Neutrino Theory

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Abstract. In this talk I give an overview of current approaches in neutrino theory and the key experimental questions which will enable progress to be made.

1. Introduction

In the three active neutrino paradigm, the lepton mixing matrix can be parameterised as in Fig.1 in terms of three angles θ_{ij}, one oscillation phase δ and (if neutrinos are Majorana particles) two Majorana phases α_i.

$$U_{MNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\alpha_1/2} & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Atmospheric Reactor Solar Majorana

Figure 1. The lepton mixing matrix with phases factorizes into a matrix product of four matrices, associated with the physics of Atmospheric neutrino oscillations, Reactor neutrino oscillations, Solar neutrino oscillations and a Majorana phase matrix.

Ignoring the phases, the lepton mixing angles can be visualised as the Euler angles in Fig.2. The mass squared ordering is not yet determined uniquely for the atmospheric mass squared splitting, but the solar neutrino data requires $m_2^2 > m_1^2$, as shown in Fig.3. The absolute scale of neutrino masses is not fixed by oscillation data and the lightest neutrino mass may vary from about $0.0 - 0.2$ eV where the upper limit comes from cosmology. The current best fit values for the lepton angles and neutrino mass squared differences are given in Figs.4,5. Note that the reactor angle θ_{13} is not currently measured but its value is only inferred. The prospects for its future determination have been projected as in Fig.6.

2. Why go beyond the Standard Model?

It has been one of the long standing goals of theories of particle physics beyond the Standard Model (SM) to predict quark and lepton masses and mixings. With the discovery of neutrino mass and mixing, this quest has received a massive impetus. Indeed, perhaps the greatest advance in particle physics over the past decade has been the discovery of neutrino mass and

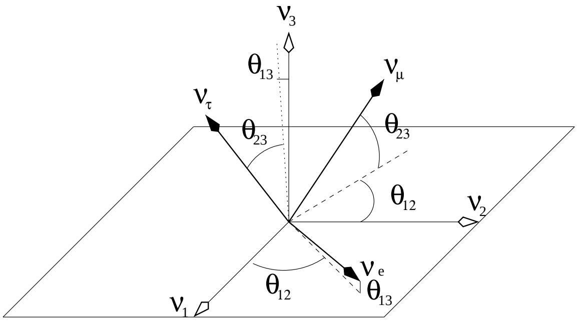

Figure 2. The relation between the neutrino weak eigenstates ν_e, ν_μ, and ν_τ and the neutrino mass eigenstates ν_1, ν_2, and ν_3 in terms of the three mixing angles θ_{12}, θ_{13}, θ_{23}. Ignoring phases, these are just the Euler angles respresenting the rotation of one orthogonal basis into another.

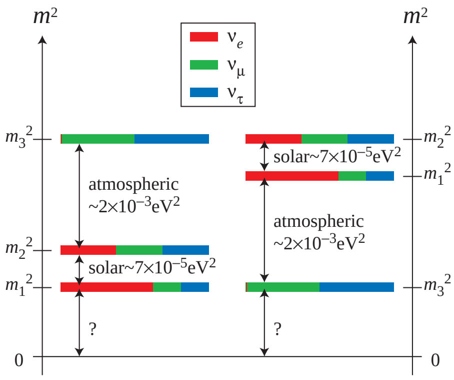

Figure 3. Alternative neutrino mass patterns that are consistent with neutrino oscillation explanations of the atmospheric and solar data. The pattern on the left (right) is called the normal (inverted) pattern. The coloured bands represent the probability of finding a particular weak eigenstate ν_e, ν_μ, and ν_τ in a particular mass eigenstate.

$$\theta_{12} = 34.4 \pm 1.0 \; \left({}^{+3.2}_{-2.9}\right)^\circ$$
$$\theta_{23} = 42.8\,{}^{+4.7}_{-2.9} \; \left({}^{+10.7}_{-7.3}\right)^\circ$$
$$\theta_{13} = 5.6\,{}^{+3.0}_{-2.7} \; (\leq 12.5)^\circ$$

Figure 4. The best fit lepton mixing angles with 1σ error (3σ error) from [1].

$$\Delta m^2_{21} = 7.59 \pm 0.20 \; \left({}^{+0.61}_{-0.69}\right) \times 10^{-5} \text{ eV}^2$$
$$\Delta m^2_{31} = \begin{cases} -2.36 \pm 0.11 \; (\pm 0.37) \times 10^{-3} \text{ eV}^2 \\ +2.46 \pm 0.12 \; (\pm 0.37) \times 10^{-3} \text{ eV}^2 \end{cases}$$

Figure 5. The best fit neutrino mass squared differences with 1σ error (and 3σ error) from [1].

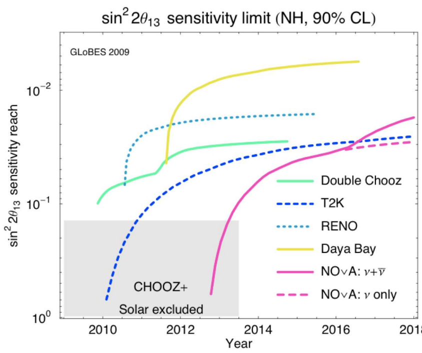

Figure 6. The future $\sin^2 2\theta_{13}$ sensitivity limit (normal hierarchy, 90% CL) from [2].

mixing involving two large mixing angles commonly known as the atmospheric angle θ_{23} and the solar angle θ_{12}, while the remaining mixing angle θ_{13}, although unmeasured, is constrained to be relatively small. The largeness of the two large lepton mixing angles contrasts sharply with the smallness of the quark mixing angles, and this observation, together with the smallness of neutrino masses, provides new and tantalizing clues in the search for the origin of quark and lepton flavour. However, before trying to address such questions, it is worth recalling why neutrino mass forces us to go beyond the SM.

Neutrino mass is zero in the SM for three independent reasons:

(i) There are no right-handed neutrinos ν_R.

(ii) There are only Higgs doublets of $SU(2)_L$.

(iii) There are only renormalizable terms.

In the SM these conditions all apply and so neutrinos are massless with ν_e, ν_μ, ν_τ distinguished by separate lepton numbers L_e, L_μ, L_τ. Neutrinos and antineutrinos are distinguished by total conserved lepton number $L = L_e + L_\mu + L_\tau$. To generate neutrino mass we must relax one or more of these conditions. For example, by adding right-handed neutrinos the Higgs mechanism of the Standard Model can give neutrinos the same type of mass as the electron mass or other charged lepton and quark masses. It is clear that the *status quo* of staying within the SM, as it is usually defined, is not an option, but in what direction should we go?

3. Neutrino Mass Models

The rest of this talk will be organized according to the plan in Fig.7. The plan in Fig.7 contains key experimental questions (in blue), leading in particular theoretical directions (in red), starting from the top left hand corner with the question "LSND?"

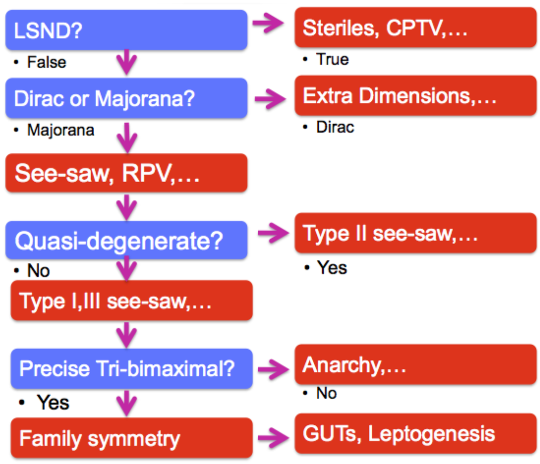

Figure 7. Plan of the talk.

3.1. LSND?

The antineutrino results from MiniBOONE may support the LSND result, but the neutrino results are consistent with the three active neutrino oscillation paradigm [3]. If LSND were correct then this could imply either sterile neutrinos and/or CPT violation, or something more exotic. For the remainder of this talk we shall focus on models based on three active neutrinos, which are well supported by atmospheric and solar neutrino experiments [4].

3.2. Dirac or Majorana?

Majorana neutrino masses are of the form $m^{\nu}_{LL}\overline{\nu_L}\nu^c_L$ where ν_L is a left-handed neutrino field and ν^c_L is the CP conjugate of a left-handed neutrino field, in other words a right-handed antineutrino field. Such Majorana masses are possible since both the neutrino and the antineutrino are electrically neutral. Such Majorana neutrino masses violate total lepton number L conservation, so the neutrino is equal to its own antiparticle. If we introduce right-handed neutrino fields then there are two sorts of additional neutrino mass terms that are possible. There are additional Majorana masses of the form $M^{\nu}_{RR}\overline{\nu_R}\nu^c_R$. In addition there are Dirac masses of the form $m^{\nu}_{LR}\overline{\nu_L}\nu_R$. Such Dirac mass terms conserve total lepton number L, but violate separate lepton numbers L_e, L_μ, L_τ. The question of "Dirac or Majorana?" is a key experimental question which could be decided by the experiments which measure neutrino masses directly.

3.3. Majorana neutrinos: see-saw mechanisms and R-parity violating SUSY

We have already remarked that neutrinos, being electrically neutral, allow the possibility of Majorana neutrino masses. However such masses are forbidden in the SM since neutrinos form part of a lepton doublet L, and the Higgs field also forms a doublet H, and $SU(2)_L \times U(1)_Y$ gauge invariance forbids a Yukawa interaction like HLL. So, if we want to obtain Majorana masses, we must go beyond the SM.

One possibility is to introduce Higgs triplets Δ such that a Yukawa interaction like ΔLL is allowed. However the limit from the SM ρ parameter implies that the Higgs triplet should have a VEV $\langle\Delta\rangle < 8$ GeV. One big advantage is that the Higgs triplets may be discovered at the LHC and so this mechanism of neutrino mass generation is directly testable [8].

Another possibility, originally suggested by Weinberg, is that neutrino Majorana masses originate from operators $HHLL$ involving two Higgs doublets and two lepton doublets, which, being higher order, must be suppressed by some large mass scale(s) M. When the Higgs doublets get their VEVs Majorana neutrino masses result: $m^{\nu}_{LL} = \lambda_\nu\langle H\rangle^2/M$. This is nice because the large Higgs VEV $\langle H\rangle \approx 175$ GeV can lead to small neutrino masses providing that the mass scale M is high enough. E.g. if M is equal to the GUT scale $1.75.10^{16}$ GeV then $m^{\nu}_{LL} = \lambda_\nu 1.75.10^{-3}$ eV. To obtain larger neutrino masses we need to reduce M below the GUT scale (since we cannot make λ_ν too large otherwise it becomes non-perturbative).

Typically in physics whenever we see a large mass scale M associated with a non-renormalizable operator we tend to associate it with tree level exchange of some heavy particle or particles of mass M in order to make the high energy theory renormalizable once again. This idea leads directly to the see-saw mechanism where the exchanged particles can either couple to HL, in which case they must be either fermionic singlets (right-handed neutrinos) or fermionic triplets, or they can couple to LL and HH, in which case they must be scalar triplets. These three possibilities have been called the type I [9], III [11] and II [10] see-saw mechanisms, respectively. If the coupling λ_ν is very small (for some reason) then M could even be lowered to the TeV scale and the see-saw scale could be probed at the LHC [12], however the see-saw mechanism then no longer solves the problem of the smallness of neutrino masses. Further types of see-saw mechanism have also been considered including the inverse see-saw mechanism [13] and the linear see-saw mechanism [14].

There are other ways to generate Majorana neutrino masses which lie outside of the above discussion. One possibility is to introduce additional Higgs singlets and triplets in such a way as to allow neutrino Majorana masses to be generated at either one [15] or two [16] loops. Another possibility is within the framework of R-parity violating Supersymmetry [17] in which the sneutrinos $\tilde{\nu}$ get small VEVs inducing a mixing between neutrinos and neutralinos χ leading to Majorana neutrino masses $m_{LL} \approx \langle\tilde{\nu}\rangle^2/M_\chi$, where for example $\langle\tilde{\nu}\rangle \approx$ MeV, $M_\chi \approx$ TeV leads to $m_{LL} \approx$ eV. A viable spectrum of neutrino masses and mixings can be achieved at the one loop level leading to distinctive predictions at the LHC such as [18]:

$$\tan^2\theta_{23} \approx \frac{BR(\chi^0_1 \to \mu W)}{BR(\chi^0_1 \to \tau W)}. \tag{1}$$

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3.4. Quasi-degenerate?

This key experimental question may be decided by the same experiments as will also determine the nature of neutrino mass (Dirac or Majorana). Although not a theorem, it seems that a hierarchical spectrum could indicate a type I see-saw mechanism, while a (quasi) degenerate spectrum could imply a type II see-saw mechanism. It is possible that a type II see-saw mechanism could naturally explain the degenerate mass scale with the degeneracy enforced by an $SO(3)$ family symmetry, while the type I see-saw part could be responsible for the small neutrino mass splittings and the (TB) mixing [19]. An A_4 model of quasi-degenerate neutrinos with TB mixing of this kind was considered recently in [20], leading to a rather precise relationship between the neutrinoless double beta decay observable m_{ee} and the lightest neutrino mass, as shown in Fig.8. Different such relationships are predicted by the linear and inverse see-saw mechanisms [21].

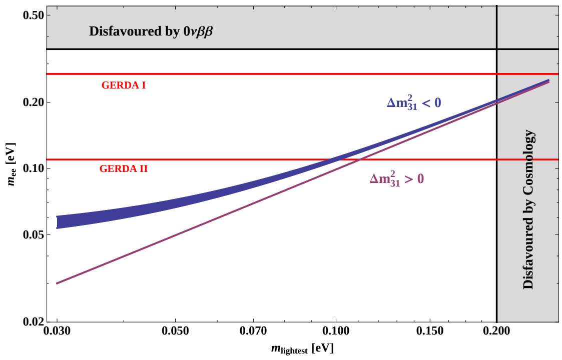

Figure 8. This plot shows the rather precise relationship between the neutrinoless double beta decay observable m_{ee} and the lightest neutrino mass, from the model in [20].

3.5. Very precise tri-bimaximal mixing?

It is a striking fact that current data on lepton mixing is (approximately) consistent with the so-called tri-bimaximal (TB) mixing pattern [22],

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} P_{Maj}, \tag{2}$$

where P_{Maj} is the diagonal phase matrix involving the two observable Majorana phases. However there is no convincing reason to expect exact TB mixing, and in general we expect deviations. These deviations can be parametrized by three parameters r, s, a defined as [23]:

$$\sin\theta_{13} = \frac{r}{\sqrt{2}}, \quad \sin\theta_{12} = \frac{1}{\sqrt{3}}(1+s), \quad \sin\theta_{23} = \frac{1}{\sqrt{2}}(1+a). \tag{3}$$

The global fits of the conventional mixing angles [1] can be translated into the 1σ ranges:

$$0.07 < r < 0.21, \quad -0.05 < s < 0.003, \quad -0.09 < a < 0.04. \tag{4}$$

Clearly a non-zero value of r, if confirmed, would rule out TB mixing. However it is possible to preserve the good predictions that $s = a = 0$, by postulating a modified form of mixing matrix called tri-bimaximal-reactor (TBR) mixing [24], where only r is allowed to be non-zero.

3.6. Family Symmetry
Let us expand the neutrino mass matrix in the diagonal charged lepton basis, assuming exact TB mixing, as $M^\nu_{TB} = U_{TB}\mathrm{diag}(m_1, m_2, m_3)U^T_{TB}$ leading to (absorbing the Majorana phases in m_i):

$$M^\nu_{TB} = m_1\Phi_1\Phi_1^T + m_2\Phi_2\Phi_2^T + m_3\Phi_3\Phi_3^T \quad (5)$$

where $\Phi_1^T = \frac{1}{\sqrt{6}}(2, -1, 1)$, $\Phi_2^T = \frac{1}{\sqrt{3}}(1, 1, -1)$, $\Phi_3^T = \frac{1}{\sqrt{2}}(0, 1, 1)$, are the respective columns of U_{TB} and m_i are the physical neutrino masses. In the neutrino flavour basis (i.e. diagonal charged lepton mass basis), it has been shown that the above TB neutrino mass matrix is invariant under S, U transformations:

$$M^\nu_{TB} = SM^\nu_{TB}S^T = UM^\nu_{TB}U^T \ . \quad (6)$$

A very straightforward argument [25] shows that this neutrino flavour symmetry group has only four elements corresponding to Klein's four-group $Z_2^S \times Z_2^U$. By contrast the diagonal charged lepton mass matrix (in this basis) satisfies a diagonal phase symmetry T. The matrices S, T, U form the generators of the group S_4 in the triplet representation, while the A_4 subgroup is generated by S, T. Some candidate family symmetries G_f are shown in Fig.9.

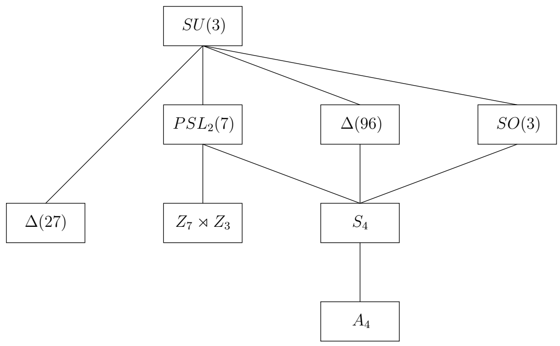

Figure 9. Some subgroups of $SU(3)$ that contain triplet representations and have been used as candidate family symmetries G_f.

3.7. Direct vs Indirect Models and Form Dominance
As discussed in [25], the flavour symmetry of the neutrino mass matrix may originate from two quite distinct classes of models. The first class of models, which we call direct models, are based on a family symmetry $G_f = S_4$, or a closely related family symmetry as discussed below, some of whose generators are directly preserved in the lepton sector and are manifested as part of the observed flavour symmetry. The second class of models, which we call indirect models, are based on some more general family symmetry G_f which is completely broken in the neutrino sector, while the observed neutrino flavour symmetry $Z_2^S \times Z_2^U$ in the neutrino flavour basis emerges as an accidental symmetry which is an indirect effect of the family symmetry G_f. In such indirect models the flavons responsible for the neutrino masses break G_f completely so that none of the generators of G_f survive in the observed flavour symmetry $Z_2^S \times Z_2^U$.

In the direct models, the symmetry of the neutrino mass matrix in the neutrino flavour basis (henceforth called the neutrino mass matrix for brevity) is a remnant of the $G_f = S_4$ symmetry of the Lagrangian, where the generators S, U are preserved in the neutrino sector, while the diagonal generator T is preserved in the charged lepton sector. For direct models, a larger family symmetry G_f which contains S_4 as a subgroup is also possible e.g. $G_f = PSL(2, 7)$ [26]. Typically direct models satisfy form dominance [27], and require flavon F-term vacuum alignment, permitting an $SU(5)$ type unification [28] typically based in A_4 family symmetry [29]. Such minimal A_4 models lead to neutrino mass sum rules between the three masses m_i, resulting in/from a simplified mass matrix in Eq.5. A_4 may result from 6D orbifold models [30] and recently an $A_4 \times SU(5)$ SUSY GUT model has been constructed in 6D [31], while a similar model in 8D enables vacuum alignment to be elegantly achieved by boundary conditions [32].

In the indirect models [25] the idea is that the three columns of $U_{TB}\ \Phi_i$ are promoted to new Higgs fields called "flavons" whose VEVs break the family symmetry, with the particular vacuum alignments along the directions Φ_i. In the indirect models the underlying family symmetry of the Lagrangian G_f is completely broken, and the flavour symmetry of the neutrino mass matrix $Z_2^S \times Z_2^U$ emerges entirely as an accidental symmetry, due to the presence of flavons with particular vacuum alignments proportional to the columns of U_{TB}, where such flavons only appear quadratically in effective Majorana Lagrangian [25]. Such vacuum alignments can be elegantly achieved using D-term vacuum alignment, which allows the large classes of discrete family symmetry G_f, namely the $\Delta(3n^2)$ and $\Delta(6n^2)$ groups [25]. The indirect models satisfy natural form dominance since each column of the Dirac mass matrix corresponds to a different flavon VEV. In the limit $m_1 \ll m_2 < m_3$ FD reduces to constrained sequential dominance (CSD)[33]. Examples of discrete symmetries used in the indirect approach can be found in [34].

Explicitly, the TB form of the neutrino mass matrix in Eq.5 is obtained from the see-saw mechanism in these models as follows. In the diagonal right-handed neutrino mass basis we may write $M^{\nu}_{RR} = \mathrm{diag}(M_A, M_B, M_C)$ and the Dirac mass matrix as $M^{\nu}_{LR} = (A, B, C)$ where A, B, C are three column vectors. Then the type I see-saw formula $M^{\nu} = M^{\nu}_{LR}(M^{\nu}_{RR})^{-1}(M^{\nu}_{LR})^T$ gives

$$M^{\nu} = \frac{AA^T}{M_A} + \frac{BB^T}{M_B} + \frac{CC^T}{M_C}. \tag{7}$$

By comparing Eq.7 to the TB form in Eq.5 it is clear that TB mixing will be achieved if $A \propto \Phi_3$, $B \propto \Phi_2$, $C \propto \Phi_1$, with each of $m_{3,2,1}$ originating from a particular right-handed neutrino of mass $M_{A,B,C}$, respectively. This mechanism allows a completely general neutrino mass spectrum and, since the resulting M^{ν} is form diagonalizable, it is referred to as form dominance (FD) [27]. For example, the direct A_4 see-saw models [28] satisfy FD [27], where each column corresponds to a linear combination of flavon VEVs.

A more natural possibility, called Natural FD, arises when each column arises from a separate flavon VEV, and this possibility corresponds to the case of indirect models. For example, if $m_1 \ll m_2 < m_3$ then the precise form of C becomes irrelevant, and in this case FD reduces to constrained sequential dominance (CSD)[33]. The CSD mechanism has been applied in this case to the class of indirect models with Natural FD based on the family symmetries $SO(3)$ [33, 36] and $SU(3)$ [35], and their discrete subgroups [34].

3.8. GUTs and leptogenesis

Finally we have reached the end of the decision tree, with the possibility of an all-encompassing unified theory of flavour based on GUTs and/or strings. Such theories could also include a family symmetry in order to account for the TB mixing. There are many possibilities for the choice of family symmetry and GUT symmetry. Examples include the Pati-Salam gauge group $SU(4)_{PS} \times SU(2)_L \times SU(2)_R$ in combination with $SU(3)$ [35], $SO(3)$ [33, 36], A_4 [37] or Δ_{27} [38]. Other examples are based on $SU(5)$ GUTs in combination with A_4 [39], T' [40] or S_4 [41].

In typical Family Symmetry $\otimes$ GUT models the origin of the quark mixing angles derives predominantly from the down quark sector, which in turn is closely related to the charged lepton sector. In order to reconcile the down quark and charged lepton masses, simple ansatze, such as the Georgi-Jarlskog hypothesis [42], lead to very simple approximate expectations for the charged lepton mixing angles such as $\theta^e_{12} \approx \lambda/3$, $\theta^e_{23} \approx \lambda^2$, $\theta^e_{13} \approx \lambda^3$, where $\lambda \approx 0.22$ is the Wolfenstein parameter from the quark mixing matrix. If the family symmetry enforces accurate TB mixing in the neutrino sector, then $\theta^e_{12} \approx \lambda/3$ charged lepton corrections will cause deviations from TB mixing in the physical lepton mixing angles, and lead to a sum rule relation [33, 43, 44], which can be conveniently expressed as [23] $s \approx r\cos\delta$ where $r \approx \lambda/3$ and δ is the observable CP violating oscillation phase, with RG corrections of less than one degree [45]. Such sum rules can be tested in future high precision neutrino oscillation experiments [46].

Note that in such a GUT-flavour framework, one expects the charged lepton corrections to the neutrino mixing angles to be less than of order $\theta^e_{12}/\sqrt{2}$ (where typically θ^e_{12} is a third of the Cabibbo angle) plus perhaps a further 1^o from renormalization group (RG) corrections. Thus such theoretical corrections cannot account for an observed reactor angle as large as 8^o, corresponding to $r = 0.2$, starting from the hypothesis of exact TB neutrino mixing.

Another interesting observation is that all models which satisfy form dominance will have zero leptogenesis [47]. This includes all models which account for tri-bimaximal mixing due to a discrete family symmetry. One way to obtain non-zero leptogenesis is to consider indirect models based on constrained sequential dominance, where the almost decoupled right-handed neutrino is the lightest one. The couplings of such a lightest right-handed neutrino are unconstrained and may allow successful leptogenesis [47].

4. Conclusion

Over the past dozen years there has been a revolution in our understanding of neutrino physics. Yet, despite this progress, it must be admitted that we still do not understand the origin or nature of neutrino mass and mixing. Neutrino mass and mixing clearly requires new physics beyond the SM, but in which direction should we go? There are many roads for model building, but we have seen that answers to key experimental questions will provide the sign posts *en route* to a unified theory of flavour. For example, if neutrinos have Majorana masses then this might signal some sort of see-saw mechanism at work. If these Majorana masses are sufficiently large then an indication might come soon from neutrinoless double beta decay experiments. In certain models such measurements would be directly related to the mass of the lightest physical neutrino mass.

One intriguing observation is that lepton mixing conforms to the tri-bimaximal pattern, at least approximately. This suggests an underlying non-Abelian discrete family symmetry that might unlock the long-standing flavour puzzle. However, even if tri-bimaximal mixing is accurately realised in the neutrino sector, due to a discrete family symmetry, when the model is embedded into a GUT theory, the charged lepton mixing corrections are typically related to quark mixing angles, leading to predicted deviations from tri-bimaximal mixing described by the sum rules $s \approx r\cos\delta$ where $r \approx \lambda/3$. Measurement of the parameters r, s, a which measure the deviation from tri-bimaximal mixing, will clearly be an important goal of the next generation of neutrino experiments.

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