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Analysis of heat flow in the post-quasi-static approximation for gravitational collapse in five dimension

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Abstract In this work, a generalized framework of the post-quasistatic approximation in higher dimensional non-comoving coordinates is presented. We study the evolution of adiabatically radiating and dissipative fluid configuration in higher dimensional post-quasi-static approximation. An iterative method for describing self-gravitating spheres is developed for this purpose. Dissipation is described by free-streaming radiation and heat flux. We match the higher dimensional interior solution, in non-comoving coordinates, with the corresponding Vaidya exterior solution. The generalized form of post-quasistatic approximation leads to a system of higher dimensional surface equations. The surface equations are of significant importance in the understanding of the physical phenomenon like luminosity, Doppler shift and red-shift at the boundary surface of gravitating sources.

1 Introduction

Spherically symmetric solutions in general relativity (GR) are important in the study of compact objects. The gravitational fields of astronomical bodies can be modeled by using spherically symmetric solutions to the Einstein field equations (EFEs). Indeed, most studied exact solutions to EFEs are spherically symmetric. If the metric components of spherical symmetry are static then exterior space time is taken as Schwarzschild solution [1]. Reissner [2], Weyl [3] and Nordström [4] developed the Reissner–Nordström solution to describe impact of electromagnetic field on gravitating system. The exact vacuum solution of EFEs that describes a rotating, stationary, axially symmetric black hole was discovered by Roy Kerr [5]. This solution describes a black hole because it describes the spacetime generated by a singularity with a curvature hidden by a horizon. Myers and Perry investigated the Schwarzschild, Reissner–Nordström, and Kerr solutions for higher dimensional spacetimes [6]. Shen and Tan [7] discussed Wyman's solution in higher dimensions. Chatterjee [8] obtained an exterior solution for spherically symmetric Kaluza–Klein (KK) type metric.

The development of GR to higher dimensions has gained a lot of attention in recent years. The five dimension and higher manifolds presented by KK are used in various gravity theories that extends Einstein's GR. After a couple of decades of the introduction of special relativity, Kaluza [9] and Klein [10] postulated the existence of an extra spatial dimension that can be considered as an extension to relativistic theory. Their motivation for doing so was to give a unified description of electromagnetism and gravity in terms of a five-dimensional metric. Chodos and Detweiler [11] proposed the time-independent spherically symmetric (in the typical three spatial dimensions) solution to the five-dimensional vacuum Einstein equations, under the condition that a killing vector exists in the fifth dimension. The considerable work in this domain after KK was presented by Wesson's [12], he studied the properties of matter in KK theories. The space-time matter theories [13] have become increasingly prominent in gravitation and cosmology in the fifth dimension. Liu and Overduin [14] investigated light deflection and time delay results for massless test particles in higher dimensions. Rahaman et al. [15] studied the usual solar system phenomenon, such as the perihelion shift, light bending, gravitational red-shift, gravitational time delay and motion of test particles that are compatible with the existence of higher spatial dimensions. Bars and Terning [16] introduced the extra time dimension. The solution was based on gauge symmetry. They developed the general framework by using the extra time dimension coordinate and found that the results are consistent with standard models of general rel-

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ativity. The higher dimensional gravastars were also studied by Rahaman et al. [17]. Moreover, many researchers have worked on higher derivative gravities in connection to extension of GR [18–26].

The Rosen's bimetric field equations in higher dimensions for the static spherically symmetric space-time with charged anisotropic fluid distribution were solved by Pandya and Hasmani [27]. Singh et al. [28] investigated two distinct cosmological models with massive strings in five dimensional relativistic theories. The first produces a five-dimensional model of the Universe, while the second produces the vacuum Universe. The properties of the model Universe are investigated and compared to the properties of the four-dimensional model. Baro et al. [29] investigated a model of the universe that is isotropic throughout its evolution, non-sharing and free from the initial singularity.

Debnath and Chakraborty [30] discussed the gravitational collapse of spherically symmetric inhomogeneous dust in higher dimensional space-time. The appearance of a naked singularity has been studied in both non-marginal bound and marginally bound cases. They found that naked singularity is possible in all dimensions ($n \geq 4$) for non-marginal cases. By examining the existence of radial null geodesic through the singularity in the marginally bound situation, they have clearly come to the conclusion that naked singularity may only appear for ($n \leq 5$). Debnath [31] used a higher-dimensional extension of the quasi-spherical Szekeres metrics with a non-zero cosmic constant to study gravitational collapse in ($n + 2$) dimensions. They discovered that the possibility of a naked singularity depends on the initial density of a space-time with more than five dimensions. The results are comparable to the collapse in Tolman–Bondi–Lemaitre space-times with spherically symmetric space-times. Yamada and Shinkai [32] investigated the gravitational collapse of collisionless particles in spheroidal structures in both four and five dimensions of space-time using numerical methods. The collapsing behaviors are quite similar to the cases in four-dimension, but they also found that five-dimensional collapses proceed rapidly than four-dimensional collapses. Khan et al. [33] presented the five-dimensional spherically symmetric anisotropic collapse with a positive cosmological constant. They employ the Schwarzschild–de Sitter and five-dimensional spherically symmetric metrics for the inner and outer regions, respectively. They found that the entire collapse process is impacted by the cosmological constant. The collapse process is slowed down by the cosmological constant.

Barnafoldi et al. [34] investigate the higher dimensional neutron star with compactified fifth-dimensional excitations. They showed that neutron stars with hyperon or extra-dimensional cores are remarkably similar objects in a simple model of a compact star. The Tolman–Oppenheimer–Volkoff (TOV) equation produces a comparable structure with a clearly defined stability area in the extra-dimensional case, where the lowest KK modes may be observed. Additionally, they introduced a new dimension, which contributed about the emergence of new stability areas and the presence of many stable hybrid star configurations. The double neutrino shower from SN 1987A supports this conclusion. Paul [35] studied the relativistic solutions of higher dimension compact star in hydrostatic equilibrium with spherically symmetric space-time. He found that the presence of higher dimensions directly affects the star's central density. The square of the dimensions of space and time causes the density of the star's core to increase roughly proportionally. As a result, if a star is surrounded in dimensions other than the standard four of space-time, its centre density is relatively higher for a given radius. It is also obvious that for a given radius, more space-time dimensions than four allow for a more massive compact star. Chattopadhyay and Paul [36] studied compact stars in hydrostatic equilibrium in higher dimensions using a pseudo-spheroidal space-time geometry. They found that the centre density of the star increases linearly with the square of the space-time dimensions. As a result, if a star is immersed in more dimensions than the typical four-dimensional space-time, its core density will be relatively higher for a given radius. Bhar et al. [37] provided evidence for the existence of higher dimensional anisotropic compact stars in noncommutative space-time. They found that the physical behaviors of the model parameters, such as matter-energy density, radial and transverse pressures, anisotropy, and other characteristics, are generally consistent throughout the stellar structure. They also mention that as one goes to higher dimensions, the central densities abruptly decrease, and that the measure of anisotropy gradually increases, reaching its maximum at five dimensions. A star's central density is greatest in four dimensions and lowest in higher dimensions.

Numerical techniques enable mathematicians to study systems that are complicated to handle analytically [38]. Numerical models have proven to be beneficial in the investigation of strong field scenarios and for revealing unexpected occurrences in GR [39]. Nonetheless, it is clearly more straight forward to solve ordinary differential equations (ODEs) than partial differential equations in general. However, numerical solutions frequently make it difficult to express general, qualitative, aspects of this process. The suggested method produces a system of ODEs for quantities determined at the fluid distribution's boundary surface (BS) starting from any interior static spherically symmetric seed solution to the EFEs. The static limit of the numerical solution, which simulates dynamical self-gravitating spheres, is the initial seed solution.

The pioneer work of Oppenheimer and Snyder [40] urged researchers to explore relativistic aspects of gravitating source their formation and inner structure. The motivation for such interest is based on the fact that the relativistic collapse of massive stars is one of the main visible process in which GR is predicted to play a vital role. However, self-gravitating compact objects may experience periods of extreme dynamical activity as they evolve over time. The static or quasi-static (QS) approximation is unreliable for some phenomena, such as the origination of neutron stars as a result of quick collapse. In such conditions, it is necessary to consider concepts that describe departures from equilibrium. Herrera et al. [41] initially presented the post-quasistatic (PQS) approximations essence using radiative Bondi approach. Herrera and his coworkers [42] have made considerable use of it. In Bondi approach, the concept of QS approximation is absent: the system proceeds immediately from static to PQS evolution. The PQS approximation depends on "effective" variables, such as effective pressure and energy density [43]. Because the effective variables of the QS approximation correspond to the physical variables. This approximation can be assumed as an iterative technique, with each successive step representing a greater deviation from equilibrium. More precisely, the authors [39] employed a method for modelling the evolution of compact objects that does not necessitate complete integration of the EFEs with respect to the time coordinate.

The purpose of this work is to investigate the evolution of compact objects in the PQS regime in five dimensions. The fluid distribution under consideration is supposed to be radiative in which free streaming of radiations induced dissipative effects. The emission of photons and/or neutrino particles causes dissipation, which is a common procedure in the evolution of compact objects. In fact, neutrino emission appears to be the only viable technique for removing the majority of the binding energy from a collapsing star. However, two other approximations, diffusion and streaming out are frequently employed in the analysis of radiative transport within compact objects. The diffusion approximation assumes that likewise thermal conduction, the energy flux of radiation is approximately equal to the temperature gradient. This assumption is generally viable because the mean free path of the particles responsible for energy transmission in star interiors is usually quite short in comparison to the object's normal length. A star, such as the sun, has a mean free path of massless particles photons on the order of 2 cm. The mean free path of trapped neutrinos is less than the size of the star core in compact cores with densities of about 10^{12} g cm^{-3} [44,45]. In addition, data from the 1987A supernova show that during the emission phase, the predominant radiation transport regime is closer to the diffusion approximation than the streaming out limit [46].

In this article, we will discuss physical variables such as $\rho, P, \epsilon, \omega$ and Q. These physical variables predicted to play a vital role in the evolution of self-gravitating objects. We will use noncomoving coordinates, which means that the velocity of any fluid element has to be taken into account as a relevant physical variable, despite the fact that using co-moving coordinates is the most common method to solve EFEs [47,48]. The layout of this work is as follows. We describe the conventions and provide the higher dimensional EFEs in Sect. 2. The Methodology of this paper is discussed in Sect. 3. Finally, conclusion and discussion is presented in Sect. 4 that are followed by a list of references.

2 The Einstein field equations in higher dimension

We consider non-static spherically symmetric distributions of a collapsing fluid confined by a spherical surface $\sum$, where dissipation occurs due to free-streaming radiation and/or heat flow. By using five dimensional Schwarzchild-like coordinates [49], the metric is then written as

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(dr^2 + sin^2 d\phi^2) - e^{\mu}dw^2, \quad (1)$$

where ν, λ and μ are general functions of time and radial coordinates. The spacetime coordinates are $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \phi, x^4 = h$. The corresponding EFEs in tensorial form are:

$$G^{\nu}_{\mu} = -8\pi T^{\nu}_{\mu}. \quad (2)$$

That leads to following set of equations

$$-8\pi T^0_0 = -\frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} + \frac{(\mu' - \lambda')}{r} + \frac{((\mu')^2 - \mu'\lambda')}{4} + \frac{\mu''}{2}\right) + e^{-\nu}\left(\frac{\dot{\mu}\dot{\lambda}}{4}\right), \quad (3)$$

$$-8\pi T^1_1 = -\frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} + \frac{(\nu' + \mu')}{r} + \frac{\mu'\nu'}{4}\right) - \frac{e^{-\nu}}{4}\left(2\ddot{\mu} + (\dot{\mu})^2 - \dot{\mu}\dot{\nu}\right), \quad (4)$$

$$-8\pi T^2_2 = -8\pi T^3_3 = \frac{1}{4}\Big[- e^{-\nu}\Big\{2(\ddot{\mu} + \ddot{\lambda}) + \dot{\mu}(\dot{\mu} - \dot{\nu} + \dot{\lambda}) + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\Big\} + e^{-\lambda}\Big\{2(\nu'' + \mu'') + (\nu')^2 + (\mu')^2 - \nu'\lambda' - \mu'(\lambda' + \nu')\Big\}\Big] + \frac{2}{r}\left(\nu' - \lambda' + \mu'\right), \quad (5)$$

$$-8\pi T^4_4 = \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu''}{2} + \frac{(\nu')^2}{4} - \frac{\nu'\lambda'}{4} + \frac{(\nu' - \lambda')}{r}\right) + \frac{e^{-\nu}}{4}\left(2\ddot{\lambda} + (\dot{\lambda})^2 - \dot{\nu}\dot{\lambda}\right), \quad (6)$$

$$-8\pi T_{01} = \left(\frac{\mu'\dot{\mu}}{4} - \frac{\dot{\mu}\nu'}{4} - \frac{\mu'\dot{\lambda}}{4} + \frac{\dot{\mu}'}{2} - \frac{\dot{\lambda}}{r}\right), \quad (7)$$

where (.) and ($\prime$) denote partial differentiation in terms of t and r respectively. We use the Bondi technique to give physical meaning to the components of energy stress tensor T^{μ}_{ν} .

Thus, in the accordance with Bondi, we will introduce Minkowski coordinates (τ, x, y, z, h), in five dimension as

$$d\tau = e^{\nu/2}dt, \quad dx = e^{\lambda/2}dr, \quad dy = rd\theta, \quad dz = rsin\theta d\phi, \quad dh = e^{\mu/2}dw.$$

Then, using a bar to represent the higher dimensional Minkowski coordinates of the energy stress tensor, we obtain

$$\bar{T}^0_0 = T^0_0; \quad \bar{T}^1_1 = T^1_1; \quad \bar{T}^2_2 = T^2_2; \quad \bar{T}^3_3 = T^3_3; \quad \bar{T}^4_4 = T^4_4; \quad \bar{T}_{01} = e^{-\frac{(\nu+\lambda)}{2}}T_{01}.$$

Now, we assume that the physical content of space, as seen by an observer moving relative to these coordinates with proper velocity ω in the radial direction, consists of an isotropic fluid with energy density ρ, isotropic pressure P, radial heat flux $\hat{q}$ and unpolarized radiation with energy density $\hat{\epsilon}$ traveling in the radial direction. The five-dimensional covariant tensor in Minkowski coordinates is defined as

$$\begin{bmatrix} \rho+\hat{\epsilon} & -\hat{q}-\hat{\epsilon} & 0 & 0 & 0 \\ -\hat{q}-\hat{\epsilon} & P+\hat{\epsilon} & 0 & 0 & 0 \\ 0 & 0 & P & 0 & 0 \\ 0 & 0 & 0 & P & 0 \\ 0 & 0 & 0 & 0 & P \end{bmatrix}$$

The five-dimensional Lorentz transformation then demonstrates that

$$T^0_0 = \bar{T}^0_0 = \frac{\rho + P\omega^2}{1-\omega^2} + \frac{2Q\omega e^{\frac{\lambda}{2}}}{\sqrt{(1-\omega^2)}} + \epsilon, \tag{8}$$

$$T^1_1 = \bar{T}^1_1 = -\frac{P + \rho\omega^2}{1-\omega^2} - \frac{2Q\omega e^{\frac{\lambda}{2}}}{\sqrt{(1-\omega^2)}} - \epsilon, \tag{9}$$

$$T^2_2 = \bar{T}^2_2 = T^3_3 = \bar{T}^3_3 = T^4_4 = \bar{T}^4_4 = -P, \tag{10}$$

$$T_{01} = e^{\frac{(\nu+\lambda)}{2}}\bar{T}_{01} = -\frac{(\rho+P)\omega e^{\frac{(\nu+\lambda)}{2}}}{1-\omega^2} - \frac{Qe^{\frac{\nu}{2}}e^{\lambda}(1+\omega^2)}{\sqrt{(1-\omega^2)}} - e^{\frac{(\nu+\lambda)}{2}}\epsilon, \tag{11}$$

with

$$Q \equiv \frac{\hat{q}e^{\frac{-\lambda}{2}}}{\sqrt{(1-\omega^2)}}, \tag{12}$$

and

$$\epsilon \equiv \hat{\epsilon}\frac{(\omega+1)}{(1-\omega)}. \tag{13}$$

It is worth noting that in (t, r, θ, ϕ, w) system, the coordinate velocity $\frac{dr}{dt}$ is associated with the proper velocity ω by

$$\omega = \frac{dr}{dt}e^{\frac{(\lambda-\nu)}{2}}. \tag{14}$$

Applying Lorentz transformed Eqs. (8–11), in field Eqs. (3–7), we obtain

$$\frac{\rho + P\omega^2}{1-\omega^2} + \frac{2Q\omega e^{\frac{\lambda}{2}}}{\sqrt{(1-\omega^2)}} + \epsilon = -\frac{1}{8\pi}\Bigg\{ + e^{-\lambda}\bigg(\frac{1}{r^2} + \frac{(\mu'-\lambda')}{r} + \frac{\big((\mu')^2 - \mu'\lambda'\big)}{4} + \frac{\mu''}{2}\bigg) - \frac{1}{r^2} + e^{-\nu}\bigg(\frac{\dot{\mu}\dot{\lambda}}{4}\bigg)\Bigg\}, \tag{15}$$

$$\frac{P + \rho\omega^2}{1-\omega^2} + \frac{2Q\omega e^{\frac{\lambda}{2}}}{\sqrt{(1-\omega^2)}} + \epsilon = -\frac{1}{8\pi}\Bigg\{\frac{1}{r^2} - e^{-\lambda}\bigg(\frac{1}{r^2} + \frac{(\nu'+\mu')}{r} + \frac{\mu'\nu'}{4}\bigg) + \frac{e^{-\nu}}{4}\Big(2\ddot{\mu} + (\dot{\mu})^2 - \dot{\mu}\dot{\nu}\Big)\Bigg\}, \tag{16}$$

$$P = \frac{-1}{32\pi}\Bigg[e^{-\nu}\bigg(2\big(\ddot{\mu}+\ddot{\lambda}\big) + \dot{\mu}\big(\dot{\mu}-\dot{\nu}+\dot{\lambda}\big) + \dot{\lambda}\big(\dot{\lambda}-\dot{\nu}\big)\bigg) - e^{-\lambda}\bigg(2\big(\nu''+\mu''\big) + (\nu')^2 + (\mu')^2 - \nu'\lambda' - \mu'\big(\lambda'+\nu'\big)\bigg)\Bigg] - \frac{2}{r}\Big(\nu' - \lambda' + \mu'\Big), \tag{17}$$

$$\frac{(\rho+P)\omega e^{\frac{(\nu+\lambda)}{2}}}{1-\omega^2} + \frac{Qe^{\frac{\nu}{2}}e^{\lambda}(1+\omega^2)}{\sqrt{(1-\omega^2)}} + e^{\frac{(\nu+\lambda)}{2}}\epsilon = -\frac{1}{8\pi}\Bigg\{\bigg(-\frac{\mu'\dot{\mu}}{4} + \frac{\dot{\mu}\nu'}{4} + \frac{\mu'\dot{\lambda}}{4} - \frac{\dot{\mu}'}{2} + \frac{\dot{\lambda}}{r}\bigg)\Bigg\}. \tag{18}$$

Equations (15–18) yields a system for P, ρ, ω, Q and ϵ for defined functions ν, μ and λ. when all of these physical variables are non-zero, the system is under-determined, and two equations of state must be presented. In general, a transport equation must be considered whenever $Q \neq 0$. The system is closed in the original Bondi [50] case of isotropic fluid (radial pressure is equal to tangential pressure) and free streaming regime, $Q = 0$. The system of equations is over-determined for the adiabatic case when $\epsilon = Q = 0$, isotropic fluid and a constraint on the physical variable appears. The corresponding Vaidya exterior geometry for higher dimension is considered as [51]

$$ds^2 = \bigg(1 - \frac{2M(u)}{R^2}\bigg)du^2 + 2dudR - R^2\Big(d\theta_1^2 + sin^2\theta_1\Big(d\theta_2^2 + sin\theta_2^2 d\theta_2^3\Big)\Big), \tag{19}$$

where $M(u)$ represent the total mass of the system inside the BS denoted as Σ and u denotes the retarded time. At the BS and outside it, the two coordinate systems (t, r, θ, ϕ, h) and $(u, R, \theta_1, \theta_2, \phi)$ are connected by

$$u = t - r - 2Mln\left(\frac{r}{2M} - 1\right), \quad R = r_\Sigma, \tag{20}$$

following necessary and sufficient conditions for two metrics (1) and (19) shall be fulfilled to match smoothly.

$$e^{\nu_\Sigma} = e^{-\lambda_\Sigma} = e^{-\mu_\Sigma} = 1 - \frac{2M}{R_\Sigma^2}, \tag{21}$$

$$[P]_\Sigma = 0. \tag{22}$$

The fluid in this study is considered to be isotropic and dissipative in the form of free streaming radiations and/or heat flow, where ϵ is the radiation density and q is the heat flow are defined as

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu - P g_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu + \epsilon l_\nu l_\mu, \tag{23}$$

with

$$u^\mu = \left(\frac{-e^{\nu/2}}{\sqrt{(1-\omega^2)}}, \frac{\omega e^{-\lambda/2}}{\sqrt{(1-\omega^2)}}, 0, 0, 0\right), \tag{24}$$

$$l^\mu = \left(e^{-\nu/2}, \; e^{-\lambda/2}, 0, 0, 0\right), \tag{25}$$

where the fluid's five velocity is represented by u_μ, a five dimensional null outgoing vector is denoted by l_μ, and

$$q^u = Q\left(\omega e^{\frac{\lambda-\nu}{2}}, 1, 0, 0, 0\right). \tag{26}$$

After some computations, the radial component of conservation law is used to calculate $T^\mu_{\nu;\mu} = 0$, we obtain following equation

$$P' = -\left(\frac{\nu' + \mu'}{2}\right)\left(\rho + P\right), \tag{27}$$

which represents the static case of the TOV equation.

3 The methodology

While dealing with self-gravitating compact objects, the most basic scenario is static equilibrium. This shows that $\omega = \epsilon = Q = 0$, all time-dependent derivatives vanish and a modified TOV equation is obtained. The QS regime, the hydrostatic time scale, which is the typical time scale on which the sphere responds to small changes in the hydrostatic equilibrium, is very long in comparison to the slow rate of change of the sphere. As a result, the system is constantly near to hydrostatic equilibrium in QS regime. Its evolution can be seen as a series of static models linked together by (18). This theory is sensible because the hydrostatic time scale is relatively short for several stages of a star's life. It is approximately 4.5 s for a white dwarf, 27 min for the Sun and 10^{-4} s for a neutron star with a mass of one solar mass and a radius of 10 km [52]. Any of the star configurations mentioned above have been observed to change over the period of time that are unusually long in comparison to their respective hydrostatic time scales. As was previously stated, this approximation is no longer accurate in some crucial instances and departures from quasi-equilibrium must be taken into account. We will discuss such departures, in the PQS approximation given in following subsections.

3.1 The effective variables and PQS approximation in higher dimension

The effective variables for the PQS approximation are defined as follows:

$$\tilde{\rho} = T^0_0 = \frac{\rho + P\omega^2}{1-\omega^2} + \frac{2Q\omega e^{\frac{\lambda}{2}}}{\sqrt{(1-\omega^2)}} + \epsilon, \tag{28}$$

$$\tilde{P} = -T^1_1 = \frac{P + \rho\omega^2}{1-\omega^2} + \frac{2Q\omega e^{\frac{\lambda}{2}}}{\sqrt{(1-\omega^2)}} + \epsilon. \tag{29}$$

The effective variables in the QS regime satisfy the TOV Eq. (27) as the corresponding physical variables. As a result, effective and physical variables have the same radial dependency in a QS condition (and likely in a static one as well).

The corresponding mass function is defined as

$$m = \int_0^r 4\pi r^2 \tilde{\rho} dr. \tag{30}$$

Substituting Eq. (29) into Eq. (16):

$$\begin{aligned} \nu = \nu_\Sigma &+ \int_{r_\Sigma}^r \frac{2(8\pi \tilde{P} r^4 (r^2 - 2m) - 2r^3 m' + 6r^2 m - 12m^2 + 4rmm')}{(2r^3(r^2 - 5m) + 12rm^2 + r^4 m' - 2r^2 mm')} \\ &+ \frac{r^2}{r^2 - 2m}\left(\frac{\dot{m}}{(r^2 - 2m)} + \frac{4\dot{m}^2}{(r^2 - 2m)^2}\right) dr, \end{aligned} \tag{31}$$

$$\begin{aligned} \mu = \mu_\Sigma &+ \int_{r_\Sigma}^r \frac{2(8\pi \tilde{P} r^4 (r^2 - 2m) + 2r^3 m' - 2r^2 m + 4m^2 - 4rmm')}{(2r^3(r^2 - 3m) + 4rm^2 - r^4 m' + 2r^2 mm')} \\ &+ \frac{r^2}{r^2 - 2m}\left(\frac{\dot{m}}{(r^2 - 2m)} + \frac{4\dot{m}^2}{(r^2 - 2m)^2}\right) dr. \end{aligned} \tag{32}$$

The radial dependency of metric functions is completely determined for a given radial dependency of effective variables. Now, we will discuss the PQS regime as one that corresponds to a system that is not in equilibrium (or quasi-equilibrium), however effective pressure and energy density have the same radial dependency as the associated physical variables in an equilibrium (or quasi-equilibrium) state. As an alternative, metric functions with the same radial dependence as those in the static or QS regime define the system in the PQS regime. The logic behind this formulation is simple: we seek a regime that, while not in an equilibrium condition, represents the closest possible scenario to a QS evolution.

3.2 The algorithm in higher dimension

The approach we are going to use is outlined below

1. Consider an analytic interior (seed) solution to the EFEs, which represents a fluid distribution of matter in an equilibrium state, given as $\rho_{st} = \rho(r)$; $P_{st} = P(r)$.
2. Assume that effective pressure $\tilde{P}$ and energy density $\tilde{\rho}$ are dependent on the same r as P_{st} and ρ_{st}.
3. One may obtain m, μ and ν up to some t functions using equations (30), (31) and (32), as well as the radial dependence of $\tilde{P}$ and $\tilde{\rho}$, which will be explored in more detail below.
4. For these t functions, there are three ODEs, which are characterized as surface equations.
   - Evaluate Eq. (14) on $r = r_\Sigma$.
   - The equation that illustrates the relationship between the energy flux ($\hat{E}$) and mass loss rate along the BS.

$$\hat{E} = 8\pi r_\Sigma^5(\hat{\epsilon_\Sigma} + \hat{q}_\Sigma), \tag{35}$$

$$\hat{S} = m'_\Sigma r_\Sigma(3 + r_\Sigma^2 - 2m_\Sigma) + 26m_\Sigma - 16r_\Sigma^2, \tag{36}$$

$$\hat{B} = \frac{3r_\Sigma^2 - 12(r^2{}_\Sigma - 2m_\Sigma) - 12\pi r_\Sigma\rho_\Sigma - 6r_\Sigma + 8\pi r^3{}_\Sigma\rho_\Sigma - 6\pi r^3{}_\Sigma\rho_\Sigma\Omega^2}{3r_\Sigma(r_\Sigma^2 - 2m_\Sigma)}. \tag{37}$$

   - Determine non-static TOV equation on $r = r_\Sigma$.
5. The additional information is needed to close the given system of surface equations to determine some physical variables on the BS.
6. Once it has been closed, the system of surface equations can be integrated for any given set of initial conditions.
7. These two functions are completely determined by substituting the integration results in the expressions for m, μ and ν.
8. The EFEs develop a system of equations for physical variables after appropriately defining metric functions, can be obtained for any kind of fluid distribution.

3.3 The surface equations in higher dimension

The system of surface equations is the critical point in the algorithm, as should be obvious from the preceding. For this, dimensionless variables are introduced as

$$A = \frac{r_\Sigma}{m_\Sigma(0)}, \quad F = 1 - \frac{2M}{A^2}, \quad M = \frac{m_\Sigma}{m_\Sigma(0)},$$
$$\beta = \frac{t}{m_\Sigma(0)}, \quad \Omega = \omega_\Sigma.$$

We obtained the first surface equation with the total initial mass $m_\Sigma(0)$ by evaluating Eq. (14) at $r = r_\Sigma$. As a result,

$$\frac{dA}{d\beta} = F\Omega, \tag{33}$$

by using junction conditions, one may then obtain from (15), (18) and (21) computed at $r = r_\Sigma$, yields

$$\frac{dM}{d\beta} = \frac{-F^2}{\hat{S}}\Big((1+\Omega)\hat{E} - \frac{\Omega\rho_\Sigma}{2} + \Omega\rho_\Sigma\hat{B}\Big), \tag{34}$$

where

The gravitational redshift and Doppler shift are represented on the right of Eq. (34). The observer's perceived luminosity at infinity is then defined as

$$L = -\frac{dM}{d\beta}. \tag{38}$$

The second surface equation is

$$\frac{dF}{d\beta} = \frac{2(1-F)F\Omega}{A} + \frac{2L}{A}. \tag{39}$$

Evaluating the law of conservation $T^\mu_{\nu;\mu} = 0$ at the BS yields the third surface equation, we obtained

$$\tilde{P} + (\tilde{\rho} + \tilde{P})\Big(\frac{\nu' + \mu'}{2}\Big) = \frac{e^{-\nu}}{4\pi r(r^2 - 2m)} \Big(2\ddot{m} + \frac{7\dot{m}^2}{r^2 - 2m} - \dot{m}\dot{\nu}\Big) + \frac{2}{r}(P - \tilde{P}). \tag{40}$$

Evaluate Eq. (40) at the BS, the result is

$$\frac{d\Omega}{d\beta} = \Omega^2\Bigg[\Bigg\{\frac{4\pi A^5\tilde{\rho}_\Sigma F - 4\pi A^6\tilde{\rho}_\Sigma\Omega^2F^2 - 3A^2Fm_\Sigma + 6Fm_\Sigma^2 - 2A^3\tilde{\rho}_\Sigma\Omega F}{A^5 - 5A^3m_\Sigma + 6Am_\Sigma^2 + 2\pi A^4\tilde{\rho}_\Sigma - 4\pi A^4m_\Sigma\tilde{\rho}_\Sigma}\Bigg\} + 6\pi\tilde{\rho}_\Sigma + 16\pi^2\tilde{\rho}_\Sigma^2\Omega^2\Bigg] - \frac{AF}{2\tilde{\rho}_\Sigma}\Bigg[R + \frac{2}{A}\Big(\tilde{\rho}_\Sigma\Omega^2 + \frac{\bar{E}(1+\Omega)}{4\pi r_\Sigma^2}\Big)\Bigg], \tag{41}$$

where

$$R = \Big[\tilde{P} + (\tilde{\rho} + \tilde{P})\Big(\frac{\nu' + \mu'}{2}\Big)\Big]_{\Sigma}, \tag{42}$$

$$\bar{E} = \hat{E}(1 + \Omega). \tag{43}$$

According to Eq. (33) this implies $\Omega = 0$, we get

$$\frac{d^2 A}{d\beta^2} = F\frac{d\Omega}{d\beta}. \tag{44}$$

By applying $\Omega = 0$ on Eq. (41), we obtain

$$\frac{d\Omega}{d\beta} = -\frac{F}{\tilde{\rho}_{\Sigma}}\Big[\frac{AR}{2} + \frac{\hat{E}}{4\pi r_{\Sigma}^2}\Big]. \tag{45}$$

It is observe that the radius of the sphere tends to decrease with a positive energy flux $\hat{E}$. In other words, it favors object compactification, which is understandable. We can see the opposite effect occurs by reserving the signs of these quantities. Now the sphere will only be bounce at its surface for a positive energy flux for $\Omega = 0$, we get

$$\frac{d\Omega}{d\beta} \geq 0. \tag{46}$$

According to Eq. (45) this implies

$$-\frac{AR}{2}(\Omega = 0) \geq 0. \tag{47}$$

This equation has a physical meaning, which is as follows. The expression of R is defined in Eq. (42), is divided into two parts for non-radiating, static configurations. The first and second terms describe the hydrodynamical and gravitational forces, respectively. The resulting force as r increases is exactly $-\frac{AR}{2}$; if force is positive, there is a net acceleration outward, and vice versa. This result is generalised in Eq. (47) for general non-static configurations in five dimensions.

4 Conclusion and discussion

In this paper, a five-dimensional Schwarzschild-like non-moving coordinate system is used to develop a general framework for discussing relativistic collapse of spherical systems. To analyze astrophysical scenarios in higher dimensions, mathematicians used the KK theory, the M theory, string theory, and superstring theory. In this work, we considered the five-dimensional Schwarzchild like non comoving coordinates. The fifth dimension represents spatial coordinates in a radial direction. The higher dimension PQS approximation is developed using the five dimensional geometry. A general framework for the higher dimensional PQS approximation is developed, the physical features of the stellar structure of gravitational objects are analyzed. As a starting point, we used an interior (analytical) seed solution to the EFEs. The proposed method yields a set of ODEs for quantities estimated at the BS. The numerical solution allows for the simulation of self-gravitating spheres.

The inner fluid distribution is assumed to be isotropically configured, with heat flow and radiation factor that cause dissipative effects within the gravitating system. Dissipation is a phase of giant star evolution caused by the emission of massless particles. In fact, neutrino emission appears to be the only viable method for removing the bulk of the binding energy from a collapsing star, resulting in the formation of a neutron star or black hole. Outer space is considered as Vaidya spacetime for smooth matching at the boundary of sphere. In the discussion of departure from equilibrium there are three possible situations which are (a) static equilibrium, (b) QS equilibrium and (c) PQS equilibrium.

- Static Equilibrium: In case of static equilibrium all the components of the EFEs have radial dependency.
- QS Equilibrium: The system is predicted to evolve slowly enough in this regime to be deemed in equilibrium at any given time. This indicates that the compact object evolves very slowly, on a time scale significantly longer than the time it normally takes for the sphere to react to a small perturbation of hydrostatic equilibrium. The system is assumed to be static between two small variations of time due to very slow evolutionary process.
- PQS Equilibrium: The system which is not in the state of equilibrium or departure from equilibrium is known as PQS equilibrium.

The TOV equation is derived using a higher dimension conservation law. We developed Eq. (40) by using modified TOV Eq. (27) and effective variables. The TOV equation is satisfied by effective variables such as effective pressure and energy density, as well as physical variables. In the static and QS equilibrium, the effective and physical variables have the same radial dependency. This process is iterative, with each successive step reflecting a deeper understanding of the deviation from the equilibrium state.

Motivated by the fact that non-comoving coordinates are frequently used in gravitational collapse research, requiring the definition of the PQS approximation. This method is based on "effective" variables as well as a heuristic approach to the latter, the rationale and justification for which is revealed in the context of the PQS approximation in the five-dimensional regime.

In this work, we restricted ourselves to the five-dimensional PQS level. In higher dimension, we developed a system of surface equations using the PQS approximation algorithm. We established a higher-dimensional surface equation system and studied realistic features of stars like Doppler shift, gravitational redshift, and total mass loss rate, all of which are related to total mass loss energy flux $\hat{E}$ over the BS.

In higher dimensional spherically symmetric gravitational collapse is observed for non-comoving coordinate system in PQS approximation. The effects of heat flux and unpolarized radiation in isotropic conditions were studied by considering five dimensional Vaidya outer space. The discussion of this phenomena is not available in higher dimensional space-time in any previous work. To comprehend the nature of gravitational collapse in five dimensions, a general framework for the PQS regime must be developed, which necessitates the solution of nonlinear differential equations. Gravitational collapse is a well-known energy-dissipating process that dominates star formation and stellar evolution. We considered dissipation, which is an important factor in the gravitational collapse process. The dissipative model is described by the five-dimensional null outgoing vector in diffusion approximation.

In higher dimensional space-time, compact stars, neutron stars, and hybrid stars exist. These astrophysical phenomena motivate us to study the higher dimensional gravitational collapse. In literature, the gravitational collapse in the PQS approximation for higher dimensions has not been modeled.

In this article, we developed the higher dimensional general framework for the PQS approximation. For concrete applications of the PQS approximation, we can use three different types of models: (a) Schwarzschild-type model; (b) Lemaitre–Florides-type model; and (c) Tolman type-VI model. These models indicate the relativistic gravitational effects due to the discontinuity of the radial pressure at the BS. The evolution of the gravitational objects becomes more and more “dynamic,” stressing once again their roles in describing departures from equilibrium. This work can be extended to the PQS regime’s gravitational collapse in co-moving coordinates.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: It is a theoretical study, so no data is submitted.]

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