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Gravitational complexity factor of anisotropic polytropes in coincident gauge $f(\mathbb{Q})$ gravity

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Abstract This work presents a novel characterization of complexity for gravitationally bound astrophysical configurations arising from two factors: (i) inhomogeneity and (ii) anisotropy in the complex arrangement of stellar structures. For this purpose, we employ the non-metricity-motivated gravitational model with a linear choice of coincident gauge $f(\mathbb{Q})$ gravity, given by $f(\mathbb{Q}) = \beta_0\mathbb{Q} + \beta_1$, where β_0 and β_1 are model parameters. Our analysis begins by postulating that a fluid distribution exhibiting density uniformity and pressure isotropy is characterized by a minimal (or zero) gravitational complexity factor. The novelty of this study lies in its ability to study the effect of non-metricity on the intricate mechanism of dense-matter static stars while also considering the effectiveness of the complexity factor in determining fluctuations in the Tolman mass for both complex and non-complex compact structures within a non-metricity framework. The variation in the Tolman gravitational mass is caused by a suggested formulation of anisotropic pressure and density non-uniformity. It is observed that in Einstein's gravitational model, a stellar system that features both can exhibit zero complexity ($Y_{TF} = 0$) if their contributions cancel out. On the other hand, the linear $f(\mathbb{Q})$ model compels the gravitational configuration to maintain $Y_{TF} \neq 0$ due to non-metricity contributions, even when the fluid exhibits density uniformity and pressure anisotropy. Furthermore, we discuss the construction of anisotropic self-gravitating polytropes by coupling the $Y_{TF} = 0$ condition with a polytropic EoS. This underscores the importance of the zero-complexity criterion in modeling astrophysical compact systems.

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1 Introduction

Self-gravitating stars are fundamental to astrophysics, serving as the gravitational cornerstones of galaxies and the universe. Investigating the properties of self-gravitating compact stars is a highly complex and engaging field that blends nuclear physics, astronomy, and cosmology [1,2]. Understanding the behavior of dense-matter stellar astrophysical entities like neutron stars and black holes is essential for uncovering the mysteries of the cosmos [3]. Self-gravitating spherically symmetric stars provide an essential framework for modeling compact objects. By analyzing hydrostatic equilibrium, pressure anisotropy, and higher-order gravitational effects, we gain valuable insights into stellar stability, mass distribution, and exotic astrophysical structures. The equilibrium and dynamic behaviors of sense-matter formations governed by their gravity and possessing spherical symmetry are modeled by self-gravitating spherically symmetric star solutions, derived from Einstein's model or alternative gravity models. Self-gravitating astrophysical systems are characterized by the fact that their structural and dynamic behavior is primarily governed by the gravitational attraction between the matter that makes up the system [4–8]. Galaxies, globular clusters, compact stars like white dwarfs and neutron stars, and hypothetical stellar systems like quarks or strange stars are all examples of these dense configurations [9]. These systems are fundamentally complex due to the nonlinear coupling between internal matter characteristics and gravitational effects, the interaction of different physical forces, and the role that geometric and physical variables play in their equilibrium and evolution [2,10].

Multiple independent research efforts, such as those conducted by the Supernova Search Team, the Supernova Cosmology Project, WMAP, and SDSS, collectively led to the profound realization of cosmic accelerated growth [11,12].

Explanations for the accelerating cosmic expansion center around two main ideas: the influence of cosmic dark energy, a force characterized by its strong anti-gravitational pressure, or modifications to general relativity (GR) incorporating higher-order curvature effects, which affect cosmic geometry. In GR, dark energy is modeled through a negative equation of state (EoS) parameter [13]. On the other hand, extensions of Einstein's model (alternative models of gravity) go beyond this standard framework by putting forth a single mechanism that might account for both the current dark energy-dominated era and the early inflationary epoch, minimizing the need for independent solutions for both of these periods of cosmological expansion [14,15]. Modified models of gravity are a challenging field of study on account of the intricate structure of the gravitational models and the complexity of differentiating them from cosmic dark energy phenomena. These models require precise parametric choices to be viable, as they must comply with various observational evidence. Furthermore, the capabilities of modern technology are being tested to maintain internal consistency and validate theoretical predictions.

The evolution and application of mathematical simulations in compact stellar configurations have increased dramatically in recent years. These models serve various purposes, such as predicting the behavior of stellar formations like neutron stars and black holes, as well as simulating gravitational interactions in astronomical events. Understanding the basic principles of gravity has been a longstanding and key aim of theoretical physics. Despite general relativity being a highly successful theory for understanding gravitational interactions, there are compelling reasons for the need to extend it. Researchers are actively investigating a spectrum of altered gravitational models, which include those introducing new dynamical scalar and vector fields [16], theories based on massive gravitons [17], string theory-inspired brane-world models [18], and geometric frameworks that move beyond Riemannian geometry. This divergence from Riemannian geometry leads to theories such as those based on Einstein–Cartan geometry [19], different types of Palatini or metric-affine theories [20], such as $f(R)$ models [21], and theories where the evolution is solely driven by torsion and non-metricity effects [22,23]. This approach emphasizes that choosing torsion or non-metricity as the underlying geometric framework leads to two distinct, yet observationally equivalent, formulations of gravity namely, the teleparallel equivalent of GR [24] and symmetric teleparallel gravity [25]. Despite having different fundamental ideas, both representations produce dynamical predictions that are identical to those of GR.

Recently, physicists and cosmologists have shown great interest in $f(\mathbb{Q})$ gravity as an intriguing alternative to Einstein's GR [26,27]. It offers a potential solution to persistent theoretical challenges in relativistic astrophysics. As a modification of gravity, $f(\mathbb{Q})$ gravity extends the gravitational action to include a dependence on a non-metric field, alongside the metric. The relativistic action for $f(\mathbb{Q})$ gravity is described by an analytic function $f(\mathbb{Q})$ of the non-metric field, allowing anomalies relative to the generic relativistic gravity phenomena predicted by GR and introducing an extended set of independent parameters [28,29]. These modifications provide fresh perspectives for exploring astrophysical and cosmic behavior. One of the key features of the $f(\mathbb{Q})$ theory is that it allows gravity to be separated from inertial effects, unlike GR. A significant strength of $(f(\mathbb{Q}))$ gravity is that, unlike $f(R)$ gravity, it does not exhibit the pathologies associated with fourth-order equations of motion. This is because it leads to second-order equations of motion [30]. Therefore, the development of $f(\mathbb{Q})$ theory offers a fresh foundation for various modified models of gravity. The $f(\mathbb{Q})$ theory has seen a surge in applied research, with studies covering diverse topics such as cosmic science [31–33], singularity-free stellar solutions [26], bounce cosmologies [34], spherically symmetric and static black hole metrics [35], wormhole spacetime models [36–38], energy constraints [39], and the Newtonian limit [40].

To simplify the description of matter configurations in modeling relativistic compact structures, the assumption of isotropic pressure, which implies equal principal stresses, is commonly employed. This assumption simplifies the equations governing the stellar structure, making isotropic models easier to solve and more straightforward to interpret. In contrast, anisotropic models allow the pressure to vary in direction, distinguishing between radial and tangential components. In the extremely dense interiors of compact stars, such as neutron stars, the pressures in different directions are not necessarily equal because of the influence of strong nuclear and gravitational forces. This anisotropy significantly affects the equilibrium and stability of compact stars, enabling more accurate predictions of their mass, radius, and other physical characteristics, especially in light of recent observations of highly massive neutron stars. Both isotropic and anisotropic models contribute to our understanding of stellar physics. Although isotropic models remain valuable for studying less extreme stars and serve as foundational tools in astrophysics, anisotropic models are essential for accurately describing compact objects under extreme conditions. They provide refined insight into the interior properties of such stars, the upper limits of their mass, and the physical effects related to magnetic fields. In certain cases, anisotropy is crucial for avoiding central singularities or achieving equilibrium configurations that are stable against radial perturbations, features that are not achievable within purely isotropic frameworks.

Over the past few decades, theoretical research has shown that local anisotropies can emerge in certain density ranges due to various physical processes, even though observational data often support the adoption of an isotropic pressure model, which is a hallmark Pascalian property of fluids. In general, it is well known that the assumption of isotropic pressure represents a basic assumption, while its anisotropic nature appears to be highly critical in characterizing the fluid configuration under various conditions. This is very helpful in the analysis of dense matter relativistic objects, resulting in several intriguing astrophysical scenarios. Furthermore, it was recently demonstrated by Herrera [41] that the presence of certain thermodynamic variables, including dissipative flux components, matter density non-uniformity, and the development of a shearing nature of fluid makes the isotropic pressure condition unstable. As a result, pressure isotropy transitions into anisotropy due to these factors. This suggests that a fluid system, initially having pressure isotropy during stellar formation will likely develop anisotropy as it evolves under expected conditions. Thus, the gravitational anisotropy in the hydrostatic equilibrium equation essentially refers to the disparity between the transverse and radial pressures ($P_{\perp} - P_r$). This disparity in stellar fluids is caused by several physical phenomena. It was theoretically demonstrated by Ruderman [42] that the nuclear matter within dense-matter fluids featuring densities $> 10^{15}$ g/cm^3, relativistic interactions can cause anisotropic effects. Factors responsible for generating pressure anisotropy include the interaction between the interior core and exterior crust, particularly during the formation of a superfluid, superconductive states and boson stars, which can significantly alter stellar configurations [43–46]. Moreover, pressure anisotropy is also characterized by physical features like the presence of solid cores, viscosity, and transitions involving exotic phases of matter, including pion condensation [47].

The complex construction of self-gravitating cosmic structures suggests that even minor internal variations can lead to significant changes in their physical properties, offering crucial insights into the universe and its evolution. To examine small fluctuations in thermodynamic observables, it is essential to formulate a gravitational complexity factor (GCF) that successfully captures the multifaceted makeup and the underlying physics of the building blocks of stars. It is important to emphasize that complexity in stable, static self-gravitating systems arise from the interplay of multiple components, which can lead to an increase in structural intricacy. For years, numerous parameters across various research domains have been employed to assess the complexity of different systems [48–50]. Most existing formulations are based on theoretical notions such as entropy and structural information, based on the hypothesis that complexity should, in some way, measure an essential characteristic characterizing the interior configuration of a configuration. Physicists typically illustrate minimal complexity using two extremes: the orderly structure of a perfect crystal and the complete randomness of an ideal gas, both regarded as simple systems. The complete order of a perfect crystal, characterized by atoms arranged in specific symmetrical patterns, leads to a probability distribution strongly biased towards a perfectly symmetrical state, signifying low information content. Unlike the crystal, the ideal gas represents a state of full randomness, with each microstate has the same likelihood, thus encoding the highest possible information. Since both of these fundamental structures exhibit extreme levels of order or information, it becomes evident that the concept of complexity must incorporate additional factors beyond just order and information. Disequilibrium, a concept introduced by the authors [51–53], is used to assess the complexity of a structure by quantifying its deviation from an equiprobable distribution of attainable states. They concluded that ideal gases and perfect crystals have zero complexity, interpreting complexity through the joint contribution of information and disequilibrium.

Understanding the internal structure of self-gravitating compact fluids is significantly enhanced through the concept of the GCF. The typical physical variables examined in this approach include pressure, heat emission, and energy density. The inclusion of these physical parameters increases the level of GCF within the stellar fluids. On the other hand, it should be emphasized that matter density alone is insufficient for determining the term GCF, the pressure component of the stress-energy tensor (SET) must also be considered. Therefore, after identifying the limitations of existing complexity models used to analyze dense stellar structures, Herrera [54] proposed a novel framework incorporating fluid variables such as pressure, energy density, and other relevant factors. Subsequently, this innovative formulation of the GCF was adapted to describe non-static, dense-matter fluids within the classical GR framework [55]. Several extended theories of gravity have employed the GCF to study the physical properties of relativistic systems under different theoretical frameworks [56–62]. The underlying principle of this new conception of GCF is that a fluid with isotropic pressure and homogeneous matter density provides a model for a fundamentally simple system.

Our objective is to carry out a thorough investigation of how the presence of non-metricity influences the GCF associated with dense-matter fluid configurations and to explore the potential applications of zero-GCF in modeling polytropic self-gravitating systems. For this purpose, we employ a widely recognized non-metricity-inspired model of gravitation, known as $f(\mathbb{Q})$ gravity. The following sections provide a structured presentation of this work. We begin by reviewing the coincidence gauge representation of $f(\mathbb{Q})$ theory and its gravitational equations for a dense-matter configuration featuring spherical symmetry in Sect. 2, we briefly discuss the notion of non-metricity-inspired structure scalars for self-gravitating stars in Sect. 3. In Sect. 4, we outline the mathematical framework for constructing compact stellar solutions satisfying zero GCF under the $f(\mathbb{Q}$ gravity scheme. Moreover, the construction of self-gravitating polytropes from the coupling of zero GCF plus the polytropic EoS is also discussed in Sect. 5. The concluding section offers final reflections and potential outlooks.

2 Coincident gauge framework of $f(\mathbb{Q})$ gravity

The fundamental geometrical entity within $f(\mathbb{Q})$ theory is the non-metricity tensor, $\mathbb{Q}_{\alpha\mu\nu}$, instead of the GR curvature tensor or torsion (as in teleparallel gravity). The coincident gauge removes the connection, thus leaving the metric as the single fundamental variable. In generic metric-affine gravity, the affine connection and the metric are treated as independent variables. Correspondingly, the fundamental variables associated with $f(\mathbb{Q})$ gravity are

1. The metric tensor $g_{\mu\nu}$
2. The affine connection $\Gamma^{\alpha}{}_{\mu\nu}$, a key element, defines the covariant derivative.

The non-metricity tensor, central to $f(\mathbb{Q})$ gravity, is formulated as

$$\mathbb{Q}_{\alpha\mu\nu} := \nabla_{\alpha} g_{\mu\nu} = \partial_{\alpha} g_{\mu\nu} - \Gamma^{\beta}{}_{\alpha\mu} g_{\beta\nu} - \Gamma^{\beta}{}_{\alpha\nu} g_{\mu\beta}. \tag{1}$$

This determines how much the connection deviates from being metric compatible. An alternative way to define the aforementioned relationship in terms of the metric inverse is as

$$\mathbb{Q}_{\alpha}{}^{\mu\nu} = -\nabla_{\alpha} g^{\mu\nu}.$$

The relationship between the non-metricity and the disformation term provided by

$$L^{\alpha}{}_{\mu\nu} := \frac{1}{2}\mathbb{Q}^{\alpha}{}_{\mu\nu} - \mathbb{Q}_{(\mu}{}^{\alpha}{}_{\nu)},$$

while the contortion part of the connection is defined as

$$K^{\alpha}{}_{\mu\nu} := \frac{1}{2}T^{\alpha}{}_{\mu\nu} - T_{(\mu}{}^{\alpha}{}_{\nu)},$$

where $T^{\alpha}{}_{\mu\nu} = 2\Gamma^{\alpha}{}_{[\mu\nu]}$. Therefore, the full affine connection splits into three pieces

$$\Gamma^{\alpha}{}_{\mu\nu} := \left\{{}^{\alpha}{}_{\mu\nu}\right\} + L^{\alpha}{}_{\mu\nu} + K^{\alpha}{}_{\mu\nu},$$

where $\left\{{}^{\alpha}{}_{\mu\nu}\right\}$ corresponds to the Levi–Civita part. Next, we define the non-metricity conjugate as

$$P^{\alpha}{}_{\mu\nu} = -\frac{1}{4}\mathbb{Q}^{\alpha}{}_{\mu\nu} + \frac{1}{2}\mathbb{Q}_{(\mu}{}^{\alpha}{}_{\nu)} + \frac{1}{4}(\mathbb{Q}^{\alpha} - \bar{\mathbb{Q}}^{\alpha})g_{\mu\nu} - \frac{1}{4}\delta^{\alpha}{}_{(\mu}\mathbb{Q}_{\nu)},$$

with traces

$$\mathbb{Q}^{\mu} = \mathbb{Q}^{\mu\alpha}{}_{\alpha}, \quad \bar{\mathbb{Q}}_{\rho} = \bar{\mathbb{Q}}^{\alpha}{}_{\mu\alpha}.$$

The scalar quantity $\mathbb{Q}$ is given by

$$\mathbb{Q} = -\mathbb{Q}_{\alpha\mu\nu} P^{\alpha\mu\nu}.$$

The action for $f(\mathbb{Q})$ gravity, incorporating Lagrange multipliers, reads [63]

$$S = \int d^4x\sqrt{-g}\mathscr{L}_{\rm gravity} + \int d^4x\sqrt{-g}\mathscr{L}_{\rm matter}, \tag{2}$$

where

$$\mathscr{L}_{\rm gravity} = \frac{1}{2}f(\mathbb{Q}) + \lambda_{\alpha}{}^{\beta\mu\nu}R^{\alpha}{}_{\beta\mu\nu} + \lambda_{\alpha}{}^{\mu\nu}T^{\alpha}{}_{\mu\nu},$$

where $f(\mathbb{Q})$ is an analytic function of non-metricity, $\lambda_{\alpha}{}^{\beta\mu\nu}$ denotes the Lagrange multipliers, and $g = det(g_{\mu\nu})$. The variation of the non-metricity-supplemented action (1) with respect to $g_{\mu\nu}$ provides

$$\begin{aligned}\mathcal{X}_{\mu\nu} :=& -\frac{2}{\sqrt{-g}}\nabla_{\alpha}\left(\sqrt{-g}f_{\mathbb{Q}}P^{\alpha}{}_{\mu\nu}\right)\\ &+\left(P_{\nu}{}^{\alpha\beta}\mathbb{Q}_{\mu\alpha\beta} - 2P^{\alpha\beta}{}_{\mu}\mathbb{Q}_{\alpha\beta\nu}\right)\\ &\times f_{\mathbb{Q}} + \frac{1}{2}g_{\mu\nu}f = 8\pi\mathring{T}_{\mu\nu},\end{aligned} \tag{3}$$

where $f_{\mathbb{Q}} := \partial_{\mathbb{Q}}f(\mathbb{Q})$. Equation (3) can be rewritten as

$$\mathcal{X}_{\mu\nu} := 2f_{\mathbb{Q}\mathbb{Q}}P^{\alpha}{}_{\mu\nu}\partial_{\alpha}\mathbb{Q} + \frac{1}{2}\left(f - \mathbb{Q}f_{\mathbb{Q}}\right) + f_{\mathbb{Q}}G_{\mu\nu} = 8\pi\mathring{T}_{\mu\nu}. \tag{4}$$

It is easily shown that the Einstein gravitational equations can be recovered under the limit $f(\mathbb{Q}) \to \mathbb{Q}$. To characterize the energy-matter distribution of the relativistic self-gravitating configuration, the SET, $\mathring{T}_{\mu\nu}$, is defined as

$$\mathring{T}_{\mu\nu} := \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathscr{L}_{\rm matter})}{\delta g^{\mu\nu}},$$

whereas the variation of (1) with respect to the connection gives

$$\sqrt{-g}f_{\mathbb{Q}}P_{\alpha}{}^{\mu\nu} = \nabla_{\beta}\lambda_{\alpha}{}^{\nu\mu\beta} + \lambda_{\alpha}{}^{\mu\nu} - H_{\alpha}{}^{\mu\nu},$$
$$H_{\alpha}{}^{\mu\nu} = -\frac{\delta(\mathscr{L}_{\rm matter})}{2\delta\Gamma^{\alpha}{}_{\mu\nu}}.$$

The skew-symmetric nature of the indices μ and ν leads to the following result

$$\nabla_\mu \nabla_\nu \left(H_\alpha{}^{\mu\nu} + \sqrt{-g} f_{\mathbb{Q}} P_\alpha{}^{\mu\nu} \right) = 0,$$

which for $\nabla_\mu \nabla_\nu H_\alpha{}^{\mu\nu} = 0$, reduces to

$$\nabla_\mu \nabla_\nu \left(\sqrt{-g} f_{\mathbb{Q}} P_\alpha{}^{\mu\nu} \right) = 0.$$

For an arbitrary frame of transformation $\varepsilon(x)$, the curvature-less and torsion-less geometric connection allows us to formulate $\Gamma_\alpha{}^{\mu\nu}$ as

$$\Gamma^\alpha_{\ \mu\nu} = \left(\frac{\partial x^\alpha}{\partial \varepsilon^\beta} \right) \partial_\mu \partial_\nu \varepsilon^\beta. \tag{5}$$

Thus, within the coincident gauge framework ($\Gamma^\alpha_{\ \mu\nu} = 0$), $\mathbb{Q}_{\alpha\mu\nu}$ has the following formulation

$$\mathbb{Q}_{\alpha\mu\nu} = \partial_\alpha g_{\mu\nu}. \tag{6}$$

In general metric-affine gravity, where the connection and metric are treated independently, the coincident gauge simplifies the theory by avoiding the complexities introduced by connection variations. The metric becomes the single fundamental variable when the coincident gauge reduces the relationship. The equations of motion become more manageable when the affine connection is reduced to zero because the covariant derivatives reduce to ordinary derivatives. Equation (5) defines an inertial relationship, requiring the affine connection to be determined without gravity [63]. However, under the coincident gauge the off-diagonal components of the field equations strongly constrain the function $f(\mathbb{Q})$, leading to complex functional forms. The interior configuration of a static, dense-matter gravitational system is described by the following metric

We work within a static spacetime framework, expressed in curvature coordinates $(x^0, x^1, x^2, x^3) \equiv (t, r, \theta, \phi)$, defined as

$$ds^2 = e^a dt^2 - e^b dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right), \tag{7}$$

with $(x^0, x^1, x^2, x^3) := (t, r, \theta, \phi)$ and $a := a(r)$, $b := b(r)$. The non-metricity scalar, corresponding to the metric mentioned above, reads as

$$\mathbb{Q} = -\frac{2}{r^2} \left(1 + r a' \right) e^{-b}. \tag{8}$$

Under spherical symmetry, the SET for an anisotropic fluid can be defined as a diagonal matrix

$$\overset{\circ}{T}{}^\mu_{\ \nu} = \begin{pmatrix} \overset{\circ}{\rho} & 0 & 0 & 0 \\ 0 & -\overset{\circ}{P}_r & 0 & 0 \\ 0 & 0 & -\overset{\circ}{P}_\perp & 0 \\ 0 & 0 & 0 & -\overset{\circ}{P}_\perp \end{pmatrix}. \tag{9}$$

This implies that $\overset{\circ}{T}{}^0_{\ 0} = \overset{\circ}{\rho}$ (energy density), $\overset{\circ}{T}{}^1_{\ 1} = -\overset{\circ}{P}_r$ (radial pressure), and $\overset{\circ}{T}{}^2_{\ 2} = \overset{\circ}{T}{}^3_{\ 3} = -\overset{\circ}{P}_\perp$ (tangential pressure) with $\overset{\circ}{P}_r \neq \overset{\circ}{P}_\perp$. The coupling of the anisotropic SET (9) with the metric (7) provides the following nonzero components of the $f(\mathbb{Q})$ model

$$\begin{aligned} \mathcal{X}_{00} = 8\pi \overset{\circ}{T}_{00} : \frac{e^{a-b}}{2r^2} &\left\{ 2r\mathbb{Q}' f_{\mathbb{Q}\mathbb{Q}} (e^a - 1) \right. \\ &\left. + \left[(e^b - 1)(2 + ra') + (1 + e^b) r b' \right] f_{\mathbb{Q}} + r^2 e^b f \right\} \\ = 8\pi \overset{\circ}{T}_{00}, \end{aligned} \tag{10}$$

$$\begin{aligned} \mathcal{X}_{11} = 8\pi \overset{\circ}{T}_{11} : -\frac{1}{2r^2} &\left\{ 2r\mathbb{Q}' f_{\mathbb{Q}\mathbb{Q}} (e^a - 1) \right. \\ &\left. + \left[(e^b - 1)(2 + ra' + rb') - 2ra' \right] f_{\mathbb{Q}} + r^2 e^b f \right\} \\ = 8\pi \overset{\circ}{T}_{11} \end{aligned} \tag{11}$$

$$\begin{aligned} \mathcal{X}_{22} = 8\pi \overset{\circ}{T}_{22} : -\frac{1}{2r^2} &\left\{ -2r\mathbb{Q}' f_{\mathbb{Q}\mathbb{Q}} + f_{\mathbb{Q}} \left[2a'(e^b - 2) \right. \right. \\ &\left. \left. - ra'^2 + b'(2e^b + ra') - 2ra'' \right] f_{\mathbb{Q}} + r^2 e^b f \right\} \\ = 8\pi \overset{\circ}{T}_{22}, \end{aligned} \tag{12}$$

$$\mathcal{X}_{12} = 8\pi \overset{\circ}{T}_{21} : \frac{\mathbb{Q}'}{2} \cot\theta f_{\mathbb{Q}\mathbb{Q}} = 0, \tag{13}$$

where $\mathcal{X}_{22} = \mathcal{X}_{33}$ and $\mathcal{X}_{12} = \mathcal{X}_{21}$ due to spherical symmetry.

3 Structure scalars in $f(\mathbb{Q})$ gravity

If the affine connection is zero in the chosen coordinate system, then $f(\mathbb{Q})$ gravity only admits vacuum solutions, meaning the SET $T_{\mu\nu}$ must be zero.

$$\begin{aligned} \mathcal{X}_{00} = 8\pi \overset{\circ}{T}_{00} : \frac{f}{2} - f_{\mathbb{Q}} \left\{ \mathbb{Q} + \frac{1}{r^2} + \frac{e^{-b}}{r} (a' + b') \right\} \\ = -8\pi \overset{\circ}{\rho}, \end{aligned} \tag{14}$$

$$\mathcal{X}_{11} = 8\pi \overset{\circ}{T}_{11} : \frac{f}{2} - f_{\mathbb{Q}} \left(\mathbb{Q} + \frac{1}{r^2} \right) = 8\pi \overset{\circ}{P}_r, \tag{15}$$

$$\mathcal{X}_{22} = 8\pi \overset{\circ}{T}_{22} : \frac{f}{2} - f_{\mathbb{Q}} \left\{ \frac{\mathbb{Q}}{2} - e^{-b} \left[\frac{a''}{2} + \left(\frac{a'}{4} + \frac{1}{2r} \right) (a' - b') \right] \right\} = 8\pi \overset{\circ}{P}_t, \quad (16)$$

$$\mathcal{X}_{12} = 8\pi \overset{\circ}{T}_{21} : \frac{\mathbb{Q}'}{2} \cot\theta f_{\mathbb{Q}\mathbb{Q}} = 0. \quad (17)$$

In other words, if we assume a vanishing affine connection in the spherically symmetric coordinate system and impose vacuum solutions within $f(\mathbb{Q})$ theory, the non-diagonal components of the field equations lead to the conclusion that $f_{\mathbb{Q}\mathbb{Q}} = 0$. It follows that $f(\mathbb{Q})$ is required to be linear, as nonlinear forms like $f(\mathbb{Q}) = \mathbb{Q}^2$ result in inconsistencies within the gravitational equations. As a result, nonlinear $f(\mathbb{Q})$ functions are not well-suited for obtaining viable solutions to the stellar equations. In this direction, some exact dense-matter solutions based on $f(\mathbb{Q})$ gravity using the coincident gauge restriction have been explored in [64]. These solutions describe that when $\mathbb{Q}$ is constant ($\mathbb{Q} = \mathbb{Q}_0$), $\mathbb{Q}_0$ acts as a cosmological constant in Schwarzschild-like solutions, suggesting that $f(\mathbb{Q})$ gravity can model dark energy. Therefore,

$$f(\mathbb{Q}) = \beta_0 \mathbb{Q} + \beta_1, \quad (18)$$

where β_0 and β_1 are integration constants. A comprehensive account of the compatibility between those as mentioned above, linear functional form of $f(\mathbb{Q})$ and the coincident gauge within a static and spherically symmetric spacetime is presented in [65]. Now, substituting Eqs. (7) and (18) into the gravitational system (14)–(16), provides the following set of differential equations

$$8\pi \overset{\circ}{\rho} = \frac{1}{2r^2} \left[2\beta_0 - 2\beta_0 (1 - rb') e^{-b} - r^2 \beta_1 \right], \quad (19)$$

$$8\pi \overset{\circ}{P}_r = \frac{1}{2r^2} \left[2\beta_0 + 2\beta_0 (1 + ra') e^{-b} - r^2 \beta_1 \right], \quad (20)$$

$$8\pi \overset{\circ}{P}_t = \frac{e^{-b}}{4r} \left[2\beta_1 r e^b + \beta_0 (ra' + 2r)(a' - b') + 2\beta_0 r a'' \right]. \quad (21)$$

The hydrostatic equilibrium equation can be easily derived from the preceding expressions, and it takes the following form

$$\frac{d\overset{\circ}{P}_r}{dr} = \frac{2}{r} (\overset{\circ}{P}_r - \overset{\circ}{P}_\perp) - \frac{a'}{r} (\overset{\circ}{\rho} + \overset{\circ}{P}_r), \quad (22)$$

where

$$a' = \frac{16\pi r^3 \overset{\circ}{P}_r - \beta_1 r^3 + 4\beta_0 m}{2r(r - 2m)}. \quad (23)$$

Within the context of the above-mentioned linear formulation of $f(\mathbb{Q})$ gravity, the mass function corresponding to the dense-matter astrophysical configuration is defined as

$$m(r) = \left(\frac{r}{2} \right) R^{\phi}_{\theta\phi\theta} = \frac{r}{2} \left(1 - e^{-b} \right),$$

or identically,

$$m(r) = \frac{4\pi}{\beta_0} \int_0^r x^2 \overset{\circ}{\rho} dx + \frac{\beta_1 r^3}{12\beta_0}. \quad (24)$$

An alternative concept of GCF, derived from the orthogonal decomposition of the Riemann tensor and yielding structural scalars, was proposed by Herrera [54]. These scalar functions were initially defined in [66], and have several applications in the study of extremely dense, self-gravitating stars. Within the framework of classical GR, they successfully formulated a full array of equations, based on the orthogonal decomposition of the Riemann tensor, that govern the physical features and development of self-gravitating stellar fluids, exhibiting anisotropic stresses in terms of different scalar functions. Additionally, they have proved that these scalar quantities are intrinsically linked to key characteristics of the matter configuration, including matter density, density non-uniformity, pressure anisotropy, dissipative flux, and Tolman mass. All possible solutions of the gravitational equations of motion can be explicitly written in terms of these scalars for a spherical, relativistic, dense matter distribution. The representation of the tensor $R_{\mu\omega\nu\upsilon}$ as a set of tensorial entities is well recognized [54,67].

$$Y_{\mu\nu} = R_{\mu\omega\nu\upsilon} U^{\omega} U^{\upsilon},$$

$$X_{\mu\nu} =^{*} R^{*}_{\mu\omega\nu\upsilon} U^{\omega} U^{\upsilon} = \frac{1}{2} \xi_{\mu\omega}{}^{\varsigma\epsilon} R^{*}_{\varsigma\epsilon\nu\upsilon} U^{\omega} U^{\upsilon},$$

where where $R^{*}_{\mu\nu\omega\upsilon} = \frac{1}{2} \xi_{\varsigma\epsilon\omega\upsilon} R_{\mu\nu}{}^{\varsigma\epsilon}$. We can rewrite the expressions for the quantities $Y_{\mu\nu}$ and $X_{\mu\nu}$ by separating them into their trace and trace-free components.

$$Y_{\mu\nu} = \frac{1}{3} Y_T h_{\mu\nu} + Y_{TF} \left(\mathcal{X}_\mu \mathcal{X}_\nu - \frac{1}{3} h_{\mu\nu} \right),$$

$$X_{\mu\nu} = \frac{1}{3} X_T h_{\mu\nu} + X_{TF} \left(\mathcal{X}_\mu \mathcal{X}_\nu - \frac{1}{3} h_{\mu\nu} \right), \quad (25)$$

here $h_{\mu\nu}$ is the projection tensor, while $\mathcal{X}^\mu = \left(0, e^{-a/2}, 0, 0\right)$ is the unit four vector such that

$$U^\mu \mathcal{X}_\mu = 0, \quad \mathcal{X}^\mu \mathcal{X}_\mu = 1.$$

We can express structure scalars as

$$X_T = 8\pi \beta_0 \overset{\circ}{\rho} - \frac{\beta_1}{2}, \quad (26)$$

$$X_{TF} = \frac{1}{2\beta_0}(\Pi + \beta_1) - E, \tag{27}$$

or, alternatively

$$Y_T = 4\pi \left\{ \frac{1}{2\beta_0}\left(\mathring{\rho} + 3\mathring{P}_r - 2\Pi\right) - 3\beta_1 \right\}, \tag{28}$$

$$Y_{TF} = \frac{1}{2\beta_0}(\Pi + \beta_1) + E, \tag{29}$$

where E denotes the Weyl curvature scalar. Self-gravitating systems, which are controlled by their gravitational field, are essential to astronomy because they are the first phases of the creation of compact objects like black holes [58], neutron stars [68–70], wormholes [36–38], and white dwarfs. Gravitational collapse causes these systems to form extremely compact and dense structures. Herrera's formalism of structure scalars is a useful tool for studying the internal structure and evolution of such objects. The scalars mentioned above capture key characteristics of the system's physical makeup, including dissipative processes, density inhomogeneity, and anisotropy. To comprehend the intricate behavior and internal dynamics of compact astrophysical objects, structure scalars are an effective tool. The value of GCF in terms of density non-uniformity, and anisotropic pressure reads

$$Y_{TF} = \frac{1}{\beta_0}(8\pi\Pi + \beta_1) - \frac{1}{2\beta_0 r^3}\int_0^r \left(4\pi\mathring{\rho}' + \frac{\beta_1}{2}\right) x^3 dx. \tag{30}$$

Therefore, using Eqs. (27) and (29), we can write

$$\frac{1}{\beta_0}(\Pi + \beta_1) = Y_{TF} + X_{TF}. \tag{31}$$

which determines the local anisotropy associated with self-gravitating dense-matter object. It can further be shown that Eq. (30) yields the Tolman mass in the following form (see [54] for details)

$$m_T = \left(\frac{r}{\mathrm{R}}\right)^3 M_T + r^3 \int_r^{\mathrm{R}} \frac{e^{(a+b)/2}}{s} Y_{TF} ds. \tag{32}$$

In this formulation, M_T refers to the total Tolman mass attributed to the anisotropic self-gravitating star within radius R.

4 Zero gravitational complexity constraint

In this section, we develop spherically symmetric stellar models describing self-gravitating polytropes by employing the constraint $Y_{TF} = 0$. Then, by employing the zero-GCF condition ($Y_{TF} = 0$), we have

$$8\pi\Pi + \beta_1 = \frac{1}{r^3}\int_0^r \left(4\pi\mathring{\rho}' + \frac{\beta_1}{2}\right) x^3 dx. \tag{33}$$

For a static, anisotropic dense-matter fluid configuration, the $f(\mathbb{Q})$ equations of motion (Eqs. (19)–(21)) result in a set of three ordinary differential equations (ODEs) involving five unknown functions: $a, b, \mathring{\rho}, \mathring{P}_r$, and $\mathring{P}_\perp$. As a result, even after applying the $Y_{TF} = 0$ condition, the self-gravitating star system remains underdetermined, requiring one more condition to solve the system. It is important to note that the null GCF requirement implies either uniform matter density with pressure isotropy, or non-uniform matter density accompanied by pressure anisotropy. It should also be observed that Eq. (33) can be regarded as a non-local EoS, somewhat analogous to that suggested by the authors [71]. We will now present some examples purely for the sake of illustration.

4.1 Phenomenological metric ansatz

To explicitly solve the set of gravitational equations based on linear $f(\mathbb{Q})$ gravity, we consider a specific metric ansatz. This ansatz is particularly useful in modeling dense-matter stellar spheres, such as neutron stars with anisotropic fluid configuration. This was originally proposed by Gokhroo and Mehra to examine the behavior of anisotropic gas spheres with variable matter density. It provides a physical basis for understanding the significant redshifts observed in quasars. The Gokhroo and Mehra ansatz assumes a specific form of the matter density, given by

$$\rho = \rho_0\left(1 - \frac{Kr^2}{a^2}\right) \tag{34}$$

Here, K is a constant with $K \in (0, 1)$, a is the radius of the anisotropic sphere, and $\rho_0 = \rho(r = 0)$ represents the central density. The matter density vanishes at the center of the sphere if we set $K = 1$. Substituting Eq. (34) in Eq. (24), we get

$$m(r) = \frac{4\pi\rho_0 r^3}{3\beta_0}\left(1 - \frac{3Kr^2}{5a^2}\right) + \frac{\beta_1 r^3}{12\beta_0}, \tag{35}$$

which can be expressed in an alternative form as

$$m(r) = \frac{\alpha r^3}{2\beta_0}\left(1 - \frac{3Kr^2}{5a^2}\right) + \frac{\beta_1 r^3}{12\beta_0}, \tag{36}$$

where $8\pi\rho_0/3 = \alpha$. Next, the combination of Eqs. (24) and (36) produces

$$e^{-b(r)} = 1 - \frac{1}{\beta_0}\left\{\alpha r^2\left(1 - \frac{3Kr^2}{5a^2}\right) - \frac{\beta_1 r^3}{6}\right\}. \tag{37}$$

$$8\pi(\overset{\circ}{P}_r - \overset{\circ}{P}_\perp) = -\beta_0 e^{-b} \times \left\{\frac{a''}{2} + \left(\frac{a'}{2}\right)^2 - \frac{b'}{2}\left(\frac{a'}{2} + \frac{1}{r}\right) - \frac{b'}{2r} - \frac{1}{r^2}\right\} - \frac{1}{r^2}. \tag{38}$$

Let us introduce the functions $s(r)$ and $u(r)$, defined as

$$e^{b(r)} = \exp\left\{\int\left(2s - \frac{2}{r}\right)dr\right\}, \tag{39}$$

$$e^{-a(r)} = u(r). \tag{40}$$

As a result, Eq. (38) transforms into the following form:

$$u' + P(r)u = Q(r), \tag{41}$$

where

$$P(r) = 2s + \frac{2s'}{s} - \frac{6}{r} + \frac{2}{sr^2} \quad \text{and} \quad Q(r) = -\frac{2}{\beta_0 u}\left(8\pi\Pi + \frac{1}{r^2}\right). \tag{42}$$

The solution of Eq. (39) is defined as

$$u(r) = e^{-\int P(r)dr}\left\{\int e^{\int P(r)dr} Q(r)dr + C\right\}, \tag{43}$$

where C is an integration constant. Finally, Eqs. (39) and (40) allow us to rewrite the metric in the following form

$$ds^2 = e^{\int\left(2s - \frac{2}{r}\right)dr} dt^2 - \frac{\int e^{\int P(r)dr} Q(r)dr + C}{e^{\int P(r)dr}} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{44}$$

Consequently, the thermodynamic quantities associated with anisotropic sphere read

$$8\pi\overset{\circ}{\rho} = \frac{2m'\beta_0}{r^2} - \frac{\beta_1}{2}, \tag{45}$$

$$8\pi\overset{\circ}{P}_r = \frac{2\beta_0}{r^2}\left\{\frac{(r-2m)s}{r^2} + \frac{m}{r} - 1\right\} + \frac{\beta_1}{2}, \tag{46}$$

$$8\pi\overset{\circ}{P}_t = \frac{2\beta_0}{r^2}\left\{rs' + rs^2 - \frac{s}{r^2} + \frac{1}{r^3}\right\}(r-2m) + \frac{2\beta_0}{r^2}(m - m'r) + \beta_1. \tag{47}$$

The above results are constructed within the framework of $f(Q)$ gravity using the Gokhroo and Mehra ansatz. This ansatz enables the generation of anisotropic solutions suitable for describing dense stellar configurations. In the context of $f(Q)$ gravity, these models allow us to explore deviations from GR due to non-metricity effects, which influence the pressure, density, and anisotropy profiles of the stellar interior.

4.2 Anisotropic polytropes with zero gravitational complexity

Anisotropic polytropes characterize a class of self-gravitating stellar models with an anisotropic matter distribution. The theoretical modeling of polytropes rests on the polytropic EoS, which, in the context of an anisotropic stellar configuration, takes the form

$$\overset{\circ}{P}_r = K\rho^{1+\frac{1}{n}}. \tag{48}$$

where K is an arbitrary constant and n is the polytropic index. The polytropic EoS holds significant importance in the astrophysical exploration of stellar remnants, whether in a Newtonian or general relativistic context. In recent decades, significant detailed considerations have been made of polytropic astrophysical models composed of anisotropic matter. Herrera and Barreto [72] proposed a generic framework for modeling anisotropic relativistic Newtonian polytropes in GR. In particular, they examined the effects of anisotropy on the configuration of self-gravitating stars. Furthermore, Herrera et al. [73] examined the effects of small deviations in matter density and anisotropic pressure on cracking in self-gravitating stellar systems satisfying a polytropic EoS. Two polytropes were considered, both exhibiting cracking under certain parameter variations.

Now, to find an explicit solution for the anisotropic self-gravitational configuration, we need an additional constraint besides the polytropic EoS. The requirement for this additional constraint is fulfilled by considering the zero-GCF condition, which is defined as

$$Y_{TF} = 0 \Rightarrow \frac{1}{\beta_0}(8\pi\Pi + \beta_1) - \frac{1}{2\beta_0 r^3}\int_0^r\left(4\pi\overset{\circ}{\rho}{}' + \frac{\beta_1}{2}\right)x^3 dx = 0. \tag{49}$$

To convert the equations into a dimensionless form, we define the following quantities.

- $\Theta = \frac{\overset{\circ}{P}_{rc}}{\overset{\circ}{\rho}_c}$, the central pressure-to-density ratio
- $\Phi_0^n = \frac{\sigma_0}{\sigma_{0c}}$, the dimensionless baryonic density
- $\overset{\circ}{P}_r = \overset{\circ}{P}_{rc}\Phi_0^{n+1}$, the radial pressure
- $\rho_0 = \rho_{0c}\Phi_0^n$, the baryonic mass density
- $\rho = \rho_{0c}\Phi_0^n + nP_{rc}\Phi_0^{n+1}$, the total energy density

For the radial coordinate, we also use the following rescaling:

$$r = \frac{\varepsilon}{\Omega}, \quad \Omega^2 = \frac{4\pi\rho_c}{\Theta(n+1)}, \tag{50}$$

and a dimensionless mass function

$$\eta(\varepsilon) = \frac{\Omega^3 m}{4\pi\rho_c}.$$

Then, by applying the chain rule, we have

$$\frac{d\overset{\circ}{P}_r}{dr} = (n+1)\Omega \overset{\circ}{P}_{rc}\Phi_0^n \frac{d\Phi_0}{d\varepsilon}.$$

Finally, the hydrostatic equilibrium equation takes the following form

$$\frac{1-2\Theta(n+1)\eta/\varepsilon}{1+\Theta\Phi_0}\left\{\varepsilon^2\frac{d\Phi_0}{d\varepsilon} + \frac{2\varepsilon\Pi\Phi_0^{-n}}{(n+1)\overset{\circ}{P}_{rc}}\right\} + \frac{\beta_1\varepsilon^3\Theta}{16\pi\overset{\circ}{P}_{rc}} + \Theta\varepsilon^3\Phi_0^{n+1} + \beta_0\eta = 0, \tag{51}$$

and

$$\frac{d\eta}{d\varepsilon} = \varepsilon^2\Phi^n. \tag{52}$$

The system of two first-order ODEs given by Eqs. (51) and (52) involve the three unknown functions Φ, η, and Π, and are parameterized by n and Θ. We continue the scheme of stellar modeling by assuming the GCF is zero, resulting in the subsequent expression consistent with the earlier notation

$$\frac{2\varepsilon}{n\overset{\circ}{\rho}_c}\frac{d\Pi}{d\varepsilon} = \Phi_0^{n-1}\frac{d\Phi_0}{d\varepsilon} - \frac{6\Pi}{n\overset{\circ}{\rho}_c} - \frac{3\varepsilon\beta_1}{4\pi\Theta\overset{\circ}{\rho}_c\varepsilon}. \tag{53}$$

We now arrive at a system of three ODEs (51), (52), and (53) involving the three unknown functions Φ, η, and Π. These equations can be integrated for any chosen pair of parameter values n and Θ, provided the physical conditions are satisfied.

Finally, let us mention that the extension of the Newtonian polytropes through the dynamics of the linear $f(\mathbb{Q})$ gravitational model presents two options. These are Eq. (48) and $\overset{\circ}{P}_r = K\overset{\circ}{\rho}_b^{\,1+\frac{1}{n}}$, with $\overset{\circ}{\rho}_b$ being the baryonic mass density. A detailed explanation of this second possibility can be found in [74]. The expressions analogous to Eqs. (51) and (53) for this particular case are defined as

$$\frac{1-2\Theta(n+1)\eta/\varepsilon}{1+\Theta\Phi_b}\left\{\varepsilon^2\frac{d\Phi_b}{d\varepsilon} + \frac{2\varepsilon\Pi\Phi_b^{-n}}{(n+1)\overset{\circ}{P}_{rc}}\right\} + \frac{\beta_1\varepsilon^3\Theta}{16\pi\overset{\circ}{P}_{rc}} + \Theta\varepsilon^3\Phi_b^{n+1} + \beta_b\eta = 0,$$

$$\frac{2\varepsilon}{n\overset{\circ}{\rho}_{bc}}\frac{d\Pi}{d\varepsilon} = \Phi_b^{n-1}\frac{d\Phi_b}{d\varepsilon}\left\{1 + K(n+1)\Phi_b\overset{\circ}{\rho}_{bc}^{\,1/n}\right\} - \frac{6\Pi}{n\overset{\circ}{\rho}_{bc}} - \frac{3\varepsilon\beta_1}{4\pi\Theta\overset{\circ}{\rho}_{bc}\varepsilon},$$

with $\Phi_b^n = \frac{\rho_b}{\rho_{bc}}$.

5 Conclusion

We have developed the notion of GCF for dense-matter, stationary fluids exhibiting spherical symmetry, using the framework of the non-metricity-motivated $f(\mathbb{Q})$ model of gravity. This formulation is based on the assumption that a homogeneous fluid distribution (in terms of matter density) with isotropic pressure represents the fundamental level of simplicity. In this respect, an apparent contender to measure the degree of the GCF in stellar fluids is defined in terms of a mathematical quantity denoted by Y_{TF}. This is obtained from a mathematical framework known as the orthogonal decomposition of the Riemann tensor. For static, dense-matter stellar fluids, Y_{TF} specifies a combination of two distinct matter variables: pressure anisotropy and non-uniformity of matter density. The presence of these thermodynamic quantities gives rise to the concept of the GCF within self-gravitating stellar objects in Einstein's GR. However, we have shown that in the linear formulation of $f(\mathbb{Q})$ gravity, non-metricity terms also contribute to the complexity of stellar fluids and modify the form of Y_{TF}. Some important highlights of this investigation are as follows.

- Contributions from non-uniformity of matter density, nonmetricity $f(\mathbb{Q})$ terms, and pressure anisotropy are encoded in the scalar function Y_{TF} in a well-defined way. This reflects the fact that the isotropic pressure in stellar fluids corresponds to the lowest level of GCF.
- For electrically charged relativistic fluids, the GCF incorporates the combined effects of the charge and the higher-curvature corrections characteristic of $f(\mathbb{Q})$ gravity.
- In the context of a general non-static matter configuration with dissipation, the GCF is composed of contributions from the matter density non-uniformity, the anisotropy of pressure, and the dissipative fluxes within the relativistic fluid within the dynamics of $f(\mathbb{Q})$ gravity.
- This scalar reflects the impact of anisotropic pressure, non-metricity matter density, and non-uniformity on the Tolman mass within a compact fluid distribution.
- The nullity of Y_{TF} in non-static contexts ensures the stability of the shear-free condition [75]. This shear-free condition is the relativistic counterpart of homologous evolution, which appears to be a more systematic mode of stellar evolution (further explanation in [76]). More-

over, replacing the homologous condition with a quasi-homologous one, in combination with minimal GCF, leads to various non-static stellar models exhibiting both dissipative and non-dissipative characteristics [77].

We have demonstrated distinct closed-form stellar models that successfully fulfill the condition of zero GCF. These models highlight the significance of vanishing GCF in describing stellar fluids influenced by non-metricity effects arising from $f(\mathbb{Q})$ gravity. To develop physically viable dense-matter fluid solutions, several additional conditions can be applied in addition to the zero GCF. These include the adoption of a non-local EoS [71], the vanishing of tangential pressure, and the vanishing of radial pressure. Furthermore, the Finch–SKea metric ansatz [78], Tolman–Kuchowicz metric [79], the embedding class one condition [70], and the anisotropic ansatz are presented in [80]. The notion of zero-GCF also found several applications in different astrophysical systems such as dark energy stellar structures [81], dark matter objects [82], black holes [83], and gravitational decoupled stellar fluids.

The employed non-metricity-inspired $f(\mathbb{Q})$ gravity theory enhances the underlying geometry, providing a robust platform to analyze the extensions to GR, especially under intense gravitational conditions. It is demonstrated that the coupling of linear $f(\mathbb{Q})$ gravity and the zero GCF offers a significant theoretical platform for modeling astrophysical structures with anisotropic fluid configuration. Our findings demonstrate the potential of non-metricity as a crucial component in obtaining explicit solutions that describe dense-matter configurations of self-gravitating star systems, encouraging further study in light of continuous advancements in gravitational waves and observational astrophysics. This method could potentially be expanded in the future studies to explore more generalized $f(\mathbb{Q})$ models, alternative equations of state, and the impact of angular momentum and magnetization effects on these gravitational configurations.

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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