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Existence of non-Abelian vortices with product gauge groups

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Abstract

In this paper we establish several sharp existence and uniqueness theorems for some non-Abelian vortex models arising in supersymmetric gauge field theories. We prove these results by studying a family of systems of elliptic equations with exponential nonlinear terms in both doubly periodic-domain and planar cases. In the doubly periodic-domain case we obtain some necessary and sufficient conditions, each explicitly expressed in terms of a single inequality interestingly relating the vortex numbers, to coupling parameters and size of the domain, for the existence of solutions to these systems. In the planar case we establish the existence results for any vortex numbers and coupling parameters. Sharp decay estimates for the planar solutions are also obtained. Furthermore, the solutions are unique, which give rise to the quantized integrals in all cases.

Keywords: Non-Abelian vortices; Supersymmetric gauge theories; BPS equations; Existence; Uniqueness

1. Introduction

Vortices are important objects in various branches of physics [36] including condensed matter physics [1,28], particle physics [25], string theory and cosmology [23,29,48]. It is well

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known there admit the Abrikosov–Nielsen–Olesen vortices [1,37] for the classical Abelian Higgs model, whose static limit is also known as the Ginzburg–Landau model for superconductivity [17]. The first rigorous existence result for vortex configurations were established by Taubes [45,46] for the Ginzburg–Landau model [17]. Since then various analytic methods for studying the existence of vortices and other topological solitons have been developed [28,44,50].

During the last ten years much attention has been concentrated to vortices in non-Abelian gauge field theories since they are related to the fundamental puzzle in theoretical physics, quark confinement or color confinement [18,42]. In fact, in their famous work [38] Seiberg and Witten use non-Abelian color charged monopoles and vortices to interpret quark confinement. Motivated by the importance of non-Abelian vortices in the understanding of monopole and quark confinement, a wide class of non-Abelian gauge theories were developed in [3,21, 22,39]. See [4,7–12,15,31,40,43] for more recent progress and [14,30,41,47] for surveys on this topic. In these theories there arise many interesting and challenging systems of elliptic partial differential equations. It is interesting to carry out a rigorous analysis for these partial differential equations from both physical and mathematical points of view. In this respect, we cite the work [6,32–35], where a series of existence and uniqueness results were established.

The purpose of this paper is to establish sharp existence theories for the non-Abelian vortex model with product moduli proposed in [7], and for the Yang–Mills–Higgs model with gauge groups $U(1) \times SO(2M)$ and $U(1) \times SU(N)$ in [11,12,19]. We recall that, by the approach of moduli matrix [13,26], a series of systems of non-Abelian BPS vortex equations were obtained in [7,11,12,19]. For each of these systems we establish sharp existence, uniqueness, asymptotic behavior and quantized integral results. In particular, for these models over doubly periodic domain we obtain some necessary and sufficient condition, each explicitly expressed in a single inequality interestingly relating the vortex numbers, to coupling parameters and size of the domain, for the existence of solutions. Over the full plane, we obtain existence and uniqueness results for any vortex numbers and coupling parameters. Furthermore, the explicit decay estimates for planar solutions are established. Our approach is based on the direct minimization methods recently developed in [32,33].

The rest of our paper is organized as follows. In Section 2 we review a system of vortex equations from the model proposed in [7] and state a sharp existence result Theorem 2.1 for this problem over a doubly periodic-domain and the full plane. In Sections 3 and 4 we prove Theorem 2.1 for the doubly periodic-domain and planar cases, respectively. Sections 5 and 6 are devoted to establishing existence results for the BPS vortex equations [11,19] arising in the Yang–Mills–Higgs model with gauge groups $U(1) \times SO(2M)$ and $U(1) \times SU(N)$ separately.

2. Non-Abelian vortex model with product gauge group

In this section we consider the non-Abelian vortex model proposed in [7] with gauge group G, where

$$G = \frac{U(1) \times SU(n) \times SU(r)}{\mathbb{Z}_k},$$

and k is the least common multiple of n and r. Following [7], the action density of the model reads as

$$\mathcal{S} = \frac{1}{4g_0^2}\big(F_{\mu\nu}^{(0)}\big)^2 + \frac{1}{4g_n^2}\big(F_{\mu\nu}^{(a)}\big)^2 + \frac{1}{4g_r^2}\big(F_{\mu\nu}^{(b)}\big)^2 + \big|\mathcal{D}_\mu q^{(1)}\big|^2 + \big|\mathcal{D}_\mu \tilde{q}^{(2)}\big|^2 + \frac{g_0^2}{2}\big(\lambda_1\big[q^{(1)}q^{(1)\dagger}\big] + \lambda_2\big[\tilde{q}^{(2)}\tilde{q}^{(2)\dagger}\big] - \xi\big)^2 + \frac{g_n^2}{2}\big(q^{(1)}t^a q^{(1)\dagger}\big)^2 + \frac{g_r^2}{2}\big(\tilde{q}^{(2)}t^b\tilde{q}^{(2)\dagger}\big)^2, \tag{2.1}$$

where

$$F_{\mu\nu}^{(0)} = \partial_\mu A_\nu^{(0)} - \partial_\nu A_\mu^{(0)}, \tag{2.2}$$

$$F_{\mu\nu}^{(a)} = \partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)} + \mathrm{i}\big[A_\mu^{(a)}, A_\nu^{(a)}\big],$$

$$F_{\mu\nu}^{(b)} = \partial_\mu A_\nu^{(b)} - \partial_\nu A_\mu^{(b)} + \mathrm{i}\big[A_\mu^{(b)}, A_\nu^{(b)}\big], \tag{2.3}$$

$$\mathcal{D}_\mu q^{(1)} = \partial_\mu q^{(1)} - \mathrm{i}A_\mu^{(a)} q^{(1)}, \qquad \mathcal{D}_\mu \tilde{q}^{(2)} = \partial_\mu \tilde{q}^{(2)} - \mathrm{i}A_\mu^{(b)}\tilde{q}^{(2)}, \tag{2.4}$$

$$\lambda_1 = \frac{r}{\sqrt{2nr(n+r)}}, \qquad \lambda_2 = \frac{n}{\sqrt{2nr(n+r)}}, \tag{2.5}$$

$A_\mu^{(0)}$ is the gauge field of $U(1)$; $A_\mu^{(a)}$ ($A_\mu^{(b)}$) are the gauge fields of $SU(n)$ ($SU(r)$); t^a (t^b) are the standard generators of $SU(n)$ ($SU(r)$); $q^{(1)}, \tilde{q}^{(2)}$ are the quark fields; $\xi > 0$; g_0, g_n, g_r are the coupling constants with respect to $U(1)$, $SU(n)$ and $SU(r)$, respectively.

Via a Bogomol'nyi reduction [5], one obtains the following BPS equations [7]

$$\mathcal{D}_1 q^{(1)} \pm \mathrm{i}\mathcal{D}_2 q^{(1)} = 0, \tag{2.6}$$

$$\mathcal{D}_1 \tilde{q}^{(2)} \pm \mathrm{i}\mathcal{D}_2 \tilde{q}^{(2)} = 0, \tag{2.7}$$

$$F_{12}^{(0)} \pm g_0^2\big(\lambda_1 \operatorname{Tr}\big[q^{(1)}q^{(1)\dagger}\big] + \lambda_2 \operatorname{Tr}\big[\tilde{q}^{(2)}\tilde{q}^{(2)\dagger}\big] - \xi\big) = 0, \tag{2.8}$$

$$F_{12}^{(a)} \pm g_n^2\big(q^{(1)}t^a q^{(1)\dagger}\big) = 0, \tag{2.9}$$

$$F_{12}^{(b)} \pm g_r^2\big(\tilde{q}^{(2)}t^b\tilde{q}^{(2)\dagger}\big) = 0. \tag{2.10}$$

In the following we only consider Eqs. (2.6)–(2.10) with the upper sign, since the lower sign case can be treated similarly. To simplify the above equations we use the ansatz in [7]

$$q^{(1)} = S_n^{-1}\mathrm{e}^{-r\psi}H_0^{(n)}(z), \qquad \tilde{q}^{(2)} = S_r^{-1}\mathrm{e}^{-n\psi}H_0^{(r)}(z), \tag{2.11}$$

$$\sqrt{2nr(n+r)}\big(A_1^{(0)} + \mathrm{i}A_2^{(0)}\big) = -2\mathrm{i}\bar{\partial}\psi, \tag{2.12}$$

$$A_1^{(n)} + \mathrm{i}A_2^{(n)} = -2\mathrm{i}S_n^{-1}\bar{\partial}S_n, \qquad A_1^{(r)} + \mathrm{i}A_2^{(r)} = -2\mathrm{i}S_r^{-1}\bar{\partial}S_r, \tag{2.13}$$

where $H_0^{(n)}(z)$ and $H_0^{(r)}(z)$ are $n\times n$ and $r\times r$ matrices (called moduli matrices [13,26,27]), respectively, holomorphic in z, $S_n(S_r)$ are regular $SL(n,\mathbb{C})(SL(r,\mathbb{C}))$ matrices, ψ is a real function.

Under the above ansatz, the matter equations (2.6)–(2.7) are satisfied naturally. Let

$$\Omega_n = S_n S_n^\dagger, \qquad \Omega_r = S_r S_r^\dagger. \tag{2.14}$$

Then the gauge field equations (2.8)–(2.10) can be expressed as

$$\partial\bar{\partial}\psi = \frac{g_0^2}{8nr(n+r)}\big(r\mathrm{e}^{-2r\psi}\operatorname{Tr}\big[\Omega_n^{-1}H_0^{(n)}H_0^{(n)\dagger}\big] + n\mathrm{e}^{-2n\psi}\operatorname{Tr}\big[\Omega_r^{-1}H_0^{(r)}H_0^{(r)\dagger}\big] - \xi\sqrt{2nr(n+r)}\big), \tag{2.15}$$

$$\partial\big(\Omega_n^{-1}\bar{\partial}\Omega_n\big) = \frac{g_n^2}{4}\mathrm{e}^{-2r\psi}\left(\Omega_n^{-1}H_0^{(n)}H_0^{(n)\dagger} - \frac{1}{n}\operatorname{Tr}\big[\Omega_n^{-1}H_0^{(n)}H_0^{(n)\dagger}\big]\mathbf{1}_{n\times n}\right), \tag{2.16}$$

$$\partial\big(\Omega_r^{-1}\bar{\partial}\Omega_r\big) = \frac{g_r^2}{4}\mathrm{e}^{-2n\psi}\left(\Omega_r^{-1}H_0^{(r)}H_0^{(r)\dagger} - \frac{1}{r}\operatorname{Tr}\big[\Omega_r^{-1}H_0^{(r)}H_0^{(r)\dagger}\big]\mathbf{1}_{r\times r}\right), \tag{2.17}$$

which are called master equations for vortices [13].

To reduce Eqs. (2.15)–(2.17) one takes the ansatz in [19],

$$\Omega_n = \operatorname{diag}\{e^{(n-1)\chi}, e^{-\chi}, \ldots, e^{-\chi}\}, \qquad \Omega_r = \operatorname{diag}\{e^{(r-1)\tilde{\chi}}, e^{-\tilde{\chi}}, \ldots, e^{-\tilde{\chi}}\}, \tag{2.18}$$

where χ, $\tilde{\chi}$ are real functions.

Without loss of generality, we choose the moduli matrices $H_0^{(n)}(z)$ and $H_0^{(r)}(z)$ as

$$H_0^{(n)}(z) = \rho_1 \left(\prod_{i=1}^{3} P_i(z)\right)^{2r} \operatorname{diag}\{(P_2(z))^{n-1}, (P_2(z))^{-1}, \ldots, (P_2(z))^{-1}\}, \tag{2.19}$$

$$H_0^{(r)}(z) = \rho_2 \left(\prod_{i=1}^{3} P_i(z)\right)^{2n} \operatorname{diag}\{(P_3(z))^{r-1}, (P_3(z))^{-1}, \ldots, (P_3(z))^{-1}\}, \tag{2.20}$$

where

$$P_i(z) = \prod_{s=1}^{n_i} (z - z_{is}), \quad i = 1, 2, 3, \tag{2.21}$$

$\rho_1, \rho_2 \in \mathbb{C}$, $\rho_1\rho_2 \neq 0$, and $n_i \geqslant 0$ are integers.

Inserting (2.18)–(2.20) into (2.15)–(2.17) we arrive at

$$\begin{aligned} \partial\bar{\partial}\psi = {} & \frac{g_0^2}{8nr(n+r)} \big\{ r|\rho_1|^2 M_r(z) e^{-2r\psi} \big(|P_2(z)|^{2[n-1]} e^{-[n-1]\chi} + [n-1] |P_2(z)|^{-2} e^{\chi} \big) \\ & + n|\rho_2|^2 M_n(z) e^{-2n\psi} \big(|P_3(z)|^{2[r-1]} e^{-[r-1]\tilde{\chi}} + [r-1] |P_3(z)|^{-2} e^{\tilde{\chi}} \big) \\ & - \xi \sqrt{2nr(n+r)} \big\}, \end{aligned} \tag{2.22}$$

$$\partial\bar{\partial}\chi = \frac{g_n^2|\rho_1|^2}{4n} \big\{ M_r(z) e^{-2r\psi} \big(|P_2(z)|^{2[n-1]} e^{-[n-1]\chi} - |P_2(z)|^{-2} e^{\chi} \big) \big\}, \tag{2.23}$$

$$\partial\bar{\partial}\tilde{\chi} = \frac{g_r^2|\rho_2|^2}{4r} \big\{ M_n(z) e^{-2n\psi} \big(|P_3(z)|^{2[r-1]} e^{-[r-1]\tilde{\chi}} - |P_3(z)|^{-2} e^{\tilde{\chi}} \big) \big\}, \tag{2.24}$$

where

$$M_r(z) \equiv \left(\prod_{i=1}^{3} |P_i(z)|^2 \right)^{2r}, \qquad M_n(z) \equiv \left(\prod_{i=1}^{3} |P_i(z)|^2 \right)^{2n},$$

and the vacuum manifold is given by

$$|\rho_1|^2 + |\rho_2|^2 = \sqrt{\frac{2(n+r)}{nr}}\,\xi, \quad \xi > 0, \ \rho_1\rho_2 \neq 0. \tag{2.25}$$

Letting

$$\begin{aligned} & u_1 \equiv -\psi + \sum_{i=1}^{3} \sum_{s=1}^{n_i} \ln|z - z_{is}|^2, \qquad u_2 \equiv -\chi + \sum_{s=1}^{n_2} \ln|z - z_{2s}|^2, \\ & u_3 \equiv -\tilde{\chi} + \sum_{s=1}^{n_3} \ln|z - z_{3s}|^2, \end{aligned} \tag{2.26}$$

we reduce (2.22)–(2.24) into

$$\Delta u_1 = \frac{g_0^2}{2nr(n+r)}\Big\{r|\rho_1|^2\big(\mathrm{e}^{2ru_1+(n-1)u_2}+[n-1]\mathrm{e}^{2ru_1-u_2}\big) + n|\rho_2|^2\big(\mathrm{e}^{2nu_1+(r-1)u_3}+[r-1]\mathrm{e}^{2nu_1-u_3}\big) - \xi\sqrt{2nr(n+r)}\Big\} + 4\pi\sum_{i=1}^{3}\sum_{s=1}^{n_i}\delta_{p_{is}}, \tag{2.27}$$

$$\Delta u_2 = \frac{g_n^2|\rho_1|^2}{n}\big(\mathrm{e}^{2ru_1+(n-1)u_2}-\mathrm{e}^{2ru_1-u_2}\big) + 4\pi\sum_{s=1}^{n_2}\delta_{p_{2s}}, \tag{2.28}$$

$$\Delta u_3 = \frac{g_r^2|\rho_2|^2}{r}\big(\mathrm{e}^{2nu_1+(r-1)u_3}-\mathrm{e}^{2nu_1-u_3}\big) + 4\pi\sum_{s=1}^{n_3}\delta_{p_{3s}}, \tag{2.29}$$

where z_{is} is rewritten as p_{is}.

For the system (2.27)–(2.29), we consider two cases. In the first case we consider the problem over a doubly periodic-domain Ω, governing multiple vortices hosted in Ω such that the field configurations are subject to the 't Hooft boundary condition [24,49,50] under which periodicity is achieved modulo gauge transformations. In the second case we consider the problem over $\mathbb{R}^2$ with the boundary condition

$$u_i \to 0, \qquad |x| \to \infty, \quad i = 1, 2, 3. \tag{2.30}$$

Our main results for (2.27)–(2.29) read as follows.

Theorem 2.1. *Consider the problem (2.27)–(2.29) with any distribution of points $p_{i1}, \ldots, p_{in_i}$, and $n_i \geqslant 0$ are integers, $i = 1, 2, 3$. For any $\rho_1, \rho_2 \in \mathbb{C}$ and $\xi > 0$ satisfying (2.25), and any coupling parameters $g_0, g_n, g_r > 0$, we have the following conclusion*:

Over a doubly periodic-domain Ω, there exists a solution for (2.27)–(2.29) if and only if

$$\frac{2(n+r)(n_1+n_2+n_3)}{g_0^2} + \frac{(n-1)n_2}{g_n^2} + \frac{(r-1)n_3}{g_r^2} < \frac{\xi|\Omega|}{4\pi}\sqrt{\frac{2(n+r)}{nr}}. \tag{2.31}$$

Moreover, if a solution exists, it must be unique.

Over $\mathbb{R}^2$, there exits a unique solution for the problem (2.27)–(2.29) with the boundary condition (2.30). Furthermore, the solution satisfies the following exponential decay estimate at infinity

$$\sum_{i=1}^{3}\big(u_i^2 + |\nabla u_i|^2\big) \leqslant O\big(\mathrm{e}^{-\sqrt{2}\sigma_0(1-\varepsilon)|x|}\big), \tag{2.32}$$

where $\varepsilon \in (0, 1)$ is an arbitrary parameter, σ_0 is a positive constant defined by

$$\sigma_0^2 \equiv \min\left\{\frac{(r|\rho_1|^2 + n|\rho_2|^2)g_0^2}{n+r}, g_n^2|\rho_1|^2, g_r^2|\rho_2|^2\right\}. \tag{2.33}$$

In both cases, there hold the quantized integrals

$$\int\Big\{r|\rho_1|^2\big(\mathrm{e}^{2ru_1+(n-1)u_2}+[n-1]\mathrm{e}^{2ru_1-u_2}\big) + n|\rho_2|^2\big(\mathrm{e}^{2nu_1+(r-1)u_3}+[r-1]\mathrm{e}^{2nu_1-u_3}\big) - \sqrt{2nr(n+r)}\xi\Big\}\,\mathrm{d}x = -\frac{8nr(n+r)\pi(n_1+n_2+n_3)}{g_0^2}, \tag{2.34}$$

$$|\rho_1|^2 \int \left(e^{2ru_1+(n-1)u_2} - e^{2ru_1-u_2}\right) dx = -\frac{4\pi n n_2}{g_n^2}, \tag{2.35}$$

$$|\rho_2|^2 \int \left(e^{2nu_1+(r-1)u_3} - e^{2nu_1-u_3}\right) dx = -\frac{4\pi r n_3}{g_r^2}, \tag{2.36}$$

where the integrals are taken over either the domain Ω or $\mathbb{R}^2$.

3. Existence of doubly periodic solutions

In this section we prove Theorem 2.1 for the doubly periodic case. We will use the direct minimization procedure developed in [32].

Let u_1^0 be the solution of the problem (see [2])

$$\Delta u_1^0 = 4\pi \sum_{i=1}^{3} \sum_{s=1}^{n_i} \delta_{p_{is}} - \frac{4\pi(n_1+n_2+n_3)}{|\Omega|},$$

$$\int_\Omega u_1^0 \, dx = 0,$$

and u_i^0 be the solution of the problem (see [2])

$$\Delta u_i^0 = 4\pi \sum_{s=1}^{n_i} \delta_{p_{is}} - \frac{4\pi n_i}{|\Omega|},$$

$$\int_\Omega u_i^0 \, dx = 0, \quad i = 2, 3.$$

Setting $u_i = u_i^0 + v_i$, $i = 1, 2, 3$, we may reformulate (2.27)–(2.29) as

$$\begin{aligned}\Delta v_1 = {} & \tilde{g}_0^2 \left\{ r|\rho_1|^2 \left[e^{2r(u_1^0+v_1)+(n-1)(u_2^0+v_2)} + (n-1) e^{2r(u_1^0+v_1)-u_2^0-v_2} \right] \right. \\ & \left. + n|\rho_2|^2 \left[e^{2n(u_1^0+v_1)+(r-1)(u_3^0+v_3)} + (r-1) e^{2n(u_1^0+v_1)-u_3^0-v_3} \right] - \tilde{\xi} \right\} \\ & + \frac{4\pi(n_1+n_2+n_3)}{|\Omega|},\end{aligned} \tag{3.1}$$

$$\Delta v_2 = \frac{g_n^2|\rho_1|^2}{n} \left(e^{2r(u_1^0+v_1)+(n-1)(u_2^0+v_2)} - e^{2r(u_1^0+v_1)-u_2^0-v_2} \right) + \frac{4\pi n_2}{|\Omega|}, \tag{3.2}$$

$$\Delta v_3 = \frac{g_r^2|\rho_2|^2}{r} \left(e^{2n(u_1^0+v_1)+(r-1)(u_3^0+v_3)} - e^{2n(u_1^0+v_1)-u_3^0-v_3} \right) + \frac{4\pi n_3}{|\Omega|}, \tag{3.3}$$

where we use the notation

$$\tilde{g}_0^2 \equiv \frac{\tilde{g}_0^2}{2nr(n+r)}, \qquad \tilde{\xi} \equiv \sqrt{2nr(n+r)}\,\xi. \tag{3.4}$$

Noting (2.25) and (3.4), one has

$$nr\left(|\rho_1|^2 + |\rho_1|^2\right) = \tilde{\xi}. \tag{3.5}$$

Our function space is the Sobolev space $W^{1,2}(\Omega)$, which is composed of scalar or vector-valued Ω periodic L^2 functions whose derivatives also belong to $L^2(\Omega)$. We may easily check

that Eqs. (3.1)–(3.3) are the Euler–Lagrange equations of the following functional

$$\begin{aligned} I(v_1,v_2,v_3) = {} & \frac{1}{\tilde{g}_0^2}\|\nabla v_1\|_2^2 + \frac{n(n-1)}{2g_n^2}\|\nabla v_2\|_2^2 + \frac{r(r-1)}{2g_r^2}\|\nabla v_3\|_2^2 \\ & + |\rho_1|^2 \int_\Omega \left[\mathrm{e}^{2r(u_1^0+v_1)+(n-1)(u_2^0+v_2)} + (n-1)\mathrm{e}^{2r(u_1^0+v_1)-u_2^0-v_2}\right]\mathrm{d}x \\ & + |\rho_2|^2 \int_\Omega \left[\mathrm{e}^{2n(u_1^0+v_1)+(r-1)(u_3^0+v_3)} + (r-1)\mathrm{e}^{2n(u_1^0+v_1)-u_3^0-v_3}\right]\mathrm{d}x \\ & + \left(\frac{8\pi(n_1+n_2+n_3)}{\tilde{g}_0^2|\Omega|} - 2\tilde{\xi}\right)\int_\Omega v_1\,\mathrm{d}x \\ & + \frac{4\pi n(n-1)n_2}{g_n^2|\Omega|}\int_\Omega v_2\,\mathrm{d}x + \frac{4\pi r(r-1)n_3}{g_r^2|\Omega|}\int_\Omega v_3\,\mathrm{d}x. \end{aligned} \tag{3.6}$$

We first show that the condition (2.31) is necessary for the existence of solutions to (2.27)–(2.29). If (v_1,v_2,v_3) is a solution of (3.1)–(3.3), integrating Eqs. (3.1)–(3.3) over the domain Ω, we have

$$\begin{aligned} & r|\rho_1|^2\left[\int_\Omega \mathrm{e}^{2r(u_1^0+v_1)+(n-1)(u_2^0+v_2)}\,\mathrm{d}x + (n-1)\int_\Omega \mathrm{e}^{2r(u_1^0+v_1)-u_2^0-v_2}\,\mathrm{d}x\right] \\ & \quad + n|\rho_2|^2\left[\int_\Omega \mathrm{e}^{2n(u_1^0+v_1)+(r-1)(u_3^0+v_3)}\,\mathrm{d}x + (r-1)\int_\Omega \mathrm{e}^{2n(u_1^0+v_1)-u_3^0-v_3}\,\mathrm{d}x\right] \\ & = \tilde{\xi}|\Omega| - \frac{4\pi(n_1+n_2+n_3)}{\tilde{g}_0^2}, \\ & |\rho_1|^2\left(\int_\Omega \mathrm{e}^{2r(u_1^0+v_1)+(n-1)(u_2^0+v_2)}\,\mathrm{d}x - \int_\Omega \mathrm{e}^{2r(u_1^0+v_1)-u_2^0-v_2}\,\mathrm{d}x\right) = -\frac{4n\pi n_2}{g_n^2}, \\ & |\rho_2|^2\left(\int_\Omega \mathrm{e}^{2n(u_1^0+v_1)+(r-1)(u_3^0+v_3)}\,\mathrm{d}x - \int_\Omega \mathrm{e}^{2n(u_1^0+v_1)-u_3^0-v_3}\,\mathrm{d}x\right) = -\frac{4r\pi n_3}{g_r^2}, \end{aligned}$$

which imply

$$\begin{aligned} & |\rho_1|^2\int_\Omega \mathrm{e}^{2r(u_1^0+v_1)+(n-1)(u_2^0+v_2)}\,\mathrm{d}x + |\rho_2|^2\int_\Omega \mathrm{e}^{2n(u_1^0+v_1)+(r-1)(u_3^0+v_3)}\,\mathrm{d}x \\ & \quad = \frac{\tilde{\xi}|\Omega|}{nr} - \frac{4\pi(n_1+n_2+n_3)}{nr\tilde{g}_0^2} - \frac{4(n-1)\pi n_2}{g_n^2} - \frac{4(r-1)\pi n_3}{g_r^2}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} & |\rho_1|^2\int_\Omega \mathrm{e}^{2r(u_1^0+v_1)+(n-1)(u_2^0+v_2)}\,\mathrm{d}x + |\rho_2|^2\int_\Omega \mathrm{e}^{2n(u_1^0+v_1)-u_3^0-v_3}\,\mathrm{d}x \\ & \quad = \frac{\tilde{\xi}|\Omega|}{nr} - \frac{4\pi(n_1+n_2+n_3)}{nr\tilde{g}_0^2} - \frac{4(n-1)\pi n_2}{g_n^2} + \frac{4\pi n_3}{g_r^2}, \end{aligned} \tag{3.8}$$

$$|\rho_1|^2 \int_\Omega \mathrm{e}^{2r(u_1^0+v_1)-u_2^0-v_2}\,\mathrm{d}x + |\rho_2|^2 \int_\Omega \mathrm{e}^{2n(u_1^0+v_1)+(r-1)(u_3^0+v_3)}\,\mathrm{d}x$$
$$= \frac{\tilde{\xi}|\Omega|}{nr} - \frac{4\pi(n_1+n_2+n_3)}{nr\tilde{g}_0^2} + \frac{4\pi n_2}{g_n^2} - \frac{4(r-1)\pi n_3}{g_r^2}, \tag{3.9}$$

$$|\rho_1|^2 \int_\Omega \mathrm{e}^{2r(u_1^0+v_1)-u_2^0-v_2}\,\mathrm{d}x + |\rho_2|^2 \int_\Omega \mathrm{e}^{2n(u_1^0+v_1)-u_3^0-v_3}\,\mathrm{d}x$$
$$= \frac{\tilde{\xi}|\Omega|}{nr} - \frac{4\pi(n_1+n_2+n_3)}{nr\tilde{g}_0^2} + \frac{4\pi n_2}{g_n^2} + \frac{4\pi n_3}{g_r^2}. \tag{3.10}$$

Then all the right-hand sides of (3.7)–(3.10) must be positive, which with (3.4) give the necessity of the condition (2.31).

In the sequel we show that the condition (2.31) is also sufficient for the existence of solutions to (2.27)–(2.29). In other words, we prove that under the condition (2.31), the functional I admits a unique critical point, which solves (3.1)–(3.3).

There is a decomposition for the space $W^{1,2}(\Omega)$

$$W^{1,2}(\Omega) = \mathbb{R} \oplus \dot{W}^{1,2}(\Omega),$$

where

$$\dot{W}^{1,2}(\Omega) = \left\{ w \in W^{1,2}(\Omega) \,\middle|\, \int_\Omega w\,\mathrm{d}x = 0 \right\}$$

is a closed subspace of $W^{1,2}(\Omega)$. Then, for any $v \in W^{1,2}(\Omega)$, we have

$$v = c + w, \quad c \in \mathbb{R},\ w \in \dot{W}^{1,2}(\Omega). \tag{3.11}$$

By the Trudinger–Moser inequality [2,16]

$$\int_\Omega \mathrm{e}^w\,\mathrm{d}x \leqslant C \exp\left(\frac{1}{16\pi} \int_\Omega |\nabla w|^2\,\mathrm{d}x \right), \quad \forall w \in \dot{W}^{1,2}(\Omega), \tag{3.12}$$

we see that the functional I is a C^1 functional and weakly lower semi-continuous.

Using the decomposition formula (3.11) for $v_i \in W^{1,2}(\Omega)$, we have

$$v_i = c_i + w_i, \quad c_i \in \mathbb{R},\ w_i \in \dot{W}^{1,2}(\Omega),\ i = 1,2,3. \tag{3.13}$$

Then it follows from Jensen's inequality that

$$I(v_1,v_2,v_3) - \frac{1}{\tilde{g}_0^2}\|\nabla w_1\|_2^2 - \frac{n(n-1)}{2g_n^2}\|\nabla w_2\|_2^2 - \frac{r(r-1)}{2g_r^2}\|\nabla w_3\|_2^2$$
$$= |\rho_1|^2 \mathrm{e}^{2rc_1+(n-1)c_2} \int_\Omega \mathrm{e}^{2r(u_1^0+w_1)+(n-1)(u_2^0+w_2)}\,\mathrm{d}x$$
$$+ (n-1)|\rho_1|^2 \mathrm{e}^{2rc_1-c_2} \int_\Omega \mathrm{e}^{2r(u_1^0+w_1)-u_2^0-w_2}\,\mathrm{d}x$$
$$+ |\rho_2|^2 \mathrm{e}^{2nc_1+(r-1)c_3} \int_\Omega \mathrm{e}^{2n(u_1^0+w_1)+(r-1)(u_3^0+w_3)}\,\mathrm{d}x$$

$$
\begin{aligned}
&+ (r-1)|\rho_2|^2 \mathrm{e}^{2nc_1-c_3} \int_{\Omega} \mathrm{e}^{2n(u_1^0+w_1)-u_3^0-w_3}\,\mathrm{d}x \\
&- \left(2\tilde{\xi}|\Omega| - \frac{8\pi[n_1+n_2+n_3]}{\tilde{g}_0^2}\right)c_1 + \frac{4\pi n(n-1)n_2}{g_n^2}c_2 + \frac{4\pi r(r-1)n_3}{g_r^2}c_3 \\
&\geqslant |\Omega|\big\{|\rho_1|^2\big(\mathrm{e}^{2rc_1+(n-1)c_2} + [n-1]\mathrm{e}^{2rc_1-c_2}\big) + |\rho_2|^2\big(\mathrm{e}^{2nc_1+(r-1)c_3} + [r-1]\mathrm{e}^{2nc_1-c_3}\big)\big\} \\
&- \left(2\tilde{\xi}|\Omega| - \frac{8\pi[n_1+n_2+n_3]}{\tilde{g}_0^2}\right)c_1 + \frac{4\pi n(n-1)n_2}{g_n^2}c_2 + \frac{4\pi r(r-1)n_3}{g_r^2}c_3 \\
&= \big\{|\Omega||\rho_1|^2\mathrm{e}^{2rc_1+(n-1)c_2} - K_1\big(2rc_1+[n-1]c_2\big) \\
&\quad + (n-1)\big(|\Omega||\rho_1|^2\mathrm{e}^{2rc_1-c_2} - K_2[2rc_1-c_2]\big) \\
&\quad + |\Omega||\rho_2|^2\mathrm{e}^{2nc_1+(r-1)c_3} - K_1\big(2nc_1+[r-1]c_3\big) \\
&\quad + (r-1)\big(|\Omega||\rho_2|^2\mathrm{e}^{2nc_1-c_3} - K_3[2nc_1-c_3]\big)\big\},
\end{aligned}
\tag{3.14}
$$

where

$$
K_1 \equiv \frac{\tilde{\xi}|\Omega|}{2nr} - \frac{2\pi(n_1+n_2+n_3)}{nr\tilde{g}_0^2} - \frac{2\pi(n-1)n_2}{g_n^2} - \frac{2\pi(r-1)n_3}{g_r^2}, \tag{3.15}
$$

$$
K_2 \equiv \frac{\tilde{\xi}|\Omega|}{2nr} - \frac{2\pi(n_1+n_2+n_3)}{nr\tilde{g}_0^2} + \frac{2\pi(n+1)n_2}{g_n^2} - \frac{2\pi(r-1)n_3}{g_r^2}, \tag{3.16}
$$

$$
K_3 \equiv \frac{\tilde{\xi}|\Omega|}{2nr} - \frac{2\pi(n_1+n_2+n_3)}{nr\tilde{g}_0^2} - \frac{2\pi(n-1)n_2}{g_n^2} + \frac{2\pi(r+1)n_3}{g_r^2}. \tag{3.17}
$$

It is worth noting that the rearrangement of the right-hand sides of (3.14) is crucial for the subsequent treatment of the functional I.

We observe that under the condition (2.31) K_i $(i=1,2,3)$ defined by (3.15)–(3.17) are all positive. Then, from (3.14) we obtain

$$
\begin{aligned}
I(v_1,v_2,v_3) \geqslant{}& \frac{1}{\tilde{g}_0^2}\|\nabla w_1\|_2^2 + \frac{n(n-1)}{2g_n^2}\|\nabla w_2\|_2^2 + \frac{r(r-1)}{2g_r^2}\|\nabla w_3\|_2^2 + K_1 \ln\frac{|\Omega||\rho_1|^2}{K_1} \\
&+ K_1 \ln\frac{|\Omega||\rho_2|^2}{K_1} + (n-1)K_2\ln\frac{|\Omega||\rho_1|^2}{K_2} + (r-1)K_3\ln\frac{|\Omega||\rho_2|^2}{K_3}.
\end{aligned}
\tag{3.18}
$$

Therefore, it follows from (3.18) that the functional I is bounded from below and the minimization problem

$$
\eta_0 \equiv \inf\big\{I(v_1,v_2,v_3) \,\big|\, (v_1,v_2,v_3) \in W^{1,2}(\Omega)\big\} \tag{3.19}
$$

is well-defined.

Now choose a minimizing sequence of $\{(v_1^{(k)}, v_2^{(k)}, v_3^{(k)})\}$ of (3.19). We use the decomposition formula (3.11) for $v_i^{(k)}$ to get $v_i^{(k)} = c_i^{(k)} + w_i^{(k)}$, $i=1,2,3$. In view of the fact that the function $f(t) = \delta\mathrm{e}^t - \eta t$, with $\delta, \eta > 0$, satisfies the property $f(t) \to +\infty$ as $t \to \pm\infty$, we conclude from (3.14) that $\{c_i^{(k)}\}$ $(i=1,2,3)$ are all bounded.

Using (3.18) we see that $\{\nabla w_i^{(k)}\}$ $(i=1,2,3)$ are all bounded in $L^2(\Omega)$, which with the Poincaré inequality imply that $\{w_i^{(k)}\}$ $(i=1,2,3)$ are all bounded in $W^{1,2}(\Omega)$. Therefore, the

sequence $\{v_i^{(k)}\}$ $(i = 1, 2, 3)$ are all bounded in $W^{1,2}(\Omega)$. Consequently, there exists a subsequence of $\{v_i^{(k)}\}$, still denoted by $\{v_i^{(k)}\}$, such that $v_i^{(k)} \to \hat{v}_i$, weakly in $W^{1,2}(\Omega)$ as $k \to \infty$ for some $\hat{v}_i \in W^{1,2}(\Omega)$, $i = 1, 2, 3$.

Noting that the functional I is weakly lower semi-continuous, we conclude that $(\hat{v}_1, \hat{v}_2, \hat{v}_3)$ is a critical point of I. Naturally, $(\hat{v}_1, \hat{v}_2, \hat{v}_3)$ is a weak solution of Eqs. (3.1)–(3.3).

We easily check that the functional I is strictly convex. Then it admits at most one critical point, which implies the uniqueness of the doubly periodic solutions to Eqs. (3.1)–(3.3).

To get the quantized integrals (2.34)–(2.36) over Ω, we just need to integrate Eqs. (3.1)–(3.3) over Ω.

4. Existence and asymptotic behavior of planar solution

In this section we prove the existence result for (2.27)–(2.29) over the full plane with the boundary condition (2.30).

As in [28] we take the background functions

$$u_1^0(x) = -\sum_{i=1}^{3}\sum_{s=1}^{n_i} \ln\bigl(1 + \lambda|x - p_{is}|^{-2}\bigr), \tag{4.1}$$

$$u_i^0(x) = -\sum_{s=1}^{n_i} \ln\bigl(1 + \lambda|x - p_{is}|^{-2}\bigr), \quad i = 2, 3, \tag{4.2}$$

where $\lambda > 0$ is a parameter. Then we see that

$$\Delta u_1^0 = -h_1 + 4\pi \sum_{i=1}^{3}\sum_{s=1}^{n_i} \delta_{p_{is}}, \qquad \Delta u_i^0 = -h_i + 4\pi \sum_{s=1}^{n_i} \delta_{p_{is}}, \quad i = 2, 3, \tag{4.3}$$

where

$$h_1(x) = \sum_{i=1}^{3}\sum_{s=1}^{n_i} \frac{4\lambda}{(\lambda + |x - p_{is}|^2)^2}, \qquad h_i = \sum_{s=1}^{n_i} \frac{4\lambda}{(\lambda + |x - p_{is}|^2)^2}, \quad i = 2, 3. \tag{4.4}$$

Let

$$u_i = u_i^0 + v_i, \quad i = 1, 2, 3. \tag{4.5}$$

Then we recast Eqs. (2.27)–(2.29) into

$$\begin{aligned}\Delta v_1 = {} & \tilde{g}_0^2\bigl\{r|\rho_1|^2\bigl[e^{2r(u_1^0+v_1)+(n-1)(u_2^0+v_2)} + (n-1)e^{2r(u_1^0+v_1)-u_2^0-v_2}\bigr] \\ & + n|\rho_2|^2\bigl[e^{2n(u_1^0+v_1)+(r-1)(u_3^0+v_3)} + (r-1)e^{2n(u_1^0+v_1)-u_3^0-v_3}\bigr] - \tilde{\xi}\bigr\} + h_1,\end{aligned} \tag{4.6}$$

$$\Delta v_2 = \frac{g_n^2|\rho_1|^2}{n}\bigl(e^{2r(u_1^0+v_1)+(n-1)(u_2^0+v_2)} - e^{2r(u_1^0+v_1)-u_2^0-v_2}\bigr) + h_2, \tag{4.7}$$

$$\Delta v_3 = \frac{g_r^2|\rho_2|^2}{r}\bigl(e^{2n(u_1^0+v_1)+(r-1)(u_3^0+v_3)} - e^{2n(u_1^0+v_1)-u_3^0-v_3}\bigr) + h_3, \tag{4.8}$$

where we used the notation (3.4).

Our function space here is $W^{1,2}(\mathbb{R}^2)$. It is easy to see that Eqs. (4.6)–(4.8) are the Euler–Lagrange equations of the following functional

$$
\begin{aligned}
I(v_1,v_2,v_3) &= \frac{1}{\tilde{g}_0^2}\|\nabla v_1\|_2^2 + \frac{n(n-1)}{2g_n^2}\|\nabla v_2\|_2^2 + \frac{r(r-1)}{2g_r^2}\|\nabla v_3\|_2^2 \\
&\quad + |\rho_1|^2 \int_{\mathbb{R}^2} \Big\{ e^{2ru_1^0+(n-1)u_2^0}\big[e^{2rv_1+(n-1)v_2}-1\big] - \big(2rv_1+[n-1]v_2\big) \\
&\quad + (n-1)\big(e^{2ru_1^0-u_2^0}\big[e^{2rv_1-v_2}-1\big] - [2rv_1-v_2]\big)\Big\}\,dx \\
&\quad + |\rho_2|^2 \int_{\mathbb{R}^2} \Big\{ e^{2nu_1^0+(r-1)u_3^0}\big[e^{2nv_1+(r-1)v_3}-1\big] - \big(2nv_1+[r-1]v_3\big) \\
&\quad + (r-1)\big(e^{2nu_1^0-u_3^0}\big[e^{2nv_1-v_3}-1\big] - [2nv_1-v_3]\big)\Big\}\,dx \\
&\quad + \frac{2}{\tilde{g}_0^2}\int_{\mathbb{R}^2} h_1v_1\,dx + \frac{n(n-1)}{g_n^2}\int_{\mathbb{R}^2} h_2v_2\,dx + \frac{r(r-1)}{g_r^2}\int_{\mathbb{R}^2} h_3v_3\,dx. \qquad (4.9)
\end{aligned}
$$

We easily check that the functional I is C^1 and strictly convex over $W^{1,2}(\mathbb{R}^2)$. Then we can solve Eqs. (4.6)–(4.8) by finding the critical points of the functional (4.9). To this end, we use a direct approach developed in [32].

Our first step is to show I is coercive. A direct computation gives

$$
\begin{aligned}
&\big(DI(v_1,v_2,v_3)\big)(v_1,v_2,v_3) - \frac{2}{\tilde{g}_0^2}\|\nabla v_1\|_2^2 - \frac{n(n-1)}{g_n^2}\|\nabla v_2\|_2^2 - \frac{r(r-1)}{g_r^2}\|\nabla v_3\|_2^2 \\
&= 2\int_{\mathbb{R}^2} \Big\{ r|\rho_1|^2\big(e^{2r(u_1^0+v_1)+(n-1)(u_2^0+v_2)} + [n-1]e^{2r(u_1^0+v_1)-u_2^0-v_2} - n\big) \\
&\quad + n|\rho_2|^2\big(e^{2n(u_1^0+v_1)+(r-1)(u_3^0+v_3)} + [r-1]e^{2n(u_1^0+v_1)-u_3^0-v_3} - r\big)\Big\}v_1\,dx \\
&\quad + (n-1)|\rho_1|^2\int_{\mathbb{R}^2}\big(e^{2r(u_1^0+v_1)+(n-1)(u_2^0+v_2)} - e^{2r(u_1^0+v_1)-u_2^0-v_2}\big)v_2\,dx \\
&\quad + (r-1)|\rho_2|^2\int_{\mathbb{R}^2}\big(e^{2n(u_1^0+v_1)+(r-1)(u_3^0+v_3)} - e^{2n(u_1^0+v_1)-u_3^0-v_3}\big)v_3\,dx \\
&\quad + \frac{2}{\tilde{g}_0^2}\int_{\mathbb{R}^2} h_1v_1\,dx + \frac{n(n-1)}{g_n^2}\int_{\mathbb{R}^2} h_2v_2\,dx + \frac{r(r-1)}{g_r^2}\int_{\mathbb{R}^2} h_3v_3\,dx \\
&= |\rho_1|^2\int_{\mathbb{R}^2}\big(e^{2ru_1^0+(n-1)u_2^0+2rv_1+(n-1)v_2} - 1 + X_1\big)\big(2rv_1+[n-1]v_2\big)\,dx \\
&\quad + (n-1)|\rho_1|^2\int_{\mathbb{R}^2}\big(e^{2ru_1^0-u_2^0+2rv_1-v_2} - 1 + X_2\big)(2rv_1-v_2)\,dx \\
&\quad + |\rho_2|^2\int_{\mathbb{R}^2}\big(e^{2nu_1^0+(r-1)u_3^0+2nv_1+(r-1)v_3} - 1 + X_3\big)\big(2nv_1+[r-1]v_3\big)\,dx \\
&\quad + (r-1)|\rho_2|^2\int_{\mathbb{R}^2}\big(e^{2nu_1^0-u_3^0+2nv_1-v_3} - 1 + X_4\big)(2nv_1-v_3)\,dx, \qquad (4.10)
\end{aligned}
$$

where

$$X_1 \equiv \frac{1}{2|\rho_1|^2}\left(\frac{h_1}{nr\tilde{g}_0^2} + \frac{(n-1)h_2}{g_n^2} + \frac{(r-1)h_3}{g_r^2}\right), \tag{4.11}$$

$$X_2 \equiv \frac{1}{2|\rho_1|^2}\left(\frac{h_1}{nr\tilde{g}_0^2} - \frac{(n+1)h_2}{g_n^2} + \frac{(r-1)h_3}{g_r^2}\right), \tag{4.12}$$

$$X_3 \equiv \frac{1}{2|\rho_2|^2}\left(\frac{h_1}{nr\tilde{g}_0^2} + \frac{(n-1)h_2}{g_n^2} + \frac{(r-1)h_3}{g_r^2}\right), \tag{4.13}$$

$$X_4 \equiv \frac{1}{2|\rho_2|^2}\left(\frac{h_1}{nr\tilde{g}_0^2} + \frac{(n-1)h_2}{g_n^2} - \frac{(r+1)h_3}{g_r^2}\right). \tag{4.14}$$

Then we need to estimate the right-hand side of (4.12). In order to do this, we introduce the notations

$$w_1^0 = 2ru_1^0 + (n-1)u_2^0, \qquad w_1 = 2rv_1 + (n-1)v_2, \tag{4.15}$$

$$w_2^0 = 2ru_1^0 - u_2^0, \qquad w_2 = 2rv_1 - v_2, \tag{4.16}$$

$$w_3^0 = 2nu_1^0 + (r-1)u_3^0, \qquad w_3 = 2nv_1 + (r-1)v_3, \tag{4.17}$$

$$w_4^0 = 2nu_1^0 - u_3^0, \qquad w_4 = 2nv_1 - v_3. \tag{4.18}$$

With the above notations, from (4.10) we obtain

$$\begin{aligned}&\big(DI(v_1,v_2,v_3)\big)(v_1,v_2,v_3)\\ &\quad\geqslant \frac{2}{\tilde{g}_0^2}\|\nabla v_1\|_2^2 + \frac{n(n-1)}{g_n^2}\|\nabla v_2\|_2^2 + \frac{r(r-1)}{g_r^2}\|\nabla v_3\|_2^2\\ &\qquad + |\rho_1|^2\big(M_1(w_1) + (n-1)M_2(w_2)\big) + |\rho_2|^2\big(M_3(w_3) + (r-1)M_4(w_4)\big),\end{aligned} \tag{4.19}$$

where

$$M_i(w_i) \equiv \int_{\mathbb{R}^2} \big(\mathrm{e}^{w_i^0 + w_i} - 1 + X_i\big)w_i\,\mathrm{d}x, \quad i = 1,2,3,4. \tag{4.20}$$

Now we estimate the general term $M_i(w_i)$ on the right-hand side of (4.19).

Let $w_+ = \max\{w, 0\}$, $w_- = \max\{-w, 0\}$. Then, we have the following decomposition

$$M_i(w_i) = M_i(w_{i+}) + M_i(-w_{i-}), \quad i = 1,2,3,4.$$

In view of the elementary inequality $\mathrm{e}^t - 1 \geqslant t$ for $t \in \mathbb{R}$ and the fact $w_i^0, X_i \in L^2(\mathbb{R}^2)$ $(i = 1,\ldots,4)$, which follows from the definition of u_j^0 $(j = 1,2,3)$, we have

$$M_i(w_{i+}) \geqslant \int_{\mathbb{R}^2} \big(w_i^0 + w_{i+} + X_i\big)w_{i+}\,\mathrm{d}x \geqslant \frac{1}{2}\int_{\mathbb{R}^2} w_{i+}^2\,\mathrm{d}x - C, \quad i = 1,\ldots,4, \tag{4.21}$$

for some constant $C > 0$.

Noting the definition of u_j^0 $(j = 1,2,3)$ and taking λ sufficiently large, we see that $X_i < \frac{1}{2}$ for $i = 1,\ldots,4$. Then, as λ is suitably large, using the inequality $1 - \mathrm{e}^{-t} \geqslant \frac{t}{1+t}$ for $t \geqslant 0$, we have

$$
\begin{aligned}
M_i(-w_{i-}) &= \int_{\mathbb{R}^2} \big(1 - \mathrm{e}^{w_i^0 - w_{i-}} - X_i\big) w_{i-}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^2} \big(1 - \mathrm{e}^{w_i^0} + \mathrm{e}^{w_i^0}\big[1 - \mathrm{e}^{-w_{i-}}\big] - X_i\big) w_{i-}\,\mathrm{d}x \\
&\geqslant \int_{\mathbb{R}^2} \left(1 - \mathrm{e}^{w_i^0} + \mathrm{e}^{w_i^0} \frac{w_{i-}}{1 + w_{i-}} - X_i\right) w_{i-}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^2} \big\{\mathrm{e}^{w_i^0} w_{i-} + \big(1 - \mathrm{e}^{w_i^0} - X_i\big)(1 + w_{i-})\big\} \frac{w_{i-}}{1 + w_{i-}}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^2} (1 - X_i) \frac{w_{i-}^2}{1 + w_{i-}}\,\mathrm{d}x + \int_{\mathbb{R}^2} \big(1 - \mathrm{e}^{w_i^0} - X_i\big) \frac{w_{i-}}{1 + w_{i-}}\,\mathrm{d}x \\
&\geqslant \frac{1}{2} \int_{\mathbb{R}^2} \frac{w_{i-}^2}{1 + w_{i-}}\,\mathrm{d}x + \int_{\mathbb{R}^2} \big(1 - \mathrm{e}^{w_i^0} - X_i\big) \frac{w_{i-}}{1 + w_{i-}}\,\mathrm{d}x \\
&\geqslant \frac{1}{4} \int_{\mathbb{R}^2} \frac{w_{i-}^2}{(1 + w_{i-})^2}\,\mathrm{d}x - C, \quad i = 1, \ldots, 4,
\end{aligned} \tag{4.22}
$$

where we have used the fact $\mathrm{e}^{w_i^0} - 1, X_i \in L^2(\mathbb{R}^2)$ $(i = 1, \ldots, 4)$. Here and what follows we use C to denote a generic positive constant, which may take different values at different places.

Hence, combining (4.21) and (4.22), we find that

$$
M_i(w_i) \geqslant \frac{1}{4} \int_{\mathbb{R}^2} \frac{w_i^2}{(1 + |w_i|)^2}\,\mathrm{d}x - C, \quad i = 1, \ldots, 4. \tag{4.23}
$$

Therefore, we conclude from (4.19) and (4.23) that

$$
\begin{aligned}
&\big(DI(v_1, v_2, v_3)\big)(v_1, v_2, v_3) \\
&\quad \geqslant \frac{2}{g_0^2} \|\nabla v_1\|_2^2 + \frac{n(n-1)}{g_n^2} \|\nabla v_2\|_2^2 + \frac{r(r-1)}{g_r^2} \|\nabla v_3\|_2^2 \\
&\qquad + \frac{|\rho_1|^2}{4} \left(\int_{\mathbb{R}^2} \frac{w_1^2}{(1 + |w_1|)^2}\,\mathrm{d}x + (n-1) \int_{\mathbb{R}^2} \frac{w_2^2}{(1 + |w_2|)^2}\,\mathrm{d}x \right) \\
&\qquad + \frac{|\rho_2|^2}{4} \left(\int_{\mathbb{R}^2} \frac{w_3^2}{(1 + |w_3|)^2}\,\mathrm{d}x + (r-1) \int_{\mathbb{R}^2} \frac{w_4^2}{(1 + |w_4|)^2}\,\mathrm{d}x \right) - C.
\end{aligned} \tag{4.24}
$$

To proceed further, we need the standard Sobolev inequality

$$
\int_{\mathbb{R}^2} v^4\,\mathrm{d}x \leqslant 2 \int_{\mathbb{R}^2} v^2\,\mathrm{d}x \int_{\mathbb{R}^2} |\nabla v|^2\,\mathrm{d}x, \quad \forall v \in W^{1,2}\big(\mathbb{R}^2\big). \tag{4.25}
$$

Using (4.25), we have

$$\begin{aligned}\left(\int_{\mathbb{R}^2}|w_i|^2\,dx\right)^2&=\left(\int_{\mathbb{R}^2}\frac{|w_i|}{1+|w_i|}\big(1+|w_i|\big)|w_i|\,dx\right)^2\\&\leqslant\int_{\mathbb{R}^2}\frac{|w_i|^2}{(1+|w_i|)^2}\,dx\int_{\mathbb{R}^2}\big(|w_i|+|w_i|^2\big)^2\,dx\\&\leqslant 4\int_{\mathbb{R}^2}\frac{|w_i|^2}{(1+|w_i|)^2}\,dx\int_{\mathbb{R}^2}|w_i|^2\,dx\left(\int_{\mathbb{R}^2}|\nabla w_i|^2\,dx+1\right)\\&\leqslant\frac{1}{2}\left(\int_{\mathbb{R}^2}|w_i|^2\,dx\right)^2+C\left(\left[\int_{\mathbb{R}^2}\frac{|w_i|^2}{(1+|w_i|)^2}\,dx\right]^4+\left[\int_{\mathbb{R}^2}|\nabla w_i|^2\,dx\right]^4+1\right),\end{aligned}\tag{4.26}$$

which implies

$$\|w_i\|_2\leqslant C\left(\int_{\mathbb{R}^2}\frac{|w_i|^2}{(1+|w_i|)^2}\,dx+\int_{\mathbb{R}^2}|\nabla w_i|^2\,dx+1\right),\quad i=1,\dots,4.\tag{4.27}$$

In view of the notations (4.15)–(4.18), we have

$$\begin{aligned}&\|v_1\|_2^2\leqslant\|w_1\|_2^2+(n-1)\|w_2\|_2^2,\qquad\|v_2\|_2^2\leqslant\|w_1\|_2^2+(n-1)\|w_2\|_2^2,\\&\|v_3\|_2^2\leqslant\|w_3\|_2^2+(r-1)\|w_4\|_2^2,\end{aligned}$$

which give

$$\sum_{i=1}^3\|v_i\|_2\leqslant C\sum_{j=1}^4\|w_j\|_2.\tag{4.28}$$

Then, from (4.27) and (4.28), we see that

$$\sum_{i=1}^3\|v_i\|_2\leqslant C\left(\sum_{i=1}^4\int_{\mathbb{R}^2}\frac{|w_i|^2}{(1+|w_i|)^2}\,dx+\sum_{j=1}^3\|\nabla v_j\|_2^2+1\right),\tag{4.29}$$

where we used the fact

$$\sum_{i=1}^4\|\nabla w_i\|_2^2\leqslant C\sum_{j=1}^3\|\nabla v_j\|_2^2.$$

Now we conclude from (4.24) and (4.29) that there exist some positive constant C_0 and C_1 such that

$$\big(DI(v_1,v_2,v_3)\big)(v_1,v_2,v_3)\geqslant C_0\sum_{i=1}^3\|v_i\|_{W^{1,2}(\mathbb{R}^2)}-C_1.\tag{4.30}$$

By the coercive lower bound (4.30), we can show that the functional I admits a critical point in $W^{1,2}(\mathbb{R}^2)$. In view of (4.30), we can take $R_0>0$ sufficiently large such that

$$\inf\left\{\big(DI(v_1,v_2,v_3)\big)(v_1,v_2,v_3)\;\Big|\;\sum_{i=1}^3\|v_i\|_{W^{1,2}(\mathbb{R}^2)}=R_0\right\}\geqslant 1$$

(say). Since I is weakly lower semi-continuous, the minimization problem

$$\eta_0 \equiv \inf\left\{ I(f, f_1, \dots, f_M) \;\Big|\; \|f\|_{W^{1,2}(\mathbb{R}^2)} + \sum_{i=1}^{M} \|f_i\|_{W^{1,2}(\mathbb{R}^2)} \leqslant R_0 \right\} \tag{4.31}$$

admits a solution, say $(\hat{v}_1, \hat{v}_2, \hat{v}_3)$. Then we show that $(\hat{v}_1, \hat{v}_2, \hat{v}_3)$ is an interior point for the minimization problem (4.31). Otherwise, if

$$\sum_{i=1}^{3} \|\hat{v}_i\|_{W^{1,2}(\mathbb{R}^2)} = R_0,$$

then

$$\begin{aligned} \lim_{t\to 0} \frac{I((1-t)(\hat{v}_1, \hat{v}_2, \hat{v}_3)) - I(\hat{v}_1, \hat{v}_2, \hat{v}_3)}{t} &= \frac{\mathrm{d}}{\mathrm{d}t} I\big((1-t)(\hat{v}_1, \hat{v}_2, \hat{v}_3)\big)\big|_{t=0} \\ &= -\big(DI(\hat{v}_1, \hat{v}_2, \hat{v}_3)\big)(\hat{v}_1, \hat{v}_2, \hat{v}_3) \leqslant -1. \end{aligned}$$

Consequently, for $t > 0$ sufficiently small, with $(v_1^t, v_2^t, v_3^t) = (1-t)(\hat{v}_1, \hat{v}_2, \hat{v}_3)$, we have

$$I\big(v_1^t, v_2^t, v_3^t\big) < I(\hat{v}_1, \hat{v}_2, \hat{v}_3) = \eta_0, \qquad \sum_{i=1}^{3} \big\|v_i^t\big\|_{W^{1,2}(\mathbb{R}^2)} = (1-t)R_0 < R_0,$$

which lead to a contradiction. Hence, we see that $(\hat{v}_1, \hat{v}_2, \hat{v}_3)$ must be an interior critical point for the problem (4.31). As a critical point of I, $(\hat{v}_1, \hat{v}_2, \hat{v}_3)$ solves Eqs. (4.6)–(4.8). It is easy to check that the functional I is strictly convex, which implies $(\hat{v}_1, \hat{v}_2, \hat{v}_3)$ is the unique critical point of I. Then the uniqueness of the solutions to the system (4.6)–(4.8) follows.

Now we study the behavior the solution at infinity. Let us denote the solution of (4.6)–(4.8) by (v_1, v_2, v_3). In view of the well-known inequality

$$\big\|\mathrm{e}^v - 1\big\|_2^2 \leqslant C \exp\big(C\|v\|^2_{W^{1,2}(\mathbb{R}^2)}\big), \quad \forall v \in W^{1,2}\big(\mathbb{R}^2\big), \tag{4.32}$$

we see that the right-hand sides of Eqs. (4.6)–(4.8) belong to $L^2(\mathbb{R}^2)$. Then it follows from the elliptic L^2-estimate that $v_i \in W^{2,2}(\mathbb{R}^2)$, $i = 1, 2, 3$, which implies the desired boundary condition $v_i \to 0$, as $|x| \to \infty$, $i = 1, 2, 3$. From the fact $v_i \in W^{2,2}(\mathbb{R}^2)$ we see that the right-hand sides of Eqs. (4.6)–(4.8) also belong to $L^p(\mathbb{R}^2)$ for any $p \geqslant 2$. Therefore, by the elliptic L^p estimate, we have $v_i \in W^{2,p}(\mathbb{R}^2)$ for any $p \geqslant 2$, $i = 1, 2, 3$. Consequently, $|\nabla v_i| \to 0$, as $|x| \to \infty$, $i = 1, 2, 3$.

In what follows we establish the exponential decay rate for this solution. Let (u_1, u_2, u_3) be the solution of (4.1)–(4.3) obtained above. We have shown that $u_i \to 0$ as $|x| \to \infty$. Let

$$R > \max\big\{|p_{is}|,\ s = 1, \dots, n_i,\ i = 1, 2, 3\big\}.$$

When $|x| > R$, we rewrite Eqs. (2.27)–(2.29) as

$$\begin{aligned} \Delta u_1 &= \tilde{g}_0^2\big\{r|\rho_1|^2\big(\mathrm{e}^{\xi_1}\big[2ru_1 + [n-1]u_2\big] + [n-1]\mathrm{e}^{\xi_2}[2ru_1 - u_2]\big) \\ &\quad + n|\rho_2|^2\big(\mathrm{e}^{\xi_3}\big[2nu_1 + [r-1]u_3\big] + [r-1]\mathrm{e}^{\xi_4}[2nu_1 - u_3]\big)\big\} \\ &= 2nr\big(r|\rho_1|^2 + n|\rho_2|^2\big)\tilde{g}_0^2 u_1 + \tilde{g}_0^2\big\{\big(2r^2|\rho_1|^2\big(\mathrm{e}^{\xi_1} + [n-1]\mathrm{e}^{\xi_2}\big) \\ &\quad + 2n^2|\rho_2|^2\big(\mathrm{e}^{\xi_3} + [r-1]\mathrm{e}^{\xi_4}\big) - 2nr\big(r|\rho_1|^2 + n|\rho_2|^2\big)\big)u_1 \\ &\quad + r(n-1)|\rho_1|^2\big(\mathrm{e}^{\xi_1} - \mathrm{e}^{\xi_2}\big)u_2 + n(r-1)|\rho_2|^2\big(\mathrm{e}^{\xi_3} - \mathrm{e}^{\xi_4}\big)u_3\big\}, \end{aligned} \tag{4.33}$$

$$\Delta u_2 = \frac{g_n^2|\rho_1|^2}{n}\big(\mathrm{e}^{\xi_1}\big[2ru_1 + [n-1]u_2\big] - \mathrm{e}^{\xi_2}[2ru_1 - u_2]\big)$$
$$= g_n^2|\rho_1|^2 u_2 + \frac{g_n^2|\rho_1|^2}{n}\big\{2r\big(\mathrm{e}^{\xi_1} - \mathrm{e}^{\xi_2}\big)u_1 + \big([n-1]\mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_2} - n\big)u_2\big\}, \tag{4.34}$$

$$\Delta u_3 = \frac{g_r^2|\rho_2|^2}{r}\big(\mathrm{e}^{\xi_3}\big[2nu_1 + [r-1]u_3\big] - \mathrm{e}^{\xi_4}[2nu_1 - u_3]\big)$$
$$= g_r^2|\rho_2|^2 u_3 + \frac{g_r^2|\rho_2|^2}{r}\big\{2n\big(\mathrm{e}^{\xi_3} - \mathrm{e}^{\xi_4}\big)u_1 + \big([r-1]\mathrm{e}^{\xi_3} + \mathrm{e}^{\xi_4} - r\big)u_3\big\}, \tag{4.35}$$

where ξ_1 lies between $2ru_1 + [n-1]u_2$ and 0, ξ_2 between $2ru_1 - u_2$ and 0, ξ_3 between $2nu_1 + [r-1]u_3$ and 0, ξ_4 between $2nu_1 - u_3$ and 0.

Let $u = u_1^2 + u_2^2 + u_3^2$. Noting that $u_i \to 0$ as $|x| \to \infty$ and (4.33)–(4.35), we have

$$\Delta u \geqslant 2\sum_{i=1}^{3} u_i \Delta u_i \geqslant 2\sigma_0^2 u - f(x)u, \tag{4.36}$$

where σ_0 is a positive constant defined by

$$\sigma_0^2 \equiv \min\big\{2nr\big(r|\rho_1|^2 + n|\rho_2|^2\big)\tilde{g}_0^2,\ g_n^2|\rho_1|^2,\ g_r^2|\rho_2|^2\big\},$$

and $f(x)$ is a function satisfies $f(x) \to 0$ as $|x| \to \infty$.

Then for any $\varepsilon \in (0,1)$, there is an $R_\varepsilon > R$ such that

$$\Delta u \geqslant 2\left(1 - \frac{\varepsilon}{2}\right)\sigma_0^2 u \quad \text{as } |x| > R_\varepsilon. \tag{4.37}$$

Since $u \to 0$ at infinity, by a comparison function argument with (4.37), we conclude that there exists a constant $C(\varepsilon) > 0$ such that

$$u \leqslant C(\varepsilon)\mathrm{e}^{-\sqrt{2}(1-\varepsilon)\sigma_0|x|} \quad \text{as } |x| > R_\varepsilon. \tag{4.38}$$

Now we turn to the decay estimates for the derivatives. Let ∂ be any one of the two partial derivatives ∂_1 and ∂_2. When $|x| > R$, a direct computation gives

$$\begin{aligned}\Delta(\partial u_1) = \tilde{g}_0^2\big\{&\big(2r^2|\rho_1|^2\big[\mathrm{e}^{2ru_1+[n-1]u_2} + [n-1]\mathrm{e}^{2ru_1-u_2}\big]\\ &+ 2n^2|\rho_2|^2\big[\mathrm{e}^{2nu_1+[r-1]u_3} + [r-1]\mathrm{e}^{2nu_1-u_3}\big]\big)\partial u_1\\ &+ r(n-1)|\rho_1|^2\big(\mathrm{e}^{2ru_1+[n-1]u_2} - \mathrm{e}^{2ru_1-u_2}\big)\partial u_2\\ &+ n(r-1)|\rho_2|^2\big(\mathrm{e}^{2nu_1+[r-1]u_3} - \mathrm{e}^{2nu_1-u_3}\big)\partial u_3\big\},\end{aligned} \tag{4.39}$$

$$\begin{aligned}\Delta(\partial u_2) = \frac{g_n^2|\rho_1|^2}{n}\big\{&2r\big(\mathrm{e}^{2ru_1+[n-1]u_2} - \mathrm{e}^{2ru_1-u_2}\big)\partial u_1\\ &+ \big([n-1]\mathrm{e}^{2ru_1+[n-1]u_2} + \mathrm{e}^{2ru_1-u_2}\big)\partial u_2\big\},\end{aligned} \tag{4.40}$$

$$\begin{aligned}\Delta(\partial u_3) = \frac{g_r^2|\rho_2|^2}{r}\big\{&2n\big(\mathrm{e}^{2nu_1+[r-1]u_3} - \mathrm{e}^{2nu_1-u_3}\big)\partial u_1\\ &+ \big([r-1]\mathrm{e}^{2nu_1+[r-1]u_3} + \mathrm{e}^{2nu_1-u_3}\big)\partial u_3\big\}.\end{aligned} \tag{4.41}$$

Let $v = (\partial u_1)^2 + (\partial u_2)^2 + (\partial u_3)^2$. In view of the fact $u_i \to 0$ as $|x| \to \infty$ and (4.39)–(4.41), we obtain

$$\Delta v \geqslant 2\sum_{i=1}^{3}(\partial u_i)\Delta(\partial u_i) \geqslant 2\sigma_0^2 v - f(x)v, \tag{4.42}$$

where σ_0 is the same with above and $f(x)$ is a function satisfies $f(x) \to 0$ as $|x| \to \infty$. Then for any $\varepsilon \in (0,1)$, there exits an $R_\varepsilon > R$ such that

$$\Delta v \geqslant 2\left(1 - \frac{\varepsilon}{2}\right)\sigma_0^2 v \quad \text{as } |x| > R_\varepsilon. \tag{4.43}$$

Noting that $v \to 0$ at infinity and by a comparison function argument with (4.43), we infer that there exists a constant $C(\varepsilon) > 0$ such that

$$v \leqslant C(\varepsilon)\mathrm{e}^{-\sqrt{2}(1-\varepsilon)\sigma_0|x|} \quad \text{as } |x| > R_\varepsilon. \tag{4.44}$$

Then from (4.38) and (4.44), we get the desired decay estimates (2.32) for the solutions.

Now we are in a position to compute the quantized integrals for the planar case. By the definition of u_i^0, we see that $|\nabla u_i^0| = O(|x|^{-3})$ at infinity, which together with (4.44) give $|\nabla v_i| = O(|x|^{-3})$ at infinity, $i = 1, 2, 3$. Therefore, it follows from the divergence theorem that

$$\int_{\mathbb{R}^2} \Delta v_i \, \mathrm{d}x = 0, \quad i = 1, 2, 3. \tag{4.45}$$

By the definition of h_i, we have

$$\int_{\mathbb{R}^2} h_1 \, \mathrm{d}x = 4\pi(n_1 + n_2 + n_3), \qquad \int_{\mathbb{R}^2} h_i \, \mathrm{d}x = 4\pi n_i, \quad i = 2, 3. \tag{4.46}$$

Now using (4.45)–(4.46) and integrating Eqs. (4.6)–(4.8) over $\mathbb{R}^2$, we get the desired quantized integrals (2.34)–(2.36) for the planar case.

5. Yang–Mills–Higgs model with gauge group $U(1) \times SO(2M)$

In this and the following sections we study the Yang–Mills–Higgs model with gauge group $U(1) \times G'$ introduced in [11,12,19]. The concrete case with $G' = SO(2M)$ and $G' = SU(N)$ will be studied this section and next section, respectively. The Lagrangian density takes the form

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\bigg(&-\frac{1}{2e^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2g^2}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + \mathcal{D}_\mu H\big(\mathcal{D}^\mu H\big)^\dagger - \frac{e^2}{4}\left|2\,\mathrm{Tr}\big(HH^\dagger t^0\big)t^0 - \frac{\xi}{N}\mathbf{1}_N\right|^2 \\ &- g^2\big|\mathrm{Tr}\big(HH^\dagger t^a\big)t^a\big|^2\bigg),\end{aligned} \tag{5.1}$$

where field strength, gauge fields and covariant derivative are defined as

$$\begin{aligned}&F_{\mu\nu} = F_{\mu\nu}^0 t^0, \qquad F_{\mu\nu}^0 = \partial_\mu A_\nu^0 - \partial_\nu A_\mu^0, \\ &\hat{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}[A_\mu, A_\nu], \qquad A_\mu = A_\mu^a t^a, \\ &\mathcal{D}_\mu = \partial_\mu + \mathrm{i}A_\mu^0 t^0 + \mathrm{i}A_\mu^a t^a,\end{aligned}$$

A_μ^0 is the gauge field of $U(1)$, A_μ^a are the gauge fields of G', t^0 and t^a are the standard generators of $U(1)$ and G'. The matter scalar fields are written as a color-flavor mixed matrix H. Here e and g are the $U(1)$ and G' coupling constants, respectively.

With a Bogomol'nyi reduction [5,28] for static vortex solutions, the following BPS vortex equations [11,12,19] can be obtained

$$\bar{\mathcal{D}}H = \bar{\partial}H + \mathrm{i}\bar{A}H = 0, \tag{5.2}$$

$$F_{12}^0 = e^2\left(\mathrm{Tr}\left(HH^\dagger t^0\right) - \frac{\xi}{\sqrt{2N}}\right), \tag{5.3}$$

$$F_{12}^a = g^2\,\mathrm{Tr}\left(HH^\dagger t^a\right), \tag{5.4}$$

where $\bar{\mathcal{D}} = \frac{1}{2}(\mathcal{D}_1 + \mathrm{i}\mathcal{D}_2)$.

The above BPS equations (5.2)–(5.4) are still difficult to approach. With the complex variable $z = x_1 + ix_2$ and the ansatz [13,14]

$$H = S^{-1}H_0(z), \qquad \bar{A} = -\mathrm{i}S^{-1}\bar{\partial}S, \tag{5.5}$$

where $S = S(z,\bar{z}) \in \mathbb{C}^* \times G'^{\mathbb{C}}$ (the complexification of the gauge group), and $H_0(z)$ is holomorphic in z called the moduli matrix [13,26,27]. The BPS equation (5.2) is verified automatically. Let $\tilde{\Omega} = SS^\dagger$. Decompose S as $S = sS'$, and $\tilde{\Omega} = \omega\Omega'$, where $\omega = |s|^2$, $\Omega' = S'S'^\dagger$. Let us define $\Omega_0 = H_0H_0^\dagger$. Then as $G' = SO(2M)$ the BPS equations (5.3)–(5.4) become

$$\bar{\partial}\partial\ln\omega = -\frac{e^2}{8M}\left(\frac{1}{\omega}\,\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right) - \xi\right), \tag{5.6}$$

$$\bar{\partial}\left(\Omega'\partial\Omega'^{-1}\right) = \frac{g^2}{8\omega}\left(\Omega_0\Omega'^{-1} - J^\dagger\left(\Omega_0\Omega'^{-1}\right)^T J\right), \tag{5.7}$$

where

$$J \equiv \begin{pmatrix} \mathbf{0}_M & \mathbf{1}_M \\ \mathbf{1}_M & \mathbf{0}_M \end{pmatrix}.$$

As in [20], we make the following ansatz to reduce Eqs. (5.6)–(5.7)

$$\Omega' = \mathrm{diag}\left\{\mathrm{e}^{\chi_1},\ldots,\mathrm{e}^{\chi_M},\mathrm{e}^{-\chi_1},\ldots,\mathrm{e}^{-\chi_M}\right\}, \quad \omega = \mathrm{e}^{\psi}. \tag{5.8}$$

Without loss of generality, we take the moduli matrix H_0 ($2M \times 2M$ matrix) as

$$H_0 = \rho\prod_{i=0}^{M}P_i(z)\,\mathrm{diag}\left\{P_1(z),\ldots,P_M(z);\left(P_1(z)\right)^{-1},\ldots,\left(P_M(z)\right)^{-1}\right\}, \tag{5.9}$$

where

$$P_0(z) = \prod_{s=1}^{n_0}(z - z_{0s}), \qquad P_i(z) = \prod_{s=1}^{n}(z - z_{is}), \quad i = 1,\ldots,M, \tag{5.10}$$

$\rho \in \mathbb{C}$, $\rho \neq 0$, and $n_0, n \geqslant 0$ are integers.

Then inserting (5.8)–(5.9) into (5.6)–(5.7), we obtain

$$\Delta\psi = \frac{e^2}{2M}\left(|\rho|^2\prod_{i=0}^{M}\left|P_i(z)\right|^2\mathrm{e}^{-\psi}\sum_{i=1}^{M}\left[\left|P_i(z)\right|^2\mathrm{e}^{-\chi_i} + \left|P_i(z)\right|^{-2}\mathrm{e}^{\chi_i}\right] - \xi\right), \tag{5.11}$$

$$\Delta\chi_i = \frac{g^2}{2M}|\rho|^2\prod_{i=0}^{M}\left|P_i(z)\right|^2\mathrm{e}^{-\psi}\left(\left|P_i(z)\right|^2\mathrm{e}^{-\chi_i} - \left|P_i(z)\right|^{-2}\mathrm{e}^{\chi_i}\right), \quad i = 1,\ldots,M, \tag{5.12}$$

where the vacuum manifold is given by

$$2M|\rho|^2 = \xi. \tag{5.13}$$

With the notation

$$u = -\psi + \sum_{i=1}^{M}\sum_{s=1}^{n} \ln|z - z_{is}|^2, \qquad u_i = -\chi_i + \sum_{s=1}^{n} \ln|z - z_{is}|^2, \quad i = 1, \dots, M, \tag{5.14}$$

and $z_{is} = p_{is}$, we reduce (5.11)–(5.12) into

$$\Delta u = \frac{e^2}{2M}\left(|\rho|^2 \sum_{i=1}^{M}\left[e^{u+u_i} + e^{u-u_i}\right] - \xi\right) + 4\pi \sum_{s=1}^{n_0} \delta_{p_{0s}} + 4\pi \sum_{i=1}^{M}\sum_{s=1}^{n} \delta_{p_{is}}, \tag{5.15}$$

$$\Delta u_j = \frac{g^2|\rho|^2}{2}\left(e^{u+u_j} - e^{u-u_j}\right) + 4\pi \sum_{s=1}^{n} \delta_{p_{js}}, \quad j = 1, \dots, M. \tag{5.16}$$

We will consider the system (5.15)–(5.16) for two cases: over a doubly periodic-domain and over the full plane with the boundary condition

$$u \to 0, \quad u_i \to 0, \quad \text{as } |x| \to \infty, \ i = 1, \dots, M. \tag{5.17}$$

For the system (5.15)–(5.16), our main results read as follows.

Theorem 5.1. *Consider the problem (5.15)–(5.16) with any distribution of points $p_{01}, \dots, p_{0n_0}$, $p_{i1}, \dots, p_{in}$, and $n_0, n \geqslant 0$ are integers, $i = 1, \dots, M$. For any $\rho \in \mathbb{C}$ and $\xi > 0$ satisfying (5.13), and any coupling parameters $e, g > 0$, we have the following conclusion*:

Over a doubly periodic-domain Ω, there exists a solution for (5.15)–(5.16) if and only if

$$\frac{n_0 + Mn}{e^2} + \frac{n}{g^2} < \frac{\xi|\Omega|}{8M\pi}. \tag{5.18}$$

Moreover, if a solution exists, it must be unique.

Over $\mathbb{R}^2$, there exists a unique solution the problem (5.15)–(5.16) satisfying the boundary condition (5.17). Furthermore, the solution satisfies the following exponential decay estimate at infinity

$$|u|^2 + |\nabla u|^2 + \sum_{i=1}^{M}\left(u_i^2 + |\nabla u_i|^2\right) \leqslant O\left(e^{-\sqrt{2}\sigma_0(1-\varepsilon)|x|}\right), \tag{5.19}$$

where $\varepsilon \in (0, 1)$ is an arbitrary parameter, σ_0 is a positive constant defined by

$$\sigma_0^2 \equiv \min\left\{M^2e^2|\rho|^2, \ g^2|\rho|^2\right\}. \tag{5.20}$$

In both cases, there hold the quantized integrals

$$\int\left(|\rho|^2 \sum_{i=1}^{M}\left[e^{u+u_i} + e^{u-u_i}\right] - \xi\right) dx = -\frac{8M\pi(n_0 + Mn)}{e^2}, \tag{5.21}$$

$$|\rho|^2 \int\left(e^{u+u_j} - e^{u-u_j}\right) dx = -\frac{\pi n}{g^2}, \quad j = 1, \dots, M, \tag{5.22}$$

where the integrals are taken over either the domain Ω or $\mathbb{R}^2$.

5.1. Doubly periodic case

In this subsection we will prove Theorem 5.1 for the doubly periodic case. We use the argument of Section 3. Let u^0 be the solution of the problem

$$\Delta u^0 = 4\pi \sum_{s=1}^{n_0} \delta_{p_{0s}} + 4\pi \sum_{i=1}^{M} \sum_{s=1}^{n} \delta_{p_{is}} - \frac{4\pi(n_0 + Mn)}{|\Omega|},$$
$$\int_\Omega u^0 \, dx = 0,$$

and u_i^0 be the solution of the problem

$$\Delta u_i^0 = 4\pi \sum_{s=1}^{n} \delta_{p_{is}} - \frac{4\pi n}{|\Omega|},$$
$$\int_\Omega u_i^0 \, dx = 0, \quad i = 1, \dots, M.$$

With $u = u^0 + v$, $u_i = u_i^0 + v_i$, $i = 1, \dots, M$, we may reformulate (5.15)–(5.16) as

$$\Delta v = \frac{e^2}{2M} \left(|\rho|^2 \sum_{i=1}^{M} \left[e^{u^0 + u_i^0 + v + v_i} + e^{u^0 - u_i^0 + v - v_i} \right] - \xi \right) + \frac{4\pi(n_0 + Mn)}{|\Omega|}, \tag{5.23}$$

$$\Delta v_j = \frac{g^2 |\rho|^2}{2} \left(e^{u^0 + u_j^0 + v + v_j} - e^{u^0 - u_j^0 + v - v_j} \right) + \frac{4\pi n}{|\Omega|}, \quad j = 1, \dots, M, \tag{5.24}$$

which are the Euler–Lagrange equations of the following functional

$$\begin{aligned} I(v, v_1, \dots, v_M) = {} & \frac{M}{e^2} \|\nabla v\|_2^2 + \frac{1}{g^2} \sum_{i=1}^{M} \|\nabla v_i\|_2^2 + \frac{8M\pi(n_0 + Mn)}{|\Omega| e^2} \int_\Omega v \, dx \\ & + \frac{8\pi n}{|\Omega| g^2} \sum_{i=1}^{M} \int_\Omega v_i \, dx \\ & + |\rho|^2 \sum_{i=1}^{M} \int_\Omega \left[e^{u^0 + u_i^0 + v + v_i} + e^{u^0 - u_i^0 + v - v_i} - 2v \right] dx. \end{aligned} \tag{5.25}$$

We first prove the necessity of the condition (5.18) for the existence of solutions to (5.15)–(5.16). Let $(v, v_1, \dots, v_M)$ be a solution of (5.23)–(5.24). Then, integrating Eqs. (5.23)–(5.24) over the domain Ω, we obtain

$$|\rho|^2 \sum_{i=1}^{M} \int_\Omega \left(e^{u^0 + u_i^0 + v + v_i} + e^{u^0 - u_i^0 + v - v_i} \right) dx = \xi |\Omega| - \frac{8M\pi(n_0 + Mn)}{e^2},$$
$$|\rho|^2 \int_\Omega \left(e^{u^0 + u_j^0 + v + v_j} - e^{u^0 - u_j^0 + v - v_j} \right) dx = -\frac{8\pi n}{g^2}, \quad j = 1, \dots, M,$$

which give

$$|\rho|^2 \sum_{i=1}^{M} \int_{\Omega} \mathrm{e}^{u^0+u_i^0+v+v_i}\,\mathrm{d}x = \frac{\xi|\Omega|}{2} - \frac{4M\pi(n_0+Mn)}{e^2} - \frac{4M\pi n}{g^2}, \tag{5.26}$$

$$|\rho|^2 \sum_{i=1}^{M} \int_{\Omega} \mathrm{e}^{u^0-u_i^0+v-v_i}\,\mathrm{d}x = \frac{\xi|\Omega|}{2} - \frac{4M\pi(n_0+Mn)}{e^2} + \frac{4M\pi n}{g^2}. \tag{5.27}$$

Then both the right-hand sides of (5.26) and (5.27) should be positive, which concludes the necessity of the condition (5.18).

In what follows we show that the condition (5.18) is also sufficient for the existence of solutions to (5.15)–(5.16). For $(v, v_1, \ldots, v_M) \in W^{1,2}(\Omega)$, using the decomposition formula (3.11) we have $v = c + w$, $v_i = c_i + w_i$, $i = 1, \ldots, M$. Then using Jensen's inequality we obtain

$$\begin{aligned}
&I(v, v_1, \ldots, v_M) - \frac{M}{e^2}\|\nabla w\|_2^2 - \frac{1}{g^2}\sum_{i=1}^{M}\|\nabla w_i\|_2^2 \\
&\quad = |\rho|^2 \sum_{i=1}^{M}\left(\mathrm{e}^{c+c_i}\int_{\Omega}\mathrm{e}^{u^0+u_i^0+w+w_i}\,\mathrm{d}x + \mathrm{e}^{c-c_i}\int_{\Omega}\mathrm{e}^{u^0-u_i^0+w-w_i}\,\mathrm{d}x - 2c\right) \\
&\qquad + \frac{8M\pi(n_0+Mn)}{e^2}c + \frac{8\pi n}{g^2}\sum_{i=1}^{M}c_i \\
&\quad \geqslant |\rho|^2|\Omega|\sum_{i=1}^{M}\left(\mathrm{e}^{c+c_i} + \mathrm{e}^{c-c_i} - 2c\right) + \frac{8M\pi(n_0+Mn)}{e^2}c + \frac{8\pi n}{g^2}\sum_{i=1}^{M}c_i \\
&\quad = \sum_{i=1}^{M}\left(|\rho|^2|\Omega|\mathrm{e}^{c+c_i} - K_1[c+c_i] + |\rho|^2|\Omega|\mathrm{e}^{c-c_i} - K_2[c-c_i]\right),
\end{aligned} \tag{5.28}$$

where

$$K_1 \equiv \frac{\xi|\Omega|}{2M} - \frac{4\pi(n_0+Mn)}{e^2} - \frac{4\pi n}{g^2}, \qquad K_2 \equiv \frac{\xi|\Omega|}{2M} - \frac{4\pi(n_0+Mn)}{e^2} + \frac{4\pi n}{g^2}.$$

Noting the condition (5.18), we see that $K_1 > 0$, $K_2 > 0$. Then, from (5.28) we obtain

$$\begin{aligned}
I(v, v_1, \ldots, v_M) \geqslant{}& \frac{M}{e^2}\|\nabla w\|_2^2 - \frac{1}{g^2}\sum_{i=1}^{M}\|\nabla w_i\|_2^2 \\
&+ M\left(K_1 \ln\frac{|\rho|^2|\Omega|}{K_1} + K_2 \ln\frac{|\rho|^2|\Omega|}{K_2}\right).
\end{aligned} \tag{5.29}$$

Hence, from (5.29), we see that the functional I is bounded from below and the minimization problem

$$\eta_0 \equiv \inf\left\{I(v, v_1, \ldots, v_M) \,\middle|\, (v, v_1, \ldots, v_M) \in W^{1,2}(\Omega)\right\}. \tag{5.30}$$

is well-defined.

Using (5.28) and a similar argument in Section 3, we can get a critical point of the problem (5.30), which is a weak solution to (5.23)–(5.24). The uniqueness of the solution follows from the strict convexity of the functional I.

The quantized integrals follow from a direct integration. Then the proof of Theorem 5.1 for the doubly periodic case is complete.

5.2. Planar case

In this subsection we consider (5.15)–(5.16) over the full plane with the boundary condition (5.17).

Let

$$u^0(x) = -\sum_{s=1}^{n_0} \ln\bigl(1+\lambda|x-p_{0s}|^{-2}\bigr) - \sum_{i=1}^{M}\sum_{s=1}^{n} \ln\bigl(1+\lambda|x-p_{is}|^{-2}\bigr), \tag{5.31}$$

$$u_i^0(x) = -\sum_{s=1}^{n} \ln\bigl(1+\lambda|x-p_{is}|^{-2}\bigr), \quad i=1,\dots,M, \tag{5.32}$$

where $\lambda > 0$ is a parameter. Then we see that

$$\Delta u^0 = -h_0 + 4\pi\sum_{s=1}^{n_0}\delta_{p_{is}} + 4\pi\sum_{i=1}^{M}\sum_{s=1}^{n}\delta_{p_{is}}, \qquad \Delta u_i^0 = -h_i + 4\pi\sum_{s=1}^{n}\delta_{p_{is}}, \quad i=1,\dots,M, \tag{5.33}$$

where

$$h_0(x) = \sum_{s=1}^{n_0}\frac{4\lambda}{(\lambda+|x-p_{0s}|^2)^2} + \sum_{i=1}^{M}\sum_{s=1}^{n}\frac{4\lambda}{(\lambda+|x-p_{is}|^2)^2},$$

$$h_i = \sum_{s=1}^{n}\frac{4\lambda}{(\lambda+|x-p_{is}|^2)^2}, \quad i=1,\dots,M.$$

Let $u = u^0 + v$, $u_i = u_i^0 + v_i$, $i=1,\dots,M$. Then we rewrite the system (5.15)–(5.17) into the following form

$$\Delta v = \frac{e^2|\rho|^2}{2M}\sum_{i=1}^{M}\bigl(\mathrm{e}^{u^0+u_i^0+v+v_i} + \mathrm{e}^{u^0-u_i^0+v-v_i} - 2\bigr) + h_0, \tag{5.34}$$

$$\Delta v_j = \frac{g^2|\rho|^2}{2}\bigl(\mathrm{e}^{u^0+u_j^0+v+v_j} - \mathrm{e}^{u^0-u_j^0+v-v_j}\bigr) + h_j, \quad j=1,\dots,M, \tag{5.35}$$

which are the Euler–Lagrange equations of the functional

$$\begin{aligned} &I(v,v_1,\dots,v_M)\\ &\quad= \frac{M}{e^2}\|\nabla v\|_2^2 + \frac{1}{g^2}\sum_{i=1}^{M}\|\nabla v_i\|_2^2 + \frac{2M}{e^2}\int_{\mathbb{R}^2} h_0 v\,\mathrm{d}x + \frac{2}{g^2}\sum_{i=1}^{M}\int_{\mathbb{R}^2} h_i v_i\,\mathrm{d}x\\ &\qquad+ |\rho|^2\sum_{i=1}^{M}\int_{\mathbb{R}^2}\bigl(\mathrm{e}^{u^0+u_i^0}\bigl[\mathrm{e}^{v+v_i}-1\bigr] - [v+v_i] + \mathrm{e}^{u^0-u_i^0}\bigl[\mathrm{e}^{v-v_i}-1\bigr] - [v-v_i]\bigr)\,\mathrm{d}x. \end{aligned} \tag{5.36}$$

Since the functional is differentiable and strictly convex, as in Section 4, to get the solution of (5.34)–(5.35), we need to show that the functional is coercive over $W^{1,2}(\mathbb{R}^2)$. Although we can follow a similar argument as in Section 4 to prove the coerciveness of the functional, here we use a new direct approach recently developed in [33].

Taking $\alpha > 0$ such that $\alpha^2 > \max\{e^2, g^2\}$, we rewrite the functional I as

$$\begin{aligned} I(v, v_1, \dots, v_M) &= M\left(\frac{1}{e^2} - \frac{1}{\alpha^2}\right)\|\nabla v\|_2^2 + \left(\frac{1}{g^2} - \frac{1}{\alpha^2}\right)\sum_{i=1}^{M}\|\nabla v_i\|_2^2 \\ &\quad + \frac{1}{\alpha^2}\sum_{i=1}^{M}\left\{\frac{1}{2}\big\|\nabla(v+v_i)\big\|_2^2\right. \\ &\quad + \alpha^2|\rho|^2\int_{\mathbb{R}^2}\left(\mathrm{e}^{u^0+u_i^0}\left[\mathrm{e}^{v+v_i}-1\right]-[v+v_i]\right)\mathrm{d}x + \alpha^2\int_{\mathbb{R}^2} H_i^1(v+v_i)\,\mathrm{d}x \\ &\quad + \frac{1}{2}\big\|\nabla(v-v_i)\big\|_2^2 + \alpha^2|\rho|^2\int_{\mathbb{R}^2}\left(\mathrm{e}^{u^0-u_i^0}\left[\mathrm{e}^{v-v_i}-1\right]-[v-v_i]\right)\mathrm{d}x \\ &\quad \left. + \alpha^2\int_{\mathbb{R}^2} H_i^2(v-v_i)\,\mathrm{d}x\right\} \\ &= M\left(\frac{1}{e^2} - \frac{1}{\alpha^2}\right)\|\nabla v\|_2^2 + \left(\frac{1}{g^2} - \frac{1}{\alpha^2}\right)\sum_{i=1}^{M}\|\nabla v_i\|_2^2 \\ &\quad + \frac{1}{\alpha^2}\sum_{i=1}^{M}\left[J_i^1(v+v_i) + J_i^2(v-v_i)\right], \end{aligned} \tag{5.37}$$

where

$$H_i^1 \equiv \frac{h_0}{e^2} + \frac{h_i}{g^2}, \qquad H_i^2 \equiv \frac{h_0}{e^2} - \frac{h_i}{g^2}, \tag{5.38}$$

$$J_i^1(w) \equiv \frac{1}{2}\|\nabla w\|_2^2 + \alpha^2|\rho|^2\int_{\mathbb{R}^2}\left(\mathrm{e}^{u^0+u_i^0}\left[\mathrm{e}^{w}-1\right]-w\right)\mathrm{d}x + \alpha^2\int_{\mathbb{R}^2} H_i^1 w\,\mathrm{d}x, \tag{5.39}$$

$$J_i^2(w) \equiv \frac{1}{2}\|\nabla w\|_2^2 + \alpha^2|\rho|^2\int_{\mathbb{R}^2}\left(\mathrm{e}^{u^0-u_i^0}\left[\mathrm{e}^{w}-1\right]-w\right)\mathrm{d}x + \alpha^2\int_{\mathbb{R}^2} H_i^2 w\,\mathrm{d}x. \tag{5.40}$$

To show the coerciveness of I, in view of (5.37), we consider a generic functional of the following form

$$J(w) = \frac{1}{2}\|\nabla w\|_2^2 + \alpha^2|\rho|^2\int_{\mathbb{R}^2}\left(\mathrm{e}^{w_0}\left[\mathrm{e}^{w}-1\right]-w\right)\mathrm{d}x + \alpha^2\int_{\mathbb{R}^2} H w\,\mathrm{d}x, \tag{5.41}$$

where w_0 (taking the place of $u^0 \pm u_i^0$) and H (taking the place of H_i^1 and H_i^2) are defined as below

$$w_0 = \sum_{s=1}^{m}\ln\left(1+\lambda|x-p_s|^{-2}\right), \qquad H = \sum_{s=1}^{m}\frac{4\lambda}{(\lambda+|x-p_s|^2)^2}, \tag{5.42}$$

where m (taking the place of $n_0 + Mn$ or n) is a positive integer.

Let $w_+ = \{w, 0\}$, $w_- = \max\{-w, 0\}$. We decompose J as

$$J(w) = J(w_+) + J(-w_-), \tag{5.43}$$

which will be estimated separately in the sequel. Denote by B_R a disc centered at the origin with radius R. From the definition of w_0, we may choose $R_0 > 0$ with

$$R_0 > 2\max\{|p_s|,\ s = 1, \dots, m\}, \tag{5.44}$$

such that, for any $\lambda > 1$, there exists a positive constant $a_0 > 1$ such that

$$\mathrm{e}^{w_0} \geqslant \frac{1}{a_0\lambda^m} \quad \text{as } |x| > R_0, \tag{5.45}$$

$$0 \leqslant \mathrm{e}^{w_0} \leqslant \frac{a_0}{\lambda^m} \quad \text{as } |x| \leqslant R_0, \tag{5.46}$$

$$\int_{B_{R_0}} \mathrm{e}^{w_0}\,\mathrm{d}x \geqslant \frac{1}{a_0\lambda^m}. \tag{5.47}$$

Let

$$G(w) = \mathrm{e}^{w_0}\big[\mathrm{e}^{w} - 1\big] - w. \tag{5.48}$$

It is easy to see that

$$\begin{aligned}\int_{\mathbb{R}^2} G(w_+)\,\mathrm{d}x &= \int_{\mathbb{R}^2} \big(\mathrm{e}^{w_0}\big[\mathrm{e}^{w_+} - 1\big] - w_+\big)\,\mathrm{d}x \\ &\geqslant \frac{1}{2}\int_{\mathbb{R}^2} \mathrm{e}^{w_0} w_+^2\,\mathrm{d}x + \int_{\mathbb{R}^2} \big(\mathrm{e}^{w_0} - 1\big) w_+\,\mathrm{d}x.\end{aligned} \tag{5.49}$$

When $|x| > R_0$, noting that $1 - \mathrm{e}^{w_0} \in L^2(\mathbb{R}^2)$, we have

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B_{R_0}} G(w_+)\,\mathrm{d}x &\geqslant \frac{1}{2a_0\lambda^m}\|w_+\|^2_{L^2(\mathbb{R}^2\setminus B_{R_0})} + \int_{\mathbb{R}^2\setminus B_{R_0}} \big(\mathrm{e}^{w_0} - 1\big) w_+\,\mathrm{d}x \\ &\geqslant \frac{1}{4a_0\lambda^m}\|w_+\|^2_{L^2(\mathbb{R}^2\setminus B_{R_0})} - C_\lambda,\end{aligned} \tag{5.50}$$

where C_λ is a generic positive constant depending only on λ.

When $|x| < R_0$, we decompose w_+ as

$$w_+ = \dot{w}_+ + \overline{w}_+, \quad \text{with } \overline{w}_+ = \frac{1}{|R_0|}\int_{B_{R_0}} w_+\,\mathrm{d}x, \quad \int_{B_{R_0}} \dot{w}_+\,\mathrm{d}x = 0.$$

Then from (5.49) and Young's inequality, we have

$$\begin{aligned}\int_{B_{R_0}} G(w_+)\,\mathrm{d}x = \frac{\overline{w}_+^2}{2}\int_{B_{R_0}} \mathrm{e}^{w_0}\,\mathrm{d}x + \frac{1}{2}\int_{B_{R_0}} \mathrm{e}^{w_0}\dot{w}_+^2\,\mathrm{d}x + \int_{B_{R_0}} \mathrm{e}^{w_0}\dot{w}_+\,\mathrm{d}x \\ + \overline{w}_+\int_{B_{R_0}} \big(\mathrm{e}^{w_0}\dot{w}_+ + \mathrm{e}^{w_0} - 1\big)\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\geqslant \frac{\overline{w}_+^2}{4}\int\limits_{B_{R_0}} \mathrm{e}^{w_0}\,\mathrm{d}x + \frac{1}{2}\int\limits_{B_{R_0}} \mathrm{e}^{w_0}\dot{w}_+^2\,\mathrm{d}x + \int\limits_{B_{R_0}} \mathrm{e}^{w_0}\dot{w}_+\,\mathrm{d}x\\
&\quad - \Big(\int\limits_{B_{R_0}} \mathrm{e}^{w_0}\,\mathrm{d}x\Big)^{-1}\Big(\int\limits_{B_{R_0}} \big[\mathrm{e}^{w_0}\dot{w}_+ + \mathrm{e}^{w_0} - 1\big]\,\mathrm{d}x\Big)^2\\
&\geqslant \frac{\overline{w}_+^2}{4}\int\limits_{B_{R_0}} \mathrm{e}^{w_0}\,\mathrm{d}x - \frac{1}{2}\Big(\int\limits_{B_{R_0}} \mathrm{e}^{w_0}\,\mathrm{d}x\Big)^{-1}\Big(\int\limits_{B_{R_0}} \mathrm{e}^{w_0}\dot{w}_+\,\mathrm{d}x\Big)^2\\
&\quad + \Big(2|B_{R_0}|\Big(\int\limits_{B_{R_0}} \mathrm{e}^{w_0}\,\mathrm{d}x\Big)^{-1} - 1\Big)\mathrm{e}^{w_0}\dot{w}_+\,\mathrm{d}x\\
&\geqslant \frac{\overline{w}_+^2}{4}\int\limits_{B_{R_0}} \mathrm{e}^{w_0}\,\mathrm{d}x - \frac{1}{2}\big\|\mathrm{e}^{w_0}\big\|_{L^\infty(B_{R_0})}\|\dot{w}_+\|_{L^2(B_{R_0})}^2\\
&\quad - C_\lambda\|\dot{w}_+\|_{L^2(B_{R_0})} - C_\lambda. \qquad (5.51)
\end{aligned}
$$

Using the Poincaré inequality

$$
\|\dot{w}_+\|_{L^2(B_{R_0})}^2 \leqslant c_0 R_0^2\|\nabla w_+\|_{L^2(B_{R_0})}^2, \qquad (5.52)
$$

we obtain

$$
\|w_+\|_{L^2(B_{R_0})}^2 \leqslant C_0 R_0^2\big(\overline{w}_+^2 + \|\nabla w_+\|_{L^2(B_{R_0})}^2\big). \qquad (5.53)
$$

By the Hölder and Young inequalities, we have

$$
\int\limits_{\mathbb{R}^2} H w_+\,\mathrm{d}x \geqslant -\frac{C}{\sqrt{\lambda}}\|w_+\|_2 \geqslant -\frac{\varepsilon}{\lambda^m}\|w_+\|_2^2 - C_\lambda, \qquad (5.54)
$$

where $\varepsilon > 0$ is small. Then we conclude from (5.45)–(5.47) and (5.50)–(5.54) that there exist positive constants b_0, b_1 such that

$$
J(w_+) \geqslant \Big(\frac{1}{2} - \frac{b_0\alpha^2|\rho|^2}{\lambda^m}\Big)\|\nabla w_+\|_2^2 + \frac{b_1\alpha^2|\rho|^2}{\lambda^m}\|w_+\|_2^2 - C_\lambda. \qquad (5.55)
$$

In what follows we estimate $J(-v_-)$. In view of the elementary inequality $1 - \mathrm{e}^{-s} \geqslant \frac{s}{1+s}$ for $s \geqslant 0$, we get

$$
\mathrm{e}^{-t} - 1 + t = \int\limits_0^t \big(1 - \mathrm{e}^{-s}\big)\,\mathrm{d}s \geqslant \int\limits_0^t \frac{s}{1+s}\,\mathrm{d}s \geqslant \frac{t^2}{2(1+t)}, \quad t \geqslant 0. \qquad (5.56)
$$

Then from the inequality (5.56) and an inequality similar to (4.26), we may obtain

$$
\begin{aligned}
\int\limits_{\mathbb{R}^2} G(-w_-)\,\mathrm{d}x &= \int\limits_{\mathbb{R}^2} \big(\mathrm{e}^{w_0}\big[\mathrm{e}^{-w_-} - 1\big] + w_-\big)\,\mathrm{d}x\\
&= \int\limits_{\mathbb{R}^2} \big(\big[\mathrm{e}^{w_0} - 1\big]\big[\mathrm{e}^{-w_-} - 1\big] + \mathrm{e}^{-w_-} - 1 + w_-\big)\,\mathrm{d}x\\
&\geqslant \frac{1}{2}\int \frac{w_-^2}{1 + w_-}\,\mathrm{d}x \geqslant \frac{\|w_-\|_2^2}{8(1 + \|\nabla w_-\|_2^2)}. \qquad (5.57)
\end{aligned}
$$

Using Hölder's inequality again, one has

$$\int_{\mathbb{R}^2} H w_- \,\mathrm{d}x \geqslant -\frac{C_0}{\sqrt{\lambda}}\|w_-\|_2. \tag{5.58}$$

Hence we infer from (5.57) and (5.58) that

$$\begin{aligned} J(-w_-) &\geqslant \frac{1}{2}\|\nabla w_-\|_2^2 + \frac{\alpha^2|\rho|^2\|w_-\|_2^2}{8(1+\|\nabla w_-\|_2^2)} - \frac{C_0\alpha^2}{\sqrt{\lambda}}\|w_-\|_2 \\ &\geqslant \frac{3}{8}\|\nabla w_-\|_2^2 + \frac{1}{8}\left(\|\nabla w_-\|_2^2 + \frac{\alpha^2|\rho|^2\|w_-\|_2^2}{1+\|\nabla w_-\|_2^2}\right) - \frac{C_0\alpha^2}{\sqrt{\lambda}}\|w_-\|_2 \\ &\geqslant \frac{3}{8}\|\nabla w_-\|_2^2 + \left(\frac{\alpha}{4} - \frac{C_0\alpha^2|\rho|^2}{\sqrt{\lambda}}\right)\|w_-\|_2 - \frac{1}{8}, \end{aligned} \tag{5.59}$$

where we have used the inequality

$$t + \frac{A^2}{1+t} \geqslant 2A - 1, \quad \forall A \geqslant 0,\ t \geqslant 0.$$

At this point, by taking λ suitably large we infer from (5.55) and (5.59) that there exist positive constants C_1, C_2 such that

$$J(w) \geqslant C_1\|w\|_{W^{1,2}(\mathbb{R}^2)} - C_2. \tag{5.60}$$

Therefore, using the estimate (5.60) on the right-hand side of (5.37), we conclude that

$$\begin{aligned} I(v, v_1, \ldots, v_M) &\geqslant \frac{C_1}{\alpha^2}\sum_{i=1}^{M}\big(\|v+v_i\|_{W^{1,2}(\mathbb{R}^2)} + \|v-v_i\|_{W^{1,2}(\mathbb{R}^2)}\big) - 2MC_2 \\ &\geqslant C_1'\left(\|v\|_{W^{1,2}(\mathbb{R}^2)} + \sum_{i=1}^{M}\|v_i\|_{W^{1,2}(\mathbb{R}^2)}\right) - C_2', \end{aligned} \tag{5.61}$$

where C_1', C_2' are two positive constants.

Now using the coercive lower bound (5.61), we can obtain a critical point for the functional I by a routing argument. The critical point is also unique since the functional I is strictly convex.

To establish the behavior at infinity, decay estimate of the solutions and the quantized integrals, we can use a similar argument as in Section 4. Then the proof of Theorem 5.1 for the planar case is complete.

6. Yang–Mills–Higgs model with gauge group $U(1) \times SU(N)$

In this section we consider the Yang–Mills–Higgs model introduced last section with gauge group $U(1) \times SU(N)$. In this case the BPS equations (5.3)–(5.4) take the form (see [11,12,19])

$$\bar\partial\partial \ln\omega = -\frac{e^2}{4N}\left(\frac{1}{\omega}\operatorname{Tr}\big(\Omega_0\Omega'^{-1}\big) - \xi\right), \tag{6.1}$$

$$\bar\partial\big(\Omega'\partial\Omega'^{-1}\big) = \frac{g^2}{4\omega}\left(\Omega_0\Omega'^{-1} - \frac{\mathbf{1}_N}{N}\operatorname{Tr}\big(\Omega_0\Omega'^{-1}\big)\right). \tag{6.2}$$

To simplify Eqs. (6.1)–(6.2), we use the ansatz [20]

$$\Omega' = \operatorname{diag}\{e^{(N-1)\chi}, e^{-\chi}, \dots, e^{-\chi}\}, \quad \omega = e^{\psi}. \tag{6.3}$$

We take the moduli matrix H_0 ($N \times N$ matrix) as

$$H_0 = \rho \prod_{i=1}^{2} P_i(z) \operatorname{diag}\{(P_2(z))^{(N-1)}, (P_2(z))^{-1}, \dots, (P_2(z))^{-1}\}, \tag{6.4}$$

where

$$P_i(z) = \prod_{k=1}^{n_i} (z - z_{ik}), \quad i = 1, 2, \tag{6.5}$$

$\rho \in \mathbb{C}$, $\rho \neq 0$ and $n_i \geqslant 0$ are integers.

Then Eqs. (6.1)–(6.2) become

$$\Delta\psi = \frac{e^2}{N}\left(|\rho|^2 \prod_{i=1}^{2} |P_i(z)|^2 e^{-\psi}\left[|P_2(z)|^{2(N-1)} e^{-(N-1)\chi} + (N-1)|P_2(z)|^{-2} e^{\chi}\right] - \xi\right), \tag{6.6}$$

$$\Delta\chi = \frac{g^2}{N}|\rho|^2 \prod_{i=1}^{2} |P_i(z)|^2 e^{-\psi}\left(|P_2(z)|^{2(N-1)} e^{-(N-1)\chi} - |P_2(z)|^{-2} e^{\chi}\right). \tag{6.7}$$

With the notation

$$u_1 = -\psi + \sum_{i=1}^{2}\sum_{ik=1}^{n_i} \ln|z - z_{ik}|^2, \qquad u_2 = -\chi + \sum_{k=1}^{n_2} \ln|z - z_{2k}|^2, \tag{6.8}$$

Eqs. (6.6)–(6.7) reduce into

$$\Delta u_1 = \frac{e^2}{N}\left\{|\rho|^2\left(e^{u_1+(N-1)u_2} + [N-1]e^{u_1-u_2}\right) - \xi\right\} + 4\pi \sum_{i=1}^{2}\sum_{s=1}^{n_i} \delta_{p_{is}}, \tag{6.9}$$

$$\Delta u_2 = \frac{g^2|\rho|^2}{N}\left(e^{u_1+(N-1)u_2} - e^{u_1-u_2}\right) + 4\pi \sum_{s=1}^{n_2} \delta_{p_{2s}}, \tag{6.10}$$

where the vacuum manifold is given by

$$N|\rho|^2 = \xi, \quad \rho \in \mathbb{C}, \ \xi > 0. \tag{6.11}$$

As previously, we consider the problem over a doubly periodic-domain and over $\mathbb{R}^2$ with the boundary condition

$$u_i \to 0, \quad |x| \to \infty, \quad i = 1, 2. \tag{6.12}$$

Our main results for (6.9)–(6.10) read as follows.

Theorem 6.1. *Consider the problem (6.9)–(6.10) with arbitrary distribution of points $p_{i1}, \dots, p_{in_i}$, and $n_i \geqslant 0$ are integers, $i = 1, 2$. For any $\rho \in \mathbb{C}$ and $\xi > 0$ satisfying (6.11), and any coupling parameters $e, g > 0$, we have the following conclusion*:

Over a doubly periodic-domain Ω, there exists a solution for (6.9)–(6.10) if and only if

$$\frac{n_1+n_2}{e^2}+\frac{(N-1)n_2}{g^2}<\frac{\xi|\Omega|}{4\pi N}. \tag{6.13}$$

Moreover, if a solution exists, it must be unique.

Over $\mathbb{R}^2$, there exits a unique solution for (6.9)–(6.10) satisfying the boundary condition (6.12). Furthermore, the solution satisfies the following exponential decay estimate at infinity

$$\sum_{i=1}^{2}\left(u_i^2+|\nabla u_i|^2\right)\leqslant O\left(\mathrm{e}^{-\sqrt{2}\sigma_0(1-\varepsilon)|x|}\right), \tag{6.14}$$

where $\varepsilon\in(0,1)$ is an arbitrary parameter, σ_0 is a positive constant defined by

$$\sigma_0^2\equiv\min\left\{|\rho|^2e^2,|\rho|^2g^2\right\}. \tag{6.15}$$

In both cases, there hold the quantized integrals

$$\int\left\{|\rho|^2\left(\mathrm{e}^{u_1+(N-1)u_2}+[N-1]\mathrm{e}^{u_1-u_2}\right)-\xi\right\}\mathrm{d}x=-\frac{4\pi N(n_1+n_2)}{e^2}, \tag{6.16}$$

$$\int\left(\mathrm{e}^{u_1+(N-1)u_2}-\mathrm{e}^{u_1-u_2}\right)\mathrm{d}x=-\frac{4\pi Nn_2}{|\rho|^2g^2}, \tag{6.17}$$

where the integrals are taken over either the domain Ω or $\mathbb{R}^2$.

6.1. Doubly periodic solution

In this subsection we prove Theorem 6.1 for the doubly periodic-domain case. We argue as in Section 3.

Let u^0 be the solution of the problem

$$\Delta u_1^0=4\pi\sum_{i=1}^{2}\sum_{s=1}^{n_i}\delta_{p_{is}}-\frac{4\pi(n_1+n_2)}{|\Omega|},$$

$$\int_\Omega u_1^0\,\mathrm{d}x=0,$$

and u_2^0 be the solution of the problem

$$\Delta u_2^0=4\pi\sum_{s=1}^{n_2}\delta_{p_{2s}}-\frac{4\pi n_2}{|\Omega|},$$

$$\int_\Omega u_2^0\,\mathrm{d}x=0.$$

As previous section, setting

$$u_i=u_i^0+v_i,\quad i=1,2,$$

we may rewrite (6.9)–(6.10) as

$$\Delta v_1 = \frac{e^2}{N}\left\{|\rho|^2\left(e^{u_1^0+v_1+(N-1)(u_2^0+v_2)} + [N-1]e^{u_1^0+v_1-u_2^0-v_2}\right) - \xi\right\} + \frac{4\pi(n_1+n_2)}{|\Omega|}, \tag{6.18}$$

$$\Delta v_2 = \frac{g^2|\rho|^2}{N}\left(e^{u_1^0+v_1+(N-1)(u_2^0+v_2)} - e^{u_1^0+v_1-u_2^0-v_2}\right) + \frac{4\pi n_2}{|\Omega|}. \tag{6.19}$$

We observe that Eqs. (6.18)–(6.19) are the Euler–Lagrange equations of the following functional

$$\begin{aligned} I(v_1,v_2) = {} & \frac{N}{2e^2}\|\nabla v_1\|_2^2 + \frac{N(N-1)}{2g^2}\|\nabla v_2\|_2^2 \\ & + |\rho|^2 \int_\Omega \left[e^{u_1^0+v_1+(N-1)(u_2^0+v_2)} + [N-1]e^{u_1^0+v_1-u_2^0-v_2} - Nv_1\right]dx \\ & + \frac{4\pi N(n_1+n_2)}{e^2|\Omega|}\int_\Omega v_1\,dx + \frac{4\pi N(N-1)n_2}{g^2|\Omega|}\int_\Omega v_2\,dx. \end{aligned} \tag{6.20}$$

To show the necessity of condition (6.13) for existence of solutions to (6.9)–(6.10), we integrate Eqs. (6.18)–(6.19) over the domain Ω to find

$$|\rho|^2\left[\int_\Omega e^{u_1^0+v_1+(N-1)(u_2^0+v_2)} + [N-1]e^{u_1^0+v_1-u_2^0-v_2}\,dx\right] = \xi|\Omega| - \frac{4\pi N(n_1+n_2)}{e^2},$$

$$|\rho|^2\left[\int_\Omega e^{u_1^0+v_1+(N-1)(u_2^0+v_2)} - e^{u_1^0+v_1-u_2^0-v_2}\,dx\right] = -\frac{4\pi N n_2}{g^2},$$

which conclude

$$|\rho|^2\int_\Omega e^{u_1^0+v_1+(N-1)(u_2^0+v_2)}\,dx = \frac{\xi|\Omega|}{N} - \frac{4\pi(n_1+n_2)}{e^2} - \frac{4\pi(N-1)n_2}{g^2} \equiv K_1, \tag{6.21}$$

$$|\rho|^2\int_\Omega e^{u_1^0+v_1-u_2^0-v_2}\,dx = \frac{\xi|\Omega|}{N} - \frac{4\pi(n_1+n_2)}{e^2} + \frac{4\pi n_2}{g^2} \equiv K_2. \tag{6.22}$$

Hence, if there exits a solution for (6.9)–(6.10), the right-hand sides of (6.21)–(6.22) should be positive, which implies the necessity of the condition (6.13).

In what follows we prove that the condition (6.13) is also sufficient for the existence of solutions to (6.9)–(6.10).

We see that the functional I is a C^1 functional and weakly lower semi-continuous. As previous section, to find the critical point of I, we need to show the coerciveness of I.

Decompose $v_i \in W^{1,2}(\Omega)$ as

$$v_i = c_i + w_i, \quad c_i \in \mathbb{R},\ w_i \in \dot{W}^{1,2}(\Omega),\ i = 1,2.$$

Using Jensen's inequality we find

$$\begin{aligned} & I(v_1,v_2) - \frac{N}{2e^2}\|\nabla v_1\|_2^2 - \frac{N(N-1)}{2g^2}\|\nabla v_2\|_2^2 \\ & = |\rho|^2\left\{e^{c_1+(N-1)c_2}\int_\Omega e^{(u_1^0+w_1)+(N-1)(u_2^0+w_2)}\,dx + (N-1)e^{c_1-c_2}\int_\Omega e^{u_1^0+w_1-u_2^0-w_2}\,dx\right\} \end{aligned}$$

$$
\begin{aligned}
&- \left(\xi|\Omega| - \frac{4N\pi[n_1+n_2]}{e^2}\right)c_1 + \frac{4\pi N(N-1)n_2}{g^2}c_2 \\
&\geqslant |\rho|^2|\Omega|\left(e^{c_1+(N-1)c_2} + [N-1]e^{c_1-c_2}\right) - \left(\xi|\Omega| - \frac{4\pi N[n_1+n_2]}{e^2}\right)c_1 \\
&\quad + \frac{4\pi N(N-1)n_2}{g^2}c_2 \\
&= \left\{|\rho|^2|\Omega|e^{c_1+(N-1)c_2} - K_1\left(c_1+[N-1]c_2\right)\right. \\
&\quad \left. + (N-1)\left(|\rho|^2|\Omega|e^{c_1-c_2} - K_2[c_1-c_2]\right)\right\}, \qquad (6.23)
\end{aligned}
$$

where K_1 and K_2 are defined by (6.21) and (6.22), respectively.

We see that both K_1 and K_2 are positive under the condition (6.13). Then, from (6.23) we obtain

$$
\begin{aligned}
I(v_1,v_2) \geqslant{}& \frac{2N}{e^2}\|\nabla w_1\|_2^2 + \frac{2N(N-1)}{g^2}\|\nabla w_2\|_2^2 + K_1 \ln\frac{|\rho|^2|\Omega|}{K_1} \\
&+ (N-1)K_2 \ln\frac{|\rho|^2|\Omega|}{K_2}, \qquad (6.24)
\end{aligned}
$$

which implies the functional I is bounded from below and the minimization problem

$$
\eta_0 \equiv \inf\left\{I(v_1,v_2) \;\middle|\; (v_1,v_2) \in W^{1,2}(\Omega)\right\} \qquad (6.25)
$$

is well-defined.

Now we may use a similar argument as Section 3 to get the existence of a critical point of I, which is also unique since I is strictly convex.

To show the quantized integrals over Ω, it is sufficient to integrate Eqs. (6.18)–(6.19) over Ω.

6.2. *Planar solution*

In this subsection we prove the existence result for (6.9)–(6.10) over the full plane with the boundary condition (6.12). We use a similar argument as in Section 4.

We introduce the background functions

$$
u_1^0(x) = -\sum_{i=1}^{2}\sum_{s=1}^{n_i} \ln\left(1+\lambda|x-p_{is}|^{-2}\right), \qquad u_2^0(x) = -\sum_{s=1}^{n_2} \ln\left(1+\lambda|x-p_{2s}|^{-2}\right),
$$

where $\lambda > 0$ is a parameter. Then we see that

$$
\Delta u_1^0 = -h_1 + 4\pi\sum_{i=1}^{2}\sum_{s=1}^{n_i}\delta_{p_{is}}, \qquad \Delta u_2^0 = -h_2 + 4\pi\sum_{s=1}^{n_2}\delta_{p_{2s}},
$$

where

$$
h_1(x) = \sum_{i=1}^{2}\sum_{s=1}^{n_i}\frac{4\lambda}{(\lambda+|x-p_{is}|^2)^2}, \qquad h_2 = \sum_{s=1}^{n_2}\frac{4\lambda}{(\lambda+|x-p_{2s}|^2)^2}.
$$

With $u_i = u_i^0 + v_i$, $i = 1, 2$, Eqs. (6.9)–(6.10) can be written as

$$\Delta v_1 = \frac{e^2}{N}\Big\{|\rho|^2\big(\mathrm{e}^{u_1^0+v_1+(N-1)(u_2^0+v_2)} + [N-1]\mathrm{e}^{u_1^0+v_1-u_2^0-v_2}\big) - \xi\Big\} + h_1, \tag{6.26}$$

$$\Delta v_2 = \frac{g^2|\rho|^2}{N}\big(\mathrm{e}^{u_1^0+v_1+(N-1)(u_2^0+v_2)} - \mathrm{e}^{u_1^0+v_1-u_2^0-v_2}\big) + h_2. \tag{6.27}$$

Obviously, Eqs. (6.26)–(6.27) are the Euler–Lagrange equations of the following functional

$$\begin{aligned} I(v_1,v_2) = {} & \frac{N}{2e^2}\|\nabla v_1\|_2^2 + \frac{N(N-1)}{2g^2}\|\nabla v_2\|_2^2 \\ & + |\rho|^2\int_{\mathbb{R}^2}\Big\{\mathrm{e}^{u_1^0+(N-1)u_2^0}\big(\mathrm{e}^{v_1+(N-1)v_2}-1\big) - \big(v_1+[N-1]v_2\big)\Big\}\,\mathrm{d}x \\ & + [N-1]|\rho|^2\int_{\mathbb{R}^2}\Big\{\mathrm{e}^{u_1^0-u_2^0}\big(\mathrm{e}^{v_1-v_2}-1\big) - (v_1-v_2)\Big\}\,\mathrm{d}x \\ & + \frac{N}{e^2}\int_{\mathbb{R}^2} h_1 v_1\,\mathrm{d}x + \frac{N(N-1)}{g^2}\int_{\mathbb{R}^2} h_2 v_2\,\mathrm{d}x. \end{aligned} \tag{6.28}$$

We observe that the functional I is C^1 and strictly convex over $W^{1,2}(\mathbb{R}^2)$. To solve (6.26)–(6.27), as in Section 4, we just need to find the critical points of the functional (6.28).

Then we need to show the coerciveness of I. A simple computation leads to

$$\begin{aligned} & \big(DI(v_1,v_2)\big)(v_1,v_2) - \frac{N}{e^2}\|\nabla v_1\|_2^2 - \frac{N(N-1)}{g^2}\|\nabla v_2\|_2^2 \\ & = |\rho|^2\int_{\mathbb{R}^2}\Big\{\big(\mathrm{e}^{u_1^0+v_1+(N-1)(u_2^0+v_2)} + [N-1]\mathrm{e}^{u_1^0+v_1-u_2^0-v_2} - N\big)v_1\Big\}\,\mathrm{d}x \\ & \quad + (N-1)|\rho|^2\int_{\mathbb{R}^2}\big(\mathrm{e}^{u_1^0+v_1+(N-1)(u_2^0+v_2)} - \mathrm{e}^{u_1^0+v_1-u_2^0-v_2}\big)v_2\,\mathrm{d}x \\ & \quad + \frac{N}{e^2}\int_{\mathbb{R}^2} h_1 v_1\,\mathrm{d}x + \frac{N(N-1)}{g^2}\int_{\mathbb{R}^2} h_2 v_2\,\mathrm{d}x \\ & = |\rho|^2\int_{\mathbb{R}^2}\big(\mathrm{e}^{u_1^0+(N-1)u_2^0+v_1+(N-1)v_2} - 1 + X_1\big)\big(v_1+[N-1]v_2\big)\,\mathrm{d}x \\ & \quad + (N-1)|\rho|^2\int_{\mathbb{R}^2}\big(\mathrm{e}^{u_1^0-u_2^0+v_1-v_2} - 1 + X_2\big)(v_1-v_2)\,\mathrm{d}x, \end{aligned} \tag{6.29}$$

where

$$X_1 \equiv \frac{1}{|\rho|^2}\left(\frac{h_1}{e^2} + \frac{(N-1)h_2}{g^2}\right), \qquad X_2 \equiv \frac{1}{|\rho|^2}\left(\frac{h_1}{e^2} - \frac{(N-1)h_2}{g^2}\right).$$

Now estimating the right-hand side of (6.29) as that in Section 4, we obtain that there exist some positive constant C_0 and C_1 such that

$$\big(DI(v_1,v_2)\big)(v_1,v_2) \geqslant C_0(\|v_1\|_{W^{1,2}(\mathbb{R}^2)} + \|v_2\|_{W^{1,2}(\mathbb{R}^2)}) - C_1. \tag{6.30}$$

Then the existence of critical point follows a standard argument. Hence we see that the system (6.18)–(6.19) admits a solution, which is also unique since I is strictly convex.

The behavior at infinity, the decay estimates of the solution and the quantized integrals can be established as in Section 4. Then the proof of Theorem 6.1 for the planar case is complete.

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