and draw the Dynkin diagram of the simple restricted roots to find the coset corresponding to the maximal scalar manifold. The maximal scalar manifold in four dimensions of Heterotic supergravity is $SL(2;\mathbb{R})/U(1) \otimes SO(6, 6+N)/SO(6) \times SO(6+N)$.

In three dimensions all vectors can be dualized and applying the above procedure to find the maximal scalar manifold of three-dimensional Heterotic supergravity results in the coset $SO(8, 8+N)/SO(8) \times SO(8+N)$.

Coset Construction

Having recognized the coset G/K we need to check that the scalars in the D-dimensional Lagrangian indeed parameterize the coset G/K. We therefore have to show that a coset construction based on the coset found by recognizing the positive restricted roots gives the same scalar Lagrangian as obtained from the dimensional reduction. We first assume $D > 4$. For more details see [D].

The noncompact Cartan generators associated with the dilatons are written H_i and are assembled in a $(\mathrm{d}+1)$-dimensional vector $\vec{H}$ (we incorporate the dilaton already present in ten dimensions, this gives rise to some technicalities, which we avoid in this discussion; see [D]). We identify with every positive restricted root vector a Lie algebra element. We therefore make the following identification:

$$\begin{aligned}
\vec{B}_{ij} &\leftrightarrow E_{ij}\,,\ i<j\,, & [\vec{H},E_{ij}] &= \vec{B}_{ij}E_{ij}\,,\\
\vec{A}_{ij} &\leftrightarrow R_{ij}=-R_{ji}\,, & [\vec{H},R_{ij}] &= \vec{A}_{ij}R_{ij}\,,\\
\vec{C}_{i} &\leftrightarrow Y_{iI}\,,\ 1\leq I\leq N\,, & [\vec{H},Y_{iI}] &= \vec{C}_{i}Y_{iI}\,.
\end{aligned} \tag{3.3.24}$$

The generators E_{ij}, R_{ij} and Y_{iI} as defined by equations 3.3.24 span the positive restricted root subalgebra $\mathfrak{n}$ of $\mathfrak{g}$.

From equations 3.3.23 and using the Jacobi identities we see that the commutation rules are:

$$\begin{aligned}
&[E_{ij},E_{kl}]=\delta_{jk}E_{il}-\delta_{il}E_{kj}\,, && [E_{ij},R_{kl}]=-\delta_{ik}R_{jl}+\delta_{il}R_{jk}\,,\\
&[E_{ij},Y_{kK}]=-\delta_{ik}Y_{jK}\,, && [Y_{iI},Y_{jJ}]=M_{IJ}R_{ij}\,,\\
&[Y_{iI},R_{kl}]=0\,, && [R_{ij},R_{kl}]=0\,,
\end{aligned} \tag{3.3.25}$$

where the matrix M_{IJ} can not be fixed by the Jacobi equations since it involves a choice of basis in the subspace spanned by the Y_{iI}.

Using the vector representation of $SO(\mathrm{d},\mathrm{d}{+}N)$ we see that we can take $M_{IJ}=\delta_{IJ}$ [D]. The coset Lagrangian can be constructed using the representative $\mathcal{V}$ defined by $\mathcal{V}=\mathcal{V}_1\mathcal{V}_2\mathcal{V}_3\Omega$ with:

$$\begin{aligned}
\mathcal{V}_1 &= \exp\Big(\tfrac{1}{2}\vec{\Phi}\cdot\vec{H}\Big)\,,\\
\mathcal{V}_2 &= \cdots U_{24}U_{23}\cdots U_{14}U_{13}U_{12}\,,\ U_{ij}=\exp\Big(A_{ij}E_{ij}\Big)\ \text{no sum}\,,\\
\mathcal{V}_3 &= \exp\Big(\sum_{i<j}B_{ij}R_{ij}\Big)\,,\\
\Omega &= \exp\Big(\sum_{iI}A_{iI}Y_{iI}\Big)\,,
\end{aligned} \tag{3.3.26}$$

where A_{ij}, B_{ij} and A_{iI} are the axions and $\vec{\Phi}$ are the dilatons. From this one finds:

$$\begin{aligned}
d\mathcal{V}_1\mathcal{V}_1{}^{-1} &= \tfrac{1}{2}\mathrm{d}\vec{\Phi}\cdot\vec{H}\,,\\
\mathcal{V}_1\mathrm{d}\mathcal{V}_2\mathcal{V}_2{}^{-1}\mathcal{V}_1{}^{-1} &= \sum_{i<j}F_{ij}\mathrm{e}^{\frac{1}{2}\vec{\Phi}\cdot\vec{B}_{ij}}E_{ij}\,,\\
\mathcal{V}_1\mathcal{V}_2\mathrm{d}\mathcal{V}_3\mathcal{V}_3{}^{-1}\mathcal{V}_2{}^{-1}\mathcal{V}_1{}^{-1} &= \sum_{i<j}\sum_{mn}\mathrm{e}^{\frac{1}{2}\vec{A}_{ij}\cdot\vec{\Phi}}\mathrm{d}B_{mn}\Gamma_{mi}\Gamma_{nj}\,,\\
\mathcal{V}_1\mathcal{V}_2\mathcal{V}_3 d\Omega\Omega^{-1}\left(\mathcal{V}_1\mathcal{V}_2\mathcal{V}_3\right)^{-1} &= \sum_{Ii}\mathrm{e}^{\frac{1}{2}\vec{C}_{iI}\cdot\vec{\Phi}}F^{I}_{i}Y_{Ii}+\tfrac{1}{2}\sum_{Imij}\mathrm{e}^{\frac{1}{2}\vec{A}_{ij}\cdot\vec{\Phi}}A^{I}_{m}\Gamma_{mi}F^{I}_{j}R_{ij},.
\end{aligned} \tag{3.3.27}$$

From 3.3.27 one calculates the scalar action:

$$S_{G/K} = \tfrac{1}{8} \int d^D x \, e \mathrm{Tr}\left(\partial^\mu \mathcal{M} \partial_\mu \mathcal{M}^{-1}\right) . \tag{3.3.28}$$

where the trace is in a representation and $\mathcal{M} = \mathcal{V}^\# \mathcal{V}$, where # denotes the generalized transpose (see appendix B).

Though $\mathcal{M}$ is in a representation of the group G, the trace in the action is in a Lie algebra representation. Using that the Cartan involution θ is an automorphism we rewrite the action 3.3.28 as:

$$\begin{aligned} S_{G/K} &= -\tfrac{1}{4} \int \mathrm{d}^D x e \left(\mathrm{Tr}(\partial_\mu \mathcal{V}\mathcal{V}^{-1} \partial^\mu \mathcal{V}\mathcal{V}^{-1}) + \mathrm{Tr}(\partial_\mu \mathcal{V}\mathcal{V}^{-1} (\partial^\mu \mathcal{V}\mathcal{V}^{-1})^\#)\right) \\ &= -\tfrac{1}{2} \int \mathrm{d}^D x e \mathrm{Tr}(\partial_\mu \mathcal{V}\mathcal{V}^{-1} \mathbb{P} \partial^\mu \mathcal{V}\mathcal{V}^{-1}) , \end{aligned} \tag{3.3.29}$$

where $\mathbb{P} : \mathfrak{g} \to \mathfrak{g}$ denotes the projection operator defined by

$$\mathbb{P} : x \mapsto \tfrac{1}{2}(\mathbb{1} - \theta) x , \tag{3.3.30}$$

which is indeed a projection operator since $\theta^2 = 1$ and on $\mathfrak{k}$ we have $\mathbb{P} = 0$.

Putting all parts together, the scalar Lagrangian is precisely the same as obtained by dimensional reduction. There are some technicalities we have avoided; there is a one-dimensional subspace in the (d + 1)-dimensional space in which $\vec{\Phi}$ lives that decouples from the rest. One can see the decoupling needs to occur since the dimension of $SO(\mathrm{d}, \mathrm{d}+N)/SO(\mathrm{d}) \times SO(\mathrm{d}+N)$ is d whereas the dilaton vector is (d + 1)-dimensional. See [D] for more details on how to deal with this technicality and to obtain the correct scalar Lagrangian.

One concludes that a circle-by-circle reduction admits a relatively easy way to find the coset G/K parameterized by the scalars.

We now prove the claim made at the beginning of this section, that the inner product between the dilaton coupling vectors is proportional to the inner product of the restricted root vectors. We use a basis H_i for the noncompact part of the Cartan subalgebra such that $\mathrm{Tr}_{\mathrm{ad}}(H_i H_j) = \delta_{ij}$. We see from 3.3.24 that the components of a dilaton coupling vector $\vec{c}$ are $(\vec{c})_i = \gamma(H_i)$, where γ is the restricted root associated with $\vec{c}$. Using the Cartan–Killing metric $B(,)$, there is for every restricted root μ a unique noncompact Cartan generator H_μ such that $\mu(h) = B(h, H_\mu)$ for all noncompact Cartan generators h. We define $\langle \mu, \lambda \rangle = B(H_\mu, H_\lambda) = \mathrm{Tr}_{\mathrm{ad}}(H_\mu H_\lambda)$. We have $H_\mu = \sum_i c_i H_i$ and the c_i are given by $\mu_i = \mu(H_i)$. Hence $\mathrm{Tr}_{\mathrm{ad}}(H_\mu H_\lambda) = \sum_i \mu_i \lambda_i$. For semisimple Lie algebras the trace in any representation can be taken proportional to the Cartan–Killing metric. In the construction of the coset we use $\mathrm{Tr}(H_i H_j) \sim \delta_{ij}$ and hence the claim is proved.

3.4 Group Manifold Reductions

A group manifold is a Lie group. The group manifolds of semisimple G have much structure and admit an explicit analysis.

Higher-Dimensional Vacuum

The higher-dimensional vacuum, around which the fluctuations are considered, is for group manifold reductions in general not known. One only requires that the geometry is $Y_{\mathrm{d}} \times G$ where Y_{d} is a maximally symmetric space and G is a Lie group. In many cases one does not even require that Y_{d} is a maximally symmetric space; the reduction is performed and the lower-dimensional theory in practice always admits a maximally symmetric vacuum, which one can (try to) uplift to a solution of the higher-dimensional theory.

The fluxes of the fields are not specified; the vacuum value is absorbed into the Kaluza–Klein Ansatz. If one finds in the lower-dimensional theory that some field does not have the trivial value as solution, then its original higher-dimensional solution around which was perturbed, was also nontrivial. Since the vacuum is not known, except the internal geometry of G, the expansion and truncation involve a slight modification of the programme advocated in section 3.2.

A group contains natural parameters through its structure constants and therefore one expects that the structure constants appear in the lower-dimensional theory as massive parameters. Hence we expect to arrive at a lower-dimensional gauged supergravity possibly with a scalar potential.

We now discuss the geometry of G [6]. The group G acts on itself in two ways; by left multiplication $L_a : g \mapsto ag$ and right multiplication $R_a : g \mapsto ga$ for $g, a \in G$. There exist $\dim G$ left-invariant independent vector fields τ_α, $\alpha = 1, 2, \ldots, \dim G$, i.e. they are invariant under the left-translations L_a. There exists a dual basis σ^α, $\alpha = 1, 2, \ldots, \dim G$ of one-forms satisfying $\sigma^\alpha(\tau_\beta) = \delta^\alpha_\beta$ and the Maurer–Cartan equation

$$\mathrm{d}\sigma^\alpha = -\tfrac{1}{2} f^\alpha{}_{\beta\gamma} \sigma^\beta \wedge \sigma^\gamma \,, \tag{3.4.1}$$

where $f^\alpha{}_{\beta\gamma}$ are the structure constants of G determined by the Lie bracket $[\tau_\beta, \tau_\gamma] = \tau_\alpha f^\alpha{}_{\beta\gamma}$. The one-forms σ^α are left-invariant.

In a similar way there exist right-invariant vector fields and right-invariant one-forms. The structure constants determined by the Lie bracket between two right-invariant vector fields are different (for more details see e.g. [6]). There are no bi-invariant vector-fields.

From the left-invariant one-forms one can construct the metric $g = M_{\alpha\beta}\sigma^\alpha \otimes \sigma^\beta$, which is a left-invariant metric for any nondegenerate constant $M_{\alpha\beta}$. If one chooses for $M_{\alpha\beta}$ the Cartan–Killing metric of G the metric is bi-invariant. This choice gives

[6] In appendix B.5 we give more details on the geometry of compact groups.

the 'most symmetric' geometry. We therefore take the bi-invariant metric to be the vacuum metric on the group manifold G. The isometry group is thus $G \times G$ and the Killing vectors of the left-action of G are the right-invariant vector fields and vice versa (see [6] for a proof of this statement).

The existence of the left-invariant vector fields has a topological consequence; the group manifold G admits nowhere vanishing vector fields. Due to a theorem of Hopf this can only hold if the Euler characteristic $\chi(G)$ vanishes. This is important when we discuss gauge transformations of gauge fields on G.

Expansion and Truncation of Fluctuations

The left-invariant one-forms and vector fields form a basis for the fields[7] living on the group manifold G. A one-form ω on TG^* can thus be expanded as $\omega(y) = \omega_\alpha(y)\sigma^\alpha(y)$. Acting on ω with left-translation we see $(L_g^*\omega)(gy) = (L_g^*\omega_\alpha)(gy)(L_g^*\sigma^\alpha)(gy) = \omega_\alpha(gy)$ $\sigma^\alpha(y)$ by left-invariance of σ^α. The one-form is thus left-invariant if and only if ω_α is constant over G. Similarly a p-form $\omega^{(p)} = \omega_{\alpha_1\cdots\alpha_p}\sigma_1^\alpha \wedge \cdots \wedge \sigma_p^\alpha$ is left-invariant if and only if the $\omega_{\alpha_1\cdots\alpha_p}$ are constants. The coefficients $\omega_{\alpha_1\cdots\alpha_p}$ are functions on G and can thus be expanded as in 3.2.11. We now wish to find a suitable truncation.

By the singlet analysis of section 3.1 we see that restricting to the singlets of the isometry group $G \times G$ is consistent. This means that we keep both the left-invariant and right-invariant vector fields as Killing vectors. However this is in general not possible. One problem is that there are for general G no bi-invariant one-forms and hence little fields survive the truncation.

A second problem is that the left-invariant Killing vectors are not right-invariant. This can be seen as follows. The left-invariant vector fields generate right-translations. If X is a left-invariant vector field generating a right-translation, then the variation of a left-invariant vector field Y under the right-translation generated by X is $\mathcal{L}_X(Y) = [X, Y]$, which is in general nonzero.

A third problem related to the second is explained in [99]; the inner product of a left-invariant vector field with a right-invariant vector field is a scalar function that in general does depend on the coordinates on G and is thus not invariant under left- and right-translations. This scalar function does appear in the equations of motion of the lower-dimensional fields, which are independent of the coordinates on G and thus we obtain a contradicting set of equations of motion for the lower-dimensional fields.

We therefore discard the possibility of restricting the fluctuations to the full isometry group. The aforementioned problems are solved if we truncate to the fluctuations that are invariant under the subgroup of the isometry group generated by the left-translations. The isometry group is still $G \times G$ but we truncate the Killing vectors appearing in the metric components $g_{\mu\alpha}$ to the left-invariant Killing vectors. Hence

[7] We ignore the issue of whether it is possible to define fermions on G and restrict ourselves to bosonic fields.

a Lie derivative along a right-invariant vector fields annihilates all the fluctuations surviving the truncation.

Kaluza–Klein Ansatz

The most general left-invariant metric h on G is $h = G_{\alpha\beta}\sigma^\alpha \otimes \sigma^\beta$, where $G_{\alpha\beta}$ is a nondegenerate symmetric matrix that is constant over G. This motivates the following Kaluza–Klein Ansatz for the metric:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + G_{\alpha\beta}(\sigma^\alpha - V^\alpha)(\sigma^\beta - V^\beta)\,. \tag{3.4.2}$$

The V^α are the Kaluza–Klein gauge vectors and only the σ^α depend on the coordinates y^α.

We know show that the off-diagonal components of the metric match up with 3.2.27. We write $\sigma^\alpha = \sum_\beta S_{\alpha\beta}\mathrm{d}y^\beta$, where $S_{\alpha\beta}$ is a y-dependent matrix. The left-invariant vector fields dual to the σ^α are given by $\tau_\alpha = \sum_\beta S^{-1}_{\beta\alpha}\partial_\beta$. The vacuum metric on G can be chosen $G_0 = \delta_{\alpha\beta}\sigma^\alpha \otimes \sigma^\beta = \sum_\gamma S_{\gamma\alpha}\Sigma_{\gamma\beta}\mathrm{d}y^\alpha \otimes \mathrm{d}y^\beta$ since G is compact. The left-invariant vector fields are the Killing vectors and hence the off-diagonal components of the metric 3.4.2 with respect to the basis $(\mathrm{d}x^\mu, \mathrm{d}y^\alpha)$ become to zeroth order in x

$$g_{\mu\alpha} = \sum_\beta V^\beta_\mu (\tau_\beta)_\alpha = \sum_{\beta\gamma} V^\beta_\mu (\tau_\beta)^\gamma (G_0)_{\alpha\gamma} = \sum_\beta V^\beta_\mu (\sigma^\beta)_\alpha\,, \tag{3.4.3}$$

which matches with 3.2.27.

The Kaluza–Klein Ansatz for the vielbein is given by

$$\hat{e}^a = e^a\,,\quad \hat{e}^m = E^m_\alpha(\sigma^\alpha - V^\alpha)\,, \tag{3.4.4}$$

where E^m_α is such that $E^m_\alpha \delta_{mn} E^n_\beta = G_{\alpha\beta}$. The index m is a tangent space index and $\alpha, \beta, \ldots$ are curved indices.

The Kaluza–Klein Ansatz for other fields than the metric is given by an expansion in the vielbeins and the only dependence on y^α is encoded in the Maurer–Cartan one-forms in the vielbeins. The lower-dimensional fields are defined as the coefficients in the expansion with respect to the basis $\{e^a, h^\alpha \equiv \sigma^\alpha - V^\alpha\}$. Defining the lower-dimensional fields in this way guarantees that the lower-dimensional fields transform in the adjoint representation of G, the truncated isometry group. The Kaluza–Klein vectors V^α transform as gauge fields $\delta V^\alpha = -\mathrm{D}\lambda^\alpha$, where λ^α is a parameter in the adjoint of G and D is the G-covariant derivative.

3.4.1 Heterotic Supergravity on a Group Manifold

We now give an example of a group manifold reduction and perform a dimensional reduction over a group manifold of ten-dimensional dualized Heterotic supergravity to four dimensions and analyze the lower-dimensional symmetries. The analysis is based on [C]. We begin with the action 3.3.2:

$$S_{\text{2-form}} = \int \mathrm{e}^{-\phi}\Big(R\star\mathbf{1} + \star\mathrm{d}\phi\wedge\mathrm{d}\phi - \tfrac{1}{2}\star H^{(3)}\wedge H^{(3)} - \mathrm{Tr}\star F(A)\wedge F(A)\Big)\,. \quad (3.4.5)$$

where the field strengths are defined as in section 2.3.12. We dualize the two-form B to a six-form; it is in this formalism that we can see many features of the group manifold reduction procedure. We obtain the dualized action $S_{\text{6-form}} = S_{\text{SG}} + S_{\text{YM}}$, where

$$\begin{aligned} S_{\text{SG}} &= \int \mathrm{e}^{-\phi}(R\star\mathbf{1} + \star\mathrm{d}\phi\wedge\mathrm{d}\phi) - \tfrac{1}{2}\mathrm{e}^{\phi}\star H^{(7)}\wedge H^{(7)}\,, \\ S_{\text{YM}} &= -\int \mathrm{e}^{-\phi}\mathrm{Tr}\star F(A)\wedge F(A) - B^{(6)}\wedge\mathrm{Tr}\,F(A)\wedge F(A)\,, \end{aligned} \quad (3.4.6)$$

and where $H^{(7)} = \mathrm{d}B^{(6)}$. The gauge transformations of $B^{(6)}$ are $\delta B^{(6)} = \mathrm{d}\Lambda^{(5)}$. The gauge transformations are reduced in a similar fashion as the fields.

The volume of G is normalized as $\int_G \sigma^1\wedge\ldots\wedge\sigma^6 = 1$. The compactness of G implies that the structure constants are traceless, $f^\alpha{}_{\alpha\beta} = 0$.

The basis elements h^α satisfy

$$\mathrm{d}h^\alpha = -F(V)^\alpha - f^\alpha{}_{\beta\gamma}h^\beta\wedge V^\gamma - \tfrac{1}{2}f^\alpha{}_{\beta\gamma}h^\beta\wedge h^\gamma\,. \quad (3.4.7)$$

where the Kaluza–Klein field strengths $F(V)^\alpha$ are given by

$$F(V)^\alpha = \mathrm{d}V^\alpha + [V,V]^\alpha = \mathrm{d}V^\alpha + \tfrac{1}{2}f^\alpha{}_{\beta\gamma}V^\beta\wedge V^\gamma\,. \quad (3.4.8)$$

The reduction Ansätze for the other fields are given by[8]:

$$\hat{\phi} = \phi + \tfrac{1}{2}\ln|\det G_{\alpha\beta}|\,, \quad (3.4.9)$$

$$\begin{aligned} \hat{B}^{(6)} &= \tfrac{1}{2!}B^{(4)}{}_{\alpha_1\alpha_2}h^{\alpha_1}h^{\alpha_2} + \tfrac{1}{3!}B^{(3)}{}_{\alpha_1\ldots\alpha_3}h^{\alpha_1}\cdots h^{\alpha_3} + \tfrac{1}{4!}B^{(2)}{}_{\alpha_1\ldots\alpha_4}h^{\alpha_1}\cdots h^{\alpha_4} \\ &\quad + \tfrac{1}{5!}B^{(1)}{}_{\alpha_1\ldots\alpha_5}h^{\alpha_1}\cdots h^{\alpha_5} + \tfrac{1}{6!}B^{(0)}{}_{\alpha_1\ldots\alpha_6}h^{\alpha_1}\cdots h^{\alpha_6}\,. \end{aligned} \quad (3.4.10)$$

$$\hat{A}^I = A^I{}_a e^a + A^I{}_\alpha h^\alpha\,. \quad (3.4.11)$$

For the m-forms $B^{(m)}$ we used the shorthand notation

$$B^{(m)}{}_{\alpha_1\ldots\alpha_{6-m}} = \tfrac{1}{m!}B^{(m)}{}_{a_1\ldots a_m\alpha_1\ldots\alpha_{6-m}}e^{a_1}\cdots e^{a_m}\,, \quad m = 0,1,\ldots,6\,, \quad (3.4.12)$$

From the Ansatz (3.4.10) one obtains the seven-form field strength $\hat{H}^{(7)} = \mathrm{d}\hat{B}^{(6)}$. Substituting the expression for $\hat{B}$ gives:

$$\hat{H}^{(7)} = \sum_{m=1}^{4}\tfrac{1}{m!(7-m)!}H^{(m)}{}_{a_1\ldots a_m\alpha_1\ldots\alpha_{7-m}}e^{a_1}\cdots e^{a_m}h^{\alpha_1}\cdots h^{\alpha_{7-m}}\,. \quad (3.4.13)$$

[8] We omit the $\wedge$-symbols in this section to simplify the formulae.

The reduction of S_{SG} gives the four-dimensional action $S = S_1 + S_2$ with:

$$\begin{aligned} S_1 &= \int \mathrm{e}^{-\phi}\Big(R\star\mathbf{1} + \tfrac{1}{2}\star\mathrm{d}\phi\,\mathrm{d}\phi - \tfrac{1}{2}G_{\alpha\beta}\star F(V)^\alpha F(V)^\beta \\ &\qquad\qquad + \tfrac{1}{4}\star\mathrm{D}G_{\alpha\beta}\mathrm{D}G^{\alpha\beta} - V_1(G)\star\mathbf{1}\Big)\,, \\ S_2 &= -\tfrac{1}{2}\int \mathrm{e}^{\phi}\,\mathrm{det}G_{\alpha\beta}\Big(\tfrac{1}{3!}\star H^{(4)\alpha_1\ldots\alpha_3}H^{(4)}{}_{\alpha_1\ldots\alpha_3} + \tfrac{1}{4!}\star H^{(3)\alpha_1\ldots\alpha_4}H^{(3)}{}_{\alpha_1\ldots\alpha_4} \\ &\qquad + \tfrac{1}{5!}\star H^{(2)\alpha_1\ldots\alpha_5}H^{(2)}{}_{\alpha_1\ldots\alpha_5} + \tfrac{1}{6!}\star H^{(1)\alpha_1\ldots\alpha_6}H^{(1)}{}_{\alpha_1\ldots\alpha_6}\Big)\,. \end{aligned} \tag{3.4.14}$$

The covariant derivative D is covariant with respect to gauge transformations associated with right-translations. The contraction in the H^2-terms is with $G^{\alpha\beta}$. The scalar potential $V_1(G)$ due to reduction of the gravitational sector is given by [116]

$$V_1(G) = \tfrac{1}{4}G_{\alpha_1\alpha_2}G^{\beta_1\beta_2}G^{\gamma_1\gamma_2}f^{\alpha_1}{}_{\beta_1\gamma_1}f^{\alpha_2}{}_{\beta_2\gamma_2} + \tfrac{1}{2}G^{\alpha_1\alpha_2}f^{\beta}{}_{\alpha_1\gamma}f^{\gamma}{}_{\alpha_2\beta}. \tag{3.4.15}$$

The expressions for the curvatures $H^{(p)}$ are

$$H^{(1)}{}_{\alpha_1\ldots\alpha_6} = \mathrm{D}B^{(0)}{}_{\alpha_1\ldots\alpha_6} + 15f^{\beta}{}_{[\alpha_1\alpha_2}B^{(1)}{}_{\alpha_3\ldots\alpha_6]\beta}\,, \tag{3.4.16a}$$

$$H^{(2)}{}_{\alpha_1\ldots\alpha_5} = \mathrm{D}B^{(1)}{}_{\alpha_1\ldots\alpha_5} + B^{(0)}_{\alpha_1\ldots\alpha_5\beta}F(V)^\beta + 10f^{\beta}{}_{[\alpha_1\alpha_2}B^{(2)}{}_{\alpha_3\ldots\alpha_5]\beta}\,, \tag{3.4.16b}$$

$$H^{(3)}{}_{\alpha_1\ldots\alpha_4} = \mathrm{D}B^{(2)}{}_{\alpha_1\ldots\alpha_4} + B^{(1)}{}_{\alpha_1\ldots\alpha_4\beta}F(V)^\beta + 6f^{\beta}{}_{[\alpha_1\alpha_2}B^{(3)}{}_{\alpha_3\alpha_4]\beta}\,, \tag{3.4.16c}$$

$$H^{(4)}{}_{\alpha_1\ldots\alpha_3} = \mathrm{D}B^{(3)}{}_{\alpha_1\ldots\alpha_3} + B^{(2)}{}_{\alpha_1\ldots\alpha_3\beta}F(V)^\beta + 3f^{\beta}{}_{[\alpha_1\alpha_2}B^{(4)}{}_{\alpha_3]\beta}\,. \tag{3.4.16d}$$

The gauge transformations of $B^{(n)}$ are

$$\begin{aligned} \delta B^{(n)}{}_{\alpha_1\ldots\alpha_{6-n}} &= \mathrm{D}\Lambda^{(n-1)}{}_{\alpha_1\ldots\alpha_{6-n}} - \Lambda^{(n-2)}{}_{\alpha_1\ldots\alpha_{6-n}\beta}F(V)^\beta \\ &\quad - \tfrac{1}{2}(6-n)(5-n)f^{\beta}{}_{[\alpha_1\alpha_2}\Lambda^{(n)}{}_{\alpha_3\ldots\alpha_{6-n}]\beta}\,. \end{aligned} \tag{3.4.17}$$

At some points in the analysis it is more convenient to redefine the p-form gauge fields by dualizing to upper internal indices:

$$B^{(n)}{}_{\alpha_1\ldots\alpha_{6-n}} \equiv \tfrac{1}{n!}\,\tilde{\varepsilon}_{\alpha_1\ldots\alpha_{6-n}\,\beta_1\ldots\beta_n}\tilde{B}^{(n)\beta_1\ldots\beta_n}\,, \tag{3.4.18}$$

where $\tilde{\epsilon}_{\alpha_1\cdots\alpha_6}$ denotes the six-dimensional completely antisymmetric alternating symbol[9]. The fields $\tilde{B}^{(n)}$ have the correct transformation properties under gauge transformations due to the unimodularity of G, which in turn is guaranteed by compactness of G [C]. The corresponding curvatures for the fields $\tilde{B}^{(n)}$ are:

$$\begin{aligned} \tilde{H}^{(n)\alpha_1\ldots\alpha_{n-1}} &= \mathrm{D}\tilde{B}^{(n-1)\alpha_1\ldots\alpha_{n-1}} + (n-1)(-1)^n\,\tilde{B}^{(n-2)[\alpha_1\ldots\alpha_{n-2}}F(V)^{\alpha_{n-1}]} \\ &\quad + \tfrac{n-1}{2}\,f^{[\alpha_1}{}_{\beta_1\beta_2}\tilde{B}^{(n)\alpha_2\ldots\alpha_{n-1}]\beta_1\beta_2}\,. \end{aligned} \tag{3.4.19}$$

[9]The object $\tilde{\epsilon}_{\alpha_1\cdots\alpha_6}$ is thus the six-dimensional Levi–Civitá symbol; see appendix A.

The Yang–Mills field strengths are defined by: $\hat{F}(\hat{A})^I = d\hat{A}^I + (\hat{A}\wedge\hat{A})^I$. Using 3.4.11 one finds[10]:

$$\hat{F}(\hat{A})^I = G^I + \bar{\mathrm{D}}A^I{}_\alpha h^\alpha + \tfrac{1}{2}\mathcal{F}^I{}_{\alpha_1\alpha_2}\, h^{\alpha_1}h^{\alpha_2}\,, \tag{3.4.20}$$

where we use the following definitions:

$$F(A)^I = \mathrm{d}A^I + \tfrac{1}{2}f^I{}_{JK}A^JA^K\,, \tag{3.4.21a}$$
$$G^I = F(A)^I - A^I{}_\alpha F^\alpha(V)\,, \tag{3.4.21b}$$
$$\bar{\mathrm{D}}A^I{}_\alpha = \mathrm{d}A^I{}_\alpha + f^\beta{}_{\alpha\gamma}V^\gamma A^I{}_\beta + f^I{}_{JK}A^JA^K{}_\alpha\,, \tag{3.4.21c}$$
$$\mathcal{F}^I{}_{\alpha_1\alpha_2} = f^I{}_{JK}A^J{}_{\alpha_1}A^K{}_{\alpha_2} - A^I{}_\gamma f^\gamma{}_{\alpha_1\alpha_2}\,. \tag{3.4.21d}$$

The ten-dimensional Chern–Simons form $\hat{C}$ can be rewritten as

$$\hat{C} = \mathrm{Tr}\,(\hat{A}\,\mathrm{d}\hat{A} + \tfrac{2}{3}\hat{A}\,\hat{A}\,\hat{A}) = \tfrac{1}{2}\hat{A}^I\,\hat{F}(\hat{A})^I - \tfrac{1}{12}f^I{}_{JK}\hat{A}^I\,\hat{A}^J\,\hat{A}^K\,. \tag{3.4.22}$$

Substitution of the Ansatz (3.4.11) gives

$$\hat{C} = C^{(3)} + C^{(2)}{}_\alpha h^\alpha + \tfrac{1}{2!}C^{(1)}{}_{\alpha_1\alpha_2}h^{\alpha_1}h^{\alpha_2} + \tfrac{1}{3!}C^{(0)}{}_{\alpha_1\alpha_2\alpha_3}h^{\alpha_1}h^{\alpha_2}h^{\alpha_3}\,, \tag{3.4.23}$$

where

$$C^{(3)} = \tfrac{1}{2}(A^IG^I - \tfrac{1}{6}f^I{}_{JK}A^IA^JA^K)\,, \tag{3.4.24a}$$
$$C^{(2)}{}_\alpha = \tfrac{1}{2}(A^I\bar{\mathrm{D}}A^I{}_\alpha + A^I{}_\alpha G^I - \tfrac{1}{2}f^I{}_{JK}A^IA^JA^K{}_\alpha)\,, \tag{3.4.24b}$$
$$C^{(1)}{}_{\alpha_1\alpha_2} = -\tfrac{1}{2}A^IA^I{}_\gamma f^\gamma{}_{\alpha_1\alpha_2} - A^I{}_{[\alpha_1}\bar{\mathrm{D}}A^I{}_{\alpha_2]}\,, \tag{3.4.24c}$$
$$C^{(0)}{}_{\alpha_1\alpha_2\alpha_3} = f^I{}_{JK}A^I{}_{\alpha_1}A^J{}_{\alpha_2}A^K{}_{\alpha_3} - \tfrac{3}{2}A^I{}_{[\alpha_1}f^\delta{}_{\alpha_2\alpha_3]}A^I{}_\delta\,. \tag{3.4.24d}$$

The reduction of the ten-dimensional action S_{YM} is facilitated by using $\mathrm{d}\hat{C} = \frac{1}{2}\hat{F}^I\hat{F}^I$. For the topological term in S_{YM} we write

$$\hat{\mathcal{L}}_{\mathrm{CS}} = \hat{B}^{(6)}\wedge\mathrm{Tr}\,(\hat{F}\wedge\hat{F}) = -\hat{H}^{(7)}\wedge\hat{C}^{(3)} + \text{ total derivative,} \tag{3.4.25}$$

which gives the following contribution to the four-dimensional action:

$$\mathcal{L}_{\mathrm{CS}} = -\tilde{H}^{(1)}C^{(3)} - \tilde{H}^{(2)\alpha}C^{(2)}{}_\alpha - \tfrac{1}{2!}\tilde{H}^{(3)\alpha_1\alpha_2}C^{(1)}{}_{\alpha_1\alpha_2} - \tfrac{1}{3!}\tilde{H}^{(4)\alpha_1\alpha_2\alpha_3}C^{(0)}{}_{\alpha_1\alpha_2\alpha_3}\,. \tag{3.4.26}$$

[10] In the following $\bar{\mathrm{D}}$ stands for a covariant derivative that is covariant with respect to both the Scherk–Schwarz and Yang–Mills gauge transformations. It should be kept in mind that A^I is the one-form corresponding to the gauge vector, while $A^I{}_\alpha$ are scalar fields. The internal indices $\alpha, \beta, \ldots$ are therefore always written explicitly. Any repeated Yang–Mills index $I, J, \ldots$ is summed over.

After a partial integration one obtains

$$\begin{aligned}\mathcal{L}_{\text{CS}} =&\tfrac{1}{2}\tilde{B}^{(0)}G^I G^I - \tilde{B}^{(1)\alpha}G^I\bar{\text{D}}A^I{}_\alpha + \tfrac{1}{2}(G^I\mathcal{F}^I{}_{\alpha_1\alpha_2} - \bar{\text{D}}A^I{}_{\alpha_1}\bar{\text{D}}A^I{}_{\alpha_2})\tilde{B}^{(2)\alpha_1\alpha_2}\\ &- \tfrac{1}{2}\tilde{B}^{(3)\alpha_1\alpha_2\alpha_3}\bar{\text{D}}A^I{}_{\alpha_1}\mathcal{F}^I{}_{\alpha_2\alpha_3} + \tfrac{1}{8}\tilde{B}^{(4)\alpha_1\ldots\alpha_4}\mathcal{F}^I{}_{\alpha_1\alpha_2}\mathcal{F}^I{}_{\alpha_3\alpha_4}\,. \end{aligned}\tag{3.4.27}$$

We now give the complete four-dimensional Lagrangian with Yang–Mills fields, in the Einstein frame and in terms of the redefined fields 3.4.18:

$$\mathcal{L} =\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4\,, \tag{3.4.28}$$

$$\begin{aligned}\mathcal{L}_1 =&\sqrt{-g}\,\Big(R - \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{4}\text{e}^{-\phi}\,F(V)_{\mu\nu}{}^\alpha G_{\alpha\beta}F(V)^{\mu\nu\,\beta}\\ &+ \tfrac{1}{4}\text{D}_\mu G_{\alpha\beta}\text{D}^\mu G^{\alpha\beta} - \text{e}^{\phi}\,V_1(G)\Big)\,,\end{aligned}\tag{3.4.29}$$

$$\begin{aligned}\mathcal{L}_2 =&-\sqrt{-g}\,\Big(\tfrac{1}{2}\text{e}^{2\phi}\,\partial_\mu\tilde{B}^{(0)}\partial^\mu\tilde{B}^{(0)} + \tfrac{1}{4}\text{e}^{\phi}\,\tilde{H}^{(2)}{}_{\mu\nu}{}^\alpha\tilde{H}^{(2)\mu\nu\,\beta}G_{\alpha\beta}\\ &+ \tfrac{1}{4!}\tilde{H}^{(3)}{}_{\mu\nu\lambda}{}^{\alpha_1\alpha_2}\tilde{H}^{(3)\mu\nu\lambda\beta_1\beta_2}G_{\alpha_1\beta_1}G_{\alpha_2\beta_2}\\ &+ \tfrac{1}{2\cdot3!4!}\text{e}^{-\phi}\tilde{H}^{(4)}{}_{\mu\nu\lambda\rho}{}^{\alpha_1\alpha_2\alpha_3}\tilde{H}^{(4)\mu\nu\lambda\rho\beta_1\beta_2\beta_3}G_{\alpha_1\beta_1}G_{\alpha_2\beta_2}G_{\alpha_3\beta_3}\Big)\,,\end{aligned}\tag{3.4.30}$$

$$\begin{aligned}\mathcal{L}_3 =&-\sqrt{-g}\,\text{e}^{-\phi}\Big(\tfrac{1}{4}G^I{}_{\mu\nu}G^{I\,\mu\nu} + \tfrac{1}{2}\text{e}^{\phi}\,\bar{\text{D}}_\mu A^I{}_\alpha\bar{\text{D}}^\mu A^I{}_\beta G^{\alpha\beta}\\ &+ \tfrac{1}{4}\text{e}^{2\phi}\,\mathcal{F}^I{}_{\alpha_1\alpha_2}\mathcal{F}^I{}_{\beta_1\beta_2}G^{\alpha_1\beta_1}G^{\alpha_2\beta_2}\Big)\,,\end{aligned}\tag{3.4.31}$$

$$\begin{aligned}\mathcal{L}_4 =&-\tilde{\epsilon}^{\mu\nu\lambda\rho}\Big(\tfrac{1}{8}\tilde{B}^{(0)}G^I{}_{\mu\nu}G^I{}_{\lambda\rho} - \tfrac{1}{2}\tilde{B}^{(1)}{}_\mu{}^\alpha\,G^I{}_{\nu\lambda}\bar{\text{D}}_\rho A^I{}_\alpha\\ &+ \tfrac{1}{4}\tilde{B}^{(2)}{}_{\lambda\rho}{}^{\alpha\beta}\big(\tfrac{1}{2}G^I{}_{\mu\nu}\mathcal{F}^I{}_{\alpha\beta} - \bar{\text{D}}_\mu A^I{}_\alpha\bar{\text{D}}_\nu A^I{}_\beta\big)\\ &- \tfrac{1}{2\cdot3!}\tilde{B}^{(3)}{}_{\mu\nu\lambda}{}^{\alpha\beta\gamma}\bar{\text{D}}_\rho A^I{}_\alpha\mathcal{F}^I{}_{\beta\gamma} + \tfrac{1}{8\cdot4!}\tilde{B}^{(4)}{}_{\mu\nu\lambda\rho}{}^{\alpha\beta\gamma\delta}\mathcal{F}^I{}_{\alpha\beta}\mathcal{F}^I{}_{\gamma\delta}\Big)\,.\end{aligned}\tag{3.4.32}$$

3.4.2 Stückelberg Symmetries and Cohomology

We now analyze the system of gauge transformations 3.4.17 and the presence of mass-terms in the Lagrangian S_2 in equation 3.4.14. The parameters $\Lambda^{(n)}$ of the gauge transformations 3.4.17 generate shift symmetries. In analyzing which fields are physical we can restrict ourselves to the shift-symmetries. The shift-symmetries are called Stückelberg symmetries after the German physicist who introduced a formalism for dealing with massive gauge fields [117] (he is best known for introducing the concept of baryon conservation in 1939).

For a generic field Φ with p internal indices the local shift symmetry acts as:

$$\delta\Phi_{\alpha_1\ldots\alpha_p} = f^\beta{}_{[\alpha_1\alpha_2}\Lambda_{\alpha_3\ldots\alpha_p]\beta}\,. \tag{3.4.33}$$

The mass term for a field Φ is constructed from (see equations 3.4.16)

$$(m[\Phi])_{\alpha_1\ldots\alpha_{p+1}} = f^\beta{}_{[\alpha_1\alpha_2}\Phi_{\alpha_3\ldots\alpha_{p+1}]\beta}\,. \tag{3.4.34}$$

We associate with each Φ an element of the set $\Lambda_L^{(p)}$ of left-invariant p-forms on G:

$$\Phi_{\alpha_1\ldots\alpha_p} \mapsto \Phi^{(p)} = \tfrac{1}{p!}\Phi_{\alpha_1\ldots\alpha_p}\sigma^{\alpha_1}\ldots\sigma^{\alpha_p}\,. \tag{3.4.35}$$

For the exterior derivatives $\mathrm{d}_p : \Lambda_L^{(p)} \to \Lambda_L^{(p+1)}$, we define the images $Z^{(p)} \equiv \mathrm{Im}\,(\mathrm{d}_{p-1})$, the kernels $\Omega^{(p)} = \mathrm{Ker}\,(\mathrm{d}_p)$ and the quotients $H^{(p)} = \Omega^{(p)}/Z^{(p)}$. The equations (3.4.33) and (3.4.34) describe a cohomology problem for the left-invariant forms:

$$\delta\Phi^{(p)} = \mathrm{d}\Lambda^{(p-1)},\;\; m[\Phi^{(p)}] = \mathrm{d}\Phi^{(p)}\,. \tag{3.4.36}$$

Due to a theorem by Chevalley and Eilenberg [118] this cohomology problem is equivalent to the de Rahm cohomology on the group manifold G if G is compact and connected. In particular we have $\dim H^{(p)} = b^p$ where b^p is the pth Betti number of G. Since d_p is a homomorphism we have

$$Z^{(p)} \cong \Lambda_L^{(p-1)}/\Omega^{(p-1)}\,, \tag{3.4.37}$$

giving us the recurrence relation

$$\dim Z^{(p+1)} = \dim\Lambda_L^{(p)} - \dim Z^{(p)} - b^p = \binom{6}{p} - b^p - \dim Z^{(p)}\,. \tag{3.4.38}$$

The recurrence relation 3.4.38 can be solved starting from dim $Z^{(0)} = 0$:

$$\begin{aligned}
\dim Z^{(1)} &= 1 - b^0\,,\\
\dim Z^{(2)} &= 5 - b^1 + b^0\,,\\
\dim Z^{(3)} &= 10 - b^2 + b^1 - b^0\,,\\
\dim Z^{(4)} &= 10 - b^3 + b^2 - b^1 + b^0\,,\\
\dim Z^{(5)} &= 5 - b^4 + b^3 - b^2 + b^1 - b^0 = 5 - \chi(G) + b^6 - b^5\,,\\
\dim Z^{(6)} &= 1 - b^5 + b^4 - b^3 + b^2 - b^1 + b^0 = 1 + \chi(G) - b^6\,,
\end{aligned} \tag{3.4.39}$$

where $\chi(G)$ is the Euler characteristic of the six-dimensional G

$$\chi(G) = \sum_{r=0}^{6}(-1)^r b^r\,. \tag{3.4.40}$$

Since the group manifold is compact and connected we have Poincaré duality: $b^p = b^{6-p}$. On compact connected six-dimensional Lie groups $b^0 = 1$ and thus:

$$2b^1 - 2b^2 + b^3 = 2\,. \tag{3.4.41}$$

field	features	dimension	DOF
$B^{(4)}$	gauge	$\dim Z^{(2)}$	0
	massive	$\dim Z^{(3)}$	0
	massless	b^2	0
$B^{(3)}$	gauge	$\dim Z^{(3)}$	0
	massive	$\dim Z^{(4)}$	1
	massless	b^3	0
$B^{(2)}$	gauge	$\dim Z^{(4)}$	0
	massive	$\dim Z^{(5)}$	3
	massless	b^4	1
$B^{(1)}$	gauge	$\dim Z^{(5)}$	0
	massive	$\dim Z^{(6)}$	3
	massless	b^5	2
$B^{(0)}$	gauge	$\dim Z^{(6)}$	0
	massive	0	1
	massless	b^6	1

Table 3.4.1: Result of the analysis of the shift symmetries of the Kalb–Ramond fields $B^{(n)}$. Each $B^{(n)}$ splits in three parts: gauge degrees of freedom, massive and massless components. The third column shows how the dimension of the various spaces $Z^{(p)}$ and $H^{(p)}$ determines the number of components of $B^{(n)}$. DOF indicates the number of degrees of freedom for each choice of the $6-n$ internal indices.

For odd-dimensional compact connected Lie groups a relation like 3.4.41 cannot be deduced since due to Poincaré duality the Euler characteristic vanishes identically.

We draw the following conclusions: (1) The mass terms 3.4.34 are invariant under the shift symmetries. (2) A field can be gauged away if $\Phi^{(p)} \in Z^{(p)}$. These fields do not have mass terms[11]. (3) If and only if a field $\Phi^{(p)}$ has a mass term, there is a $\Phi^{(p+1)}$ that can be gauged away, i.e. $\Phi^{(p+1)}$ is 'eaten' by $\Phi^{(p)}$. (5) Physical massless fields are elements of $H^{(p)}$ for some p. (6) The number of massive fields in $\Lambda_L^{(p)}$ is $\dim Z^{(p+1)}$.

In this context the fields $B^{(n)}$ in four dimensions are interpreted as elements of $\Lambda_L^{(6-n)}$, for $n = 0, \ldots, 4$. The implications for these fields are presented in Table 3.4.1. The total number of physical degrees of freedom in four dimensions is obtained by taking, for each row of Table 3.4.1, the product of the dimension in the third, and the number of degrees of freedom in the fourth column, and by summing these

[11]Having no mass term does not imply the field is massless since the notion of a mass depends on the space-time symmetry group.

products. Using (3.4.39) one finds that this sum is $28 + \chi(G)$. As noted before the Euler characteristic vanishes since a group manifold admits a set of nowhere vanishing vector fields (the left-invariant vector fields for example). Hence the total number of degrees of freedom of the fields $B^{(n)}$, $n = 0, \ldots, 4$ in four dimensions is 28, in correspondence with the number of degrees of freedom of the ten-dimensional two-form (or equivalently six-form).

The equation of motion of a massless $B^{(3)}$ is $\mathrm{D}B^{(3)} = 0$. From this and the above cohomology analysis we conclude that we can give a vacuum expectation value (vev) to a $B^{(3)}$ if $b^3 \neq 0$. If the group G is nonabelian then $b^3 \neq 0$, since for any compact nonabelian group there is a nonzero harmonic 3-form given by

$$\Omega_3 = \mathrm{Tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg), \quad g \in G\,. \tag{3.4.42}$$

In the case of $B^{(4)}$ the equation of motion is algebraic, it sets, in the absence of Yang–Mills fields, certain components of $H^{(4)}$ to zero. The b^2 massless components in table 3.4.1 do not appear in the action at all, and never give rise to fluxes.

If G is semisimple, $b^1 = b^5 = 0$. This follows from the more general theorem proved in appendix B.5 that on a semisimple compact group the first and the one but last Betti numbers vanish; $b^1 = b^{n-1} = 0$. Combining this with $b^0 = b^6 = 1$ and $\chi(G) = 0$, we see that all vectors $B^{(1)}$ can be gauged away.

Examples for Six-Dimensional Compact Groups

The only simple compact Lie groups with dimension not exceeding 6 are $SU(2)$ and $SO(3)$ the difference between the two being that the first is the double cover of the latter. Since both are connected, the Chevalley–Eilenberg theorem applies to both and concerning the Betti numbers they are indistinguishable.

Some examples of compact six-dimensional groups are constructed using $SU(2)$ and $U(1)$: $G = SU(2) \times SU(2)$, $G = SU(2) \times U(1)^3$, $G = U(1)^6$. The Betti numbers of $SU(2)$ are $(b^0, b^1, b^2, b^3) = (1, 0, 0, 1)$ and for $U(1)^n$ we have $b^k = \binom{n}{k}$.

To calculate the Betti numbers of the product of two compact manifolds we use the Künneth formula (see appendix B or e.g. [5]). The Betti numbers for the three groups mentioned above are put in table 3.4.2. Using the equations 3.4.39 it is a matter of plugging in the numbers to find the values of $\dim Z^{(n)}$ in the table 3.4.1 for these groups. In table 3.4.3 we have presented the result.

From table 3.4.3 we see that for the group $U(1)^6$ all fields $B^{(n)}$ are massless. This is in agreement with the fact that the toroidal reductions of ungauged supergravities without fluxes results in ungauged supergravities, in which all fields are massless.

We claim that the list in 3.4.3 exhausts all six-dimensional compact connected Lie groups except for the trivial substitution of an $SU(2)$-factor by an $SO(3)$-factor. We conclude this section with proving this claim.

Group	b^0	b^1	b^2	b^3	b^4	b^5	b^6
$SU(2) \times SU(2)$	1	0	0	2	0	0	1
$SU(2) \times U(1)^3$	1	3	3	2	3	3	1
$U(1)^6$	1	6	15	20	15	6	1

Table 3.4.2: The Betti numbers for a few six-dimensional compact connected Lie groups.

field	features	dimension for the groups:			DOF
		$SU(2) \times SU(2)$	$SU(2) \times U(1)^3$	$U(1)^6$	
$B^{(4)}$	gauge	6	3	0	0
	massive	9	9	0	0
	massless	0	3	15	0
$B^{(3)}$	gauge	9	9	0	0
	massive	9	9	0	1
	massless	2	2	20	0
$B^{(2)}$	gauge	9	9	0	0
	massive	6	0	0	3
	massless	0	3	15	1
$B^{(1)}$	gauge	6	3	0	0
	massive	0	0	0	3
	massless	0	3	6	2
$B^{(0)}$	gauge	0	0	0	0
	massive	0	0	0	0
	massless	1	1	0	1

Table 3.4.3: Result of the analysis of the shift-symmetries of the Kalb–Ramond fields $B^{(n)}$ for the group $SU(2) \times SU(2)$, $SU(2) \times U(1)^3$ and $U(1)^6$, in the reduction of $B^{(6)}$ from 10 to 4 dimensions. See the caption of Table 3.4.1 for more details.

We first prove that the list in 3.4.3 exhausts all connected compact six-dimensional Lie groups that have a reductive Lie algebra. A reductive Lie algebra is by definition a Lie algebra $\mathfrak{g}$ such that for every ideal $\mathfrak{a}$ in $\mathfrak{g}$ there is an ideal $\mathfrak{b}$ in $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$. It follows that a reductive Lie algebra is the direct sum of its derived algebra $\mathfrak{g}'$ and its center such that $\mathfrak{g}'$ is semisimple [94]. From this we see that table 3.4.3 contains all possible connected compact six-dimensional Lie groups with reductive Lie algebra. The proof is finished if we can prove that any compact Lie group has a reductive Lie algebra. In order to do this we first show that every representation of a compact Lie group is unitary.

A compact Lie group H admits a measure μ that is invariant under left-translations: $\mu(gh) = \mu(h)$. A way to see this is that the wedge product of all left-invariant one-forms σ^a on H is a left-invariant volume form. Since the group H is compact we can integrate over the whole group. Suppose we have a representation ρ of H in a vector space V that has an inner product $\langle , \rangle_0$. Then we define the inner product $\langle , \rangle$

$$\langle v, w \rangle = \int_H \mathrm{d}\mu(h) \langle \rho(h)^{-1} v, \rho(h)^{-1} w \rangle_0 \,, \quad v, w \in V \,. \tag{3.4.43}$$

With respect to the inner product $\langle , \rangle$ the representation ρ is unitary;

$$\begin{aligned} \langle \rho(g) v, \rho(g) w \rangle &= \int_H \mathrm{d}\mu(h) \langle \rho(h)^{-1} \rho(g) v, \rho(h)^{-1} \rho(g) w \rangle_0 \\ &= \int_H \mathrm{d}\mu(h) \langle \rho(g^{-1}h)^{-1} v, \rho(g^{-1}h)^{-1} w \rangle_0 \\ &= \int_H \mathrm{d}\mu(g^{-1}h) \langle \rho(g^{-1}h)^{-1} v, \rho(g^{-1}h)^{-1} w \rangle_0 \\ &= \langle v, w \rangle \,. \end{aligned} \tag{3.4.44}$$

Hence the representation ρ is equivalent to a unitary representation. This we use to prove reductivity of the Lie algebra.

The unitarity of the representations of the Lie group H ensures that every representation of the Lie algebra $\mathfrak{h}$ of H is equivalent to an anti-Hermitian representation; in particular the adjoint representation of $\mathfrak{h}$. Let $\mathfrak{a}$ be an ideal in $\mathfrak{h}$ and define $\mathfrak{a}^\perp$ to be the orthogonal complement of $\mathfrak{a}$ with respect to the inner product $\langle , \rangle$ in which the adjoint representation is anti-Hermitian. Then $\mathfrak{a}^\perp$ is also an ideal, since for $b \in \mathfrak{a}^\perp$, $a \in \mathfrak{a}$ and $l \in \mathfrak{h}$ we have $\langle a, [l, b] \rangle = \langle a, \mathrm{ad}l(b) \rangle = -\langle \mathrm{ad}l(a), b \rangle = 0$ since $[l, a] \in \mathfrak{a}$ and thus $[l, b] \in \mathfrak{a}^\perp$.

If V is any vector space equipped with an inner product $\langle , \rangle_V$, then for any subspace W we have $V = W \oplus W^\perp$, where $W^\perp$ is the orthogonal complement of W with respect to the inner product $\langle , \rangle_V$ the Lie algebra. Hence $\mathfrak{h}$ decomposes uniquely into $\mathfrak{a}$ and $\mathfrak{a}^\perp$ and both $\mathfrak{a}$ and $\mathfrak{a}^\perp$ are ideals. Hence $\mathfrak{h}$ is reductive. This completes the proof.

3.5 Scherk–Schwarz Reductions

There are two types of Scherk–Schwarz reductions. Both are generalizations of the toroidal reductions. Whereas in toroidal reductions the Kaluza–Klein Ansatz is such that all fields do not depend on the coordinates on the torus, in the Scherk–Schwarz reductions they do. The dependence on the internal coordinates can be done in two ways, making the difference between the so-called Scherk–Schwarz I and Scherk–Schwarz II reductions. In this section we briefly comment on these types of reductions. For more details and examples we refer to the literature, e.g. [49, 116, 119–125] and references therein.

Scherk–Schwarz I Reductions

Scherk–Schwarz I reductions use a global symmetry of the higher-dimensional theory [120]. We call the global symmetry group G and we denote the coordinates on the torus y_i with the identification $y_i \equiv y_i + 2\pi R_i$. We assemble the coordinates into a vector $\vec{y}$.

For each coordinate y_i we introduce the vector $\vec{e}_i$ with components $(\vec{e}_i)_j = 2\pi R_i \delta_{ij}$. The y-dependence of a field ϕ, which takes values in a representation of G, is such that going around the torus in one direction the field returns to its value up to a global symmetry transformation; $\phi(\vec{y} + \vec{e}_i) = g_i \phi(\vec{y})$, where $g_i \in G$ and $g_i\phi$ denotes the action of g_i on ϕ and the dependence on the lower-dimensional coordinates is suppressed. Since $\vec{e}_i + \vec{e}_j = e_j + \vec{e}_i$ we have $g_i g_j = g_j g_i$, and hence the g_i form an abelian discrete subgroup of G.

The Kaluza–Klein Ansatz for the field ϕ is

$$\hat{\phi}(x^\mu, \vec{y}) = g(\vec{y})\phi(x^\mu)\,, \tag{3.5.1}$$

where the $g(\vec{y})$ parameterizes a subgroup of G such that $g(\vec{e}_i) = g_i$.

The group G is a global symmetry group and thus the Lagrangian is a singlet under G. Hence the Lagrangian is independent of y. This can only be arranged if all derivatives of a field depends in the same way on y as the field itself; $\partial_{y_i}\left(g(\vec{y})\phi(x)\right) = g(\vec{y})\tilde{\phi}_i(x)$ for some $\tilde{\phi}(x)_i$. Hence we need [120]

$$g(\vec{y})^{-1}\partial_{y_i} g(\vec{y}) = C^i\,, \tag{3.5.2}$$

where C^i is a y-independent element of the Lie algebra of G. In terms of the C^i the Kaluza–Klein Ansatz of the field ϕ is

$$\hat{\phi}(x^\mu, \vec{y}) = \exp\Big(C^i y_i\Big)\phi(x)\,. \tag{3.5.3}$$

The objects C_i give rise to mass terms in the lower-dimensional supergravity and thus the Scherk–Schwarz I reductions result in a lower-dimensional gauged supergravity.

If the abelian subgroup generated by the C^i commutes with the higher-dimensional supersymmetry algebra no supersymmetry is broken. More generally, the lower-dimensional supersymmetry is determined by the subalgebra of the higher-dimensional supersymmetry algebra that commutes with the subalgebra generated by the elements C^i.

Scherk–Schwarz II Reductions

Scherk–Schwarz II reductions use the higher-dimensional symmetry of general coordinate transformations to give the fields a nontrivial dependence on the internal coordinates y_i [116]. One introduces matrices $U(y)_\alpha^\beta$ and $\tilde{U}(y)_\beta^\alpha$ such that a field with p lower internal indices and q upper internal indices has the following Kaluza–Klein Ansatz (suppressing lower-dimensional indices)

$$\hat{\phi}_{\alpha_1\ldots\alpha_p}{}^{\beta_1\ldots\beta_q}(x,y) = U(y)_{\alpha_1}^{\gamma_1}\cdots U(y)_{\alpha_p}^{\gamma_p}\tilde{U}(y)_{\delta_1}^{\beta_1}\ldots\tilde{U}(y)_{\delta_q}^{\beta_q}\phi_{\gamma_1\ldots\gamma_p}{}^{\delta_1\ldots\delta_q}(x)\,. \tag{3.5.4}$$

Consistency requires $\tilde{U}(y)_\alpha^\gamma U(y)_\gamma^\beta = \delta_\alpha^\beta$ and that the following relation holds [116]:

$$\tilde{U}_\alpha^{\delta_1}\tilde{U}_\beta^{\delta_2}\left(\partial_{\delta_1}U_{\delta_2}^\gamma - \partial_{\delta_2}U_{\delta_1}^\gamma\right) = -f_{\alpha\beta}^\gamma\,, \tag{3.5.5}$$

for some constants $f_{\alpha\beta}^\gamma$ and $\partial_\alpha = \partial_{y_\alpha}$.

If we introduce the one-forms $\tilde{\sigma}^\alpha = \tilde{U}(y)_\beta^\alpha \mathrm{d}y^\beta$ equation 3.5.5 is equivalent to

$$\mathrm{d}\tilde{\sigma}^\alpha = -\tfrac{1}{2}f_{\gamma\delta}^\alpha\tilde{\sigma}^\gamma\wedge\tilde{\sigma}^\delta\,, \tag{3.5.6}$$

and the Kaluza–Klein Ansatz 3.5.4 is for a p-form on the internal manifold equivalent to

$$\hat{\phi}_{\alpha_p\ldots\alpha_p}(x,y)\mathrm{d}y^{\alpha_1}\wedge\cdots\wedge\mathrm{d}y^{\alpha_p} = \phi(x)_{\alpha_p\ldots\alpha_p}\tilde{\sigma}^{\alpha_1}\wedge\cdots\wedge\tilde{\sigma}^{\alpha_p}\,. \tag{3.5.7}$$

Hence a Scherk–Schwarz II reduction looks equivalent to a group manifold reduction where the group G has structure constants $f_{\gamma\delta}^\alpha$.

There is however a global issue; nothing forbids the structure constants appearing on the right-hand side of equation 3.5.5 to be the structure constants of a noncompact group (the structure constants have to be traceless though, $f_{\beta\alpha}^\alpha = 0$ [116] but this does not mean that the group is compact). To obtain a compact manifold one divides by a noncompact discrete subgroup Γ such that G/Γ is compact [121]. Then locally one can use the Kaluza–Klein Ansatz 3.5.4.

3.6 Other Reductions and Consistency

In Kaluza–Klein reductions the internal manifold is a symmetric space but symmetric spaces are just one special class of manifolds that can be used for dimensional reductions. To obtain other dimensional reductions one can, for example, try to generalize to the spaces that are obtained by modding a symmetric space G/K out by a discrete subgroup Γ of G, for example $T^n/\mathbb{Z}_n$ (an n-cube with the boundaries not identified).

There are dimensional reductions that require a totally new approach, such as reducing over Joyce Manifolds, Calabi–Yau Manifolds or manifolds with some 'structure', such as G_2-structures. In this thesis we do not treat them. We refer to the vast amount of literature on this subject, e.g. [126–137] and references therein.

When considering reductions and their consistency there is a subtle issue one has to keep in mind. In the preceding sections it was greatly emphasized that the truncation to a finite set of fields should be a consistent one. But if one is interested in an effective low-energy approximation of a supergravity theory one can go around the difficult issue of consistency. If the states with light masses couple to states with heavy masses, the heavy states can be produced via an interaction *at high energies*. So an inconsistent truncation is truly wrong at high energies but at energies below a certain cutoff scale the original theory can in some cases be safely described by a truncated theory where some interactions have been discarded. Of course one cannot truncate at will, in some cases a blind truncation will lead to a theory that is no longer a good approximation to the mother theory. If a truncation to a low-mass sector does not work, the physically right approach is to integrate out the massive modes. See for a more technical explanation on this issue [138, 139].

If one wants to be able revert the action of dimensional reduction and uplift a lower-dimensional solution, one needs a consistent reduction. Recall the definition of a consistent reduction from section 3.2.3: a reduction is consistent if every lower-dimensional solution can be uplifted to a solution of the higher-dimensional theory.

A feature of Kaluza–Klein reductions is that the explicit expression for the higher-dimensional fields is given by means of the Kaluza–Klein Ansatz. For other reductions the precise expressions may not be known; for example the metric on many Calabi–Yau manifolds is not known but one does know that the Ricci tensor vanishes. On the one hand it is appealing that without knowing the precise expressions for the fields still many statements can be made about the lower-dimensional action and even the Lagrangian can be constructed, but on the other hand the explicitness of Kaluza–Klein theories, admitting a full quantitative analysis, is also appealing.

Chapter 4

$\mathcal{N} = 4$ $d = 4$ Supergravity and its Scalar Potential

In the previous chapter we have investigated the relation between higher-dimensional supergravities and lower-dimensional supergravities. One can ask the question whether all four-dimensional supergravities can be deduced from a ten-dimensional supergravity. The answer is by the time of writing of this thesis still not known. To obtain an answer both the dimensional reductions of ten-dimensional supergravity and the features of four-dimensional supergravity need to be investigated.

For any theory the most important objects are the solutions of the equations of motion. Within the solutions the most prominent place is taken by the vacuum solution(s). Most vacuum solutions are characterized by the vanishing of all fields except the metric and the scalars, of which the latter take constant values. The scalars contribute to the curvature through the scalar potential. The value of the scalars at the vacuum determine how much supersymmetry is broken. Therefore the vacua of a theory can be studied by the scalars and their potential.

In four dimensional $\mathcal{N} = 4$ supergravity it is possible to introduce so-called $SU(1,1)$-angles. These angles influence the scalar potential and hence the vacua of the theory. A ten-dimensional origin of the $SU(1,1)$-angles is not known.

In this chapter we study $\mathcal{N} = 4$ $d = 4$ supergravity and we focus on the potential of the theory. A premature discussion on $\mathcal{N} = 4$ $d = 4$ supergravity was given in section 2.4.2, which we extend in this chapter. In section 4.1 we introduce the four-dimensional $\mathcal{N} = 4$ supergravity multiplet, give its (gauged and ungauged) action and introduce the $SU(1,1)$-angles. In section 4.2 we introduce the matter multiplets and in section 4.2.2 we study the symplectic embedding. The symplectic embedding provides a technique for determining the coupling between the scalars and the vectors in a supergravity. In section 4.3 we briefly discuss the matter coupled gauged $\mathcal{N} = 4$

$d = 4$ supergravity and in section 4.4 we introduce the scalar potential of $\mathcal{N} = 4$ $d = 4$ supergravity. In sections 4.5 and 4.6 we study the potential for semisimple and CSO-gaugings. We always assume that the number of space-time dimensions is four, unless otherwise stated.

4.1 The Pure $\mathcal{N} = 4$ Supergravity

In this section we review the basics of pure $\mathcal{N} = 4$ $d = 4$ supergravity. We do not consider the construction of the theory, which has been done in the end of the seventies, see e.g. [58, 59, 140].

The supergravity multiplet of $\mathcal{N} = 4$ supergravity consists of a metric, four gravitini, six vectors, four spin-1/2 fermions and two scalars. All fields carry a representation of $\mathfrak{su}(4) \cong \mathfrak{so}(6)$; the metric and the scalars are singlets under $\mathfrak{su}(4)$, the fermions are in the vector representation **4** of $\mathfrak{su}(4)$ and the vector fields are in the real **6** representation of $\mathfrak{su}(4)$, which is the same as the vector representation of $\mathfrak{so}(6)$.

The Scalar Manifold $SU(1,1)/U(1)$

The two scalars ϕ^α, $\alpha = 1, 2$, parameterize an $SU(1,1)/U(1)$ coset. We define ϕ_α by $\phi_1 = (\phi^1)^*$ and $\phi_2 = -(\phi^2)^*$. We introduce an $SU(1,1)$-matrix $\mathcal{V}$ by:

$$\mathcal{V} = \begin{pmatrix} \phi^1 & \phi_2 \\ \phi^2 & \phi_1 \end{pmatrix} ; \qquad |\phi^1|^2 - |\phi^2|^2 = \phi^\alpha \phi_\alpha = 1 , \tag{4.1.1}$$

and which satisfies $\mathcal{V}^\dagger \eta_{1,1} \mathcal{V} = \eta_{1,1}$ where

$$\eta_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{4.1.2}$$

We decompose the Lie algebra $\mathfrak{su}(1,1)$ into its compact part and its noncompact part; $\mathfrak{su}(1,1) = \mathfrak{u}(1) \oplus \mathfrak{u}(1)^\perp$. In the vector representation the subalgebra $\mathfrak{u}(1)$ is spanned by $t_1 = i\sigma_3$, whereas $\mathfrak{u}(1)^\perp$ is spanned by $t_2 = \sigma_1$ and $t_3 = \sigma_2$, where the σ_i are the Pauli-matrices (see Appendix A). We call $\mathbb{P}$ the projection onto $\mathfrak{u}(1)^\perp$.

We find

$$\mathbb{P}\mathcal{V}^{-1}\partial_\mu \mathcal{V} = \begin{pmatrix} 0 & -\phi_2 \partial_\mu \phi_1 + \phi_1 \partial_\mu \phi_2 \\ -\phi^2 \partial_\mu \phi^1 + \phi^1 \partial_\mu \phi^2 & 0 \end{pmatrix} , \tag{4.1.3}$$

and thus for the $SU(1,1)$-scalars we find the kinetic Lagrangian:

$$\tfrac{1}{2}\mathrm{Tr}\left(\mathcal{V}^{-1}\partial_\mu \mathcal{V}\mathbb{P}\mathcal{V}^{-1}\partial^\mu \mathcal{V}\right) = \partial_\mu \phi_\alpha \partial^\mu \phi^\alpha + \phi^\alpha \partial_\mu \phi_\alpha \phi^\beta \partial^\mu \phi_\beta . \tag{4.1.4}$$

The kinetic term 4.1.4 is the kinetic term of the $SU(1,1)$-scalars of $\mathcal{N} = 4$ supergravity found in the literature [36, 59].

Using a $U(1)$-transformation we can make ϕ^1 real. Hence we take the following parametrization of $SU(1,1)/U(1)$

$$\phi^1 = \frac{1}{\sqrt{1-r^2}}\,, \quad \phi^2 = \frac{r\mathrm{e}^{i\chi}}{\sqrt{1-r^2}}\,. \tag{4.1.5}$$

The parameter r is restricted to the interval $[0,1)$ and χ is an angular parameter running from 0 to 2π. Substituting the parametrization 4.1.5 into the kinetic term 4.1.4 gives rise to the scalar kinetic Lagrangian:

$$\mathcal{L}[r,\chi] = -\frac{1}{(1-r^2)^2}\left(\partial_\mu r \partial^\mu r + r^2 \partial_\mu \chi \partial^\mu \chi\right)\,. \tag{4.1.6}$$

We now briefly show the connection with other formulations of the scalar manifold $SU(1,1)/U(1)$. We first put $z = r\mathrm{e}^{-i\chi}$ and define

$$\tau = i\frac{1-z}{1+z}\,, \tag{4.1.7}$$

and obtain the scalar Lagrangian

$$\mathcal{L}[\tau] = -\frac{1}{4}\frac{\partial_\mu \tau \partial^\mu \bar{\tau}}{(\mathrm{Im}\tau)^2}\,. \tag{4.1.8}$$

The action 4.1.8 is invariant under Möbius-transformations: $\tau \mapsto (a\tau+b)(c\tau+d)^{-1}$, where $a,b,c,d \in \mathbb{R}$ and $ad-bc=1$. Putting $\tau = \sigma + i\mathrm{e}^{-\phi}$ we obtain the Lagrangian of the coset $SL(2;\mathbb{R})/SO(2)$

$$\mathcal{L}[\sigma,\phi] = -\frac{1}{4}\left(\partial_\mu \phi \partial^\mu \phi + \mathrm{e}^{2\phi}\partial_\mu \sigma \partial^\mu \sigma\right)\,. \tag{4.1.9}$$

In the following we use that formalism that gives the most elegant result.

The Bosonic Lagrangian

Ungauged $\mathcal{N}=4$ supergravity admits a formulation in which $SU(4) \cong SO(6)$ is a global symmetry. Using the $SU(1,1)$-variables σ, ϕ the bosonic Lagrangian reads in the $SO(6)$-formulation

$$\begin{aligned} e^{-1}\mathcal{L} =& R(\omega) - \tfrac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi + \mathrm{e}^{2\phi}\partial_\mu \sigma \partial^\mu \sigma\right) \\ & - \tfrac{1}{4}\mathrm{e}^{-\phi}\sum_{a=1}^{6} F^a_{\mu\nu}F^{\mu\nu,a} - \tfrac{1}{8}\sigma \epsilon^{\mu\nu\rho\sigma}\sum_{a=1}^{6} F^a_{\mu\nu}F^a_{\rho\sigma}\,, \end{aligned} \tag{4.1.10}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$ are the abelian field strengths. The global $SO(6)$-symmetry rotates the field strengths $F^a_{\mu\nu}$ and the fermions. To go to the $SU(4)$-formulation one uses the 't Hooft symbols (see appendix A.3.2) to rewrite the field strengths as: $F^{ij}_{\mu\nu}=\frac{1}{2}(G_a)^{ij}F^a_{\mu\nu}$.

The six vectors A^a_μ can be promoted to nonabelian gauge fields by making a six-dimensional subgroup H of $SO(6)$ a local symmetry group. The gauged theory has no longer a global $SO(6)$-symmetry, but only the local H-symmetry. We associate with every gauge field A^a_μ a generator T_a such that the structure constants are given by: $[T_a,T_b]=f_{ab}{}^cT_c$. The tensors $f_{abc}\equiv f_{ab}{}^d\delta_{dc}$ have to be completely antisymmetric. To remain supersymmetric the Lagrangian 4.1.10 acquires a potential. The potential $V(\sigma,\phi)$ is given by:

$$V(\sigma,\phi)=-\mathrm{e}^{\phi}\sum_{a,b,c}(f_{ab}{}^c)^2\,. \tag{4.1.11}$$

The potential 4.1.11 is unbounded from below. In theories that are described in flat space-times a potential that is unbounded from below is a problem, since there is no stable vacuum with the lowest energy. For theories that involve gravity the stability of an anti-de Sitter vacuum is guaranteed if the Breitenlohner–Freedman bound is satisfied [141, 142].

The $SU(1,1)$-angles

There exists a formulation of ungauged pure $\mathcal{N}=4$ supergravity in which the global $SO(6)$-symmetry is broken; all vector fields are coupled to the $SU(1,1)$-scalars in another way. To see what happens it is convenient to rewrite the scalars ϕ and σ into the complex scalar z as defined above; $\sigma+i\mathrm{e}^{-\phi}=i(1-z)(1+z)^{-1}$. The different couplings are then obtained by putting $z\mapsto \mathrm{e}^{-2i\alpha_a}z$ in each coupling to the gauge field A^a_μ. The Lagrangian is given by [36, 143]

$$\begin{aligned} e^{-1}\mathcal{L}=&R(\omega)-\frac{2}{(1-|z|^2)^2}\partial_\mu z\partial^\mu\bar{z}-\tfrac{1}{4}\sum_{a=1}^{6}\frac{1-|z|^2}{|1+z\mathrm{e}^{-2i\alpha_a}|^2}F^a_{\mu\nu}F^{\mu\nu,a}\\ &-\tfrac{1}{4}\epsilon^{\mu\nu\rho\sigma}\sum_{a=1}^{6}\frac{\mathrm{Im}(z\mathrm{e}^{-2i\alpha_a})}{|1+z\mathrm{e}^{-2i\alpha_a}|^2}F^a_{\mu\nu}F^a_{\rho\sigma}\,. \end{aligned} \tag{4.1.12}$$

The factor of 2 for the $SU(1,1)$-angles α_a is for later convenience. One can show that the action 4.1.12 is related to the action 4.1.10 by a duality rotation, which we discuss in section 4.2.2.

It is clear that the gauging is affected by the introduction of the $SU(1,1)$-angles. To have local H-symmetry, the $SU(1,1)$-angles need to be chosen such that they respect the H-symmetry. This is discussed in more detail in the sections 4.5.1 and 4.6.2. For now we remark that if the structure constant $f_{ab}{}^c$ is nonzero, then $\alpha_a=$

$\alpha_b = \alpha_c$. The potential that arises from the gauging becomes [36, 143]:

$$V(z) = -\frac{|1+z\mathrm{e}^{-2i\alpha_a}|^2}{1-|z|^2}\sum_{a,b,c}(f_{ab}{}^c)^2 - \tfrac{1}{36}\sum_{a,\ldots,f}\epsilon^{abcdef}f_{ab}{}^c f_{de}{}^f \sin(\alpha_a-\alpha_d)\,. \quad (4.1.13)$$

The potential 4.1.13 can have an extremum [36], which was also noticed by [144].

The fact that due to the introduction of the $SU(1,1)$-angles the potential of pure $\mathcal{N}=4$ supergravity can have an extremum, makes the $SU(1,1)$-angles appealing. However, to the knowledge of the author it has never been shown how to obtain the $SU(1,1)$-angles from a dimensional reduction of a ten-dimensional supergravity. Therefore there seems to be no string theoretical origin of the $SU(1,1)$-angles.

4.2 The Vector Multiplet

For $\mathcal{N}=4$ supersymmetry there is only one matter multiplet that can be coupled to the supergravity multiplet; the vector multiplet, which is the subject of this section. We first consider in general the $\mathcal{N}=4$ supersymmetric Yang–Mills theory, then we consider the symplectic embedding, which provides us with useful information about the coupling to the supergravity multiplet. In particular, we construct the coupling of the scalars to the vector fields when all $SU(1,1)$-angles are equal.

4.2.1 $\mathcal{N}=4$ $d=4$ Super-Yang–Mills

The $\mathcal{N}=4$ Super-Yang–Mills theory can be obtained from a dimensional reduction of the ten-dimensional $\mathcal{N}=1$ Super-Yang–Mills theory. In Super-Yang–Mills theory gravity is not included and hence the theory is formulated in a Minkowski space-time.

We first discuss important aspects of the ten-dimensional $\mathcal{N}=1$ Super-Yang–Mills theory and subsequently discuss the dimensional reduction to four dimensions. The conventions, notations and some useful formulas for spinors are explained in section appendix C.

The Ten-Dimensional Theory

In ten dimensions the nonabelian $\mathcal{N}=1$ Super-Yang–Mills multiplet consists of a gauge potential A_μ and a Majorana–Weyl spinor ψ satisfying $\Gamma_{11}\psi = P_+\psi$, where P_+ is a chiral projection operator. Both fields take values in a compact Lie algebra $\mathfrak{g}$. The Lagrangian is given by:

$$\mathcal{L}_{SYM} = \int \mathrm{d}^{10}x\ \left(\tfrac{1}{4}\mathrm{Tr}_{\mathrm{ad}}\left(F_{\mu\nu}F^{\mu\nu}\right) + \tfrac{i}{2}\mathrm{Tr}_{\mathrm{ad}}\bar{\psi}\Gamma^\mu \mathrm{D}_\mu\psi\right)\,, \quad (4.2.1)$$

where $\bar{\psi}=\psi^T\mathcal{C}$, with $\mathcal{C}$ the antisymmetric charge conjugation matrix, and where

$$\mathrm{D}_\mu\psi=\partial_\mu\psi+g_{YM}[A_\mu,\psi]\,,\quad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+g_{YM}[A_\mu,A_\nu]\,. \tag{4.2.2}$$

The field strength $F_{\mu\nu}$ satisfies the Bianchi identity $\mathrm{D}_{[\mu}F_{\nu\rho]}=0$. An infinitesimal Yang–Mills transformation with parameter Λ taking values in $\mathfrak{g}$ acts on the fields as:

$$\delta_\Lambda\psi=[\Lambda,\psi]\,,\quad \delta_\Lambda A_\mu=-\tfrac{1}{g_{YM}}\mathrm{D}_\mu\Lambda\,. \tag{4.2.3}$$

The awkward looking sign in front of the Lagrangian 4.2.1 is due to the fact that for compact Lie algebras the trace in the adjoint representation is negative definite. The Lagrangian 4.2.1 is real due to the identity $\mathcal{C}=\mathcal{B}\mathcal{C}^*\mathcal{B}$, where $\mathcal{B}$ is the symmetric complex conjugation matrix. The Lagrangian 4.2.1 is invariant under the supersymmetry transformations:

$$\delta_\epsilon A_\mu=i\bar{\epsilon}\Gamma_\mu\psi\,,\quad \delta_\epsilon\psi=-\tfrac{1}{2}\Gamma_{\mu\nu}F^{\mu\nu}\epsilon\,. \tag{4.2.4}$$

The i in the supersymmetry transformation of A_μ is needed to keep the field A_μ real. Using the Fierz identity for two Majorana–Weyl fermions ϵ_1,ϵ_2 of positive chirality

$$\epsilon_2\bar{\epsilon}_1-\epsilon_2\bar{\epsilon}_1=-\tfrac{1}{8}\bar{\epsilon}_1\Gamma^\mu\epsilon_2\Gamma_\mu P_- -\tfrac{1}{920}\bar{\epsilon}_1\Gamma^{\mu_1\cdots\mu_5}\epsilon_2\Gamma_{\mu_1\cdots\mu_5}P_-\,, \tag{4.2.5}$$

we find that the supersymmetry algebra is given by:

$$\begin{aligned}
[\delta_{\epsilon_1},\delta_{\epsilon_2}]\psi &= -2i\bar{\epsilon}_1\Gamma^\mu\epsilon_2\partial_\mu\psi+[-2ig_{YM}\bar{\epsilon}_1\Gamma^\sigma A_\sigma\epsilon_2,\psi]+\tfrac{7i}{8}\bar{\epsilon}_1\Gamma^\rho\epsilon_2\Gamma_\rho\Gamma^\mu\mathrm{D}_\mu\psi\,,\\
&\quad -\tfrac{i}{960}\bar{\epsilon}_1\Gamma^{\nu_1\cdots\nu_5}\epsilon_2\Gamma_{\nu_1\cdots\nu_5}\Gamma^\mu\mathrm{D}_\mu\psi\\
[\delta_{\epsilon_1},\delta_{\epsilon_2}]A_\mu &= -2i\bar{\epsilon}_1\Gamma^\nu\epsilon_2\partial_\nu A_\mu+\mathrm{D}_\mu\left(2i\bar{\epsilon}_1\Gamma^\sigma A_\sigma\epsilon_2\right)\,.
\end{aligned} \tag{4.2.6}$$

The supersymmetry algebra closes up to gauge transformations with gauge parameter $\Lambda=-2ig_{YM}\bar{\epsilon}_1\Gamma^\rho\epsilon_2A_\rho$ and up to the equation of motion of the fermion, $\Gamma^\mu\mathrm{D}_\mu\psi=0$.

The Reduction

We now perform a toroidal reduction to obtain the four dimensional Super-Yang–Mills $\mathcal{N}=4$ multiplet. All ten-dimensional fields and indices are hatted. The ten-dimensional space-time indices $\hat{\mu},\hat{\nu},\ldots$ are split into $0\le\mu,\nu,\ldots\le3$ and $1\le a,b,\ldots\le6$. The coordinates on the four-dimensional space-time are denoted x^μ and the coordinates on the six-dimensional torus are denoted by y^a. All lower-dimensional fields are independent of the coordinates y^a.

The reduction Ansatz for the gauge potential is given by $\hat{A}=A_\mu(x)\mathrm{d}x^\mu+Z_a(x)\mathrm{d}y^a$. From this we find the following components of the ten-dimensional field strength $\hat{F}_{\hat{\mu}\hat{\nu}}$:

$$\begin{aligned}
\hat{F}_{\mu\nu} &= F_{\mu\nu}=F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+g_{YM}[A_\mu,A_\nu]\,,\\
\hat{F}_{\mu a} &= \mathrm{D}_\mu Z_a=\partial_\mu Z_a+g_{YM}[A_\mu,Z_a]\,,\\
\hat{F}_{ab} &= g_{YM}[Z_a,Z_b]\,.
\end{aligned} \tag{4.2.7}$$

The reduction of the fermions is done by choosing a particular representation of the Clifford algebra of $\mathfrak{so}(1,9)$. We take:

$$\hat{\Gamma}_\mu = \gamma_\mu \otimes \mathbb{1}_{8\times 8}\,, \quad \hat{\Gamma}_a = \gamma_5 \otimes \Gamma_a\,, \tag{4.2.8}$$

where γ_μ represent the Clifford algebra of $\mathfrak{so}(1,3)$ and Γ_a represent the Clifford algebra of $\mathfrak{so}(6)$ by means of the 't Hooft symbols G_a as explained in appendix C.3.

The Clifford algebra of $\mathfrak{so}(1,3)$ can be taken to be real $\gamma_\mu^* = \gamma_\mu$ and an explicit real representation is given in C.2. Hence $B_-^{(4)} \sim \mathbb{1}_{4\times 4}$ and we choose a basis for the fermions such that $B_-^{(4)} = -i\mathbb{1}_{4\times 4}$. The charge conjugation matrix of $\mathfrak{so}(1,3)$, denoted C, we therefore take to be $C = i\gamma_0$. The chirality matrix $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ is purely imaginary.

For the charge conjugation matrix $\mathcal{C}$ and the complex conjugation matrix $\mathcal{B}$ we take the decomposition:

$$\mathcal{C} = i\gamma_0 \otimes \tilde{C}\,, \qquad \mathcal{B} = -i\mathbb{1}_{4\times 4} \otimes \tilde{C}\,, \tag{4.2.9}$$

where $\tilde{C}$ is the symmetric charge conjugation matrix of $\mathfrak{so}(6)$. The explicit representation of $\tilde{C}$ is given by equation C.3.2. The chirality matrix is given by $\hat{\Gamma}_{11} = \gamma_5 \otimes \Gamma_7$. The explicit representation of Γ_7 is given by equation C.3.3.

The spinors $\hat{\psi}$ are decomposed as:

$$\hat{\psi}(x,y) = \sum_{i=1}^{8} \psi^{(i)}(x) \otimes \theta^i\,. \tag{4.2.10}$$

The spinors θ^i are the real basis spinors given by: $(\theta^i)_j = \delta_{ij}$ and hence a ten-dimensional spinor $\hat{\psi}$ is effectively written as:

$$\hat{\psi} = \begin{pmatrix} \psi^{(1)} \\ \vdots \\ \psi^{(8)} \end{pmatrix}\,. \tag{4.2.11}$$

Imposing the Majorana constraint $\hat{\psi}^* = i\mathcal{B}\psi$ gives:

$$(\psi^{(1)})^* = \psi^{(5)}\,,\ (\psi^{(2)})^* = \psi^{(6)}\,,\ (\psi^{(3)})^* = \psi^{(7)}\,,\ (\psi^{(4)})^* = \psi^{(8)}\,. \tag{4.2.12}$$

Therefore we write the first four $\psi^{(i)}$ with a upper index as ψ^i, $i = 1,2,3,4$, and the last four with a lower index ψ_i, $i = 1,2,3,4$; we then have $\psi^{i*} = \psi_i$. We thus decompose the ten-dimensional spinor $\hat{\psi}$ as a doublet of a quartet of fermions, where one doublet is the complex conjugate of the other;

$$\hat{\psi} = \begin{pmatrix} \psi^i \\ \psi_i \end{pmatrix}\,. \tag{4.2.13}$$

Imposing the chirality constraint $\hat{\Gamma}_{11}\hat{\psi}=\hat{\psi}$ gives:

$$\gamma_5\psi^i=\psi^i\,,\quad \gamma_5\psi_i=-\psi_i\,, \tag{4.2.14}$$

which is consistent with $\gamma_5^*=-\gamma_5$.

We define for the four-dimensional spinor ψ^i the conjugates $\bar{\psi}^i=\psi^{iT}\gamma_0$ and for ψ_i similarly $\bar{\psi}_i=\psi_i^T\gamma_0$. With this definition we have $(\bar{\psi}_i)^*=\bar{\psi}^i$ and

$$\bar{\hat{\psi}}=i(\bar{\psi}_i,\bar{\psi}^i)\,. \tag{4.2.15}$$

Under the action of $SU(4)$ the spinors $\bar{\psi}^i$ and ψ^i both transform in the **4** and $\bar{\psi}_i$ and ψ_i transform in the $\bar{\mathbf{4}}$.

Putting the expressions together one finds:

$$\begin{aligned}\mathcal{L}_{SYM}=&\mathrm{Tr}_{\mathrm{ad}}\Big(\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}\sum_{a=1}^{6}\mathrm{D}_\mu Z_a\mathrm{D}^\mu Z_a-\tfrac{1}{2}\bar{\psi}_i\gamma^\mu\mathrm{D}_\mu\psi^i-\tfrac{1}{2}\bar{\psi}^i\gamma^\mu\mathrm{D}_\mu\psi_i\\&+\tfrac{1}{4}g_{YM}^2\sum_{a,b=1}^{6}[Z_a,Z_b][Z_a,Z_b]+g_{YM}\sum_{a=1}^{6}Z_a\mathrm{Im}\left(\bar{\psi}^iG_{a,ij}\psi^j\right)\Big)\end{aligned} \tag{4.2.16}$$

When we use the 't Hooft symbols G_a^{ij} to define

$$\phi^{ij}=\tfrac{1}{2}\sum_a G_a^{ij}Z_a \tag{4.2.17}$$

and its complex conjugate $\phi_{ij}=\phi^{ij*}$ and use the properties of the 't Hooft symbols (see appendix A.3.2) we obtain the result:

$$\begin{aligned}\mathcal{L}_{SYM}=&\mathrm{Tr}_{\mathrm{ad}}\Big(\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}\mathrm{D}_\mu\phi_{ij}\mathrm{D}^\mu\phi^{ij}-\tfrac{1}{2}\bar{\psi}_i\gamma^\mu\mathrm{D}_\mu\psi^i-\tfrac{1}{2}\bar{\psi}^i\gamma^\mu\mathrm{D}_\mu\psi_i\\&+\frac{g_{YM}^2}{4}[\phi_{ij},\phi_{kl}][\phi^{ij},\phi^{kl}]+ig_{YM}\left(\bar{\psi}_i\phi^{ij}\psi_j-\bar{\psi}^i\phi_{ij}\psi^j\right)\Big)\,.\end{aligned} \tag{4.2.18}$$

The supersymmetry transformation under which the action 4.2.18 is invariant are given by:

$$\begin{aligned}\delta A_\mu&=-\bar{\epsilon}^i\gamma_\mu\psi_i-\bar{\epsilon}_i\gamma_\mu\psi^i\,,\\ \delta\phi^{ij}&=-i\left(\bar{\epsilon}^i\psi^j-\epsilon^j\psi^i+\epsilon^{ijkl}\bar{\epsilon}_k\psi_l\right)\,,\\ \delta\psi^i&=-\tfrac{1}{2}F^{\mu\nu}\gamma_{\mu\nu}\epsilon^i+2i\left(\mathrm{D}_\mu\phi^{ij}\right)\gamma^\mu\epsilon_j+2[\phi^{ik},\phi_{kj}]\epsilon^j\,.\end{aligned} \tag{4.2.19}$$

The supersymmetry transformation rules 4.2.19 can be found by reducing the ten-dimensional supersymmetry transformation rules 4.2.4.

The action 4.2.18 contains a scalar potential that is proportional to g_{YM}^2. When we take the abelian limit - that is, setting the derived algebra $\mathfrak{g}'$ to zero - the scalar potential vanishes. In the abelian limit only the free theory survives and the Lagrangian becomes a sum of kinetic terms.

To obtain matter coupled $\mathcal{N}=4$ supergravity the free theory is coupled to the ungauged $\mathcal{N}=4$ supergravity. To gauge the coupled theory one uses the global symmetry group G of which a subgroup H is promoted to a local symmetry. The gauge fields of the supergravity multiplet and the gauge fields from the Super-Yang–Mills multiplet are then promoted to the Yang–Mills fields of the group H and all derivatives are made covariant with respect to H.

The gauging procedure in general breaks supersymmetry and to restore supersymmetry one has to modify the supersymmetry transformation rules and the Lagrangian. The modification of the Lagrangian involves adding a potential. The appearance of a scalar potential looks natural in the light of the action 4.2.18, since this action has a local gauge symmetry and is supersymmetric (though globally supersymmetric). The potential in gauged matter coupled supergravity is also proportional to the square of the coupling constant of the gauge group H. However, the two potentials are different in nature; the potential of 4.2.18 is contained in the potential of gauged matter coupled $\mathcal{N}=4$ supergravity.

4.2.2 The Symplectic Embedding

In ungauged supergravity all gauge fields are abelian gauge fields. The gauge fields couple to the scalars in the following way:

$$\mathcal{L} = -f(\Phi) \star F \wedge F + g(\Phi) F \wedge F \,, \tag{4.2.20}$$

where f and g are functions depending on the scalars, collectively denoted Φ. The functions f and g can be found when the scalar manifold is known by a trick called the 'symplectic embedding'. We now present how the symplectic embedding works and apply it to ungauged $\mathcal{N}=4$ supergravity, where the scalar manifold is $SU(1,1)/U(1) \otimes SO(6,n)/SO(6) \times SO(n)$. This section is based on [36, 72]. To make the presentation less spoilt by a large number of indices, we first introduce a compact and more abstract notation. We then show how the symplectic groups arise as duality groups, rotating equivalent but different Lagrangians into each other. Then we use the duality group to obtain the coupling of the vector fields to the scalars.

We introduce N abelian gauge field strengths $F^I_{\mu\nu}$, $I = 1, \ldots, N$. The scalars Φ parameterize a coset G/K. The most general Lagrangian for the field strengths $F^I_{\mu\nu}$ coupled to the scalars Φ is[1]

$$\mathcal{L}_{vec}[\Phi, F] = -\tfrac{1}{4}\gamma_{IJ} F^I_{\mu\nu} F^{J\mu\nu} + \tfrac{1}{8}\theta_{IJ}\epsilon^{\mu\nu\lambda\rho} F^I_{\mu\nu} F^J_{\lambda\rho} \,, \tag{4.2.21}$$

[1] The role of gravity is immaterial and hence we will take a flat space-time Lagrangian. The extension to curved space-times is straightforward.

where γ_{IJ} and θ_{IJ} are Φ-dependent symmetric matrices.

The field strengths F^I are two-forms and the space of two-forms we denote Ω_2. Let us define the linear transformation j on Ω_2 by

$$(jF)_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}\,, \tag{4.2.22}$$

for any $F \in \Omega_2$. We have $j^2 = -1$.

We assemble the field strengths F^I into the vector $\mathcal{F} = (F^1, \ldots, F^N)$, which is an element of $\mathcal{W} = \mathrm{I\!R}^N \otimes \Omega_2$. We equip $\mathcal{W}$ with the inner product $(,)$ defined by

$$(\mathcal{X}, \mathcal{Y}) = \sum_{I=1}^{N} \mathcal{X}^I_{\mu\nu}\mathcal{Y}^{I,\mu\nu}, \quad \mathcal{X}, \mathcal{Y} \in \mathcal{W}\,. \tag{4.2.23}$$

The linear transformation j is extended to the linear transformation $\mathcal{J} = \mathbb{1} \otimes j$ on $\mathcal{W}$. Since $\mathcal{J}$ squares to minus the identity on $\mathcal{W}$ it has eigenvalues $+i$ and $-i$ and therefore we pass to the complex extension of $\mathcal{W}$, denoted $\mathcal{W}^{\mathbb{C}}$. The inner product $(,)$ is extended to a bilinear form on $\mathcal{W}^{\mathbb{C}}$ by $\mathbb{C}$-linearity; if $U, V, X, Y \in \mathcal{W}$, then on $\mathcal{W}^{\mathbb{C}}$ we have

$$(U + iV, X + iY) = (U, X) + i\,(V, X) + i\,(U, Y) - (V, Y)\;. \tag{4.2.24}$$

The vector space $\mathcal{W}^{\mathbb{C}}$ splits into the direct sum of the eigenspaces of $\mathcal{J}$. We define for any $\mathcal{F} \in \mathcal{W}_{\mathbb{C}}$ the projections $\mathcal{F}^{\pm}$ onto the $\pm i$-eigenspaces of $\mathcal{J}$ by

$$\mathcal{F} = \mathcal{F}^+ + \mathcal{F}^-, \quad \mathcal{F}^{\pm} = \tfrac{1}{2}\left(\mathcal{F} \mp i\mathcal{J}\mathcal{F}\right), \quad \mathcal{J}\mathcal{F}^{\pm} = \pm i\mathcal{F}^{\pm}\,. \tag{4.2.25}$$

With respect to the inner product $(,)$ the $\pm i$-eigenspaces of $\mathcal{J}$ are orthogonal to each other.

We introduce the symmetric kinetic matrix K and its complex conjugate $\bar{K}$ by

$$K = i\gamma + \theta\,, \quad \bar{K} = -i\gamma + \theta\,, \tag{4.2.26}$$

where the components of γ and θ are γ_{IJ} and θ_{IJ} respectively. We put $\mathcal{K} = K \otimes \mathbb{1}$. If $\mathcal{J}\mathcal{X} = \pm i\mathcal{X}$ then we have $\mathcal{J}\mathcal{K}\mathcal{X} = \pm i\mathcal{K}\mathcal{X}$.

With these definitions the Lagrangian 4.2.21 can be written as

$$\mathcal{L}_{vec}[\Phi, \mathcal{F}] = \frac{i}{4}\left(\mathcal{F}^+, \mathcal{K}\mathcal{F}^+\right) - \frac{i}{4}\left(\mathcal{F}^-, \bar{\mathcal{K}}\mathcal{F}^-\right)\;. \tag{4.2.27}$$

We define $\mathcal{G}$ by

$$\mathcal{J}\mathcal{G} \equiv 2\frac{\partial \mathcal{L}_{vec}}{\partial \mathcal{F}} = (-\gamma \otimes \mathbb{1} + \theta \otimes j)\mathcal{F}\,, \tag{4.2.28}$$

which gives $\mathcal{G} = (\gamma \otimes j + \theta \otimes \mathbb{1})\,\mathcal{F}$. One finds the following $\mathcal{J}$-eigenspace decomposition of $\mathcal{G}$:

$$\mathcal{G}^+ = \mathcal{K}\mathcal{F}^+\,, \quad \mathcal{G}^- = \bar{\mathcal{K}}\mathcal{F}^-\,. \tag{4.2.29}$$

The equation of motion and Bianchi identity now read

$$\partial^\mu (j\mathcal{F})_{\mu\nu} = 0\,, \quad \partial^\mu (j\mathcal{G})_{\mu\nu} = 0\,. \qquad (4.2.30)$$

The Bianchi equation and the equation of motion are invariant under $GL(2N, \mathbb{R})$-transformations;

$$\begin{pmatrix} \mathcal{F}^+_{\mu\nu} \\ \mathcal{G}^+_{\mu\nu} \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathcal{F}^+_{\mu\nu} \\ \mathcal{K}\mathcal{F}^+_{\mu\nu} \end{pmatrix}\,, \qquad (4.2.31)$$

where A, B, C, D are $N \times N$-matrices. However for general $GL(2N, \mathbb{R})$-rotations the transformation of $\mathcal{G}^\pm$ is inconsistent with its definition 4.2.28. We now show that when we restrict to $Sp(N, \mathbb{R})$-rotations and transform the scalars, using G-transformations, the inconsistency can be cured. We thus show that an isometry of the scalar manifold G/K accompanied by an $Sp(N, \mathbb{R})$-rotation on the electric and magnetic field strengths, gives rise to an equivalent set of equations of motions and Bianchi equations. The combination of the isometry of G/K with the $Sp(N, \mathbb{R})$-rotation is called a duality transformation. The duality transformations do not describe symmetries of the Lagrangian, but symmetries of the equations of motion and Bianchi identities.

A duality transformation acts on the vector fields as a $GL(2N, \mathbb{R})$-transformation mapping $\mathcal{F}, \mathcal{G}$ to $\mathcal{F}', \mathcal{G}'$ respectively such that the following relation holds:

$$\mathcal{J}\mathcal{G}' = 2\frac{\partial \mathcal{L}'_{vec}}{\partial \mathcal{F}'}\,, \qquad (4.2.32)$$

where $\mathcal{L}'_{vec}$ is the new vector Lagrangian: $\mathcal{L}'_{vec} = \mathcal{L}_{vec}[\Phi', \mathcal{F}']$. Imposing the constraint 4.2.32 gives

$$K' = (C + DK)(A + BK)^{-1}\,, \qquad (4.2.33)$$

where the A, B, C, D are the $N \times N$-matrix as in equation 4.2.31. Transformations of the kind 4.2.33 are called fractional linear transformation and they form a group.

What singles out the symplectic group is the requirement that the new kinetic matrix K' is again symmetric and that the duality rotations form a group, called the duality group. Working out the requirement of symmetry of K' gives

$$0 = A^T C + K^T (B^T C - D^T A) + K^T B^T D K - \text{Transpose}\,. \qquad (4.2.34)$$

Since K is an arbitrary complex symmetric matrix, we find $A^T C = C^T A$, $D^T B = B^T D$ and $A^T D - C^T B = \sigma \mathbb{1}$, $\sigma \in \mathbb{R}$. The constraints can be summarized by

$$\Lambda^T \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \Lambda = \sigma \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}\,, \quad \Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\,. \qquad (4.2.35)$$

The number σ must equal 1 in order that the duality transformations describe a group. The duality group is then fixed and is $Sp(N, \mathbb{R})$. We have thus proved that any theory involving scalars and N abelian field strengths the maximal symmetry group of the equations of motion and Bianchi identities is $Sp(N,\mathbb{R})$. The $Sp(N,\mathbb{R})$-rotation is fixed by the isometry of the scalar manifold. Hence we can see the duality rotations as an embedding map from the set of isometries of the scalar manifold to the group $Sp(N,\mathbb{R})$.

We now present a way to find the kinetic matrix K. The idea is to embed the isometry group G into $Sp(N,\mathbb{R})$ such that the compact subgroup K gets embedded into the compact subgroup of $Sp(N,\mathbb{R})$.

The group $Sp(N,\mathbb{R})$ is isomorphic to the group $USp(N,N) \equiv Sp(N,\mathbb{C}) \cap U(N,N)$. A complex $2N \times 2N$-matrix M describes an element of $USp(N,N)$ if

$$M^T \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} M = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \; ; \; M^\dagger \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} M = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} . \tag{4.2.36}$$

The general form of $M \in USp(N,N)$ is

$$M = \begin{pmatrix} T & V^* \\ V & T^* \end{pmatrix} , \quad \text{where } \mathrm{T}^\dagger \mathrm{T} - \mathrm{V}^\dagger \mathrm{V} = \mathbb{1} \text{ and } \mathrm{T}^\dagger \mathrm{V}^* = \mathrm{V}^\dagger \mathrm{T}^*, \tag{4.2.37}$$

If Λ is a $Sp(N,\mathbb{R})$-matrix, then the matrix M, given by

$$M = \mathcal{C}\Lambda\mathcal{C}^{-1} \text{ where, } \mathcal{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & i\mathbb{1} \\ \mathbb{1} & -i\mathbb{1} \end{pmatrix} , \tag{4.2.38}$$

is a $USp(N,N)$-matrix. Using the $USp(N,N)$-language has the advantage that the compact subgroup is 'on the diagonal'; the compact subgroup of $USp(N,N)$ is $U(N)$ and is obtained by putting $V=0$ in equation 4.2.37.

Let $\mathcal{V}(\Phi)$ be a representative of G/K, then we embed $\mathcal{V}$ into a $2N \times 2N$-matrix $\mathbb{V}(\Phi) \in USp(N,N)$, such that if $\Phi \neq \Phi'$, there is no $U(N)$-matrix W such that

$$\mathbb{V}(\Phi') = \mathbb{V}(\Phi) \begin{pmatrix} W & 0 \\ 0 & W^* \end{pmatrix} . \tag{4.2.39}$$

The $USp(N,N)$-element $\mathbb{V}(\Phi)$ describes the symplectic embedding of G/K into the coset $USp(N,N)/U(N)$, such that K is embedded in $U(N)$. We write the matrix $\mathbb{V}(\Phi)$ in block form as

$$\mathbb{V}(\Phi) = \begin{pmatrix} U_0(\Phi) & U_1^*(\Phi) \\ U_1(\Phi) & U_0^*(\Phi) \end{pmatrix} . \tag{4.2.40}$$

If we perform a duality transformation, mapping Φ to Φ', the matrix $\mathbb{V}(\Phi)$ gets rotated by a $USp(N,N)$-matrix S such that

$$S\mathbb{V}(\Phi) = \mathbb{V}(\Phi') \begin{pmatrix} W(\Phi) & 0 \\ 0 & W^*(\Phi) \end{pmatrix} , \quad S = \begin{pmatrix} T & V^* \\ V & T^* \end{pmatrix} , \tag{4.2.41}$$

where W is a $U(N)$-matrix. Re-expressing this in $Sp(N,\mathrm{I\!R})$-language, using 4.2.38 and the expression for $\mathbb{V}$ 4.2.40 one finds

$$\begin{aligned} U_0^\dagger(\Phi') + U_1^\dagger(\Phi') &= W^\dagger(\Phi)\left(U_0^\dagger(\Phi)\left(A^T + iB^T\right) + U_1^\dagger(\Phi)\left(A^T - iB^T\right)\right) , \\ U_0^\dagger(\Phi') - U_1^\dagger(\Phi') &= W^\dagger(\Phi)\left(U_0^\dagger(\Phi)\left(D^T - iC^T\right) - U_1^\dagger(\Phi)\left(D^T + iC^T\right)\right) , \end{aligned} \tag{4.2.42}$$

where A, B, C, D are given by inverting 4.2.38;

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathcal{C}^{-1} S \mathcal{C} \,. \tag{4.2.43}$$

From 4.2.42 we see that putting

$$K = i\left(U_0^\dagger + U_1^\dagger\right)^{-1}\left(U_0^\dagger - U_1^\dagger\right) \tag{4.2.44}$$

solves the transformation rule 4.2.33. Using that $\mathbb{V}$ is a $USp(N,N)$-matrix one verifies that K is symmetric. Hence we have found a way to find the kinetic matrix. The result 4.2.44 is the result derived in 1981 by Gaillard and Zumino [145].

We now apply the procedure outlined above and obtain the vector-scalar couplings in ungauged $\mathcal{N}=4$ supergravity coupled to n vectors with all $SU(1,1)$-angles equal.

There are $6+n$ vectors and hence the symplectic group of interest is $Sp(6+n,\mathrm{I\!R}) \cong USp(6+n, 6+n)$. The scalar manifold is given by:

$$G/K = \frac{SU(1,1)}{U(1)} \otimes \frac{SO(6,n)}{SO(6)\times SO(n)} \,. \tag{4.2.45}$$

Since the global symmetry group consists of the direct product of the two groups $SU(1,1)$ and $SO(6,n)$, the symplectic embedding must be such that the embedded $SU(1,1)$ commutes with the embedded $SO(6,n)$. The embedding map into $USp(6+n, 6+n)$ is denoted $\mathfrak{I}$.

The precise embedding is contained in the decompositions of the irreducible representations of $Sp(N,\mathrm{I\!R})$ into irreducible representations of the global symmetry group. The decomposition is often dictated by supersymmetry. We take the decomposition as given and proceed from there to give the scalar vector couplings. The decomposition of the vector representation $\mathbf{12+2n}$ of $Sp(6+n,\mathrm{I\!R})$ decomposes into the direct sum of two vector representations of $SO(6,n)$: $\mathbf{12+2n} \to (\mathbf{6+n}) \oplus (\mathbf{6+n})$. The decomposition of the vector representation $\mathbf{12+2n}$ of $Sp(6+n,\mathrm{I\!R})$ decomposes into the irreducible representations of $SU(1,1)$ as $\mathbf{12+2n} \to \bigoplus_{i=1}^{6+n}\mathbf{2}$, where $\mathbf{2}$ denotes the vector representation of $SU(1,1)$.

Let $\mathcal{V}_1$ be an $(6+n)\times(6+n)$ $SO(6,n)$-matrix satisfying:

$$\mathcal{V}_1^T \eta_{6,n} \mathcal{V}_1 = \eta \,, \quad \eta_{6,n} = \begin{pmatrix} -\mathbb{1}_{6\times 6} & 0 \\ 0 & \mathbb{1}_{n\times n} \end{pmatrix} . \tag{4.2.46}$$

Then the symplectic embedding of $\mathcal{V}_1$ into $Sp(6+n,\mathbb{R})$ can be taken to be:

$$\mathcal{V}_1 \hookrightarrow \begin{pmatrix} \mathcal{V}_1 & 0 \\ 0 & \mathcal{V}_1^{-T} \end{pmatrix} \in Sp(6+n,\mathbb{R}). \tag{4.2.47}$$

Hence the embedding $\mathfrak{I}(\mathcal{V}_1)$ of $\mathcal{V}_1$ into $USp(6+n,6+n)$ is

$$\mathfrak{I} : \mathcal{V}_1 \mapsto \tfrac{1}{2} \begin{pmatrix} \mathcal{V}_1 + \eta_{6,n}\mathcal{V}_1\eta_{6,n} & \mathcal{V}_1 - \eta_{6,n}\mathcal{V}_1\eta_{6,n} \\ \mathcal{V}_1 - \eta_{6,n}\mathcal{V}_1\eta_{6,n} & \mathcal{V}_1 + \eta_{6,n}\mathcal{V}_1\eta_{6,n} \end{pmatrix} . \tag{4.2.48}$$

A general $SU(1,1)$-matrix is given by

$$\mathcal{V}_2(\phi^\alpha) = \begin{pmatrix} \phi^1 & \phi^2 \\ \phi_2 & \phi_1 \end{pmatrix} ; \qquad |\phi^1|^2 - |\phi^2|^2 = \phi^\alpha\phi_\alpha = 1 . \tag{4.2.49}$$

The embedding of this matrix into $USp(6+n,6+n)$ is given by

$$\mathfrak{I}(\mathcal{V}_2(\phi^\alpha)) = \begin{pmatrix} \mathrm{Re}\phi^1\mathbb{1} + i\mathrm{Im}\phi^1\eta_{6,n} & \mathrm{Re}\phi^2\mathbb{1} - i\mathrm{Im}\phi^2\eta_{6,n} \\ \mathrm{Re}\phi^2\mathbb{1} + i\mathrm{Im}\phi^2\eta_{6,n} & \mathrm{Re}\phi^1\mathbb{1} - i\mathrm{Im}\phi^1\eta_{6,n} \end{pmatrix} , \tag{4.2.50}$$

and the embeddings commute.

The embedding is completed by putting

$$\mathbb{V}(\phi^\alpha, Z) = \mathfrak{I}(\mathcal{V}_1) \circ \mathfrak{I}(\mathcal{V}_2) = \begin{pmatrix} U_0(\phi^\alpha, Z) & U_1^*(\phi^\alpha, Z) \\ U_1(\phi^\alpha, Z) & U_0^*(\phi^\alpha, Z) \end{pmatrix} \tag{4.2.51}$$

and inserted in the master equation 4.2.44 gives:

$$K(\phi^\alpha, Z) = \frac{i}{|\phi^1 + \phi^2|^2} \left(\mathcal{V}_1\mathcal{V}_1^T\right)^{-1} + \frac{2\mathrm{Im}(\phi^1\phi^{2*})}{|\phi^1 + \phi^2|^2}\eta_{6,n} . \tag{4.2.52}$$

We now choose a particular representative for the $SO(6,n)$-scalars. We put

$$\eta_{6,n}\mathcal{V}_1^{-1} = \begin{pmatrix} X & Y \\ U & V \end{pmatrix} , \quad \mathcal{V}_1^{-T}\eta_{6,n}\mathcal{V}_{6,n}^{-1} = \eta_{6,n} , \tag{4.2.53}$$

from which we find

$$\eta_{6,n}\mathcal{V}_1^{-T}\mathcal{V}_1^{-1}\eta_{6,n} = \eta_{6,n} + 2\begin{pmatrix} X^T \\ Y^T \end{pmatrix} \begin{pmatrix} X & Y \end{pmatrix} . \tag{4.2.54}$$

We call $(X\,,Y)_a{}^R = Z_a{}^R$, with $1 \le a,b,\ldots \le 6$ and $1 \le R,S,\ldots \le 6+n$, and we define $Z^{RS} = \sum_a Z_a{}^R Z_a{}^S$. For the kinetic matrix we find

$$K(\phi^\alpha, Z_a{}^R)_{RS} = \frac{i}{|\phi^1 + \phi^2|^2} (\eta_{RS} + 2Z_{RS}) + \frac{2\mathrm{Im}(\phi^1\phi^{2*})}{|\phi^1 + \phi^2|^2}\eta_{RS} , \tag{4.2.55}$$

where η_{RS} are the components of $\eta_{6,n}$, and $Z_{RS} = Z^{TU}\eta_{RT}\eta_{US}$. If L is an $SO(6,n)$-matrix, then so is L^T and hence we have $Z_a{}^R\eta_{RS}Z_b{}^S = -\delta_{ab}$, and thus $\eta_{RS}+2Z_{RS}$ is an $SO(6,n)$-matrix.

The coupling between the scalars and the vectors is given by:

$$\mathcal{L}_{vec} = -\frac{\eta_{RS}+2Z_{RS}}{4|\phi^1+\phi^2|^2}F^R_{\mu\nu}F^{S\mu\nu} + \frac{\mathrm{Im}(\phi^1\phi^{2*})}{4|\phi^1+\phi^2|^2}\eta_{RS}F^R_{\mu\nu}F^S_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}\,, \tag{4.2.56}$$

which is the same scalar-vector coupling as found with the conformal programme by De Roo for all $SU(1,1)$-angles equal [60].

We now show that the result 4.2.56 is compatible with the Lagrangian of pure supergravity. When $n=0$ the manifold $SO(6,n)/SO(6)\times SO(n)$ becomes trivial; the scalars $Z_a{}^R$ can be put to $Z_a{}^R=\delta_a{}^R$ using the local $SO(6)$. Hence we have $\eta_{RS}+2Z_{RS}=\delta_{RS}$. When we parameterize the $SU(1,1)$-scalars as

$$\phi^1 = \frac{1}{\sqrt{1-|z|^2}}\,,\quad \phi^2 = \frac{z}{\sqrt{1-|z|^2}}\,,\quad z\in\mathbb{C},\ |z|<1\,, \tag{4.2.57}$$

we obtain the vector-scalar couplings of 4.1.12 for all $SU(1,1)$-angles equal.

We thus have obtained by using duality symmetries the vector scalar couplings. When the $SU(1,1)$-angles are nonzero the coupling 4.2.56 becomes a little different, but is a straightforward extension of 4.2.56.

4.3 Matter Coupled Gauged $\mathcal{N}=4$ Supergravity

We now discuss briefly the $\mathcal{N}=4$ matter coupled gauged supergravity, where the number of matter multiplets is n. We restrict ourselves mostly to the bosonic sector. For the full result and a more complete discussion see e.g. [36]. We do not present the Lagrangian since it is not very instructive. We also do not give the most general gaugings; our gaugings are a subsector of the most general gaugings as discussed in [146] (in terms of reference [146] our gaugings are characterized by the vanishing of the parameter $\xi_{\alpha M}$).

The field content of the theory is: one metric, four gravitini of spin 3/2, $6+n$ gauge fields, $4+4n$ fermions of spin 1/2 and $6n+2$ scalars. The scalars are the $SU(1,1)$-scalars and the $SO(6,n)$-scalars. The $SU(1,1)$-scalars can be combined in a complex scalar z as shown in section 4.1. The $SO(6,n)$-scalars are the $Z_a{}^R$, as in section 4.2.2. The kinetic term of the $SU(1,1)$-scalars is as in section 4.1 while the kinetic term of the $SO(6,n)$-scalars is expressed in terms of the symmetric $SO(6,n)$-matrix $M_{RS}=\eta_{RS}+2Z_{RS}$ as

$$\mathcal{L}_{kin}[Z_a{}^R] = \tfrac{1}{16}\mathrm{Tr}\left(\partial_\mu M^{-1}\partial^\mu M\right)\,. \tag{4.3.1}$$

The ungauged theory has a global symmetry group $G = SO(6,n) \times SU(1,1)$ of which the maximal compact subgroup $K = SO(6) \times SO(n) \times U(1)$ is a local symmetry group. The gauge fields are all abelian and the scalars parameterize the coset G/K. The $U(1)$-symmetry is used to write the $SU(1,1)$-scalars as z and the $SO(n)$-subgroup of K acts trivially on the scalars ${Z_a}^R$, while the global $SO(6,n)$ acts as ${Z_a}^R \mapsto O^R{}_S {Z_a}^S$.

The coupling of the scalars to the vector fields in the ungauged theory was discussed section 4.2.2 for equal $SU(1,1)$-angles. As in section 4.1 the extension to different $SU(1,1)$-angles can be done by replacing z with $z\mathrm{e}^{2i\alpha_R}$ for each vector field A^R_μ, or equivalently putting $\phi^1 \mapsto \phi^1 \mathrm{e}^{i\alpha_R}$ and $\phi^2 \mapsto \phi^2 \mathrm{e}^{-i\alpha_R}$ for each vector field A^R_μ.

A subgroup H of G can be promoted to a local symmetry; this destroys the global symmetry G. The number of vector fields is $6+n$ and hence at most a $(6+n)$-dimensional group can be gauged. With every gauge field A^R_μ we associate a generator T_R and we define the gauge group H by its structure constants ${f_{RS}}^T$: $[T_R, T_S] = {f_{RS}}^T T_T$. The structure constants are not arbitrary but have to satisfy

$$ {f_{RS}}^T \eta_{TU} + {f_{RU}}^T \eta_{TS} = 0 \,. \tag{4.3.2} $$

Hence the adjoint representation of the Lie algebra $\mathfrak{h}$ of the group H has to be embedded in the vector representation of $\mathfrak{so}(6,n)$. This severely restricts the possibilities of gaugings and is discussed more in detail when we investigate the scalar potential. The scalar potential factorizes in an $SU(1,1)$-part and an $SO(6,n)$-part. The potential arises when the ungauged supergravity is gauged while keeping the theory supersymmetric.

4.4 The Scalar Potential

The solutions of a theory are the most important features of a theory and within the solutions the vacua play the most dominant role. The vacua are the backgrounds around which the quantum field theory is developed.

We are mainly interested in vacua where the geometry is maximally symmetric and all fields except the metric vanish. In this siutation the number of Killing vectors is maximal. If we denote the Killing vectors by K^I, $I = 1, 2, \ldots$, the metric satisfies $(\mathcal{L}_{K^I} g)_{\mu\nu} = \nabla_\mu K^I_\nu + \nabla_\nu K^I_\mu = 0$. For a given metric the maximal number of Killing vectors is thus 10. It turns out that there are only three distinct maximally symmetric geometries . Either space-time is Minkowski or de Sitter or anti-de Sitter. In a Minkowski space-time there is no curvature and the 10 Killing vectors generate an $SO(3,1) \ltimes \mathbb{R}^4$-group. In a de Sitter space-time the curvature is positive and the Killing vectors form an $\mathfrak{so}(4,1)$ Lie algebra, while in an anti-de Sitter the curvature is negative and the Lie algebra formed by the Killing vectors is $\mathfrak{so}(3,2)$.

The configuration of the fields has to be compatible with the symmetry and therefore the Lie derivative along a Killing vector on a field has to vanish. For gauge fields one of course demands that the Lie derivative along a Killing vector on the field strength vanishes. However, working out the Lie derivatives along the Killing vectors is quite tedious for the de Sitter and anti-de Sitter vacua.

All three isometry groups contain the Lorentz group. Therefore it is a necessary condition that all fields (or field strengths for gauge fields) are invariant under local Lorentz symmetry. For a scalar this implies that it has to be constant since $\partial_a \Phi = E_a^\mu \partial_\mu \Phi$, where Φ is any scalar, is not invariant under local Lorentz transformations unless $\partial_a \Phi = 0$. The requirement that the scalars are constant is precisely what one obtains when demanding that the Lie derivative along the Killing vectors vanishes on the scalars. The fermions are in irreducible representations of the Lorentz group and hence in a Lorentz invariant configuration all fermions vanish. Along similar lines, all two-form field strengths have to vanish. Hence for $\mathcal{N} = 4$ $d = 4$ supergravity a solution is a vacuum solution if the scalars are constants, the metric describes a maximally symmetric space and all other fields (field strengths for gauge fields) vanish.

The scalar potential in the vacuum configuration V_0 is the only contribution of the nonmetric fields to the energy-momentum tensor. The vacuum potential V_0 therefore determines the geometry. If $V_0 > 0$ the geometry is de Sitter, if $V_0 < 0$ the geometry is the anti-de Sitter and if $V_0 = 0$ the geometry is Minkowskian.

Recent observations have shown that the universe is at present in an accelerating phase [147–150]. Therefore our universe has (on large scale) a de Sitter geometry or is evolving into a de Sitter geometry [151]. On the other hand, there are no-go theorems that state that no dimensional reduction of a ten-dimensional supergravity theory admits a four-dimensional de Sitter vacuum [152, 153].

Within the context of string theory some attempts have been made to evade the no-go theorems, for example in [154–156], but in these cases it is a delicate issue to stabilize all the scalars. When not all scalars are stabilized, the vacuum is not stable and the theory will roll into another vacuum, which in most cases is not a de Sitter vacuum (see e.g. [157]).

In four dimensions it is possible to obtain stable de Sitter vacua from $\mathcal{N} = 2$ supergravities [158, 159]. But also in five dimensional $\mathcal{N} = 2$ supergravities stable de Sitter vacua have been constructed [160].

In order to obtain a de Sitter vacuum it seems necessary to gauge a noncompact group. It has been shown that the noncompact gaugings can in some cases be associated with a dimensional reduction of a higher dimensional supergravity over a noncompact manifold [161]. But not for all gauged supergravities a higher dimensional origin is known; for example, for the anomaly-free six-dimensional Salam–Sezgin model [162–164] and for the matter coupled $\mathcal{N} = 4$ supergravity with nontrivial $SU(1,1)$-angles no higher dimensional origin is yet known. Maybe it is not possible to obtain all lower-dimensional supergravities from a ten-dimensional supergravity.

There exist vacua where spacetime geometry is not maximally symmetric and where vector fields have nonvanishing field strengths. Examples are provided by the electrovac and magnetovac solutions [165–167]. However, we focus on maximally symmetric spacetimes.

In the following sections we try to find stable de Sitter vacua in $\mathcal{N}=4$ supergravity. We first give the potential and give various definitions that come in handy at later points in the discussion. In section 4.5 we discuss semisimple gaugings and in section 4.6 we study the potential obtained from gauging a CSO-group, the concept of which is discussed in the same section. The possible extensions of the analysis are discussed in section 4.7.

The Potential of $\mathcal{N}=4$ Supergravity

We first give the potential and then discuss the ingredients. The potential is given by

$$\begin{aligned} V({Z_a}^R,\phi^\alpha) =& \tfrac{1}{4}\left[Z^{RU}Z^{SV}\left(\eta^{TW}+\tfrac{2}{3}Z^{TW}\right)\right]\mathrm{Re}\Big(\Phi^*_{(R)}\Phi_{(U)}\Big)g_{(R)}g_{(U)}f_{RST}f_{UVW} \\ &+\tfrac{1}{36}Z^{RSTUVW}\mathrm{Im}\Big(\Phi^*_{(R)}\Phi_{(U)}\Big)g_{(R)}g_{(U)}f_{RST}f_{UVW}\,, \end{aligned} \tag{4.4.1}$$

where $Z^{RS}={Z_a}^R{Z_a}^S$ and $Z^{RSTUVW}=\epsilon^{abcdef}{Z_a}^R{Z_b}^S{Z_c}^T{Z_d}^U{Z_e}^V{Z_f}^W$ are $SO(6)$-invariant combinations of the $SO(6,n)$-scalars. Therefore the scalar potential is $SO(6)$-invariant. The compact subgroup $SO(6)\times SO(n)$ acts on the $SO(6,n)$-scalars ${Z_a}^R$ only through the first $SO(6)$ subgroups and hence the potential is invariant under the local symmetry group $SO(6)\times SO(6)$.

The structure constants are not arbitrary; $f_{RST}\equiv\eta_{RU}{f^U}_{ST}$ is completely antisymmetric, which is equivalent to the constraint 4.3.2. The coupling constants are contained in the numbers $g_{(R)}$. For every subgroup of the gauge group we allow different coupling constants. With every generator T_R we associate a coupling constant $g_{(R)}$, such that if T_R and T_S belong to the same subgroup, then $g_{(R)}=g_{(S)}$.

The $SU(1,1)$-scalars are contained in $\Phi_{(R)}$ through

$$\Phi_{(R)}=\mathrm{e}^{i\alpha_R}\phi^1+\mathrm{e}^{-i\alpha_R}\phi^2\,. \tag{4.4.2}$$

If all α_R angles are the same we have $\mathrm{Im}\Phi^*_{(R)}\Phi_{(U)}=0$ and the last term in the potential 4.4.1 vanishes. In this situation the symmetry group of the potential is enlarged to $O(6)$.

4.5 The Potential with Semisimple Gaugings

In this section we study the potential with semisimple gaugings. Most of the discussion can be found in references [A,B]. Before we study the potential we analyze which gauge groups are allowed.

4.5.1 Semisimple Gaugings

Let us denote the gauge group with H. If we demand that H is semisimple, H is the direct product of simple groups: $H = H_1 \otimes \ldots \otimes H_n$. We denote the Lie algebras of the simple factors H_i by $\mathfrak{h}_i$; $\mathfrak{h}$ is the direct sum of the $\mathfrak{h}_i$. A basis of $\mathfrak{h}$ is given by $T_1, T_2, \ldots$ such that we have $[T_R, T_S] = f_{RS}{}^T T_T$. The coefficients $f_{RS}{}^T$ are real and are called the structure constants of $\mathfrak{h}$.

With each gauge field A_μ^R we associate a generator T_R in the gauge algebra and an $SU(1,1)$-angle α_R. The gauge group rotates the gauge fields associated to the same factor into each other. All the generators that can be obtained by rotating the generator T_R need to have the same $SU(1,1)$-angle α_R for the gauge group to be a symmetry. Hence along the gauge orbit of T_R, denoted by $\Gamma[T_R]$ and defined by

$$\Gamma\left[T_R\right] = \left\{ \mathrm{e}^{\mathrm{ad}A}(T_R) | A \in \mathfrak{h} \right\} , \tag{4.5.1}$$

the $SU(1,1)$-angle has to be constant. If $\Gamma[T_R] \cap \Gamma[T_S] \neq 0$ we need $\alpha_R = \alpha_S$. For semisimple groups the gauge orbits are the simple factors and hence with each simple factor we associate a single $SU(1,1)$-element.

Let us define a nondegenerate symmetric bilinear form $\Omega : \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$ on $\mathfrak{h}$ by

$$\Omega(T_R, T_S) = \eta_{RS} \, . \tag{4.5.2}$$

The relation 4.3.2 then reads $\Omega([T_R, T_S], T_U) = \Omega(T_R, [T_S, T_U])$ and Ω is thus an invariant bilinear symmetric nondegenerate form on $\mathfrak{h}$. From now on we abbreviate 'symmetric nondegenerate bilinear form' as 'metric'. A group H can be a gauge group if Ω is an invariant metric on the Lie algebra of H.

For simple complex Lie algebras there is up to a multiplicative constant just one invariant metric. In appendix B we prove this statement. Since the Cartan–Killing metric is invariant, all invariant metrics on a simple complex Lie algebra are proportional to the Cartan–Killing metric. For real Lie algebras of which the complex extension is simple, the same result holds. This can be seen by going to the complex extension; if the real Lie algebra would have more than one invariant metric, so would the complex Lie algebra, which is a contradiction.

However, there exist real Lie algebras of which the complex extension is not simple and in this case the complex extension is the direct sum of two identical simple complex Lie algebras [94]. The real Lie algebra is then complex; it is a complex Lie algebra, written as a real direct sum of the real and imaginary part[2]. An example is given by $\mathfrak{so}(1,3)$, for which we have $\mathfrak{so}(1,3)^{\mathbb{C}} = \mathfrak{so}(3)^{\mathbb{C}} \oplus \mathfrak{so}(3)^{\mathbb{C}}$ and $\mathfrak{so}(1,3) \cong \mathfrak{so}(3) \oplus_{\mathbb{R}} i\mathfrak{so}(3)$. One can show that for real Lie algebras of this kind there exist a two-parameter family of invariant metrics. In the discussion that follows we ignore this and take as invariant metric on the Lie algebra the metric that is proportional to

[2] Note that the decomposition into the real and imaginary part is not unique.

Real Forms	$\dim \mathfrak{k}_i$	$\dim \mathfrak{p}_i$	Real Forms	$\dim \mathfrak{k}_i$	$\dim \mathfrak{p}_i$
$\mathfrak{su}(2)$	3	0	$\mathfrak{so}(1,3)$	3	3
$\mathfrak{sl}(2,\mathbb{R})$	1	2	$\mathfrak{so}(5)$	10	0
$\mathfrak{su}(3)$	8	0	$\mathfrak{so}(1,4)$	6	4
$\mathfrak{su}(2,1)$	4	4	$\mathfrak{so}(2,3)$	4	6
$\mathfrak{sl}(3,\mathbb{R})$	3	5			

Table 4.5.1: The real simple Lie algebras of dimension not exceeding twelve. For every real simple Lie algebra the dimension of the compact subalgebra $\dim \mathfrak{k}_i$ and of the noncompact part $\dim \mathfrak{p}_i$ are denoted.

the Cartan–Killing metric. Only $\mathfrak{so}(1,3)$ appears in our discussion as a complex Lie algebra and the extension to a two-parameter family of invariant metrics is postponed and left for later research.

For semisimple Lie algebras consisting of n simple factors that are not complex, the space of invariant metrics is n-dimensional. The Cartan–Killing metric of each simple factor $\mathfrak{h}_i$ has to be proportional to the metric Ω.

To make contact with string theory we will take 6 vector multiplets coupled to $\mathcal{N}=4$ supergravity. This corresponds to ten-dimensional $\mathcal{N}=1$ supergravity where the Yang–Mills fields are truncated away. With the choice of six additional vector multiplets the total number of gauge vectors is twelve and hence the gauge group H can at most be 12-dimensional.

Every simple Lie algebra $\mathfrak{h}_i$ can be decomposed as $\mathfrak{h}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$, where $\mathfrak{k}_i$ is the compact part and where $\mathfrak{p}_i$ is the noncompact part. In table 4.5.1 we have put up to isomorphisms the real simple Lie algebras of dimension less than twelve. The relevant isomorphisms are:

$$\begin{aligned}
&\mathfrak{su}(2) \cong \mathfrak{so}(3) \cong \mathfrak{sp}(1)\,, && \mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} \\
&\mathfrak{su}(1,1) \cong \mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{sp}(1,\mathbb{R}) \cong \mathfrak{so}(1,2)\,, && \mathfrak{so}(2,3) \cong \mathfrak{sp}(2,\mathbb{R}) \\
&\mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})\,, && \mathfrak{so}(1,4) \cong \mathfrak{sp}(1,1)\,.
\end{aligned} \tag{4.5.3}$$

The real Lie algebra $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ is obtained by writing the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ as the direct sum of its real part and its complex part $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} \cong \mathfrak{sl}(2,\mathbb{R}) \oplus_{\mathbb{R}} i\mathfrak{sl}(2,\mathbb{R})$ (also see appendix B).

The Cartan–Killing metric is negative definite on the compact part and positive definite on the noncompact part. Therefore the dimensions of each $\mathfrak{k}_i$ and $\mathfrak{p}_i$ cannot exceed six. From table 4.5.1 we see that the simple algebras that can be used for gauging are: $\mathfrak{su}(2)$, $\mathfrak{so}(2,3)$, $\mathfrak{so}(1,4)$, $\mathfrak{sl}(8,\mathbb{R})$, $\mathfrak{su}(1,1)$, $\mathfrak{su}(1,2)$, $\mathfrak{so}(1,3)$. In this list only $\mathfrak{so}(1,3)$ is a complex Lie algebra. For reasons of simplicity, we ignore the fact that $\mathfrak{so}(1,3)$ admits a two-parameter family of invariant metrics and fix the invariant metric of $\mathfrak{so}(1,3)$ to be the Cartan–Killing metric.

The gauge Lie algebra can be split into a part on which Ω, (defined in equation 4.5.2) is positive definite, denoted $\mathfrak{h}_+$, and a part on which Ω is negative definite, denoted $\mathfrak{h}_-$. For every simple factor there are two distinct ways how to embed them into the gauge group. For the noncompact simple algebras $\mathfrak{h}_i$ either $\mathfrak{k}_i$ lies in $\mathfrak{h}_+$ and $\mathfrak{p}_i$ in $\mathfrak{h}_-$ or $\mathfrak{p}_i$ lies in $\mathfrak{h}_+$ and $\mathfrak{k}_i$ in $\mathfrak{h}_-$. The compact simple Lie algebras $\mathfrak{h}_i$ have to be embedded either completely in $\mathfrak{h}_+$ or completely in $\mathfrak{h}_-$. A semisimple gauging is determined by the choice of simple factors and the choice of embedding in the gauge algebra.

Taking into account the different ways to embed the simple groups into the gauge group, the possible Lie algebras that can be the gauge algebra $\mathfrak{h}$ are:

$$\begin{aligned} &\mathfrak{so}(2,3)\,, \quad \mathfrak{so}(1,4)\,, && 4\times\mathfrak{su}(2)\,, \quad 4\times\mathfrak{sl}(2,\mathbb{R})\,, \\ &\mathfrak{su}(2)\oplus 3\times\mathfrak{sl}(2,\mathbb{R})\,, && 2\times\mathfrak{su}(2)\oplus 2\times\mathfrak{sl}(2,\mathbb{R})\,, \\ &2\times\mathfrak{su}(2)\oplus\mathfrak{so}(1,3)\,, && \mathfrak{su}(2,1)\oplus\mathfrak{sl}(2,\mathbb{R})\,, \\ &\mathfrak{sl}(3,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})\,, && \mathfrak{so}(1,3)\oplus 2\times\mathfrak{sl}(2,\mathbb{R})\,, \end{aligned} \tag{4.5.4}$$

and subalgebras of these algebras. The notation $n\times\mathfrak{h}_i$ means the direct sum of n copies of $\mathfrak{h}_i$.

4.5.2 Extrema in the $SU(1,1)$-Sector

We now wish to find extrema of the potential with respect to the $SU(1,1)$-scalars. The discussion is based on reference [A]. For the $SU(1,1)$-scalars we use the parametrization 4.1.5.

For every factor $\mathfrak{h}_i$ we denote the structure constants $f_{RS}^{(i)\,T}$, the $SU(1,1)$-angle α_i, the coupling constants g_i and in a similar way we write $\Phi_i = \mathrm{e}^{i\alpha_i}\phi^1 + \mathrm{e}^{-i\alpha_i}\phi^2$. The potential 4.4.1 can be written as a sum over the factors;

$$V = \sum_{i,j} V_{ij}R^{(ij)} + W_{ij}I^{(ij)}\,, \tag{4.5.5}$$

where

$$\begin{aligned} R^{(ij)} &= \mathrm{Re} g_i g_j \Phi_i^* \Phi_j = \frac{g_i g_j}{1-r^2}\left[\cos(\alpha_i-\alpha_j)(1+r^2) - 2r\cos(\alpha_i+\alpha_j+\chi)\right]\,, \\ I^{(ij)} &= \mathrm{Im} g_i g_j \Phi_i^* \Phi_j = -g_i g_j \sin(\alpha_i-\alpha_j)\,. \end{aligned} \tag{4.5.6}$$

The V_{ij} and W_{ij} contain the structure constants and the fields ${Z_a}^R$ and are symmetric, respectively antisymmetric in the indices ij and are given by:

$$\begin{aligned} V_{ij} &= \tfrac{1}{4} Z^{RU} Z^{SV}\left(\eta^{TW} + \tfrac{2}{3} Z^{TW}\right) f^{(i)}_{RST} f^{(j)}_{UVW} \\ W_{ij} &= \tfrac{1}{36} Z^{RSTUVW} f^{(i)}_{RST} f^{(j)}_{UVW} \end{aligned} \tag{4.5.7}$$

We define:

$$\begin{aligned} C_{\pm} &= \sum_{i,j} g_i g_j \cos(\alpha_i \pm \alpha_j) V_{ij} \,, \\ S_{+} &= \sum_{i,j} g_i g_j \sin(\alpha_i + \alpha_j) V_{ij} \,, \\ T_{-} &= \sum_{i,j} g_i g_j \sin(\alpha_i - \alpha_j) W_{ij} \,, \\ \Delta &= C_-^2 - C_+^2 - S_+^2 \,,\ \varepsilon = \mathrm{sgn} C_- \,. \end{aligned} \tag{4.5.8}$$

Equipped with these definitions the potential can be written as

$$V = C_- \frac{1+r^2}{1-r^2} - \frac{2r}{1-r^2}\left(C_+ \cos\chi - S_+ \sin\chi\right) - T_- . \tag{4.5.9}$$

The potential only has an extremum in the $SU(1,1)$-sector if $\Delta > 0$ [A]. The scalars r, χ are at the extremum given by:

$$\cos\chi_0 = \frac{\varepsilon C_+}{\sqrt{C_+^2 + S_+^2}} \,, \quad \sin\chi_0 = -\frac{\varepsilon S_+}{\sqrt{C_+^2 + S_+^2}} \,, \quad r_0 = \frac{|C_-| - \Delta}{\sqrt{C_+^2 + S_+^2}} \,, \tag{4.5.10}$$

and the value of the potential is given by

$$V_0 = \varepsilon \Delta - T_- \,. \tag{4.5.11}$$

The potential at the extremum does not depend on the absolute values of the α_i but on the differences $\alpha_i - \alpha_j$. If all α_i are the same, we have $\Delta = 0$ implying $r_0 = 1$, which lies outside the range of r, and the potential has no extremum. This result is a generalization of the work Freedman and Schwarz in the pure $\mathcal{N}=4$ $d=4$ supergravity [140].

If $\varepsilon > 0$ the extremum is a minimum[A]. In the limit $r \to 1$ the potential becomes singular and at the point $r = 0$ the potential and its derivatives are well defined.

To proceed we need to find an extremum of the potential with respect to the $SO(6,6)$-scalars, such that the conditions $\Delta > 0$ and $\varepsilon > 0$ are satisfied. In the following we study for different semisimple gaugings what extremum exists and calculate the value of the potential, ε and Δ and investigate the stability.

4.5.3 Extrema in the $SO(6,6)$-sector

If all α_i are the same, no minimum exists in the $SU(1,1)$-sector and therefore we discard all gaugings where the gauge algebra is simple. The gauge algebras $\mathfrak{so}(3,2)$ and $\mathfrak{so}(1,4)$ are thus left out of consideration.

To make a distinction between the two different embeddings for every simple factor of the gauge algebra, we have the following notation. If the simple factor is embedded with its compact subalgebra $\mathfrak{k}_i$ into $\mathfrak{h}_-$, where Ω is negative definite, we denote the simple factor by $\mathfrak{h}_{i-}$. If the simple factor is embedded with its compact part in $\mathfrak{h}_+$, where Ω is positive definite, we denote it $\mathfrak{h}_{i+}$.

It was shown in reference [A] that the point where ${Z_a}^R$ is given by

$$ {Z_a}^R = {\delta_a}^R \,,\, R \leq 6 \,, \quad {Z_a}^R = 0 \,,\, 6 < R \leq 12 \,, \tag{4.5.12} $$

corresponds in all cases that are studied to an extremum, if an extremum exists. Therefore we restrict the analysis to this point. This point corresponds mathematically to the identity of the manifold $SO(6,6)/SO(6) \times SO(6)$ and the representative of this point can be chosen to be the identity element of $SO(6,6)$. Physically this points corresponds to turning off the matter fields. The point is invariant under $SO(6)$-transformations, but not under $SO(6,6)$-transformations that are not $SO(6)$-transformations since these transformations mix matter fields and supergravity fields. We refer to this point as Z_0.

We split the indices $R, S, \ldots$ up in capital letter $A, B, \ldots$ from the beginning of the alphabet running from 1 to 6, and middle-alphabet capitals $I, J, \ldots$ running from 7 to 12. With this convention the point Z_0 is given by: ${Z_a}^A = {\delta_a}^A$, ${Z_a}^I = 0$. The normalization of the generators is chosen such that in the vector representation $|\mathrm{tr}(T_R T_S)| = 2\delta_{RS}$.

In the point Z_0 we have the following simplifications:

$$ \begin{aligned} V_{ij}(Z_0) &= \delta_{ij} \left(-\tfrac{1}{12} \sum_{ABC} f^{(i)}_{ABC} f^{(i)}_{ABC} + \tfrac{1}{4} \sum_{ABI} f^{(i)}_{ABI} f^{(i)}_{ABI} \right) , \\ C_-(Z_0) &= \sum_i g_i^2 V_{ii}(Z_0) \,, \\ \Delta(Z_0) &= 2 \sum_{i,j} V_{ii}(Z_0) V_{jj}(Z_0) \left(g_i g_j \sin(\alpha_i - \alpha_j) \right)^2 \,. \end{aligned} \tag{4.5.13} $$

The functions V_{ij} are thus nonzero only if $i = j$ at the point Z_0 and the diagonal values can be calculated for each simple factor $\mathfrak{h}_i$. The obtained results are displayed in table 4.5.2 and the details can be found in reference [B].

In order to check that the point Z_0 is a minimum one uses a parametrization of the ${Z_a}^R$ and then differentiate the potential with respect to the parameters and evaluate at Z_0. The result then has to vanish in order for the point Z_0 to be an extremum. To check for stability, one evaluates the second derivatives at Z_0 and the matrix of second derivatives has to be positive definite to be a minimum.

There exist different parameterizations of the ${Z_a}^R$, for example one can take:

Simple Factor	$V_{ii}(Z_0)$	Simple Factor	$V_{ii}(Z_0)$
$\mathfrak{so}(3)_-$	$-\frac{1}{2}$	$\mathfrak{so}(3)_+$	0
$\mathfrak{so}(3,1)_-$	$-\frac{1}{2}$	$\mathfrak{so}(3,1)_+$	$\frac{3}{2}$
$\mathfrak{sl}(3,\mathbb{R})_-$	$-\frac{1}{2}$	$\mathfrak{sl}(3,\mathbb{R})_+$	$\frac{15}{2}$
$\mathfrak{su}(2,1)_-$	-2	$\mathfrak{su}(2,1)_+$	6
$\mathfrak{su}(1,1)_-$	0	$\mathfrak{su}(1,1)_+$	$\frac{1}{2}$

Table 4.5.2: The functions V_{ii} at the point Z_0.

${Z_a}^R = (X,\, Y)_a{}^R$, where X and Y are 6×6-matrices given by:

$$\begin{aligned} X &= \tfrac{1}{2}\left(G + G^{-1} + BG^{-1} - G^{-1}B - BG^{-1}B\right)\,, \\ Y &= \tfrac{1}{2}\left(G - G^{-1} - BG^{-1} - G^{-1}B - BG^{-1}B\right)\,, \end{aligned} \qquad (4.5.14)$$

in which G is a symmetric 6×6-matrix and B is an antisymmetric 6×6-matrix. The X and Y solve the $SO(6,6)$-constraint ${Z_a}^R\eta_{RS}{Z_b}^S = -\delta_{ab}$ since $XX^T - YY^T = \mathbb{1}$. The independent coordinate one can take to be $P \equiv G + B$. Every arbitrary[3] P can be decomposed into a symmetric part corresponding to G and an antisymmetric part corresponding to B.

Working out the derivatives is straightforward but elaborate and not very insightful. Therefore we do not give the full expressions of the first and second derivatives. In reference [B] the expressions can be found and evaluating at Z_0 is easiest done on a computer.

The results are presented in table 4.5.3. A tachyonic mode corresponds to a mode with imaginary mass; for every negative eigenvalue of the matrix of second derivatives there is a tachyonic mode. If tachyonic modes are present the vacuum is instable. In four cases both the value of the potential and the presence of tachyonic modes in the matter scalars ${Z_a}^R$ depends on the choice of the $SU(1,1)$-scalars. In reference [B] it is shown that for these four cases the value of the potential is positive if no matter tachyons are present. However, for these four cases $C_- < 0$ and there are tachyons in the $SU(1,1)$-sector. From table 4.5.3 we conclude that the point Z_0 does not correspond to a stable vacuum, neither de Sitter nor anti-de Sitter nor Minkowski for semisimple gaugings with 6 matter multiplets.

The Vector Fields

We now briefly discuss the masses of the vectors and the signs of the kinetic terms.

[3] The matrix P is almost arbitrary; the determinant of $P + P^T$ cannot be zero.

Gauging	ε_0	signV_0	Matter Tachyons
$SO(2,1)^3_+ \times SO(3)$	$+1$	$+1$	Y
$SO(2,1)^2_+ \times SO(2,1)^2_-$	$+1$	$+1$	Y
$SO(3,1)_+ \times SO(2,1)_+ \times SO(2,1)_-$	$+1$	$+1$	Y
$SO(3,1)^2_+$	$+1$	$+1$	Y
$SO(3)^2_+ \times SO(3)^2_-$	-1	± 1	Y/N
$SO(3)_- \times SO(3)_+ \times SO(3,1)_-$	-1	± 1	Y/N
$SO(3,1)^2_-$	-1	± 1	Y/N
$Sl(3,\mathbb{R})_- \times SO(3)_-$	-1	± 1	Y/N
$SU(2,1)_+ \times SO(2,1)_+$	$+1$	$+1$	Y

Table 4.5.3: Result of the analysis of semisimple gaugings. Only the gaugings with $\Delta_0 > 0$ that give rise to an extremum with respect to the matter scalars at Z_0 are listed. The sign of C_- at Z_0 is denoted ε_0 and the point Z_0 is only a minimum with respect to the $SU(1,1)$-scalars if $\varepsilon_0 = +1$. If the sign of the potential at Z_0 depends on the $SU(1,1)$-angles this is denoted ± 1. The third column is used to indicate whether there are tachyonic modes present in the matter sector; a Y means that tachyons are present and N means no tachyons are present. If the presence of the tachyonic modes in the matter sector depends on the choice of the $SU(1,1)$-angles we denote Y/N.

In reference [B] it was shown that the vectors associated with noncompact groups acquire a mass. This is sensible since the noncompact groups do not leave the points Z_0 invariant. Hence the vector fields associated with noncompact gauge groups acquire a mass through a Higgs mechanism.

The vector kinetic term can be found from equation 4.2.56 and the discussion in section 4.3:

$$\mathcal{L}_{vec-kin} = -\frac{\eta_{RS} + 2Z_{RS}}{4|\Phi_{(R)}|^2} F^R_{\mu\nu} F^{S\mu\nu} \,. \tag{4.5.15}$$

Evaluating the kinetic term at Z_0 gives:

$$\mathcal{L}_{vec-kin} = -\frac{\delta_{RS}}{4|\Phi_{(R)}|^2} F^R_{\mu\nu} F^{S\mu\nu} \,. \tag{4.5.16}$$

Hence there are no wrong-sign kinetic terms and the theory has no ghosts.

4.6 The Potential with CSO-Gaugings

Most gaugings in field theories involve semisimple gauge groups. The reason is that the trace in any representation of the gauge group is an invariant symmetric bilinear form on the Lie algebra and no components of the field strength get projected out in calculating the kinetic term $\sim \mathrm{Tr}F^2$. But a group manifold reduction of Heterotic supergravity results in a gauged supergravity with a gauge group that is not semisimple [112]. Hence for supergravities even the gauge groups that are not semisimple are interesting, see e.g. [168–174].

A class of groups that are not semisimple is the class of CSO-groups. The CSO-groups are contractions of the special orthogonal groups, the SO-groups. In the following we explain what a CSO-group is and give a theorem that is useful when considering gaugings with CSO-groups.

4.6.1 The CSO-type Algebras: an Introduction

We define the group $CSO(p,q,r)$ as the connected real Lie group with real Lie algebra $\mathfrak{cso}(p,q,r)$. We now show in detail how the Lie algebra $\mathfrak{cso}(p,q,r)$ is defined.

Consider the Lie algebra $\mathfrak{so}(p,q+r)$, which has the vector representation as a faithful representation. In the vector representation the Lie algebra $\mathfrak{so}(p,q+r)$ admits a set of basis elements $J_{AB}=-J_{BA}$, $1\leq A,B\leq p+q+r$ satisfying the commutation relation:

$$[J_{AB},J_{CD}]=\eta_{BC}J_{AD}+\eta_{AD}J_{BC}-\eta_{AC}J_{BD}-\eta_{BD}J_{AC}\,, \tag{4.6.1}$$

where η_{AB} are the components of the diagonal metric $\eta_{p,q+r}$, with p eigenvalues $+1$ and $q+r$ eigenvalues -1.

We split the indices $A,B,\ldots$ into indices $I,J,\ldots$ running from 1 to $p+q$ and indices $a,b,\ldots$ running from $p+q+1$ to $p+q+r$. The Lie algebra $\mathfrak{so}(p,q+r)$ splits as a vector space direct sum $\mathfrak{so}(p,q+r)=\mathfrak{so}(p,q)\oplus\mathcal{V}\oplus\mathcal{Z}$, where the elements J_{IJ} span the $\mathfrak{so}(p,q)$ subalgebra, the elements $J_{Ia}=-J_{aI}$ span the subspace $\mathcal{V}$ and the elements J_{ab} span the subalgebra $\mathcal{Z}$. The subspace $\mathcal{V}$ consists of r copies of the vector representation of the subalgebra $\mathfrak{so}(p,q)$, whereas the subalgebra $\mathcal{Z}$ consists of singlet representations of $\mathfrak{so}(p,q)$. The commutation relations are schematically given by:

$$\begin{aligned}
&[\mathfrak{so}(p,q)\,,\mathcal{V}]\subset\mathcal{V}\,, && [\mathcal{V}\,,\mathcal{V}]\subset\mathcal{Z}\oplus\mathfrak{so}(p,q)\,,\\
&[\mathfrak{so}(p,q)\,,\mathcal{Z}]\subset 0\,, && [\mathcal{Z}\,,\mathcal{V}]\subset\mathcal{V}\,,\\
&[\mathfrak{so}(p,q)\,,\mathfrak{so}(p,q)]\subset\mathfrak{so}(p,q)\,, && [\mathcal{Z}\,,\mathcal{Z}]\subset\mathcal{Z}\,.
\end{aligned} \tag{4.6.2}$$

We define for any real number ξ a linear map $T_\xi:\mathfrak{so}(p,q+r)\to\mathfrak{so}(p,q+r)$ by its action on the subspaces:

$$\begin{aligned} x \in \mathfrak{so}(p,q)\,, &\quad T_\xi : x \mapsto x\,, \\ x \in \mathcal{V}\,, &\quad T_\xi : x \mapsto \xi x\,, \\ x \in \mathcal{Z}\,, &\quad T_\xi : x \mapsto \xi^2 x\,. \end{aligned} \tag{4.6.3}$$

If $\xi \neq 0, \infty$ the map T_ξ is a bijection. The maps T_0 and T_∞ give rise to so-called contracted Lie algebras.

We define the limits $T_0(\mathfrak{so}(p,q)) = \mathfrak{s} \cong \mathfrak{so}(p,q)$, $T_0(\mathcal{V}) = \mathfrak{r}$ and $T_0(\mathcal{Z}) = \mathfrak{z}$. The Lie algebra $\mathfrak{cso}(p,q,r)$ is defined as $T_0(\mathfrak{so}(p,q+r))$. Hence we have $\mathfrak{cso}(p,q,r) = \mathfrak{so}(p,q) \oplus \mathfrak{r} \oplus \mathfrak{z}$ and the commutation rules are of the form

$$[\mathfrak{s}\,,\mathfrak{s}] \subset \mathfrak{s}\,, \qquad [\mathfrak{r}\,,\mathfrak{r}] \subset \mathfrak{z}\,, \qquad [\mathfrak{s}\,,\mathfrak{r}] \subset \mathfrak{r}\,, \qquad [\mathfrak{r}\,,\mathfrak{z}] = [\mathfrak{s}\,,\mathfrak{z}] = [\mathfrak{z}\,,\mathfrak{z}] = 0\,. \tag{4.6.4}$$

We mention some special cases and properties. If $r = 0$ the construction is trivial and therefore we take $r > 0$. If $p + q = 1$ we have $\mathfrak{s} = 0$ and if $p + q = r = 1$ also $\mathfrak{z} = 0$ and we have $\mathfrak{cso}(1,0,1) \cong \mathfrak{cso}(0,1,1) \cong \mathfrak{u}(1)$. If $p + q = 2$ the Lie algebra $\mathfrak{s}$ is abelian and if $p+q > 2$ the Lie algebra $\mathfrak{s}$ is semisimple and the vector representation is irreducible. Hence if $p + q > 2$ we have $[\mathfrak{s}, \mathfrak{r}] \cong \mathfrak{r}$. If $r = 1$ we have $\mathfrak{z} = 0$ and the Lie algebra is an Inönü–Wigner contraction. If $r > 1$ the subalgebra $\mathfrak{z}$ is nontrivial and is contained in the center of $\mathfrak{cso}(p,q,r)$. If the center of a Lie algebra is nonzero, the adjoint representation is not faithful.

From the construction follows a convenient set of basis elements of $\mathfrak{cso}(p,q,r)$. The elements $S_{IJ} = -S_{JI}$ are the basis elements of the subalgebra $\mathfrak{s}$, the elements v_{Ia} are the basis elements of $\mathfrak{r}$ and the elements $z_{ab} = -z_{ba}$ are the basis elements of $\mathfrak{z}$. The only nonzero commutation relations are:

$$\begin{aligned} [S_{IJ}, S_{KL}] &= \eta_{JK} S_{IL} - \eta_{IK} S_{JL} - \eta_{JL} S_{IK} + \eta_{IL} S_{JK}\,, \\ [S_{IJ}, v_{Ka}] &= \eta_{JK} v_{Ia} - \delta_{IK} v_{Ja}\,, \\ [v_{Ia}, v_{Jb}] &= \eta_{IJ} Z_{ab}\,. \end{aligned} \tag{4.6.5}$$

The numbers η_{IJ} are the elements of the metric $\eta_{p,q}$. The commutation relations 4.6.5 can also be taken as the definition of the Lie algebra $\mathfrak{cso}(p,q,r)$.

4.6.2 Gaugings with CSO-algebras

Having outlined the construction of CSO-algebras, we now analyze their gaugings. We first work out the constraint 4.3.2 and investigate how the $SU(1,1)$-angles can be chosen.

As explained in section 4.5.1, the constraint 4.3.2 implies the existence of an invariant metric Ω on the gauge algebra. Hence for a CSO-type algebra to be a gauge algebra it has to admit an invariant metric. It turns out that demanding invariance and nondegeneracy of a bilinear form Ω on $\mathfrak{cso}(p,q,r)$ is in all but a few cases impossible. We state the result in a theorem:

Theorem 4.6.1. The Lie algebra $\mathfrak{cso}(p,q,r)$ with $r>0$ admits an invariant nondegenerate symmetric bilinear form (i.e. an invariant metric) only if (1) $p+q+r=2$ or (2) $p+q+r=4$.

The proof of this theorem is postponed to section 4.8. In the proof we have listed the most general invariant metrics up to a multiplicative factor for each of the Lie algebras $\mathfrak{cso}(p,q,r)$ with $p+q+r$. From theorem 4.6.1 follows that only 5 $\mathfrak{cso}$-type algebras are interesting for gaugings - $\mathfrak{cso}(1,0,1)$ is abelian and does not give rise to a potential. Hence the only $\mathfrak{cso}$-type Lie algebras that can be used for gauging a nonabelian algebra are $\mathfrak{cso}(p,q,r)$ with $p+q+r=4$.

We now turn to the $SU(1,1)$-angles. As in section 4.5.1 we investigate the gauge orbits. For the algebras $\mathfrak{cso}(2,0,2)$, $\mathfrak{cso}(1,1,2)$ the gauge orbit of $\mathfrak{s}$, which is one-dimensional, is $\mathfrak{s}\oplus\mathfrak{r}$ and the gauge-orbit of $\mathfrak{r}$ is $\mathfrak{r}\oplus\mathfrak{z}$. For the algebras $\mathfrak{cso}(3,0,1)$ and $\mathfrak{cso}(2,1,1)$ the gauge orbit of every element of $\mathfrak{s}$ is the whole Lie algebra. Finally, for $\mathfrak{cso}(1,0,3)$ the gauge orbit of each element $\mathfrak{r}$ is contained in $\mathfrak{r}\oplus\mathfrak{z}$ and all gauge orbits overlap. Hence for all CSO-type algebras under consideration the $SU(1,1)$-angles have to be constant over the whole Lie algebra.

Analogous to the semisimple gaugings, we need at least two factors to have an extremum in the $SU(1,1)$-sector.

To embed a Lie algebra $\mathfrak{g}$ into the gauge algebra $\mathfrak{h}$, the metric Ω has to be diagonalized and brought into a form with eigenvalues $\pm\lambda$ and the Lie algebra is split into the eigenspaces of Ω: $\mathfrak{g}=\mathfrak{g}_\lambda\oplus\mathfrak{g}_{-\lambda}$. There are two inequivalent embeddings: either $\mathfrak{g}_{\pm\lambda}\subset\mathfrak{h}_\pm$ or $\mathfrak{g}_{\pm\lambda}\subset\mathfrak{h}_\mp$. In the first case we denote the embedding of $\mathfrak{g}$ into the gauge algebra by $\mathfrak{g}_-$ and in the second case we write $\mathfrak{g}_+$. Recall that the subspace $\mathfrak{h}_\pm$ is the subspace of the gauge algebra on which the metric η_{RS} is ±1.

In the basis of the Lie algebra where the invariant metric is diagonal with eigenvalues $\pm\lambda$, the structure constants are calculated, which are then used to evaluate the potential and its derivatives at Z_0. We demonstrate this procedure by an example below.

4.6.3 Analysis of $\mathfrak{cso}(3,0,1)\oplus\mathfrak{cso}(3,0,1)$-Gaugings

The invariant metric and commutation relations of $\mathfrak{cso}(3,0,1)$ is given by equations 4.8.13 and 4.8.12 in section 4.8. We use the same notation as in section 4.8. The metric 4.8.13 has eigenvalues $\lambda_\pm(a)=\frac{1}{2}(a\pm\sqrt{a^2+4})$, each with multiplicity three. The eigenvectors with eigenvalues λ_+ are $\lambda_+t_i+v_i$. The eigenvectors with eigenvalues λ_- are $\lambda_-t_i+v_i$.

We define

$$T_i^+=\lambda_+t_i+v_i\,,\quad T_i^-=\tfrac{\lambda_+}{\lambda_-}(\lambda_+t_i+v_i)\,, \tag{4.6.6}$$

from which we find $\Omega(T_i^\pm, T_j^\pm) = \pm\delta_{ij}(\lambda_+^3 + \lambda_+)$ and $\Omega(T_i^+, T_j^-) = 0$ so Ω is in the required form. If we use the indices $1 \le a, b, \ldots \le 3$ for the generators T_1^+, T_2^+, T_3^+ and the indices $1 \le \alpha, \beta, \ldots \le 3$ for the generators T_1^-, T_2^-, T_3^-, we find after a suitable rescaling the structure constants:

$$
\begin{aligned}
f_{ab}{}^{c} &= (\lambda_+^2 + 2)\epsilon_{abc}\,, & f_{ab}{}^{\alpha} &= \epsilon_{ab\alpha}\,,\\
f_{a\alpha}{}^{\beta} &= \lambda_+^2\epsilon_{a\alpha\beta}\,, & f_{\alpha\beta}{}^{\gamma} &= -(2\lambda_+^2 + 1)\epsilon_{\alpha\beta\gamma}\,,
\end{aligned}
\tag{4.6.7}
$$

where ϵ_{xyz} is the three-dimensional totally antisymmetric alternating symbol.

The embedding of the generators $T_i^\pm$ into the gauge algebra, with generators T_i,can be done as follows. For the gauging $CSO(3,0,1)_- \otimes CSO(3,0,1)_-$ we take $T_i^- = T_i$ and $T_i^+ = T_{i+6}$ with for the first subgroup and $T_i^- = T_{3+i}$ and $T_i^+ = T_{i+9}$ with $i = 1, 2, 3$ for the second subgroup. For the gauging $CSO(3,0,1)_+ \otimes CSO(3,0,1)_+$ we take $T_i^+ = T_i$ and $T_i^- = T_{i+6}$ for the first subgroup and $T_{i+3}^+ = T_i$ and $T_i^- = T_{i+9}$ for the second subgroup. Finally, for the gauging $CSO(3,0,1)_+ \otimes CSO(3,0,1)_-$ we take $T_i^+ = T_i$ and $T_i^- = T_{i+6}$ for the $CSO(3,0,1)_+$ subgroup and $T_{i+3}^+ = T_i$ and $T_i^- = T_{i+9}$ for the $CSO(3,0,1)_-$ subgroup. One checks that with the chosen embeddings the tensors $f_{RS}{}^{T}\eta_{UT}$ are totally antisymmetric in R, S and U. Hence the structure constants satisfy 4.3.2. For example, for the gauging $CSO(3,0,1)_+ \otimes CSO(3,0,1)_+$ one finds the tensors:

$$
\begin{aligned}
f^{(1)}_{ABC} &= -(\lambda_+(a_1)^2 + 2)\delta^{123}_{ABC}\,, & f^{(2)}_{ABC} &= -(\lambda_+(a_2)^2 + 2)\delta^{456}_{ABC}\,,\\
f^{(1)}_{ABI} &= \lambda_+(a_1)^2\delta^{123}_{AB(I-6)}\,, & f^{(2)}_{ABI} &= \lambda_+(a_2)^2\delta^{456}_{AB(I-6)}\,,\\
f^{(1)}_{AIJ} &= \delta^{789}_{(A+6)IJ}\,, & f^{(2)}_{AIJ} &= \delta^{10,11,12}_{(A+6)IJ}\,,\\
f^{(1)}_{IJK} &= -(2\lambda_+(a_1)^2 +)\delta^{789}_{IJK}\,, & f^{(2)}_{IJK} &= -(2\lambda_+(a_2)^2 + 1)\delta^{10,11,12}_{IJK}\,,
\end{aligned}
\tag{4.6.8}
$$

where $1 \le A, B, \ldots \le 6$ are the indices for $\mathfrak{h}_-$ and $7 \le I, J, \ldots \le 12$ are the indices for $\mathfrak{h}_+$ and δ^{ijk}_{abc} is the totally antisymmetric Kronecker delta, see appendix A.

The value of $V(Z_0)$ is the same for both $CSO(3,0,1)_+$ and $CSO(3,0,1)_-$ and given by:

$$
V(Z_0)_{ij} = -\tfrac{1}{2}\delta_{ij}(a_i^2 + 6)\lambda_+(a_i)^2\,. \tag{4.6.9}
$$

From 4.6.9 follows that for the gaugings $CSO(3,0,1) \otimes CSO(3,0,1)$ we have $C_-(Z_0) < 0$ and $\Delta(Z_0) > 0$. Hence an extremum in the $SU(1,1)$-sector exists and is a maximum, that is, unstable.

In reference [B] general formulas are given for calculating the first derivatives of the potential at Z_0 with respect to 36 independent parameters P_{ab} with $1 \le a, b \le 6$, describing the $SO(6,6)/SO(6) \times SO(6)$-coset. These general formulas can be summarized as:

$$\begin{aligned}
\frac{\partial V}{\partial P_{ab}}(Z_0) &= \frac{2}{\sqrt{\Delta(Z_0)}}\sum_{ij}\frac{\partial V_{ii}}{\partial P_{ab}}(Z_0)V(Z_0)_{jj}a_{ij}^2 - \sum_{ij}a_{ij}\frac{\partial W_{ij}}{\partial P_{ab}}(Z_0)\,,\\
\frac{\partial V_{ii}}{\partial P_{ab}}(Z_0) &= \sum_{BJ} f^{(i)}_{(a+6)BJ}f^{(i)}_{bBJ}\,,\\
\frac{\partial W_{ij}}{\partial P_{ab}}(Z_0) &= \frac{1}{12}\epsilon^{bBCDEF}\left[f^{(i)}_{(a+6)BC}f^{(j)}_{DEF} - (i\leftrightarrow j)\right]\,,
\end{aligned} \tag{4.6.10}$$

where $a_{ij} = g_i g_j \sin(\alpha_i - \alpha_j)$. Applying the formulas 4.6.10 we find that not all derivatives of the potential vanish at Z_0 unless $a_{ij} = 0$. Hence the point Z_0 is not an extremum with respect to the $SO(6,6)$-scalars when an extremum exists for the $SU(1,1)$-scalars.

From the above we can draw a conclusions. The free parameters in the invariant metric give rise to inequivalent embeddings and different potentials, potentially with different properties.

In reference [E] the question whether other CSO-gauging gives rise to a stable de Sitter vacuum is treated in more generality.

4.7 Extensions

The search for stable de Sitter vacua as presented in the previous section can be extended in a few directions. We briefly comment on the possible extensions.

The number of matter multiplets can be chosen to be different. From the dimensional reduction of Heterotic supergravity over a torus one obtains a theory with 28 abelian vector fields, since only the abelian subalgebra of the Yang–Mills gauge group survives. Therefore it seems natural to take 22 matter multiplets. This extension allows for gauging other semisimple groups, for example G_2-gaugings.

The analysis of section 4.5 ignored the existence of a two-parameter family of invariant metrics on $\mathfrak{so}(1,3)$. The two parameters can be captured in an overall scaling parameter and an angle θ. The overall scaling is unimportant, while the angle θ gives physical effects. The value of $C_-(Z_0)$ and $\Delta(Z_0)$ can be tuned with the θ's of all $\mathfrak{so}(1,3)$ subalgebras of the gauge algebra but even whether a potential has an extremum in Z_0 depends on θ. It needs to be investigated what all consequences are of the angle θ.

The analysis of section 4.6 is incomplete since not all gaugings using CSO-gaugings are investigated. Performing a group manifold reduction of Heterotic supergravity one obtains a gauged supergravity with a gauge group that is not semisimple [112]; from a reduction over an $SU(2)\times SU(2)$-manifold the gauge group is a CSO-group[C]. This motivates the study of CSO-gaugings, which is performed in [E].

In this line of thinking, one can even try using other nonsemisimple groups than the CSO-groups. The nonsemisimple groups are not classified, which impedes a systematic analysis. However, the nonsemisimple Lie algebras have a simple structure; due to a theorem of Leví (see e.g. [76]) every Lie algebra $\mathfrak{g}$ is the direct vector space sum of a semisimple subalgebra $\mathfrak{l}$ and a solvable ideal $\mathfrak{s}$. It follows that the adjoint action of $\mathfrak{l}$ on $\mathfrak{s}$ is a representation of the semisimple Lie algebra $\mathfrak{l}$. The representations of semisimple Lie algebras are direct sums of irreducible representation, which are classified for the semisimple Lie algebras. This suggests that fixing a number N, one can construct using representation theory of semisimple Lie algebras all nonsemisimple Lie algebras with dimension up to N. The drawback of this programme is that the number of possibilities grows rapidly with N. The nonsemisimple gaugings can provide interesting phenomena, but a complete discussion seems not feasible.

An extension in another direction is given by the recent developments following from the work of de Wit, Samtleben and Trigiante [175–177]. They found a scheme that enables a more systematic treatment of maximal gauged supergravities, on which the work of Schön and Weidner on gauged $\mathcal{N} = 4$ supergravity is based. The idea can be explained as follows. In constructing a gauged supergravity one first finds the ungauged version of the theory, writes down the action and symmetry-variations of the fields and tries to gauge a subgroup of the global-symmetry group, to get local interactions. However, one generality is lost due to the writing down of the action: one needs to choose which gauge fields are electric and which are magnetic. In five dimensions the situation is clear, since any massless two-form can be traded for a massless gauge vector by dualization. Hence by choosing a symplectic gauge one looses generality. This loss can be restored by first introducing the local interaction by gauging and afterwards choosing a symplectic gauge. One therefore introduces for every gauge field a magnetic dual and auxiliary tensor fields and then gauges a subgroup of the global-symmetry group. The equations of motion of the auxiliary tensors and extra gauge fields can be solved for to achieve a gauged supergravity theory.

For $\mathcal{N} = 4$ supergravity this programme has been worked out by Schön and Weidner [146] and the most general gauging is then described by parameters $f_{\alpha KLM}$ and $\xi_{\alpha M}$, where $\alpha = 1, 2$ and $1 \leq K, L, M \leq 6 + n$, which have to satisfy a set of linear and quadratic equations. The gaugings in this thesis correspond to the gaugings with $\xi_{\alpha M} = 0$. The gaugings with $\xi_{\alpha M} \neq 0$ are physically inequivalent to those with $\xi_{\alpha M} = 0$. Unfortunately, both the authors of [146] and this thesis have not yet managed to solve the equations for $\xi_{\alpha M}$ and $f_{\alpha KLM}$ if $\xi_{\alpha M} \neq 0$ in the general case.

4.8 Proof of Theorem 4.6.1

The proof consists of two parts. In the first part we prove for all but the CSO-algebras listed in 4.6.1 that no invariant metric exists. We do this by assuming a bilinear form Ω is invariant and then prove it is degenerate. In the second part we give for CSO-algebras listed in 4.6.1 the invariant metrics.

The first part uses the concepts of isotropic subspaces and Witt-indices. For a bilinear symmetric form B on a real vector space V, an isotropic subspace is a subspace W of V on which B vanishes. A maximal isotropic subspace is an isotropic subspace with a maximal dimension; any other subspace with a larger dimension is not isotropic. One can show that all isotropic subspaces are related by a nonsingular linear transformation; one therefore speaks of *the* maximal isotropic subspace. The dimension of the maximal isotropic subspace is the Witt-index of the pair (B, V) and is denoted m_W.

If B is nondegenerate and the dimension of V is n, one can always choose a basis in which B has the matrix form

$$B = \begin{pmatrix} \mathbb{1}_{p\times p} & 0 & \\ 0 & 0 & \mathbb{1}_{r\times r} \\ 0 & \mathbb{1}_{r\times r} & 0 \end{pmatrix}\,, \quad \text{for } p, r \text{ with } p+2r=n\,. \tag{4.8.1}$$

This clearly shows that the Witt-index is r. Hence we have the inequality: $m_W \leq [n/2]$.

If the center $\mathfrak{z}$ is nonzero we have $[\mathfrak{r},\mathfrak{r}] = \mathfrak{z}$, that is, for every $z \in \mathfrak{z}$ there are $v, w \in \mathfrak{r}$ such that $[v, w] = z$. Hence if z, z', with $z = [v, w]$ and $v, w \in \mathfrak{r}$, we have $\Omega(z, z') = \Omega([v, w], z') = \Omega(v, [w, z']) = 0$ and hence the center $\mathfrak{z}$ is contained in the maximal isotropic subspace. Hence if the dimension of $\mathfrak{z}$ exceeds half the dimension of the Lie algebra, any invariant symmetric bilinear form is necessarily degenerate.

Part I

We split part I in six different cases. For every case we assume an invariant symmetric bilinear form Ω exists and prove degeneracy. We use the same decomposition as in section 4.6.1, $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r} \oplus \mathfrak{z}$, with $\mathfrak{g}$ a CSO-type Lie algebra, and the commutation relations 4.6.5, which we for convenience list again:

$$\begin{aligned} [S_{IJ}, S_{KL}] &= \eta_{JK} S_{IL} - \eta_{IK} S_{JL} - \eta_{JL} S_{IK} + \eta_{IL} S_{JK} \\ [S_{IJ}, v_{Ka}] &= \eta_{JK} v_{Ia} - \delta_{IK} v_{Ja} \\ [v_{Ia}, v_{Jb}] &= \eta_{IJ} Z_{ab}\,. \end{aligned} \tag{4.8.2}$$

<u>$\mathfrak{cso}(p, q, r)$ **with** $p+q > 2$ **and** $r > 1$</u>

We have $[\mathfrak{s},\mathfrak{s}]=\mathfrak{s}$, $[\mathfrak{r},\mathfrak{r}]=\mathfrak{z}$ and $[\mathfrak{s},\mathfrak{r}]=\mathfrak{r}$. We prove that $\mathfrak{z}$ is perpendicular to the whole algebra with respect to Ω, which implies that Ω is degenerate.

The center $\mathfrak{z}$ is perpendicular to itself since it is nonzero and thus defines an isotropic subspace. For every $v\in\mathfrak{r}$ there are $j\in\mathfrak{s}$ and $w\in\mathfrak{r}$ such that $[j,w]=v$. Hence for such v and $z\in\mathfrak{z}$ we have $\Omega(v,z)=\Omega([j,w],z)=\Omega(j,[w,z])=0$ and Ω is zero on $\mathfrak{z}\times\mathfrak{r}$. Since $\mathfrak{s}$ is semisimple a similar argument shows that Ω is zero on $\mathfrak{z}\times\mathfrak{s}$ and then $\mathfrak{z}$ is orthogonal to the whole Lie algebra with respect to Ω.

$\mathfrak{cso}(p,q,r)$ with $p+q=1$ and $r>3$

We have $\mathfrak{s}=0$ and $\dim\mathfrak{r}=r$ and $\dim\mathfrak{s}=r(r-1)/2$. The dimension of the center, which is contained in the maximal isotropic subspace, becomes too large for Ω to be nondegenerate if $r(r-1)/2>r(r+1)/4$. It follows that if $r>3$ there is no invariant metric.

$\mathfrak{cso}(p,q,r)$ with $p+q=1$ and $r=2$

From the commutation relations 4.8.2 we see that we can choose a basis e,f,z such that the only nonzero commutator is $[e,f]=z$. We have $\Omega(z,z)=0$, but also $\Omega(e,z)=\Omega(e,[e,f])=\Omega([e,e],f)=0$. Similarly $\Omega(z,f)=0$ and thus z is perpendicular to the whole algebra and Ω is degenerate.

$\mathfrak{cso}(p,q,r)$ with $p+q=2$ and $r=1$

The Lie algebras $\mathfrak{cso}(1,1,1)$ and $\mathfrak{cso}(2,0,1)$ have zero center and hence $[\mathfrak{r},\mathfrak{r}]=0$. For every $x\in\mathfrak{r}$ there are $y\in\mathfrak{r}$ and $A\in\mathfrak{s}$ such that $x=[A,y]$. Therefore we have for such x,y,A and $v\in\mathfrak{r}:\ \Omega(x,v)=\Omega([A,y],v)=\Omega(A,[y,v])=0$. Thus $\mathfrak{r}$ is an isotropic subspace of dimension 2, whereas the dimension of the Lie algebra is 3.

$\mathfrak{cso}(p,q,r)$ with $p+q=2$ and $r>2$

We choose a basis $\{j,e_a,f_a,z_{ab}\}$, where $j\in\mathfrak{s}$, $e_a,f_a\in\mathfrak{r}$ and $z_{ab}=-z_{ba}\in\mathfrak{z}$ and $1\le a,b\le r$. In terms of the basis elements in 4.8.2 we have $j=J_{12}$, $e_a=v_{1a}$, $f=v_{2a}$. The only nonzero commutation relations are

$$[j,e_a]=f_a\,,\quad [j,f_a]=\sigma e_a\,,\quad [f_a,f_b]=\sigma z_{ab}\quad [e_a,e_b]=z_{ab}\,, \tag{4.8.3}$$

where $\sigma=+1$ for $\mathfrak{cso}(1,1,r)$ and $\sigma=-1$ for $\mathfrak{cso}(2,0,r)$.

From the commutation relations 4.8.3 one deduces that the subspace spanned by the elements e_a and z_{ab} defines an isotropic subspace of dimension $r(r+1)/2$. The dimension of this isotropic subspace exceeds half the dimension of the Lie algebra if $r>2$.

$\mathfrak{cso}(p,q,r)$ with $p+q>3$ and $r=1$

The Lie algebras in this class have zero center and hence $[\mathfrak{r},\mathfrak{r}]=0$. We have $\mathfrak{r}=[\mathfrak{s},\mathfrak{r}]$, $\mathfrak{s}=[\mathfrak{s},\mathfrak{s}]$ and $\mathfrak{s}$ is semisimple. It follows that Ω is zero on $\mathfrak{r}\times\mathfrak{r}$ and Ω coincides with the Cartan–Killing metric of $\mathfrak{s}$ on $\mathfrak{s}\times\mathfrak{s}$. Hence we are interested in Ω on $\mathfrak{r}\times\mathfrak{s}$.

From 4.8.2 we see that we can choose a basis $\{S_{IJ}, v_I\}$, where $1\leq I,J\leq p+q$, and the only nonzero commutation relations are:

$$\begin{aligned}[S_{IJ},S_{KL}] &= \eta_{JK}S_{IL}-\eta_{IK}S_{JL}-\eta_{JL}S_{IK}+\eta_{IL}S_{JK}\\ [S_{IJ},v_K] &= \eta_{JK}v_I-\delta_{IK}v_J\,.\end{aligned} \tag{4.8.4}$$

We define $\Omega_{IJK}=\Omega(v_I,S_{JK})=-\Omega_{IKJ}$. Invariance requires $\Omega([S_{IJ},v_K],S_{LM})=-\Omega(v_K,[S_{IJ},S_{LM}])$, from which we obtain:

$$\eta_{JK}\Omega_{ILM}-\eta_{IK}\Omega_{KLM}=-\eta_{JL}\Omega_{KIM}-\eta_{IM}\Omega_{KJL}+\eta_{IL}\Omega_{KJM}+\eta_{JM}\Omega_{KIL}\,. \tag{4.8.5}$$

Contracting equation 4.8.5 with $\eta^{IK}\eta^{JL}$ we obtain:

$$\eta^{IJ}\Omega_{IJK}=0\,,\forall K\,. \tag{4.8.6}$$

Contracting 4.8.5 with η^{IK} and using 4.8.6 we find:

$$-(p+q)\Omega_{IJK}=\Omega_{IJK}+\Omega_{KIJ}+\Omega_{JIK}\,. \tag{4.8.7}$$

Writing out 4.8.7 three times with the indices cyclically permuted and adding the three expressions we find the result:

$$(p+q-3)\left(\Omega_{IJK}+\Omega_{JKI}+\Omega_{KIJ}\right)=0\,. \tag{4.8.8}$$

Since we assumed $p+q>3$ the cyclic sum of Ω_{IJK} has to vanish.

Using the relation $[S_{IJ},v^J]=v_I$, where no sum is taken over the repeated index J and where $v^J=\eta^{JK}v_K$, and requiring $\Omega([S_{IJ},v^J],S_{KL})=-\Omega(v^J,[S_{IJ},S_{KL}])$ we obtain:

$$\Omega_{IJK}+\Omega_{JIK}+\Omega_{KJI}=0\,. \tag{4.8.9}$$

Combining 4.8.9 and the vanishing of the cyclic sum we see that $\Omega_{IJK}=0$. Hence the subspace $\mathfrak{r}$ is orthogonal to the whole Lie algebra with respect to Ω and Ω is degenerate. This concludes part I.

Part II

We now give for the Lie algebras listed in 4.6.1 the most general invariant metric up to a multiplicative constant.

The Lie algebra $\mathfrak{cso}(1,0,1)$ is abelian and hence any metric is invariant.

For the Lie algebras $\mathfrak{cso}(2,0,2)$ and $\mathfrak{cso}(1,1,2)$ we use the ordered basis $\beta = \{j, e_1, e_2, f_1, f_2, z\}$ with the only nonzero commutation relations

$$[j, e_a] = f_a\,, \quad [j, f_a] = \sigma e_a\,, \quad [f_a, f_b] = \sigma z\,, \quad [e_a, e_b] = z\,, \tag{4.8.10}$$

where $\sigma = +1$ for $\mathfrak{cso}(1,1,2)$ and $\sigma = -1$ for $\mathfrak{cso}(2,0,2)$.

In the basis β the invariant metric can be written in matrix form as:

$$\Omega = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\,, \quad a \in \mathbb{R}\,, \tag{4.8.11}$$

for both $\mathfrak{cso}(1,1,2)$ and $\mathfrak{cso}(2,0,2)$. The eigenvalues are $-1, -1, +1, +1, \frac{1}{2}(a+\sqrt{a^2+4})$, $\frac{1}{2}(a-\sqrt{a^2+4})$ and the signature is $+++---$.

For the Lie algebras $\mathfrak{cso}(2,1,1)$ and $\mathfrak{cso}(3,0,1)$ we use the ordered basis $\beta = \{t_1, t_2, t_3, v_1, v_2, v_3\}$ such that the commutation relations are

$$[t_i, t_j] = \epsilon_{ijk}\eta^{kl} t_l\,, \quad [t_i, v_j] = \epsilon_{ijk}\eta^{kl} v_l, \quad [v_i, v_j] = 0\,, \tag{4.8.12}$$

where ϵ_{ijk} is the three-dimensional alternating symbol and η^{ij} is the inverse of $\eta_{1,2}$ for $\mathfrak{cso}(2,1,1)$ and the inverse of $\eta_{3,0}$ for $\mathfrak{cso}(3,0,1)$. The matrix $\eta_{p,q}$ is diagonal with p eigenvalues $+1$ and q eigenvalues -1.

With respect to the ordered basis β the invariant metric is given by

$$\Omega = \begin{pmatrix} a\eta & \eta \\ \eta & 0 \end{pmatrix}\,, \tag{4.8.13}$$

where each entry is a 3×3-matrix. The eigenvalues are $\lambda_\pm = \frac{1}{2}(a \pm \sqrt{a^2+4})$, both with multiplicity three and the signature is $---+++$.

For the Lie algebra $\mathfrak{cso}(1,0,3)$ we use the ordered basis $\beta = \{v_1, v_2, v_3, z_1, z_2, z_3\}$ such that the commutation relations are

$$[v_i, v_j] = \tfrac{1}{2}\epsilon_{ijk} z_k\,, \quad [v_i, z_j] = [z_i, z_j] = 0\,, \tag{4.8.14}$$

where a summation is understood for every repeated index. The invariant metric is given in matrix form with respect to the basis β by:

$$\Omega = \begin{pmatrix} A_{3\times 3} & \mathbb{1}_{3\times 3} \\ \mathbb{1}_{3\times 3} & 0 \end{pmatrix}\,, \tag{4.8.15}$$

where $A_{3\times 3}$ is an undetermined 3×3-matrix. Since $\det\Omega = -1$ there are no null vectors. We use an LU-decomposition to find

$$\det(\Omega - \lambda \mathbb{1}_{6\times 6}) = -\lambda^3 \det\Big(A - (\lambda - \frac{1}{\lambda})\mathbb{1}\Big)\,. \tag{4.8.16}$$

Hence if μ_1, μ_2, μ_3 are the eigenvalues of A, then $\lambda_i = \frac{1}{2}\Big(\mu_i \pm \sqrt{\mu_i^2 + 4}\Big)$, are the eigenvalues of Ω. Hence the signature is $+++---$.

Chapter 5

Conclusions, Discussion and Developments

In this chapter we summarize and discuss the important conclusions and try to give directions for future developments. In particular we try to answer the questions posed in the introduction. As to be expected from a thesis about a small piece of a big jigsaw puzzle, there is not a single clear conclusion. Therefore we gather the main results that are presented in the thesis.

We have shown how the degrees of freedom that result from a group manifold reduction of a six-form gauge field can be analyzed using the cohomology of Chevalley and Eilenberg. No degrees of freedom are lost since a group manifold has Euler characteristic zero. In the same analysis we showed that all compact connected Lie groups of dimension smaller than seven are products of $SO(3)$, $SU(2)$ and $U(1)$.

We have shown a technique to find the global symmetry group of a dimensionally reduced ten-dimensional supergravity. The technique is an extension of the work of Pope and Lü [114]. Although the result is generally known, the technique is new and explains how the coset structure of the scalars arises when one goes down in dimensions. We have however not investigated what happens when the higher-dimensional theory already has scalars that parameterize a coset.

We have found no stable vacuum for semisimple gaugings of $\mathcal{N} = 4$ supergravity coupled to 6 vector multiplets at the identity point of the scalar manifold $SO(6,6)/SO(6) \times SO(6)$. The identity point of a coset G/K is that point that is identified with the compact subgroup K of G. For many gaugings the identity point corresponds to an extremum of the scalar potential.

We have ignored one aspect in the semisimple gauging of $\mathcal{N} = 4$ supergravity coupled to 6 vector multiplets. There exists a two-parameter family of invariant metrics on $\mathfrak{so}(1,3)$. We leave it for future research to discuss whether this allows stable vacua at the identity point of $SO(6,6)$.

We have shown that the only the CSO-groups that can be a subgroup of the gauge group in $\mathcal{N}=4$ supergravity are $CSO(p,q,r)$ with $p+q+r=4$. We have shown in the same section that there is a straightforward way to investigate the CSO-gaugings by working out how the adjoint representation of a $\mathfrak{cso}$-algebra can be embedded in the vector representation of $\mathfrak{so}(6,6)$. It is worth remarking that the only CSO-groups and SO-groups that can be used for gauging $\mathcal{N}=8$ are the $CSO(p,q,r)$ with $p+q+r=8$ [170–172, 178–184].

The CSO-groups are just one family of nonsemisimple groups. To the knowledge of the author there is no special reason to gauge CSO-groups or semisimple groups in $\mathcal{N}=4$ supergravity. Hence gauging a group in $\mathcal{N}=4$ is always ad hoc; why not another group? Since the number of coupled vector multiplets is arbitrary, the dimension of the gauged group is arbitrary. This makes the search for stable de Sitter vacua in $\mathcal{N}=4$ supergravities by gauging different groups a bit like a lottery.

The above mentioned conclusions are the main conclusions that we wish the reader who wants to remember something, remembers. We now turn to the questions posed in the Introduction. We give the answers, if any are found, in the same order as the questions are listed.

-1- The role of the $SU(1,1)$-angles is still obscure. Their presence breaks a few symmetries. Firstly, in the ungauged supergravity the global symmetry $SO(6,n)$ is broken. Secondly, in the gauged theory the symmetry group of the potential is broken from an $O(6)$ to an $SO(6)$ in presence of the $SU(1,1)$-angles. We have not obtained the $SU(1,1)$-angles by a toroidal or group manifold dimensional reduction. Although the $SU(1,1)$-angles can be seen as a subset of the full set of parameters that determine a gauging of $\mathcal{N}=4$ supergravity [146], this does not fix a higher-dimensional origin.

-2- A Lie group G can be used to gauge $\mathcal{N}=4$ supergravity if and only if its Lie algebra $\mathfrak{g}$ admits an invariant metric with n_+ positive eigenvalues and n_- negative eigenvalues such that $\min(\mathrm{n}_-,\mathrm{n}_+)\leq 6$.

-3- The answer is already known in the literature, see [185], and can be summarized as follows. If the scalars in a supergravity parameterize a coset G/K, the kinetic term is determined by the invariant metric on G/K. The invariant metric is invariant and Riemannian only if K is the maximal compact subgroup of G. Note that we cannot divide out the noncompact subgroup since the noncompact part is not a subgroup. Hence the only coset G/K that does not give rise to ghosts for the scalars is the coset where K is the maximal compact subgroup of G.

-4- If no fluxes are present the isometry group of the internal manifold of a dimensional reduction is a subgroup of the global symmetry group. This can be seen by performing a dimensional reduction of the isometries acting on the fields. The gauge symmetries that are present in the higher-dimensional theory can give rise to an enlarged symmetry group. For example, when a theory with a two-form gauge field present is reduced over $SU(2)$ the symmetry group is enlarged from $SU(2)$ to $CSO(3,0,1)$.

When fluxes are taken into account, the full isometry group is in general no longer a symmetry of the theory. In the general case the symmetry group is smaller. A dimensional reduction with fluxes gives rise to a gauged supergravity, in which the global symmetry group of the ungauged theory is broken to a smaller local symmetry group.

-5- The last question is the hardest to answer. The fate of string/supergravity theories is simply not known. The richness of the theories is so large that many more years are needed to get an overview of the interplay between string theories and supergravity theories and their solutions.

In the year 2007 the LHC will start looking for supersymmetry. If supersymmetry is found, this will be a great triumph for string/supergravity theory and will motivate more intensive research. If supersymmetry is not found, this still does not mean supersymmetry does not exist since the energies above which supersymmetry has to be seen, depend strongly on the models and scenarios that are used. However, if supersymmetry is not found, superstrings and supergravities will become a more outback area of physics, isolated from the other physical disciplines. It does not mean that string/supergravity theories are useless, to the contrary, we have learned much (theoretical) physics and mathematics while investigating them.

We wish to conclude this final chapter by posing some questions for future research:

-a- Can dimensional reductions that are not Kaluza–Klein reductions, such as orbifold reductions, give rise to parameters in a lower-dimensional theory that resemble the $SU(1,1)$-angles?

-b- If one performs a toroidal reduction of a theory that has coset scalars, what coset do the lower-dimensional scalars parameterize?

-c- Can statements be made about global properties of the scalar potential of $\mathcal{N}=4$ supergravity coupled to n vector multiplets, such as the existence of global or local minima?

-d- Can the potential of $\mathcal{N}=4$ supergravity drive the present-day acceleration of the universe?

-e- Can general solutions to the parameters $\xi_{\alpha M}$ and $f_{\alpha KLM}$ of [146] that determine the most general gauging of $\mathcal{N}=4$ supergravity be found?

Some of these question may find an answer easily, others might never be answered. And there are many other questions of course, but as always in science, posing the right questions is harder than giving an answer to a given question.

Appendix A

Conventions

A.1 General Notation

$[p]$	: the greatest integer smaller than p.
$\mu, \nu, \ldots$	: curved space-time indices; indices w.r.t. a coordinate basis.
$\alpha, \beta, \ldots$	: curved indices for internal space, as used in dimensional reductions.
$a, b, \ldots$	: flat space-time indices; indices w.r.t. an orthonormal frame.
$m, n, \ldots$	: flat indices for internal space, as used in dimensional reductions.
$\hat{X}$	: a higher-dimensional X, where X can be anything; fields, coordinates, indices.
$X_{[abcd\ldots]}$	: antisymmetrization with weight one; $X_{[a_1\ldots a_p]} = \frac{1}{p!}(X_{a_1\ldots a_p} \pm \text{other permutations})$.
$\delta^{a_1\ldots a_p}_{b_1\ldots b_p}$	: antisymmetric Kronecker symbol; $\delta^{a_1\ldots a_p}_{b_1\ldots b_p} = \delta^{a_1}_{b_1}\cdots\delta^{a_p}_{b_p} \pm$ other permutations
$\mathcal{L}_X$	: the Lie derivative along the vector field X.

Einstein Convention

The Einstein convention is used throughout the text; any index that appears twice in a expression is summed over if it appears once as upper index and once as a lower index. Occasionally additional summation signs are used. An index that appears twice as upper index or twice as lower index is not summed over unless otherwise stated.

A.2 Differential Geometry Conventions

A manifold is generically denoted by $\mathcal{M}$, its tangent space by $T\mathcal{M}$ and the dual of a vector space V is denoted by V^*. A (p,q)-tensor is a multilinear map from $\underbrace{T\mathcal{M}\times\cdots\times T\mathcal{M}}_{q\text{ times}}$ to $\underbrace{T\mathcal{M}\times\cdots\times T\mathcal{M}}_{p\text{ times}}$. Multilinearity means that if $f_1,\ldots,f_q$ are functions and $X_1, \ldots, X_q$ are vector fields and T is a (p,q)-tensor, then

$$T(f_1 X_1, \ldots, f_q X_q) = f_1 \ldots f_q T(X_1, \ldots, X_q) \,. \tag{A.2.1}$$

Therefore the covariant derivative ∇ is not a tensor, but $\nabla \wedge \nabla$ is.

A.2.1 Forms

For clearness we have tried to write down all $\wedge$-symbols where this was possible without making an unreadable format. The exterior derivative is written as d. If ω is a p-form we sometimes make this explicit by writing $\omega^{(p)}$. We use the following conventions for forms:

$$\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu \,, \tag{A.2.2a}$$

$$\omega^{(p)} = \tfrac{1}{p!}\omega_{\mu_1 \ldots \mu_p} \mathrm{d}x^{\mu_1} \wedge \ldots \mathrm{d}x^{\mu_p} \,, \tag{A.2.2b}$$

$$\mathrm{d}\omega^{(p)} = \tfrac{1}{p!}\partial_\nu \omega_{\mu_1 \ldots \mu_p} \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \mathrm{d}x^{\mu_p} \,, \tag{A.2.2c}$$

$$\omega^{(p)} \wedge \chi^{(q)} = \upsilon^{(p+q)} \Rightarrow \upsilon_{\mu_1 \ldots \mu_{p+q}} = \tfrac{(p+q)!}{p!q!}\omega_{[\mu_1 \ldots \mu_p}\chi_{\mu_{p+1} \ldots \mu_{p+q}]} \,. \tag{A.2.2d}$$

We define a vector-valued p-form on $\mathcal{M}$ as an element of the tensor product of the space of p-forms and a vector space V. Thus any vector-valued form Ω can be written as $\Omega = v_a \otimes \omega^a$ where ω^a are forms on $\mathcal{M}$ and v^a are vectors in a vector space V.

A.2.2 Metric, Levi–Civitá and Hodge dual

The Minkowski metric η_{ab} has signature $(- + + + \ldots)$. The Levi–Civitá object is written $\tilde{\epsilon}_{\mu_1 \ldots \mu_p}$ and takes values $0, \pm 1$. It is totally antisymmetric and we have $\epsilon_{0123\ldots} = +1$. The Levi–Civitá tensor is written $\epsilon_{\mu_1 \ldots \mu_p}$ and is defined using the vielbeins e^a_μ;

$$\epsilon_{\mu_1 \ldots \mu_p} = e^{a_1}_{\mu_1} \cdots e^{a_p}_{\mu_p} \tilde{\epsilon}_{a_1 \ldots a_p} = \det(e^a_\mu)\tilde{\epsilon}_{\mu_1 \ldots \mu_p} \,, \tag{A.2.3}$$

and hence takes values $0, \pm\det(e^a_\mu)$. The Levi–Civitá tensor transforms as a tensor under general coordinate transformations and indices can be raised and lowered using the metric $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$. When there is no relation with a space-time (for example with groups) and no confusion is possible, the notation $\epsilon_{abcd\ldots}$ is also used for the Levi–Civitá symbol, which also goes under the name of completely antisymmetric alternating symbol.

The Hodge dual of a p-form $\omega^{(p)}$ on an n-dimensional space-time is an $(n-p)$-form $\upsilon^{(n-p)}$ defined by

$$\upsilon_{\mu_1 \ldots \mu_{n-p}} = \tfrac{1}{p!}\epsilon_{\mu_1 \ldots \mu_{n-p}}{}^{\nu_1 \ldots \nu_p}\omega_{\nu_1 \ldots \nu_p} \,, \tag{A.2.4}$$

and denoted by $\star\omega^{(p)}$.

A.2.3 Convenient Formulae

$$\delta R_{\mu\nu}{}^{ab}(\omega) = \mathrm{D}_\mu \delta\omega_\nu{}^{ab} - \mathrm{D}_\nu \delta\omega_\mu{}^{ab}\,, \tag{A.2.5a}$$

$$\mathrm{D}_\mu\left(eE^\mu_a\right) = -eE^\mu_d E^\nu_a \tau_{\mu\nu}{}^d\,, \tag{A.2.5b}$$

$$\mathrm{D}_\mu\left(eE^\mu_{[a}E^\nu_{b]}\right) = -e\left(E^\mu_d E^\lambda_{[a}E^\nu_{b]} + \tfrac{1}{2}E^\nu_d E^\mu_{[a}E^\lambda_{b]}\right)\tau_{\mu\lambda}{}^d\,, \tag{A.2.5c}$$

$$\begin{aligned}\mathrm{D}_\mu\left(eE^\mu_{[a}E^\nu_b E^\lambda_{c]}\right) = -e\Big(&E^\mu_d E^\sigma_{[a}E^\nu_b E^\lambda_{c]} + \tfrac{1}{2}E^\nu_d E^\mu_{[a}E^\sigma_b E^\lambda_{c]} \\ &+ \tfrac{1}{2}E^\lambda_d E^\mu_{[a}E^\nu_b E^\sigma_{c]}\Big)\tau_{\mu\sigma}{}^d\,,\end{aligned} \tag{A.2.5d}$$

$$\star\omega^{(p)} \wedge \zeta^{(p)} = \tfrac{1}{p!}\sqrt{-g}\,\omega_{\mu_1\ldots\mu_p}\zeta^{\mu_1\ldots\mu_p}\,, \tag{A.2.5e}$$

$$\tfrac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\;V^\mu\right) = \nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma_{\mu\nu}^{\;\mu}V^\nu\,, \tag{A.2.5f}$$

$$\epsilon_{\mu_1\ldots\mu_p\lambda_1\ldots\lambda_{d-p}}\epsilon^{\mu_1\ldots\mu_p\nu_1\ldots\nu_{d-p}} = -\,p!\,\delta^{\nu_1\ldots\nu_{d-p}}_{\lambda_1\ldots\lambda_{d-p}}\,, \tag{A.2.5g}$$

$$\mathcal{L}_X(Y^\mu\partial_\mu) = [X,Y] = (X^\nu\partial_\nu Y^\mu - Y^\nu\partial_\nu X^\mu)\partial_\mu\,, \tag{A.2.5h}$$

$$\mathcal{L}_X(\omega_\mu \mathrm{d}x^\mu) = (X^\nu\partial_\nu\omega_\mu X^\nu\partial_\mu\omega_\nu)\mathrm{d}x^\mu\,, \tag{A.2.5i}$$

$$\left(\mathcal{L}_X g\right)_{\mu\nu} = X^\rho\partial_\rho g_{\mu\nu} + X^\rho\partial_\mu g_{\rho\nu} + X^\rho\partial_\nu g_{\mu\rho}\,. \tag{A.2.5j}$$

Weyl Rescalings

Weyl rescalings are rescalings of the metric. Under a rescaling of the metric $g_{\mu\nu} \mapsto g'_{\mu\nu} = \mathrm{e}^{2\alpha\phi}g_{\mu\nu}$, where α is a number and ϕ is a space-time function, the Ricci scalar transforms as $R[g] \mapsto R'[g']$, where in d dimensions we have

$$R'[g'] = \mathrm{e}^{-2\alpha\phi}\left(R[g] - \alpha^2(d-1)(d-2)\partial_\mu\phi\partial^\mu\phi - 2\alpha(d-1)\Box\phi\right)\,. \tag{A.2.6}$$

Note that $\sqrt{-g'} = \mathrm{e}^{\alpha d\phi}$.

A.3 Group Theoretical Conventions

In our conventions an inner product is always positive definite and nondegenerate, whereas a metric is symmetric and nondegenerate and need not be positive definite.

If $A: V \to V$ is a linear transformation in a vector space V and $\langle,\rangle$ denotes an inner product, then the Hermitian conjugate of A, denoted $A^\dagger$ is defined as $\langle u, Av\rangle = \langle A^\dagger u, v\rangle$ for all $u, v \in V$. If we equip V with an orthonormal basis $\{e_i\}_{i=1}^n$ with respect to which A has the components A_{ij}, we have $(A^\dagger)_{ij} = A^*_{ji}$. With respect to the orthonormal basis we define the transpose of A, being the linear transformation A^T defined by $(A^T)_{ij} = A_{ji}$. It is important to note that the Hermitian conjugate is defined with respect to an inner product whereas the transpose is defined with respect to a basis.

Groups are denoted by Roman letters starting with a capital. The associated Lie algebra uses the same letters but small and in Gothic style. We mention a few important groups and their Lie algebras:

(1) $GL(n;\mathbb{R})$: The Lie group of all invertible real $n \times n$-matrices. The Lie algebra is denoted for $\mathfrak{gl}(n;\mathbb{R})$ and contains all $n \times n$ matrices. Similar for $GL(n;\mathbb{C})$ and $\mathfrak{gl}(n;\mathbb{C})$.

(2) $SL(n;\mathbb{R})$: The subgroup of $GL(n;\mathbb{R})$ in which all $n \times n$-matrices have unit determinant. The Lie algebra is denoted $\mathfrak{sl}(n;\mathbb{R})$ and consists of all real traceless $n \times n$-matrices. Similar for $SL(n;\mathbb{C})$ and $\mathfrak{sl}(n,\mathbb{C})$.

(3) $SO(p,q)$: The subgroup of $GL(p+q;\mathbb{R})$ where the matrices O satisfy $O^T\eta_{p,q}O = \eta_{p,q}$ and $\det O = 1$, where $\eta_{p,q}$ is defined by:

$$\begin{pmatrix} \mathbb{1}_{p\times p} & 0 \\ 0 & -\mathbb{1}_{q\times q} \end{pmatrix}. \tag{A.3.1}$$

The Lie algebra is denoted by $\mathfrak{so}(p,q)$ and consist of all real $(p+q)\times(p+q)$-matrices X satisfying $X^T\eta_{p,q} + \eta_{p,q}X = 0$.

(4) $U(n)$: The subgroup of $GL(n;\mathbb{C})$ of which the elements preserve the standard inner product on $\mathbb{C}^n$: $U \in U(n) \Leftrightarrow \langle Uv, Uw\rangle = \langle v, w\rangle$, $\forall v, w \in \mathbb{C}^n$. This implies $U^\dagger U = UU^\dagger = \mathbb{1}$. The Lie algebra of $U(n)$ is denoted $\mathfrak{u}(n)$ and consists of the anti-Hermitian complex $n \times n$ matrices.

(5) The group $SU(n)$ is a subgroup of $U(n)$ where the matrices have unit determinant. The Lie algebra is denoted $\mathfrak{su}(n)$ and consists of the traceless anti-Hermitian complex $n \times n$ matrices.

(6) The group $Sp(n,\mathbb{R})$ ($Sp(n,\mathbb{C})$) is the subgroup of $GL(2n,\mathbb{R})$ ($GL(2n,\mathbb{C})$ respectively) leaving the matrix J_n invariant, where:

$$J_n \equiv \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0 \end{pmatrix}. \tag{A.3.2}$$

The group $Sp(p,q)$ is the defined as

$$\begin{aligned} &Sp(p,q) = \left\{X \in Sp(p+q,\mathbb{C}) | X^\dagger Q_{p,q} X = Q_{p,q}\right\} , \\ &\text{where} \;\; \mathrm{Q_{p,q}} = \begin{pmatrix} \eta_{p,q} & 0 \\ 0 & \eta_{p,q} \end{pmatrix} , \end{aligned} \tag{A.3.3}$$

and can thus be seen as the cross section of $Sp(p+q,\mathbb{C})$ with $U(2p,2q)$.

A.3.1 Pauli Matrices

The three standard Pauli matrices are denoted σ_i, $i = 1, 2, 3$ and are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.3.4}$$

They satisfy the fundamental relations

$$\begin{array}{ll} \sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k\,, & \{\sigma_i, \sigma_j\} = 2\delta_{ij}\,, \\ [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k\,, & \sigma_i^\dagger = \sigma_i\,, \\ \mathrm{Tr}\sigma_i = 0\,, & \mathrm{Tr}\sigma_i\sigma_j = 2\delta_{ij}\,. \end{array} \tag{A.3.5}$$

A.3.2 Representations of $\mathfrak{su}(4)$ and $SU(4)$

The Lie algebra $\mathfrak{su}(4)$ is 15-dimensional, simple and corresponds in the Dynkin classification to A3. It is isomorphic to $\mathfrak{so}(6)$, which corresponds to D3 in the Dynkin classification.

The vector representation is carried by $\mathbb{C}^4$, the space of all four-dimensional complex vectors. We equip $\mathbb{C}^4$ with the standard inner product $\langle v, w\rangle = \sum_i (v^i)^* w^i$. The matrices $A = (A^i{}_j)$ representing elements of $\mathfrak{su}(4)$ satisfy

$$\langle Av, w\rangle + \langle v, Aw\rangle = 0 \Leftrightarrow A^\dagger + A = 0\,, \quad \mathrm{Tr}A = 0\,. \tag{A.3.6}$$

The vector representation is the defining representation of $\mathfrak{su}(4)$.

We use the following notation for complex vectors; the vector v_i with index down is obtained from v^i by complex conjugation $v_i = (v^i)^*$. Note that from the point of view of the vector representation, the vectors v_i are co-vectors and are in the dual space $(\mathbb{C}^4)^*$. For general $\mathfrak{su}(4)$-tensors we state that all upper indices go down and all lower indices go up with complex conjugation; $(A^i{}_j)^* = A_i{}^j$. The inner product $\langle , \rangle$ can then be rewritten as $\langle v, w\rangle = v_i w^i$ and the requirement that a matrix A is anti-Hermitian can be written as $A^i{}_j = -A_j{}^i$.

The vector representation, denoted **4**, can be conjugated, i.e. one can view the complex vectors v_i as carriers of the representation. The conjugated representation is denoted $\bar{\mathbf{4}}$ and a matrix $A = (A^i{}_j)$ in the **4** representation acts in the conjugated representation as $\bar{A} : v_i \mapsto A_i{}^j v_j$. The representations **4** and $\bar{\mathbf{4}}$ are not equivalent, that is, there is no matrix U such that $\bar{A} = UAU^{-1}$.

By taking tensor products of the **4** and $\bar{\mathbf{4}}$ representation and decomposing these products into irreducible representations, one obtains all irreducible representations of $\mathfrak{su}(4)$. A way to do this is to use Young tableaus (see e.g. [186] for an introduction to Young tableaus).

The vector representation **4** is in terms of Young tableaus represented by a single box, whereas the $\bar{\mathbf{4}}$ representation is represented by a column of 3 boxes. The tensor product of **4** and **4** can be shown to give: $\mathbf{4}\otimes\mathbf{4} = \mathbf{6}\oplus\mathbf{10}$. The product of the vector and co-vector representation should contain the adjoint, since the matrices $A = (A^i{}_j)$ are contained in this product, and indeed $\mathbf{4}\otimes\bar{\mathbf{4}} = \mathbf{1}\oplus\mathbf{15}$, and the **15** representation is the adjoint representation. The singlet representation **1** in the decomposition corresponds to the trace.

The **6** is a real representation, which means that it is equivalent to the conjugated representation, $\bar{\mathbf{6}} = \mathbf{6}$. The vectors in this representation are antisymmetric rank two tensors $T^{ij} = -T^{ji}$. If $A^i{}_j$ denotes an anti-Hermitian traceless 4×4-matrix, then T^{ij} transforms as

$$T^{ij} \mapsto A^i{}_k T^{kj} + A^j{}_k T^{ik} = (A^i{}_k \delta^j{}_l + \delta^i{}_k A^j{}_l) T^{kl} \,. \tag{A.3.7}$$

The representations of the group $SU(4)$ can be obtained by exponentiating the representations of the algebra $\mathfrak{su}(4)$. The notation for the representations is the same, which causes no misunderstandings since every representation of the Lie algebra can be exponentiated to a representation of the Lie group and by going to the derived representation one goes from any group representation to a Lie algebra representation.

The $\mathbf{6}^-$ representation of $SU(4)$

The fields ϕ^{ij} as introduced in section 2.4.2 are in the $\mathbf{6}^-$ representation of $SU(4)$. We now explain what this means.

The matrices in the vector representation of $SU(4)$ satisfy $UU^\dagger = U^\dagger U = \mathbb{1}$ and have unit determinant. With our conventions on complex four vectors this can be translated to

$$U^i{}_j U_k{}^j = \delta^i_k \,, \quad \epsilon_{ijkl} U^i{}_a U^j{}_b U^k{}_c U^l{}_d = \epsilon_{abcd} \,, \tag{A.3.8}$$

where ϵ_{abcd} is the completely antisymmetric alternating symbol; $\epsilon_{abcd} = +(-)1$ if $abcd$ is an even (respectively odd) permutation of 1234 and otherwise it is zero.

The transformation of an antisymmetric two tensor T^{ij} in the **6** representation of $SU(4)$ reads:

$$T^{ij} \mapsto U^i{}_k U^j{}_l T^{kl} \,, \tag{A.3.9}$$

which is the exponentiated version of A.3.7 and in which U is a 4×4 $SU(4)$ matrix.

The clue is that the representation **6** is a representation over the complex numbers; one is allowed to build complex linear combinations of antisymmetric two-tensors. But since the **6** representation is real, one can restrict to a real subspace by imposing a suitable reality constraint, which is preserved under the $SU(4)$ transformations. The suitable constraint turns out to be self-duality or equivalently, anti-self-duality.

An antisymmetric two-tensor is called self-dual if

$$\tfrac{1}{2}\epsilon_{ijkl} T^{kl} = T_{ij} = (T^{ij})^* \,, \tag{A.3.10}$$

and anti-self-dual if

$$\tfrac{1}{2}\epsilon_{ijkl}T^{kl} = -T_{ij} = -(T^{ij})^* \, . \tag{A.3.11}$$

The unimodularity (having unit determinant) of the 4×4 $SU(4)$-matrices ensure that (anti-) self-duality is maintained under $SU(4)$ transformations.

Since we have

$$T^{ij} = \tfrac{1}{2}\left(T^{ij} + \tfrac{1}{2}\epsilon^{ijkl}Tkl\right) + \tfrac{1}{2}\left(T^{ij} - \tfrac{1}{2}\epsilon^{ijkl}Tkl\right) \, , \tag{A.3.12}$$

every antisymmetric two tensor can be written uniquely as the sum of an anti-self-dual and a self-dual part.

The (anti-) self-dual two-tensors form a real vector space; taking real linear combinations preserves the constraints A.3.11 and A.3.10. But if T^{ij} is self-dual, then iT^{ij} is anti-self-dual, hence taking complex combinations destroys the duality properties.

With $\mathbf{6}^+$ ($\mathbf{6}^-$) we denote the real vector space of all antisymmetric self-dual (anti-self-dual) two-tensors contained in the $\mathbf{6}$ representation of $SU(4)$. We can write:

$$\mathbf{6} = \mathbf{6}^+ \oplus_{\mathbb{R}} \mathbf{6}^- \, , \tag{A.3.13}$$

where the subscript $\mathbb{R}$ reminds us of the fact that we are dealing with two real vector spaces; we have written a complex vector space as a direct sum of two real vector spaces.

Looking at A.3.13 one might feel suspicious since $6 + 6 = 12 \neq 6$, however this is wrong counting. The representation $\mathbf{6}$ comprises a complex six-dimensional vector space, which has real dimension 12, and both the representations $\mathbf{6}^\pm$ comprise a six-dimensional real vector space.

For the representation $\mathbf{6}^-$ one can find a convenient basis in terms of 't Hooft symbols $(G_a)_{ij} = (G_a)^{ij*}$. Since they are rank two tensors we can represent them by matrices (but they are not!) and a convenient basis is:

$$\begin{aligned}
(G_1)_{ij} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_{ij} , \quad (G_2)_{ij} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}_{ij} , \\
(G_3)_{ij} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}_{ij} , \quad (G_4)_{ij} = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}_{ij} , \\
(G_5)_{ij} &= \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}_{ij} , \quad (G_6)_{ij} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}_{ij} .
\end{aligned} \tag{A.3.14}$$

The 't Hooft symbols satisfy

$$(G_a)_{ij}(G_b)^{ij} = 4\delta_{ab}\,, \tag{A.3.15a}$$

$$(G_a)_{ik}(G_b)^{kj} + (G_b)_{ik}(G_a)^{kj} = -2\delta_i^j\delta_{ab}\,, \tag{A.3.15b}$$

$$(G_a)_{im_1}(G_b)^{m_1m_2}(G_c)_{m_2m_3}(G_d)^{m_3m_4}(G_e)_{m_4m_5}(G_f)^{m_5j} = i\delta_i^j\epsilon_{abcdef}\,. \tag{A.3.15c}$$

Since the 't Hooft symbols form a complete basis, any element ϕ_{ij} of $\mathbf{6}^-$ can uniquely be written as $\phi_{ij} = \sum_a Z_a(G_a)_{ij}$, with $\phi_{ij}(G_a)^{ij} = \phi^{ij}(G_a)_{ij} = 4Z_a \in \mathbb{R}$. Hence we have the relations:

$$\begin{aligned} \tfrac{1}{2}\sum_{a=1}^{6}(G_a)_{ij}(G_a)^{kl} &= \delta_i^k\delta_j^l - \delta_j^k\delta_i^l \\ \tfrac{1}{2}\sum_{a=1}^{6}(G_a)_{ij}(G_a)_{kl} &= -\epsilon_{ijkl}. \end{aligned} \tag{A.3.16}$$

If we write $\phi_{ij} = \sum_a Z_a(G_a)_{ij}$ then under a $SU(4)$ transformation, the Z_a form the components of a vector in the vector representation of $SO(6) \cong SU(4)/\mathbb{Z}_2$.

Taking a look at equation A.3.15b one can already guess that there must be a relation between the 't Hooft symbols and the spinor representation of $SO(6)$. Introducing the matrices

$$\Gamma_a = \begin{pmatrix} 0 & i(G_a)_{ij} \\ i(G_a)^{ij} & 0 \end{pmatrix}\,, \tag{A.3.17}$$

which are seen to satisfy $\Gamma_a\Gamma_b + \Gamma_b\Gamma_a = 2\delta_{ab}\mathbb{1}_{8\times 8}$, one concludes that the 't Hooft symbols provide us with an explicit representation of the spinor representation of $SO(6)$. In appendix C we mention more on the spinor representation of $SO(6)$.

Appendix B

Some Lie Group and Lie Algebra Theory

In this appendix we review some group theoretical aspects that are used throughout the thesis. Since we cannot cover the whole subject of Lie groups and Lie algebras in every detail we refer the reader to the existing vast amount of literature on finite-group theory, Lie groups and Lie algebras. Though finite groups are little used in this thesis, they are conceptually easier and more important, they share many properties with Lie groups. For introductions in finite-group theory we refer to [186, 187]. For introductions into Lie groups and Lie algebras we refer to [73, 188–190] and to the appendices of many books on particle physics. Books on Lie groups and Lie algebras that go a little beyond an introduction are for example [76, 93, 94, 113, 191, 192].

B.1 Basic Definitions and Properties

A **Lie group** G is a smooth manifold that has a group structure; there is a product $\cdot :$ $G \times G \to G$, satisfying: (1) there is an identity element $e \in G$ such that $a \cdot e = e \cdot a = a$ $\forall a \in G$, (2) for every $a \in G$ there is an element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$, (3) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, $\forall a, b, c \in G$, (4) the product and the map $a \mapsto a^{-1}$ are smooth. It is an easy exercise to deduce that the identity element is unique. The product symbol $\cdot$ is often omitted.

A (Lie) **subgroup** H of G is a submanifold of G that is a group. Thus for all $a, b \in H \subset G$ we have $ab \in H$ and $a^{-1} \in H$. In particular, every subgroup contains the identity element.

A Lie group is **abelian** if $ab = ba$, $\forall a, b \in G$. Every abelian Lie group of dimension n is diffeomorphic to $\mathbb{R}^p \times T^{n-p}$.

A Lie group is **compact** if it is compact as a manifold. Compact groups are special since one can integrate over them.

The tangent space of a group G at the identity element e can be identified with the Lie algebra $\mathfrak{g}$ of G but the Lie algebra also admits a more algebraic definition. We use the algebraic approach to define Lie algebras. For the relation of a group to its Lie algebra we refer to the literature mentioned in the beginning of this appendix.

Although a Lie algebra need not be finite-dimensional, we restrict to the finite-dimensional Lie algebras. For an introduction into infinite-dimensional Lie algebras see e.g. [193]. A finite-dimensional **Lie algebra** $\mathfrak{g}$ is a finite-dimensional vector space over a field $I\!\!K$ (we only consider $I\!\!K = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$) with a bilinear product $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying: (1) $[a,a] = 0$, $\forall a \in \mathfrak{g}$, (2) $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$. From (1) follows $[a,b] = -[b,a]$. Note that the product $[,]$ is non-associative due to the identity (2), which is called the **Jacobi identity**. If for the field $I\!\!K$ we take $I\!\!K = \mathbb{R}$ ($I\!\!K = \mathbb{C}$) the Lie algebra is called a real (complex) Lie algebra. An example of a real Lie algebra is $\mathfrak{gl}(N,\mathbb{R})$, the set of all $N \times N$ matrices with real entries equipped with the product $[a,b] = ab - ba$, where ab denotes the ordinary matrix product of a and b. The reader not familiar with Lie algebras is encouraged to check the following statements for the Lie algebra $\mathfrak{gl}(N,\mathbb{R})$.

A (Lie) **subalgebra** $\mathfrak{h}$ of $\mathfrak{g}$ is a vector subspace of $\mathfrak{g}$ that is a Lie algebra; we have $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$. For example, the real upper triangular $N \times N$-matrices form a subalgebra $\mathfrak{gl}(N,\mathbb{R})^+$ of $\mathfrak{gl}(N,\mathbb{R})$.

An **ideal** $\mathfrak{i}$ of $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ satisfying $[\mathfrak{i},\mathfrak{g}] \subset \mathfrak{i}$. If $\mathfrak{i}$ is an ideal in $\mathfrak{g}$ we can form the **coset Lie algebra** $\mathfrak{g}/\mathfrak{i}$ and every element in $\mathfrak{g}/\mathfrak{i}$ is of the form $a + \mathfrak{i}$ where $a \in \mathfrak{g}$. The coset Lie algebra $\mathfrak{g}/\mathfrak{i}$ is indeed a Lie algebra;

$$[x + \mathfrak{i}, y + \mathfrak{i}] = [x,y] + \mathfrak{i}\,, \ \forall x,y \in \mathfrak{g}\,. \tag{B.1.1}$$

If $x' = x + i_1, y' = y + i_2$ with $i_1, i_2 \in \mathfrak{i}$ we have $[x' + \mathfrak{i}, y' + \mathfrak{i}] = [x',y'] + \mathfrak{i} = [x,y] + \mathfrak{i}$ so that the product is well defined.

The **derived subalgebra** $\mathfrak{g}'$ of a Lie algebra $\mathfrak{g}$ is the subalgebra $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$. It is easy to check that $\mathfrak{g}'$ is a subalgebra and an ideal.

The **center** of a Lie algebra $\mathfrak{g}$ is the subalgebra of all elements that commute with every other element and is denoted $Z(\mathfrak{g})$. If $\mathfrak{k}$ and $\mathfrak{h}$ are subalgebras of $\mathfrak{g}$, the **centralizer** of $\mathfrak{k}$ in $\mathfrak{h}$ consists of those elements of $\mathfrak{h}$ that commute with every element of $\mathfrak{k}$ and is denoted $Z_{\mathfrak{h}}(\mathfrak{k})$.

A Lie group is **reductive** if the Lie algebra $\mathfrak{g}$ is such that for every ideal $\mathfrak{a}$ there exists another ideal $\mathfrak{b}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$. This implies that for a reductive group the Lie algebra is of the form $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \oplus Z(\mathfrak{g})$ [94]. It is proved in section 3.4.2 that every compact group is reductive.

A Lie algebra $\mathfrak{g}$ is **abelian** if the derived subalgebra vanishes; $[\mathfrak{g},\mathfrak{g}] = 0$. The Lie algebra of an abelian Lie group is an abelian Lie algebra and vice versa. An abelian subalgebra is called a **torus** and the largest abelian subalgebra is called a

maximal torus. This definition is sensible since the Lie algebra is assumed to be finite-dimensional and since one can prove that two different maximal tori are related by an isomorphism.

As remarked before, the tangent space of a group at the identity can be identified with the Lie algebra. For matrix Lie groups, the Lie algebra is obtained by differentiating the Lie group elements with respect to the coordinates on the Lie group. From the Lie algebra one obtains the Lie group elements in a neighborhood of the origin, i.e. the identity, by exponentiation, which is a well-defined operation for matrices (when one restricts to a neighborhood sufficiently close to the identity). If we choose an element $x \in \mathfrak{g}$, then the abelian subgroup X of G given by $X = \{\exp(tx) | t \in I \subset \mathbb{R}\}$, where I is that interval on which $\exp(tx)$ is well defined, is clearly connected. It follows that exponentiating the Lie algebra elements defines a map from the Lie algebra to the component of G that is connected to the identity, called the **identity component** and denoted G_0.

Let $\mathfrak{g}^0 = \mathfrak{g}$ and for $k > 0$ put $\mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k]$. The series $\{\mathfrak{g}^0, \mathfrak{g}^1, \ldots\}$ is a series of ideals. If for finite n the series terminates, that is $\mathfrak{g}^n = 0$, then the Lie algebra $\mathfrak{g}$ is **solvable**. For example, the Lie algebra $\mathfrak{gl}(N, \mathbb{R})^+$ is solvable.

We define the following series of ideals of a Lie algebra $\mathfrak{g}$: $\{\mathfrak{g}_0, \mathfrak{g}_1, \ldots\}$ by $\mathfrak{g}_0 = \mathfrak{g}$ and if $k > 0$, $\mathfrak{g}_{k+1} = [\mathfrak{g}, \mathfrak{g}_k]$. The reader is warned that we now use lower indices instead of the upper indices used to define solvable Lie algebras; $\mathfrak{g}_k \neq \mathfrak{g}^k$ in general. If the series $\{\mathfrak{g}_k\}_{k \geq 0}$ terminates for a finite n the Lie algebra $\mathfrak{g}$ is said to be **nilpotent**.

A Lie algebra $\mathfrak{g}$ is called **simple** if it is nonabelian and the only ideals are $\mathfrak{g}$ and the trivial subalgebra $\{0\}$. Since $\mathfrak{g}'$ is an ideal we have $\mathfrak{g}' = \mathfrak{g}$ if $\mathfrak{g}$ is simple.

A Lie algebra is **semisimple** if it contains no solvable ideals. Every semisimple Lie algebra $\mathfrak{g}$ is the direct sum of simple subalgebras: $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ and $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ if $i \neq j$ and $\mathfrak{g}'_j = \mathfrak{g}_j$, hence $\mathfrak{g}' = \mathfrak{g}$. In particular, every simple Lie algebra is semisimple.

A **derivation** on a Lie algebra is a linear map $D : \mathfrak{g} \to \mathfrak{g}$ satisfying $D([a, b]) = [D(a), b] + [a, D(b)]\ \forall a, b \in \mathfrak{g}$. The set of all derivations on $\mathfrak{g}$ can be made into a Lie algebra by defining $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ and this Lie algebra is denoted $\mathrm{Der}(\mathfrak{g})$. For any $a \in \mathfrak{g}$ we can define a derivation called the **adjoint** of a by defining $\mathrm{ad}a : b \mapsto [a, b]$. The adjoint maps form a Lie algebra, $\mathrm{ad}a \circ \mathrm{ad}b - \mathrm{ad}b \circ \mathrm{ad}a = \mathrm{ad}([a, b])$. All derivations that can be written as the adjoint map of an element of $\mathfrak{g}$ are called inner derivations. For semisimple Lie algebras all derivations are inner derivations. A derivation that is not inner is called outer.

The **Cartan–Killing form** $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is defined as $B(x, y) = \mathrm{tr}_{\mathrm{ad}}(\mathrm{ad}x \circ \mathrm{ad}y)$. It is a theorem due to Cartan that the Cartan–Killing form is nondegenerate if and only if the Lie algebra is semisimple. Note that a bilinear form B is nondegenerate if and only if from $B(u, x) = 0$, $\forall x \in \mathfrak{g}$, it follows that $u = 0$. The Cartan–Killing form is an **invariant bilinear form** on $\mathfrak{g}$; $B([a, b], c) + B(b, [a, c]) = 0$.

If the Cartan–Killing form is (strictly) negative definite a Lie algebra is called **compact**. Thus only semisimple Lie algebras can be compact. The Lie group generated by a compact Lie algebra is compact, but not vice versa.

A **Lie algebra homomorphism**, or just short homomorphism, from a Lie algebra $\mathfrak{g}$ to a Lie algebra $\mathfrak{k}$ is a linear map $\phi : \mathfrak{g} \to \mathfrak{k}$ such that $\phi([a,b]) = [\phi(a), \phi(b)], \forall a, b \in \mathfrak{g}$. If $\mathfrak{i}$ is an ideal in $\mathfrak{g}$ then the projection $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$, which maps $a \in \mathfrak{g}$ to $a + \mathfrak{i}$, is a homomorphism. If a homomorphism is one-to-one it is called an isomorphism. If an isomorphism maps $\mathfrak{g}$ to $\mathfrak{g}$ itself it is called an automorphism.

It is often useful to think of a Lie algebra as a set of matrices satisfying some commutation relations. The rigorous justification is due to **Ado's Theorem**, which states that every finite-dimensional Lie algebra is isomorphic to a Lie algebra of matrices. In other words, every finite-dimensional Lie algebra over the field $I\!\!K$ is a subalgebra of $\mathfrak{gl}(N, I\!\!K)$ for some N.

The Lie group G has a natural action on its Lie algebra $\mathfrak{g}$, called the **Adjoint** action (written with capital A). For $g \in G$ we write $\mathrm{Ad}g : x \mapsto \mathrm{Ad}g(x)$ $x \in \mathfrak{g}$ and the linear map $\mathrm{Ad}g$ is an automorphism. For matrix groups $A \in G$ acts on the matrix $b \in \mathfrak{g}$ as $\mathrm{Ad}A : b \mapsto AbA^{-1}$, i.e. by conjugation. Due to Ado's theorem we can always denote the Adjoint action of the group on its Lie algebra by a conjugation: $\mathrm{Ad}g(x) = U(g)xU(g)^{-1}$ for a matrix $U(g)$ associated with the Adjoint action of $g \in G$. Hence the map $\mathrm{Ad}g$ is a Lie algebra homomorphism for every $g \in G$.

B.2 Representations

A **representation** of a Lie algebra $\mathfrak{g}$ is a pair (ρ, V) consisting of a vector space V, often called the **module** or **$\mathfrak{g}$-module**, and a homomorphism $\rho : \mathfrak{g} \to \mathrm{End(V)}$, where End(V) denotes the Lie algebra of all linear maps from V to V with product $[a,b] = a \circ b - b \circ a,\ a, b \in \mathrm{End(V)}$. A similar definition holds for a representation of a Lie group, but then End(V) has to be replaced by $GL(V)$, the Lie group of all nonsingular matrices acting on V. In the remainder we are most concerned with Lie algebra representations but most results and definitions have a straightforward analogue for Lie group representations.

A representation is called **faithful** if the homomorphism $\rho : \mathfrak{g} \to \mathrm{End(V)}$ has a trivial kernel; $\rho(x) = 0 \Leftrightarrow x = 0$. Ado's Theorem equivalently states that every finite-dimensional Lie algebra admits a finite-dimensional faithful representation.

A representation (ρ, V) is **reducible** if there is a nontrivial proper subspace $W \subset V$ such that $\rho(\mathfrak{g})W \subset W$; W is called an invariant subspace. A representation (ρ, V) is **fully reducible** if for any invariant subspace W there is an invariant subspace W' such that $V = W \oplus W'$. A representation (ρ, V) is **irreducible** if there is no proper nontrivial invariant subspace. Hence if (ρ, V) is an irreducible representation and $x \in V$ is a nonzero vector, then $x + \rho(\mathfrak{g})x = V$ as vector spaces, i.e. the whole module can be obtained from one single vector. For semisimple Lie algebras every finite-dimensional representation is fully reducible. If ρ is a representation of $\mathfrak{g}$ then $\mathrm{Ker}\rho$ is an ideal, and hence for simple $\mathfrak{g}$ either the representation is trivial or faithful.

Irreducible representations are often not denoted as a pair, but just with either ρ or with V. In mathematical literature one often denotes an irreducible representation with the homomorphism ρ, whereas in physics literature one uses the vector space V.

Important examples of representations are given by $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}\mathfrak{g}$ and $\mathrm{Ad} : G \to GL(\mathfrak{g})$, where the first is a Lie algebra representation and the second is a representation of the Lie group G associated with the Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ is simple, these representations are irreducible and faithful.

If ρ is a representation of a connected Lie group G with Lie algebra $\mathfrak{g}$ we define the **derived representation** $\mathrm{d}\rho$ of $\mathfrak{g}$ as follows:

$$g \in G,\ g = \exp X \Rightarrow \rho(g) = \exp \mathrm{d}\rho(X)\,. \tag{B.2.1}$$

The dimensions of the modules associated with ρ and $\mathrm{d}\rho$ are the same. The derived representation can thus be obtained by 'differentiating' the representation ρ.

Two representations (ρ, V) and (ρ', V') of a Lie algebra $\mathfrak{g}$ are **equivalent** if there is a nonsingular map $S : V \to V'$ such that $\rho'(x) = S\rho(x)S^{-1}$ for all $x \in \mathfrak{g}$. **Schur's Lemma** consists of two parts; the first part states that if for two irreducible representations (ρ, V) and (ρ', V') there exists a map $T : V \to V'$ such that $T \circ \rho(x) = \rho'(x) \circ T$ for all $x \in \mathfrak{g}$, then either $T \equiv 0$ or T is nonsingular and the representations are equivalent. The second part states that if for an irreducible representation (ρ, V) there exist a matrix C that commutes with all representing matrices $\rho(x)$, $x \in \mathfrak{g}$, then C is a multiple of the identity.

If (ρ, V) and (ρ', V') are two representations of the Lie algebra $\mathfrak{g}$, then the **tensor product** $(\rho \otimes \rho', V \otimes V')$ of the two representations is defined by the following action of a Lie algebra element $x \in \mathfrak{g}$ on an element $v \otimes w$ of $V \otimes V'$:

$$(\rho \otimes \rho')(x)(v \otimes w) = \rho(x)v \otimes w + v \otimes \rho'(x)w = (\rho(x) \otimes \mathbb{1} + \mathbb{1} \otimes \rho'(x))(v \otimes w) \tag{B.2.2}$$

One checks that B.2.2 indeed defines a representation.

If (ρ,V) is a representation of $\mathfrak{g}$ then the trace in V is an invariant form on $\mathfrak{g}$ by the cyclic property of the trace. We now prove the claim made in section 4.5.1 that up to multiplicative constant, there is only one invariant metric on a complex simple Lie algebra.

Suppose ρ is an irreducible complex representation of $\mathfrak{g}$ with module V. Then we define the representation $\tilde{\rho}$ of $\mathfrak{g}$ in the dual space V^* of V by

$$\forall\, x \in \mathfrak{g},\ \omega \in V^*,\ v \in V\ :\ (\tilde{\rho}(x)\omega)(v) = -\omega(\rho(x)v)\,. \tag{B.2.3}$$

This defines a representation and furthermore, $\tilde{\rho}$ is an irreducible representation.

Suppose that B is any bilinear form on V. The bilinear form induces a map from V to V^*, denoted β by: if $v, w \in V$ then $\beta v(w) = B(v, w)$. B is invariant if $B(\rho(x)v, w) = -B(v, \rho(x)w)$ for all $v, w \in V$ and $x \in \mathfrak{g}$. If B is invariant, we have the identity

$$(\beta \circ \rho(x))(v) = (\rho(x) \circ \beta)(v)\,. \tag{B.2.4}$$

Hence any invariant bilinear form B induces an intertwining map from V to V^*. By Schurs' lemma either $B \equiv 0$ or there exists a basis of elements $e_1, e_2, \ldots$ in V such that $B(e_i, e_j) = \delta_{ij}$.

When we consider the adjoint representation it follows that for simple complex Lie algebras there is up to a multiplicative constant just one single invariant metric that can be put into diagonal form with eigenvalues $+1$. (**QED**)

From the claim one can prove that for a semisimple complex Lie algebra $\mathfrak{g}$ consisting of n simple ideals, the space of invariant forms on $\mathfrak{g}$ is n-dimensional; the trace in any representation is completely fixed by giving n proportionality factors.

B.3 Classification of Complex Simple Lie Algebras

To understand the classification of real simple Lie algebras in section B.4 it is useful to understand the classification of complex simple Lie algebras. We therefore briefly review the classification of complex simple Lie algebras.

Let $\mathfrak{B}$ be a subalgebra of the complex Lie algebra $\mathfrak{g}$. The **normalizer** of $\mathfrak{B}$, denoted $N(\mathfrak{B})$ is the subalgebra of $\mathfrak{g}$ containing all elements $x \in \mathfrak{g}$ satisfying $[x, \mathfrak{B}] \subset \mathfrak{B}$. A subalgebra $\mathfrak{h}$ is called a Cartan subalgebra of $\mathfrak{g}$ if (1) $\mathfrak{h}$ is nilpotent and (2) $\mathfrak{h}$ is self-normalizing; $N(\mathfrak{h}) = \mathfrak{h}$. Up to isomorphism there is just one Cartan subalgebra.

If $\mathfrak{g}$ is semisimple the Cartan subalgebra $\mathfrak{h}$ coincides with the maximal torus in $\mathfrak{g}$. The Cartan–Killing form is nondegenerate on $\mathfrak{h} \times \mathfrak{h}$. The dimension of $\mathfrak{h}$ is called the rank of the Lie algebra and we generically denote it by r.

Since the elements of $\mathfrak{h}$ commute, it is possible to simultaneously diagonalize the elements of ad$\mathfrak{h}$ and to decompose the Lie algebra into the eigenspaces. Reading off an eigenvalue corresponding to a particular eigenvector gives rise to a linear functional on $\mathfrak{h}$. We now formalize the idea; a nonzero linear functional α on $\mathfrak{h}$ is a **root** if the subspace $\mathfrak{g}_\alpha$ defined by

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h) x\} \tag{B.3.1}$$

is nonzero. The roots form a subset Δ of the dual $\mathfrak{h}^*$ of the Cartan subalgebra. The eigenspace decomposition reads:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \,. \tag{B.3.2}$$

The decomposition B.3.2 is called a **root-decomposition**.

The set Δ is called a **root system** and it has the following properties

(1) The roots in Δ span the whole of $\mathfrak{h}^*$.

(2) If $\alpha \in \Delta$ then $-\alpha \in \Delta$.

(3) The root system Δ is reduced; if $\alpha \in \Delta$ the only multiples of α in Δ are 0 and $\pm\alpha$

(4) $\alpha \equiv 0 \Leftrightarrow \mathfrak{g}_\alpha = \mathfrak{h}$.

(5) If α, β and $\alpha + \beta$ are roots, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. If α, β are roots but $\alpha + \beta$ is no root, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$.

(6) The eigenspaces $\mathfrak{g}_\alpha$ for $\alpha \neq 0$ are one-dimensional.

(7) If $\alpha, \beta \in \Delta$ and $\alpha + \beta \neq 0$, then the Cartan–Killing form vanishes on $\mathfrak{g}_\alpha \times \mathfrak{g}_\beta$.

Since the roots span the space $\mathfrak{h}^*$ and the Cartan–Killing form is nondegenerate on $\mathfrak{h}$ there is for every $\alpha \in \Delta$ a unique element $h_\alpha \in \mathfrak{h}$ such that for all $k \in \mathfrak{h}$

$$\alpha(k) = B(k, h_\alpha)\,. \tag{B.3.3}$$

Thus there is a one-to-one relation with the roots and the elements of the Cartan subalgebra, which we use to give Δ a symmetric bilinear form $\langle , \rangle$:

$$\alpha, \beta \in \Delta: \quad \langle \alpha, \beta \rangle = B(h_\alpha, h_\beta)\,. \tag{B.3.4}$$

Since $\langle , \rangle$ is not positive definite, it does not define an inner product. Later we will make it into an inner product on a real subspace of $\mathfrak{h}$.

Using a fixed basis $\{h_1, \ldots, h_r\}$ in $\mathfrak{h}$ we can introduce an ordering in Δ. We call a root α positive if the first nonzero number in the series $\alpha(h_1), \ldots, \alpha(h_r)$ is greater than zero. The set of positive roots is called Δ^+. A root is either positive or negative. A root is called **simple** if it is positive and it is not the sum of two other positive roots. The set of simple roots is called Π. The number of simple roots equals the rank of the Lie algebra. We denote the simple roots by α_i for $i = 1, \ldots, r$; $\Pi = \{\alpha_1, \ldots, \alpha_r\}$. For the set of simple roots one can prove

(1) The simple roots α_i are linearly independent and span $\mathfrak{h}^*$.

(2) For $\alpha_i, \alpha_j \in \Pi$: $\langle \alpha_i, \alpha_j \rangle \leq 0$.

(3) If α_i and α_j are simple roots, then $\alpha_i - \alpha_j$ is no root.

(4) Every positive root is a sum of simple roots with nonnegative integer coefficients: $\alpha \in \Delta^+$; $\alpha = \sum_i n_i \alpha_i$ with n_i nonnegative integers.

(5) We define the real subspace $\mathfrak{h}_0^*$ by $\mathfrak{h}_0^* = \oplus_{i=1}^r \mathbb{R}\alpha_i$, i.e. $\mathfrak{h}_0^*$ is the real vector space spanned by the simple root vectors. The bilinear symmetric form $\langle , \rangle$ on Δ restricts to a positive definite real symmetric bilinear form on $\mathfrak{h}_0^*$ and hence defines an inner product on $\mathfrak{h}_0^*$. Thus the simple root vectors span a real vector space with the Euclidean inner product $\langle , \rangle$.

Let us now fix a nonzero root $\alpha \in \Delta$. Then there exist $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$ and $h_\alpha \in \mathfrak{h}$ such that

$$[e_\alpha, f_\alpha] = h_\alpha\,, \quad [h_\alpha, e_\alpha] = 2e_\alpha\,, \quad [h_\alpha, f_\alpha] = -2f_\alpha\,. \tag{B.3.5}$$

The Lie algebra spanned by the elements $e_\alpha, f_\alpha, h_\alpha$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2)^{\mathbb{C}}$ for which all representations are known and treated in most quantum mechanics textbooks. Now let us fix a second root β and consider the string of roots $S_{\alpha\beta} \equiv \{\beta + k\alpha | k = 0, \pm1, \pm2, \ldots; \beta + k\alpha \in \Delta\}$. Since we assumed $\mathfrak{g}$ to be finite, the string is finite and the root space $\{g_\gamma | \gamma \in S_{\alpha\beta}\}$ associated with the string $S_{\alpha\beta}$ forms a representation of $\mathfrak{sl}(2,\mathbb{C})$. Analyzing these strings for the simple roots and the associated $\mathfrak{sl}(2,\mathbb{C})$-representations gives the structure of the simple complex Lie algebras.

The **Cartan matrix** A_{ij} is defined as follows:

$$A_{ij} = 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}\,. \tag{B.3.6}$$

One can deduce the following properties of the Cartan matrix[1]:

(1) $A_{ii} = 2$ and if $i \neq j$ $A_{ij} \leq 0$.

(2) A_{ij} is an integer, and if $i \neq j$ then $A_{ij} = 0, -1, -2, -3$.

(3) The determinant of A_{ij} is positive; $\det A_{ij} > 0$.

(4) We can introduce Lie algebra elements e_i, f_i and h_i for every simple root α_i such that:

$$\begin{aligned} [h_i, h_j] &= 0\,, & [h_i, e_j] &= A_{ji} e_j\,, \quad \text{no sum,} \\ [e_i, f_j] &= \delta_{ij} h_i\,, & [h_i, f_j] &= -A_{ji} f_j\,, \quad \text{no sum.} \end{aligned} \tag{B.3.7}$$

(5) From the Cartan matrix one can deduce all commutation relations and from the elements $[e_{i_1}[\ldots[e_{i_{p-1}}, e_{i_p}]\ldots]]$ for $p \geq 1$ one obtains all positive root generators of the Lie algebra $\mathfrak{g}$. Similarly every negative root generator can be written as a linear sum of the elements $[f_{i_1}[\ldots[f_{i_{p-1}}, f_{i_p}]\ldots]]$ for $p \geq 1$.

The classification of complex simple Lie algebras is thus complete if one can classify all Cartan matrices or equivalently all simple root systems Π. An elegant way of doing this and presenting the result is by using Dynkin diagrams. A **Dynkin diagram** is constructed by using for every simple root a 'blob' that is connected by a number of lines. The number of lines between blob i and j equals $A_{ij}A_{ji}$ and thus the number

[1] It is here that one uses the properties of the $\mathfrak{sl}(2,\mathbb{C})$-representations

of lines is minimally 0 if $\langle \alpha_i, \alpha_j \rangle = 0$ and maximally 3. If $A_{ij}A_{ji} > 1$ the simple roots α_i and α_j have different lengths;

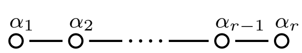
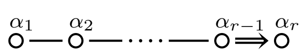
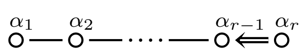
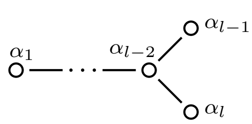
$$\frac{\langle \alpha_i, \alpha_i \rangle}{\langle \alpha_j, \alpha_j \rangle} = \frac{A_{ji}}{A_{ij}}, \tag{B.3.8}$$

and in these cases an arrow is drawn on the lines connecting blob i and j pointing in the direction of the longer root.

From the properties of the Cartan matrix one deduces rules for a Dynkin diagram to be a valid Dynkin diagram. Using these rules one can pictorially classify all complex simple Lie algebras by classifying all valid Dynkin diagrams. We do not give the rules but display the result in table B.3.1.

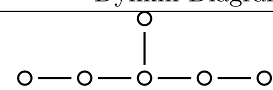
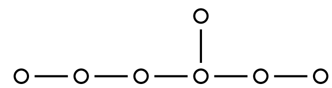
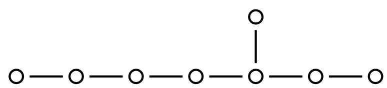
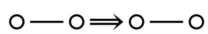
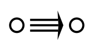
There are four infinite series of Lie algebras; (1) the A_n-series, which consists of the $\mathfrak{sl}(n-1,\mathbb{C})$-algebras, (2) the B_n-series consisting of the $\mathfrak{so}(2n+1)^{\mathbb{C}}$-algebras, (3) the C_n-series comprising the $\mathfrak{sp}(n,\mathbb{C})$-algebras and the (4) D_n-series, which consists of the $\mathfrak{so}(2n)^{\mathbb{C}}$-algebras. But besides the four 'classical Lie algebras' there are five exceptional algebras; $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$. This finishes our discussion on the classification of complex simple Lie algebras.

Table B.3.1: Dynkin Diagrams of the Complex Simple Lie Algebras

Dynkin Diagram	Name
	A_r: $\mathfrak{sl}(r-1,\mathbb{C}) \cong \mathfrak{su}(r-1,\mathbb{C})$
	B_r: $\mathfrak{so}(2r+1)^{\mathbb{C}}$
	C_r: $\mathfrak{sp}(r,\mathbb{C})$
	D_r: $\mathfrak{so}(2r)^{\mathbb{C}}$

Continued on next page

Table B.3.1: continued

Dynkin Diagram	Name
	$\mathfrak{e}_6$
	$\mathfrak{e}_7$
	$\mathfrak{e}_8$
	$\mathfrak{f}_4$
	$\mathfrak{g}_2$

B.4 Classification of Real Simple Lie Algebras

We now discuss the classification of real simple Lie algebras. The classification of real simple Lie algebras is used in chapter 3. We do not prove the statements and refer to [94, 113] for the proofs.

A **real form** of a complex semisimple Lie algebra $\mathfrak{g}_c$ is a real semisimple Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}_c \cong (\mathfrak{g}_c)_{\mathbb{R}} \equiv \mathfrak{g} \oplus_{\mathbb{R}} i\mathfrak{g}$, where the subscript $\mathbb{R}$ on the $\oplus$-sign indicates that the direct sum is a direct sum of real vector spaces and hence only real combinations of the two summands can be taken. A nice way to think of a real form of a complex Lie algebra is by comparing it with a real line in $\mathbb{C}$; every line through the origin in $\mathbb{C}$ defines a real vector subspace and there are an infinite number of such subspaces all related by rotation. For Lie algebras there are an infinite number of real forms, but if we count two real forms that are isomorphic as one, there are only a finite number of real forms for every complex semisimple Lie algebra.

If $\mathfrak{g}$ is a real simple Lie algebra, then for the complexification $\mathfrak{g}_c$ there are two options: (1) $\mathfrak{g}_c = \mathfrak{s} \oplus \mathfrak{s}$ where $\mathfrak{s}$ is a simple complex Lie algebra, (2) $\mathfrak{g}_c$ is simple as a complex Lie algebra. An example of the first option is provided by $\mathfrak{so}(1,3)$; $\mathfrak{so}(1,3)^{\mathbb{C}} = \mathfrak{so}(3)^{\mathbb{C}} \oplus \mathfrak{so}(3)^{\mathbb{C}}$. On the other hand, if $\mathfrak{g}_c$ is a complex simple Lie algebra, then every real form is a simple real Lie algebra. Therefore classifying all real forms of simple complex Lie algebras gives a classification of all simple real Lie algebras.

Having written $\mathfrak{g}_c$ as a real Lie algebra $\mathfrak{g}_r = (\mathfrak{g}_c)_{\mathbb{R}}$ we see that $\mathfrak{g}_r$ admits an involutive (i.e. squaring to one) automorphism that in $\mathfrak{g}_c$ acts as complex conjugation. This motivates the introduction of a more general and abstract involutive automorphism called the Cartan involution. A **Cartan involution** of a real semisimple Lie algebra $\mathfrak{g}$ is an involutive automorphism $\theta : \mathfrak{g} \to \mathfrak{g}$ such that the bilinear form B_θ defined by $B_\theta(x, y) \equiv -B(x, \theta y)$, is positive definite, where $B(,)$ is the Cartan–Killing form.

The bilinear form B_θ is symmetric and that the adjoint of $\mathrm{ad}X$ with respect to B_θ is given by $-\mathrm{ad}\theta X$;

$$B_\theta(\mathrm{ad}X(Y), Z) = B_\theta(Y, (\mathrm{ad}X)^\dagger(Z)) \Leftrightarrow (\mathrm{ad}X)^\dagger = -\mathrm{ad}\theta X\,. \tag{B.4.1}$$

Since θ is involutive, the Lie algebra splits into a positive eigenspace $\mathfrak{k}$, and a negative eigenspace $\mathfrak{p}$, such that $\theta|_{\mathfrak{k}} = +\mathbb{1}$, $\theta_{\mathfrak{p}} = -\mathbb{1}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Since θ is a homomorphism, θ introduces a $\mathbb{Z}_2$-grading in $\mathfrak{g}$:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}. \tag{B.4.2}$$

The eigenspace $\mathfrak{k}$ is thus a Lie subalgebra on which the Cartan–Killing form is negative definite. From equations B.4.2 it follows that the Cartan–Killing form vanishes on $\mathfrak{k} \times \mathfrak{p}$ and hence is nondegenerate when restricted to either $\mathfrak{p}$ or $\mathfrak{k}$.

With every real form $\mathfrak{g}$ we can associate a coset G/K by taking G to be the simply connected group G with Lie algebra $\mathfrak{g}$ and K to be the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. The coset G/K is a symmetric space and the tangent space at any point is isomorphic to $\mathfrak{p}$. The inner product B_θ gives G/K a Riemannian structure.

The action of the Cartan involution can be extended to the group G. Let $g \in G$, then $g = \exp X$ for some (unique) $X \in \mathfrak{g}$ and write $X = \log g$. Then the **generalized transpose** of g is denoted $g^\#$ and defined by

$$g^\# = \exp(-\theta \log g)\,. \tag{B.4.3}$$

When restricted to the subgroup K, the action of # is the same as inverting; if $k \in K$ then $k^\# k = k k^\# = e$. The push-forward of # is $-\theta$.

The elements of $\mathrm{ad}\mathfrak{p}$ are Hermitian linear operators with respect to B_θ. Hence $\mathfrak{g}$ admits an eigenspace decomposition with respect to a maximal torus $\mathfrak{a}$ in $\mathfrak{p}$ similar to the root space decomposition of a complex Lie algebra with respect to its Cartan subalgebra. The elements of $\mathfrak{a}$ are called the noncompact Cartan generators. For μ in

$\mathfrak{a}^*$ we define the subspace $\mathfrak{g}_\mu = \{x \in \mathfrak{g} | [A, x] = \mu(A)x, \forall A \in \mathfrak{a}\}$ and a nonzero element μ of $\mathfrak{a}^*$ for which $\mathfrak{g}_\mu$ is nonzero is called a **restricted root**. The set of restricted roots is denoted Σ. The eigenspace decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$ reads:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda \, , \quad \mathfrak{g}_0 = \mathfrak{a} \oplus Z_{\mathfrak{k}}(\mathfrak{a}) \, , \tag{B.4.4}$$

where $Z_{\mathfrak{k}}(\mathfrak{a})$ denotes the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, i.e. the set of all elements $\mathfrak{k}$ that commute with $\mathfrak{a}$.

The restricted roots have some properties that are similar to those of the roots of the complex simple Lie algebras:

(1) The restricted roots in Σ span a subspace of $\mathfrak{h}^*$.

(2) $\mu \in \Sigma \Leftrightarrow -\mu \in \Sigma$.

(3) If μ, ν and $\mu + \nu$ are restricted roots, then $[\mathfrak{g}_\mu, \mathfrak{g}_\nu] \subset \mathfrak{g}_{\mu+\nu}$. If μ, ν are restricted roots but $\mu + \nu$ is no restricted root, then $[\mathfrak{g}_\mu, \mathfrak{g}_\nu] = 0$.

(4) If μ is a restricted root then the only multiples of μ that can also be restricted roots are $\pm 2\mu$, $\pm 1\mu$ and $\pm \frac{1}{2}\mu$, but if 2μ is a restricted root, then $\frac{1}{2}\mu$ is not a restricted root and vice versa.

(5) If $\mu, \nu \in \Delta$ and $\mu - \nu \neq 0$, then the inner product B_θ vanishes on $\mathfrak{g}_\mu \times \mathfrak{g}_\nu$.

(6) The inner product B_θ vanishes on $\mathfrak{a} \times Z_{\mathfrak{k}}(\mathfrak{a})$ and is nondegenerate on $\mathfrak{a}$.

(7) The dimension of the root space $\mathfrak{g}_\mu$, called the multiplicity of μ, can exceed 1; $m(\mu) \equiv \dim \mathfrak{g}_\mu \geq 1$.

Although the properties are similar, important differences occur; especially properties (4) and (7) are important differences.

Since B_θ and B are nondegenerate on $\mathfrak{a}$ there exists for every $\mu \in \Sigma$ a unique nonzero $A_\mu \in \mathfrak{a}$ such that $B(A_\mu, A) = \mu(A)$ for all $A \in \mathfrak{a}$. Hence we can define an inner product on Σ through $\langle \mu, \nu \rangle = B(A_\mu, A_\nu)$.

The set of restricted roots can be given an ordering $>$ in a similar way as the roots of a complex simple Lie algebra. The set of positive restricted roots with respect to this ordering is denoted Σ^+. A restricted root is simple if it is positive and it is not the sum of two other positive restricted roots. We denote the set of simple restricted roots by Σ_0. If we denote $l = \dim \mathfrak{a}$ then there are l simple restricted roots λ_i that span $\mathfrak{a}^*$. For the simple restricted roots one can define the integer-valued Cartan matrix $A_{ij} = 2\frac{\langle \lambda_i, \lambda_j \rangle}{\langle \lambda_j, \lambda_j \rangle}$ and draw a Dynkin diagram in an analogous way as for the simple roots of a simple complex Lie algebra.

We define $\mathfrak{n} = \bigoplus_{\mu \in \Sigma^+} \mathfrak{g}_\mu$. The subalgebra $\mathfrak{n}$ is nilpotent and the subalgebra $\mathfrak{a} \oplus \mathfrak{n}$ is solvable. The **Iwasawa decomposition** of a real simple Lie algebra reads:

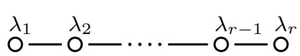
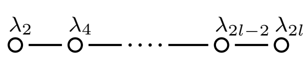
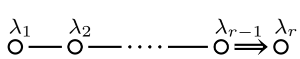
$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}. \tag{B.4.5}$$

The Iwasawa decomposition provides us with a parametrization of the coset G/K through $G/K \cong \exp \mathfrak{a} \exp \mathfrak{n}$. Due to this parametrization the dilatonic coupling vectors of section 3.3.2 are identified with the positive restricted roots.

The simple real Lie algebras can be classified using the above concepts plus some other material, which is not needed to understand the text in this thesis and which is therefore omitted in this appendix. To characterize a real form uniquely we need to specify the Dynkin diagram for Σ_0 (or equivalently the Cartan matrix) and the multiplicities $m(\lambda_i)$ and $m(2\lambda_i)$ for all simple restricted roots $\lambda_i \in \Sigma_0$. In table B.4.1 we present the Dynkin diagrams of the set of simple restricted roots for all possible simple real algebras $\mathfrak{g}$ that are not of the form $\tilde{\mathfrak{g}} \oplus_{\mathbb{R}} i\tilde{\mathfrak{g}}$ and give the associated maximally noncompact coset G/K, where G is the simply connected real Lie group with Lie algebra $\mathfrak{g}$ and K is the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. Table B.4.1 is of use for recognizing the coset structure of the scalars in dimensionally reduced supergravity theories as explained in section 3.3.2.

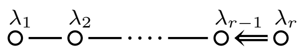
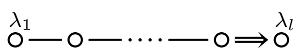
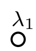
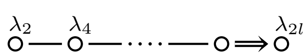
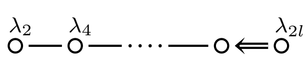
The multiplicities $m(\lambda_i)$ and $m(2\lambda_i)$ for the simple restricted roots are presented in table B.4.2 for the simple real Lie algebras. The labelling of the simple restricted roots as in table B.4.1 is also used in table B.4.2. The number r is the rank of the complexified Lie algebra.

Table B.4.1: Restricted root diagrams and associated cosets G/K.

Restricted Root Diagram	Type
	$\mathbf{A}\boldsymbol{I}$:$SL(n;\mathbb{R})/SO(n)$ $l = r = n - 1$
	$\mathbf{A}\boldsymbol{II}$:$SU^*(2n)/Sp(n)$ $l = 2r - l = n - 1$
	$\mathbf{A}\boldsymbol{III}$:$SU(p,q)/S(U_p \times U_q)$ $l = min(p,q)$, $r = p + q - 1$ If $2 \leq l \leq r/2$

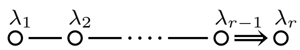
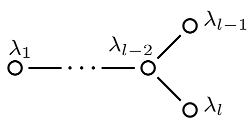
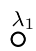
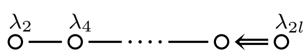
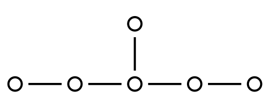
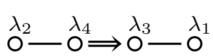
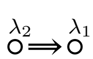
Continued on next page

Table B.4.1: continued

<table>
<tr><th>Restricted Root Diagram</th><th>Type</th></tr>
<tr><td></td><td>If $r = 2l - 1$</td></tr>
<tr><td></td><td>$\mathbf{A}\boldsymbol{IV}$:$SU(n,1)/SU(n)$
$l = r + 1 = n$</td></tr>
<tr><td></td><td>$\mathbf{B}\boldsymbol{I}$:$\frac{SO(p,q)}{SO(p)\times SO(q)}$; $p+q$ odd
$2 \leq l = min(p,q) \leq r$</td></tr>
<tr><td></td><td>$\mathbf{B}\boldsymbol{II}$:$SO(2p,1)/SO(2p)$
$l = 1$, $r = p$</td></tr>
<tr><td></td><td>$\mathbf{C}\boldsymbol{I}$:$Sp(n,\mathbb{R})/U(n)$
$l = r = n$</td></tr>
<tr><td></td><td>$\mathbf{C}\boldsymbol{II}$:$\frac{Sp(p,q)}{Sp(p)\times Sp(q)}$
$l = min(p,q)$
If $1 \leq l \leq \frac{1}{2}(r-l)$</td></tr>
<tr><td></td><td>If $2 \leq l = r/2$</td></tr>
<tr><td></td><td>$\mathbf{D}\boldsymbol{I}$:$\frac{SO(p,q)}{SO(p)\times SO(q)}$, $p+q$ even
$l = min(p,q)$;
If $2 \leq l \leq r-2$</td></tr>
</table>

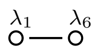
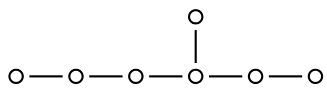
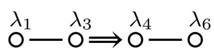
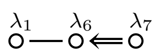
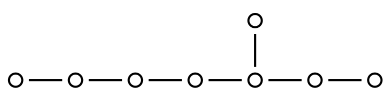
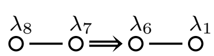
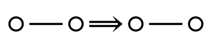
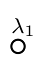
Continued on next page

Table B.4.1: continued

Restricted Root Diagram	Type
	If $r = l + 1$.
	If $r = l$.
	**D*II***: $\frac{SO(2r-1,1)}{SO(2r-1)}$, $r \neq 3$ $l = 1$
	**D*III***: $\frac{SO^*(2n)}{U(n)}$ $l = [n/2]$ If $r = 2l$
	If $r = 2l + 1$
	**E*I***: $E_{6(6)}/Sp(4)$. $l = r = 6$
	**E*II***: $\frac{E_{6(2)}}{SU(6) \times SU(2)}$. $l = 4$
	**E*III***: $\frac{E_{6(-14)}}{SO(10) \times U(1)}$. $l = 2$

Continued on next page

Table B.4.1: continued

Restricted Root Diagram	Type
	**E*IV***: $\frac{E_{6(-24)}}{F_4}$. $l = 2$
	**E*V***: $E_{7(7)}/SU(8)$. $l = r = 7$
	**E*VI***: $\frac{E_{7(-5)}}{SO(12)\times SU(2)}$. $l = 4$
	**E*VII***: $\frac{E_{7(-25)}}{E_6\times U(1)}$. $l = 3$
	**E*VIII***: $E_{8(8)}/SO(16)$. $l = r = 8$
	**E*IX***: $\frac{E_{8(-24)}}{E_7\times SU(2)}$. $l = 4$
	**F*I***: $\frac{F_{4(4)}}{Sp(3)\times SU(2)}$. $r = l = 4$
	**F*II***: $F_{4(-20)}/SO(9)$. $l = 1$

Continued on next page

Table B.4.1: continued

Restricted Root Diagram	Type
	$\mathbf{G}$: $\frac{G_{2(2)}}{SU(2)\times SU(2)}$. $l = r = 2$

Table B.4.2: Multiplicities of the restricted simple roots

Type		m_{λ_i}	$m_{2\lambda_i}$
$\mathbf{A}\boldsymbol{I}$	$\forall i$	1	0
$\mathbf{A}\boldsymbol{II}$	$\forall i$	4	0
$\mathbf{A}\boldsymbol{III}$; $2 \leq l \leq \frac{r}{2}$	$i < l$	2	0
	$i = l$	$2(r - 2l + 1)$	1
$\mathbf{A}\boldsymbol{III}$; $r = 2l - 1$	$i < l$	2	0
	$i = l$	1	0
$\mathbf{A}\boldsymbol{IV}$		$2(r - 1)$	1
$\mathbf{B}\boldsymbol{I}$	$i < l$	1	0
	$i = l$	$2(r - l) + 1$	0
$\mathbf{B}\boldsymbol{II}$	$\forall i$	$2r - 1$	0
$\mathbf{C}\boldsymbol{I}$	$\forall i$	1	0
$\mathbf{C}\boldsymbol{II}$; $1 \leq l \leq \frac{1}{2}(r - 1)$	$i < 2l$	4	0
	$i = 2l$	$4(r - 2l)$	3
$\mathbf{C}\boldsymbol{II}$; $2 \leq l = \frac{1}{2}r$	$i < 2l$	4	0
	$i = 2l$	3	0
$\mathbf{D}\boldsymbol{I}$; $2 \leq l \leq r - 2$	$i < l$	1	0
	$i = l$	$2(r - l)$	0
$\mathbf{D}\boldsymbol{I}$; $l = r - 1$	$i < l$	1	0
	$i = l$	2	0
$\mathbf{D}\boldsymbol{I}$; $l = r$	$\forall i$	1	0
$\mathbf{D}\boldsymbol{II}$	$\forall i$	1	0
$\mathbf{D}\boldsymbol{III}$; $r = 2l$	$i < 2l$	4	0
	$i = 2l$	1	0
$\mathbf{D}\boldsymbol{III}$; $r = 2l + 1$	$i < 2l$	4	1
$\mathbf{E}\boldsymbol{I}$	$\forall i$	1	0
$\mathbf{E}\boldsymbol{II}$	$i = 2, 4$	1	0
	$i = 1, 3$	2	0
$\mathbf{E}\boldsymbol{III}$	$i = 1$	8	1
	$i = 2$	6	0

Continued on next page

Table B.4.2: continued

Type		m_λ	$m_{2\lambda}$
E*IV*	$\forall i$	8	0
E*V*	$\forall i$	1	0
E*VI*	$i = 1, 3$	1	0
	$i = 2, 4$	4	0
E*VII*	$i = 1, 6$	8	0
	$i = 7$	1	0
E*VIII*	$\forall i$	1	0
E*IX*	$i = 1, 6$	8	0
	$i = 7, 8$	1	0
F*I*	$\forall i$	1	0
F*II*	$\forall i$	8	7
G*I*	$\forall i$	1	0

B.5 Elements of Geometry on Compact Lie Groups

In this section we briefly review some differential geometry on semisimple compact Lie groups that is used in section 3.4. Some concepts are already explained in the thesis and hence are not repeated here or only briefly. We prove little and the reader interested in the proofs is referred to the literature [5,6,8,113,118]. In the end of this section when we discuss cohomological properties of compact Lie groups, we give a few easy and (hopefully) insightful examples for the reader not familiar with cohomology on compact Lie groups.

A group acts on itself in two ways; by **left multiplication** $L_a : g \mapsto ag$ for $a, g \in G$ and **right multiplication** $R_a : g \mapsto ga$ for $a, g \in G$. A vector field X is **left-invariant** if $(L_{a*}X)_{ag} = X_{ag}$ and **right-invariant** if $(R_{a*}X)_{ag} = X_{ag}$, where L_{a*} (R_{a*}) is the push-forward of L_a (R_a) and X_p denotes the vector field X at p.

The tangent space of the identity TG_e is isomorphic to the Lie algebra of G, which we denote by $\mathfrak{g}$. Let us fix a basis of vectors $t_1, \ldots, t_n$ for TG_e. We define for each t_a a vector field τ_a by putting $(\tau_a)_x = L_{x*}t_a$ for any $x \in G$. The vector fields τ_a are left-invariant. On the other hand, a left-invariant vector field is completely determined by its value at the identity. Hence the left-invariant vector fields are in one-to-one correspondence with the Lie algebra.

The Lie bracket between two left-invariant forms is again left-invariant, which implies that there should be constants ${C_{ab}}^c$ such that

$$[\tau_a, \tau_b] = {C_{ab}}^c \tau_c \,. \tag{B.5.1}$$

A **left-invariant** p**-form** $\omega^{(p)}$ satisfies $L_g^* \omega_{ga}^{(p)} = \omega_a^{(p)}$ where L_g^* is the pull-back of

L_g. Suppose we define a basis $\{s_1, \ldots, s_n\}$ of $\mathfrak{g}^*$ such that $s^i(t_j) = \delta^i_j$. But $TG^*_e \cong \mathfrak{g}^*$ and hence we can identify with every s^a a one-form at the identity $(\sigma^a)_0$. For any other $g \in G$ we define $(\sigma^a)_g = L_{g^{-1}*}(\sigma^a)_0$ and hence the σ^a are globally defined left-invariant one-forms dual to the left-invariant vector fields τ_b satisfying $\sigma^a(\tau_b) = \delta^a_b$. They satisfy the Maurer–Cartan equation

$$\mathrm{d}\sigma^a + \tfrac{1}{2} C_{bc}{}^a \sigma^b \wedge \sigma^c = 0\,. \tag{B.5.2}$$

The algebra of left-invariant forms is finite-dimensional and generated by the left-invariant one-forms over $\mathbb{R}$; every left-invariant p-form $\omega^{(p)}$ can be written as $\omega^{(p)} = \omega^{(p)}_{a_1 \ldots a_p} \sigma^{a_1} \wedge \ldots \wedge \sigma^{a_p}$ where the $\omega^{(p)}_{a_1 \ldots a_p}$ are real constants.

A **left-invariant metric** g on G satisfies $g(L_{h*}X, L_{h*}Y)|_{ha} = g(X,Y)|_a$ for all vector fields X, Y and all $h, a \in G$. If we take any positive-definite metric η on $\mathfrak{g} \cong TG_e$, the metric η can be extended to a left-invariant metric g on G by putting:

$$g = \eta_{ab} \sigma^a \otimes \sigma^b\,, \quad \eta_{ab} = \eta(t_a, t_b)\,. \tag{B.5.3}$$

A similar discussion holds for a right-invariant metric and if G is simple, the only left-invariant metric that is also right-invariant is the one where the metric η is proportional to the Cartan–Killing metric of $\mathfrak{g}$.

A left-invariant (right-invariant) vector-field translates right-translations (left-translations). Under an infinitesimal *left-translation* a field Φ varies as $\delta\Phi = \mathcal{L}_X \Phi$, where $\mathcal{L}_X$ denotes the Lie derivative along X and $X = \epsilon^a X_a$ is a *right-invariant* vector field. Hence an alternative way of stating left-invariance is that the Lie derivative along a right-invariant vector field vanishes. Since left-multiplication and right-multiplication commute, the commutator of a left-invariant vector field with a right-invariant vector field vanishes. This is consistent with the invariance of the vector fields, since the variation under an infinitesimal left-translation of a left-invariant vector field X_L is given by $0 = \delta X_L = \mathcal{L}_{Y_R}(X_L) = [Y_R, X_L]$ for a right-invariant vector field Y_R.

On a compact Lie group there is up to multiplicative constant a unique bi-invariant integration measure μ:

$$\int_G \mathrm{d}\mu(x) f(ax) = \int_G \mathrm{d}\mu(x) f(xa) = \int_G \mathrm{d}\mu(x) f(x)\,. \tag{B.5.4}$$

The measure μ is called the **Haar measure**. Since $\mu' = \sigma^1 \wedge \ldots \wedge \sigma^n$, where $n = \dim G$, is a left-invariant form of maximal rank we can take μ' to be the integration measure. If we define $\mu = \mu' / \int_G \sigma^1 \wedge \ldots \wedge \sigma^n$ then the volume of G is normalized to 1 with respect to the measure μ.

If the Lie group G is equipped with the bi-invariant metric we have for any p-form on G:

$$L_{g*} \star \omega^{(p)} = \star L_{g*} \omega^{(p)}\,. \tag{B.5.5}$$

Hence if ω is left-invariant then also $\star\omega$.

A form ω is **closed** if $\mathrm{d}\omega = 0$ and **exact** if there is a form λ such that $\omega = \mathrm{d}\lambda$. Any exact form is also closed.

We call Λ^p the space of all p-forms on G and $\Lambda = \oplus_{p=0}^{n} \Lambda^p$ the **exterior algebra** with product $\wedge$. Every Λ^p is a vector space. Within each Λ^p we define two subspaces: Ω^p is the subspace of all closed p-forms and Z^p is the space of all exact p-forms. We have $Z^p \subset \Omega^p$ since $\mathrm{d}^2 = 0$.

We define an equivalence relation $\sim$ in Ω^p by $\omega \sim \eta$ if $\omega - \eta = \mathrm{d}\lambda$ with $\lambda \in \Lambda^{p-1}$; i.e. two closed p-forms are equivalent if they differ an exact p-form. The vector space obtained by dividing out this equivalence relation is called the **pth cohomology class** and denoted H^p;

$$H^p \equiv \frac{\Omega^p}{Z^p}. \tag{B.5.6}$$

We define the **cohomology algebra** H by $H = \oplus_{p=0}^{n} H^p$ and equip H with the product $\wedge$.

The **pth Betti number** is the dimension of the real vector space H^p and is denoted b^p; $b^p = \dim_{\mathbb{R}} H^p$. The Betti numbers are so-called topological invariants[2]. If one changes the manifold $\mathcal{M}$ in a smooth way - that is, no cutting and pasting allowed - then the Betti numbers stay the same. As an example, a tea cup has the same Betti numbers as a donut, since one can deform a tea cup in a continuous way to a donut. But a solid sphere cannot be deformed in a continuous way into a donut, since to get the ear of the cup we need to pinch a hole in the solid sphere.

From the Betti numbers one calculates the **Euler characteristic**:

$$\chi = \sum_{r=0}^{n} (-1)^r b^r \,. \tag{B.5.7}$$

The Euler characteristic is also a topological invariant.

Due to a theorem of Hopf and Poincaré the Euler characteristic of a compact manifold vanishes if there exists a nowhere vanishing vector field. Since a compact Lie group admits a set of nowhere vanishing vector fields by means of the left-invariant vector fields τ_a (they are nowhere vanishing because the duals σ^b exist) it has vanishing Euler characteristic.

On a compact n-dimensional manifold the rth and $(n-r)$th Betti numbers are related by **Poincaré duality**: $b^r = b^{n-r}$. Thus odd-dimensional compact manifolds always have vanishing Euler characteristic.

If a compact Lie group G is connected we have $b^0 = b^n = 1$. A zero-form with vanishing exterior derivative is given by a constant, but if the manifold is not connected, then on every connected piece the constant may take different values.

The **theorem of Chevalley and Eilenberg** states that in calculating the Betti numbers on a compact Lie group one can restrict to the left-invariant forms. That is,

[2]Note that we are using a broad definition of geometry; geometry in our sense also includes topological aspects.

the cohomology algebra of $\Lambda(G)$, which is infinite-dimensional, gives the same Betti numbers as the cohomology algebra generated by the left-invariant forms, which is finite-dimensional.

As an example, we calculate the Betti numbers of $SU(2)$. We choose a basis $\{t_1, t_2, t_3\}$ of $\mathfrak{su}(2)$ such that the structure constants are given by $[t_a, t_b] = \epsilon_{abc} t_c$. Since $SU(2)$ is connected we have $b^0 = b^3 = 1$. To obtain b^1 we take a left-invariant one-form $\omega = \omega_a \sigma^a$ with constant ω^a. Demanding that it is closed gives: $\omega_a \epsilon_{abc} = 0$, from which it follows that $\omega_a = 0$ and hence $\Omega^1 = 0$ and thus $b^2 = b^1 = 0$.

As another example we extend this result and show that on any compact semi-simple Lie group the first Betti number always vanishes. Suppose $\omega = \omega_a \sigma^a$ is closed, then $\omega_a C_{bc}{}^a = 0$ for all b, c. It follows that ω, seen as a linear functional on the Lie algebra $\mathfrak{g}$, vanishes on the derived algebra $[\mathfrak{g}, \mathfrak{g}]$. But for semisimple Lie groups the derived algebra coincides with the Lie algebra: $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and hence $\omega_a = 0$ and hence $\Omega^1 = 0$. Hence $b^1 = b^{n-1} = 0$.

To calculate the Betti numbers of the Cartesian product of two compact manifolds M and N one uses the **Künneth formula**:

$$b^k(M \times N) = \sum_{p+q=k} b^p(M) b^q(N) \,. \tag{B.5.8}$$

As an example, we calculate the Betti numbers of an n-torus T^n, which is the Cartesian product of n circles. The Betti numbers of a circle are easily found using Poincaré duality and connectedness of S^1: $b^0 = b^1 = 1$. Hence we find

$$b^k(T^n) = \sum_{m_1 + \ldots + m_n = k; m_i = 0,1} 1 = \binom{n}{k} \,. \tag{B.5.9}$$

Every compact group is reductive (a proof is given in section 3.4.2) and hence the Lie algebra is of the form: $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z(\mathfrak{g})$ where $Z(\mathfrak{g})$ is the center. The center corresponds precisely to the toroidal directions of G; $\exp \mathfrak{g} \cong U(1)^{\dim Z(\mathfrak{g})} \times \exp([\mathfrak{g}, \mathfrak{g}])$. Using this it is not too hard to prove that for a compact group G we have $b^1 = \dim Z(\mathfrak{g})$.

From the Künneth formula it follows that the Euler characteristic of the product of two compact manifolds is simply the product of the Betti numbers:

$$\sum_{k=0} (-1)^k b^k(M \times N) = \sum_k \sum_{p+q=k} (-1)^{p+q} b^p(M) b^q(N) = \chi(M)\chi(N) \,. \tag{B.5.10}$$

Since the Euler characteristic of a circle vanishes ($b^0 - b^1 = 1 - 1 = 0$), the Euler characteristic of an n-torus vanishes. This is in agreement with the fact that an n-torus is isomorphic to the compact Lie group $U(1)^n$, which admits n nowhere vanishing left-invariant vector fields.

Appendix C

Spinors and Clifford Algebras

In this appendix we give a short summary on general spinor representations and give a few specific properties of spinors of $SO(1,9)$, $SO(1,3)$ and $SO(6)$ together with the conventions we used for spinors and Γ-matrices. This appendix is based on [33, 45]. Some finite-group theory is used; the reader not familiar with the theory of finite groups is referred to the literature, e.g. [186, 187].

C.1 General Spinor Representations

In essence, spinors are nothing more than the vectors in specific representations of the Lie algebras $\mathfrak{so}(p,q)$ or, equivalently, of the Lie groups $SO(p,q)$. These representations are called spinor representations and they cannot be obtained from tensor products of the fundamental representation. The fields that are described by spinors are called fermions. To obtain spinor representations we first construct representations of the Clifford algebra, from which we can extract the spinor representations.

C.1.1 Clifford Algebra Representations

In a flat space-time[1] with signature (t,s), where s (t) denotes the number of positive (negative) eigenvalues of the metric η_{ab}, the Clifford algebra is an algebra generated by $d=s+t$ objects Γ_a, called generators, with the multiplication rule:

$$\Gamma_a\Gamma_b+\Gamma_b\Gamma_a=2\eta_{ab}\,. \tag{C.1.1}$$

We restrict ourselves first to space-times with signature $(0,s+t)$, since multiplying t generators Γ_a's by i we obtain a Clifford algebra for signature (t,s).

[1] We first develop the theory for flat space-times and later make the link with curved space-times.

A (complex) representation of the Clifford algebra is provided once we have a (complex) matrix representation for every generator Γ_a such that C.1.1 is satisfied. By multiplying these matrices the whole representation is generated.

In a first step we build a finite group from the generators, since then we can apply the well-known representation theory of finite groups. To this end, we first introduce some notation: $\Gamma_{a_1 \ldots a_p}$ is zero unless the a_i are all different and in then $\Gamma_{a_1 \ldots a_p} = \Gamma_{a_1} \cdots \Gamma_{a_p}$[2]. The generators Γ_a generate the finite group C_d with elements:

$$C_d = \{\pm 1, \pm\Gamma_\mu, \pm\Gamma_{\mu_1\mu_2}, \ldots, \pm\Gamma_{\mu_1\ldots\mu_d}\} \,. \tag{C.1.2}$$

The product between two elements of C_d is again in C_d and is completely determined by the relation C.1.1. The number of elements in C_d, called the order of C_d and denoted $\#C_d$, is given by

$$\#C_d = 2\sum_{p=0}^{d} \binom{d}{p} = 2^{d+1} \,. \tag{C.1.3}$$

Before we go into more detail we need to choose whether d is even or odd.

d Even

The number of inequivalent irreducible representations of a finite group equals the number of conjugacy classes. For even d the conjugacy classes of C_d are found to be

$$[+1], [-1], [\Gamma_a], [\Gamma_{a_1a_2}], \ldots, [\Gamma_{a_1\ldots a_d}] \,, \tag{C.1.4}$$

giving $2^d + 1$ inequivalent irreducible representations of C_d.

Now we prove the following lemma (it is little proved in the literature, hence we provide a simple proof).

Lemma C.1.1. The number of inequivalent irreducible one-dimensional representations of a finite group G is given by the order G divided by the order of the commutator subgroup $Com(G)$, which is by definition generated by the elements $aba^{-1}b^{-1}$, for all $a, b \in G$.

Proof. Since $gaba^{-1}b^{-1}g^{-1} = gag^{-1}gbg^{-1}(gag^{-1})^{-1}(gbg^{-1})^{-1}$ the commutator subgroup is a self-conjugate subgroup: $gCom(G)g^{-1} = Com(G)$, $\forall g \in G$. Hence $G/Com(G)$ is a group. The elements of $G/Com(G)$ are the equivalence classes defined in G by $g \sim h$ if and only if there is $a \in Com(G)$ such that $g = ah$.

[2] It is sometimes convenient to think of $\Gamma_{a_a \ldots a_p}$ as the antisymmetrized product, but then we need to sum products, which is not defined in a group.

Take two elements $x, y \in G/Com(G)$: $xy \sim xyy^{-1}x^{-1}yx \sim yx$ and hence $G/Com(G)$ is an abelian group. Hence all irreducible representations are one-dimensional[3]. Since $G/Com(G)$ is abelian, the number of conjugacy classes in $G/Com(G)$ equals the order of $G/Com(G)$. Therefore there are $\#(G/Com(G)) = \#G/\#Com(G)$ one-dimensional inequivalent irreducible representations of $G/Com(G)$. The lemma is thus proved if we have proved that there is a one-to-one correspondence between the irreducible representations of $G/Com(G)$ and the one-dimensional irreducible representations of G.

Let ρ be an irreducible representation of $G/Com(G)$. Then it is one-dimensional and we can extend it to an irreducible one-dimensional representation of G (it is irreducible since it is one-dimensional) if we put $\rho = 1$ on $Com(G)$. But if we have a one-dimensional irreducible representation σ of G then automatically we have $\sigma(aba^{-1}b^{-1}) = 1$. Thus $\sigma = 1$ on $Com(G)$ and σ is an irreducible representation of $G/Com(G)$.

□

For C_d we have $Com(C_d) = \{-1, +1\}$ and thus there is precisely one irreducible representation of C_d that is not one-dimensional and we denote it $\rho_s(C_d)$. For spinor representations we do not need the one-dimensional representations and hence we discard them.

To get the dimension n of $\rho_s(C_d)$ we use that for a finite group G we have $\#G = \sum_\rho (n_\rho)^2$, where n_ρ is the dimension of the representation ρ and the sum is over all inequivalent irreducible representations. For C_d we have

$$2^{d+1} = 2^d \cdot 1^2 + n^2 \Rightarrow n = 2^{(d/2)} \,. \tag{C.1.5}$$

As an aside, for $d = 2$ one finds $n = 2$ and the representation is generated by two out of the three Pauli matrices. The reader is motivated to check the statements of section C.1.2 for this Clifford algebra.

d Odd

The conjugacy classes are now given by:

$$[+1], [-1], [\Gamma_a], \ldots, [\Gamma_{a_1 \ldots a_{d-1}}], [\Gamma_{a_1 \ldots a_d}], [-\Gamma_{a_1 \ldots a_d}] \,. \tag{C.1.6}$$

Hence there are $2^d + 2$ irreducible inequivalent representations. We have $Com(C_d) = \{-1, +1\}$ and hence there are 2 inequivalent irreducible representations of dimension greater than one. The element $\Gamma_{12\cdots d}$ commutes with every other element and according to Schur's lemma is diagonal, so we write $\Gamma_{12\cdots d} = c\mathbb{1}$ and we have $\Gamma_{12\cdots d-1} = c\Gamma_d$,

[3] The reader is warned that this statement follows from Schurs' Lemma, which only holds for *complex* representations.

which can be squared to give:

$$(\Gamma_{12\cdots d-1})^2 = (-1)^{\frac{d(d-1)}{2}} = (-1)^{\frac{d-1}{2}} = c^2\,. \qquad \text{(C.1.7)}$$

The second equality follows since d is odd. Hence if $d = 1(mod 4)$ we have $c = \pm 1$ and if $d = 3(mod 4)$ we have $c = \pm i$. The two choices correspond to the two inequivalent irreducible representations of dimension greater than one. So, in odd dimensions, the last matrix Γ_d is given by a multiple of the product of the others, which form a Clifford algebra for $d' = d - 1$, which is even. Hence the Clifford algebra representations are obtained from the unique Clifford algebra in one dimension lower and hence the dimension of the representation is $n = 2^{\frac{d-1}{2}}$. So for any dimension (odd or even) we have $n = 2^{[\frac{d}{2}]}$.

C.1.2 Properties of Clifford Algebra Representations

We now briefly turn to some special properties of the Clifford algebra representations while focussing on the case d even, since the case d odd is quickly recovered from the Clifford algebra representation of one dimension lower. We focus on zero ($t = 0$) and one ($t = 1$) time-like directions, i.e. on space-time signatures $(0, d)$ and $(1, d - 1)$, since these are of special interest to us. For $t = 1$ the direction $a = 1$ correspond to the time-like direction, hence $(\Gamma_1)^2 = -\mathbb{1}$.

For a finite group every representation is equivalent to a unitary representation and hence we may assume the Γ-matrices to be unitary. From relation C.1.1 we have:

$$\begin{aligned} \Gamma_a^\dagger &= \Gamma_a\,, \quad t = 0\,, \\ \Gamma_a^\dagger &= \Gamma_1 \Gamma_a \Gamma_1\,, \quad t = 1\,. \end{aligned} \qquad \text{(C.1.8)}$$

The matrices[4] Γ_a^T and $-\Gamma_a^T$ also represent the Clifford algebra and hence by uniqueness of the representation $\rho_s(C_d)$ there should be matrices $C_\pm$ such that

$$\Gamma_a^T = C_+ \Gamma_a C_+^{-1}\,, \quad -\Gamma_a^T = C_- \Gamma_a C_-^{-1}\,. \qquad \text{(C.1.9)}$$

The matrices $C_\pm$ are called charge conjugation matrices[5].

The matrices $C_\pm$ have symmetry and unitarity properties that are independent of a choice of basis and using a particular basis one can show (see e.g. [45]):

$$C_\pm^T = (-1)^{\frac{d(d\mp 2)}{8}} C_\pm\,, \quad C_\pm^\dagger C_\pm = \mathbb{1}\,. \qquad \text{(C.1.10)}$$

Using equation C.1.10 one can obtain the symmetry properties of $C\Gamma_{a_1 \ldots a_p}$[6].

[4] The matrix representing Γ_a are also denoted Γ_a to avoid clumsy notation.
[5] The choice of name is not very satisfactory. See also [45].
[6] The Γ-matrices itself have no definite symmetry properties.

The matrices $\pm\Gamma_a^*$ also represent the Clifford algebra and hence by uniqueness of the representation $\rho_s(C_d)$ there should be matrices $B_\pm$ such that

$$\Gamma_a^* = B_+ \Gamma_a B_+^{-1}\,, \quad -\Gamma_a^* = B_- \Gamma_a B_-^{-1}\,. \tag{C.1.11}$$

Since $\Gamma_a^* = (\Gamma^\dagger)^T$ we have:

$$\begin{aligned} \Gamma_a C_\pm^{-1} B_\pm &= C_\pm^{-1} B_\pm \Gamma_a\,, \quad t=0\,,\\ B_\mp^{-1}\left(C_\pm \Gamma_1\right)^T \Gamma_a &= \Gamma_a B_\mp^{-1}\left(C_\pm\Gamma_1\right)^T\,, \quad t=1\,. \end{aligned} \tag{C.1.12}$$

Using Schur's lemma we see that $C_\pm^{-1}B_\pm$ and $B_\mp^{-1}\left(C_\pm\Gamma_1\right)^T$ must be diagonal for $t=0$ respectively $t=1$. Since $B_\pm$ is defined up to a phase, we can choose:

$$\begin{aligned} B_\pm &= C_\pm\,, \quad t=0\,,\\ B_\pm^T &= C_\mp\Gamma_1\,, \quad t=1\,. \end{aligned} \tag{C.1.13}$$

Since $C_\pm$ and Γ_1 are unitary, so are $B_\pm$. From the unitarity of $B_\pm$ and the above equations one concludes:

$$\begin{aligned} B_\pm B_\pm^* &= (-1)^{\frac{d(d\mp2)}{8}}\,\mathbb{1}\,, \quad t=0\,,\\ B_\pm B_\pm^* &= \mp(-1)^{\frac{d(d\pm2)}{8}}\,\mathbb{1}\,, \quad t=1\,. \end{aligned} \tag{C.1.14}$$

The chirality matrix is important for the discussion on spinor representations. In d dimensions the chirality matrix is denoted Γ_{d+1} and defined by:

$$\begin{aligned} \Gamma_{d+1} &= (-i)^{\frac{d}{2}}\Gamma_1\cdots\Gamma_d\,, \quad t=0\,,\\ \Gamma_{d+1} &= (-i)^{\frac{d+2}{2}}\Gamma_1\cdots\Gamma_d\,, \quad t=1\,. \end{aligned} \tag{C.1.15}$$

In odd dimensions Γ_{d+1} is proportional to the identity. But in even dimensions it has the important properties that it (1) is Hermitian $\Gamma_{d+1}^\dagger = \Gamma_{d+1}$, (2) squares to one $(\Gamma_{d+1})^2 = 1$ - so it is also unitary - (3) anticommutes with all generators of the Clifford algebra; $\Gamma_\mu\Gamma_{d+1} = -\Gamma_{d+1}\Gamma_\mu$.

We conclude this section on the properties of the Clifford algebra by discussing the Clifford algebra in curved space-times. The Γ-matrices Γ_μ for a curved space-time are obtained from the Minkowskian Γ-matrices Γ_a by contracting with vielbeins: $\Gamma_\mu = e_\mu^a\Gamma_a$. The curved index μ on Γ_μ can be raised with the inverse metric $g^{\mu\nu}$. The matrices $\Gamma_{\mu_1\ldots\mu_p}$ thus transform as tensors under general coordinate transformations. The Minkowskian Γ-matrices Γ_a transform under local Lorentz transformations, but how we discuss in the next section.

C.1.3 Introducing the Spinors

Given a representation of the Clifford algebra in d dimensions, we can form the $\frac{1}{2}d(d-1)$ elements $\Sigma^{ab} = \frac{1}{2}\Gamma^{ab}$, which satisfy

$$[\Sigma^{ab}, \Sigma^{cd}] = \Sigma^{ab}\Sigma^{cd} - \Sigma^{cd}\Sigma^{ab} = \eta^{bc}\Sigma^{ad} + \eta^{ad}\Sigma^{bc} - \eta^{ac}\Sigma^{bd} - \eta^{bd}\Sigma^{ac}\,. \qquad \text{(C.1.16)}$$

Hence the elements Σ^{ab} form a representation of the Lie algebra $\mathfrak{so}(d)$ if $t = 0$ or $\mathfrak{so}(1, d-1)$ if $t = 1$. This representation might be reducible so we investigate how reducible this representation is and split this representation into irreducibles. The 'vectors' in these (irreducible) representations are called (irreducible) spinors and one can prove that no tensor product of any number of vector representations of $\mathfrak{so}(d)$ or $\mathfrak{so}(1, d)$ gives a spinor representation. Even more, one can prove that by taking tensor products of the vector and spinor representations, one obtains *all* representations of the Lie algebra $\mathfrak{so}(d)$ or $\mathfrak{so}(1, d-1)$ (see e.g. [76]).

There are two conditions to be imposed on a spinor to render irreducibility. The first condition is a chirality condition while the second condition is a reality condition.

We first discuss how to obtain the irreducible spinors by applying the above mentioned conditions and then discuss some properties of spinors and their bilinears. Again it is assumed that d is even and we only make a few statements about odd d.

Chiral Fermions

The matrix Γ_{d+1} commutes with all generators $\Sigma_{\mu\nu}$ and hence the spinors can be decomposed into eigenspinors of Γ_{d+1}. Since Γ_{d+1} squares to one its eigenvalues are ± 1 and we can form the projection operators $P_\pm = \frac{1}{2}(\mathbb{1} \pm \Gamma_{d+1})$. The spinor space V splits into two subspaces $V_\pm$ of equal dimension and with zero intersection; $V = V_+ \oplus V_-$ and $\Gamma_{d+1}|_{V_\pm} = \pm\mathbb{1}$.

If a fermion satisfies $\Gamma_{d+1}\psi = \psi$ we call it a fermion of positive chirality and if $\Gamma_{d+1}\psi = -\psi$ we call it a fermion of negative chirality.

In odd dimensions the matrix Γ_{d+1} is a multiple of the identity and thus every spinor has the same chirality.

A chiral fermion - i.e. a fermion of definite chirality - has to be massless. This can be seen as follows. The Dirac equation for a spin $1/2$ spinor ψ with mass m is

$$(\Gamma_\mu g^{\mu\nu}\partial_\nu - m\mathbb{1})\psi = 0\,. \qquad \text{(C.1.17)}$$

Acting on equation C.1.17 with Γ_{d+1} and using $\{\Gamma_\mu, \Gamma_{d+1}\} = 0$ gives

$$(\Gamma_\mu g^{\mu\nu}\partial_\nu + m\mathbb{1})\psi = 0\,, \qquad \text{(C.1.18)}$$

and hence $m\psi = 0$, which can only be satisfied for a nontrivial fermion if $m = 0$.

Chiral fermions (spinors) are also called Weyl fermions (spinors).

Majorana Fermions

The spinor representations are by construction complex (since we used Schurs' Lemma to obtain the representations of the group C_d). Let us investigate the possibility to obtain real representations. We look for a condition of the form $\psi^* = L\psi$ for some matrix L. We now prove that L should be a multiple of $B_\pm$.

The condition $\psi^* = L\psi$ should be invariant under the transformations $\psi \mapsto \Gamma_{\mu\nu}\psi$, which implies that $B_\pm^{-1}L$ commutes with all the $\Gamma_{\mu\nu}$. However, the $\Gamma_{\mu\nu}$ do not generate the whole group C_d (for even d) and we cannot use Schurs' Lemma directly. But the $\Gamma_{\mu\nu}$ generate the Lie subalgebra $\mathfrak{so}(1, d-1)$ for $t = 1$ and $\mathfrak{so}(d)$ for $t = 0$. Hence we can use Schurs' Lemma applied to the spinor representation and acting on *irreducible* spinors the matrix $B_\pm^{-1}L$ should render scalar multiplication: $B_\pm^{-1}L\psi = \alpha\psi$ for a complex number α. This proves the claim and so we put

$$\psi^* = \alpha B_\pm \psi \, . \tag{C.1.19}$$

Demanding $\psi^{**} = \psi$ gives $|\alpha|^2 B_\pm B_\pm^* = \mathbb{1}$ and hence using C.1.14 we have $|\alpha| = 1$ and $B_\pm B_\pm^* = \mathbb{1}$. This thus puts restrictions on the choice of $B_\pm$ and even on space-time dimensions and signatures. For example, if $d = 4$ and $t = 0$ we have $B_\pm B_\pm^* = -\mathbb{1}$ and no reality condition can be imposed on the spinors.

If it is possible to impose C.1.19 the spinor space V splits into two real subspaces. A spinor satisfying $\psi^* = \alpha B_+ \psi$ is called a Majorana spinor and the field it describes is a Majorana fermion and likewise, the spinors satisfying $\psi^* = \alpha B_- \psi$ are called pseudo-Majorana spinors and they describe pseudo-Majorana fermions.

A pseudo-Majorana fermion is necessarily massless since taking the complex conjugate of C.1.17 we obtain:

$$0 = \left(-B_- \Gamma_\mu \partial^\mu B_-^{-1} - m\mathbb{1}\right) B_- \psi = -B_- \left(\Gamma_\mu \partial^\mu + m\mathbb{1}\right) \psi \, . \tag{C.1.20}$$

Hence $m\psi = 0$ and for nontrivial ψ this implies $m = 0$.

Now let us comment on the complex number α. Its value can be fixed using physical arguments; one can fix α by demanding that the action is Hermitian or that the supersymmetry algebra gives rise to a positive definite Hamiltonian. In this thesis we mainly deal with Minkowskian signatures and α can be fixed to be $+i$ following the discussion in Appendix A of [45].

In odd dimensions one uses the representation of the Clifford algebra of one dimension less. The last Γ-matrix is given by $\Gamma_d = c\Gamma_{012\ldots d-1}$, where c is determined up to a sign by $(\Gamma_d)^2 = 1$ and thus depends on t and d. But we have either $c = \pm i$ or $c = \pm 1$. Using $\Gamma_d^* = c^* B_\pm \Gamma_{012\ldots d-1} B_\pm^{-1}$ one sees that in odd dimension we can either work with B_+ if $c = \pm 1$ or with B_- if $c = \pm i$.

Majorana–Weyl Spinors

Sometimes it is possible to impose both a reality condition and a chirality condition. Hence we require $\psi = P_\pm\psi$ and $\psi^* = \alpha B_\pm \psi$. Consistency requires $B_\pm P_\pm = (P_\pm)^* B_\pm$. Using C.1.15 and the fact that we are working with d even, we obtain

$$\begin{aligned} \Gamma^*_{d+1} &= (-1)^{\frac{d}{2}} B_\pm \Gamma_{d+1} B^{-1}_\pm \,, \;\; t=0\,, \\ \Gamma^*_{d+1} &= (-1)^{\frac{d+2}{2}} B_\pm \Gamma_{d+1} B^{-1}_\pm \,, \;\; t=1\,. \end{aligned} \tag{C.1.21}$$

We see that the chirality condition and the reality condition can only imposed simultaneously if $d/2$ is even for $t = 0$ and if $\frac{d+2}{2}$ for $t = 1$, but the reality constraint ($B_\pm B^*_\pm = \mathbb{1}$) and chirality constraint (d even) still have to be satisfied too. A spinor that is both (pseudo-) Majorana and Weyl, is called a Majorana–Weyl spinor.

When Majorana–Weyl spinors exist, both B_+ and B_- can be used to impose the reality condition C.1.19. One can show that the choice of $B_\pm$ can be traded for the choice of $B_\mp$ by simply redefining α (see e.g. [45]).

Dirac and Majorana Conjugates

In space-times with signature $t = 1$, the most interesting for our purposes, one defines the Dirac conjugate $\bar{\psi}_D$ of a spinor ψ by

$$\bar{\psi}_D \equiv \psi^\dagger \Gamma_1 \alpha^{-1}\,. \tag{C.1.22}$$

The Majorana conjugate $\bar{\psi}_M$ is defined as

$$\bar{\psi}_M = \psi^T C_\pm\,. \tag{C.1.23}$$

Hence the reality condition C.1.19 can be rephrased as $\bar{\psi}_M = \bar{\psi}_D$.

Counting Degrees of Freedom

We now show briefly how the count on-shell degrees of freedom for fermions in space-times with signature $t = 1$. As usual, to count degrees of freedom we ignore any interactions and consider a freely moving particle.

Suppose ψ describes a spin 1/2 fermion and ψ satisfies the massless Dirac equation: $\Gamma_\mu \partial^\mu \psi = 0$. We choose a frame in which the particle has momentum $k_\mu = (E, E, 0, \ldots, 0)$, which is a null-vector; $k^2 = 0$. In momentum-space the Dirac equation reads $-E\Gamma_1(\mathbb{1} - \Gamma_1\Gamma_2)\psi = 0$. But $\frac{1}{2}(\mathbb{1} - \Gamma_1\Gamma_2)$ is a projection operator and we have $\mathbb{1} = \frac{1}{2}(\mathbb{1} - \Gamma_1\Gamma_2) + \frac{1}{2}(\mathbb{1} + \Gamma_1\Gamma_2)$ so that the equation of motion projects out half of the off-shell degrees of freedom for a massless fermion.

Now consider a massive spin 1/2 fermion λ satisfying $(\Gamma_\mu \partial^\mu - m\mathbb{1})\lambda = 0$. We can go to the rest frame in which the momentum reads $k_\mu = (m, 0, \ldots, 0)$. The

Dimension	$c(d)$	pM, M, W, MW
2	0	MW
4	2	M or W
6	4	W
8	8	pM or W
10	8	MW
12	32	M or W

Table C.1.1: Properties of spin 1/2 fermions in even space-times with signature $t = 1$. The on-shell degrees of freedom are called $c(d)$ and pM, M, W and MW mean pseudo-Majorana, Majorana, Weyl and Majorana–Weyl respectively.

Dirac equation in momentum-space reads $(im\Gamma_1 - m\mathbb{1})\lambda = 0$. Again we recognize in $\frac{1}{2}(\mathbb{1} - i\Gamma_1)$ a projection operator that precisely splits the spinor space in two parts of equal size since $\mathbb{1} = \frac{1}{2}(\mathbb{1} - i\Gamma_1) + \frac{1}{2}(\mathbb{1} + i\Gamma_1)$. Hence also the massive Dirac equation projects out half of the off-shell degrees of freedom.

Now we need to know the off-shell degrees of freedom for a spin 1/2 fermion. But this is just the number of independent *real* components of a spinor. Since the representation $\rho_s(C_d)$ of the Clifford algebra for even d is of complex dimension $2^{d/2}$, an unconstrained spinor has $2 \cdot 2^{d/2}$ real components. A Weyl fermion and a (pseudo-)Majorana fermion thus have $2^{d/2}$ real components for even d. A Majorana–Weyl fermion thus has $\frac{1}{2}2^{d/2}$ real components. For even d we present the on-shell degrees of freedom, denoted $c(d)$, in table C.1.3.

Local Lorentz Transformations

The local Lorentz transformations are already described briefly in section 2.1.2. We now provide some extra details.

We fix $t = 1$, take $a = 1$ to be the time-like direction and we take for the space-like directions the indices $i, j, k, \ldots$ to run from 2 to d. For the collective of tangent space indices we take the indices $a, b, \ldots$ running from 1 to d. The Σ_{1i} are Hermitian while the Σ_{ij} are anti-Hermitian.

A fermion ψ transforms under infinitesimal local Lorentz transformation as

$$\delta\psi = \omega^{ab}\Sigma_{ab}\psi\,, \tag{C.1.24}$$

where $\omega^{ab} = -\omega_{ba}$ parameterize the local Lorentz transformations and can depend on the space-time coordinates.

The Dirac conjugate transforms as

$$\delta\bar{\psi}_D = -\bar{\psi}_D\omega^{ab}\Sigma_{ab}\,. \tag{C.1.25}$$

Hence $\bar{\psi}_D\chi$ is Lorentz-invariant.

The Γ-matrices are linear transformations in the spinor space and hence if we rotate the spinors $\psi \mapsto U\psi$ the Γ-matrices should transform as matrices: $\Gamma_a \mapsto U\Gamma_a U^{-1}$. But the Γ-matrices Γ_a are not just matrices; they also transform as co-vectors under local Lorentz transformations. Hence under an infinitesimal local Lorentz transformation with parameters ω^{ab} the Γ-matrices should transform both as a matrix and as a co-vector, that is, as follows

$$\delta\Gamma^a = \omega^{cd}\left(J_{cd}\right)^a{}_b\Gamma^b + \omega^{cd}\Sigma_{cd}\Gamma_a - \Gamma_a\omega^{cd}\Sigma_{cd}\,, \tag{C.1.26}$$

where the J_{cd} are the matrices in the vector representation of $\mathfrak{so}(1, d-1)$ as defined in section 2.1.2: $(\Sigma_{cd})^a{}_b = \delta^a_c\eta_{db} - \delta^a_d\eta_{bc}$. Using this explicit expression and the identity $\Gamma^{ab}\Gamma^c = \Gamma^{abc} - \eta^{bc}\Gamma^a + \eta^{ac}\Gamma^b$ the above expression for the variation of the matrix Γ^a vanishes; $\delta\Gamma^a = 0$.

From the transformation rules of the spinors and the Γ-matrices under local Lorentz transformations we see that the bilinear $\bar{\psi}_D\Gamma^{a_1\ldots a_p}{}_{b_1\ldots b_q}\chi$ transforms as a (p, q)-tensor under local Lorentz transformations.

The group generated by the Σ_{ab} is not the group $SO(1, d-1)$ but its double cover, which is called $Spin(1, d-1)$. More generally, for space-time signatures (t, s) the generators Σ_{ab} generate the double cover of $SO(t, s)$, which is called $Spin(t, s)$.

Grassmann Numbers

Fermions are fields that describe particles with half-integer spin. Fermions should obey Pauli statistics and this implies that the components of a fermion should be Grassmann numbers, i.e. anticommuting numbers. Writing spinorial indices down, we thus have $\psi_\alpha\chi_\beta = -\chi_\beta\psi_\alpha$.

Taking Hermitian conjugates of products of Grassmann numbers a and b is defined by $(ab)^\dagger = b^\dagger a^\dagger$, whereas complex conjugation is defined by $(ab)^* = a^*b^*$.

For Majorana fermions one writes $\bar{\psi} = \bar{\psi}_D$ and we have:

$$\bar{\psi}\Gamma_{\mu_1\ldots\mu_p}\chi = \sum_{\alpha,\beta}\psi_\alpha\left(C_\pm\Gamma_{\mu_1\ldots\mu_p}\right)_{\alpha\beta}\chi_\beta\,. \tag{C.1.27}$$

Using C.1.10 we obtain

$$\left(C_\pm\Gamma_{\mu_1\ldots\mu_p}\right)^T = (\pm 1)^p(-1)^{\frac{p(p-1)}{2}}(-1)^{\frac{d(d\mp 2)}{8}}C_\pm\Gamma_{\mu_1\ldots\mu_p}\,, \tag{C.1.28}$$

from which it follows that

$$\psi C_\pm\Gamma_{\mu_1\ldots\mu_p}\chi = -(\pm 1)^p(-1)^{\frac{p(p-1)}{2}}(-1)^{\frac{d(d\mp 2)}{8}}\chi C_\pm\Gamma_{\mu_1\ldots\mu_p}\psi\,. \tag{C.1.29}$$

The extra minus sign is due to the Grassmanian nature of the spinors. Equation C.1.29 implies that, in a given dimension and for a fixed choice of $C_\pm$, for some values of p the bilinears $\bar{\psi}\Gamma_{\mu_1\ldots\mu_p}\psi$ vanish when using Majorana fermions.

Fierz Rearrangements

Given the nontrivial representation $\rho_s(C_d)$ of the Clifford algebra, the set of matrices

$$\{\mathbb{1}, \Gamma_\mu, \Gamma_{\mu_1\mu_2}, \ldots, \Gamma_{\mu_1\ldots\mu_d}\} \tag{C.1.30}$$

form a set of 2^d matrices that are linearly independent and form a basis for the set of complex $2^{d/2} \times 2^{d/2}$-matrices. Let us prove this claim.

First we observe that $\mathrm{Tr}\Gamma_\mu = -\mathrm{Tr}C_-\Gamma C_-^{-1} = -\mathrm{Tr}\Gamma$ and thus $\mathrm{Tr}\Gamma_\mu = 0$. Using the cyclic property of the trace we see that $\mathrm{Tr}\Gamma_{\mu_1\ldots\mu_p}$ vanishes for even $p \geq 2$;

$$\mathrm{Tr}\Gamma_{\mu_1\ldots\mu_p} = \mathrm{Tr}\Gamma_{\mu_2\ldots\mu_p\mu_1} = (-1)^{p-1}\mathrm{Tr}\Gamma_{\mu_1\ldots\mu_p}\,. \tag{C.1.31}$$

Now take p odd and use $\mathrm{Tr}X = \mathrm{Tr}X^T$ for any matrix X to obtain:

$$\mathrm{Tr}\Gamma_{\mu_1\ldots\mu_p} = (-1)^p\mathrm{Tr}C_-\Gamma_{\mu_p\ldots\mu_p}C_-^{-1} = -\mathrm{Tr}\Gamma_{\mu_p\ldots\mu_1}\,, \tag{C.1.32a}$$

$$\mathrm{Tr}\Gamma_{\mu_1\ldots\mu_p} = \mathrm{Tr}C_+\Gamma_{\mu_p\ldots\mu_1}C_+^{-1} = \mathrm{Tr}\Gamma_{\mu_p\ldots\mu_p}\,. \tag{C.1.32b}$$

But we should have C.1.32a=C.1.32b and thus $\mathrm{Tr}\Gamma_{\mu_1\ldots\mu_p} = 0$ for all $p > 0$. Hence we have the identity

$$\mathrm{Tr}\Gamma_{\mu_1\ldots\mu_p}\Gamma^{\nu_1\ldots\nu_q} = (-1)^{[\frac{p}{2}]}\delta_{p,q}\delta^{\nu_1\ldots\nu_q}_{\mu_1\ldots\mu_p}\mathrm{Tr}\mathbb{1}\,. \tag{C.1.33}$$

Where we used $(-1)^{\frac{p(p-1)}{2}} = (-1)^{[\frac{p}{2}]}$. The identity C.1.33 implies that all $\Gamma_{\mu_1\ldots\mu_p}$ are linearly independent; if $\sum c_{\mu_1\ldots\mu_p}\Gamma_{\mu_1\ldots\mu_p}$ has to vanish then by taking trace with $\Gamma_{\nu_1\ldots\nu_q}$ the coefficient $c_{\nu_1\ldots\nu_1}$ has to vanish and hence all coefficients $c_{\mu_1\ldots\mu_p}$ vanish. Since there are 2^d of the matrices $\Gamma_{\mu_1\ldots\mu_p}$ they form a complete basis for the $2^{d/2}\times 2^{d/2}$-matrices.

The bilinear $\psi\bar{\chi}$ is a linear transformation mapping a spinor λ to the spinor $(\bar{\chi}\lambda)\psi$ and hence is a matrix and can thus be expanded in terms of the $\Gamma_{\mu_1\ldots\mu_p}$ where the coefficients are found by taking appropriate traces. If we write

$$\psi\bar{\chi} = \sum c^{\mu_1\ldots\mu_p}\Gamma_{\mu_1\ldots\mu_p}\,, \tag{C.1.34}$$

we have

$$c^{\mu_1\ldots\mu_p} = \frac{1}{2^{d/2}}\frac{(-1)^{[p/2]}}{p!}\mathrm{Tr}(\psi\bar{\chi}\Gamma^{\mu_1\ldots\mu_p}) = -\frac{1}{2^{d/2}}\frac{(-1)^{[p/2]}}{p!}\bar{\chi}\Gamma^{\mu_1\ldots\mu_p}\psi\,. \tag{C.1.35}$$

An equation of this kind is called a Fierz rearrangement or a Fierz identity.

C.2 Spinor Representation of $\mathfrak{so}(1,3)$

In Minkowski space-times it is often conventional to let the space-time indices run from 0 to $d-1$ and to take $\mu = 0$ to be the time-like direction.

From the previous section we obtain: $B_\pm B_\pm^* = \pm\mathbb{1} C_\pm^T = -C_\pm$ and no Majorana–Weyl fermions exist. One can choose which charge conjugation matrix to work with. When discussing four-dimensional supergravities we have chosen to work with C_- and with $B_+ = (-C_-\Gamma_0)^T = -C_-\Gamma_0$. Often the suffixes $+$ and $-$ have then been dropped.

The matrices $C\Gamma_{\mu_1\ldots\mu_p}$ are symmetric for $p = 1,2$ and antisymmetric for $p = 0,3,4$. Hence for Majorana fermions ψ, χ we have the following swop identities:

$$\bar{\psi}\Gamma_{\mu_1\ldots\mu_p}\chi = (-1)^{\left[\frac{p+1}{2}\right]}\bar{\chi}\Gamma_{\mu_1\ldots\mu_p}\psi \,. \tag{C.2.1}$$

The chirality matrix $\Gamma_5 = i\Gamma_{0123}$ satisfies: $\Gamma_5^T = C\Gamma_5 C^{-1}$ and $\Gamma_5^* = C\Gamma_5 C^{-1}$. Hence Γ_5 is indeed Hermitian.

An explicit representation is found by putting:

$$\begin{aligned} \Gamma_0 &= i\sigma_1 \otimes \mathbb{1}\,, \quad \Gamma_1 = \sigma_2 \otimes \mathbb{1}\,, \\ \Gamma_3 &= \sigma_3 \otimes \sigma_1\,, \quad \Gamma_4 = \sigma_3 \otimes \sigma_2\,, \end{aligned} \tag{C.2.2}$$

where the σ_i are the standard Pauli matrices and $\mathbb{1}$ is the 2×2 unit matrix. In this representation we have $\Gamma_5 = \sigma_3 \otimes \sigma_3$, which is given explicitly by

$$\Gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{C.2.3}$$

Since Γ_5 is diagonal, this basis is sometimes called a chiral representation.

There exists a real representation given by:

$$\begin{aligned} \Gamma_0 &= i\sigma_2 \otimes \mathbb{1}\,, \quad \Gamma_1 = \sigma_3 \otimes \mathbb{1}\,, \\ \Gamma_3 &= \sigma_1 \otimes \sigma_1\,, \quad \Gamma_4 = \sigma_1 \otimes \sigma_3\,. \end{aligned} \tag{C.2.4}$$

C.3 Spinor Representation of $\mathfrak{so}(6)$

In d-dimensional space-times with Euclidean signature $t = 0$ it is conventional to let the space-time indices run from 1 to d.

From section C.1 one finds that for the spinor representation of $\mathfrak{so}(6)$ we have: $C_\pm^T = \mp C_\pm$, $B_\pm B_\pm^* = \mp\mathbb{1}$ and no Majorana–Weyl fermions exist.

A nice representation can be given explicitly in terms of the 't Hooft symbols introduced in section A.3.2:

$$\Gamma_a = \begin{pmatrix} 0 & i(G_a)^{ij} \\ i(G_a)_{ij} & 0 \end{pmatrix} . \tag{C.3.1}$$

The Γ-matrices defined in this way satisfy $\Gamma_a^* = \Gamma_a^T$ and the symmetric charge conjugation matrix C_- is found to be given by:

$$C_- = B_- = \begin{pmatrix} 0 & \delta^{ij} \\ \delta_{ij} & 0 \end{pmatrix} . \tag{C.3.2}$$

The chirality matrix is found to be

$$\Gamma_7 = i\Gamma_{123456} = \begin{pmatrix} \delta^i{}_j & 0 \\ 0 & -\delta_i{}^j \end{pmatrix} , \tag{C.3.3}$$

which satisfies $\Gamma_7^T = -C_\pm \Gamma_7 C_\pm^{-1}$.

C.4 Spinor Representation of $\mathfrak{so}(1,9)$

From section C.1 one finds that for the spinor representation of $\mathfrak{so}(1,9)$ we have: $C_\pm^T = \pm C_\pm$, $B_\pm B_\pm^* = \mathbb{1}$ and Majorana–Weyl fermions exist.

We have the following symmetry properties:

$$\begin{aligned} (C_-\Gamma_{\mu_1\ldots\mu_p})^T &= -(-1)^{\frac{p(p+1)}{2}} C_-\Gamma_{\mu_1\ldots\mu_p} , \\ (C_+\Gamma_{\mu_1\ldots\mu_p})^T &= (-1)^{\frac{p(p-1)}{2}} C_+\Gamma_{\mu_1\ldots\mu_p} . \end{aligned} \tag{C.4.1}$$

We have chosen to work with C_- for spinors in ten dimensions and the Majorana conjugate is defined by $\bar\psi = \psi^T C_-$.

Since $P_\pm^T = C_- P_\mp C_-^{-1}$ the bilinear $\bar\psi \Gamma_{\mu_1 \cdot \mu_p}\chi$ vanishes when (1) ψ and χ have the same chirality and p is even or when (2) ψ and χ have different chirality and p is odd.

Since the fermions are Majorana–Weyl, an appropriate basis of matrices for performing Fierz transformations is given by

$$\{P_\pm, \Gamma_\mu P_\pm, \Gamma_{\mu\nu} P_\pm, \Gamma_{\mu\nu\rho} P_\pm, \Gamma_{\mu\nu\lambda\rho} P_\pm, \Gamma_{\mu\nu\lambda\rho\sigma} P_\pm\} . \tag{C.4.2}$$

We have $\mathrm{Tr} P_\pm = 16$. For example, if ψ and χ both have positive chirality we have

$$\chi\bar\psi = -\tfrac{1}{16}\bar\psi\Gamma^\mu\chi\Gamma_\mu P_- - \tfrac{1}{64}\bar\psi\Gamma^{\mu\nu\lambda}\chi\Gamma_{\mu\nu\lambda} P_- - \tfrac{1}{1920}\bar\psi\Gamma^{\mu\nu\lambda\rho\sigma}\chi\Gamma_{\mu\nu\lambda\rho\sigma} P_- . \tag{C.4.3}$$

Some useful identities are

$$\begin{aligned} \mathrm{Tr}\Gamma^{\mu_1\cdots\mu_p}\Gamma_{\nu_1\cdot\nu_p} &= 16\delta^{\mu\cdots\mu_p}_{\nu\cdots\nu_p}\,, \\ \Gamma^{\mu}\Gamma_{\rho}\Gamma_{\mu} &= -8\Gamma_{\rho}\,, \\ \Gamma^{\mu\nu}\Gamma^{\rho}\Gamma_{\nu} &= 7\Gamma^{\rho}\Gamma_{\mu} - 16\delta^{\rho}_{\mu}\,, \\ \Gamma^{\mu}\Gamma_{\nu_1\cdots\nu_5}\Gamma_{\mu} &= 0\,. \end{aligned} \tag{C.4.4}$$

Using the Γ-matrices of $\mathfrak{so}(1,3)$ and $\mathfrak{so}(6)$ the Γ-matrices of $\mathfrak{so}(1,9)$ can be constructed by taking direct products. To this purpose we write the $\mathfrak{so}(1,3)$ Γ-matrices with small γ and we write the $\mathfrak{so}(6)$ Γ-matrices with Greek capital Γ, while writing capital hatted Γ's for the $\mathfrak{so}(1,9)$ Γ matrices. The matrices $\hat{\Gamma}$ can be represented by

$$\begin{aligned} \hat{\Gamma}_{\mu} &= \gamma_{\mu} \otimes \mathbb{1}\,, \quad \mu = 0,1,2,3 \\ \hat{\Gamma}_{\mu} &= \gamma_5 \otimes \Gamma_{\mu}\,, \quad \mu = 4,\ldots,9\,. \end{aligned} \tag{C.4.5}$$

The charge conjugation matrix $\mathcal{C} = C_-^{(10)}$ of $\mathfrak{so}(1,9)$ can be taken to be given in terms of the $\mathfrak{so}(1,3)$ and $\mathfrak{so}(6)$ charge conjugation matrices $C_-^{(4)}$ and $C_-^{(6)}$ respectively through:

$$\mathcal{C} = C_-^{(4)} \otimes C_-^{(6)}\,. \tag{C.4.6}$$

One checks that $\mathcal{C}$ is antisymmetric and that we have

$$\begin{aligned} (\gamma_{\mu} \otimes \mathbb{1})^T &= -C_-^{(4)} \otimes C_-^{(6)} \cdot \gamma_{\mu} \otimes \mathbb{1} \cdot (C_-^{(4)} \otimes C_-^{(6)})^{-1} = \gamma_{\mu}^T \otimes \mathbb{1}\,, \\ (\gamma_5 \otimes \Gamma_{\mu})^T &= -C_-^{(4)} \otimes C_-^{(6)} \cdot \gamma_5 \otimes \Gamma_{\mu} \cdot (C_-^{(4)} \otimes C_-^{(6)})^{-1} = \gamma_5^T \otimes \Gamma_{\mu}^T\,, \end{aligned} \tag{C.4.7}$$

where we used $\gamma_5^T = C_-^{(4)}\gamma_5 C_-^{(4)-1}$.

Similarly for the $\mathfrak{so}(1,9)$ B_+-matrix, denoted $\mathcal{B}$, one finds

$$\mathcal{B} = -\mathcal{C}\hat{\Gamma}_0 = C_-^{(4)} \otimes C_-^{(6)} \cdot \gamma_0 \otimes \mathbb{1} = B_+^{(4)} \otimes B_-^{(6)}\,, \tag{C.4.8}$$

from which it follows that

$$\mathcal{B}\mathcal{B}^* = \mathbb{1}\,, \quad \mathcal{B}^T = \mathcal{B}\,, \quad \mathcal{B}\mathcal{C}^*\mathcal{B} = \mathcal{C}\,. \tag{C.4.9}$$

We finish this discussion on $\mathfrak{so}(1,9)$ spinors by mentioning a few on the spinors of $\mathfrak{so}(1,10)$ since they are of interest for eleven-dimensional supergravity.

In eleven dimensions there are clearly no Weyl fermions, but there are Majorana fermions. The swopping properties are the same as in ten dimensions. This is due to the fact that the eleven-dimensional spinor representations are similar to ten-dimensional spinor representation, and hence one can use the same charge conjugation matrix. The eleventh Γ-matrix is given by plus or minus the ten-dimensional chirality matrix. The choice for plus or minus corresponds to the two inequivalent representations of the Clifford algebra in eleven dimensions.

Appendix D

Publications

- [A] M. de Roo, D. B. Westra and S. Panda, *De Sitter solutions in N = 4 matter coupled supergravity*, JHEP **0302** (2003) 003 [arXiv:hep-th/0212216].
- [B] M. de Roo, D. B. Westra, S. Panda and M. Trigiante, *Potential and mass-matrix in gauged N = 4 supergravity*, JHEP **0311** (2003) 022 [arXiv:hep-th/0310187].
- [C] M. de Roo, M. G. C. Eenink, D. B. Westra and S. Panda, *Group manifold reduction of dual N = 1 d = 10 supergravity*, JHEP **0506** (2005) 077 [arXiv:hep-th/0503059].
- [D] D. B. Westra and W. Chemissany, *Coset symmetries in dimensionally reduced heterotic supergravity*, JHEP **0602** (2006) 004 [arXiv:hep-th/0510137].
- [E] M. de Roo, S. Panda and D. B. Westra, *Gauging CSO groups in N = 4 supergravity*, To Appear,
- [F] J. Hartong, A. Ploegh, T. Van Riet and D. B. Westra, *Dynamics of Generalized Assisted Inflation*, Class. Quantum Grav. **23** (2006), [arXiv:gr-qc/0602077],
- [G] T. Van Riet, J. Rosseel and D. B. Westra, *Scaling Cosmologies of Gauged Maximal Supergravity*. To Appear.

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Nederlandse Samenvatting

Dit proefschrift behandelt enkele aspecten van zogeheten vierdimensionale $\mathcal{N} = 4$ supergravitatie theorieën en daarbij wordt de nadruk gelegd op de rol die symmetrieën spelen. In de hieronder volgende tekst zal ik kort proberen uit te leggen wat de hiervoor genoemde zin betekent en kort uitleggen welke onderzoekingen in het proefschrift staan beschreven.

De theoretische natuurkunde probeert de natuur om ons heen op een quantitatieve manier zo correct mogelijk te beschrijven en voorspellingen te doen. Een belangrijk ingrediënt hierbij is wiskunde, de taal voor het doen van berekeningen. Om op een correcte manier een voorspelling te kunnen doen (bijvoorbeeld, voorspellen hoe zwaar een elektron is), is het natuurlijk van belang dat de wiskunde erachter kloppend is; er mogen niet voor een bepaalde grootheid twee of meerdere antwoorden zijn, maar ook is het onwenselijk dat het antwoord volledig onzinnig is. Er moet uit de theorie op eenduidige manier één quantitatief zinvol antwoord volgen. Wiskundige consistentie is dus een belangrijke eis voor natuurkundige theorieën. Voor een leek klinkt het misschien iets als een ver-van-mijn-bed show, want hoe krijg je nu ergens twee antwoorden uit een theorie, of hoe kan een antwoord nou niet zinvol zijn? De natuurkunde heeft in ieder geval de laatste honderd jaar laten zien dat zulke situaties voorkomen. Sommige grootheden bleken verkeerd voorspeld te kunnen worden; uit sommige theorieën volgde namelijk dat het elektron oneindig zwaar zou zijn - hier bleek het probleem vooral daarin te liggen dat de natuurkunde achter de massa van het elektron nog niet correct begrepen was. Wiskundige consistentie is dus niet gegarandeerd, maar moet geverifieerd worden! Dit is de reden waarom binnen de theoretische natuurkunde en dus ook in dit proefschrift veel aandacht aan de wiskunde wordt besteed.

De theoretische natuurkunde is de afgelopen honderd jaar in een stroomversnelling terecht gekomen. Einstein ontwikkelde de theorieën van speciale en algemene relativiteitstheorie in het begin van de twintigste eeuw. Met deze theorieën werd onze huidige visie op ruimte en tijd bepaald; ruimte en tijd waren niet langer statische altijd aanwezige entiteiten, maar werden dynamische objecten, die konden veranderen volgens nauwkeurig beschreven regels. Ook werd in de twintigste eeuw de quantummechanica ontwikkeld, die fundamenteel nieuwe inzichten opleverde over de kleinste bouwstenen van de natuur, de elementaire deeltjes, zoals het elektron. Na de ontwikkeling van de quantumveldentheorie, die gezien kan worden als een huwelijk tussen speciale relativiteitstheorie en quantummechanica, was het mogelijk om het Standaardmodel te formuleren.

Het Standaardmodel beschrijft eigenschappen van de elementaire deeltjes en de wisselwerkingen tussen de elementaire deeltjes en is een hoeksteen geworden in de hedendaagse natuurkunde. Volgens het model zijn er twee typen deeltjes, bosonen en fermionen, en deze twee typen hebben duidelijk verschillende eigenschappen. De voorspellingen van het Standaardmodel werden in de beginjaren geplaagd door (wiskundige en natuurkundige) inconsistenties, maar deze konden uiteindelijk worden opgelost[1] en de quantitatieve antwoorden behoren tot de waarden die het meest nauwkeurig experimenteel zijn geverifieerd binnen de gehele natuurwetenschappen.

Met de komst van het Standaardmodel is de natuurkunde nog niet af. Om maar eens één van de grootste tekortkomingen te noemen, het model bevat niet de zwaartekracht, de kracht die ons aan de aarde vastbindt. Om de zwaartekracht in te bouwen in het Standaardmodel blijkt geen gemakkelijke opgave te zijn. Vele natuurkundigen denken dan ook dat er fundamenteel nieuwe ideeën nodig zijn om een theorie te maken die zowel dezelfde voorspellingen kan doen als het Standaardmodel alsmede de zwaartekracht bevatten. Een kanshebber om zo'n nieuw idee te zijn is de snarentheorie. De snarentheorie heeft als aanname dat de fundamentele deeltjes niet een punt zijn, maar een dimensie hebben; de fundamentele deeltjes worden voorgesteld als kleine ééndimensionale objecten, de snaren. Het blijkt dat de snarentheorie op een natuurlijke wijze zwaartekracht bevat. Een bijzondere eigenschap van snarentheorie is dat er alleen wiskundige consistentie is als de theorie leeft in tien dimensies. Dit klinkt misschien eerst als een reden om de theorie meteen af te keuren, maar er zijn manieren om zes dimensies 'weg te moffelen', zodat er effectief vier over blijven, drie ruimtelijke en een tijdsachtige (zie ook hieronder en hoofdstuk 3 van dit proefschrift). Er blijken vijf verschillende manieren te zijn om snarentheorie consistent te formuleren in tien dimensies. Deze ontdekking wierp eerst een smet op het ambitieuze plan om een 'theorie van alles' te vinden, want een theorie van alles zou toch uniek moeten zijn? Echter, in de jaren negentig van de afgelopen eeuw werd duidelijk dat de vijf verschillende snarentheorieën via dualiteiten waren verbonden.

Een praktisch nadeel van de snarentheorie is dat het doen van berekeningen erg lastig is. De complexiteit is zo groot dat veel moeilijke wiskunde nodig is voor zelfs de kleinste berekeningen. Echter door een lage-energie limiet te nemen kun je een theorie bouwen die dan niet meer de echte snarentheorie is, maar wel een goede benadering onder bepaalde omstandigheden, namelijk bij energieën die laag genoeg zijn. In deze benaderingen is het gemakkelijker om berekeningen te doen. Na het nemen van deze lage-energie limiet vind je als resultaat een supergravitatie theorie.

[1] De Nederlanders Gerard 't Hooft en Martinus Veltman hebben hieraan bijgedragen en ontvingen hiervoor in 1999 de Nobelprijs.

Een supergravitatie theorie is een klassieke veldentheorie die zowel zwaartekracht bevat als supersymmetrie. De laatstgenoemde eigenschap, de supersymmetrie, zegt dat er een symmetrie is in de theorie die fermionen en bosonen in elkaar overvoert. De eis dat een theorie zwaartekracht en supersymmetrie bevat blijkt restrictief te zijn. In tien dimensies zijn vijf verschillende consistente supergravitatie theorieën (ik reken hier Romans type IIA supergravitatie niet mee omdat deze 'massief' is; zie hoofdstuk 2), precies in overeenkomst met de vijf consistente snarentheorieën.

Door deze tiendimensionale supergravitatie theorieën te zetten op een geometrie $X_4 \times Y_6$, waarbij Y_6 een zesdimensionale compacte ruimte is en X_4 een nog nader te bepalen vierdimensionale ruimte, die de rol moet gaan spelen van onze vierdimensionale ruimtetijd, verkrijgt men vierdimensionale theorieën. Deze constructie heet dimensionele reductie en hoofdstuk 3 van dit proefschrift is gewijd aan deze constructie. Men kan ook direkt in vier dimensies beginnen met het construeren van een theorie die zwaartekracht en supersymmetrie bevat. Op deze manier zijn er twee verschillende manieren om vierdimensionale supergravitatie theorieën te verkrijgen en het lijkt natuurlijk te veronderstellen dat er een één-op-één relatie zou moeten zijn tussen deze twee constructies. Echter, de theorieën die in vier dimensies zijn geconstrueerd bevatten enkele eigenschappen die nog niet zijn begrepen vanuit een tiendimensionaal oogpunt, zoals de zogenaamde $SU(1,1)$-hoeken.

De eis van supersymmetrie bepaalt veel eigenschappen van een theorie; de structuur ligt grotendeels vast. Des te meer supersymmetrie - de hoeveelheid wordt uitgedrukt in een positief geheel getal $\mathcal{N}$; in vier dimensies $0 \leq \mathcal{N} \leq 8$ -, des te minder vrijheden er zijn in de theorie. Supersymmetrie legt ook andere symmetrieën op aan een theorie. Door deze symmetrieën goed te begrijpen kunnen veel eigenschappen van een supergravitatie theorie begrepen worden.

Het begrip symmetrie wordt vaak intuïtief goed begrepen; als we een cirkel om zijn middelpunt draaien ziet de cirkel er nog steeds hetzelfde uit en als we een vierkant spiegelen om een van zijn diagonalen ziet het vierkant er hetzelfde uit. De symmetrie beschreven voor de cirkel is een voorbeeld van een continue symmetrie, de hoek waarover men roteert, kan op een continue wijze gevarieerd worden. De symmetrie beschreven voor het vierkant is een voorbeeld van een discrete symmetrie, er kan niet een beetje gespiegeld worden, of men spiegelt wel, of niet. De wiskunde om symmetrieën te beschrijven heet groepentheorie en hierin maakt men onderscheid tussen discrete groepen, die discrete symmetrieën beschrijven, en Lie groepen, die continue symmetrieën begrijpen. In dit proefschrift ligt de nadruk op de continue symmetrieën en derhalve is een gedetailleerde appendix toegevoegd over de theorie van Lie groepen.

In dit proefschrift wordt groepentheorie vooral gebruikt om de scalaire sector van $\mathcal{N} = 4$ supergravitatie theorieën te analyseren. De scalairen in een theorie bepalen in grote mate wat voor soort 'vacua' de theorie heeft. Een vacuüm van een theorie is een oplossing van een theorie voor lege ruimte zonder materie erin. Men zou dus kunnen zeggen dat in dit proefschrift voornamelijk lege ruimte is onderzocht. In een eerste lezing lijkt dat niet zo interessant, maar lege ruimte is overal om ons heen; het heelal is voornamelijk leeg, maar ook het menselijk lichaam bestaat voornamelijk uit lege ruimte - de afstand tussen het elektron en het proton in het waterstof-atoom is erg veel groter dan de doorsnede van het elektron of proton. Lege ruimte kan namelijk door de aanwezigheid van de scalairen[2] en de daarmee geassocieerde energie gekromd zijn. De kromming van de lege ruimte in het heelal bepaalt hoe snel het heelal uitdijt.

Het merendeel van de astronomische waarnemingen doet ons concluderen dat het heelal een begin heeft gehad waarin de kromming van de ruimte erg sterk, de temperatuur erg hoog en de dichtheid erg hoog was. In deze periode dijde het heelal hard uit, als bij een explosie. Dit hete en knallende begin wordt ook wel de Big Bang genoemd. Na deze periode koelde het heelal af en dijde het uit en na verloop van tijd werden er sterrenstelsel en planeten gevormd. Men zou verwachten dat het heelal steeds langzamer zou gaan expanderen, maar recente astronomische waarnemingen tonen aan dat dat niet het geval is. Ten eerste, om in overeenstemming te zijn met waarnemingen moet er na de Big Bang nog ergens een periode zijn geweest waarin het heelal versneld uitdijde en ten tweede, het heelal blijkt nu alweer te versnellen. Deze versnelde expansie, ook wel inflatie genoemd, van het heelal kan men onder andere verklaren door te veronderstellen dat er scalairen zijn die bijdragen aan de totale energie in het heelal. Zodoende kunnen scalairen de uitdijing aandrijven. Nu geeft het Standaardmodel weinig kandidaten voor de scalairen die de versnelde uitdijing kunnen aandrijven, maar snarentheorie voorspelt een plethora aan scalairen. Dit is één van de motivaties geweest om de scalairen van $\mathcal{N} = 4$ supergravitatie en de scalar potentiaal te onderzoeken.

Een versnelde uitdijing van het heelal is mogelijk in $\mathcal{N} = 4$ supergravitatie als de scalairen in een minimum van de potentiaal zitten waar de waarde van de potentiaal positief is. Zo'n vacuüm oplossing heet dan een de Sitter vacuüm. Binnen een grote klasse van mogelijke scalar potentialen in $\mathcal{N} = 4$ supergravitatie hebben we gezocht naar de Sitter vacua en hebben er geen gevonden. Hoewel dit het een en ander uitsluit, is het niet mogelijk om te beweren dat $\mathcal{N} = 4$ supergravitatie geen de Sitter oplossingen toestaat. In artikel [E] hebben we een andere mogelijkheid beschreven om een versneld uitdijend heelal te krijgen binnen $\mathcal{N} = 4$ supergravitaties, namelijk via zogenaamde scaling solutions. We vonden een stabiele scaling solution waarin het heelal noch versneld noch vertraagd uitdijde; bij de gevonden oplossing is de uitdijingsnelheid constant.

Hoewel we geen oplossingen met een versneld uitdijende heelal hebben gevonden, is er wel het een en ander duidelijk geworden over de structuur van de scalar potentialen in $\mathcal{N} = 4$ supergravitaties. Natuurlijk zijn er nog vele vragen over; $\mathcal{N} = 4$ supergravitaties bieden nog voldoende mogelijkheden voor toekomstig onderzoek.

Middels bovenstaande tekst hoop ik te hebben uitgelegd dat het gebruik maken

[2]Van scalairen kun je geen sterren of planeten bouwen en worden daarom soms niet tot materie gerekend, daarom noemt men lege ruimte waarin scalairen zitten nog steeds leeg.

van de symmetrieën van een theorie er veel aspecten kunnen worden begrepen. In dit proefschrift is de theorie die onder de loep is genomen vierdimensionale $\mathcal{N} = 4$ supergravitatie. Ik heb ook geprobeerd in het kort een indruk te geven waar de symmetrieën, en dus de groepentheorie, gebruikt is. Hiermee hoop ik dan in ieder geval de titel van dit proefschrift te hebben uitgelegd.

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