

## A NEW APPROACH TO DYNAMIC APERTURE PROBLEMS

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We develop the theory of a passive magnetic system intended to suppress nonlinear orbit distortion in high-energy proton storage rings. The system is designed to immediately reduce "Collins distortion functions," which describe the size of nonlinear orbit distortion in first-order perturbation theory. Such a scheme could permit one significantly to decrease the physical aperture of a storage ring over most—but not necessarily all—of its length. This work was motivated by design needs of the proposed Superconducting Super Collider (SSC).

### 1. GENERAL PHILOSOPHY

Concerns about the high cost of the proposed Superconducting Super Collider (SSC) and about discouraging results from single-beam numerical simulation<sup>1</sup> of the SSC Reference Design,<sup>2</sup> have recently generated heightened interest in the general problem of predicting the dynamic aperture of a high-energy proton (nonradiative) storage ring. Loosely speaking, dynamic aperture is the region of phase space within which beam can be injected without risk of being lost soon afterward. The small dynamic aperture predicted by orbit tracking of the SSC Reference Design is due largely to the rich multipole content expected from proposed designs for SSC dipoles.

Hopes for an increase in SSC dynamic aperture are fueling theoretical research in two areas: numerical simulation,<sup>3</sup> and analytic perturbation theory.<sup>3,4</sup> However, these are problematical. Simulation can, in principle, tell us when a design cannot work, but no one knows at present how to turn this into a concrete prescription for design modifications. High-order perturbation theory is also not yet sufficiently intuitive to suggest specific remedies for orbital difficulties. Perturbation theory is attractive at this time mainly because of the hope that it may be able to provide the same sort of information as numerical simulation, in greater detail, in much less computing time.

Recently, we proposed<sup>5</sup> an example of a third, intermediate approach to dynamic aperture problems: a passive magnetic correction scheme, suggested by first-order perturbation theory, to be applied when orbit tracking indicates trouble, and to be tested with further orbit tracking. This would be a *designer's tool*, whereas simulation and perturbation theory are only suited at present primarily for *screening*. In this paper, we analyze our proposal in detail.

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Even if this particular example is ultimately impractical, we feel strongly that the general philosophy of this intermediate approach is sound and should be pursued further.

Here is a quick sketch of our ideas: Our scheme is based on a simple reformulation of first-order perturbation theory due to Collins.<sup>6</sup> In this formulation, Collins writes the first-order corrections to the betatron amplitudes  $I_x$  and  $I_y$  (constants in zeroth order) as sums of terms of the form

$$I_x^{|m|/2} I_y^{|n|/2} d(s) \exp [i(nv_x + mv_y)(2\pi s/C)] \exp i\phi, \quad (1)$$

where  $m$  and  $n$  are integers, the  $v$ 's are unperturbed tunes,  $C$  is the storage ring circumference,  $\phi$  is an arbitrary constant phase, and the complex periodic functions—a sort of basis for first-order nonlinear orbit distortions—generically written as  $d$ , are called “distortion functions.” As a working principle, we adopt Collins's viewpoint that a well-behaved storage ring has small  $d$ 's, although this is not yet a theorem in any sense.

Let  $N$  be the total number of dipole magnets (presumed to be the most important sources of unwanted higher multipoles) in the storage ring. If the error multipoles on these magnets are random and uncorrelated, with vanishing averages, then, as we shall show, the mean square magnitude of any particular distortion function is roughly (ignoring variation in beta functions) proportional to  $N$ . Our idea is to decrease distortion functions by partitioning the accelerator into  $M$  sectors, with correction magnets in each sector powered so that the distortion functions in any sector receive contributions only from the magnets *in that sector*. In this way, mean square distortion functions become roughly proportional to  $N/M$  instead of  $N$ . That is, we expect reduction by a factor  $1/M$  times some position-dependent factor  $\xi$  that is to be determined numerically.

In the next section, we shall develop in detail the general theory of uncorrected sextupole distortion functions. In Section 3, we develop in detail the mechanics of our correction scheme. In Section 4, we prove a theorem that tells us how much improvement to expect in certain regions of the storage ring and a theorem that places an important restriction on the positioning of correction magnets. In Section 5, we apply our scheme to sextupole distortion in a (somewhat modified) model lattice for SSC magnet design evaluation that Gelfand<sup>2</sup> developed for use with the orbit-tracking program TEVLAT.<sup>7</sup> In Section 6, we offer some concluding remarks.

As we shall see, our scheme significantly reduces distortion functions away from the correction magnets, perhaps permitting a reduction in physical aperture throughout most of the storage ring. Reduction can be much more modest near the correction elements. For this reason, we have not tracked orbits in a model lattice corrected in this way, since TEVLAT at present assumes a position-independent physical aperture.

Ideally, one would like to correct nonlinear orbit distortion *actively*, using model orbit tracking directly—or even post construction measurements of real orbits in the real storage ring—to determine correction strengths. Active and passive schemes alike might be improved by combination with magnet shuffling.<sup>8</sup>

Originally, we had wanted to develop a completely optimized procedure for

reducing distortion functions with a restricted number of correction magnets. To this end, we explored schemes that determined correction strengths by minimizing quadratic figures of merit of the form

$$\sum_d \oint ds |d(s)|^2 W_d(s) \quad (2)$$

(in an obvious notation), where the  $W$ 's are positive weight functions. The  $W$ 's might be chosen, for example, to emphasize regions with some beta function large. However, we eventually abandoned this idea, for two reasons:

- (i) It required a great deal of computing time. One must invert an  $L \times L$  matrix in order to arrange for  $L$  normal sextupole magnets to reduce the five normal sextupole distortion functions by minimizing Eq. (2). By contrast, the  $M$ -sector scheme detailed below, with  $12M = L$ , requires inversion of  $L/12$  matrices, each only  $12 \times 12$ ; thus, computing time grows only linearly with  $L$ .
- (ii) We know of no way, short of large-scale calculation, to make even a crude preliminary estimate of the extent of the distortion-function reduction in this least-squares scheme. In particular, we therefore could not a priori justify to ourselves the expected large expenditure of computing time. By contrast, in the scheme described below, we can say with confidence that reduction will be substantial in certain regions of the storage ring.

Of course, there may be some variation on this minimization theme that does not suffer from these defects.

## 2. DEFINITION OF DISTORTION FUNCTIONS

We derive Collins's<sup>6</sup> formulation of first-order betatron perturbation theory assuming, for simplicity, that nonlinearities are due only to normal sextupole magnets and ignoring skew quadrupole magnets. This is the first discussion of this formulation in the published literature (even though some of the manipulations below are standard). The general Hamiltonian for this problem is

$$H = \frac{1}{2}(p_x^2 + k_x(s)x^2) + \frac{1}{2}(p_y^2 + k_y(s)y^2) + \frac{eB_y''(s)}{6E}(x^3 - 3xy^2), \quad (3)$$

where most notation here and below is standard<sup>9</sup> ( $B_y''$  is  $\partial^2 B_y / \partial x^2$  on the design orbit). The usual canonical action and angle variables for this system are

$$I_x = (2\beta_x)^{-1}[x^2 + (\beta_x p_x - \frac{1}{2}\beta_x' x)^2], \quad (4)$$

$$\Theta_x = -\Phi_x(s) - \arctan [\beta_x p_x - \frac{1}{2}\beta_x' x]/x,$$

where  $\Phi_x$  is defined by

$$\Phi_x(s) = \int_0^s \frac{ds'}{\beta_x(s')} - (2\pi\nu_x/C)s \quad (5)$$

and similarly for  $y$ . (The reference point  $s = 0$  is arbitrary.) In terms of Eq. 4, Eq. 3 becomes

$$H = I_x(2\pi\nu_x/C) + I_y(2\pi\nu_y/C) + \frac{eB_y''}{3E\sqrt{2}} \{ \frac{1}{2}(I_x\beta_x)^{3/2} \cos 3(\Phi_x + \Theta_x) \\ + \frac{3}{2}(I_x\beta_x)^{3/2} \cos (\Phi_x + \Theta_x) - 3(I_x\beta_x)^{1/2}(I_y\beta_y) \cos (\Phi_x + \Theta_x) \\ - 3(I_x\beta_x)^{1/2}(I_y\beta_y) \cos [\Phi_x + 2\Phi_y + (\Theta_x + 2\Theta_y)] \\ - 3(I_x\beta_x)^{1/2}(I_y\beta_y) \cos [\Phi_x - 2\Phi_y + (\Theta_x - 2\Theta_y)] \}. \quad (6)$$

To first order in the sextupole strengths, the actions  $I_x$  and  $I_y$  are given by  $I_x^0 + \delta I_x$  and  $I_y^0 + \delta I_y$ , where  $I_x^0$  and  $I_y^0$  are independent of  $s$ , and where  $\delta I_x$  satisfies

$$\delta \dot{I}_x = - \frac{\partial H}{\partial \Theta_x} \Big|_{\Theta_x = \Phi_x + 2\pi\nu_x s/C, \Theta_y = \Phi_y + 2\pi\nu_y s/C}, \quad (7)$$

and similarly for  $\delta I_y$ . Since the general differential equation

$$\left( \frac{d}{ds} + i\omega \right) d(s) = R(s) \quad (8)$$

has, for periodic  $R(s)$ , the unique periodic solution

$$d(s) = \int_0^C G_\omega(s, s') R(s') ds', \quad (9)$$

with

$$G_\omega(s, s') = e^{i\omega(s' - s)} (1 - e^{i\omega C})^{-1} \cdot \begin{cases} e^{i\omega C} & 0 < s' < s < C \\ 1 & 0 < s < s' < C \end{cases}, \quad (10)$$

we can integrate Eq. (7) to obtain

$$\delta I_x = \frac{e}{6E\sqrt{2}} \text{Im} \{ \frac{1}{2} d_{30}(s) (I_x^0)^{3/2} \exp 3i(\Phi_x + 2\pi\nu_x s/C) \\ + \frac{3}{2} d_{10}(s) (I_x^0)^{3/2} \exp i(\Phi_x + 2\pi\nu_x s/C) \\ - 3\bar{d}_{10}(s) (I_x^0)^{1/2} I_y^0 \exp i(\Phi_x + 2\pi\nu_x s/C) \\ - 3d_{12}(s) (I_x^0)^{1/2} I_y^0 \exp i[\Phi_x + 2\phi_y + 2\pi(\nu_x + 2\nu_y)s/C] \\ - 3d_{1-2}(s) (I_x^0)^{1/2} I_y^0 \exp i[\Phi_x - 2\phi_y + 2\pi(\nu_x - 2\nu_y)s/C] \}, \quad (11)$$

where, for example  $d_{12}$  is defined by Eq. 9 with  $\omega = 2\pi(\nu_x + 2\nu_y)/C$ , and  $R(s)$  is given by  $B_y''\beta_x^{1/2}\beta_y \cdot \exp i(\Phi_x + 2\Phi_y)$ , etc. In a similar fashion,  $\delta I_y$  is formed from  $d_{12}$  and  $d_{1-2}$  alone.

In this way, first-order perturbation theory has been decomposed into the effects of initial conditions ( $I^0$ 's and  $\phi$ 's) and intrinsic properties ( $\nu$ 's and  $d$ 's) of the storage ring. The periodic functions  $\bar{d}_{10}$  and the  $d$ 's are the normal sextupole distortion functions. Note that these functions are normalized differently here than in Ref. 6. However, the equations that follow are not sensitive to normalizations.

### 3. DEFINITION OF THE CORRECTION SCHEME

Imagine now partitioning the storage ring into  $M$  nonoverlapping sectors. For simplicity we imagine them to be of equal length, although this is not necessary. According to Eq. 8, the contribution of any magnet within a sector to a distortion function outside that sector is a constant times  $\exp(-i\omega s)$ , for some  $\omega$ . If, in the sector in question, we place two real correction magnets (of the appropriate multipolarity) for every one complex distortion function that we want to reduce, we can ensure that all the constants add to zero. In this way, we ensure that no magnet contributes to any designated distortion function outside that magnet's own sector. This procedure is formally equivalent to an "orbit bump" calculation.

In the case of  $d_{30}$ , for example, the cancellation of these constants takes the specific form

$$0 = \int ds \beta_x^{3/2}(s) B_y''(s) \exp \left[ 3i \int_0^s \frac{ds'}{\beta_x(s')} \right] \quad (12)$$

where the integration is restricted to the sector in question. The cancellation conditions for the other distortion functions are similar. If we imagine that the sextupole moments are lumped,

$$B_y''(s) = \sum_i \delta(s - s_i) b_i + \sum_i \delta(s - s_j^c) b_j^c, \quad (13)$$

where the  $b^c$ 's correspond to correction magnets, then Eq. (12) becomes

$$\begin{aligned} & \sum_j b_j^c \beta_x^{3/2}(s_j^c) \exp \left[ 3i \int_0^{s_j^c} \frac{ds'}{\beta_x(s')} \right], \\ &= - \sum_i b_i \beta_x^{3/2}(s_i) \exp \left[ 3i \int_0^{s_i} \frac{ds'}{\beta_x(s')} \right]. \end{aligned} \quad (14)$$

The sum in Eq. (14) is, of course, restricted to locations within the sector in question.

We now explain why this scheme leads to the  $1/M$  scaling law claimed in Section 1. We suppose, for fixed  $\{s_i\}$  (the centers of dipoles, for example), that the various  $b_i$ , whose distorting effects we want to suppress, are independent random variables, with zero mean and common mean square  $\sigma^2$ . Then the mean square magnitude of any sextupole distortion function is a sum of contributions due to the various  $s_i$ , with no cross terms, with each contribution proportional to  $\sigma^2$ . If there were no correction magnets at all, then we would have, for example,

$$\langle |d_{12}(s)|^2 \rangle = \frac{\sigma^2}{4 \sin^2 \left[ \frac{\pi}{2} (v_x + 2v_y) \right]} \sum_i \beta_x(s_i) \beta_y^2(s_i), \quad (15)$$

using Eq. (10). Ignoring variation in the  $\beta$ 's, this is proportional to  $N$ , as claimed earlier.

When correction magnets are in place, the contribution of any  $s_i$  to any  $\langle |d|^2 \rangle$  must take into account the correction strengths that the preexisting random

sextupole at  $s_1$  induces according to Eq. (14), etc. The additions that arise in this way are formed from the beta functions at the  $s_i$  in question, from betatron phase advances from some reference point to  $s_i$ , and from the elements of the inverse of the matrix of coefficients, in Eq. (12), of the  $b_j^c$  in  $s_i$ 's sector.

How do these ingredients vary from one  $s_i$  to another? If the uncorrected storage ring is composed of identical (up to random multipole errors) cells—as it is in the model discussed in Section 5—then beta functions repeat from cell to cell. Further, if betatron phase advances per cell are close to simple fractions of  $2\pi$  ( $\pi/3$  in the model of Section 5), then, when exponentiated, the betatron phase advances from a fixed reference point [ $s = 0$  in Eq. (14)] repeat after several cells. Thus, in any sector, the full contributions of different random multipoles to some  $\langle |d(s)|^2 \rangle$  at any  $s$  very nearly repeat after a number of cells that is independent of  $M$ .

[According to Eq. (10), this last statement is complicated by a phase shift when the random-multipole  $s_i$  passes the point  $s$  at which  $\langle |d|^2 \rangle$  is evaluated. Strictly speaking, we should say that the contributions very nearly repeat as  $s_i$  moves away from  $s$  in either direction.]

Finally, if correction magnets are placed in sectors in a way that is independent of  $M$  in some sense (for example, four per cell in the first three cells, as discussed below), then it follows from the foregoing that any  $\langle |d(s)|^2 \rangle$  is roughly proportional to the number of cells per sector—itself proportional to  $1/M$ —times an  $s$ -dependent factor that is independent of  $M$  as long as  $s$  is taken to refer to the same point *relative to the correction system* (for example, in the middle of the seventh cell from the start of the sector, in the configuration discussed below) for all  $M$ . We report on a check of this scaling law in Section 5.

Notice that we have ignored sextupole magnets that are deliberately included in designs in order to cancel the storage ring's natural (quadrupole-induced) chromaticity. The strengths of these chromaticity correction magnets are *not* uncorrelated random numbers. In the original SSC model that we modified for the discussion in Section 5, they repeat from cell to cell (with strengths comparable to those of the random sextupoles), so that their contributions to distortion functions add coherently. With betatron phase advances close to  $60^\circ$  per cell, it follows from Eqs. (9) and (10) that their contributions to sextupole distortion functions nearly cancel in groups of six cells (for  $d_{10}$ ,  $\bar{d}_{10}$ , and  $d_{1-2}$ ) or two cells (for  $d_{30}$  and  $d_{12}$ ). Thus, the total chromaticity correction contribution to any distortion function in any sector is comparable to the contribution of just a few cells of random sextupoles; therefore, any associated non- $1/M$  piece in the scaling law should be negligible unless the total number of cells in a sector is small. For this reason, standard chromaticity correction magnets are ignored in the calculations described in Section 5.

A real storage ring may also need correction magnets to cancel chromaticity induced by random, construction-related sextupole inaccuracies. Accordingly, in the calculations of Section 5, we include two such magnets per sector, beyond the minimal ten needed to reduce the five complex normal sextupole distortion functions. Sector-by-sector cancellation of chromaticity induced by random fields

requires

$$\begin{aligned}\sum_j \beta_x(s_j^c) b_j^c &= - \sum_i \beta_x(s_i) b_1, \\ \sum_j \beta_y(s_j^c) b_j^c &= - \sum_i \beta_y(s_i) b_i,\end{aligned}\tag{16}$$

where each sum covers only one sector at a time and where standard natural-chromatically-compensating sextupoles are not included.

#### 4. TWO THEOREMS

The model storage ring discussed in the next section consists of 480 identical (up to random sextupole moments) cells, each with approximately  $60^\circ$  horizontal and vertical betatron phase advances per cell. The number 480 was chosen for its many factors that could serve as the number  $M$  of structurally identical correction sectors. (This is a variant of an SSC model due to Gelfand, which has 481 cells.) For programming simplicity, we imagined each sector beginning with several cells in which identical numbers of correction magnets are placed in identical positions, followed by a long string of cells containing no corrections at all. Of course, other configurations are possible.

The following theorem describes in quantitative terms the distortion function reduction to be expected from this sort of arrangement, at least in certain parts of the storage ring.

*Theorem 1.* Call the correction end of a sector its *head*, and the other end its *tail*. Assume that the random sextupole moments are concentrated in Dirac delta functions, as in Eq. (13), with assumptions on the  $b_i$  as stated in the previous section. (We state the theorem in terms of sextupoles only for concreteness; an analogous theorem can be proved for any multipole moment.) Number the discrete random-sextupole sites in a sector from the tail. Then, with corrections, ignoring variation in the beta functions, the mean absolute square of any sextupole distortion function, between random sextupoles  $K$  and  $K + 1$  in any sector, is at most equal to  $4(K/N) \sin^2 \omega C/2$  times the mean absolute square of that distortion function with all correction magnets absent [ $\omega$  is the combination of frequencies appearing in the definition of the  $d$ 's according to Eqs. (9) and (10)], as long as random magnet  $K + 1$  is not in a correction cell.

This is significant because even if our scheme does not markedly decrease distortion functions everywhere, we may still be reassured that distortion functions can reliably be reduced by at most a factor  $4/M$  over a substantial fraction of the storage ring. For example, if  $M$  is 24 and  $\omega C/2$  is nearly  $\pi/2 \bmod 2\pi$  (as it is in our model for  $d_{30}$  and  $d_{12}$ ), and if the correction magnets occupy the first three cells in any sector, then, on the average, distortion functions in 85% of the ring can reliably be reduced in absolute square by anywhere from 0 to  $4(17/480) \cong 0.14$ , roughly in steps of  $4/480 \cong 0.008$ . Of course, one would like to have tightly focused orbits everywhere. Failing that,

perhaps our scheme can enable one to cut SSC cost by reducing physical aperture in, say 85% of the machine.

*Proof of Theorem 1.* By the definition of our correction scheme, the corrected distortion function in question vanishes identically between the last magnet at the tail of a sector and the first magnet in the adjacent head of the next sector. Integrating the defining equation [Eq. (8)] into the tail of a sector, past the  $K$  tailmost random sextupoles, yields  $e^{i\alpha s}$  times

$$\sum_{i=1}^K (\text{phase})_i \beta_x^{m/2}(s_i) \beta_y^{n/2}(s_i) b_i, \quad (17)$$

for some integers  $m$  and  $n$ , so that its mean absolute square is

$$\sigma^2 \sum_{i=1}^K \beta_x^m(s_i) \beta_y^n(s_i). \quad (18)$$

Comparison with Eq. (15) and its generalization to the other  $d$ 's, yields the desired result.

*Theorem 2.* In this scheme, when the betatron phase advances per cell are *exactly*  $60^\circ$ , a necessary condition for linear independence of the equations that determine the correction strengths (the  $b^c$ 's) is that every correction cell at the head of a sector contain at least four correction magnets.

Thus, with nearly  $60^\circ$  phase advances per cell, as in the model of the next section, we cannot spread out sextupole corrections too much without paying the price of very large correction strengths, since they would arise from inversion of a nearly singular matrix.

Note, incidentally, that linear independence may impose different constraints on correction placement in the case of multipoles other than sextupole, and also when the phase advances per cell differ significantly from  $60^\circ$ .

*Proof of Theorem 2.* The real and imaginary parts of the linear conditions  $d_{30} = d_{12} = 0$  have coefficients of the  $b^c$  that change sign from one correction cell to the next, when phase advances are close to  $60^\circ$ . If there were fewer than four correction magnets per cell, then the real coefficients in these four real equations, restricted to any one cell, would be linearly dependent, since any four  $n$ -tuples with  $n \leq 3$  are linearly dependent. Linear dependence of  $d_{30} = d_{12} = 0$  would follow immediately, since only the signs change in moving to other correction cells.

Had we applied that same viewpoint to  $d_{10}$ ,  $\bar{d}_{10}$ , and the complex conjugate of  $d_{1-2}$ , we would have spoken about three complex conditions whose coefficients would each have been multiplied by the same  $\exp(\pm i\pi/3)$  in moving from cell to cell. Real linear dependence of these six (real and imaginary) real conditions *sectorwise* would imply only complex linear independence of these three complex conditions *cellwise*, which would imply at least three correction magnets per cell. Similarly, linear independence of the two chromaticity constraints (in which coefficients of the  $b^c$  in adjacent cells are identical) would require at least two corrections per cell. Thus, the strongest constraints come from the argument in the preceding paragraph.

## 5. NUMERICAL RESULTS

Theorem 1, above, justifies our “ $\xi/M$ ” intuition outside cells that contain correction elements. In this section, we report a check of our intuition globally, including the correction cells.

In particular, we have applied our scheme to the normal sextupole distortion functions in a model SSC containing 480 cells, as shown in Fig. 1. The random sextupole moments are concentrated in the centers of the dipoles, with

$$\sigma/B\rho \approx 0.011 \quad (19)$$

(Gaussian distribution), where  $B\rho$  is the magnetic rigidity of a 20-TeV proton. Following Theorem 2, four correction magnets are placed in each of the first three cells in each sector: at the centers of each of the two dipoles, and at the centers of the  $D$  and  $F$  quadrupoles.

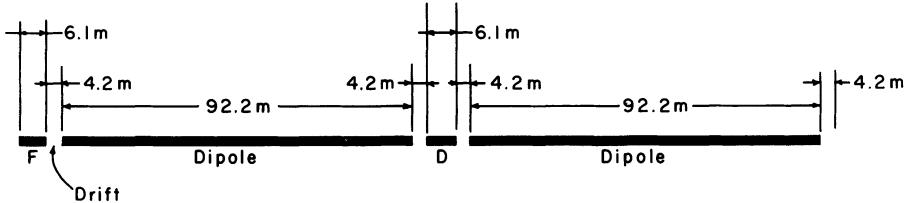


FIGURE 1 Standard cell in our version of N. Gelfand's SSC model. Lengths are approximate.  $F$  and  $D$  quadrupoles have strength of  $\sim 0.04 \text{ m}^{-2}$ . A dipole has bend angle of  $\sim 6.5 \text{ mrad}$ .

For a particular choice of the 960 random sextupole moments, we have computed the ratios of the maximum values of the absolute squares of the five normal sextupole distortion functions with corrections to their maximum absolute values without corrections, for  $M$  equal to every factor of 480; and we have least-squares fit each ratio to the functional form  $k/M$ . The results for  $k$  are, approximately, 24.1, 41.5, 16.9, 9.2, and 10.2 for  $d_{10}$ ,  $d_{30}$ ,  $d_{12}$ ,  $d_{1-2}$ , and  $\bar{d}_{10}$ , respectively. The computed with-correction-to-without-correction ratio and the  $k/M$  fit for  $d_{10}$  are shown together in Figs. 2 and 3. Agreement is encouraging, for such a rough theory.

Strictly speaking, this is not a fully statistical calculation, since we have used only one configuration of 960 random magnets. Presumably, fluctuation effects can average out for large values of  $N/M$ . For large  $M$ , on the other hand, only a small number of random magnets contribute in any one sector, so that fluctuations may be important.

Note that the data point for  $M = 1$  in Fig. 2 is far below the smooth fit. This is true for the other  $d$ 's as well. The computer program that generated our random sextupoles constrained their sum to be zero, for reasons unrelated to the present project. Presumably, this had its most dramatic effect for  $M = 1$ , which ought to be the sector multiplicity most sensitive to biases connecting *all* the magnets. Accordingly,  $M = 1$  was omitted from the least-squares fitting procedure.

Our results indicate that *global* reduction of *all* sextupole distortion functions is not achieved until  $M$  is significantly greater than about 40, at which point one

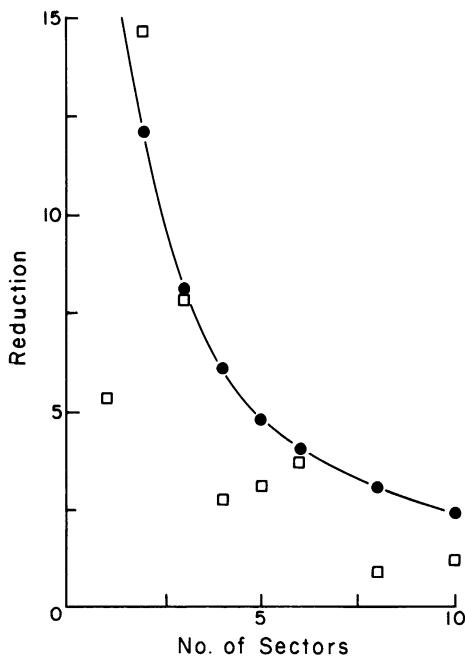


FIGURE 2 Ratio of maximum square modulus of distortion function  $d_{10}$  with corrections to maximum square modulus without, as a function of the number  $M$  of correction sectors. Plotted results were obtained by numerical computation (squares) and by least-squares fit (ignoring  $M = 1$ ) to  $(\text{const.})/M$ .

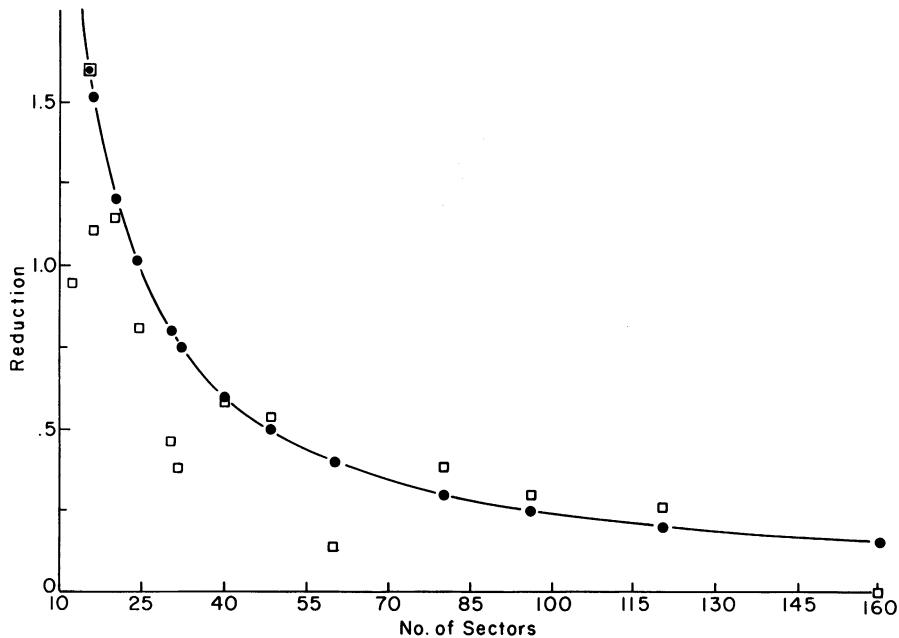


FIGURE 3 Same as Fig. 2, extended to larger  $M$ .

worries about the cost of so many correction elements in so many sectors. Significant reduction of distortion over a large (but not exhaustive) fraction of the ring, as in Theorem 1, may be the best that one can afford to do with this scheme.

Ultimately, one should submit a lattice corrected in this way to a numerical orbit-tracking program with physical apertures that are *not* constant throughout the model machine. Moreover, since the small dynamic aperture of the SSC Reference Design appears to be due not only to sextupole errors, but also at the same time to skew quadrupole, octupole, and even decapole,<sup>1</sup> a fair test of this scheme should also include correction magnets for some of these multipoles as well.

## 6. CONCLUDING REMARKS

Candidly, our scheme may turn out to be of little use for a variety of reasons:

- (i) Corrections introduced in this way to help in lowest-order perturbation theory may lead to new problems in higher orders.
- (ii) If it should prove necessary to repeat this scheme for many multipoles beyond sextupole, the expense could prove prohibitive.
- (iii) A variable aperture can introduce technical problems (involving, for example, collective beam instabilities) that we have not explored.
- (iv) If our scheme is to be useful not only in storage, but also during acceleration, it requires knowledge of time-dependent multipole errors, which may pose an excessive measurement burden.

However, these issues obscure our main point: The dynamic aperture problem in high-energy proton storage rings is highly nonlinear and thereby deeply complicated. One may certainly seek a solution in rigorous derivation and/or enhanced numerics. As a spur to the imagination, I urge, in addition, a third approach: *Guess a cure (even a questionable one), develop it analytically at least until it's well defined, and then try it out on a storage ring computer simulation. It can't hurt.*

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