

Introduction to String Theory and String Compactifications

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Abstract. Basics of some topics on perturbative and non-perturbative string theory are reviewed. After a mathematical survey of the Standard Model of particle physics and GUTs, the bosonic string kinematics for the free case and with interaction is described. The effective action of the bosonic string and the spectrum is also discussed. T-duality in closed and open strings and the definition of D-brane are surveyed. Five perturbative superstring theories and their spectra is briefly outlined. Calabi-Yau three-fold compactifications of heterotic strings and their relation to some four-dimensional physics are given. Finally, non-perturbative issues like S-duality, M-theory and F-theory are also reviewed.

1. Introduction

At present the Standard Model (SM) of particle physics is the best and most sensible theory which summarizes our understanding of the basic components of matter and their interactions in an unified scheme. However there is the known ‘aesthetic’ problem arising in the SM of particles, that include the hierarchy problem, the abundance of free parameters and the apparent arbitrariness of the flavor and gauge groups, etc. The SM is for this reason commonly regarded as the low energy effective description of a more fundamental theory, which solves these problems (for a nice review on this subject, see [1]). On the other hand, it is also widely recognized that Quantum Mechanics and General Relativity (GR) cannot be reconciliated in the context of a perturbative quantum field theory of point particles. Hence the nonrenormalizability of GR is also regarded as a evidence that it is just an effective field theory and new physics associated to some fast degrees of freedom should exist at higher energies (for a review, see for instance [2]). String theory proposes that these fast degrees of freedom are precisely the strings at the perturbative level and at the non-perturbative level the relevant degrees of freedom are higher-dimensional extended objects called D-branes (dual degrees of freedom).

At the perturbative level String Theory has intriguing generic predictions such as: (i) Spacetime supersymmetry, (ii) General Relativity and (iii) Yang-Mills fields. These subjects interesting by themselves are deeply interconnected in a rich way in string theory.

Though the study of theories involving D-branes has produced great number of results, it is still at an exploratory stage of the whole structure of the string theory. Therefore the theory is far of being completed and it is necessary to explore the structure of the theory before we can give concrete physical predictions to make contact with collider experiments and/or astrophysical

observations. However many aspects of theoretic character, necessary in order to make of string theory a physical theory, are quickly in progress. The purpose of these lectures are to overview the basic ideas in order to understand these progresses following strongly the set of lectures given in Ref. [3].

2. Basic Facts on Field Theory

First we overview the basic structure of GR and Yang-Mills (YM) theory in four dimensions. They are very different theories. GR for instance, is the dynamical theory of the spacetime metric $g_{\mu\nu}$ while quantum YM theories and in general, Quantum Field Theory (QFT) describes the dynamical building blocks of matter in a fixed spacetime background. Here we survey basics aspects of GR and YM theory following closely Ref. [4].

2.1. General Relativity

The pure gravitational field is described by a pseudo-Riemannian metric $g_{\mu\nu}$ with $\mu, \nu = 0, 1, 2, 3$ (on a four-dimensional space-time manifold M) satisfying the vacuum Einstein equations with a cosmological constant Λ ,

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \quad (1)$$

Einstein equations can be derived from the Einstein-Hilbert action

$$S_{GR} = \frac{1}{16\pi G_N} \int_M d^4x \sqrt{-g} (R - 2\Lambda), \quad (2)$$

where G_N is the Newton's constant. This constant together with \hbar and c , determines the Planck scale where gravitational effects in the quantum theory are relevant. The mass scale termed Planck mass is $M_{Pl} = \sqrt{\frac{\hbar c}{G_N}} = 1.2 \times 10^{-5} \text{ grams}$ or equivalently the Planck length $L_{Pl} = \frac{\hbar}{M_{Pl} c} \approx 10^{-33}$ centimeters.

2.2. Gauge Theories

Classical Gauge Theories

If one wants to formulate the gauge theory on a fixed pseudo-Riemannian manifold spite of the metric $g_{\mu\nu}$ we require from an additional structure on the spacetime *i.e.* a connection $A \in \Gamma(T^*E \otimes \mathcal{G})$ on a G -principal bundle on M : $G \rightarrow E \xrightarrow{\pi} M$, where G is the SM gauge group, $G = \text{SU}_C(3) \times \text{SU}_L(2) \times \text{U}_Y(1)$. As usual, the gauge field $A_\mu(x)$ given by the connection one-form has associated the field strength $F_{\mu\nu} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + i f_{bc}^a A_\mu^a A_\nu^b$ with f_{bc}^a being the structure constants of G . Given any representation \mathbf{R} of G one can construct the *associated vector bundle* $V_{\mathbf{R}}$.

The Yang-Mills action is given by

$$S_{YM} = -\frac{1}{4g_{YM}^2} \int_M g^{\mu\mu'} g^{\nu\nu'} Tr_{\mathbf{R}} F_{\mu\nu} F_{\mu'\nu'}, \quad (3)$$

where $Tr_{\mathbf{R}}$ denotes the trace in the adjoint representation of G .

Now we want to introduce fermions. The chiral fermions are sections of the the chiral spin bundles \hat{S}_\pm over spacetime manifold with *Spin* structure M , *i.e.* $\hat{S} \xrightarrow{\pi} M$, where $\hat{S} = \hat{S}_+ \oplus \hat{S}_-$. The fibers are the Clifford modules constructed with the Dirac matrices Γ^μ . Dirac operator is $\mathcal{D} \equiv \Gamma^\mu D_\mu : \Gamma(\hat{S}) \rightarrow \Gamma(\hat{S})$ with D_μ being the spacetime covariant derivative. In even dimensions Dirac operator decomposes as: $\mathcal{D} = \mathcal{D}_+ \oplus \mathcal{D}_-$ where $\mathcal{D}_\pm : \Gamma(\hat{S}_\pm) \rightarrow \Gamma(\hat{S}_\mp)$ with

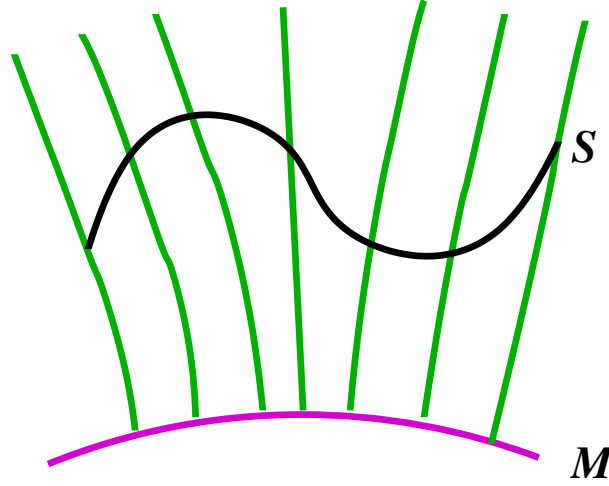


Figure 1. Spin bundle

$$\mathcal{D}_- \psi_+ = 0, \quad \mathcal{D}_+ \psi_- = 0, \quad (4)$$

and $\psi_{\pm} \in \Gamma(\widehat{S}_{\pm})$.

The possibility to add a mass term in the above equation implies that that mass should be of order one in mass Planck unities M_{Pl} . But the mass of the low energies particle m should be much lower than the Planck mass M_{Pl} , *i.e.* $m \ll M_{Pl}$. A very nice solution can be given by introducing gauge fields in *complex* representations of the gauge group G . In that case the fermions should be sections of the original spin bundle \widehat{S} coupled to the associated vector bundle $V_{\mathbf{R}}$. If $\mathbf{R} \not\cong \mathbf{R}^*$ then the corresponding bundles are not isomorphic $V_{\mathbf{R}} \not\cong V_{\mathbf{R}^*}$. So we have four possibilities:

$$\begin{aligned} W_+ &= \widehat{S}_+ \otimes V_{\mathbf{R}}, & W_- &= \widehat{S}_- \otimes V_{\mathbf{R}}, \\ \widetilde{W}_+ &= \widehat{S}_+ \otimes V_{\mathbf{R}^*}, & \widetilde{W}_- &= \widehat{S}_- \otimes V_{\mathbf{R}^*}. \end{aligned} \quad (5)$$

CPT theorem implies that the fermions with different chirality are given by

$$\psi_+ \in \Gamma(\widehat{S}_+ \otimes V_{\mathbf{R}}), \quad \widetilde{\psi}_- \in \Gamma(\widehat{S}_- \otimes V_{\mathbf{R}^*}). \quad (6)$$

This explains why the mass term are not allowed in the right-hand of Eq. (4).

Quantum Gauge Theories

In Standard Model (SM) leptons and quarks are left-handed Weyl fermions. One generation of them are organized as follows:

$$\begin{pmatrix} u \\ d \end{pmatrix}_{\frac{1}{6}}, \quad \bar{u}_{\frac{1}{3}}, \quad \bar{d}_{-\frac{2}{3}}, \quad \begin{pmatrix} \nu \\ e^- \end{pmatrix}_{-\frac{1}{2}}, \quad e_1^+, \quad \nu_0. \quad (7)$$

More precisely they are organized in irreps of the gauge group: $SU(3)_C \times SU(2)_L \times U(1)_Y$

$$[(\mathbf{3}, \mathbf{2})_{\frac{1}{6}} \oplus (\mathbf{3}^*, \mathbf{1})_{\frac{1}{3}} \oplus (\mathbf{3}^*, \mathbf{1})_{-\frac{2}{3}} \oplus (\mathbf{1}, \mathbf{2})_{-\frac{1}{2}} \oplus (\mathbf{1}, \mathbf{1})_1] \oplus 3(\mathbf{1}, \mathbf{1})_0, \quad (8)$$

where (\cdot, \cdot) stands for the irreps of $SU(3) \times SU(2)$ respectively working around $E \sim 1 \text{ TeV}$.

Now define $\mathbf{R} = \oplus_{i=1}^{15} \mathbf{R}_i$ and $\mathbf{R}^* = \oplus_{i=1}^{15} \mathbf{R}_i^*$ where \mathbf{R}_i and \mathbf{R}_i^* are irreducible complex representations of the gauge group of the SM. Define the formal difference

$$\Delta \equiv \mathbf{U} \ominus \mathbf{U}^* \quad (9)$$

between the general complex representation of a particle $\mathbf{U} = \mathbf{U}_0 \oplus \mathbf{R}$ and its corresponding complex conjugated $\mathbf{U}^* = \mathbf{U}_0 \oplus \mathbf{R}^*$ with \mathbf{U}_0 being a *real* irreducible representation (irrep). Thus in the computation of Δ only the complex representations are relevant, *i.e.* $\Delta = \mathbf{R} \ominus \mathbf{R}^*$.

At $E < M_W \sim 10^2 GeV$ gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ breaks down to $U(1)_{em}$. With the *Higgs field* ϕ transforming as: $(\mathbf{1}, \mathbf{2})_{-\frac{1}{2}}$ has a potential $V(\phi) \sim m^2 \phi^2 + \lambda \phi^4$. Then, masses of fermions in SM are obtained from spontaneously symmetry breaking through the Yukawa couplings:

$$Q_L U \phi, \quad Q_L D \phi^*, \quad L E \phi, \quad (\mathbf{3}, \mathbf{2})_{\frac{1}{6}} \otimes (\mathbf{3}^*, \mathbf{1})_{\frac{1}{3}} \otimes (\mathbf{1}, \mathbf{2})_{-\frac{1}{2}}. \quad (10)$$

The Spontaneous Symmetry Breaking mechanism leads to consider theories with higher dimensional gauge groups \overline{G} than the SM groups and recuperate the latter by symmetry breaking.

2.3. Grand Unified Theories (GUT's)

One of the original proposals (Georgi and Glashow, 1973) for GUT's was the so called SU(5) GUT, where the gauge group \overline{G} is SU(5) and it breaks down to the SM group [1, 5]. In this model one generation of leptons and quarks is organized in terms of irreps of the SU(5) as follows:

$$\mathbf{10} \oplus \mathbf{5}^* : \left(\begin{array}{ccccc} 0 & \bar{u} & \bar{u} & u & d \\ & 0 & \bar{u} & u & d \\ & & 0 & u & d \\ & & & 0 & e^+ \\ & & & & 0 \end{array} \right) \oplus \left(\begin{array}{c} \bar{d} \\ \bar{d} \\ \bar{d} \\ \nu \\ e^- \end{array} \right), \quad (11)$$

where $\mathbf{5}$ is the fundamental and $\mathbf{5}^*$ is the anti-fundamental representations of SU(5), $\mathbf{10}$ is the anti-symmetric part of the representation $\mathbf{5} \otimes \mathbf{5}$ and $\mathbf{10}^*$ is its complex conjugated

$$\mathbf{10} \rightarrow (\mathbf{3}, \mathbf{2})_{\frac{1}{6}} \oplus (\mathbf{3}^*, \mathbf{1})_{\frac{1}{3}} \oplus (\mathbf{1}, \mathbf{1})_1 \quad (12)$$

and

$$\mathbf{5}^* \rightarrow (\mathbf{3}^*, \mathbf{1})_{-\frac{2}{3}} \oplus (\mathbf{1}, \mathbf{2})_{-\frac{1}{2}}. \quad (13)$$

In this case

$$\Delta = 3 \left(\mathbf{5}^* \oplus \mathbf{10} \ominus \mathbf{5} \oplus \mathbf{10}^* \right), \quad (14)$$

where the '3' in the front part stands for the mysterious number of generations of quarks and leptons. We will come back later to comment about this number.

[Remark: SM gauge group can be embedded in the SU(5) group in the form

$$\left(\begin{array}{cc} SU(3) & X, Y \\ X, Y & SU(2) \end{array} \right). \quad (15)$$

X and Y are a new color triplet of massive gauge bosons. This leads to a prediction of the proton decay: $p \rightarrow \pi^0 e^+$.]

$SU(5)$ is by itself a non-trivial maximal subgroup of $SO(10)$. The GUT with gauge group $SO(10)$ is another candidate for a unified model. The decomposition of irreps of $SO(10)$ in terms of irreps of $SU(5)$ is as follows: the fundamental representation of $SO(10)$ $\mathbf{10}$ decomposes under $SU(5)$ irreps as $\mathbf{10} = \mathbf{5} \oplus \mathbf{5}^*$. $SO(10)$ has two complex conjugated spinor representations of 16 dimensions, they are: $\mathbf{16}$ and $\mathbf{16}^*$. They can be decomposed under $SU(5)$ irreps as, $\mathbf{16} = \mathbf{1} \oplus \mathbf{5}^* \oplus \mathbf{10}$ and $\mathbf{16}^* = \mathbf{1} \oplus \mathbf{5} \oplus \mathbf{10}^*$. Then computation of Δ yields

$$\Delta = 3 \left(\mathbf{16} \ominus \mathbf{16}^* \right). \quad (16)$$

Higher dimensionality group like the exceptional group E_6 is the next candidate for a GUT. This group has complex representations which are: $\mathbf{27}$ and $\mathbf{27}^*$. Under $SO(10)$ irreps, these representations decompose into the spinor, vector and identity irreps: *i.e.* $\mathbf{27} = \mathbf{16} \oplus \mathbf{10} \oplus \mathbf{1}$. Vector representation is real. Thus Δ is computed easily to get

$$\Delta = 3 \left(\mathbf{27} \ominus \mathbf{27}^* \right). \quad (17)$$

Bigger exceptional groups like E_8 only has real representations and therefore $\Delta = 0$.

The SM and GUTs are thus unable to answer the arbitrariness of the number of families of lepton and quarks (basically the ‘3’ arising in Eqs. (14), (16) and (17)) as well as the arbitrariness of the gauge group. The hierarchy of lepton and quarks masses, the existence of the Higgs mechanism and the abundance of free parameters are ‘aesthetic problems’ as they don’t contradict any experiment. However it is clear that the explanation of the origin has to come of somewhere beyond SM and GUTs. In the last 15 years we have learned that string theory has the necessary ingredients to solve these potential problems and it is a serious candidate to provide us with a complete unified theory of all known fundamental interactions of nature. In these lectures we attempt to give the very basic notions of some topics of perturbative and non-perturbative string theory.

Theoretical Questions

- Gauge interaction disconnected from gravity. Then we would like to incorporate gravity. Can this really be done? and how?
- Are all interactions described together in an unified setup? Or do they remain as intrinsically different, up to arbitrary energies?
- Why are there two different scales, M_W and M_P ? Why are there so widely separated? Are they related in any way, and if so, which?
- Why M_W , which is fixed by the mass of the Higgs scalar, so not modified by quantum loops of stuff related to physics at the scale M_P ? Power counting would suggest that the natural value of the corrections is of order of M_P^2 , which would then push M_W up to M_P .
- Are there other scales between M_W and M_P ? or there is just a big desert in energies in between?
- Arbitrariness of the gauge group. Why there are three factors? Why these fermion representations? Why stop up to E_6 ? if we have more possibilities:

$$SU(5) \subset SO(10) \subset E_6 \subset E_7 \subset E_8. \quad (18)$$

Why there are three families of leptons and quarks? How are these features determined by an underlying microscopic theory that includes gravity?

- Are global symmetries of SM exact symmetries of the underlying theory? Or just accidental symmetries? is baryon number really conserved? Why is the proton stable, and if not what new physics mediates its decay?
- Why are there four spacetime dimensions? Is it true that there are just four dimensions? Does this follow from any consistency condition of the theory supposedly underlying gauge and gravitational interactions?
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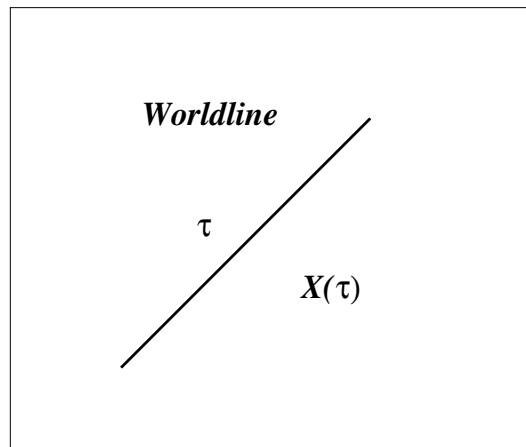
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3. Perturbative Bosonic String Theory

In this section we overview some basic aspects of bosonic strings (for reviews, see [6, 7, 8, 9, 10, 11, 12, 13]). We focus mainly in the description of the spectrum of the perturbative theory in the light-cone gauge, string interactions, and finally, the effective action.

Minkowski Spacetime M



First of all consider, as usual, the action of a relativistic point particle. It is given by $S = -m \int d\tau \sqrt{-\dot{X}^I \dot{X}^I \eta_{IJ}}$, where X^I are D embedding functions representing the coordinates of the $(D-1, 1)$ -dimensional Minkowski spacetime (the target space), $\dot{X}^I \equiv \frac{dX^I}{d\tau}$ and m can be identified with the mass of the point particle. This action is proportional to the length of the world-line of the relativistic particle. This action is invariant under: (i) Reparametrizations: $\tau \rightarrow \tau' = \tau'(\tau)$, (ii) Poincaré transformations: $X'^I(\tau) = \Lambda^I_J X^J(\tau) + a^I$.

In analogy with the relativistic point particle, the action describing the dynamics of a string (one-dimensional object) moving in a $(D-1, 1)$ -dimensional Minkowski spacetime with metric η_{IJ} (the target space) is proportional to the area \mathbf{A} of the worldsheet Σ . We know from the theory of surfaces that such an area is given by $\mathbf{A} = \int \sqrt{\det(-g)}$, where g is the induced metric (with signature $(-, +)$) on the worldsheet Σ . The background metric will be denoted by η_{IJ} and $\sigma^a = (\tau, \sigma)$ with $a = 0, 1$ are the local coordinates on the worldsheet. η_{IJ} and g_{ab} are related by

$g_{ab} = \eta_{IJ} \partial_a X^I \partial_b X^J$ with $I, J = 0, 1, \dots, D-1$. Thus the classical action of a relativistic string is given by the Nambu-Goto action

$$S_{NG}[X^I] = -T \int_{\Sigma} d\tau d\sigma \sqrt{-\det(\partial_a X^I \partial_b X^J \eta_{IJ})}, \quad (19)$$

where $T = \frac{1}{2\pi\alpha'}$ (where $\alpha' = \ell_S^2$, with ℓ_S being the string length) is the string tension, X^I are D functions representing the embedding of the worldsheet Σ into the target space M . Now introduce a metric h describing the intrinsic worldsheet geometry, we get a classically equivalent action to the Nambu-Goto action. This is the Polyakov action (originally proposed by Brink, di Vecchia, Howe and Zumino)

$$S_P[X^I, h_{ab}] = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \sqrt{-h} h^{ab} \partial_a X^I \partial_b X^J \eta_{IJ}, \quad (20)$$

where the X^I 's are D scalar fields on the worldsheet Σ . Such a fields can be interpreted as the coordinates of spacetime M (target space), $h = \det(h^{ab})$ and $h_{ab} = \partial_a X^I \partial_b X^J \eta_{IJ}$. Polyakov action has the following symmetries:

- Poincaré invariance:

$$\begin{aligned} X^I(\tau, \sigma) &= \Lambda^I_J X^J(\tau, \sigma) + a^I, \\ h'_{ab}(\tau, \sigma) &= h_{ab}(\tau, \sigma). \end{aligned} \quad (21)$$

- Worldsheet diffeomorphism invariance:

$$\begin{aligned} X'^I(\tau', \sigma') &= X^I(\tau, \sigma), \\ h'_{ab}(\tau', \sigma') &= \frac{\partial\sigma^c}{\partial\sigma'^a} \frac{\partial\sigma^d}{\partial\sigma'^b} h_{cd}(\tau, \sigma). \end{aligned} \quad (22)$$

- Weyl invariance:

$$\begin{aligned} X'^I(\tau, \sigma) &= X^I(\tau, \sigma), \\ h'_{ab}(\tau, \sigma) &= e^{[2\omega(\tau, \sigma)]} h_{ab}(\tau, \sigma), \end{aligned} \quad (23)$$

for arbitrary $\omega(\tau, \sigma)$.

The energy-momentum tensor of the two-dimensional theory is given by

$$\begin{aligned} T^{ab} &:= \frac{1}{\sqrt{-h}} \frac{\delta S_P}{\delta h_{ab}} \\ &= \frac{1}{4\pi\alpha'} \left(\partial^a X^I \partial^b X_I - \frac{1}{2} h^{ab} h^{cd} \partial_c X^I \partial_d X_I \right). \end{aligned} \quad (24)$$

Invariance under worldsheet diffeomorphisms implies that it should be conserved *i.e.* $\nabla_a T^{ab} = 0$, while the Weyl invariance gives its traceless condition, $T^a_a = 0$.

In order to get equations of motion from the Polyakov action we vary his action:

$$\begin{aligned} \delta S_P &= \frac{1}{2\pi\alpha'} \int_{-\infty}^{\infty} d\tau \int_0^{\ell_S} d\sigma \delta X^I \partial_a \left[\sqrt{-h} h^{ab} \partial_b X^I \right] \\ &\quad - \frac{1}{2\pi\alpha'} \int_{-\infty}^{\infty} d\tau \sqrt{-h} \delta X^I \partial^\sigma X_I \Big|_{\sigma=0}^{\sigma=\ell_S}. \end{aligned} \quad (25)$$

The equation of motion associated with Polyakov action is given by

$$\partial_a \left(\sqrt{-h} h^{ab} \partial_b X^I \right) = 0, \quad (26)$$

satisfying the Virasoro constraints: $T_{ab} = 0$ and $T_a^a = 0$. Whose solutions should satisfy the following boundary conditions:

Closed Strings:

- Newmann:

$$\begin{aligned} X^I(\tau, 0) &= X^I(\tau, \ell_S), \\ \partial^\sigma X^I(\tau, 0) &= \partial^\sigma X^I(\tau, \ell_S). \end{aligned} \quad (27)$$

Open strings:

- Newmann:

$$\partial^\sigma X^I(\tau, \sigma)|_{\sigma=0}^{\sigma=\ell_S} = 0. \quad (28)$$

- Dirichlet:

$$\partial^\tau X^I(\tau, \sigma)|_{\sigma=0}^{\sigma=\ell_S} = 0. \quad (29)$$

Here $\ell_S = \pi$ is the characteristic length of the open string. The variation of S_P with respect to h^{ab} leads to the constraint equations: $T_{ab} = 0$. From now on we will work in the *conformal gauge*. In this gauge: $h_{ab} = \eta_{ab}$ and equations of motion (26) reduce to the Laplace equation in the flat worldsheet whose solutions can be written as linear superposition of plane waves.

The Closed String

For the closed string the boundary condition $X^I(\tau, \sigma) = X^I(\tau, \sigma + 2\pi)$, leads to the general solution of Eq. (26)

$$X^I(\tau, \sigma) = X_0^I + \frac{1}{\pi T} P^I \tau + \frac{i}{2\sqrt{\pi T}} \sum_{n \neq 0} \frac{1}{n} \left\{ \alpha_n^I \exp(-i2n(\tau - \sigma)) + \tilde{\alpha}_n^I \exp(-i2n(\tau + \sigma)) \right\}, \quad (30)$$

where X_0^I and P^I are the position and momentum of the center-of-mass of the string and α_n^I and $\tilde{\alpha}_n^I$ satisfy the conditions $\alpha_n^{I*} = \alpha_{-n}^I$ (left-movers) and $\tilde{\alpha}_n^{I*} = \tilde{\alpha}_{-n}^I$ (right-movers).

The Open String

For the open string the corresponding boundary condition is $\partial_\sigma X^I|_0^{\ell=\pi} = 0$ (this is the only boundary condition which is Lorentz invariant) and the solution is given by

$$X^I(\tau, \sigma) = X_0^I + \frac{1}{\pi T} P^I \tau + \frac{i}{\sqrt{\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^I \exp(-in\tau) \cos(n\sigma) \quad (31)$$

with the matching condition $\alpha_n^I = \tilde{\alpha}_{-n}^I$.

3.1. Quantization

The quantization of the closed bosonic string can be carried over, as usual, by using the Dirac prescription to the center-of-mass and oscillator variables in the form

$$\begin{aligned} [X_0^I, P^J] &= i\eta^{IJ}, \\ [\alpha_m^I, \alpha_n^J] &= [\tilde{\alpha}_m^I, \tilde{\alpha}_n^J] = m\delta_{m+n,0}\eta^{IJ}, \\ [\alpha_m^I, \tilde{\alpha}_n^J] &= 0. \end{aligned} \tag{32}$$

One can identify $(\alpha_n^I, \tilde{\alpha}_n^I)$ for $n > 0$, with the annihilation operators and the corresponding operators $(\alpha_{-n}^I, \tilde{\alpha}_{-n}^I)$ with the creation ones. In order to specify the physical states we first denote the center of mass state given by $|P^I\rangle$. The vacuum state $|0, P^I\rangle$ is defined by: $\alpha_m^I|0, P^I\rangle = 0$ with $m > 0$ and $P^I|0, P^I\rangle = p^I|0, P^I\rangle$ and similar for the right movings (here $|0, P^I\rangle = |P^I\rangle \otimes |0\rangle$). For the zero modes these states have negative norm (ghosts). However one can choose a suitable gauge where ghosts decouple from the Hilbert space for a given critical dimension D_c of the target space i.e. for the bosonic string $D_c = D = 26$.

Light-cone Quantization

Now we turn out to work in the so called *light-cone gauge*. In this gauge it is possible to solve explicitly the Virasoro constraints: $T_{ab} = 0$. This is done by removing the light-cone coordinates $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1})$ leaving only the transverse coordinates X^i representing the physical degrees of freedom (with $i, j = 1, 2, \dots, D - 2$). For the closed string one takes $X^+ = \alpha' p^+ \tau$. In this gauge the Virasoro constraints are explicitly solved. Thus the independent variables are $(X_0^-, P^+, X_0^j, P^j, \alpha_n^j, \tilde{\alpha}_n^j)$. Operators α_n^- and $\tilde{\alpha}_n^-$ can be written in terms of α_n^j and $\tilde{\alpha}_n^j$ respectively as follows: $\alpha_n^- = \frac{1}{\sqrt{2\alpha'P^+}}(\sum_{m=-\infty}^{\infty} : \alpha_{n-m}^i \alpha_m^i : - 2A\delta_n)$ and $\tilde{\alpha}_n^- = \frac{1}{\sqrt{2\alpha'P^+}}(\sum_{m=-\infty}^{\infty} : \tilde{\alpha}_{n-m}^i \tilde{\alpha}_m^i : - 2A\delta_n)$. For the open string we get $\alpha_n^- = \frac{1}{2\sqrt{2\alpha'P^+}}(\sum_{m=-\infty}^{\infty} : \alpha_{n-m}^i \alpha_m^i : - 2A\delta_n)$. Here $:\cdot:$ stands for the normal ordering and A is its associated constant representing the zero-point energy.

In this gauge the Hamiltonian is given by

$$H = \frac{1}{2}(P^i)^2 + N - A \text{ (open string)}, \quad H = (P^i)^2 + N_L + N_R - 2A \text{ (closed string)} \tag{33}$$

where N is the operator number of particles, $N_L = \sum_{m=-\infty}^{\infty} : \alpha_{-m} \alpha_m :$, and $N_R = \sum_{m=-\infty}^{\infty} : \tilde{\alpha}_{-m} \tilde{\alpha}_m :$. The *mass-shell condition* is given by $\alpha' M^2 = (N - A)$ (open string) and $\alpha' M^2 = 2(N_L + N_R - 2A)$ (closed string). For the open string, for instance, Lorentz invariance implies that the first excited state is massless and therefore $A = 1$. In the light-cone gauge A takes the form $A = -\frac{D-2}{2} \sum_{n=1}^{\infty} n$. From the fact $\sum_{n=1}^{\infty} n^{-s} = \zeta(s)$, where ζ is the Riemann's zeta function (which converges for $s > 1$ and has a unique analytic continuation at $s = -1$, where it takes the value $-\frac{1}{12}$) then $A = \frac{D-2}{24}$ and therefore $D = 26$.

3.2. Spectrum of the Bosonic String

Open Strings

For the open string, the ground state includes a tachyon since $\alpha' M^2 = -1$. The first excited state $N = 1$ is given by a massless vector field in 26 dimensions. The second excitation level is given by the massive states $\alpha_{-2}^i |0, P\rangle$ and $\alpha_{-1}^i \alpha_{-1}^j |0, P\rangle$ which are in irreducible

representations of the little group $SO(25)$ of $SO(1, 25)$. For higher values of N i.e. $N \geq 3$ there is an infinite tower of massive states. In the following table we summarize this spectrum:

$N = 0$	$ P, 0\rangle$	$-\frac{1}{4}P^2 = -1$	Tachyon
$N = 1$	$\alpha_{-1}^j P, 0\rangle$	$-\frac{1}{4}P^2 = 0$	Massless Vect
$N = 2$	$\alpha_{-1}^i \alpha_{-1}^j P, 0\rangle$	$-\frac{1}{4}P^2 = 1$	spin 2
	$\alpha_{-2}^j P, 0\rangle$	$-\frac{1}{4}P^2 = 1$	massive

Closed Strings

The spectrum of the closed string can be obtained from the combination of the left-moving states and the right-moving ones. The ground state ($N_L = N_R = 0$) is given by $\alpha' M^2 = -4$. That means that the ground state includes again a tachyon. The first excited state ($N_L = 1 = N_R$) is massless and it is given by $\alpha_{-1}^i \tilde{\alpha}_{-1}^j |0, P\rangle$. This state can be naturally decomposed into irreducible representations of the little group $SO(24)$ as follows

$$\begin{aligned} \alpha_{-1}^i \tilde{\alpha}_{-1}^j |0, P\rangle &= \alpha_{-1}^{[i} \tilde{\alpha}_{-1}^{j]} |0, P\rangle + \left(\alpha_{-1}^{(i} \tilde{\alpha}_{-1}^{j)} - \frac{1}{D-2} \delta^{ij} \alpha_{-1}^k \tilde{\alpha}_{-1}^k \right) |0, P\rangle \\ &+ \frac{1}{D-2} \delta^{ij} \alpha_{-1}^k \tilde{\alpha}_{-1}^k |0, P\rangle. \end{aligned} \quad (34)$$

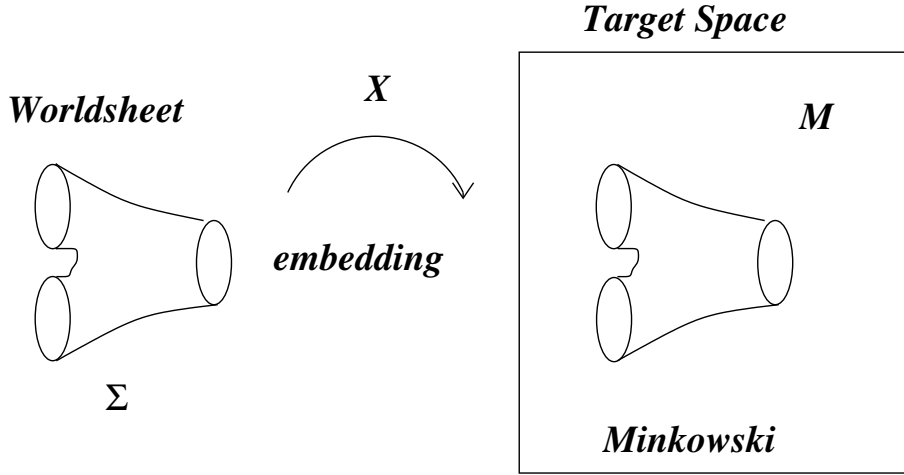
The first term of the rhs is interpreted as a spin 2 massless particle G_{ij} (*graviton*). The second term is a range 2 anti-symmetric tensor B_{ij} (*Kalb-Ramond field*). While the last term is a scalar field Φ (*dilaton*). Higher excited massive states are combinations of irreducible representations of the corresponding little group $SO(25)$ of the Lorentz group $SO(25, 1)$.

3.3. Interacting Strings and the Effective Action

Interacting Strings

So far we have described the *free* propagation of a closed (or open) bosonic string. In what follows we consider the interaction of these strings. Here we focus in the closed string case, the open case requires from further definitions. The interaction of strings at the perturbative level is just the extension of the technique of Feynman diagrams for point particles to extended objects. The vacuum-vacuum amplitude \mathcal{A} is given by

$$\mathcal{A} \sim \int \mathcal{D}h_{ab} \mathcal{D}X^I \exp \left(i S_P[X^I, h_{ab}] \right). \quad (35)$$



The interacting case requires to sum over all loop diagrams. In the closed string case it means that we have to sum over all compact orientable surfaces with non-trivial boundary ($\partial\Sigma \neq \emptyset$). In two dimensions these surfaces are completely characterized by their number of holes g (the genus) and boundaries b . The relevant topological invariant is the Euler number

$$\chi(\Sigma) = \frac{1}{4\pi} \int_{\Sigma} d^2\sigma \sqrt{-h} R^{(2)} + \frac{1}{2\pi} \int_{\partial\Sigma} ds K,$$

where $R^{(2)}$ is the scalar curvature and K is the geodesic curvature of the worldsheet Σ . In order to include the interaction of strings, the generalization of the Polyakov action consistent with its symmetries is given by

$$S = S_P[X^I, h_{ab}] + \frac{\Phi(X^I)}{2\pi} \left[\frac{1}{2\pi} \int_{\Sigma} d^2\sigma \sqrt{-h} R^{(2)} + \int_{\partial\Sigma} ds K \right], \tag{36}$$

where $\Phi(X)$ is a scalar background field and represents the gravitational coupling constant of the two-dimensional Einstein-Hilbert Lagrangian. If we define the string coupling constant by $g_S \equiv e^{\Phi}$, then Eq. (35) for the case of closed strings generalizes to

$$\mathcal{A} \sim \sum_{\chi} g_S^{\chi(\Sigma)} \int \mathcal{D}h_{ab} \mathcal{D}X^I \exp \left(i S_P[X^I, h_{ab}] \right). \tag{37}$$

The amplitude defined on-shell correspond to $g = 0$ and the rest ($g \geq 1$) corresponds to g -loop corrections.

The definition of correlation functions of operators requires of the idea of *vertex operators* \mathcal{V}_{Λ} . These operators are defined as

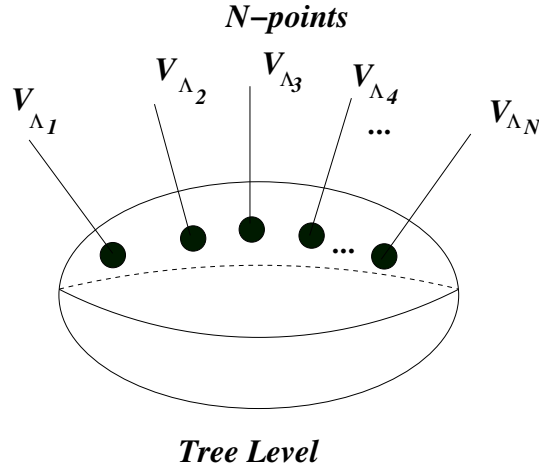
$$\mathcal{V}_{\Lambda}(k) = \int d^2\sigma \sqrt{-h} \mathcal{W}_{\Lambda}(\sigma, \tau) \exp(ik \cdot X), \tag{38}$$

where $\mathcal{W}_{\Lambda}(\sigma, \tau)$ (with Λ being a generic massless field of the bosonic spectra) is a local operator assigned to some specific state Λ of the spectrum of the theory. For instance for the tachyon ($\Lambda = T$) it is given by $\mathcal{W}_T(\sigma, \tau) \sim \partial_a X_I \partial^a X^I$. While that for the graviton G with polarization ζ_{IJ} it is given by $\mathcal{W}_G(\sigma, \tau) = \zeta_{IJ} \partial_a X^I \partial^a X^J$. Local operators \mathcal{V}_{Λ} are diffeomorphism and conformal invariant and therefore more convenient to define scattering amplitudes. In perturbation theory the scattering amplitude is given by

$$\mathcal{A}(\Lambda_1, k_1; \dots \Lambda_N, k_N) \sim \sum_{\chi} g_S^{\chi(\Sigma)} \int \mathcal{D}h \mathcal{D}X \exp \left(i S_P[X^I, h_{ab}] \right) \prod_{i=1}^N \mathcal{V}_{\Lambda_i}(k_i). \tag{39}$$

This scattering amplitude is, of course, proportional to the correlation function of the product of N invariant operators $\mathcal{V}_{\Lambda_i}(k_i)$ as follows

$$\mathcal{A}(\Lambda_1, k_1; \dots; \Lambda_N, k_N) \propto \left\langle \prod_{i=1}^N \mathcal{V}_{\Lambda_i}(k_i) \right\rangle. \quad (40)$$



3.4. Effective String Actions

In order to make contact with the spacetime physics we now describe how the spacetime equations of motion come from conformal invariance conditions for the non-linear sigma model in curved spaces. The immediate generalization of the Polyakov action is

$$S = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \sqrt{-h} h^{ab} \partial_a X^I \partial_b X^J G_{IJ}(X), \quad (41)$$

where $G_{IJ}(X)$ is an arbitrary background metric of the curved target space M . The perturbation of this metric $G_{IJ}(X) = \eta_{IJ} + h_{IJ}(X)$ in the partition function $Z \sim \exp(-S[X^I, \eta_{IJ} + h_{IJ}])$, leads to an expansion in powers of h_{IJ} . This partition function can be easily interpreted as containing the information of the interaction of the string with a coherent state of gravitons with invariant vertex operator $\mathcal{V}_G(k) = \int d^2\sigma \sqrt{-h} \mathcal{W}_G(\sigma, \tau) \exp(ik \cdot X)$ where $\mathcal{W}_G = h^{ab} \partial_a X^I \partial_b X^J h_{IJ}(X)$.

On the other hand, the Polyakov action can be generalized to be consistent with all symmetries and with the massless spectrum of the bosonic closed strings in the form of a non-linear sigma model

$$\hat{S} = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \sqrt{-h} \left[\left(h^{ab} G_{IJ}(X) + i\varepsilon^{ab} B_{IJ}(X) \right) \partial_a X^I \partial_b X^J + \alpha' \Phi(X) R^{(2)} \right], \quad (42)$$

where $G_{IJ}(X)$ is the target space curved metric, $B_{IJ}(X)$ is an anti-symmetric field, also called the *Kalb-Ramond* field, and $\Phi(X)$ is the scalar field called the *dilaton* field. From the viewpoint of the two-dimensional non-linear sigma model these background fields can be regarded as *coupling constants* and the renormalization group techniques become applied. The computation of the quantum conformal anomaly by using the dimensional regularization method, leads to express the energy-momentum trace as a linear combination

$$T_a^a = -\frac{1}{2\alpha'} \beta_{IJ}^G h^{ab} \partial_a X^I \partial_b X^J - \frac{i}{2\alpha'} \beta_{IJ}^B \varepsilon^{ab} \partial_a X^I \partial_b X^J - \frac{1}{2} \beta^\Phi R^{(2)}, \quad (43)$$

where β are the one-loop beta functions associated with each coupling constant or background field. They are explicitly computed and give

$$\beta_{IJ}^G = \alpha' \left(R_{IJ} + 2\nabla_I \nabla_J \Phi - \frac{1}{4} H_{IKL} H_J^{KL} \right) + O(\alpha'^2), \quad (44)$$

$$\beta_{IJ}^B = \alpha' \left(-\frac{1}{2} \nabla^K H_{KIJ} + \nabla^K \Phi H_{KIJ} \right) + O(\alpha'^2), \quad (45)$$

$$\beta^\Phi = \alpha' \left(\frac{D-26}{6\alpha'} - \frac{1}{2} \nabla^2 \Phi + \nabla_K \Phi \nabla^K \Phi - \frac{1}{24} H_{IJK} H^{IJK} \right) + O(\alpha'^2), \quad (46)$$

where $H_{IJK} = \partial_I B_{JK} + \partial_J B_{KI} + \partial_K B_{IJ}$. Weyl invariance at the quantum level implies the vanishing of the conformal anomaly and therefore the vanishing of each beta function separately. This leads to three coupled field equations for the background fields. These conditions for these fields can be regarded as equations of motion derivable from the spacetime action in D dimensions in the ‘string frame’

$$S = \frac{1}{2\kappa_0^2} \int_X d^D x \sqrt{-G} e^{-2\Phi} \left(R + 4\nabla_I \Phi \nabla^I \Phi - \frac{1}{12} H_{IJK} H^{IJK} - \frac{2(D-26)}{3\alpha'} + O(\alpha') \right), \quad (47)$$

where κ_0 is a normalization constant.

It is interesting to see that a redefinition of background metric under the conformal transformation in D dimensions: $\tilde{G}_{IJ}(X) = \exp(2\varpi(X)) G_{IJ}$ with $\varpi = \frac{2}{D-2}(\Phi_0 - \Phi)$, leads to the background action in the ‘Einstein frame’

$$\begin{aligned} \hat{S} = \frac{1}{2\kappa^2} \int d^D x \sqrt{-\tilde{G}} & \left(\tilde{R} - \frac{4}{D-2} \nabla_I \tilde{\Phi} \nabla^I \tilde{\Phi} - \frac{1}{12} e^{-8\tilde{\Phi}/(D-2)} H_{IJK} H^{IJK} \right. \\ & \left. - \frac{2(D-26)}{3\alpha'} e^{4\tilde{\Phi}/(D-2)} + O(\alpha') \right), \end{aligned} \quad (48)$$

where $\tilde{R} = e^{-2\varpi} [R - 2(D-1)\nabla^2 \varpi - (D-2)(D-1)\partial_I \varpi \partial^I \varpi]$ and $\tilde{\Phi} = \Phi - \Phi_0$. The form of this action will be of extreme importance later when we describe the strong/weak coupling duality in effective supergravity actions of the different superstring theories types. In the above action $\kappa \equiv \kappa_0 e^{\Phi_0} = \kappa_0 \cdot g_S$ is the gravitational coupling constant in D dimensions, *i.e.* $\kappa = \sqrt{8\pi G_N}$.

A very close procedure can be performed for the open string in order to compute its effective low energy action. For gauge fields with constant field strength F_{IJ} the action is given by the Dirac-Born-Infeld action

$$S_O = -T \int d^D x e^{-\Phi} \sqrt{-\det(G_{IJ} + B_{IJ} + 2\pi\alpha' F_{IJ})}. \quad (49)$$

4. T-duality and D-branes

This section has the purpose of introducing basic ideas about T-duality in closed and open string theory. The open string case leads in a natural way to the definition of D-branes (for reviews of D-branes see [7, 14, 15]). These objects are of extreme importance since they constitute the solitonic degrees of freedom which realize the strong/weak coupling duality in superstring theory. This duality is also known as string S-duality. T and S dualities relate the five perturbative superstring

theories discussed previously and their compactifications in diverse dimensions. Moreover, the strong coupling limit of HE and Type II string theory (and their compactifications) suggest that there is an eleven-dimensional theory which has the eleven-dimensional supergravity as low energy limit. This prospect of theory is widely known as M-theory. Compactifications to diverse lower dimensions than ten gives more evidence of the existence of this theory. The fundamental degrees of freedom of this unified theory are still unknown, but macroscopically it includes membranes and fivebranes. ‘Matrix Theory’ is an attempt to give the dof’s of M-theory. The proposal is that these degrees of freedom are the D0-branes. The worldvolume effective theory of a gas of N D0-branes is a $SU(N)$ quantum mechanics. Large N -limit reproduces the description of membranes and fivebranes and some other results of eleven dimensional supergravity (for some reviews the reader can consult [16, 17]).

D-branes also, are very important tools to study the strong coupling of supersymmetric theories in various dimensions. Different properties as chirality, dualities etc. are encoded in the engineering of brane configurations. The moduli space of these susy gauge theories is described by the Higgs and the Coulomb branches of the corresponding brane configuration. Many field theory results were understood in terms of a geometrical language and many generalizations have been established motivated by the brane engineering (more about this topic can be found in Ref. [18]). In these sections we will discuss some of these interesting topics.

4.1. Toroidal Compactification, T-duality and D-branes

D-branes are, despite of the dual fundamental degrees of freedom in string theory, extremely interesting and useful tools to study nonperturbative properties of string and field theories (for some reviews see [14, 15]). Non-perturbative properties of supersymmetric gauge theories can be better understanding as the world-volume effective theory of some configurations of intersecting D-branes (for a review see [18]). D-branes also are very important to connect gauge theories with gravity. This is the starting point of the AdS/CFT correspondence or Maldacena’s conjecture [19]. Roughly speaking D-branes are static solutions of string equations which satisfy Dirichlet boundary conditions. That means that open strings can end on them. To explain these objects we follow the traditional way, by using T-duality on open strings we will see that Neumann conditions are turned out into the Dirichlet ones. To motivate the subject we first consider T-duality in closed bosonic string theory.

T-duality in Closed Strings

The general solution of Eq. (26) in the conformal gauge can be written as $X^I(\sigma, \tau) = X_R^I(\sigma^-) + X_L^I(\sigma^+)$, where $\sigma^\pm = \sigma \pm \tau$. Now, take one coordinate, say X^{25} and compactify it on a circle of radius R . Thus we have that X^{25} can be identified with $X^{25} + 2\pi Rm$ where m is called the *winding number*. The general solution for X^{25} with the above compactification condition is

$$\begin{aligned} X_R^{25}(\sigma^-) &= X_{0R}^{25} + \sqrt{\frac{\alpha'}{2}} P_R^{25}(\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{l \neq 0} \frac{1}{l} \alpha_{R,l}^{25} \exp\left(-il(\tau - \sigma)\right), \\ X_L^{25}(\sigma^+) &= X_{0L}^{25} + \sqrt{\frac{\alpha'}{2}} P_L^{25}(\tau + \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{l} \alpha_{L,l}^{25} \exp\left(-il(\tau + \sigma)\right), \end{aligned} \quad (50)$$

where

$$P_{R,L}^{25} = \frac{1}{\sqrt{2}} \left(\frac{\sqrt{\alpha'}}{R} n \mp \frac{R}{\sqrt{\alpha'}} m \right). \quad (51)$$

Here n and m are integers representing the discrete Kaluza-Klein momentum and the winding number, respectively. The latter has not analogous in field theory. While the canonical momentum is given by $P^{25} = \frac{1}{\sqrt{2\alpha'}}(P_L^{25} + P_R^{25})$. Now, by the mass shell condition, the mass of the perturbative states is given by $M^2 = M_L^2 + M_R^2$, with

$$M_{L,R}^2 = -\frac{1}{2}P^I P_I = \frac{1}{2}(P_{L,R}^{25})^2 + \frac{2}{\alpha'}(N_{L,R} - 1). \tag{52}$$

We can see that for all states with $m \neq 0$, as $R \rightarrow \infty$ the mass become infinity, while $m = 0$ implies that the states take all values for n and form a continuum. At the case when $R \rightarrow 0$, for states with $n \neq 0$, mass become infinity. However in the limit $R \rightarrow 0$ for $n = 0$ states with all values of m produce a continuum in the spectrum. So, in this limit the compactified dimension disappears. For this reason, we can say that the mass spectrum of the theories at radius R and $\frac{\alpha'}{R}$ are identical when we interchange $n \leftrightarrow m$. This symmetry is known as *T-duality*.

The importance of T-duality lies in the fact that the T-duality transformation is a parity transformation acting on the left and right moving degrees of freedom. It leaves invariant the left movers and changes the sign of the right movers (see Eq. (52))

$$P_L^{25} \rightarrow P_L^{25}, \quad P_R^{25} \rightarrow -P_R^{25}. \tag{53}$$

The action of T-duality transformation must leave invariant the entire theory (at all order in perturbation theory). Thus, all kind of interacting states in certain theory should correspond to those states belonging to the dual theory. In this context, also the vertex operators are invariant. For instance the tachyonic vertex operators are

$$\mathcal{V}_L = \exp(iP_L^{25} X_L^{25}), \quad \mathcal{V}_R = \exp(iP_R^{25} X_R^{25}). \tag{54}$$

Under T-duality, $X_L^{25} \rightarrow X_L^{25}$ and $X_R^{25} \rightarrow -X_R^{25}$; and from the general solution Eq. (30), we get: $\alpha_{R,i}^{25} \rightarrow -\alpha_{R,i}^{25}$, $X_{0R}^{25} \rightarrow -X_{0R}^{25}$. Thus, T-duality interchanges $n \leftrightarrow m$ (Kaluza-Klein modes \leftrightarrow winding number) and $R \leftrightarrow \frac{\alpha'}{R}$ in closed string theory.

T-duality in Open Strings

Now, consider *open strings* with Neumann boundary conditions. Take again the 25th coordinate and compactify it on a circle of radius R , but keeping Neumann conditions. As in the case of closed string, center of mass momentum takes only discrete values $P^{25} = \frac{n}{R}$. While there is not analogous for the winding number. So, when $R \rightarrow 0$ all states with nonzero momentum go to infinity mass, and do not form a continuum. This behavior is similar as in field theory, but now there is something new. The general solutions are

$$\begin{aligned} X_R^{25} &= \frac{X_0^{25}}{2} - \frac{a}{2} + \alpha' P^{25}(\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{l \neq 0} \frac{1}{l} \alpha_l^{25} \exp(-i2l(\tau - \sigma)), \\ X_L^{25} &= \frac{X_0^{25}}{2} + \frac{a}{2} + \alpha' P^{25}(\tau + \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{l \neq 0} \frac{1}{l} \alpha_l^{25} \exp(-i2l(\tau + \sigma)), \end{aligned} \tag{55}$$

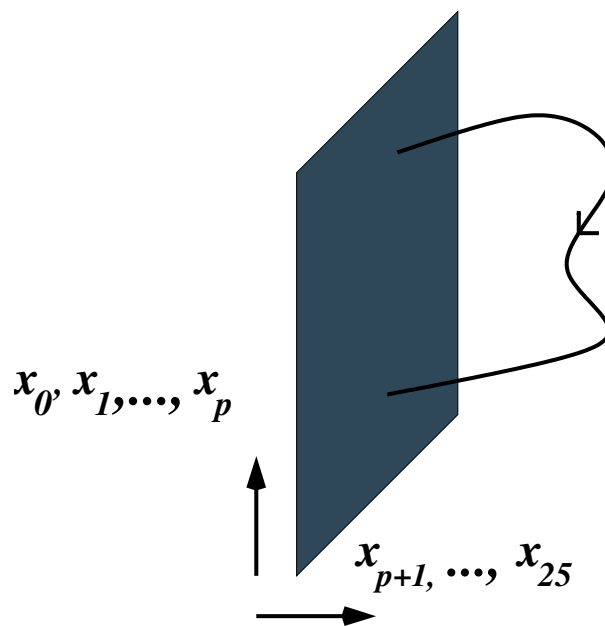
where a is a constant. Thus, $X^{25}(\sigma, \tau) = X_R^{25}(\sigma^-) + X_L^{25}(\sigma^+) = X_0^{25} + \frac{2\alpha'n}{R}\tau + \text{oscillator terms}$. Taking the limit $R \rightarrow 0$, only the $n = 0$ mode survives. Because of this, the string seems to move in 25 spacetime dimensions. In other words, the strings vibrate in 24 transversal directions. T-duality provides a new T-dual coordinate defined by $\tilde{X}^{25}(\sigma, \tau) = X_L^{25}(\sigma^+) - X_R^{25}(\sigma^-)$. Now,

taking $\tilde{R} = \frac{\alpha'}{R}$ we have $\tilde{X}^{25}(\sigma, \tau) = a + 2\tilde{R}\sigma n + \text{oscillator terms}$. Using the boundary conditions at $\sigma = 0, \pi$ one has $\tilde{X}^{25}(\sigma, \tau) |_{\sigma=0} = a$ and $\tilde{X}^{25}(\sigma, \tau) |_{\sigma=\pi} = a + 2\pi\tilde{R}n$. Thus, we started with an open bosonic string theory with Neumann boundary conditions, and T-duality and a compactification on a circle in the 25th dimension, give us Dirichlet boundary conditions in such a coordinate. We can visualize this saying that an open string has its endpoints fixed at a hyperplane with 24 dimensions.

Strings with $n = 0$ lie on a 24 dimensional plane space (D24-brane). Strings with $n = 1$ has one endpoint at a hyperplane and the other at a different hyperplane which is separated from the first one by a factor equal to $2\pi\tilde{R}$, and so on. But if we compactify q of the X^i directions over a T^q torus ($i = 1, \dots, q$). Thus, after T-dualizing them we have strings with endpoints fixed at hyperplane with $25 - q$ dimensions, the D(25 - q)-brane.

Summarizing: the system of open strings moving freely in spacetime with q compactified dimensions on T^q is equivalent, under T-duality, to strings whose endpoints are fixed at a D p -brane (with $p = 25 - q$) *i.e.* obeying Neumann boundary conditions in the X^i longitudinal directions ($i = 1, \dots, p$) and Dirichlet ones in the transverse coordinates X^m ($m = p + 1, \dots, 25$).

D p -brane



The effect of T-dualizing a coordinate is to change the nature of the boundary conditions, from Neumann to Dirichlet and vice versa. If one dualize a longitudinal coordinate this coordinate will satisfies the Dirichlet condition and a D p -brane becomes a D($p - 1$)-brane. But if the dualized coordinate is one of the transverse coordinates the D p -brane becomes a D($p + 1$)-brane.

T-duality also acts conversely. We can think to begin with a closed string theory, and compactify it on to a circle in the 25th coordinate, and then by imposing Dirichlet conditions, obtain a D-brane. This is precisely what occurs in Type II theory, a theory of closed strings.

Spectrum and Wilson Lines

Now, we will see how does emerges a gauge field on the D p -brane worldvolume. Again, for the mass shell condition for open bosonic strings and because T-duality one gets: $M^2 = (\frac{n}{\alpha'}\tilde{R})^2 + \frac{1}{\alpha'}(N - 1)$, and consequently, the following spectrum:

- The groundstate ($N = 0, n = 0$) has a tachyon field of mass: $M^2 = -\frac{1}{\alpha'}$.
- The massless state ($N = 1, n = 0$) implies that the gauge boson $\alpha_{-1}^I | 0\rangle$ ($U(1)$ gauge boson) lies on the D24-brane world-volume.
- There is a scalar $\alpha_{-1}^{25} | 0\rangle$ which has a *vev* (vacuum expectation value) and parametrizes the position \tilde{X}^{25} of the D-brane after T-dualizing.

Thus, we can say in general, there is a $U(1)$ gauge theory over the worldvolume of the Dp -brane.

Consider now an *orientable open string*. The endpoints of the string carry global charge under a non-Abelian gauge group. For Type II theories the gauge group is $U(N)$. One endpoint transforms under the fundamental representation \mathbf{N} of $U(N)$ and the other one, under its complex conjugate representation (the anti-fundamental one) \mathbf{N}^* .

The ground state wave function is specified by the momentum of the center of mass and by the charges of the endpoints. This implies the existence of a basis $|k; ij\rangle$ called *Chan-Paton basis*. States $|k; ij\rangle$ of the Chan-Paton basis are those states which carry charge 1 under the i^{th} $U(1)$ generator and -1 under the j^{th} $U(1)$ generator. So, we can decompose the wave function for ground state as $|k; a\rangle = \sum_{i,j=1}^N |k; ij\rangle \lambda_{ij}^a$ where λ_{ij}^a are called *Chan-Paton factors*. From this, we see that it is possible to add degrees of freedom to endpoints of the string, that are precisely the Chan-Paton factors.

This is consistent with the theory, because the Chan-Paton factors have a Hamiltonian which do not possesses dynamical structure. So, if one endpoint to the string is prepared in a certain state, it always will remains the same. It can be deduced from this, that $\lambda^a \rightarrow U \lambda^a U^{-1}$ with $U \in U(N)$. Thus, the worldsheet theory is symmetric under $U(N)$, and this global symmetry is a gauge symmetry in spacetime. So the vector state at massless level $\alpha_{-1}^I |k, a\rangle$ is a $U(N)$ gauge boson.

When we have a gauge configuration with non-trivial line integral around a compactified dimension (i.e a circle), we said there is a Wilson line. In case of open strings with gauge group $U(N)$, a toroidal compactification of the 25^{th} dimension on a circle of radius R . If we choice a background field A^{25} given by $A^{25} = \frac{1}{2\pi R} \text{diag}(\theta_1, \dots, \theta_N)$ a Wilson line appears. Moreover, if $\theta_i = 0$ for $i = 1, \dots, l$ and $\theta_j \neq 0$ for $j = l + 1, \dots, N$, then gauge group is broken: $U(N) \rightarrow U(l) \times U(1)^{N-l}$. It is possible to deduce that θ_i plays the role of a Higgs field. Because string states with Chan-Paton quantum numbers $|ij\rangle$ have charges 1 under i^{th} $U(1)$ factor (and -1 under j^{th} $U(1)$ factor) and neutral with all others; canonical momentum is given now by $P_{(ij)}^{25} \Rightarrow \frac{n}{R} + \frac{(\theta_j - \theta_i)}{2\pi R}$. Returning to the mass shell condition it results,

$$M_{ij}^2 = \left(\frac{n}{R} + \frac{\theta_j - \theta_i}{2\pi R} \right)^2 + \frac{1}{\alpha'}(N - 1). \tag{56}$$

Massless states ($N = 1, n = 0$) are those in where $i = j$ (diagonal terms) or for which $\theta_j = \theta_i$ ($i \neq j$). Now, T-dualizing we have: $\tilde{X}_{ij}^{25}(\sigma, \tau) = a + (2n + \frac{\theta_j - \theta_i}{\pi})\tilde{R}\sigma + \text{oscillator terms}$. Taking $a = \theta_i \tilde{R}$, $\tilde{X}_{ij}^{25}(0, \tau) = \theta_i \tilde{R}$ and $\tilde{X}_{ij}^{25}(\pi, \tau) = 2\pi n \tilde{R} + \theta_j \tilde{R}$. This give us a set of N D-branes whose positions are given by $\theta_j \tilde{R}$, and each set is separated from its initial positions ($\theta_j = 0$) by a factor equal to $2\pi \tilde{R}$. Open strings with both endpoints on the same D-brane gives massless gauge bosons. The set of N D-branes give us $U(1)^N$ gauge group. An open string with one endpoint in one D-brane, and the other endpoint in a different D-brane, yields a massive state with $M \sim (\theta_j - \theta_i) \tilde{R}$. Mass decreases when two different D-branes approximate to each other, and is zero as the distance between them become zero. When all D-branes take up the same position, the gauge group is enhanced from $U(1)^N$ to $U(N)$. On the D-brane worldvolume there are also scalar fields in the adjoint representation of the gauge group $U(N)$. The scalars parametrize the transverse positions of the D-brane in the target space M .

4.2. D-Brane Action

With the massless spectrum on the D-brane worldvolume it is possible to construct a low energy effective action. Open strings massless fields are interacting with the closed strings massless spectrum from the **NS-NS** sector. Let ξ^a (with $a = 0, \dots, p$) be the worldvolume coordinates on W . The effective action is the gauge invariant action well known as the Dirac-Born-Infeld (DBI)-action

$$S_D = -T_p \int_W d^{p+1}\xi e^{-\Phi} \sqrt{\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})}, \quad (57)$$

where T_p is the tension of the D-brane, G_{ab} is the worldvolume induced metric, B_{ab} is the induced antisymmetric field, F_{ab} is the Abelian field strength on W and Φ is the dilaton field.

For N D-branes the massless fields turns out to be $N \times N$ matrices and the action turns out to be non-Abelian DBI-action (for a nice review about the Born-Infeld action in string theory see [20])

$$S_D = -T_p \int_W d^{p+1}\xi e^{-\Phi} \text{Tr} \left(\sqrt{\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})} + O([X^m, X^n]^2) \right), \quad (58)$$

where $m, n = p + 1, \dots, 25$. The scalar fields X^m representing the transverse positions become $N \times N$ matrices and so, the spacetime become a noncommutative spacetime.

4.3. Ramond-Ramond Charges

D-branes are coupled to Ramond-Ramond (RR) fields G_p [14]. The complete effective action on the D-brane world-volume W which taking into account this coupling is

$$S_D = -T_p \int_W d^{p+1}\xi \left\{ e^{-\Phi} \sqrt{\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})} + i\mu_p \int_W \sum_p C_{(p+1)} \text{Tr} \left(e^{2\pi\alpha'(F+B)} \right) \right\}, \quad (59)$$

where μ_p is the RR charge. RR charges can be computed by considering the anomalous behavior of the action at intersections of D-branes. Thus RR charge is given by

$$Q_{RR} = ch(j!E) \sqrt{\widehat{A}(TM)}, \quad (60)$$

where $j : W \hookrightarrow M$. Here E is the Chan-Paton bundle over M , $\widehat{A}(TM)$ is the genus of the spacetime manifold M . This gives an ample evidence that the RR charges take values not in a cohomology theory, but in fact, in a K-Theory. This result was developed by Witten in Ref. [21] in the context of non-BPS brane configurations worked out by Sen [22].

Finally, RR charges and RR fields do admit a classification in terms of topological K-theory. The inclusion of a B -field turns out the effective theory non-commutative and a suitable generalization of the topological K-theory is needed. The *right* generalization seems to be the K-Homology and the K-theory of C^* algebras.

5. Perturbative Superstring Theory

In bosonic string theory there are two bold problems. The first one is the presence of tachyons in the spectrum. The second one is that there are no spacetime fermions. Here is where superstrings come to the rescue. A superstring is described, despite of the usual bosonic fields X^I , by Majorana-Weyl fermions $\psi_{L,R}^I$ on the worldsheet Σ . Fermion fields satisfy anticommutation

rules where the L and R denote the left and right worldsheet chiralities respectively. The action for the superstring is given by

$$L_{SS} = -\frac{1}{8\pi} \int d^2\sigma \sqrt{-h} \left[h^{\alpha\beta} \partial_\alpha X^I \partial_\beta X_I + 2i\bar{\psi}^I \gamma^\alpha \partial_\alpha \psi_I - i\bar{\chi}_\alpha \gamma^\beta \gamma^\alpha \psi^I (\partial_\beta X_I - \frac{i}{4} \bar{\chi}_\beta \psi_I) \right], \quad (61)$$

where ψ^I and χ_α are the superpartners of X^I and the vier-bein e_α^a . This action is invariant under:

(i) Worldsheet $\mathcal{N} = 1$ local supersymmetry:

$$\begin{aligned} \delta_\varepsilon X^I &= \bar{\varepsilon} \psi^I, & \delta_\varepsilon \psi^I &= \frac{1}{2} \gamma^\alpha \varepsilon \left(\partial_\alpha X^I - \frac{i}{2} \bar{\chi}_\alpha \psi^I \right), \\ \delta_\varepsilon e_\alpha^a &= \frac{i}{2} \bar{\varepsilon} \gamma^a \chi_\alpha, & \delta_\varepsilon \chi_\alpha &= 2D_\alpha \varepsilon, \end{aligned} \quad (62)$$

where $\varepsilon(\sigma, \tau)$ is a Majorana spinor and D_α is the covariant derivative with respect to the spin connection.

(ii) Weyl transformations:

$$\begin{aligned} \delta_\Lambda X^I &= 0, & \delta_\Lambda \psi^I &= -\frac{1}{2} \Lambda \psi^I, \\ \delta_\Lambda e_\alpha^a &= \Lambda e_\alpha^a, & \delta_\Lambda \chi_\alpha &= \frac{1}{2} \Lambda \chi_\alpha. \end{aligned} \quad (63)$$

(iii) Super-Weyl transformations:

$$\delta_\eta \chi_\alpha = \gamma_\alpha \eta, \quad \delta_\eta(\text{others}) = 0, \quad (64)$$

with $\eta(\sigma, \tau)$ being a Majorana spinor parameter.

(iv) Two-dimensional Lorentz transformations:

$$\begin{aligned} \delta_l X^I &= 0, & \delta_l \psi^I &= \frac{1}{2} l \bar{\gamma} \psi^I, \\ \delta_l e_\alpha^a &= l \varepsilon^a_b e_\alpha^b, & \delta_l \chi_\alpha &= \frac{1}{2} l \bar{\gamma} \chi_\alpha. \end{aligned} \quad (65)$$

(v) Worldsheet reparametrizations:

$$\begin{aligned} \delta_\xi X^I &= \xi^\beta \partial_\beta X^I, & \delta_\xi \psi^I &= \xi^\beta \partial_\beta \psi^I, \\ \delta_\xi e_\alpha^a &= \xi^\beta \partial_\beta e_\alpha^a + e_\beta^a \partial_\alpha \xi^\beta, & \delta_\xi \chi_\alpha &= \xi^\beta \partial_\beta \chi_\alpha + \chi_\beta \partial_\alpha \xi^\beta. \end{aligned} \quad (66)$$

In the superconformal gauge ($h_{ab} = \eta_{ab}$ and $\chi_a = 0$) and in light-cone coordinates it can be reduced to

$$L_{SS} = \frac{1}{2\pi} \int \left(\partial_L X^I \partial_R X_I + i\psi_R^I \partial_L \psi_{I R} + i\psi_L^I \partial_R \psi_{I L} \right). \quad (67)$$

In analogy to the bosonic case, the local dynamics of the worldsheet metric is manifestly conformal anomaly free at the quantum level if the critical spacetime dimension D is 10. Thus the string oscillates in the 8 transverse dimensions. The equation of motion for the X^I 's fields is the same that in the bosonic case (Laplace equation) and whose general solution is given by Eqs. (30) or (31). Equation of motion for the fermionic field is the Dirac equation in two dimensions. Constraints here are more involved and they are called the *super-Virasoro constraints*. However in the light-cone gauge, everything simplifies and the transverse

coordinates (eight coordinates) become the bosonic physical degrees of freedom together with their corresponding supersymmetric partners. Analogously to the bosonic case, massless states of the spectrum come into representations of the little group $SO(8)$ which is a subgroup of $SO(9,1)$, while that the massive states lie into representations of the little group $SO(9)$.

For the closed string there are two possibilities for the boundary conditions of fermions: (i) periodic boundary conditions (Ramond (**R**) sector) $\psi_{L,R}^I(\sigma) = +\psi_{L,R}^I(\sigma + 2\pi)$ and (ii) anti-periodic boundary conditions (Neveu-Schwarz (**NS**) sector) $\psi_{L,R}^I(\sigma) = -\psi_{L,R}^I(\sigma + 2\pi)$. Solutions of Dirac equation satisfying these boundary conditions are:

- **R** Sector:

$$\psi_L^I(\sigma, \tau) = \sum_{n \in \mathbb{Z}} \tilde{d}_n^I \exp\left(-in(\tau + \sigma)\right), \quad \psi_R^I(\sigma, \tau) = \sum_{n \in \mathbb{Z}} d_n^I \exp\left(-in(\tau - \sigma)\right). \quad (68)$$

- **NS** Sector:

$$\psi_L^I(\sigma, \tau) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \tilde{b}_r^I \exp\left(-ir(\tau + \sigma)\right), \quad \psi_R^I(\sigma, \tau) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} b_r^I \exp\left(-ir(\tau - \sigma)\right), \quad (69)$$

where $\tilde{d}_n^I, \tilde{b}_r^I$ and d_n^I, b_r^I are fermionic modes for the left and right movers respectively.

In the case of the fermions in the **R** sector n is integer and r is semi-integer in the **NS** sector.

The quantization of the superstring come from the promotion of the fields X^I and ψ^I to operators whose oscillator variables are operators satisfying the relations $[\alpha_n^I, \alpha_m^J]_- = n\delta_{m+n,0}\eta^{IJ}$ and $[d_n^I, d_m^J]_+ = \eta^{IJ}\delta_{m+n,0}$, $[b_r^I, b_s^J]_+ = \eta^{IJ}\delta_{r+s,0}$, where $[\cdot, \cdot]_-$ and $[\cdot, \cdot]_+$ stand for commutator and anti-commutator respectively.

The zero modes of α are diagonal in the Fock space and its eigenvalue can be identified with its momentum. For the **NS** sector there is no fermionic zero modes but they can exist for the **R** sector and they satisfy a Clifford algebra $[\psi_0^I, \psi_0^J]_+ = \eta^{IJ}$. The Hamiltonian for the closed superstring is given by $H_{L,R} = N_{L,R} + \frac{1}{2}P_{L,R}^2 - A_{L,R}$. For the **NS** sector $A = \frac{1}{2}$, while for the **R** sector $A = 0$. The spectrum of masses is given by $M^2 = M_L^2 + M_R^2$ with $\frac{1}{2}M_{L,R}^2 = N_{L,R} - A_{L,R}$.

There are five consistent superstring theories: Type IIA, IIB, Type I, $SO(32)$ and $E_8 \times E_8$ heterotic strings, represented by HO and HE respectively. In what follows of this section we briefly describe the spectrum in each one of them.

5.1. Type II Superstrings

In this case the theory consist of closed strings only. They are theories with $\mathcal{N} = 2$ spacetime supersymmetry. There are 8 scalar fields (representing the 8 transverse coordinates to the string). There are 8 left-moving and 8 right-moving Majorana-Weyl fermions.

In the **NS** sector there is still a tachyon in the ground state. But in the supersymmetric case this problem can be solved through the introduction of the called **GSO** projection. This projection eliminates the tachyon in the **NS** sector and it acts in the **R** sector as a ten-dimensional spacetime chirality operator. That means that the application of the **GSO** projection operator defines the chirality of a massless spinor in the **R** sector. Thus from the left and right moving sectors, one can construct states in four different sectors: (i) **NS-NS**, (ii) **NS-R**, (iii) **R-NS** and (iv) **R-R**. Taking into account the two types of chirality L and R one has two possibilities:

a)– The **GSO** projections on the left and right fermions produce different chirality in the ground state of the **R** sector (*Type IIA*).

b)–**GSO** projection are equal in left and right sectors and the ground states in the **R** sector, have the same chirality (*Type IIB*). Thus the spectrum for the Type IIA and IIB superstring theories is:

- *Type IIA*
 - (i) The **NS-NS** sector has a symmetric tensor field G_{IJ} (spacetime metric), an antisymmetric tensor field B_{IJ} and a scalar field Φ (dilaton).
 - (ii) In the **R-R** sector there is a vector field C_I associated with a 1-form $C_{(1)}$ ($C_I \Leftrightarrow C_{(1)}$) and a rank 3 totally antisymmetric tensor $C_{IJK} \Leftrightarrow C_{(3)}$ and by Hodge duality in ten dimensions also we have $C_{(5)}$ and $C_{(7)}$. In general the **R-R** sector consist of p -forms $F_{(p)} = dC_{(p-1)}$ (where $C_{(p)}$ are called RR fields) on the ten-dimensional spacetime M with p even i.e. $F_{(2)}, F_{(4)}, \dots, F_{(8)}$.
 - (iii) In the **NS-R** and **R-NS** sectors we have two gravitinos with opposite chirality and the supersymmetric partners of the mentioned bosonic fields.
- *Type IIB*
 - (i) In the **NS-NS** sector Type IIB theory has exactly the same spectrum than Type IIA theory.
 - (ii) On the **R-R** sector it has a scalar field $C \Leftrightarrow C_{(0)}$ (the axion field), an antisymmetric tensor field $C_{IJ} \Leftrightarrow C_{(2)}$ and a rank 4 totally antisymmetric tensor $C_{IJKL} \Leftrightarrow C_{(4)}^+$ whose field strength is self-dual i.e., $F_{(5)} = dC_{(4)}^+$ with $*F_{(5)} = +F_{(5)}$. Similar than for the case of Type IIA theory one has also the Hodge dual fields $C_{(6)}$ and $C_{(8)}$. In general, RR fields in Type IIB theory are given by p -forms $F_{(p)} = dC_{(p-1)}$ on the spacetime M with p odd i.e. $F_{(1)}, F_{(3)}, \dots, F_{(9)}$.
 - (iii) The **NS-R** and **R-NS** sectors contain two gravitinos with the same chirality and the corresponding fermionic matter.

In the next table we summarizes the spectrum of Type II theory:

	NS – NS	R – R
<i>IIA</i>	$\mathbf{8}_v \otimes \mathbf{8}_v$ $= \mathbf{1} \oplus \mathbf{26} \oplus \mathbf{35}$ G_{IJ}, B_{IJ}, Φ	$\mathbf{8}_s \otimes \mathbf{8}_c$ $= \mathbf{8}_v \oplus \mathbf{56}_t$ $C_{(1)}, C_{(3)}$
<i>IIB</i>	$\mathbf{8}_v \otimes \mathbf{8}_v$ $= \mathbf{1} \oplus \mathbf{26} \oplus \mathbf{35}$ G_{IJ}, B_{IJ}, Φ	$\mathbf{8}_s \otimes \mathbf{8}_s$ $= \mathbf{1} \oplus \mathbf{26} \oplus \mathbf{35}$ $C_{(0)}, C_{(2)}, C_{(4)}^+$
	NS – R	R – NS
<i>IIA</i>	$\mathbf{8}_v \otimes \mathbf{8}_c$ $\mathbf{8}_c \otimes \mathbf{56}_c$ $\lambda^\alpha, \psi_I^\alpha$	$\mathbf{8}_s \otimes \mathbf{8}_v$ $= \mathbf{8}_s \oplus \mathbf{56}_s$ $\lambda^{\dot{\alpha}}, \psi_I^{\dot{\alpha}}$
<i>IIB</i>	$\mathbf{8}_v \otimes \mathbf{8}_s$ $= \mathbf{8}_s \oplus \mathbf{56}_s$ $\lambda^\alpha, \psi_I^\alpha$	$\mathbf{8}_s \otimes \mathbf{8}_v$ $= \mathbf{8}_s \oplus \mathbf{56}_s$ $\lambda^\alpha, \psi_I^\alpha$

5.2. Type I Superstrings

In this case the L and R degrees of freedom are identified by the worldsheet parity operation. Type I and Type IIB theories have the same spectrum, except that in the former one the states which are not invariant under the change of orientation of the worldsheet, are projected out. This worldsheet parity Ω interchanges the left and right modes. Type I superstring theory is a

theory of breakable closed strings, thus it incorporates also open strings. The Ω operation leave invariant only one half of the spacetime supersymmetry, thus the theory is $\mathcal{N} = 1$.

The spectrum of bosonic massless states:

- (i) In the **NS-NS** sector it is: G_{IJ} (spacetime metric) and Φ (dilaton) from the closed sector and B_{IJ} is projected out.
- (ii) In the **R-R** sector there is an antisymmetric field C_{IJ} from the closed sector.
- (iii) The open string sector is necessary in order to cancel tadpole diagrams. A contribution to the spectrum come from this sector. Chan-Paton factors can be added at the boundaries of open strings. Hence the cancellation of the tadpole are needed 32 labels at each end. Therefore in the **NS-NS** sector there are 496 gauge fields in the adjoint representation of $SO(32)$.

5.3. Heterotic Superstrings

This kind of theory involves only closed strings. Thus there are left and right sectors. The left-moving sector contains a bosonic string theory and the right-moving sector contains superstrings. This theory is supersymmetric on the right sector only, thus the theory contains $\mathcal{N} = 1$ spacetime supersymmetry. The momentum at the left sector P_L lives in 26 dimensions, while P_R lives in 10 dimensions. It is natural to identify the first ten components of P_L with P_R . Consistency of the theory tell us that the extra 16 dimensions should belong to the root lattice $E_8 \times E_8$ or a \mathbf{Z}_2 -sublattice of the $SO(32)$ weight lattice.

The spectrum consists of:

- (i) A tachyon in the ground state of the left-moving sector.
- (ii) In both sectors we have the spacetime metric G_{IJ} , the antisymmetric tensor B_{IJ} , the dilaton Φ .
- (iii) There are 496 gauge fields A_I in the adjoint representation of the gauge group $E_8 \times E_8$ or $SO(32)$.

All these Types of superstring theories do admit a low energy effective description in terms of a supergravity theory. These theories involves the corresponding background massless fields of their spectra. Supergravity actions of these diverse types will be constructed later (in the third lecture)to study strong/weak coupling duality in string theory.

Finally to summarize we have:

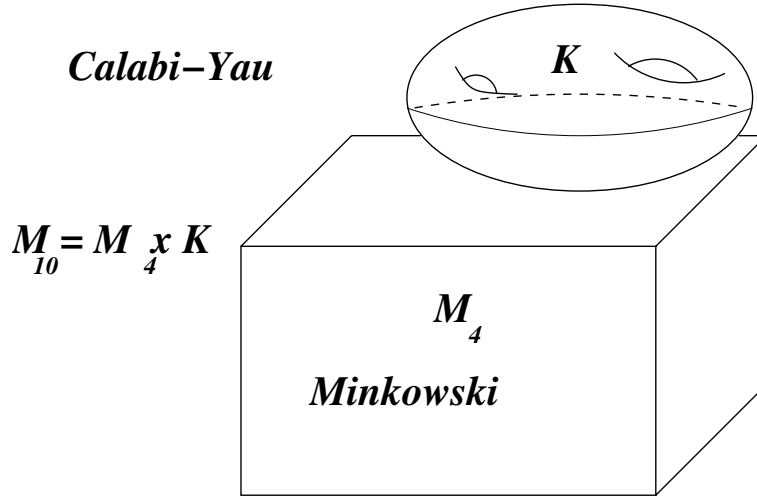
	Q's	\mathcal{N}	Sector	Spectrum
<i>IA</i>	32	2	NS-NS RR	G_{IJ}, B_{IJ}, Φ $C_{(1)}, C_{(3)}$
<i>IIB</i>	32	2	NS-NS RR	G_{IJ}, B_{IJ}, Φ $C_{(0)}, C_{(2)}, C_{(4)}^+$
<i>I</i>	16	1	NS-NS NS(OS)	G_{IJ}, Φ $A_I^a \in \mathbf{Ad}(SO(32))$
<i>HE</i>	16	1	NS-NS Gauge	G_{IJ}, B_{IJ}, Φ $A_I^a \in \mathbf{Ad}(E_8 \times E_8)$
<i>HO</i>	16	1	NS-NS Gauge	G_{IJ}, B_{IJ}, Φ $A_I^a \in \mathbf{Ad}(SO(32))$

6. Calabi-Yau Compactifications in Perturbative String Theory

In order to connect superstring theories to the observed 4-dimensional spacetime physics, we have to reduce the critical dimension $D = 10$ to four dimensions. To preserve certain supersymmetry consistent with chirality in four dimensions it is necessary to require some conditions over the ten dimensional spacetime M_{10} . Perhaps the simplest ansatz is to assume that the four-dimensional Minkowski spacetime M_4 and a six-dimensional internal manifold \mathcal{K} factorizes as (see figure)

$$M_{10} \cong M_4 \times \mathcal{K}, \tag{70}$$

where \mathcal{K} has tiny dimensions and unobservable in our present collider experiments. It is worth to say that this factorization ansatz is not unique and other possibility is the *warped compactification* of the celebrated Randall-Sundrum scenarios, which are nicely reviewed in Ref. [23].



It is useful to classify the compactifications according to how much supersymmetries is broken, because this number is related with the quantum corrections that we shall consider. We choose \mathcal{K} to be a manifold with the property that a certain number of supersymmetries are preserved¹.

We are now looking for conditions in the background which leave some supersymmetry unbroken. These conditions are given by null variations of the Fermi fields.

Consider the *diagonal* metric for ten-dimensional spacetime M_{10} given by

$$G_{IJ} = f(y)\eta_{\mu\nu} + G_{mn}(y), \tag{71}$$

where y labels the compact coordinates and $I, J = 0, \dots, 9, \mu, \nu = 0, \dots, 3, m, n = 4, \dots, 9$.

In order to preserve some supersymmetry after compactification to the four-dimensional Minkowski spacetime we impose some conditions on \mathcal{K} . These conditions can be studied from the susy conditions on the internal manifold \mathcal{K} : For $D = 10, \mathcal{N} = 1$ heterotic string theory the Fermi fields variations are:

- *gravitino*: $\delta\psi_I = (\partial_I + \frac{1}{4}\Omega_{IJK}^- \Gamma^{JK})\varepsilon,$
- *dilatino*: $\delta\xi = (\Gamma^I \partial_I \phi - \frac{1}{12}\Gamma^{IJK} H_{IJK})\varepsilon,$
- *gaugino*: $\delta\lambda = F_{IJ}\Gamma^{IJ}\varepsilon,$

¹ A cosmological constant is generated by perturbation theory. Strings propagating in a manifold \mathcal{K} in which all supersymmetries are broken destabilizes the Minkowski vacuum.

where $\varepsilon(x, y)$ is a Weyl-Majorana spinor in ten dimensions (in the spinor irrep **16**), $\Omega_{MNP}^- = \omega_{MNP} - \frac{1}{2}H_{MNP}$ and Γ^I are the Dirac matrices in ten dimensions.

The compactification ansatz $M_{10} = M_4 \times \mathcal{K}$ breaks the Lorentz group $SO(9,1)$ into $SO(3,1) \times SO(6)$. In the spinor representation **16** the Weyl-Majorana supersymmetry parameter $\varepsilon_{\alpha\beta}(x, y)$ decomposes as $\varepsilon(x, y) \rightarrow \varepsilon_{\alpha\beta}(x, y) + \varepsilon_{\alpha\beta}^*(x, y)$ under $\mathbf{16} \rightarrow (\mathbf{2}, \mathbf{4}) \oplus (\mathbf{2}^*, \mathbf{4}^*)$. The general form of $\varepsilon_{\alpha\beta}$ is $\varepsilon_{\alpha\beta}(x, y) = u_\alpha(x)\zeta_\beta(y)$ with $u_\alpha(x)$ an arbitrary Weyl spinor and $\zeta_\beta(y)$ is in a **4** or **4*** of $SO(6)$. Susy unbroken in low energy physics is associated to the remaining (local) susy parameters $\varepsilon(x, y)$ in $M_4 \times \mathcal{K}$. When we put the condition that Fermi fields variations vanish, then each internal spinor $\zeta_\beta(y)$ gives the minimal ($\mathcal{N} = 1$) $D = 4$ supersymmetric algebra.

Now, by the null Fermi fields variations we can find conditions in the background fields assuming that $H_{mnp} = 0$. These are:

- $\delta\psi_\mu = 0 \Rightarrow G_{\mu\nu} = \eta_{\mu\nu}$,
- $\delta\psi_m = 0 \Rightarrow \nabla_m \zeta = 0$.
- $\delta\xi = 0 \Rightarrow \partial_m \phi$,
- $\delta\lambda = 0 = F_{mn}\Gamma^{mn}\varepsilon$,

where Ω_{mnp}^- is the internal component of Ω_{MNP}^- .

The gravitino equation tell us that ζ_β is covariantly constant on the internal space \mathcal{K} , and it implies that \mathcal{K} is *Ricci-flat*. This is because

$$[\nabla_m, \nabla_n]\zeta = \frac{1}{4}R_{mnpq}\Gamma^{pq}\zeta = 0. \quad (72)$$

Spinor representation of $SO(6) \cong SU(4)$ are **4** and **4***. These representations don't have a singlet. If we introduce a metric with holonomy group $SU(3)$ then spinor representations should transform as $\mathbf{4} = \mathbf{1} \oplus \mathbf{3}$ and $\mathbf{4}^* = \mathbf{1} \oplus \mathbf{3}^*$. Then

$$\begin{aligned} SO(10) &\rightarrow SO(6) \times SO(4) \rightarrow SU(3) \times SO(4) \\ \mathbf{16} &\rightarrow (\mathbf{4}, \mathbf{2}) \oplus (\mathbf{4}^*, \mathbf{2}^*) \rightarrow (\mathbf{3}, \mathbf{2}) \oplus (\mathbf{3}^*, \mathbf{2}^*) \oplus (\mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2}^*). \end{aligned} \quad (73)$$

The part $(\mathbf{3}, \mathbf{2}) \oplus (\mathbf{3}^*, \mathbf{2}^*)$ is then broken while $(\mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2}^*)$ is unbroken.

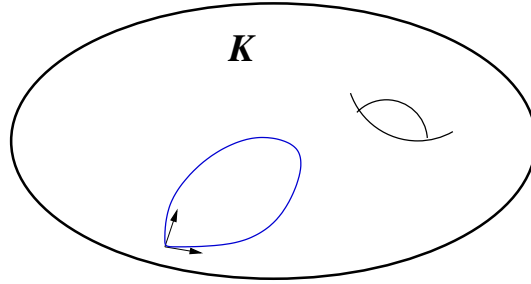
From the spectrum of the heterotic theory we know that the gravitino transform as **56_s** of $\mathbf{8}_v \otimes \mathbf{8}_c$. Then respect to $SO(6) \times SO(2)$ we can decompose it into:

$$\mathbf{8}_v \rightarrow \mathbf{6}_0 \oplus \mathbf{1}_{\pm 1}, \quad \mathbf{8}_c \rightarrow \mathbf{4}_{\frac{1}{2}} \oplus \mathbf{4}_{-\frac{1}{2}}^*. \quad (74)$$

4D gravitinos have spin $\frac{3}{2}$ and are charged with respect to $SO(2)$. Under $SU(3) \times SO(2)$, the fields obtained are:

$$\mathbf{1}_{\pm 1} \otimes (\mathbf{3}_{\frac{1}{2}} \oplus \mathbf{3}_{-\frac{1}{2}}^* \oplus \mathbf{1}_{\frac{1}{2}} \oplus \mathbf{1}_{-\frac{1}{2}}). \quad (75)$$

This is the reason, in general Γ^{pq} do not belong to $SO(6)$ but to $SU(3)$, which is a subgroup that leaves one component of the spinor ζ invariant. Thus the compact manifold \mathcal{K} must have $SU(3)$ holonomy. The first unbroken susy condition implies that the warped factor $f(y)$ in metric is 1 and the metric G_{IJ} is unwarped. Finally, the third condition implies that the dilaton is constant. This is a *Calabi-Yau* three-fold. A Calabi-Yau three-fold is also a Kähler manifold in which the first Chern class is zero *i.e.* $c_1(T\mathcal{K}) = 0$. Any Calabi-Yau manifold of $SU(3)$ holonomy possesses a unique Ricci-flat metric. When we consider $\mathcal{N} = 1$ heterotic string theory on Calabi-Yau three-fold we obtain a four-dimensional chiral theory with spacetime supersymmetry $\mathcal{N} = 1$. In fact, compactification on manifolds of $SU(3)$ holonomy preserves 1/4 of supersymmetry. If we consider $\mathcal{N} = 2$ theories (for example, Type II superstrings) in $D = 10$ dimensions, after compactification on a Calabi-Yau three-fold we obtain $\mathcal{N} = 2$ theories in $D = 4$.



Calabi–Yau

In addition to the CY-threefold structure for \mathcal{K} the unbroken susy condition $\delta\lambda^a = 0 = F_{mn}\Gamma^{mn}\varepsilon$, leads to the equations in complex coordinates

$$F_{mn} = F_{\bar{m}\bar{n}} = 0, \tag{76}$$

$$G^{m\bar{n}}F_{m\bar{n}} = 0. \tag{77}$$

These equations require to specify a gauge subbundle V of a $E_8 \times E_8$ gauge bundle over \mathcal{K} and a gauge connection A on V with curvature F . The first condition (76) tell us that the subbundle V as well as the corresponding connection should be holomorphic. The second condition (77) has a unique solution if the bundle V is stable and if it is satisfied the integrability condition $\int_{\mathcal{K}} \Omega^{n-1} \wedge c_1(V) = 0$, where Ω is the Kähler form of \mathcal{K} . Equation (77) is the celebrated Donaldson-Uhlenbeck-Yau equation for A . There is a further condition to be satisfied by the connection A , the Bianchi identity for H and F , it is given by

$$dH = trR \wedge R - \frac{1}{30}trF \wedge F. \tag{78}$$

The only solution is $trR \wedge R \propto trF \wedge F$ which implies that $c_2(T\mathcal{K}) = c_2(V)$. This situation is usually known as the *standard embedding* of the spin connection in the gauge connection and it is a method to determine the connection A on V .

Thus in the compactification of phenomenological interest of the heterotic theory with the ansatz $M_{10} = M_4 \times \mathcal{K}$, the internal space has to be a Calabi-Yau three-fold and one has to specify a stable, holomorphic vector bundle V over M_{10} (or \mathcal{K}) satisfying $c_1(V) = 0$ and $c_2(V) = c_2(TM_{10})$.

If V is a $SU(n)$ vector bundle over M_{10} the subgroups of $E_8 \times E_8$ that commutes are E_6 , $SO(10)$ and $SU(5)$ for $n = 3, 4, 5$ respectively. This leads to GUTs in four dimensions justly with the gauge groups E_6 , $SO(10)$ and $SU(5)$.

6.1. Massless Spectrum

In order to describe the impact of the characteristics of \mathcal{K} and V on the properties of the spectrum of the four dimensional theory we start by decomposing the ten-dimensional Dirac operator under $M_4 \times \mathcal{K}$ into

$$\mathcal{D}^{(10)} = \sum_{I=0}^9 \Gamma^I D_I = \mathcal{D}^{(4)} + \mathcal{D}_{\mathcal{K}}, \tag{79}$$

where $\mathcal{D}^{(4)} = \sum_{I=0}^3 \Gamma^I D_I$ and $\mathcal{D}_{\mathcal{K}} = \sum_{J=4}^9 \Gamma^J D_J$. Dirac equation in ten dimensions is

$$\mathcal{D}^{(10)}\Psi(x^I, y^J) = (\mathcal{D}^{(4)} + \mathcal{D}_{\mathcal{K}})\Psi(x^I, y^J). \tag{80}$$

Thus the spectrum of the Dirac operator $\mathcal{D}_{\mathcal{K}}$ on \mathcal{K} determines the massive spectrum of fermions in four dimensions.

In ten dimensions the Lorentz group only has *real* spinor representations and the Clifford modules decomposes as: $S^{(10)} = S_+^{(10)} \oplus S_-^{(10)}$. Positive and negative chirality are distinguished by $\Gamma^{(10)} = \Gamma^0 \Gamma^1 \dots \Gamma^9$. **CPT** theorem implies that we must take only one chirality

$$\Gamma^{(10)}\Psi = +\Psi. \quad (81)$$

Decompose the spinor representation of $SO(1,9)$ under $SO(1,3) \times SO(6)$ with $\Gamma^{(10)} = \Gamma^{(4)} \cdot \Gamma^{(6)}$ where $\Gamma^{(4)} = i\Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3$ and $\Gamma^{(6)} = -i\Gamma^4 \Gamma^5 \dots \Gamma^9$. One solution with $\Gamma^{(10)} = +1$ is given by $\Gamma^{(4)} = \Gamma^{(6)}$ and then the spin bundle decomposes under $M_4 \times \mathcal{K}$ as

$$\widehat{S}^{(10)} = \left(\widehat{S}_+^{(4)} \otimes \widehat{S}_+^{\mathcal{K}} \right) \oplus \left(\widehat{S}_-^{(4)} \otimes \widehat{S}_-^{\mathcal{K}} \right). \quad (82)$$

Now solve the Dirac equation with the ansatz $\Psi(x^I, y^J) = \sum_m \phi_m(x^I) \otimes \chi_m(y^J) = \sum_m \psi_m$ and $\mathcal{D}'_{\mathcal{K}} \chi_m = \lambda_m \chi_m$. It leads to

$$(\mathcal{D}'^{(4)} + \lambda_m) \psi_m = 0 \quad (83)$$

where $\mathcal{D}'^{(4)} = \Gamma^{(4)} \mathcal{D}^{(4)}$ and $\mathcal{D}'_{\mathcal{K}} = \Gamma^{(4)} \mathcal{D}_{\mathcal{K}}$.

$\mathcal{D}_{\mathcal{K}}$ is an elliptic operator on the compact manifold \mathcal{K} , this implies that that operator has a *finite* number of fermion zero modes. Massless fermions in four dimensions originate as zero modes of the Dirac operators $\mathcal{D}^{\mathcal{K}}$ of the internal manifold \mathcal{K} . By the Atiyah-Singer theorem, a topological invariant of \mathcal{K} containing the information of the chiral fermions on \mathcal{K} is given by the index of the Dirac operator

$$\text{Index}(\mathcal{D}_{\mathcal{K}}) = N_+^{\lambda=0} - N_-^{\lambda=0}, \quad (84)$$

for chiral fermions on \mathcal{K} with $\Gamma^{(6)} = \pm 1$. Here $N_{\pm}^{\lambda=0}$ are the number of positive and negative chiral zero modes. In $2k + 2$ dimensions this index is vanishing. We need to couple gauge fields coming from the heterotic string theory. Recall that they are $E_8 \times E_8$ valued gauge fields.

The *standard embedding* of the spin connection in the gauge connection leads to the chain of maximal subgroups: $SO(6) \times SO(10) \subset SO(16) \subset E_8$. This breaks $SO(16)$ to $SO(10)$. The computation of the Δ for this case yields $\Delta = \oplus_i \mathbf{L}_i \otimes \mathbf{R}_i$ where \mathbf{L}_i are irreps of $SO(6)$ and \mathbf{R}_i are complex irreps of $SO(10)$. These latter determine the irreps where are distributed the massless fermions of the four-dimensional theory. The former irreps \mathbf{L}_i determine the number of fermionic chiral zero modes described by the topological index $\delta = \text{Index}(\mathcal{D}_{\mathcal{K}})$. This is given by

$$\begin{aligned} \delta &= N_{\Gamma^{(6)=+1}}^{\lambda=0} - N_{\Gamma^{(6)=-1}}^{\lambda=0} \\ &= \int_{\mathcal{K}} ch(V) td(\mathcal{K}) = \frac{1}{2} \int_{\mathcal{K}} c_3(V) \end{aligned} \quad (85)$$

and from the solution $\Gamma^{(6)} = \Gamma^{(4)}$, it determines the chiral fermion families in four dimensions

$$\delta = N_{\Gamma^{(4)=+1}}^{\lambda=0} - N_{\Gamma^{(4)=-1}}^{\lambda=0}. \quad (86)$$

Thus the theory in four dimensions has $\Delta = \oplus_i \delta_i \mathbf{R}_i$ where we used

$$\mathbf{248} = (\mathbf{15}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{45}) \oplus (\mathbf{6}, \mathbf{10}) \oplus (\mathbf{4}, \mathbf{16}) \oplus (\mathbf{4}^*, \mathbf{16}^*) - \quad (87)$$

Then we have

$$\Delta = \delta \left(\mathbf{16} \ominus \mathbf{16}^* \right), \quad (88)$$

where $\delta = \chi(\mathcal{K})/2$ with $\chi(\mathcal{K})$ is the Euler number of \mathcal{K} .

6.2. Some Physics in Four Dimensions

In this section we intend to make contact with some four-dimensional physics. The development of this line of work is known as *string phenomenology*. Recent reviews of this topic at the light of string dualities is given in Ref. [24]. In the present short review we follows Ref. [7, 25].

Continuous and Discrete Symmetries

In building models coming from string theory, *there are no global internal symmetries in spacetime* (there are no continuous global symmetries in all string theories). This is because if there is an internal symmetry, there should be a vector field in the spectrum because the properties of SCFT and it has the same properties of the gauge field of that symmetry.

Take for instance Type I or Type II superstring theory. We know from Noether's theorem of the two-dimensional theory that associated with the Type I or II superconformal symmetry there is a worldsheet conserved supercharge, $Q = \frac{1}{2\pi i} \oint (dzd\theta J - d\bar{z}d\bar{\theta} \bar{J})$, where, by uses of this symmetry, J should be a $(\frac{1}{2}, 0)$ tensor superfield and \bar{J} is a $(0, \frac{1}{2})$ tensor superfield. The associated bosonic vertex operators (when we combine these tensors with the fermionic fields $\tilde{\psi}^I$ and ψ^I respectively) have the property to couple with left and (or) right-moving parts of Q , giving rise to a spacetime gauge symmetry.

The absence of internal global symmetries in spacetime physics coming from string theory may help to understand some exciting problems of particle physics like the existence of the non-zero neutrino masses, which lie in the violation of the leptonic number. This was argued recently by Witten in [26].

However there are generically some discrete symmetries in string models. For example, T-duality which is an infinite dimensional one, or some models inherited from the point group of orbifold constructions which are finite dimensional, are in fact regarded as discrete symmetries. The importance of this kind of symmetries relay in the fact that they are useful for model building, hierarchy of masses and other related problems.

P, C, T Symmetries

We will see how discrete spacetime symmetries **P**, **C** and **T** are broken in string theory. If string theory is correct, when we compactify (for example on Calabi-Yau manifolds, orbifolds, tori , etc.) one must obtain the same symmetries (or broken symmetries) as these of the SM.

- **P-symmetry**

Parity symmetry is violated by gauge interactions in SM. In string theory there are an analogous situation. Take for instance the heterotic string. The massless states in ten dimension are labeled by irreps of the little group $SO(8)$. The action of parity symmetry reverses the spinor representations $\mathbf{8}_s$ and $\mathbf{8}_s'^*$ of the left and right-moving sectors. The symmetry is realized if the corresponding gauge representations \mathbf{R} and \mathbf{R}'^* are equal.

However, \mathbf{R} is the adjoint representation while that \mathbf{R}'^* is empty. This tell us that parity symmetry is broken and the gauge couplings are chiral. But although in ten dimensions the spectrum is chiral, when we compactify to four dimensions, the spectrum could be turned out into a non-chiral one.

For example, for toroidal compactifications, the spectrum is no-chiral, but for Z_3 orbifold (compactification) the spectrum it is. Other kinds of compactifications produce chiral gauge couplings.

The chirality of the spectrum can be expressed by a topological quantity called *Index* as we saw in the last subsection. Since the index is a topological invariant quantity, it does not suffers any change under continuous transformations of the CFT.

- **C and CP Symmetry**

Charge conjugation symmetry is also broken in SM. This is because \mathbf{C} leaves spacetime invariant, but conjugates the gauge generators. As in SM, in string theories we require that conjugate representations (for example in SM, the fermions) satisfy $\mathbf{R}_\pm = \mathbf{R}_\pm^*$. From this we can see that chiral gauge couplings do not satisfy \mathbf{C} symmetry. For the orbifold example this is also true.

Consider now the \mathbf{CP} symmetry. This symmetry takes $\mathbf{R}_+ \rightarrow \mathbf{R}_-^*$. Thus, any gauge coupling satisfies it as a consequence of \mathbf{CPT} invariance. In the case of the orbifold there is a symmetry of the action which reverses X^k into ψ^k with $k = 3, 5, 7, 9$, and all the λ^I (I odd). This is a \mathbf{CP} symmetry in 4-dimensions. So, Z_3 orbifold is \mathbf{CP} symmetric.

- **CPT Symmetry**

In string theory, as in local Lorentz-invariant quantum field theory, \mathbf{CPT} symmetry is preserved. In string perturbation theory we use the θ -operation². This is defined as $\theta(X^{0,3}) = \psi^{0,3}$ and vice versa. In Euclidean time this can be represented by a π -rotation in the plane (iX^0, X^3) . In this context is clear that the action of θ is a symmetry, that reverses time and includes parity (in X^3). In order to show that this action also includes charge conjugation consider the S -matrix,

$$\langle \alpha, out | \beta, in \rangle = \langle \bar{\mathcal{V}}_\alpha \mathcal{V}_\beta \rangle, \quad (89)$$

where \mathcal{V} is the vertex operator and we are only considering vertex operators to the initial and final states. The action of θ is

$$\langle \alpha, out | \beta, in \rangle = \langle \theta \cdot \bar{\mathcal{V}}_\alpha \theta \cdot \mathcal{V}_\beta \rangle = \langle \theta \bar{\beta}, out | \theta \bar{\alpha}, in \rangle. \quad (90)$$

When we apply \mathbf{CPT} operation, we can see that it is antiunitary and it is θ combined with the conjugation of the vertex operator

$$\langle \mathbf{CPT} \cdot \beta, out | \mathbf{CPT} \cdot \alpha, in \rangle = \langle \alpha, out | \beta, in \rangle. \quad (91)$$

The manner in what we saw that \mathbf{CPT} is an exact symmetry in string theory is only applicable to the perturbative sector. However for the non-perturbative sector, we can argue that SM, or field theory is the low energy limit of the string theory, so we take \mathbf{CPT} symmetry for this low energy limit, and then we put it in 10 dimensions.

6.3. Effective Actions in Four Dimensions

First of all, it is important to emphasize that consistent four-dimensional superstring models which are chiral lead to $\mathcal{N} = 1$ supersymmetric theories. At $\mathcal{N} \geq 2$ supersymmetry spoils chirality. Thus in order to consider phenomenological string models in four dimensions we restrict ourselves to construct $\mathcal{N} = 1$ supersymmetric actions. Mostly of the material we describe here is at the review by F. Quevedo [25] and we recommend it for checking details.

The corresponding spectrum of massless particles are composed by graviton-gravitino multiplet (G_{IJ}, ψ^I) and the gauge-gaugino multiplet $(A_J^\alpha, \lambda^\alpha)$. Also there are matter and moduli fields, in the form of chiral multiplets (Z, χ) . In the case of the dilaton field Φ , it couples to the antisymmetric tensor B_{IJ} to form the linear $\mathcal{N} = 1$ multiplet (Φ, B_{IJ}, ρ) . We can

² This is basically, the same argument used in field theory to prove \mathbf{CPT} symmetry.

construct a $\mathcal{N} = 1$ chiral multiplet (S, χ_S) from the linear multiplet, where S is obtained by a duality transformation of the dilaton field. This transformation is given by $S = a + ie^\Phi$ and $\nabla_I a \equiv \varepsilon_{IJKL} \nabla^J B^{KL}$.

Thus the whole theory can be described as a $\mathcal{N} = 1$ supergravity theory coupled only to gauge and chiral multiplets. The most general Lagrangian which describes these fields depends on three functions of the chiral multiplets. These fields are: (i) $K(Z, Z^*)$, the Kähler potential which is a real function which determines the kinetic terms of the chiral fields. The corresponding Lagrangian is given by $L_{kin} = K_{ZZ^*} \partial_I Z \partial^I Z^*$. (ii) $W(Z)$, the superpotential, which is an arbitrary holomorphic function of the chiral multiplets. (iii) $f_{ab}(Z)$, the gauge kinetic function (holomorphic), and determines the gauge kinetic couplings in the corresponding Lagrangian $L_{gauge} = \text{Re} f_{ab}(Z) F_{IJ}^a F^{IJb} + \text{Im} f_{ab}(Z) F_{IJ}^a \tilde{F}^{IJb}$. This function contributes to gaugino masses.

There are another quantity called the scalar potential $V = V_F + V_D$, where $V_F(Z, Z^*) = \exp(\frac{K}{M_{Pl}^2}) \{ D_Z W K_{ZZ^*}^{-1} D_{Z^*} W^* - 3 \frac{|W|^2}{M_{Pl}^2} \}$, $D_Z W = W_Z + W \frac{K_Z}{M_{Pl}^2}$ and $V_D = (\text{Re} f^{-1})_{ab}(K_Z, T^a Z)(K_{Z^*}, T^b Z^*)$.

Thus, the problem to find an effective four-dimensional action, is to calculate the functions K, W and f_{ab} when we are giving a specific string model. To do this, we have to use all the symmetries we have and taking into account that four dimensional string models are governed by two perturbation expansions. That is, an expansion in the sigma model (controlled by the size of the extra dimensions) and the proper string perturbation (in terms of the dilaton field) of the string coupling constant g_S .

First of all we consider only couplings generated at string tree-level. For the sigma model, we also take only the tree-level expansion. Using symmetries as the four dimensional Poincaré symmetry, supersymmetry, gauge symmetries and the axionic symmetry, we can extract the dependence of the effective action on the dilaton field S . Then, at tree-level, the functions K, W, f are given by $K = -\log(S + S^*) + \hat{K}(T, U, Q)$, $W = Y_{IJK} Q^I Q^J Q^K$ and $f_{ab} = S \delta_{ab}$ with \hat{K} an undetermined function. Our purpose is to find approximated expressions to this functions in the tree-level of the string perturbation theory, but otherwise exact in the CFT.

It can be showed that by using the axionic symmetries that at all orders in sigma-model expansion, superpotential W does not depend on T and U , so it is just a function of the matter fields Q^I . Thus W does not admit any kind of corrections in the sigma model³. The superpotential W does not depends on S as well. We know that S is the string loop-counting parameter and this implies that W is also an exact expression at tree-level string perturbation theory, *i.e.*, does not admit any radiative corrections.

Now, we are interested in finding a useful expression for K . This is more difficult, because we only can calculate it for some simple cases. Take for example a Calabi-Yau compactification whit $h_{1,1} = 1$ and $h_{2,1} = 0$. This give us that $K = -\log(S + S^*) - 3\log(T + T^* + QQ^*)$. When we write the Kähler potential as an expansion in matter fields, it is possible to extract an exact tree-level expression. The expansion is given by

$$K = -\log(S + S^*) + K^M(T, T^*, U, U^*) + K^Q(T, T^*, U, U^*) QQ^* + \hat{Z}(T, T^*, U, U^*)(QQ + Q^*Q^*) + \mathcal{O}(Q^3).$$

For some (2, 2) orbifold and Calabi-Yau models, it has been computed the quantities K^M, K^Q and \hat{Z} .

Consider now the loop corrections. We have seen that the superpotential (which is an holomorphic function) does not admit radiative corrections. However, for the Kähler potential

³ The reason for this, is that the field T , related with the size of the extra dimensions, comes from the internal components of the metric and determines the form that the loop expansion of the worldsheet action takes.

this is different. We have just to calculate order by order in the loop expansions the corresponding expression for K_{loop} . On the other hand, the gauge kinetic function f_{ab} is also holomorphic and we know the expression in an exact manner for the tree level. Loop corrections to this function have a great importance due to this function determines the gauge coupling. Here we do not get expressions for these corrections, but it is important to say that there are no further corrections to f_{ab} beyond one loop, as in the standard supersymmetric theories.

In general, we have problems to determine how supersymmetry is broken at low energies. We can not solve this problem within perturbative string theory. We need work in the non-perturbative sector of the theory. But this sector, despite of many efforts and excellent results, we do not yet have a complete non-perturbative version of string theory. However there are some interesting non-perturbative mechanisms to break supersymmetry as gaugino condensation, composite goldstinos and instantons. The reader interested in these and another issues of non-perturbative string phenomenology is encouraged to consult [24, 33].

7. Non-perturbative String Theory

We have described the massless spectrum of the five consistent superstring theories in ten dimensions. Additional theories can be constructed in lower dimensions by compactification of some of the ten dimensions. Thus the ten-dimensional spacetime M_{10} looks like the product $M_{10} = \mathcal{K}^d \times \mathbb{R}^{1,9-d}$, with \mathcal{K} a suitable compact manifold or orbifold. Depending on which compact space is taken, it will be the quantity of preserved supersymmetry.

We have described the *massless spectrum* of the five consistent superstring theories in ten dimensions. Additional theories can be constructed in lower dimensions by compactifying some of the ten dimensions in a compact manifold \mathcal{K} . All five theories and their compactifications are parametrized by the *moduli space* which consist of:

- The string coupling constant g_S .
- Scalars coming from the geometry and topology of the compact manifold \mathcal{K} .
- Scalars coming from the KK compactification of the spectrum of bosonic fields in the **NS – NS** and the **RR** sectors.

Thus one can define the *string moduli space* \mathcal{M}_A , of the string theory A , as the space of all associated parameters (scalar fields).

Moreover, it can be defined a map between the moduli spaces \mathcal{M}_A and \mathcal{M}_B of the corresponding string theories A and B . This map is called *the duality map*:

$$\mathcal{S} : \mathcal{M}_A \rightarrow \mathcal{M}_B, \quad (92)$$

such that the *strong/weak* region of \mathcal{M}_A is interchanged with the *weak/strong* region of \mathcal{M}_B .

One can define another map:

$$\mathcal{T} : \mathcal{M}_A \rightarrow \mathcal{M}_A, \quad (93)$$

which interchanges the volume V_A of a \mathcal{K}_A for $\frac{1}{V_A}$. One example of the map \mathcal{T} is the equivalence, by *T-duality*, between the theories *Type IIA* compactified on \mathbf{S}^1 at radius R and the *Type IIB* theory on \mathbf{S}^1 at radius $\frac{1}{R}$. The theories HE and HO constitutes another example of the equivalence under \mathcal{T} map. In this section we will follows the Sen's review [27]. Another useful references are [28, 29, 30, 31, 32, 33]. Type IIB theory is self-dual with respect the \mathcal{S} map.

7.1. Strong-Weak Coupling String Duality

Type IIB-IIB Duality

The Type IIB theory is self-dual. In order to see that write the bosonic part of the action of Type IIB superstring theory in the *string frame* is given by

$$S_{IIB} = S_{NS} + S_R + S_{CS}, \quad (94)$$

where

$$S_{NS} = \frac{1}{2\kappa^2} \int_{M_{10}} d^{10}x \sqrt{-G_{IIB}} e^{-2\Phi} \left[R + 4\partial_I \Phi \partial^I \Phi - \frac{1}{2} |H_{(3)}|^2 \right], \quad (95)$$

$$S_{NS} = \frac{1}{2\kappa^2} \int_{M_{10}} d^{10}x \sqrt{-G_{IIB}} \left[|F_1|^2 - |\tilde{F}_{(3)}|^2 + \frac{1}{2} |\tilde{F}_{(5)}|^2 \right], \quad (96)$$

$$S_{CS} = -\frac{1}{4\kappa^2} \int_{M_{10}} C_{(4)} \wedge H_{(3)} \wedge F_{(3)}, \quad (97)$$

where $\tilde{F}_{(3)} = F_{(3)} - C_{(0)} \wedge H_{(3)}$, $\tilde{F}_{(5)} = F_{(5)} - \frac{1}{2} C_{(2)} \wedge H_{(3)} + \frac{1}{2} B_{(2)} \wedge F_{(3)}$ and $*\tilde{F}_{(5)} = \tilde{F}_{(5)}$.

Taking the following definitions:

$$G_{EIJ} = e^{(-\Phi/2)} G_{IJ}, \quad \tau \equiv C_{(0)} + ie^{(-\Phi)} \quad (98)$$

and

$$\mathcal{N}_{ij} = \frac{1}{\text{Im}(\tau)} \begin{pmatrix} |\tau|^2 & -\text{Re}(\tau) \\ -\text{Re}(\tau) & 1 \end{pmatrix}, \quad F_3^i = \begin{pmatrix} H_{(3)} \\ F_{(3)} \end{pmatrix}. \quad (99)$$

Action S_{IIB} can be equivalently written in the *Einstein frame* as:

$$S_{IIB} = \frac{1}{2\kappa^2} \int_{M_{10}} d^{10}x \sqrt{-G_{IIB}^E} \left(R_E - \frac{\partial_I \bar{\tau} \partial^I \tau}{2(\text{Im}\tau)^2} - \frac{1}{2} |F_{(1)}|^2 - \frac{\mathcal{N}_{ij}}{2} F_{(3)}^i - \frac{1}{2} |\tilde{F}_{(5)}|^2 \right) - \frac{\varepsilon_{ij}}{4\kappa^2} \int_{M_{10}} C_{(4)} \wedge F_{(3)}^i \wedge F_{(3)}^j. \quad (100)$$

This action is clearly invariant under the following $SL(2, \mathbb{R})$ symmetry

$$\tau' = \frac{a\tau + b}{c\tau + d} \quad (101)$$

with the field transformations

$$F_{(3)}'^i = \Lambda_j^i F_{(3)}^j, \quad \tilde{F}_{(5)}' = \tilde{F}_{(5)}, \\ G'_{EIJ} = G_{EIJ}, \quad \mathcal{N}' = (\Lambda^{-1})^T \mathcal{N} \Lambda^{-1} \quad (102)$$

where

$$\Lambda_j^i = \begin{pmatrix} d & c \\ b & a \end{pmatrix}, \quad (103)$$

$a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. This symmetry leads to an identification of a pair of D1-branes and the interchanging of a pair of D3-branes.

Type I-SO(32)-Heterotic Duality

In order to analyze the duality between Type I and SO(32) heterotic string theories we first recall the spectrum of both theories. These fields are the dynamical fields of a supergravity Lagrangian in ten dimensions.

Type I string theory:

- **NS – NS**: the metric G_{IJ}^I , the dilaton Φ^I .
- **R – R** sector: the antisymmetric tensor C_{IJ}^I .
- Open sector: There are 496 gauge bosons A_J^{aI} in the adjoint representation of the gauge group $\text{SO}(32)$.
- Observe that fields $C_{(0)}$, $B_{(2)}$ and $C_{(4)}$ are projected out by the orientifold projection Ω .

The effective action of Type I theory is:

$$S_I = S_c + S_o, \quad (104)$$

where

$$S_c = \frac{1}{2\kappa^2} \int_{M_{10}} d^{10}x \sqrt{-G^I} e^{-2\Phi^I} \left(R + 4(\partial\Phi^I)^2 - \frac{1}{2} |\tilde{F}_{(3)}|^2 \right), \quad (105)$$

$$S_o = -\frac{1}{2g^2} \int_{M_{10}} d^{10}x \sqrt{-G^I} e^{-\Phi^I} \text{Tr}_V(f_{IJ}^I f^{IJ}), \quad (106)$$

where $\tilde{F}_{(3)} = F_{(3)} - \frac{\kappa^2}{g^2} [\omega_{3Y}(A) - \omega_{3L}(\omega)]$ and $\omega_3 = \text{Tr}_V(A \wedge dA + \frac{2}{3} A \wedge A \wedge)$ is the Chern-Simons form.

HO string theory:

- **NS – NS**: The spacetime metric G_{IJ}^H , the dilaton field Φ^H , the antisymmetric tensor B_{IJ}^H .
- 496 gauge fields A_I^{aH} in the adjoint representation of $\text{SO}(32)$.

The effective action S_{HO} of *HO* is given by:

$$S_{HO} = \frac{1}{2\kappa^2} \int_{M_{10}} d^{10}x \sqrt{-G^H} e^{-2\Phi^H} \left(R + 4(\partial\Phi^H)^2 - \frac{1}{2} |\tilde{H}_{(3)}|^2 - \frac{\kappa^2}{g^2} \text{Tr}_V(f_{IJ}^H f^{HIJ}) \right), \quad (107)$$

where $\tilde{H}_{(3)} = dB_{(2)} - \frac{\kappa^2}{g^2} [\omega_{3Y}(A) - \omega_{3L}(\omega)]$.

Both theories have $\mathcal{N} = 1$ spacetime supersymmetry. The comparison of these two actions leads to the following identification of the fields

$$\begin{aligned} G_{IJ}^I &= e^{-\Phi^H} G_{IJ}^H, & \Phi^I &= -\Phi^H, \\ A_J^{aI} &= A_I^{aH}, & \tilde{F}_{(3)}^I &= \tilde{H}_{(3)}^H. \end{aligned} \quad (108)$$

This give us many information, the first relation tell us that the metrics of both theories are the same. The second relation interchanges $B_{(2)}$ field in the **NS-NS** sector and the $C_{(2)}$ field in the **R-R** sector. That implies interchanging of heterotic fundamental strings *F1* and Type I *D1*-branes

$$F1 \Leftrightarrow D1. \quad (109)$$

The third relation of Eq. (108) identifies the gauge fields coming from the Chan-Paton factors from the Type I side with the gauge fields coming from the 16 compactified internal dimensions of the heterotic string. Finally, the opposite sign for the dilaton relation means that the string coupling constant g_S^I is inverted $g_S^H = 1/g_S^I$ within this identification, and interchanges the strong and weak couplings of both theories leading to the explicit realization of the \mathcal{S} map.

Strong-Weak Duality Between Heterotic on \mathbf{T}^4 and Type IIA on $K3$

The statement is the following equivalence (or duality)

$$H/\mathbf{T}^4 \Leftrightarrow IIA/K3. \quad (110)$$

KK reduction of the low energy effective action of any heterotic string theory in ten dimensions compactified on a $\mathcal{K}_H = \mathbf{T}^4$ leads to a moduli space \mathcal{M}_H consisting of:

- Usual dilaton field $\Phi^{(H)}$.
- Many other scalars coming from internal components of the metric, anti-symmetric field and gauge fields. These scalars are organized by a 24×24 matrix valued field $M^{(H)}$ which satisfy:

$$M^{(H)} L M^{(H)T} = L, \quad M^{(H)T} = M^{(H)}, \quad (111)$$

where

$$L = \begin{pmatrix} & I_4 & \\ I_4 & & \\ & & -I_{16} \end{pmatrix}. \quad (112)$$

At a generic point of the moduli space where scalars contained in $M^{(H)}$ take an expectation value, there are 24 abelian gauge fields being the gauge group $U(1)^{24}$. Thus we get:

- 16 gauge fields come from Cartan subalgebra of the original gauge group.
- 8 gauge fields come from the internal components of $G_{m\mu}$ and $B_{m\mu}$ with $6 \leq m \leq 9$ and $0 \leq \mu \leq 5$.

We will denote them by $A_\mu^{(H)a}$ with $1 \leq a \leq 24$. With these fields after compactification the low energy effective theory is given by:

$$S_H = \frac{1}{(2\pi)^3} \int d^6x \sqrt{-g^{(H)}} \left[R^{(H)} - \frac{1}{2} g^{(H)\mu\nu} \partial_\mu \Phi^{(H)} \partial_\nu \Phi^{(H)} + \frac{1}{8} g^{(H)\mu\nu} \text{Tr}(\partial_\mu M^{(H)} L \partial_\nu M^{(H)} L) \right. \\ \left. - \frac{1}{4} e^{-\Phi^{(H)}/2} g^{(H)\mu\mu'} g^{(H)\nu\nu'} F_{\mu\nu}^{(H)a} (L M^{(H)} L)_{ab} F_{\mu'\nu'}^{(H)b} - \frac{1}{12} e^{-\Phi^{(H)}} g^{(H)\mu\mu'} g^{(H)\nu\nu'} g^{(H)\rho\rho'} H_{\mu\nu\rho}^{(H)} H_{\mu'\nu'\rho'}^{(H)} \right], \quad (113)$$

where $F_{\mu\nu}^{(H)a}$ is the field strength associated to $A_\mu^{(H)a}$ and $H_{\mu\nu\rho}^{(H)}$ is the field strength associated to $B_{\mu\nu}^{(H)}$ which is given by:

$$H_{\mu\nu\rho}^{(H)} = \partial_\mu B_{\nu\rho}^{(H)} + \frac{1}{2} A_\mu^{(H)a} L_{ab} F_{\nu\rho}^{(H)b} + (\text{cyclic permutations of } \mu, \nu, \rho). \quad (114)$$

This action is invariant under an $O(4, 20)$ transformation:

$$M^{(H)} \rightarrow \Omega M^{(H)} \Omega^T, \quad A_\mu^{(H)a} \rightarrow \Omega_{ab} A_\mu^{(H)b}, \\ g_{\mu\nu}^{(H)} \rightarrow g_{\mu\nu}^{(H)}, \quad B_{\mu\nu}^{(H)} \rightarrow B_{\mu\nu}^{(H)}, \quad \Phi^{(H)} \rightarrow \Phi^{(H)}, \quad (115)$$

where Ω satisfies

$$\Omega L \Omega^T = L. \quad (116)$$

Now we study the KK reduction of the low energy effective action of Type IIA string theory in ten dimensions compactified on a $K3$ manifold:

Recall the Type IIA sugra action:

$$S_{IIA} = \frac{1}{(2\pi)^7} \int_{M_{10}} d^{10}x \sqrt{-g} \left[R - \frac{1}{8} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{12} e^{-\Phi/2} g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'} H_{\mu\nu\rho} H_{\mu'\nu'\rho'} \right. \\ \left. - \frac{1}{4} e^{3\Phi/4} g^{\mu\mu'} g^{\nu\nu'} F_{\mu\nu} F_{\mu'\nu'} - \frac{1}{48} e^{\Phi/4} g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'} g^{\sigma\sigma'} G_{\mu\nu\rho\sigma} G_{\mu'\nu'\rho'\sigma'} \right. \\ \left. - \frac{1}{(48)^2} (\sqrt{-g})^{-1} \varepsilon^{\mu_0 \dots \mu_9} B_{\mu_0 \mu_1} G_{\mu_2 \dots \mu_5} G_{\mu_6 \dots \mu_9} \right], \quad (117)$$

where $F_{\mu'\nu'} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \text{cyclic permutations of } \mu, \nu, \rho$ and

$$G_{\mu\nu\rho} = \partial_\mu C_{\nu\rho\sigma} + A_\mu H_{\nu\rho\sigma} + \text{cyclic permutations of } \mu, \nu, \rho.$$

Up on compactification on $K3$ the structure of the moduli space is as follows:

- we get a new set of scalar fields from the Kahler and complex structure of $K3$. This give a total of 58 real scalar fields.
- 22 scalars $\phi^{(p)}$ by decomposing the antisymmetric tensor field along the 22 harmonic forms $\omega_{mn}^{(p)}$ in $K3$:

$$B_{mn}(x, y) \sim \sum_{p=1}^{22} \phi^{(p)}(x) \omega_{mn}^{(p)}(y) + \dots \quad (118)$$

These 80 scalars parametrizing a coset:

$$O(4, 20)/O(4) \times O(20)$$

which can be described by a matrix $M^{(A)}$ satisfying identical properties as $M^{(H)}$.

This theory has also 24 $U(1)$ gauge fields:

- 22 come from the components of C_{MNP} :

$$C_{mn\mu}(x, y) \sim \sum_{p=1}^{22} \omega_{mn}^{(p)}(y) \mathcal{A}_\mu^{(p)}(x) + \dots \quad (119)$$

- 1 comes from the original RR field A_μ .
- The last one come from dualizing $C_{\mu\nu\rho}$: $G \sim *(dA)$.

We denote these gauge fields by: $A_\mu^{(A)a}$ with $1 \leq a \leq 24$. Up on compactification, the six-dimensional effective action is:

$$\begin{aligned} S_A = & \frac{1}{(2\pi)^3} \int d^6x \sqrt{-g^{(A)}} \left[R^{(A)} - \frac{1}{2} g^{(A)\mu\nu} \partial_\mu \Phi^{(A)} \partial_\nu \Phi^{(A)} + \frac{1}{8} g^{(A)\mu\nu} \text{Tr}(\partial_\mu M^{(A)} L \partial_\nu M^{(A)} L) \right. \\ & \left. - \frac{1}{4} e^{\Phi^{(A)}/2} g^{(A)\mu\mu'} g^{(A)\nu\nu'} F_{\mu\nu}^{(A)a} (LM^{(A)}L)_{ab} F_{\mu'\nu'}^{(A)b} - \frac{1}{12} e^{-\Phi^{(A)}} g^{(A)\mu\mu'} g^{(A)\nu\nu'} g^{(A)\rho\rho'} H_{\mu\nu\rho}^{(A)} H_{\mu'\nu'\rho'}^{(A)} \right] \\ & - \frac{1}{16} (\sqrt{-g^{(A)}})^{-1} \varepsilon^{\mu\nu\rho\delta\epsilon\eta} B_{\mu\nu}^{(A)} F_{\rho\delta}^{(A)a} L_{ab} F_{\epsilon\eta}^{(A)b}, \end{aligned} \quad (120)$$

where $F_{\mu\nu}^{(A)a}$ is the field strength associated to $A_\mu^{(A)a}$ and $H_{\mu\nu\rho}^{(A)}$ is the field strength associated to $B_{\mu\nu}^{(A)}$ which is given by:

$$H_{\mu\nu\rho}^{(A)} = \partial_\mu B_{\nu\rho}^{(A)} + (\text{cyclic permutations of } \mu, \nu, \rho). \quad (121)$$

This action is invariant under an $O(4, 20)$ transformation:

$$\begin{aligned} M^{(A)} & \rightarrow \Omega M^{(A)} \Omega^T, \quad A_\mu^{(A)a} \rightarrow \Omega_{ab} A_\mu^{(A)b}, \\ g_{\mu\nu}^{(A)} & \rightarrow g_{\mu\nu}^{(A)}, \quad B_{\mu\nu}^{(A)} \rightarrow B_{\mu\nu}^{(A)}, \quad \Phi^{(A)} \rightarrow \Phi^{(A)}, \end{aligned} \quad (122)$$

where Ω satisfies

$$\Omega L \Omega^T = L. \quad (123)$$

Then equations of motion and Bianchi identities are the same if we make the following identifications:

$$\begin{aligned} g_{\mu\nu}^{(H)} & = g_{\mu\nu}^{(A)}, \quad M^{(H)} = \tilde{\Omega} M^{(A)} \tilde{\Omega}^T, \\ \Phi^{(H)} & = -\Phi^{(A)}, \quad A_\mu^{(H)a} = \tilde{\Omega}_{ab} A_\mu^{(A)b}, \\ \sqrt{-g^{(H)}} e^{-\Phi^{(H)}} H^{(H)\mu\nu\rho} & = \frac{1}{6} \varepsilon^{\mu\nu\rho\delta\epsilon\eta} H_{\delta\epsilon\eta}^{(A)}, \end{aligned} \quad (124)$$

for $\tilde{\Omega}$ arbitrary.

7.2. M-Theory

We have described how to construct dual pairs of string theories. By the uses of the \mathcal{S} and the \mathcal{T} maps a network of theories can be constructed in various dimensions all of them related by dualities. However new theories can emerge from this picture, this is the case of M-theory. M-theory (the name come from ‘mystery’, ‘magic’, ‘matrix’, ‘membrane’, etc.) was originally defined as the strong coupling limit for Type IIA string theory.

It is known that Type IIA theory can be obtained from the dimensional reduction of the eleven dimensional supergravity theory (a theory known from the 70’s years) and given by the Cremmer-Julia-Scherk Lagrangian

$$I_{11} = \frac{1}{2\kappa_{11}^2} \int_Y d^{11}x \sqrt{-G_{11}} \left(R - \frac{1}{2} |G_{(4)}|^2 \right) - \frac{1}{6} \int_Y A_{(3)} \wedge G_{(4)} \wedge G_{(4)}, \quad (125)$$

where $G_{(4)} = dA_{(3)}$ and Y is the eleven dimensional manifold. If we assume that the eleven-dimensional spacetime factorizes as $Y = M_{10} \times \mathbf{S}_R^1$, where the compact dimension has radius R . If one takes the metric ansatz

$$ds^2 = G_{MN}^{11} dx^M dx^N = G_{MN}^{10} dx^I dx^J + e^{4\Phi/3} \left[dx^{10} + A_I(x) dx^I \right]^2. \quad (126)$$

With this metric, usual Kaluza-Klein dimensional reduction leads to get the spectrum of the Type IIA superstring theory:

- the ten-dimensional metric $G_{I,J}^{10}$ in **NS – NS** sector.
- scalar field Φ in **NS – NS** sector.
- vector field $C_{(1)}$ in **RR** sector.

On the other hand $A_{(3)}$ from the eleven dimensional theory leads to the following fields in the ten dimensional theory:

- $C_{(3)}$ in **RR** sector.
- $B_{(2)}$ in **NS – NS** sector.

Thus the eleven-dimensional Lagrangian leads to the Type IIA supergravity in the weak coupling limit ($g_S \rightarrow 0$ or $R \rightarrow 0$):

$$S_{IIA} = S_{NS} + S_R + S_{CS}, \quad (127)$$

where

$$S_{NS} = \frac{1}{2\kappa^2} \int_{M_{10}} d^{10}x \sqrt{-G^{IIA}} e^{-2\Phi^{IIA}} \left(R + 4(\partial\Phi^{IIA})^2 - \frac{1}{2} |H_{(3)}|^2 \right), \quad (128)$$

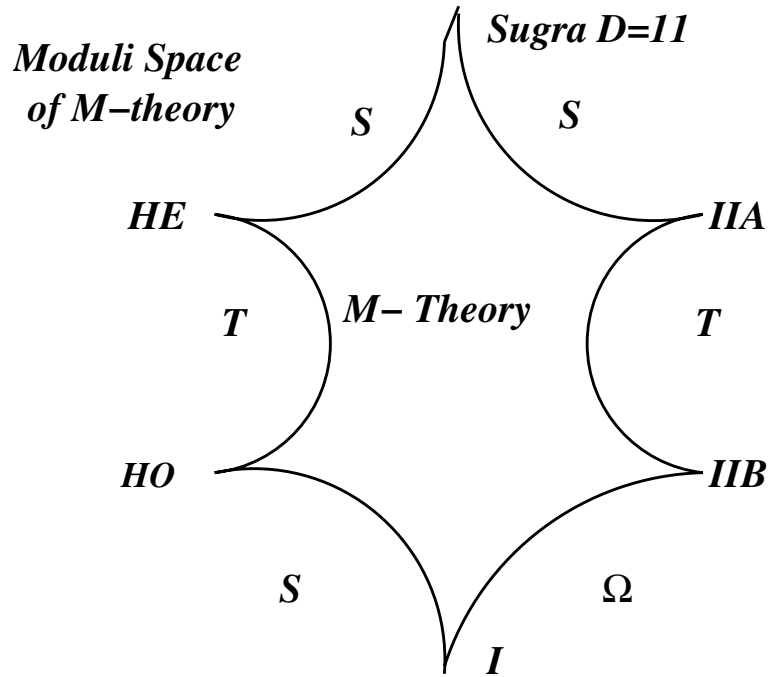
$$S_R = -\frac{1}{4\kappa^2} \int_{M_{10}} d^{10}x \sqrt{-G^{IIA}} \left(|F_{(2)}|^2 + |\tilde{F}_{(4)}|^2 \right), \quad (129)$$

$$S_{CS} = -\frac{1}{4\kappa^2} \int_{M_{10}} B_{(2)} \wedge F_{(4)} \wedge F_{(4)}, \quad (130)$$

where $H_{(3)} = dB_{(2)}$, $F_{(2)} = dC_{(1)}$ and $\tilde{F}_{(4)} = F_{(4)} - C_{(1)} \wedge H_{(3)}$ and $F_{(4)} = dC_{(3)}$.

It is conjectured that there exist an eleven dimensional fundamental theory whose low energy limit is the 11 dimensional supergravity theory and that it is the strong coupling limit of the Type IIA superstring theory.

At the present time the degrees of freedom (dof’s) are still unknown, through at the macroscopic level they should be membranes and fivebranes (also called $M2$ -branes and $M5$ -branes).



7.3. Horava-Witten Theory

Just as the M-theory compactification on \mathbf{S}_R^1 leads to the Type IIA theory, Horava and Witten realized that orbifold compactifications leads to the $E_8 \times E_8$ heterotic theory in ten dimensions HE. More precisely

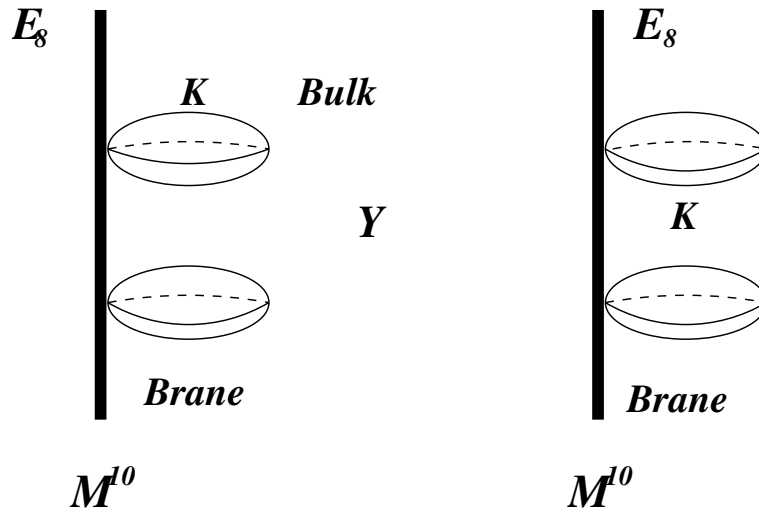
$$M - \text{theory}/(\mathbf{S}^1/\mathbf{Z}_2) \iff HE, \quad (131)$$

where $\mathbf{S}^1/\mathbf{Z}_2$ is homeomorphic to the finite interval I and the M -theory is thus defined on

$$Y = M_{10} \times I. \quad (132)$$

From the ten-dimensional point of view, this configuration is seen as two parallel planes placed at the two boundaries ∂I of I . Dimensional reduction and anomalies cancellation conditions imply that the gauge degrees of freedom should be trapped on the ten-dimensional planes M_{10} with the gauge group being E_8 in each plane. While that the gravity is propagating in the bulk and thus both copies of M_{10} 's are only connected gravitationally.

Horava–Witten Model



7.4. *F*-Theory

F-Theory was formulated by C. Vafa, looking for an analog theory to *M*-Theory for describing non-perturbative compactifications of Type IIB theory. Usually in perturbative compactifications the parameter $g_S = C + i \exp(-\Phi/2)$ is taken to be constant. *F*-theory generalizes this fact by considering variable λ . Thus *F*-theory is defined as a twelve-dimensional theory whose compactification on the elliptic fibration:

$$T^2 - \mathcal{K} \xrightarrow{\pi} B, \tag{133}$$

gives the Type IIB theory compactified on B (for a suitable space B) with the identification of $\lambda(\vec{z})$ with the modulus $\tau(\vec{z})$ of the torus T^2 . These compactifications can be related to the *M*-theory compactifications through the known \mathcal{S} mapping $\mathcal{S} : IIA \rightarrow M/S^1$ and the \mathcal{T} map between Type IIA and IIB theories. This gives

$$F/\mathcal{K} \times \mathbf{S}^1 \iff M/\mathcal{K}. \tag{134}$$

Thus the spectrum of massless states of *F*-theory compactifications can be described in terms of *M*-theory. Other interesting *F*-theory compactifications are the Calabi-Yau compactifications

$$F/CY \iff H/K3. \tag{135}$$

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