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On a symplectic quantum Howe duality

Elijah Bodish[1] · **Daniel Tubbenhauer**[2]

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Abstract

We prove a nonsemisimple quantum version of Howe's duality with the rank 2n symplectic and the rank 2 special linear group acting on the exterior algebra of type C. We also discuss the first steps towards the symplectic analog of harmonic analysis on quantum spheres, give character formulas for various fundamental modules, and construct canonical bases of the exterior algebra.

Keywords Quantum Howe duality · Tilting modules · Weyl characters · Canonical bases · Symplectic groups

Mathematics Subject Classification Primary 17B37 · 20G42; Secondary 05E10 · 20G05 · 22E46

Contents

✉ Elijah Bodish
ebodish@mit.edu

Daniel Tubbenhauer
daniel.tubbenhauer@sydney.edu.au

1 Department of Mathematics, MIT, Building 2, Office 2-178, Cambridge, MA 02139, USA

2 School of Mathematics and Statistics F07, The University of Sydney, Office Carslaw 827, Sydney, NSW 2006, Australia

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1 Introduction

In this paper we prove a nonsemisimple and quantum version of one of Howe's dualities. We work mostly over $\mathbb{Z}[q, q^{-1}]$ and specializations to any field and any quantum parameter are allowed.

1A Howe dualities

A cornerstone of modern invariant theory are *Schur–Weyl–Brauer dualities*, which allow one to play two commuting actions on tensor space against one another. See for example [4, Section 3] for a summary, using the language of this paper.

Building on earlier work as e.g. in [25], these dualities were generalized far beyond tensor space by Howe in the legendary 1995 Schur lectures [26]. Howe's goal, which was achieved magnificently, was to present a new approach to classical invariant theory.

As Howe points out, classical invariant theory, historically speaking, was mostly about group actions on symmetric algebras or, less classically, on exterior algebras. Howe reformulated these actions in the *double centralizer* (or double commutant) approach of Schur–Weyl–Brauer dualities, and the result is what we call *Howe dualities*.

An example of a Howe duality (here formulated for the Lie algebra instead of the Lie group as Howe did in [26]) reads as follows. Consider the $\mathbb{C}$-vector space $\Lambda_{\mathbb{C}} := \Lambda_{\mathbb{C}}(\mathbb{C}^m \otimes \mathbb{C}^n)$, i.e. the exterior algebra of $\mathbb{C}^m \otimes \mathbb{C}^n$. Then:

(a) There are commuting actions

$$U_{\mathbb{C}}(\mathfrak{gl}_m) \circlearrowright \Lambda_{\mathbb{C}} \circlearrowleft U_{\mathbb{C}}(\mathfrak{gl}_n).$$

(b) Let $\phi_{\mathbb{C}}$ and $\psi_{\mathbb{C}}$ be the $\mathbb{C}$-algebra homomorphisms induced by the two actions from (a). Then:

$$\phi_{\mathbb{C}}\colon U_{\mathbb{C}}(\mathfrak{gl}_m) \twoheadrightarrow \mathrm{End}_{U_{\mathbb{C}}(\mathfrak{gl}_n)}(\Lambda_{\mathbb{A}}), \quad \psi_{\mathbb{C}}\colon U_{\mathbb{C}}(\mathfrak{gl}_n) \twoheadrightarrow \mathrm{End}_{U_{\mathbb{C}}(\mathfrak{gl}_m)}(\Lambda_{\mathbb{A}}).$$

That is, the two actions generate the others centralizer.

(c) There is an explicit $U_{\mathbb{C}}(\mathfrak{gl}_m) \otimes U_{\mathbb{C}}(\mathfrak{gl}_n)^{op}$-module decomposition of $\Lambda_{\mathbb{C}}$ pairing a Weyl $U_{\mathbb{C}}(\mathfrak{gl}_m)$-module and a dual Weyl $U_{\mathbb{C}}(\mathfrak{gl}_n)$-module.

This is known as *exterior type A Howe duality* and appears in [26, Theorem 4.1.1] as one of many examples considered by Howe.

Remark 1A.1 The more classical version is *symmetric type A Howe duality*, but the above is closer to the results in this paper, so we decide to prefer exterior over symmetric type A Howe duality. ◇

As Howe explains, when working over $\mathbb{C}$, the various forms of Howe dualities are equivalent to the respective Schur–Weyl–Brauer dualities. In particular, Howe-type dualities have been of crucial importance for invariant theory every since, but are also pervasive in other fields. Most notably, representation theory, low dimensional topology, and categorification.

Let us recall two generalizations of Howe's dualities.

Remark 1A.2 We are interested in nonsemisimple quantum versions of Howe dualities and, in order to not lose focus, we will ignore other possible generalizations like super versions. ◇

A first possible generalization is to study *nonsemisimple versions* of Howe's dualities, for example, working over the ground ring $\mathbb{Z}$. Note hereby that [26] stays over $\mathbb{C}$ which makes the story semisimple and character-type arguments apply as explained masterfully by Howe. In the nonsemisimple cases such character arguments are not as powerful as over $\mathbb{C}$, and one needs some form of replacement. While (a) is straightforward and characteristic independent, replacements for Howe's arguments showing (b) and (c) are needed.

As explained for Schur–Weyl(–Brauer) dualities in [20] and for Howe dualities in [1] the theory of tilting modules in the spirit of [39] and [16] usually plays a crucial role. In particular, [1] studies exterior versions of Howe dualities using the tilting module approach. Whenever tilting theory is not available, like for the symmetric invariants, the nonsemisimple study of invariant theory tends to be much trickier, see e.g. [15], and is less accessible via Howe's approach.

Remark 1A.3 As stated above, over $\mathbb{C}$ or, more generally, in the semisimple case, Schur–Weyl–Brauer dualities and Howe dualities are equivalent. This is no longer true in the nonsemisimple situation, and [1, 20] seem to have appeared independently. ◇

A second possible generalization are *quantum versions* involving some form of quantum groups instead of Lie groups. For starters, let us assume that we work over $\mathbb{C}(q)$ for a formal parameter q. In this case quantum Howe dualities are semisimple, so Howe's character-type arguments apply. In other words, (b) and (c) are rather easy to prove. However, (a) can be quite difficult.

Part of the problem is that quantum groups, and quantizations in general, often crucially depend on the involved choices of quantization. Sometimes one needs to work hard to get (a) for quantum groups instead of Lie groups, sometimes one even needs to give up on quantum groups altogether.

First examples of quantum versions of Howe's dualities appeared in the 90s, see e.g. the work [35], and indicated that nonstandard quantizations often play a role. See also [21, 42] for the same type of phenomena. Sometimes, however, Howe's dualities quantize nicely (meaning with the standard Drinfeld–Jimbo quantum groups). In type A, see for example [31] or [14] for exterior, [40] for symmetric, [46] for exterior-symmetric and [29] for Verma versions of nicely quantized Howe dualities. But again, proving (a) might still be tricky, in particular outside of type A, as e.g. in [43].

Combining both, the nonsemisimple and quantum setting, then needs new arguments (compared to [26]) for (a), (b) and (c). Modulo this paper and our ignorance, the only known nonsemisimple quantum Howe duality is the exterior type A Howe duality from above. In the exterior type A Howe duality case it turns out that (a), (b) and (c) are rather easy to prove, but that is a coincidence since the exterior powers of the vector representation in type A are tilting regardless of the characteristic. This is implicit already in [14, 31], and has then been pointed out explicitly, and masterfully used, in [22].

1B What this paper does

The starting point of this paper is to take the "easiest" of Howe's dualities outside of type A and to prove it in the nonsemisimple and quantum case.

In [26] there are four essentially equivalent dualities listed:

(A) We have SP_{2n} acting on $\Lambda\big((\mathbb{C}^{2n})^{\otimes m}\big)$ or O_n acting on $\mathrm{Sym}\big((\mathbb{C}^{n})^{\otimes m}\big)$ with the dual action by SP_{2m}.

(B) We have O_n acting on $\Lambda\big((\mathbb{C}^n)^{\otimes m}\big)$ or SP_{2n} acting on $\mathrm{Sym}\big((\mathbb{C}^{2n})^{\otimes m}\big)$ with the dual action by SO_{2m}.

Since we want nonsemisimple versions, thus tilting theory should come in handy, the exterior cases are certainly easier than the symmetric ones, so we stay with these. Moreover, [42] suggests that quantization might be an issue in general, but we have one special case where quantization is nice due to the small number coincidence that $SP_2 \cong SL_2$, see Remark 3B.7 for a detailed discussion.

It turns out that one is lucky and *one case remains*: SP_{2n} acting on $\Lambda((\mathbb{C}^{2n})^{\otimes 1})$ with the dual action by $SP_{2\cdot 1} \cong SL_2$. This is the content of the present paper.

That is, our main result, (a version of) *symplectic quantum Howe duality*, is as follows. Let $\mathbb{A} = \mathbb{Z}[q, q^{-1}]$ be the integral form where q is a variable, and let $\Lambda_{\mathbb{A}}$ the exterior algebra of the symplectic quantum vector representation. Then:

Theorem 1B.1 *Consider the free $\mathbb{A}$-module $\Lambda_{\mathbb{A}}$.*

(a) *There are commuting actions*

$$U_{\mathbb{A}}(\mathfrak{sp}_{2n}) \circlearrowright \Lambda_{\mathbb{A}} \circlearrowleft U_{\mathbb{A}}(\mathfrak{sl}_2)^{op}.$$

(b) *Let $\phi_{\mathbb{A}}$ and $\psi_{\mathbb{A}}$ be the $\mathbb{A}$-algebra homomorphisms induced by the two actions from (a). Then:*

$$\phi_{\mathbb{A}}\colon U_{\mathbb{A}}(\mathfrak{sp}_{2n}) \twoheadrightarrow \mathrm{End}_{U_{\mathbb{A}}(\mathfrak{sl}_2)^{op}}(\Lambda_{\mathbb{A}}), \quad \psi_{\mathbb{A}}\colon U_{\mathbb{A}}(\mathfrak{sl}_2) \twoheadrightarrow \mathrm{End}_{U_{\mathbb{A}}(\mathfrak{sp}_{2n})}(\Lambda_{\mathbb{A}}).$$

That is, the two actions generate the others centralizer.

(c) *Let $\boldsymbol{U}_{\mathbb{A}} := U_{\mathbb{A}}(\mathfrak{sp}_{2n}) \otimes U_{\mathbb{A}}(\mathfrak{sl}_2)$. Then $\Lambda_{\mathbb{A}}$ is a Howe tilting $\boldsymbol{U}_{\mathbb{A}}$-module (see Definition 4A.15 for the definition), that is, $\Lambda_{\mathbb{A}}$ is tilting for $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$ and $U_{\mathbb{A}}(\mathfrak{sl}_2)$, as well as full tilting for $\mathrm{End}_{U_{\mathbb{A}}(\mathfrak{sp}_{2n})}(\Lambda_{\mathbb{A}})$ and $\mathrm{End}_{U_{\mathbb{A}}(\mathfrak{sl}_2)}(\Lambda_{\mathbb{A}})$, both separately, and satisfies*

$$\begin{aligned} \mathrm{Hom}_{U_{\mathbb{A}}(\mathfrak{sp}_{2n})}\big(\Delta_{\mathbb{A}}(\varpi_k), \Lambda_{\mathbb{A}}\big) &\cong \nabla_{\mathbb{A}}(n-k), \\ \mathrm{Hom}_{U_{\mathbb{A}}(\mathfrak{sl}_2)}\big(\Delta_{\mathbb{A}}(n-k), \Lambda_{\mathbb{A}}\big) &\cong \nabla_{\mathbb{A}}(\varpi_k). \end{aligned} \tag{1B.2}$$

Moreover, in the Grothendieck ring

$$[\Lambda_{\mathbb{A}}] = \sum_{k=0}^{n} [\Delta_{\mathbb{A}}(\varpi_k) \otimes \nabla_{\mathbb{A}}(n-k)]. \tag{1B.3}$$

Here $\Delta_{\mathbb{A}}(\varpi_k)$ and $\nabla_{\mathbb{A}}(n-k)$ denote the Weyl $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-module and dual Weyl $U_{\mathbb{A}}(\mathfrak{sl}_2)$-module of the indicated highest weights.

Note that $U_{\mathbb{A}}(\mathfrak{sl}_2)$ is isomorphic to $U_{\mathbb{A}}(\mathfrak{sp}_2)$. Thus, Theorem 1B.1 is the nonsemisimple and quantum version of symplectic Howe duality as in [26, Theorem 3.8.9.3] in the special case $m = 1$.

Remark 1B.4 In the semisimple case (1B.2) implies the $\mathbf{U}_{\mathbb{A}}$-module decomposition

$$\Lambda_{\mathbb{A}} \cong \bigoplus_{k=0}^{n} \Delta_{\mathbb{A}}(\varpi_k) \otimes \nabla_{\mathbb{A}}(n-k),$$

and (1B.3) its combinatorial shadow. Thus, we can view Theorem 1B.1.(c) as a nonsemisimple analog of (c) of the respective Howe duality. ◇

Remark 1B.5 (a) The "op" makes an appearance in Theorem 1B.1 since we have two left actions, one of $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$ and one of $U_{\mathbb{A}}(\mathfrak{sl}_2)$, instead of a left and a right action.

(b) As we will elaborate in Remark 3B.7, it is rather surprising that [26, Theorem 3.8.9.3, $m = 1$] can be quantized as in Theorem 1B.1, namely involving two standard quantum groups.
(c) The nonsemisimple but non-quantum versions of the exterior cases in (A) and (B) above are discussed in [1], see e.g. [1, Theorem 2.1].
(d) Although formulated differently, the result [43, Theorem 3.4] can be interpreted as the orthogonal version of Theorem 1B.1 corresponding to (A) for $m = 1$ above. Note that [43] works over $\mathbb{Q}(q^{1/2})$ for a variable q, i.e. the semisimple case, which is somewhat expected due to the appearance of the symmetric algebra. (The fraction power of q is related to the usage of the braiding in [43] and does not play any essential role.) Note also that [43] has a nice quantization, again due to the small number coincidence $SP_2 \cong SL_2$.

We will elaborate on these points in the main text. ◇

Our proof of Theorem 1B.1 splits into two parts: we first prove the semisimple version and then go to the nonsemisimple case. Along the way we sketch the beginning of the symplectic analog of harmonic analysis on quantum spheres, prove some consequences of our Howe duality for the characters and various multiplicities of fundamental simple, Weyl or tilting modules in type C, and we construct canonical bases of $\Lambda_{\mathbb{A}}$.

1C Further directions

Here are a few ideas to continue the story:

(a) As we explain in more details in Remark 3B.7, our quantization results can be seen as a "dual" to [42] for $m = 1$. A version for $m > 1$ could give a Howe duality approach to quantum BCD web categories as in e.g. [10] or their various applications to tilting modules as e.g. in [9]. These webs could then be compared with the $q = 1$ version of [42].
(b) Some of the most important modules in this paper are simple, Weyl and tilting modules for fundamental weights of the symplectic group. Not much is known about these modules for the nonsemisimple quantum case. Our results, particularly those in Sect. 7, already give the Weyl character of the fundamental tilting modules; in particular, we can tell when a Weyl module is simple. These results might help to generalize some of the known facts about these modules such as those that can be found in, for example, [23, 33, 38] or [24].
(c) The results of Sect. 8 suggest the existence of a categorification of our story.

Remark 1C.1 Some calculations in this paper were done using code. If the reader wants to run that code themselves, then they can find it here: https://github.com/dtubbenhauer/sp2n-fundamental-tilting-charaters https://github.com/dtubbenhauer/symplectic-diff-operators. ◇

2 The module and the two commuting actions

Throughout, and unless stated otherwise, e.g. in specific examples, we fix $n \in \mathbb{Z}_{\geq 1}$.

Remark 2.1 All colors in this paper are a visual aid only. ◇

2A Quantum exterior algebras of type C

We start by stating our quantum notation:

Notation 2A.1 For a parameter q let $\mathbb{A} := \mathbb{Z}[q, q^{-1}]$ and $\mathbb{F} := \mathrm{frac}(\mathbb{A}) = \mathbb{Q}(q)$. Working with $\mathbb{A}$ as the ground ring means working *integrally*, working with $\mathbb{F}$ makes the story *semisimple*. The specializations $\mathbb{A} \to \mathbb{C}$ given by $q \mapsto 1$ or $q \mapsto -1$ are called the *classical case*.

In general:

(a) We use the subscript $\mathbb{A}$ to denote integral versions of modules, divided power versions of algebras, or, more generally, any notion that involves $\mathbb{A}$.
(b) We use the subscript q to denote the $\mathbb{F}$ versions of modules and quantum groups, or, more generally, any notion that involves $\mathbb{F}$.
(c) Similarly, we use the subscript 1 to denote the classical counterparts. (The difference between 1 and -1 does not play an essential role for us.)
(d) We will later use arbitrary specializations which are denoted by $\mathbb{K}$.
(e) In general, we often only define $\mathbb{A}$-versions and their variants over other rings can be obtained by scalar extension, or scalar extension and specialization.

For $k \in \mathbb{Z}_{\geq 0}$ let

$$[k]_{\mathbb{A}} := (q^k - q^{-k})/(q - q^{-1}) = q^{-k+1} + q^{-k+3} + \dots + q^{k-3} + q^{k-1},$$

denote the usual quantum numbers. For $k \in \mathbb{Z}_{<0}$ let $[k]_{\mathbb{A}} = -[-k]_{\mathbb{A}}$. For $k \in \mathbb{Z}$ and $l \in \mathbb{Z}_{\geq 0}$ also define

$$[l]_{\mathbb{A}}! := [l]_{\mathbb{A}}[l-1]_{\mathbb{A}} \dots [1]_{\mathbb{A}}, \quad \begin{bmatrix} k \\ l \end{bmatrix}_{\mathbb{A}} := \frac{[k]_{\mathbb{A}}[k-1]_{\mathbb{A}} \dots [k-l+1]_{\mathbb{A}}}{[l]_{\mathbb{A}}!}.$$

We let $q_1 = \dots = q_{n-1} = q$ and $q_n = q^2$. As usual, we can then define the quantum numbers (and factorials and binomials etc.) in q_i instead of q and indicate this using a subscript i. ◇

Notation 2A.2 Using interval notation, we denote by

$$[1, -1] := \{1 < 2 < \dots < n < -n < \dots < -2 < -1\}$$

the set that we order as indicated. We will also use e.g. $\leq$ on this set, having the evident meaning. Moreover, for $i, j \in [1, -1]$ with $i \leq j$ we set

$$[i, j] := \{z \in [1, -1] \mid i \leq z \leq j\}.$$

Note that we will often write $i \in [1, n]$ and then index elements by $-i$ which, up to the order, is the same as $i \in [-n, -1]$ and using the index i. ◇

The following is the main object under study in this paper:

Definition 2A.3 Let $\Lambda_{\mathbb{A}} = \Lambda_{\mathbb{A}}(V_{\mathbb{A}})$ (the notation $\Lambda_{\mathbb{A}}(V_{\mathbb{A}})$ will be clear later) be the $\mathbb{A}$-algebra generated by elements

$$v_1, v_2, \dots, v_n, v_{-n}, \dots, v_{-2}, v_{-1},$$

modulo the relations

$$v_i^2 = 0, i \in [1, -1], \quad v_j v_i = -q \cdot v_i v_j \text{ and } v_{-i} v_{-j}$$
$$= -q \cdot v_{-j} v_{-i} \text{ for } i, j \in [1, n], i < j, v_{-i} v_j = -q \cdot v_j v_{-i} \text{ for } i, j \in [1, n], i \neq j, v_{-i} v_i$$
$$= -q^2 \cdot v_i v_{-i} + (q - q^{-1}) \sum_{k \in [1, n-i]} (-q)^{k+1} \cdot v_{i+k} v_{-(i+k)} \text{ for } i \in [1, n].$$

We call $\Lambda_{\mathbb{A}}$ the *quantum exterior algebra of type C*. ◇

Remark 2A.4 Note that the final relation in Definition 2A.3 is very different for the classical counterpart of $\Lambda_{\mathbb{A}}$ due to the factor $(q - q^{-1})$. ◇

For $k \in \mathbb{Z}_{\geq 0}$ let further $\Lambda^k_{\mathbb{A}}$ denote the $\mathbb{A}$-span of all monomials in k symbols.

Lemma 2A.5 *The $\mathbb{A}$-algebra $\Lambda_{\mathbb{A}}$ is $\mathbb{Z}_{\geq 0}$-graded, and the set $\Lambda^k_{\mathbb{A}}$ is the degree k part of $\Lambda_{\mathbb{A}}$.*

Proof The relations are homogeneous with respect to the number of appearing monomials. □

Note that the first three relations in Definition 2A.3 are saying that $\{v_i | i \in [1, n]\}$ and $\{v_i | i \in [-n, -1]\}$ span *quantum exterior algebras of type A* defined as in e.g. [7].

Remark 2A.6 The $\mathbb{A}$-algebra $\Lambda_{\mathbb{A}}$ morally is a special case of the quantum exterior algebras considered in [7] where the underlying module is the vector representation of $\mathfrak{sp}_{2n}$. By the exterior version of [48, Corollary 4.26] the $\mathbb{A}$-algebra $\Lambda_{\mathbb{A}}$ is therefore flat, i.e. $\Lambda_{\mathbb{A}}$ is free and has the same rank as the dimension of its classical counterpart. We will prove this result in our setting in Proposition 2A.9 below. The symmetric version of the $\mathbb{A}$-algebra $\Lambda_{\mathbb{A}}$ also appears in [31, Section 5], and this is where we got the idea for the generators and relations of $\Lambda_{\mathbb{A}}$.

The following is an $\mathbb{A}$-basis of $\Lambda_{\mathbb{A}}$:

Definition 2A.7 For a(n ordered) subset $S \subset [1, -1]$ with $S = \{x_1 < x_2 < \cdots < x_{|S|}\}$ we define *standard basis vectors* as

$$v_S := v_{x_1} v_{x_2} \cdots v_{x_{|S|}} \in \Lambda_{\mathbb{A}}.$$

For $i \in \{1, \ldots, n\}$ let ϵ_i be the standard basis vector of the n dimensional Euclidean space. We define the *weight* wt on these $\mathrm{wt}\, v_i = \epsilon_i$, $\mathrm{wt}\, v_{-i} = -\epsilon_i$, and $\mathrm{wt}\, v_S = \sum_{x \in S} \mathrm{wt}\, v_x$. ◇

Example 2A.8 (a) For $n = 1$ the generating relations of $\Lambda_{\mathbb{A}}$ are

$$v_1^2 = v_{-1}^2 = 0, \quad v_{-1}v_1 = q^2 \cdot v_1 v_{-1}.$$

We have $[1, -1] = 1 < -1$, and the ordered subsets $S \subset [1, -1]$ are $S_\emptyset = \emptyset$, $S_1 = \{1\}$, $S_{-1} = \{-1\}$ and $S_{1,-1} = \{1 < -1\}$. The $\mathbb{A}$-algebra $\Lambda_{\mathbb{A}}$ is free of rank four and

$$\{v_{S_\emptyset} = 1, v_{S_{\{1\}}} = v_1, v_{S_{\{-1\}}} = v_{-1}, v_{S_{\{1,-1\}}} = v_1 v_{-1}, \}$$

is a $\mathbb{A}$-basis of $\Lambda_{\mathbb{A}}$. The total rank, ordered by degree, is $1 + 2 + 1 = 4 = 2^{2 \cdot 1}$.

(b) For $n = 2$ the most complicated relation is

$$v_{-1}v_1 = q^2 \cdot v_1 v_{-1} + (q - q^{-1})(-q)^2 \cdot v_2 v_{-2}.$$

We have sixteen subsets $S \subset [1, -1]$ and the standard basis of $\Lambda_{\mathbb{A}}$ is

$$\{v_{S_\emptyset} = 1, v_{S_{\{1\}}} = v_1, \ldots, v_{S_{\{1,2,-1\}}} = v_1 v_2 v_{-1}, v_{S_{\{1,2,-2,-1\}}} = v_1 v_2 v_{-2} v_{-1}\}.$$

Note that for $n = 1$ we have that $\mathfrak{sp}_{2 \cdot 1}$ and $\mathfrak{sl}_2$ are isomorphic, and this is reflected in $\Lambda_{\mathbb{A}}$ as the most complicated relations do not play any role. ◇

The standard basis vectors form what we call the *standard basis* of $\Lambda_{\mathbb{A}}$ given by:

Proposition 2A.9 *We have the following.*

(a) *For all* $k \in \{0, 1, \dots, 2n\}$, *the sets*

$$\{v_S | S \subset [1, -1]\}, \quad \{v_S | S \subset [1, -1], |S| = k\}$$

are $\mathbb{A}$*-basis for* $\Lambda_{\mathbb{A}}$ *and* $\Lambda^k_{\mathbb{A}}$*, respectively.*

(b) $\Lambda_{\mathbb{A}}$ *is flat, i.e.* $\mathrm{rk}_{\mathbb{A}} \Lambda^k_{\mathbb{A}} = \dim_{\mathbb{C}} \Lambda^k_1 = \binom{2n}{k}$ *and thus,* $\mathrm{rk}_{\mathbb{A}} \Lambda_{\mathbb{A}} = \dim_{\mathbb{C}} \Lambda_1 = 2^{2n}$.

Proof *(a).* As it happens quite often in this context, see e.g. [43, Proof of Proposition 2.2] for a closely related usage, the result is a consequence of Bergman's diamond lemma [8, Theorem 1.2].

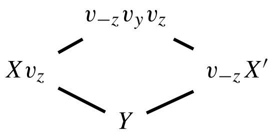
Precisely, one easily checks that, if $v_1 < \cdots < v_n < v_{-n} < \cdots < v_{-1}$, then the lexicographic order on monomials satisfies the hypotheses of the diamond lemma, and the irreducible monomials are exactly the v_S such that $S \subset [1, -1]$.

It remains to verify that one can resolve all the following ambiguities coming from the defining relations:

$$v_x v_x v_x, \quad v_{-x} v_{-x} v_{-x}, \quad v_j v_j v_i, \quad v_{-x} v_{-x} v_y, \quad v_{-i} v_{-i} v_{-j}, \quad v_{-x} v_{-x} v_x, \quad v_j v_i v_i,$$
$$v_j v_i v_k, v_{-x} v_y v_y, \quad v_{-x} v_y v_z, \quad v_{-z} v_y v_z, \quad v_{-k} v_{-i} v_{-j}, \quad v_{-z} v_{-y} v_y, \quad v_{-x} v_x v_x, \quad v_{-j} v_j v_i,$$

where $x, y, z \in [1, n]$, such that $z < y$ and $y \neq x \neq z$, and $i, j, k \in [1, n]$, such that $k < i < j$. We give one example, showing that $(v_{-z} v_y) v_z = v_{-z}(v_y v_z)$, and leave the rest of the verification as an exercise.

If X denotes a replacement of $v_{-z} v_y$ and X' one of $v_y v_z$, then we need to show that Y in



is the same for both ways to go around the diamond. To this end, the first calculation is:

$$\begin{aligned}(v_{-z} v_y) v_z &= -q \cdot v_y v_{-z} v_z = q^3 \cdot v_y v_z v_{-z} + (q - q^{-1}) \sum_{r=1}^{n-z} (-q)^{r+2} \cdot v_y v_{z+r} v_{-z-r} \\ &= -q^4 \cdot v_z v_y v_{-z} + (q - q^{-1}) \sum_{r=1}^{y-z-1} (-q)^{r+3} \cdot v_{z+r} v_y v_{-z-r} \\ &\quad + (q - q^{-1}) \sum_{r=y-z+1}^{n-z} (-q)^{r+2} \cdot v_y v_{z+r} v_{-z-r}.\end{aligned}$$

The second is:

$$\begin{aligned}v_{-z}(v_y v_z) &= -q \cdot v_{-z} v_z v_y = q^3 \cdot v_z v_{-z} v_y \\ &\quad + (q - q^{-1}) \sum_{r=1}^{n-z} (-q)^{r+2} \cdot v_{z+r} v_{-z-r} v_y = -q^4 \cdot v_z v_y v_{-z} \\ &\quad + (q - q^{-1}) \Bigg(\sum_{\substack{1 \leq r \leq n-z \\ z+r \neq y}} (-q)^{r+3} \cdot v_{z+r} v_y v_{-z-r} + (-q)^{y-z+2} \cdot v_y v_{-y} v_y \Bigg) \\ &= -q^4 \cdot v_z v_y v_{-z} + (q - q^{-1}) \Bigg(\sum_{r=1}^{y-z-1} (-q)^{r+3} \cdot v_{z+r} v_y v_{-z-r}\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{r=y-z+1}^{n-z} (-q)^{r+4} \cdot v_y v_{z+r} v_{-z-r} \Bigg) \\
&+(-q)^{y-z+2}(q-q^{-1})^2 \sum_{s=1}^{n-y} (-q)^{s+1} \cdot v_y v_{y+s} v_{-y-s} = -q^4 \cdot v_z v_y v_{-z} \\
&+(q-q^{-1}) \Bigg(\sum_{r=1}^{y-z-1} (-q)^{r+3} \cdot v_{z+r} v_y v_{-z-r} \\
&+ \sum_{r=y-z+1}^{n-z} (-q)^{r+4} \cdot v_y v_{z+r} v_{-z-r} \Bigg) \\
&+(q-q^{-1})^2 \sum_{r=y-z+1}^{n-z} (-q)^{z+r-y+1} (-q)^{y-z+2} \cdot v_y v_{z+r} v_{-z-r} \\
= {}& -q^4 v_z v_y v_{-z} + (q-q^{-1}) \\
&\Bigg(\sum_{r=1}^{y-z-1} (-q)^{r+3} \cdot v_{z+r} v_y v_{-z-r} + \sum_{r=y-z+1}^{n-z} \Big((-q)^{r+4} \\
&+(-q)^{r+3}(q-q^{-1}) \Big) \cdot v_y v_{z+r} v_{-z-r} \Bigg) \\
= {}& -q^4 v_z v_y v_{-z} + (q-q^{-1}) \Bigg(\sum_{r=1}^{y-z-1} (-q)^{r+3} \cdot v_{z+r} v_y v_{-z-r} \\
&+ \sum_{r=y-z+1}^{n-z} (-q)^{r+2} \cdot v_y v_{z+r} v_{-z-r} \Bigg).
\end{aligned}
$$

Hence, we find that the ambiguity $v_{-z} v_y v_z$ is resolvable.

(b). Directly from (a) since the classical $q = 1$ version has the claimed dimensions. □

Note that, by Proposition 2A.9, $\Lambda^k_{\mathbb{A}}$ is zero unless $k \in \{0, \ldots, 2n\}$, so we will only consider those degrees.

2B The symplectic action

The $\mathbb{A}$-algebra $\Lambda_{\mathbb{A}}$ is actually a module of the associated quantum group (we explain this first in the semisimple case only). Before we come to the definition we need the following root conventions:

Notation 2B.1 All the below are the conventions from [11, Plate III].

The Dynkin diagram that we will use is  where we have n nodes labeled, from left to right, 1 to n, and the symmetrizer is $(1, \ldots, 1, 2)$.

Let $X := \oplus_{i=1}^n \mathbb{Z}\epsilon_i$ (the ϵ_i are as in Definition 2A.7) be the weight lattice for $\mathfrak{sp}_{2n}$, and write $\Phi_{C_n} \subset X$ for the root system. We fix the following choice of simple roots:

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, \text{ for } i \in \{1, \ldots, n-1\}, \text{ and } \alpha_n = 2\epsilon_n.$$

The set of reflections through simple roots $s_i := s_{\alpha_i}$ generates the Weyl group W_{C_n}, which acts on X. The form on X is determined by $(\epsilon_i, \epsilon_j) = \delta_{i,j}$ and is W-invariant.

Furthermore, we have associated coroots using $\alpha^\vee := 2\alpha/(\alpha,\alpha)$, Cartan integers using $a_{ij} := (\alpha_i^\vee, \alpha_j)$, and fundamental weights by $\varpi_i := \epsilon_1 + \cdots + \epsilon_i$ for $i \in \{1,\ldots,n\}$. ◇

We use the following conventions for quantum groups over $\mathbb{F}$. Integral versions will be recalled later in Sect. 4.

The *quantum group* $U_q(\mathfrak{sp}_{2n})$ is defined as the $\mathbb{F}$-algebra generated by elements e_i, f_i, and $k_i^{\pm 1}$, for $i \in \{1,\ldots,n\}$, called the ($\mathfrak{sp}_{2n}$) *Chevalley generators*. We take everything modulo the relations that k_i is the two-sided inverse of k_i^{-1}, the k_i commute among each other and:

$$\begin{aligned} k_i e_j &= q^{(\alpha_i,\alpha_j)} \cdot e_j k_i, \\ k_i f_j &= q^{-(\alpha_i,\alpha_j)} \cdot f_j k_i, \\ &\quad e_i f_j - f_j e_i \\ &= \delta_{i,j} \cdot \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \\ &\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q,i} \cdot e^{1-a_{ij}-s} e_j e_i^s = 0, \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q,i} \cdot f^{1-a_{ij}-s} f_j f_i^s = 0. \end{aligned}$$

We choose a Hopf structure on $U_q(\mathfrak{sp}_{2n})$ by demanding that the k_i are group-like and:

$$\begin{aligned} \Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i, \\ S(e_i) = -k_i^{-1} e_i, \quad S(f_i) = -f_i k_i, \quad \epsilon(e_i) = 0, \quad \epsilon(f_i) = 0. \end{aligned} \tag{2B.2}$$

One can check that (2B.2) defines the structure of a Hopf algebra on $U_q(\mathfrak{sp}_{2n})$. In fact, this convention is in line with [27].

Recall that there is a $\mathbb{F}$-algebra automorphism, the *Chevalley involution*, determined by

$$\omega\colon U_q(\mathfrak{sp}_{2n}) \to U_q(\mathfrak{sp}_{2n}), \quad \omega(e_i) = f_i, \omega(f_i) = e_i, \omega(k_i^{\pm 1}) = k_i^{\mp 1}.$$

This gives:

Definition 2B.3 Given a $U_q(\mathfrak{sp}_{2n})$-module (V, ρ_V) with the indicated action ρ_V, we define the *ω-twist* of V, denoted ${}^{\omega}V$, as the $U_q(\mathfrak{sp}_{2n})$-module $(V, \rho_V \circ \omega)$. ◇

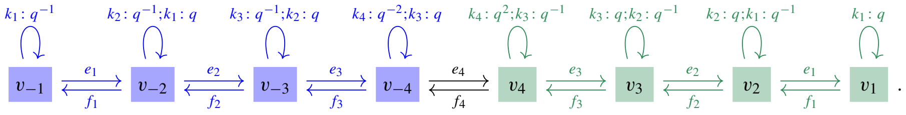
Definition 2B.4 The *vector representation* V_q of $U_q(\mathfrak{sp}_{2n})$ is the $\mathbb{F}$-vector space with basis $\{v_1, v_2, \ldots, v_n, v_{-n}, \ldots, v_{-2}, v_{-1}\}$ and $U_q(\mathfrak{sp}_{2n})$-action given by

$$\begin{gathered} k_i \centerdot v_{\pm j} = q_i^{\pm\delta_{i,\pm j}} \cdot v_{\pm j},\ j \in [-n,-1], \\ i \neq n\colon \begin{cases} e_i \centerdot v_{i+1} = v_i, \\ e_i \centerdot v_{-i} = v_{-(i+1)}, \\ e_i \centerdot v_j = 0 \text{ if } j \notin \{-i, i+1\}, \end{cases} \quad \begin{cases} f_i \centerdot v_i = v_{i+1}, \\ f_i \centerdot v_{-(i+1)} = v_{-i}, \\ f_i \centerdot v_j = 0 \text{ if } j \notin \{i, -(i+1)\}, \end{cases} \\ \begin{cases} e_n \centerdot v_{-n} = v_n, \\ e_n \centerdot v_j = 0 \text{ else}, \end{cases} \quad \begin{cases} f_n \centerdot v_n = v_{-n}, \\ f_n \centerdot v_j = 0 \text{ else}. \end{cases} \end{gathered} \tag{2B.5}$$

We call the $\mathbb{F}$-span of $\{v_1, v_2, \ldots, v_n\}$ and of $\{v_{-n}, \ldots, v_{-2}, v_{-1}\}$ *vector representations of type A* and its *dual*. ◇

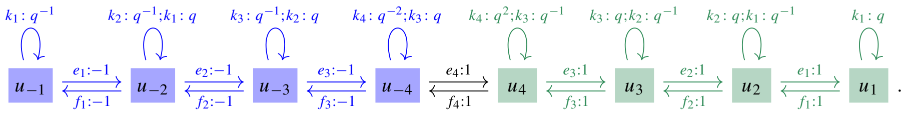
Remark 2B.6 The $U_q(\mathfrak{sp}_{2n})$-modules such that K_i acts on μ weight vectors as $q^{(\alpha_i^\vee,\mu)}$ are called *type-*1 *modules*. We will only consider $U_q(\mathfrak{sp}_{2n})$-modules of this form, see [27, Section 5.2] for background and why this restriction is harmless. ◇

Example 2B.7 Let us as usual denote the action of e_i, f_i and k_i as labeled edges, with the convention that not displayed arrows mean trivial actions.



The colors illustrate the underlying vector representations of type A. The Chevalley generators e_i and f_i always act by one, while the scalar for k_i is illustrated. ◇

Remark 2B.8 Scaling the standard basis vectors by $v_{-n+1} \mapsto -u_{-n+1}$, $v_{-n+3} \mapsto -u_{-n+3}$ all the way to $v_{-1} \mapsto -u_{-1}$ or $v_{-2} \mapsto -u_{-2}$, depending on parity, and $v_i \mapsto u_i$ otherwise, we see that V_q is isomorphic to the $U_q(\mathfrak{sp}_{2n})$-module illustrated by:



This matches the conventions one often sees in the literature, as for example in [31, Section 5], and is be a bit better to identify with the classical case. ◇

Lemma 2B.9 *There is an $U_q(\mathfrak{sp}_{2n})$-action on Λ_q determined by (2B.5) and:*

$$
\begin{aligned}
k_i \centerdot v_I &= q^{(\alpha_i, \mathrm{wt}\, I)} \cdot v_i, \quad e_i \centerdot v_{x_1} v_{x_2} \dots v_{x_m} \\
&= \sum_{j=1}^{m} q^{(\alpha_i, \mathrm{wt}\, v_1 \dots v_{j-1})} \cdot v_{x_1} \dots (e_i \centerdot v_{x_j}) \dots v_{x_m}, \; f_i \centerdot v_{x_1} v_{x_2} \dots v_{x_m} \\
&= \sum_{j=1}^{m} q^{(\alpha_i, \mathrm{wt}\, v_{x_{j+1}} \dots v_{x_m})} \cdot v_{x_1} \dots (f_i \centerdot v_{x_j}) \dots v_{x_m}.
\end{aligned}
$$

Proof We have that Λ_q is the tensor algebra in V_q modulo the symmetric algebra, that is $\Lambda_q = \Lambda_q(V_q)$. Thus, the formulas above can be obtained from the coproduct and the claim follows. □

For $S \subset [1, -1]$ we let $S_{i,i+1} := S \cap \{\pm i, \pm(i+1)\}$ and $S_n := S \cap \{\pm n\}$. We need these as these are the only places where the ith Chevalley generators act nontrivially, see (2B.5).

We can now explicitly describe the $U_q(\mathfrak{sp}_{2n})$-action from Lemma 2B.9 on the standard basis:

Lemma 2B.10 *For $i \in \{1, \dots, n-1\}$ and $i = n$ we have:*

$$
f_i \centerdot v_S = \begin{cases}
v_{(S \setminus \{i\}) \cup \{i+1\}} & \textit{if } S_{i,i+1} = \{i\}, \\
v_{(S \setminus \{-(i+1)\}) \cup \{-i\}} & \textit{if } S_{i,i+1} = \{-(i+1)\}, \\
v_{(S \setminus \{-(i+1)\}) \cup \{-i\}} + q^{-1} \cdot v_{(S \setminus \{i\}) \cup \{i+1\}} & \textit{if } S_{i,i+1} = \{i, -(i+1)\}, \\
q \cdot v_{(S \setminus \{i\}) \cup \{i+1\}} & \textit{if } S_{i,i+1} = \{i, -i\}, \\
v_{(S \setminus \{-(i+1)\}) \cup \{-i\}} & \textit{if } S_{i,i+1} = \{i+1, -(i+1)\}, \\
v_{(S \setminus \{i\}) \cup \{i+1\}} & \textit{if } S_{i,i+1} = \{i, -(i+1), -i\}, \\
v_{(S \setminus \{-(i+1)\}) \cup \{-i\}} & \textit{if } S_{i,i+1} = \{i, (i+1), -(i+1)\}, \\
0 & \textit{otherwise},
\end{cases}
$$

$$f_n \centerdot v_S = \begin{cases} v_{(S\setminus\{n\})\cup\{-n\}} & \textit{if } S_n = \{n\}, \\ 0 & \textit{otherwise}. \end{cases}$$

$$e_i \centerdot v_S = \begin{cases} v_{(S\setminus\{-i\})\}\cup\{-(i+1)\}} & \textit{if } S_{i,i+1} = \{-i\}, \\ v_{(S\setminus\{i+1\})\cup\{i\}} & \textit{if } S_{i,i+1} = \{i+1\}, \\ v_{(S\setminus\{i+1\})\cup\{i\}} + q^{-1} \cdot v_{S\setminus\{-i\})\cup\{-(i+1)\}} & \textit{if } S_{i,i+1} = \{i+1, -i\}, \\ q \cdot v_{(S\setminus\{-i\})\cup\{-(i+1)\}} & \textit{if } S_{i,i+1} = \{i, -i\}, \\ v_{(S\setminus\{i+1\})\cup\{i\}} & \textit{if } S_{i,i+1} = \{i+1, -(i+1)\}, \\ v_{(S\setminus\{-i\})\cup\{-(i+1)\}} & \textit{if } S_{i,i+1} = \{i, i+1, -i\}, \\ v_{(S\setminus\{i+1\})\cup\{i\}} & \textit{if } S_{i,i+1} = \{i+1, -(i+1), -i\}, \\ 0 & \textit{otherwise}, \end{cases}$$

$$e_n \centerdot v_S = \begin{cases} v_{(S\setminus\{-n\})\cup\{n\}} & \textit{if } S_n = \{-n\}, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof An annoying but straightforward calculation, so we only give one prototypical example. Say $S_{i,i+1} = \{i, -i\}$. Since $f_i \centerdot v_j$ is zero unless $j = i$ or $j = -(i+1)$, we immediately see that only one summand of $f_i \centerdot v_{x_1} v_{x_2} \dots v_{x_m}$ can survive, namely the one where f_i hits v_i. The vector v_i is changed to v_{i+1} and thus, S changes to $S \setminus \{i\} \cup \{i+1\}$. Moreover, due to our choice of the coproduct k_i^{-1} is involved as well and gives the q-power. □

Remark 2B.11 Note that the formula for the action of e_i is related to the formula for the action of f_i by "multiplying $\{i, i+1, -(i+1), -i\}$ by -1". In what follows we will often just present the formulas for the action of f_i, noting that the formula for e_i is similar. ◇

Lemma 2B.12 *The map $v_S \mapsto v_{-S}$ defines a $U_q(\mathfrak{sp}_{2n})$-module isomorphisms $\Lambda_q^k \to {}^{\omega}\Lambda_q^k$.*

Proof Note that $|S| = |-S|$, so the $\mathbb{F}$-linear map determined by $v_S \mapsto v_{-S}$ is an isomorphism $\Lambda^k \to \Lambda^k$. Since $\mathrm{wt}\, v_{-S} = \mathrm{wt}\, v_S$, the map clearly commutes with the actions of $k_i^{\pm 1}$, for $i \in \{1, \dots, n\}$. The lemma then follows from analyzing the descriptions of actions of e_i and f_i, for $i \in \{1, \dots, n\}$ in Lemma 2B.10. □

Remark 2B.13 By Proposition 2A.9, the map $v_S \mapsto v_{-S}$ also determines an isomorphism of $\mathbb{A}$-modules $\Lambda_{\mathbb{A}}^k \to \Lambda_{\mathbb{A}}^k$. ◇

The same as in Lemma 2B.12, of course, also holds for the whole space Λ_q.

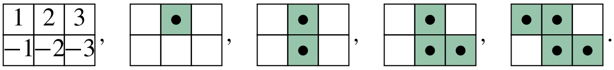
2C Dot diagrams

We now rephrase the previous results using certain decorated Young diagrams that we call *dot diagrams*. In this paper all Young diagrams are illustrated in the English convention, and we have two rows and n columns:

Definition 2C.1 A *dot diagram* is a two row rectangular Young diagram with $2n$ nodes and these nodes can be decorated with zero or one dot.

We view the nodes in the first row as numbered, from left to right, 1 to n, and in the second row -1 to $-n$. We number the columns 1 to n reading left to right, so that the nodes in the ith column contain the numbers i and $-i$. ◇

Example 2C.2 Here are a few examples of dot diagrams. The leftmost diagram illustrates the numbering convention:



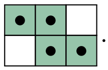
In these examples $n = 3$, which is the number of columns. The color shading is just a visual aid and has no other relevance. ◇

Remark 2C.3 Various versions of dot diagrams have been used in several papers dealing with symplectic groups. We got the idea to use dot diagrams for illustrations from [18]. ◇

Definition 2C.4 We define a *dot map* from the standard basis $\{v_S | S \subset [1, -1]\}$ to the set of dot diagrams with $2n$ nodes by assigning $S = \{x_1 < x_2 < \cdots < x_{|S|}\}$ to the dot diagram with one dot in each x_j position. ◇

Example 2C.5 The dot map is exemplified by:

$$S = \{1, 2, -3, -2\} \mapsto$$



Here $n = 3$. ◇

The following lemma will be used to silently identify dot diagrams with $S \subset [1, -1]$ or with v_S for $S \subset [1, -1]$.

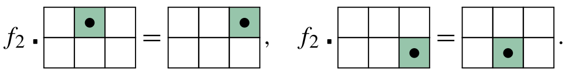
Lemma 2C.6 *The dot map is a bijection that restricts also to a bijection on the degree k part where the image are dot diagrams with k dots.*

Proof Immediate. □

Using Lemma 2C.6, we can rephrase the $U_q(\mathfrak{sp}_{2n})$-action in Lemma 2B.10 as follows (in all illustrations we have $n = 3$):

(i) In general, as soon as two dots end in the same node in the process of the action, then the result is zero. We will therefore only describe the nonzero actions.

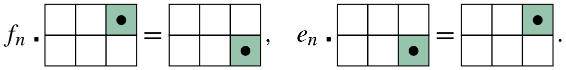
(ii) The Chevalley operators f_i for $i \in \{1, \dots, n-1\}$ move dots along the first row rightwards and along the second row leftwards. Here the operator f_i only moves dots in the ith top column and the $(i + 1)$th bottom column. For example:



No other potential dots are moved.

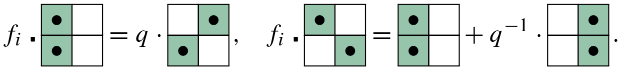
(iii) The Chevalley operators e_i for $i \in \{1, \dots, n-1\}$ do the opposite and move dots leftwards in the first row and rightwards in the second row.

(iv) The operators f_n and e_n act, for example, by



Again, no other potential dots are moved. The general picture is similar.

(v) The q comes in for the following local configurations



Here we assume that the leftmost displayed column is the ith column for $i \in \{1, \dots, n-1\}$. Similarly for the action of e_i.

Let us point out that the Chevalley operators e_i and f_i *do not change the number of dots*. We leave it to the reader to rewrite Lemma 2B.10 in terms of dot diagrams.

Remark 2C.7 We later need to keep track of additional data and to this end we will have *rainbow diagrams*, certain decorated dot diagrams, in Sect. 4B below. ◇

2D Self duality of the exterior algebra

Let $(\_)^*$ denote the $\mathbb{F}$-vector space dual. Recall that the antipode allows to extend $U_q(\mathfrak{sp}_{2n})$-actions to these duals.

It is well-known that the $U_q(\mathfrak{sp}_{2n})$-module V_q and its dual V_q^* are isomorphic. The following is an explicit isomorphism:

Lemma 2D.1 *There is an isomorphism of $U_q(\mathfrak{sp}_{2n})$-modules*

$$\varphi\colon V_q \to V_q^*, \quad v_i \mapsto (-q)^{i-1} \cdot v_{-i}^*, i \in [1,n] \quad v_{-i} \mapsto -q^{2n}(-q)^{-(i-1)} \cdot v_i^*, i \in [1,n].$$

Proof Essentially by definition, the map φ is a $\mathbb{F}$-linear bijection. Moreover, $U_q(\mathfrak{sp}_{2n})$-equivariance is checked by direct calculation. For example, if $i \in [1, n-1]$, then

$$\begin{aligned}
&\varphi(f_i \centerdot v_i) = \varphi(v_{i+1}) = (-q)^i \cdot v_{-(i+1)}^* = (-q)^{i-1} \cdot f_i \centerdot v_{-i}^* = f_i \centerdot \varphi(v_i),\\
&\varphi(f_n \centerdot v_n) = \varphi(v_{-n}) = -q^{2n}(-q)^{-(n-1)} \cdot v_{-n}^* = q(-q)^n \cdot v_{-n}^*\\
&= -(-q)^{n-1}q^2 \cdot v_{-n}^* = (-q)^{n-1} \cdot f_n \centerdot v_n^* = f_n \centerdot \varphi(v_n),
\end{aligned}$$

where we used that $S(f_i) = -f_i k_i$ (this holds also for $i = n$) and $k_i \centerdot v_i = q \cdot v_i$, as well as $k_n \centerdot v_n = q^2 \cdot v_n$. □

For $i \in [1, n]$ and $S = \{x_1, x_2, \dots, x_m\} \subset [1, -1]$ such that $x_1 < x_2 < \dots < x_m$, we define:

$$d_i := \varphi^{-1}(v_{-i}^*) = (-q)^{-(i-1)} \cdot v_i, \quad d_{-i} := \varphi^{-1}(v_i^*) = q^{-2n}(-q)^{i-1} \cdot v_{-i}, d_S := d_{x_1} d_{x_2} \dots d_{x_m}.$$

These elements will be used in Proposition 2D.4 below to describe the *dual quantum exterior algebra of type C*.

Lemma 2D.2 *The $\mathbb{A}$-algebra $\Lambda_{\mathbb{A}}$ is generated by the elements*

$$d_1, \dots, d_n, d_{-n}, \dots, d_{-1},$$

subject only to the following relations:

$$d_i^2 = 0, i \in [1, -1], \quad d_j d_i = -q^{-1} \cdot d_i d_j \textit{ and}$$

$$
\begin{aligned}
d_{-i}d_{-j} &= -q^{-1}\cdot d_{-j}d_{-i} \textit{ for } i,j\in[1,n], i<j d_{-i}d_j \\
&= -q^{-1}d_jd_{-i} \textit{ for } i,\in[1,n], i\neq j d_{-i}d_i = -q^{-2}\cdot d_id_{-i} \\
&\quad +(q^{-1}-q)\sum_{k\in[1,n-i]}(-q^{-1})^{k+1}\cdot d_{i+k}d_{-(i+k)} \textit{ for } i\in[1,n].
\end{aligned}
$$

Moreover, for all $k\in\{0,1,\dots,2n\}$*, the sets*

$$
\{d_S|S\subset[1,-1]\},\quad \{d_S|S\subset[1,-1],|S|=k\}
$$

are $\mathbb{A}$*-bases for* $\Lambda_{\mathbb{A}}$ *and* $\Lambda_{\mathbb{A}}^k$*, respectively.*

Proof Up to units in $\mathbb{A}$, the set of generators $\{v_i \mid i\in[1,-1]\}$ is equal to the set of elements $\{d_i \mid i\in[1,-1]\}$. So the latter generates $\Lambda_{\mathbb{A}}$. That the given sets are $\mathbb{A}$-bases can be proven as in Proposition 2A.9. □

Lemma 2D.3 (a) *For* $i\in\{1,\dots,n-1\}$ *and* $i=n$ *we have:*

$$
f_i\centerdot d_S = \begin{cases}
(-q)\cdot d_{(S\setminus\{i\})\cup\{i+1\}} & \textit{if } S_{i,i+1}=\{i\},\\
(-q)\cdot d_{(S\setminus\{-(i+1)\})\cup\{-i\}} & \textit{if } S_{i,i+1}=\{-(i+1)\},\\
(-q)\cdot\big(d_{(S\setminus\{-(i+1)\})\cup\{-i\}}+q^{-1}\cdot d_{(S\setminus\{i\})\cup\{i+1\}}\big) & \textit{if } S_{i,i+1}=\{i,-(i+1)\},\\
(-q)\cdot\big(q\cdot d_{(S\setminus\{i\})\cup\{i+1\}}\big) & \textit{if } S_{i,i+1}=\{i,-i\},\\
(-q)\cdot d_{(S\setminus\{-(i+1)\})\cup\{-i\}} & \textit{if } S_{i,i+1}=\{i+1,-(i+1)\},\\
(-q)\cdot d_{(S\setminus\{i\})\cup\{i+1\}} & \textit{if } S_{i,i+1}=\{i,-(i+1),-i\},\\
(-q)\cdot d_{(S\setminus\{-(i+1)\})\cup\{-i\}} & \textit{if } S_{i,i+1}=\{i,(i+1),-(i+1)\},\\
0 & \textit{otherwise.}
\end{cases}
$$

$$
f_n\centerdot d_S = \begin{cases}
(-q^2)\cdot d_{(S\setminus\{n\})\cup\{-n\}} & \textit{if } S_n=\{n\},\\
0 & \textit{otherwise.}
\end{cases}
$$

Similarly, for $e_i\centerdot d_S$ *and* $e_n\centerdot d_S$.

(b) *For* $i\in\{1,\dots,n-1\}$ *and* $i=n$ *we have:*

$$
f_i\centerdot v^*_{-S} = \begin{cases}
(-q)\cdot v^*_{-((S\setminus\{i\})\cup\{i+1\})} & \textit{if } S_{i,i+1}=\{i\},\\
(-q)\cdot v^*_{-((S\setminus\{-(i+1)\})\cup\{-i\})} & \textit{if } S_{i,i+1}=\{-(i+1)\},\\
(-q)\cdot\big(v^*_{-((S\setminus\{-(i+1)\})\cup\{-i\})}+q^{-1}\cdot v^*_{-((S\setminus\{i\})\cup\{i+1\})}\big) & \textit{if } S_{i,i+1}=\{i,-(i+1)\},\\
(-q)\cdot\big(q\cdot v^*_{-((S\setminus\{i\})\cup\{i+1\})}\big) & \textit{if } S_{i,i+1}=\{i,-i\},\\
(-q)\cdot v^*_{-((S\setminus\{-(i+1)\})\cup\{-i\})} & \textit{if } S_{i,i+1}=\{i+1,-(i+1)\},\\
(-q)\cdot v^*_{-((S\setminus\{i\})\cup\{i+1\})} & \textit{if } S_{i,i+1}=\{i,-(i+1),-i\},\\
(-q)\cdot v^*_{-((S\setminus\{-(i+1)\})\cup\{-i\})} & \textit{if } S_{i,i+1}=\{i,(i+1),-(i+1)\},\\
0 & \textit{otherwise.}
\end{cases}
$$

$$
f_n\centerdot v^*_{-S} = \begin{cases}
(-q^2)\cdot v^*_{-((S\setminus\{n\})\cup\{-n\})} & \textit{if } S_n=\{n\},\\
0 & \textit{otherwise.}
\end{cases}
$$

Similarly, for $e_i\centerdot v^*_{-S}$ *and* $e_n\centerdot v^*_{-S}$.

See also Lemma 2B.10 for the action of e_i*: the comparison between the actions of* f_i *and* e_i *therein is similar to the ones in these formulas.*

Proof As in Lemma 2B.10, and omitted. □

We leave it to the reader to formulate the dot diagram analog of Lemma 2D.3.

Proposition 2D.4 *The map*

$$\varphi_q^k\colon \Lambda_q^k \to (\Lambda_q^k)^*, \quad d_S \mapsto v_{-S}^*$$

is an isomorphism of $U_q(\mathfrak{sp}_{2n})$-modules.

Proof Proposition 2A.9 shows that $\Lambda_{\mathbb{A}}^k$ is a free $\mathbb{A}$-module with basis $\{v_S | S \subset [1,-1], |S| = k\}$. Therefore, $\mathrm{Hom}_{\mathbb{A}}(\Lambda_{\mathbb{A}}^k, \mathbb{A})$ is also free with basis $\{v_S^* | S \subset [1,-1], |S| = k\}$. Lemma 2D.2 implies that the map $\varphi_{\mathbb{A}}^k\colon \Lambda_{\mathbb{A}}^k \to \mathrm{Hom}_{\mathbb{A}}(\Lambda_{\mathbb{A}}^k, \mathbb{A})$, defined by $d_S \mapsto v_{-S}^*$, is an $\mathbb{A}$-module isomorphism. Tensoring with $\mathbb{F}$ yields a $\mathbb{F}$-vector space isomorphism $\varphi_q^k\colon \Lambda_q^k \to (\Lambda_q^k)^*$.

To see $U_q(\mathfrak{sp}_{2n})$-equivariance one uses (a) and (b) of Lemma 2D.3 to compare the actions of the Chevalley generators on the bases given by d_S and v_{-S}^*, respectively. □

2E The special linear action

We now discuss the dual action. In the classical case and for $\mathbb{C}$ as the ground field, the formula $2^{2n} = 4^n$ shows that, as $\mathbb{C}$-vector spaces, we have

$$\Lambda_1(V_1) \cong \Lambda_1(\mathbb{C}^{2n}) \cong \big(\Lambda_1(\mathbb{C}^2)\big)^{\otimes n} \cong (\mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C})^{\otimes n}.$$

There is thus a rather straightforward $U_1(\mathfrak{sl}_2)$-action on $\Lambda_1(V_1)$, simply by acting on $\mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}$ (we read this as the trivial plus vector plus trivial $U_1(\mathfrak{sl}_2)$-module) and then taking the nth tensor power. The same works in the classical case in general as pointed out in [23, Section 2.2]. We now describe the quantum version of this action.

Remark 2E.1 The classical version of the below is not the same as to Howe's approach from [26, Theorem 3.8.9.3] where differential operators play a key role. We will describe the quantum analog of differential operators later in Sect. 6. ◇

Recall that the *quantum group* $U_q(\mathfrak{sl}_2)$ is the $\mathbb{F}$-algebra generated by E, F and $K^{\pm 1}$, the $(\mathfrak{sl}_2)$ *Chevalley generators*, such that K and K^{-1} are two-sided inverses and

$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

We will additionally choose the following coproduct Δ, antipode S and counit ϵ on $U_q(\mathfrak{sl}_2)$. First, K and K^{-1} are group like, and second:

$$\begin{aligned} \Delta(E) &= 1 \otimes E + E \otimes K, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \\ S(F) &= -K^{-1}F, \quad S(E) = EK, \quad \epsilon(E) = 0, \quad \epsilon(F) = 0. \end{aligned} \tag{2E.2}$$

One easily checks that this defines the structure of a Hopf algebra on $U_q(\mathfrak{sl}_2)$. We will use this structure throughout.

Remark 2E.3 Comparing (2B.2) and (2E.2), note that we use the opposite coproduct conventions for $U_q(\mathfrak{sp}_{2n})$ and for $U_q(\mathfrak{sl}_2)$. ◇

A $U_q(\mathfrak{sl}_2)$-module is a *weight module*, if the action of K is diagonalizable. Given a finite dimensional weight $U_q(\mathfrak{sl}_2)$-module, the eigenvalues of K are contained in the set $\{\pm q^n\}_{n\in\mathbb{Z}}$, see e.g. [27, Proposition 2.3]. We will only consider weight modules for $U_q(\mathfrak{sl}_2)$ and will drop that adjective.

Definition 2E.4 Following for example [27], if all of the eigenvalues of a $U_q(\mathfrak{sl}_2)$-module are positive powers of q, then we say the module is *type-1* and if all of the eigenvalues are negative powers of q, then we say the module is *type-(-1)*. ◇

Definition 2E.5 Let σ be the $\mathbb{F}$-algebra automorphism of $U_q(\mathfrak{sl}_2)$ determined by

$$\sigma(E) = -E, \quad \sigma(F) = F, \quad \sigma(K^{\pm 1}) = -K^{\pm 1}.$$

Given a $U_q(\mathfrak{sl}_2)$-module (V, ρ_V) with action ρ_V, we define the *σ-twist of V*, denoted by ${}^{\sigma}V$, as the $U_q(\mathfrak{sl}_2)$-module $(V, \rho_V \circ \sigma)$. ◇

As one easily checks, if V is a type-ϵ $U_q(\mathfrak{sl}_2)$-module, then ${}^{\sigma}V$ is a type-$(-\epsilon)$ $U_q(\mathfrak{sl}_2)$-module.

Lemma 2E.6 *The category of finite dimensional $U_q(\mathfrak{sl}_2)$-modules is a direct sum of the category of type-1 $U_q(\mathfrak{sl}_2)$-modules and the category of type-(-1) $U_q(\mathfrak{sl}_2)$-modules. Moreover, twisting by σ induces an equivalence of categories between the two summands.*

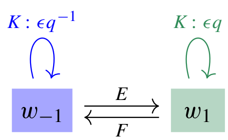
Proof See [27, Section 5.2]. □

We will use the following two $U_q(\mathfrak{sl}_2)$-modules, one will be used as a type-1 module, the other as a type-(-1) module:

Definition 2E.7 Let $\epsilon \in \{\pm 1\}$. The *type-ϵ trivial representation* ${}^{\epsilon}\mathbb{F}_*$ of $U_q(\mathfrak{sl}_2)$ is the $\mathbb{F}$-vector space with basis $\{w_0\}$ and $U_q(\mathfrak{sl}_2)$-action given by

$$E \centerdot w_0 = F \centerdot w_0 = 0, \quad K \centerdot w_0 = \epsilon \cdot w_0.$$

Moreover, the *type-ϵ vector representation* ${}^{\epsilon}W_q$ of $U_q(\mathfrak{sl}_2)$ is the $\mathbb{F}$-vector space with basis $\{w_1, w_{-1}\}$ and $U_q(\mathfrak{sl}_2)$-action given by



The picture is to be read as in Example 2B.7. ◇

Consider the $U_q(\mathfrak{sl}_2)$-module

$$X_q = {}^{1}\mathbb{F}_* \oplus {}^{-1}W_q \oplus {}^{1}\mathbb{F}_* = \mathbb{F}\{w_0, w_{-1}, w_1, w_0'\},$$

with the indicated $\mathbb{F}$-basis. The nth tensor power of X_q has an $\mathbb{F}$-basis

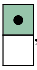
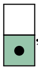
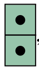
$$\big\{\vec{u} = u_1 \otimes u_2 \otimes \cdots \otimes u_n | u_i \in \{w_0, w_{-1}, w_1, w_0'\}\big\}.$$

Definition 2E.8 For each $\vec{u}$ we associate a set $S = S_{\vec{u}} \subset [1, -1]$ as follows:

$$S \cap \{\pm i\} = \begin{cases} i \text{ if } u_i = w_0, \\ -i \text{ if } u_i = w_0', \\ \pm i \text{ if } u_i = w_1, \\ \emptyset \text{ if } u_i = w_{-1}. \end{cases}$$

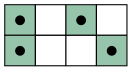
(Note that this determines S.) We use this to define an $\mathbb{F}$-linear map $\Psi\colon X_q^{\otimes n} \to \Lambda_q$ by $\vec{u} \mapsto v_{S_{\vec{u}}}$. ◇

In dot diagrams Definition 2E.8 means that the ith column of S is as follows:

$u_i = w_0 \leftrightsquigarrow$ , $u_i = w_0' \leftrightsquigarrow$ , $u_i = w_1 \leftrightsquigarrow$ , $u_i = w_{-1} \leftrightsquigarrow$ . (2E.9)

These are local pictures showing only the ith column. The two column configurations on the right are called *fully dotted* and *undotted*.

Example 2E.10 For $n = 4$ we have

$w_1 \otimes w_{-1} \otimes w_0 \otimes w_0' \leftrightsquigarrow S = \{1, 3, -4, -1\} \leftrightsquigarrow$ .

The leftmost column is fully dotted, the second column from the left is undotted. ◇

We use the following lemma to identify the tensor product basis coming from $X_q^{\otimes n}$ with dot diagrams.

Lemma 2E.11 *The map Ψ is an isomorphism of $\mathbb{F}$-vector spaces with inverse $\Psi^{-1}(v_S) = \vec{u}$, where*

$$u_i := \begin{cases} w_0 \textit{ if } S \cap \{\pm i\} = i, \\ w_0' \textit{ if } S \cap \{\pm i\} = -i, \\ w_{-1} \textit{ if } S \cap \{\pm i\} = \emptyset, \\ w_1 \textit{ if } S \cap \{\pm i\} = \pm i. \end{cases}$$

Proof A direct verification. □

The previous lemma endows Λ_q with an action of $U_q(\mathfrak{sl}_2)$:

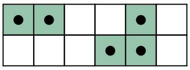
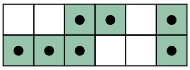
$$X \centerdot v_S := \Psi\big(X \centerdot \Psi^{-1}(v_S)\big) \text{ for all } X \in U_q(\mathfrak{sl}_2).$$

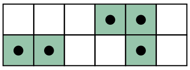
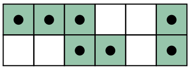
We will use this action throughout.

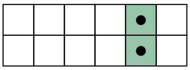
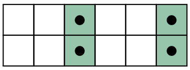
Notation 2E.12 Let $S \subset [1, -1]$. Define $S_0 := S \cap -S$, i.e. the subset of S consisting of all fully dotted columns. Also, write $S^c = [1, -1] \setminus S$ and $S_0^c := S^c \cap -S^c$, i.e. the subset of $[1, -1]$ consisting of all undotted columns.

For $k \in \{1, \dots, n\}$, and $T \subset [1, -1]$, write $T_{>k} := T \cap \{\pm(k+1), \pm(k+2), \dots, \pm n\}$. Similarly define $T_{\geq k}$, $T_{<k}$, and $T_{\leq k}$. We set $\mathbf{w}(T) := 1/2\big(|T_0| - |T_0^c|\big)$, $\mathbf{w}_{>i}(T) := 1/2\big(|(T_0)_{>i}| - |(T_0^c)_{>i}|\big)$ and $\mathbf{w}_{<i}(T) := 1/2\big(-|(T_0)_{<i}| + |(T_0^c)_{<i}|\big)$. ◇

Example 2E.13 Let $n = 6$ and $S = \{1, 2, 5, -5, -4\}$. Then $-S = \{4, 5, -5, -2, -1\}$, $S_0 = \{5, -5\}$, $S^c = \{3, 4, 6, -6, -3, -2, -1\}$, $-S^c = \{1, 2, 6, -6, -4, -3\}$, and $S_0^c = \{6, -6\}$. In dot diagrams:

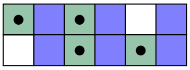
$S \rightsquigarrow$ , $S^c \rightsquigarrow$ ,

$-S \rightsquigarrow$ , $-S^c \rightsquigarrow$ ,

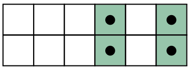
$S_0 \rightsquigarrow$ , $S_0^c \rightsquigarrow$ .

In words, the minus operation flips the dot diagram upside down, the complement fills in the open nodes with dots and removes all other dots, and the subscript zero is the part stable under mirroring upside down. ◇

Example 2E.14 Let $n = 6$ and let

$S = \{1, 3, -3, -5\} \rightsquigarrow$ ,

where we shaded the undotted columns of the dot diagram for S. Then

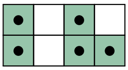
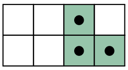
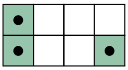
$(S_0^c)_{>3} = \{4, -4, 6, -6\} \rightsquigarrow$ .

Note that this fills in exactly the shaded parts above that are to the right of column 3. ◇

We now describe the action of $U_q(\mathfrak{sl}_2)$ on Λ_q. By (2E.9) and up to q-powers, E acts on a dot diagram by adding two dots into undotted columns while F acts by removing the two dots in fully dotted columns. In particular, E and F do not fix the number of dots, but only *preserve the number of dots modulo two*.

Example 2E.15 As an example for the action of F for $n = 4$ we have

$$
\begin{aligned}
F \centerdot w_1 \otimes w_{-1} \otimes w_1 \otimes w_0' &= (F \centerdot w_1) \otimes w_{-1} \otimes w_1 \otimes w_0' \\
&\quad + (K^{-1} \centerdot w_1) \otimes (K^{-1} \centerdot w_{-1}) \otimes (F \centerdot w_1) \otimes w_0' \\
&= w_{-1} \otimes w_{-1} \otimes w_1 \otimes w_0' \\
&\quad + (-q^{-1})(-q) \cdot w_1 \otimes w_{-1} \otimes w_{-1} \otimes w_0'
\end{aligned}
$$

$F \centerdot$  $=$  $+$ ,

These two calculations are the same, but using vectors and dot diagrams, respectively. ◇

Example 2E.15 formalized gives:

Lemma 2E.16 *The action of $U_q(\mathfrak{sl}_2)$ on Λ_q has the following explicit description.*

$$K \centerdot v_S = (-q)^{\mathbf{w}(S)} \cdot v_S, E \centerdot v_S = \sum_{\{\pm i\} \subset S_0^c} (-q)^{\mathbf{w}_{>i}(S)} \cdot v_{S \cup \{\pm i\}}, \quad F \centerdot v_S = \sum_{\{\pm i\} \subset S_0} (-q)^{\mathbf{w}_{<i}(S)} \cdot v_{S \setminus \{\pm i\}}.$$

Proof Directly from (2E.2) and the translation of $\vec{w}$ to dot diagrams. □

Lemma 2E.17 *The $(-q)^k$ weight space of the $U_q(\mathfrak{sl}_2)$-module Λ_q is exactly Λ_q^{n+k}.*

Proof Note that $|S| = |S_0| + x$, where x is the number of pairs $\pm i$ such that $|S \cap \{\pm i\}| = 1$. Similarly, $|S^c| = |S_0^c| + x$. Therefore, $2n = |S| + |S^c| = |S_0| + 2x + |S_0^c|$, so $|S_0| - |S_0^c|/2 = |S_0|/2 - (n - x - |S_0|/2) = -n + |S|$. □

Proposition 2E.18 *The two actions, the one of $U_q(\mathfrak{sp}_{2n})$ and the one of $U_q(\mathfrak{sl}_2)$, on Λ_q commute.*

Proof It suffices to show that if $S \subset [1, -1]$ and $k \in \{1, \dots, n\}$, then $x_i \centerdot (X \centerdot v_S) = X \centerdot (x_i \centerdot v_S)$, for $x_i \in \{e_i, f_i\}$ and $X \in \{E, F\}$. Using Lemma 2B.10 and Lemma 2E.16 this is a series of elementary calculations. We give for following example, and leave the rest to the reader.

Suppose $S_{k,k+1} = \{k, -(k+1)\}$ and write $T := (S \setminus \{-(k+1)\}) \cup \{-k\}$ and $U := (S \setminus \{k\}) \cup \{k+1\}$. Then

$$\begin{aligned}
E \centerdot (f_k \centerdot v_S) &= E \centerdot (v_T + q^{-1} \cdot v_U) \\
&= \sum_{\{\pm i\} \subset T_0^c} (-q)^{\mathbf{w}_{>i}(T)} \cdot v_{T \cup \{\pm i\}} + q^{-1} \sum_{\{\pm i\} \subset U_0^c} (-q)^{\mathbf{w}_{>i}(U)} \cdot v_{U \cup \{\pm i\}} \\
&= \sum_{\substack{\{\pm i\} \subset T_0^c \\ i \neq k+1}} (-q)^{\mathbf{w}_{>i}(T)} \cdot v_{T \cup \{\pm i\}} + (-q)^{\mathbf{w}_{>k+1}(T)} \cdot v_{T \cup \{\pm (k+1)\}} \\
&\quad + q^{-1}(-q)^{\mathbf{w}_{>k}(U)} \cdot v_{U \cup \{\pm k\}} + q^{-1} \sum_{\substack{\{\pm i\} \subset U_0^c \\ i \neq k}} (-q)^{\mathbf{w}_{>i}(U)} \cdot v_{U \cup \{\pm i\}} \\
&= \sum_{\substack{\{\pm i\} \subset T_0^c \\ i \neq k+1}} (-q)^{\mathbf{w}_{>i}(T)} \cdot v_{T \cup \{\pm i\}} + q^{-1} \sum_{\substack{\{\pm i\} \subset U_0^c \\ i \neq k}} (-q)^{\mathbf{w}_{>i}(U)} \cdot v_{U \cup \{\pm i\}} \\
&= \sum_{\{\pm i\} \subset S_0^c} (-q)^{\mathbf{w}_{>i}(S)} \cdot v_{T \cup \{\pm i\}} + q^{-1} \sum_{\{\pm i\} \subset S_0^c} (-q)^{\mathbf{w}_{>i}(S)} \cdot v_{U \cup \{\pm i\}} \\
&= \sum_{\{\pm i\} \subset S_0^c} (-q)^{\mathbf{w}_{>i}(S)} \cdot (v_{T \cup \{\pm i\}} + q^{-1} \cdot v_{U \cup \{\pm i\}}) \\
&= \sum_{\{\pm i\} \subset S_0^c} (-q)^{\mathbf{w}_{>i}(S)} \cdot (v_{(S \cup \{\pm i\}) \setminus \{-(k+1)\} \cup \{-k\}} + q^{-1} \cdot v_{(S \cup \{\pm i\}) \setminus \{k\} \cup \{-k\}}) \\
&= f_k \centerdot \Big(\sum_{\{\pm i\} \subset S_0^c} (-q)^{\mathbf{w}_{>i}(S)} \cdot v_{S \cup \{\pm i\}} \Big) \\
&= f_k \centerdot (E \centerdot v_S).
\end{aligned}$$

All other cases can be calculated similarly. □

3 Howe duality: part 1

We now state and prove a quantum version of [26, Theorem 3.8.9.3] for $m = 1$.

3A Semisimple classical Howe duality

We recall the classical case.

Let $V_1 = \mathbb{C}\{u_1, \dots, u_n, u_{-n}, \dots, u_{-1}\}$. One can check that

$$\begin{cases} (u_i, u_{-i}) = 1, \quad (u_{-i}, u_i) = -1, \quad \text{and} \quad (u_i, u_j) = 0 \text{ otherwise}, \\ \text{pairing matrix: } J = \begin{pmatrix} 0 & -A \\ A & 0 \end{pmatrix}, \quad A := \text{anti-diagonal matrix with1on the anti-diagonal}, \end{cases}$$

equips V_1 with a symplectic form, which we are going to use on Λ_1. Let

$$\mathfrak{sp}_{2n} = \left\{M | JM = -M^T J\right\} = \left\{\left(A\ BC\ -A^T\right) | B = B^T, C = C^T\right\} \subset \mathfrak{gl}_{2n}.$$

Then $\mathfrak{sp}_{2n}$ acts on V_1 as the linear endomorphisms $X\colon V_1 \to V_1$ such that $(Xv, w) + (v, Xw) = 0$ for all $v, w \in V$. This is the $q = 1$ version of V_q from Definition 2B.4, i.e. V_1 is the *vector representation* of $\mathfrak{sp}_{2n}$. This can be seen by using Remark 2B.8 since, in elementary matrices, we have

$$e_i \leftrightsquigarrow E_{i,i+1} - E_{-i-1,-i}, \quad e_n \leftrightsquigarrow E_{n,-n}, \quad \text{etc.},$$

so that, for example, $e_1 \centerdot u_2 = u_1$ and $e_1 \centerdot u_{-1} = -u_{-2}$.

There is an action of $\mathfrak{sp}_{2n}$ on the exterior algebra of V, denoted Λ_1, such that $X \in \mathfrak{sp}_{2n}$ acts on $\Lambda_1^1 \cong V_1$ as before and on Λ_1^k by derivations. Note that Λ_1 is the $q = 1$ version of $\Lambda_{\mathbb{A}}$ from Definition 2A.3 (when using the u_i basis vectors). In other words, Λ_1 is the exterior algebra.

The $\mathbb{C}$-vector space Λ_1 has the $\mathbb{C}$-basis

$$\{u_{A,B,C} := u_{a_1} \dots u_{a_{|A|}} u_{-b_1} \dots u_{-b_{|B|}} u_{c_1} u_{-c_1} \dots u_{c_{|C|}} u_{-c_{|C|}} \mid A \sqcup B \sqcup C \subset [1, n]\}. \tag{3A.1}$$

Note that this basis is not the classical version of the basis given by the v_S.

Let $E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ be the usual generators of $\mathfrak{sl}_2$. The $q = 1$ version of the dual action from Sect. 2E arises as follows (a precise comparison is given in Remark 3B.6). $\mathfrak{sl}_2$ acts on Λ_1 such that the E, F, and H in $\mathfrak{sl}_2$ act in the basis from (3A.1) as

$$E \centerdot u_{A,B,C} = \sum_{x \in \{1,\dots,n\} \setminus C} u_{A,B,C \cup \{x\}},$$
$$F \centerdot u_{A,B,C} = \sum_{x \in C} u_{A,B,C \setminus \{x\}},\ H \centerdot u_{A,B,C} = (-n + |A| + |B| + 2|C|) \cdot u_{A,B,C}.$$

We now use the enveloping $\mathbb{C}$-algebras $U_1(\mathfrak{sp}_{2n})$ and $U_1(\mathfrak{sl}_2)$. Let $\mathbf{U}_1 := U_1(\mathfrak{sp}_{2n}) \otimes U_1(\mathfrak{sl}_2)$. Note that a $U_1(\mathfrak{sp}_{2n})$-$U_1(\mathfrak{sl}_2)^{op}$-bimodule is the same as a $\mathbf{U}_1$-module.

Theorem 3A.2 *Consider the $\mathbb{C}$-vector space Λ_1.*

(a) *There are commuting actions*

$$U_1(\mathfrak{sp}_{2n}) \circlearrowright \Lambda_1 \circlearrowleft U_1(\mathfrak{sl}_2)^{op}.$$

as described above.

(b) *Let ϕ_1 and ψ_1 be the $\mathbb{C}$-algebra homomorphisms induced by the two actions from (a). Then:*

$$\phi_1\colon U_1(\mathfrak{sp}_{2n}) \twoheadrightarrow \mathrm{End}_{U_1(\mathfrak{sl}_2)^{op}}(\Lambda_1), \quad \psi_1\colon U_1(\mathfrak{sl}_2) \twoheadrightarrow \mathrm{End}_{U_1(\mathfrak{sp}_{2n})}(\Lambda_1).$$

That is, the two actions generate the others centralizer.

(c) *The $\mathbf{U}_1$-module Λ_1 decomposes as*

$$\Lambda_1 \cong \bigoplus_{k=0}^{n} L_1(\varpi_k) \otimes L_1(n-k).$$

Here $L_1(\varpi_k)$ and $L_1(n-k)$ denote the simple $U_1(\mathfrak{sp}_{2n})$-module and $U_1(\mathfrak{sl}_2)^{op}$-module of the indicated highest weights.

Proof This is a special case of [26, Theorem 3.8.9.3], but in a different convention, so let us sketch a proof.

First, as one easily checks, the actions of $\mathfrak{sp}_{2n}$ and $\mathfrak{sl}_2$ from above commute. This is seen by identifying E and F with multiplication and contraction by the element $\omega := \sum_{i=1}^{n} u_i u_{-i} \in \Lambda_1^2$ preserved by $\mathfrak{sp}_{2n}$. This shows (a).

Then (b) follows from (c), so we prove (c). Let x be a highest weight vector for $\mathfrak{sp}_{2n}$ and a lowest weight vector for $\mathfrak{sl}_2$. Then using that x is a lowest weight vector for $\mathfrak{sl}_2$ we get $x \in \Lambda_1^m$ for $0 \leq m \leq n$. Since x is a highest weight vector for $\mathfrak{sp}_{2n}$ in Λ_1, it follows from the explicit $\mathbb{C}$-basis in (3A.1) and the $\mathfrak{sp}_{2n}$-action on it by derivations that $x = \sum_{C \subset \{k+1,\dots,n\}} \xi_C \cdot u_{\{1,\dots,k\},\emptyset,C}$ for some $\xi_C \in \mathbb{C}$. Suppose that $C \neq \emptyset$. Then from $F \centerdot x = 0$ it is possible to find a contradiction. Thus, up to scalar all the joint highest-lowest weight vectors for $\mathfrak{sp}_{2n}$-$\mathfrak{sl}_2$ are of the form $u_1 u_2 \cdots u_k$ for $k \in \{0, 1, \dots, n\}$, and (c) follows. □

3B Semisimple quantum Howe duality

Using Proposition 6.13, we will consider Λ_q as a $U_q(\mathfrak{sp}_{2n})$-$U_q(\mathfrak{sl}_2)^{op}$-bimodule or, alternatively, as a $\mathbf{U}_q := U_q(\mathfrak{sp}_{2n}) \otimes U_q(\mathfrak{sl}_2)$-module. We use the convention that $\varpi_0 := 0$, so the irreducible with highest weight ϖ_0 is the trivial module.

Lemma 3B.1 *Let $\epsilon \in \{\pm 1\}$.*

(a) *The finite dimensional simple $\mathbf{U}_q$-modules are all of the form $L_q(\lambda) \otimes {}^{\epsilon} L_q(k)$ where λ is an integral dominant weight for $\mathfrak{sp}_{2n}$ and $k \in \mathbb{Z}_{\geq 0}$, thus, k is a dominant integral weight for $\mathfrak{sl}_2$. Their characters are given by Weyl's quantum character formula per component.*
(b) *Let V be a finite dimensional module over $\mathbf{U}_q$. Then V is completely reducible. Moreover, assume that $v \in V$ is such that:*

 (i) $K_i \centerdot v = q^{(\alpha_i^{\vee}, \lambda)} \cdot v$ *and* $(\alpha_i^{\vee}, \lambda) \in \mathbb{Z}_{\geq 0}$ *for* $i \in \{1, \dots, n\}$.
 (ii) $K \centerdot v = \epsilon q^k \cdot v$ *for some* $k \in \mathbb{Z}_{\geq 0}$.
 (iii) $e_i \centerdot v = 0$ *for* $i \in \{1, \dots, n\}$.
 (iv) $F \centerdot v = 0$.

Then the $\mathbf{U}_q$ module $L_q(\lambda) \otimes {}^{\epsilon} L_q(-k)$ is a direct summand of V.

Proof *(a).* By, for example, [27, Theorems 5.15 and 5.17] or [2, Section 6] the category of finite dimensional $\mathbf{U}_q$ modules is semisimple and has the same character combinatorics as the classical case, so the claims follow from classical theory.

(b). As in the proof of (a) this statement follows by classical theory and keeping track of quantum weight combinatorics. □

Definition 3B.2 Let V be a finite dimensional $\mathbf{U}_q$ module. We write $V_{\lambda,k}^{+}$ to denote the subspace of vectors in V satisfying the conditions in Lemma 3B.1.(b). We refer to a vector in $V_{\lambda,k}^{+}$ as a *singular vector* of weight (λ, k). ◇

Lemma 3B.3 *Let V be a finite dimensional $\mathbf{U}_q$-module. Then there integral dominant weights λ for $\mathfrak{sp}_{2n}$ and $k \in \mathbb{Z}_{\geq 0}$*

$$V \cong \bigoplus_{\lambda,k} L_q(\lambda) \otimes \left({}^{\epsilon} L_q(-k)\right)^{\oplus \dim V_{\lambda,k}^{+}}.$$

Proof Again, using [27, Theorems 5.15 and 5.17] or [2, Section 6], this follows from classical theory. □

Theorem 3B.4 *Consider the $\mathbb{F}$-vector space Λ_q.*

(a) *There are commuting actions*

$$U_q(\mathfrak{sp}_{2n}) \circlearrowright \Lambda_q \circlearrowleft U_q(\mathfrak{sl}_2)^{op}.$$

as in Sect. 2.

(b) *Let ϕ_q and ψ_q be the $\mathbb{F}$-algebra homomorphisms induced by the two actions from (a). Then:*

$$\phi_q : U_q(\mathfrak{sp}_{2n}) \twoheadrightarrow \mathrm{End}_{U_q(\mathfrak{sl}_2)^{op}}(\Lambda_q), \quad \psi_q : U_q(\mathfrak{sl}_2) \twoheadrightarrow \mathrm{End}_{U_q(\mathfrak{sp}_{2n})}(\Lambda_q).$$

That is, the two actions generate the others centralizer.

(c) *The $\boldsymbol{U}_q$-module Λ_q decomposes as*

$$\Lambda_q \cong \bigoplus_{k=0}^{n} L_q(\varpi_k) \otimes {}^{\epsilon}L_q(n-k).$$

Here $L_q(\varpi_k)$ and ${}^{\epsilon}L_q(n-k)$ denote the simple $U_q(\mathfrak{sp}_{2n})$-module and $U_q(\mathfrak{sl}_2)^{op}$-module of the indicated highest weights. Here $\epsilon = 1$ for even degrees and $\epsilon = -1$ for odd degrees.

Proof (a)+(b). The first statement is Proposition 6.13, while (b) is a consequence of (c).

(c). It is easy to see that each vector in the set

$$\{v_{\emptyset}, v_{\{1\}}, v_{\{1,2\}}, \ldots, v_{\{1,2,\ldots,n\}}\}$$

is a singular vector of weight

$$(\varpi_0, -n), (\varpi_1, -n+1), (\varpi_2, -n+2), \ldots, (\varpi_n, 0).$$

Thus, by Lemma 3B.3 we have the following inclusion as a direct summand:

$$\bigoplus_{k=0}^{n} L_q(\varpi_k) \otimes {}^{\epsilon}L_q(n-k) \subset_{\oplus} \Lambda_q. \tag{3B.5}$$

To prove equality, recall that $\dim_{\mathbb{F}} L_q(\lambda) = \dim_{\mathbb{F}} L(\lambda)$ and $\dim_{\mathbb{F}} L_q(k) = \dim_{\mathbb{F}} L(k)$, see Lemma 3B.1, and $\dim_{\mathbb{F}} \Lambda_q = \dim_{\mathbb{C}} \Lambda_1$ by Proposition 2A.9. It hence follows from Theorem 3A.2 that

$$\dim_{\mathbb{F}} \left(\bigoplus_{k=0}^{n} L_q(\varpi_k) \otimes {}^{\epsilon}L_q(n-k) \right) = \dim_{\mathbb{F}} \Lambda_q.$$

Thus, the inclusion in (3B.5) is an equality.

Note that the $U_q(\mathfrak{sp}_{2n})$-action preserves the degree while the $U_q(\mathfrak{sl}_2)$-action preserves the parity of the degree so that we really do have two separate action pairs. Since the $U_q(\mathfrak{sl}_2)$-action is on tensor powers of $X_q = {}^{1}\mathbb{F}_{*} \oplus {}^{-1}W_q \oplus {}^{1}\mathbb{F}_{*}$ the even-odd claim follows. □

Remark 3B.6 The meticulous reader might note that Theorem 3B.4 does not quite quantize the classical version in Theorem 3A.2 because of the appearance of type-1 and type-(-1) $U_q(\mathfrak{sl}_2)$-modules. This can be fixed by observing that Λ_q^{even} is a type-1 $U_q(\mathfrak{sl}_2)$-module and Λ_q^{odd} is a type-(-1) $U_q(\mathfrak{sl}_2)$-module, and then specialize these two separately.

It is actually more accurate to compare the action in Sect. 3A to the one given in Sect. 6 later on. However, the difference will not play a role for us. ◇

Remark 3B.7 The quantum $(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2k})$-duality that was given in [42, Theorem A] suggests that our quantum $(\mathfrak{sp}_{2n}, \mathfrak{sl}_2 \cong \mathfrak{sp}_2)$-duality might need a quantum symmetric pair and a quantum group instead of two quantum groups. (Note that [42, Theorem A] replaces $\mathfrak{sp}_{2n}$ by a quantum symmetric pair, but one might suspect the existence of a "dual" of [42, Theorem A] with the roles of the quantum group and the quantum symmetric pair swapped.)

Indeed, we do have a quantum symmetric pair: According to [42, Theorem A] one of the quantum objects associated to symplectic exterior Howe duality is the quantum symmetric pair $U'_q(\mathfrak{sp}_{2k})$ for the Satake-type diagram ●—○—●⋯○—●. Algebraically, $U'_q(\mathfrak{sp}_{2k})$ is the subalgebra of $U_q(\mathfrak{gl}_{2k})$ generated by

$$\begin{aligned} E_i, F_i, K_i^{\pm 1} \quad & \text{for } i \in \{1, 3, \dots, 2k-1\}, \\ B_i = F_i - K_i^{-1} ad(E_{i-1}E_{i+1}) \centerdot E_i \quad & \text{for } i \in \{2, 4, \dots, 2k\}. \end{aligned}$$

In the special case $k = 1$ we thus get the "small number coincidence"

$$U'_q(\mathfrak{sp}_{2k}) \cong U_q(\mathfrak{sp}_2) \cong U_q(\mathfrak{sl}_2),$$

and we thus have a quantum symmetric pair appearing, albeit in a roundabout way. The same is true for the orthogonal version of Theorem 3B.4, see [43, Theorem 3.4] (note that this theorem is formulated differently). That theorem corresponds to a quantum $(\mathfrak{so}_m, \mathfrak{sl}_2 \cong \mathfrak{sp}_2)$-duality where the quantum exterior is replaced by a quantum symmetric algebra.

The remaining two dualities in [42, Theorem A] come from the Satake-type diagram ○—○⋯○—○. In this case the quantum symmetric pair does not correspond to a quantum group in any known way, so a quantization might be difficult. ◇

Corollary 3B.8 *We have an isomorphism of $U_q(\mathfrak{sp}_{2n})$-modules*

$$\ker F \cap \Lambda_q^k \cong L_q(\varpi_k)$$

with distinguished highest weight vector $v_{\{1,\dots,k\}}$ for the $U_q(\mathfrak{sp}_{2n})$-action.

Proof Let $V = \bigoplus_{k \in \mathbb{Z}} V_k$ be a finite dimensional representation of $U_q(\mathfrak{sl}_2)$ such that $K \centerdot v = \epsilon q^k \cdot v$ for all $v \in V_k$. Then

$$V \cong \bigoplus_{k \in \mathbb{Z}_{\geq 0}} {}^{\epsilon}L_q(k)^{\oplus \dim_{\mathbb{F}} \ker F \cap V_{-k}}.$$

Observing that K acts on Λ_q^k as ϵq^{-n+k} we deduce that the multiplicity of $L_q(n-k)$ as a direct summand of Λ_q is equal to $\dim_{\mathbb{F}} \ker F \cap \Lambda_q^k$. By Theorem 3B.4 we find that this multiplicity is also equal to $\dim L_q(\varpi_k)$.

Since $v_1 \dots v_k \in \ker F \cap \Lambda_q^k$ there is a nonzero map $L_q(\varpi_k) \to \ker F \cap \Lambda_q^k$. The representation $L_q(\varpi_k)$ is simple, so the map is injective, hence, an isomorphism. □

4 Integral versions and Weyl filtrations

We now extend the previous results to $\mathbb{A}$.

4A Divided power quantum groups

Recall from [32] that the *divided power quantum group* $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$ is the $\mathbb{A}$-subalgebra of $U_q(\mathfrak{sp}_{2n})$ generated by the *divided power generators* $E_i^{(k)} = E_i^k/[k]_q$ and $F_i^{(k)} = F_i^k/[k]_q$

as well as an adjusted Cartan part. We will not need the Cartan part so we only refer to, for example, [2] for details.

Before we go to the most important, for the purpose of this paper, $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-modules, we state a general restriction lemma. Let $\mathrm{Res}_{\mathbb{A}}^{\mathbb{F}}$ denote the scalar restriction from $\mathbb{F}$ to $\mathbb{A}$.

Lemma 4A.1 *Let V_q and W_q be $U_q(\mathfrak{sp}_{2n})$-modules and let $\varphi \in \mathrm{Hom}_{U_q(\mathfrak{sp}_{2n})}(V_q, W_q)$. Suppose $U_{\mathbb{A}}(\mathfrak{sp}_{2n}) \subset U_q(\mathfrak{sp}_{2n})$ preserves $V_{\mathbb{A}} \subset \mathrm{Res}_{\mathbb{A}}^{\mathbb{F}} V_q$ and $W_{\mathbb{A}} \subset \mathrm{Res}_{\mathbb{A}}^{\mathbb{F}} W_q$, and that $\varphi(V_{\mathbb{A}}) \subset W_{\mathbb{A}}$. Then $\ker \varphi \cap V_{\mathbb{A}}$ is a $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-submodule of $V_{\mathbb{A}}$.*

Proof Directly from the definitions. □

Remark 4A.2 The theory below about Weyl and dual Weyl modules is standard, but mostly used over fields, see e.g. [16, 39] for the original references. Some papers however are very careful with the ground ring, for example [28] or [41], which are helpful and are main sources for the integral statements. ◇

As usual in the theory, see again [2], there are two types of $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-modules indexed by dominant integral weights $\lambda \in X^+$:

(a) The *dual Weyl modules* $\nabla_{\mathbb{A}}(\lambda)$ which we, following [2], define as induced modules where one induces a free rank one $\mathbb{A}$-module over the negative Borel subalgebra, see e.g. the arXiv version of [5] for details.
(b) The *Weyl modules* $\Delta_{\mathbb{A}}(\lambda)$ that we define, as a free $\mathbb{A}$-module, as the $\mathbb{A}$-module dual of $\nabla_{\mathbb{A}}(\lambda)$ whose action is the one on $\nabla_{\mathbb{A}}(\lambda)$ twisted by the antipode and the Chevalley involution ω, using Definition 2B.3, see e.g. [5, Section 2A] for details.

Although the above definitions are the gold standard in the field, the following construction of the Weyl module is what we actually use throughout. The next lemma also summarizes some properties of these $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-modules.

Lemma 4A.3 (a) *We have $\Delta_{\mathbb{A}}(\lambda) \cong U_{\mathbb{A}}(\mathfrak{sp}_{2n}) \centerdot v_\lambda \subset L_q(\lambda)$, where $L_q(\lambda)$ is the simple $U_q(\mathfrak{sp}_{2n})$-module of highest weight λ with highest weight vector v_λ.*
(b) *The $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-modules $\Delta_{\mathbb{A}}(\lambda)$ and $\nabla_{\mathbb{A}}(\lambda)$ are $\mathbb{A}$-free and have the same character as the simple $U_q(\mathfrak{sp}_{2n})$-module $L_q(\lambda)$ from Lemma 3B.1.*
(c) *We have* Ext-vanishing, *i.e.* $\mathrm{Ext}^i\big(\Delta_{\mathbb{A}}(\lambda), \nabla_{\mathbb{A}}(\mu)\big) = 0$ *unless $i = 0$ and $\lambda = \mu$, in which case we have*

$$\mathrm{Hom}\big(\Delta_{\mathbb{A}}(\lambda), \nabla_{\mathbb{A}}(\lambda)\big) \cong \mathbb{A}.$$

Proof All of this is well-known, see for example [2] or [5] for details, and [41] for integral formulations. For example, the comparison in (a) can be obtained from [41, Lemma 4.3 and Theorem 5.4]. □

Definition 4A.4 A $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-module is called (integrally) *tilting* if it has a Weyl and a dual Weyl filtration. ◇

Example 4A.5 Let $V_{\mathbb{A}}$ be as in Definition 2B.4, but over $\mathbb{A}$ instead of $\mathbb{F}$. For all $k \in \mathbb{Z}_{\geq 0}$, the $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-module $\Delta_{\mathbb{A}}(\varpi_1)^{\otimes k} \cong V_{\mathbb{A}}^{\otimes k}$ is tilting. In general, tensor products of Weyl modules for minuscule weights are examples of tilting modules, see e.g. [4, Proposition 2.3] for an explicit statement with a self-contained proof. ◇

More generally than the previous example:

Lemma 4A.6 *Tensor products of tilting modules are tilting.*

Proof See [37, Theorem 3.3], which can be adapted to work over $\mathbb{A}$. □

Recall that *specialization* in this context is:

Notation 4A.7 Let $\mathbb{K}$ be a field and fix $\xi \in \mathbb{K} \setminus \{0\}$. Then $q \mapsto \xi$ induces an action of $\mathbb{A}$ on $\mathbb{K}$. We will say that $\mathbb{K}$ is a *field over* $\mathbb{A}$ or a *specialization*, leaving the data of $\xi \in \mathbb{K} \setminus \{0\}$ implicit. ◇

Example 4A.8 There are essentially four different types of specializations (roughly ordered by complexity):

(a) The semisimple specializations, where ξ is not a root of unity. As an example consider the fraction field $\mathbb{K} = \mathbb{F}$ with $\xi = q$. Another example is $\mathbb{K} = \mathbb{F}_7(q)$ for generic q when we let $\xi = q$.
(b) The complex root of unity case. That is, $\mathbb{K} = \mathbb{C}$ and ξ is a root of unity.
(c) The (strictly) mixed cases. For example, $\mathbb{K} = \mathbb{F}_7$ and $\xi = 2$.
(d) Characteristic p, which, for example, could be $\mathbb{K} = \mathbb{F}_7$ and $\xi = 1$.

The only simple Lie algebra where the representation theory is understood in all of the above cases is $\mathfrak{sl}_2$; for the combinatorics of the modules see e.g. [17, Section 3.4]. Other expositions, with a focus on special cases, are e.g. [3, Section 2] or [45]. ◇

We recall the following:

Lemma 4A.9 *For all $\lambda \in X^+$ there exists a simple module $L_{\mathbb{K}}(\lambda)$, a Weyl module $\Delta_{\mathbb{K}}(\lambda)$, a dual Weyl module $\nabla_{\mathbb{K}}(\lambda)$ and an indecomposable tilting module $T_{\mathbb{K}}(\lambda)$, where λ is the highest weight. They are pairwise isomorphic as $U_{\mathbb{K}}(\mathfrak{sp}_{2n})$-modules if and only if all of the four types are isomorphic.*

Proof Again, this is well-known, see the references in the proof of Lemma 4A.3. □

Example 4A.10 Note that integral tilting $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-modules will be tilting after specialization, but not all tilting modules after specialization come from integral tilting modules. An easy example is $\Delta_{\mathbb{A}}(2\varpi_1)$ which is not integrally tilting but $\Delta_q(2\varpi_1)$ is tilting, or more generally, $\Delta_{\mathbb{K}}(2\varpi_1)$ is tilting if $\xi + \xi^{-1} \neq 0$. ◇

Similarly, the *divided power quantum group* $U_{\mathbb{A}}(\mathfrak{sl}_2)$ is the $\mathbb{A}$-subalgebra of $U_q(\mathfrak{sl}_2)$ generated by the *divided power generators* $E^{(k)} = E^k/[k]_q$ and $F^{(k)} = F^k/[k]_q$, as well as K and K^{-1}.

Remark 4A.11 When considering tilting modules for the quantum group for $\mathfrak{sl}_2$, one generally restricts to type-1 modules, since this subcategory is closed under tensor product. However, the $\mathbb{A}$ version of Lemma 2E.6 allows us to define type-(-1) Weyl modules, dual Weyl modules, and tilting modules, as σ-twists of the respective type-1 analogs, and the theory recalled above still works out. One must just take that the tensor product of two type-(-1) modules is of type-1 and the tensor product of a type-(-1) module with a type-1 module is a type-(-1) module. (In this paper, modules that are not of type-1 will only be relevant for $\mathfrak{sl}_2$.) ◇

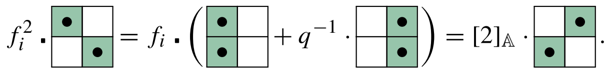
Definition 4A.12 The *vector representation* $V_{\mathbb{A}}$ of $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$ is the free $\mathbb{A}$-submodule of V_q with basis $\{v_1, v_2, \dots, v_n, v_{-n}, \dots, v_{-2}, v_{-1}\}$ and induced action. ◇

Since e.g. $e_i^2 v_j = 0$, we see that $V_{\mathbb{A}}$ is indeed a $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-module.

Proposition 4A.13 *We have the following.*

(a) *There is a* $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$*-action on* $\Lambda_{\mathbb{A}}$*, the* $\mathbb{A}$*-version of the action from Sect. 2B.*
(b) *There is a* $U_{\mathbb{A}}(\mathfrak{sl}_2)$*-action on* $\Lambda_{\mathbb{A}}$*, the* $\mathbb{A}$*-version of the action from Sect. 2E.*
(c) *The actions commute.*

Proof (a). By symmetry between the actions of e_i and f_i, it suffices to show that f_i^k acts on v_S with a scalar $[k]_{\mathbb{A}}!$ that can be canceled. Using Lemma 2B.10 this boils down to a case-by-case check where the hardest local configuration to check is



Here $i \neq n$ and the leftmost column is the ith column. Any further application of f_i annihilates the vectors, so we are done. All other cases are easier and omitted.
(b). Immediate from the construction in Sect. 2E.
(c). As in Proposition 2E.18.

□

Hence, we get a $U_{\mathbb{A}} := U_{\mathbb{A}}(\mathfrak{sp}_{2n}) \otimes U_{\mathbb{A}}(\mathfrak{sl}_2)$-action on $\Lambda_{\mathbb{A}}$. We use this action throughout.

Remark 4A.14 The following definition uses facts from the theory of *Ringel duality*, all of which can be found in e.g. [17, Appendix]. (Note that [17, Appendix] works over a field, but the parts of the theory we use work integrally.) The definition also generalizes immediately beyond the setting of this paper. ◇

Definition 4A.15 Given a $U_{\mathbb{A}}$-module M we define the *M-Schur algebras* as

$$S_{\mathbb{A}}^l = \mathrm{End}_{U_{\mathbb{A}}(\mathfrak{sp}_{2n})}(M), \quad S_{\mathbb{A}}^r = \mathrm{End}_{U_{\mathbb{A}}(\mathfrak{sl}_2)}(M).$$

We call M *Howe tilting* if it is tilting for $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$ and $U_{\mathbb{A}}(\mathfrak{sl}_2)$, as well as full tilting for $S_{\mathbb{A}}^l$ and $S_{\mathbb{A}}^r$, both separately, and satisfies

$$\mathrm{Hom}_{U_{\mathbb{A}}(\mathfrak{sp}_{2n})}\big(\Delta_{\mathbb{A}}(\lambda), M\big) \cong \nabla_{\mathbb{A}}(\lambda'), \quad \mathrm{Hom}_{U_{\mathbb{A}}(\mathfrak{sl}_2)}\big(\Delta_{\mathbb{A}}(\lambda'), M\big) \cong \nabla_{\mathbb{A}}(\lambda)$$

for the bijection $(\_)'$ determined by Ringel duality. ◇

A main ingredient for the integral version of the Howe duality from Theorem 3B.4 is to show that $\Lambda_{\mathbb{A}}$ is Howe tilting for the above $U_{\mathbb{A}}$-action on $\Lambda_{\mathbb{A}}$.

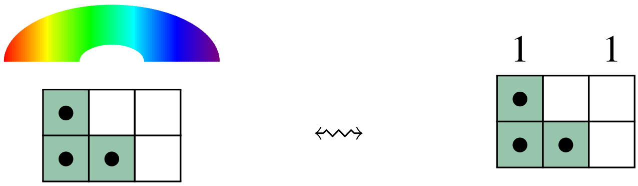
4B Canonical basis

Recall the standard basis from Definition 2A.7. We now define a canonical-type basis (a justification for that name will be given in Sect. 8 below), but we need some notation first. Before that, recall from Notation 2E.12 that $S^c = [1, -1] \setminus S$, $S_0 = S \cap -S$ (S_0 corresponds to fully dotted columns) and $S_0^c = S^c \cap -S^c$ (S_0^c corresponds to undotted columns).

Definition 4B.1 Fix $S \subset [1, -1]$. To every $x \in S_0$ we associate x^S such that either $x^S \in S_0^c$ or $x^S = x$. This is done inductively as follows: reading left-to-right, pick the first not yet considered fully dotted column $\pm x$ and pair it with the rightmost available undotted column $\pm x^S$ if the number of dots to the right of the fully dotted column is strictly smaller than the number of columns to the right, and set $x^S = x$ otherwise. Define $L(S) := \{x \in S_0 | x > 0, x \neq x^S\}$, $R(S) := \{x^S | x \in L(S)\}$ and additionally $\binom{L(S)}{k} := \big\{X \subset L(S) \,|\, |X| = k\big\}$. We also write $\pm L(S) := \{\pm x | x \in L(S)\}$ and $\pm R(S) := \{\pm x | x \in R(S)\}$. ◇

Definition 4B.2 A *rainbow diagram* is a dot diagram where every fully dotted column x with $x \neq x^S$ is paired to some undotted column via an arc on top of the diagram connecting x to x^S. ◇

Notation 4B.3 The name rainbow diagrams comes from us thinking of rainbows connecting a fully dotted column with an undotted column as for example in:



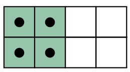
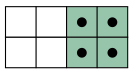
However, as drawing rainbows gets a bit demanding, we decided to simply put numbers on top of the diagram so that the same numbers indicate the start and end of a rainbow; see above for an example. ◇

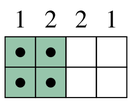
Lemma 4B.4 *The definition of x^S is well-defined such that every left endpoint of a rainbow is a fully dotted column, every right endpoint is an undotted column, and and rainbows do not intersect.*

Proof Directly from the definition. □

Example 4B.5 (a) We continue Example 2E.13. Since $S_0 = \{5, -5\}$, so $x_1 = 5$, and $S_0^c = \{3, 6, -6, -3\}$, and we get $x_1^S = 3$. We get $L(S) = \{5\}$, $R(S) = \{3\}$, $\binom{L(S)}{0} = \{\emptyset\}$ and $\binom{L(S)}{1} = \{\{5\}\}$.

(b) Consider the following example:

$S = S_0 =$ , $S^c = S_0^c =$ ,

rainbow: 

.

We can see that $\pm L(S) = \{1, 2, -2, -1\}$, the left end points of the rainbows, $\pm R(S) = \{3, 4, -4, -3\}$, the right end points of the rainbows.
More examples are given below. ◇

Definition 4B.6 Let $S \subset [1, -1]$. We define *canonical basis vectors* by

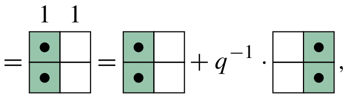
$$b_S := \sum_{p=0}^{|L(S)|} \sum_{\{x_1,\dots,x_p\} \in \binom{L(S)}{p}} q^{-p} \cdot v_{(S \setminus \{\pm x_1,\dots,\pm x_p\}) \cup \{\pm x_1^S,\dots,\pm x_p^S\}}.$$

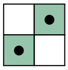
The set $\{b_S | S \subset [1, -1]\}$ is the *canonical basis*. ◇

The rainbow diagrams give the canonical basis by *jumping dots along rainbows*:

Example 4B.7 (a) For $n = 1$ we have $b_S = v_S$ for all $S \subset [1, -1]$ since in this case $S_0 = \emptyset$ or $S_0^c = \emptyset$.

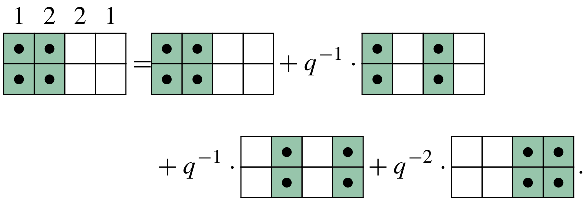
(b) For $n = 2$ we already see a difference between the standard and the canonical bases. Precisely, let $S = \{1, -1\}$ so that $S_0 = S$ and $S_0^c = \{2, -2\}$, and we have $1^S = 2$. For $T = \{2, -1\}$ we have $T_0 = \emptyset$. We get

$b_{\{1,-1\}} =$ 

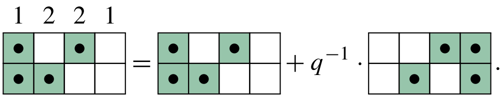
$f_1 \centerdot b_{\{1,-1\}} = [2]_{\mathbb{A}} \cdot$  $= [2]_{\mathbb{A}} \cdot b_{\{2,-1\}}$,

where we use dot diagrams for the standard basis.

(c) Continuing Example 4B.5.(b), we get that $b_{\{1,2,-2,-1\}}$ is equal to



We get that $f_2 \centerdot b_{\{1,2,-2,-1\}} = [2]_{\mathbb{A}} \cdot b_{\{1,3,-2,-1\}}$ since $b_{\{1,3,-2,-1\}}$ is



In general, $b_S = v_S$ if $S_0 = \emptyset$ or $S_0^c = \emptyset$. ◇

As often in the theory of canonical bases, the explicit form of the basis elements is less important than their properties. The same is true for us. For example, the $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-action on the canonical basis is, to a large extend, coefficient free:

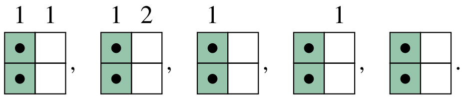
Lemma 4B.8 *For $i \in \{1, \dots, n-1\}$ and $i = n$ we have:*

$$f_i \centerdot b_S = \begin{cases} b_{(S\setminus\{i\})\cup\{i+1\}} & \text{if } S_{i,i+1} = \{i\}, \\ b_{(S\setminus\{-(i+1)\})\cup\{-i\}} & \text{if } S_{i,i+1} = \{-(i+1)\}, \\ b_{(S\setminus\{-(i+1)\})\cup\{-i\}} & \text{if } S_{i,i+1} = \{i, -(i+1)\}, \\ [2]_{\mathbb{A}} \cdot b_{(S\setminus\{i\})\cup\{i+1\}} & \text{if } S_{i,i+1} = \{i, -i\}, \\ b_{(S\setminus\{-(i+1)\})\cup\{-i\}} & \text{if } S_{i,i+1} = \{i+1, -(i+1)\}, \\ b_{(S\setminus\{i\})\cup\{i+1\}} & \text{if } S_{i,i+1} = \{i, -(i+1), -i\}, \\ b_{(S\setminus\{-(i+1)\})\cup\{-i\}} & \text{if } S_{i,i+1} = \{i, i+1, -(i+1)\}, \end{cases}$$

$$f_n \cdot b_S = \begin{cases} b_{(S\setminus\{n\})\cup\{-n\}} & \text{if } S_n = \{n\}. \end{cases}$$

Proof This is an exercise of how to work with rainbow diagrams. We therefore prove only one of the above equations, namely the one with $[2]_{\mathbb{A}}$ which is the most difficult.

In this case there are five local rainbow diagrams, where the dots are in the ith column:



The left case can be checked as in Example 4B.7, and none of the other cases can appear by the construction of rainbow diagrams, cf. Lemma 4B.4. □

Remark 4B.9 Note the missing "otherwise" when comparing Lemma 4B.8 to Lemma 2B.10. Indeed, there are other, rather complicated, coefficients such as for $f_3 \centerdot b_{\{1,2\}}$ when $n = 4$. ◇

Next, we partially describe the action of $U_{\mathbb{A}}(\mathfrak{sl}_2)$ on the canonical basis.

4C Crystal basis for fundamental Weyl modules

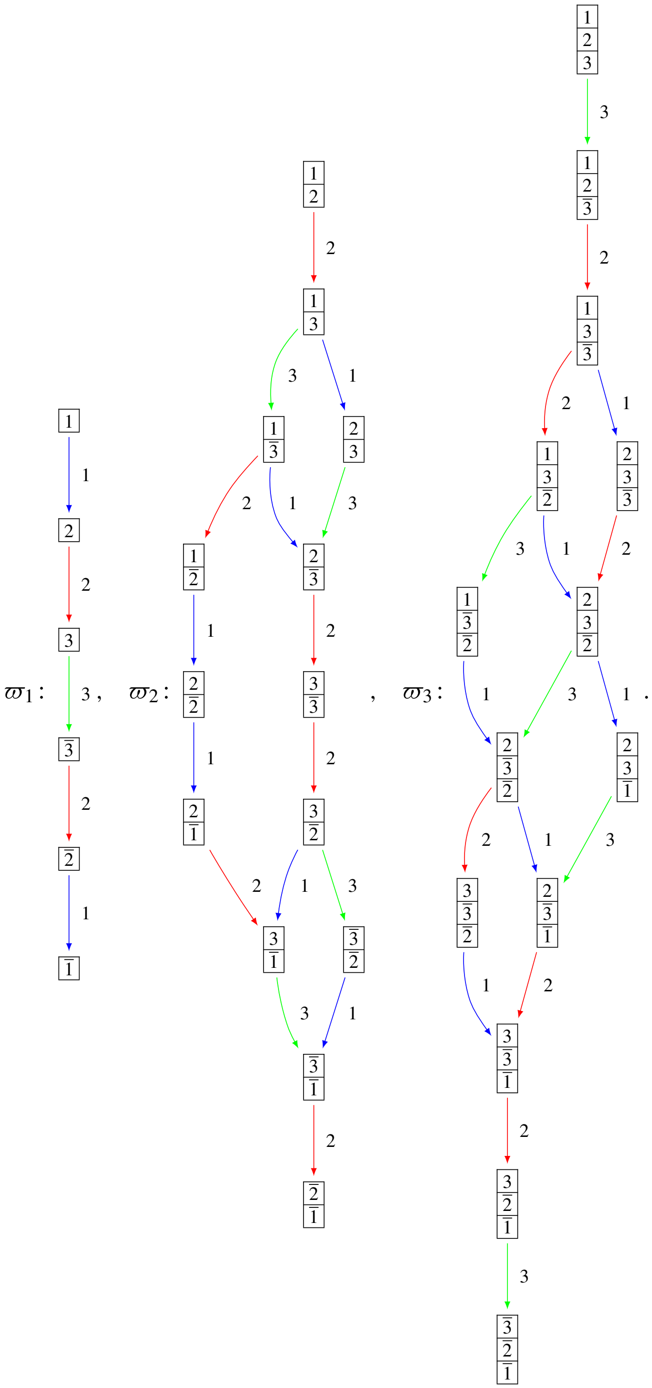
The crystal combinatorics for fundamental weights in classical types is nicely summarized in [13, Chapter 6], which is also our main source for the below.

Example 4C.1 Before we give the abstract definitions, we start with an example that the reader can keep in mind while reading the below.

Roughly, the setting is as follows:

(a) For $\mathfrak{sp}_{2n}$ the ordered underlying set of fillings is $\{1 < \cdots < n < -n < \cdots < -1\}$. Note that, as often in the theory of crystals, $-i$ is denoted $\bar{i}$ in the examples below.
(b) The crystal graph for ϖ_k is modeled on Young tableaux with one column and k rows (a *column tableaux*). That is, the vertices of this graph are Young tableaux with entries from $\{1 < 2 < \cdots < n < -n < \cdots < -2 < -1\}$ that strictly increase when reading columns from top to bottom.
(c) The crystal operator $\tilde{f}_i$, illustrated as i-colored edges, changes i to $i+1$, or $-(i+1)$ to $-i$ for $i \in \{1, \ldots, n-1\}$, and $\tilde{f}_n$ changes n to $-n$. The crystal operator $\tilde{e}_i$ does the opposite and is usually omitted from the pictures.

For $n = 3$, that is $\mathfrak{sp}_6$, the three fundamental crystals are as follows.


ϖ_1 :
ϖ_2 :
ϖ_3 :

All of these crystals were produced using SageMath, see https://doc.sagemath.org/html/en/thematic_Tutorials/lie/crystals.html (last checked early 2023). ◇

The following can be found in [13, Section 6.7 and Proposition 6.7].

Definition 4C.2 The *column tableaux model* of the $\mathfrak{sp}_{2n}$ crystal with highest weight ϖ_k is the set

$$\mathcal{CT}_{\varpi_k} := \{(i_1, \dots, i_k) \in [1, -1]^k | i_1 < i_2 < \cdots < i_k, \text{and if } i_p = x > 0, i_q = -x \text{ then } q - p \leq n - x\}.$$

The elements are called *standard column Young diagrams* and we will write them as words w with the leftmost entry corresponding to the top entry in Young diagram notation.

We further equip this set with the *weight function*

$$\mathrm{wt}(i_1, \dots, i_k) = \sum_{j=1}^{k} \mathrm{wt}\, i_j, \quad \text{where } \mathrm{wt}\, i = \epsilon_i \text{ and } \mathrm{wt}\, {-i} = -\epsilon_i.$$

We also equip this set with the *crystal operators* $\tilde{f}_i$ and $\tilde{e}_i$, for $i \in \{1, \dots, n\}$, that act as follows. Let $i \in \{1, \dots, n-1\}$, then:

$$\tilde{f}_i \centerdot w = \begin{cases} \text{change } i \text{ to } i+1 \text{ if possible,} \\ \text{else change } -(i+1) \text{ to } -i \text{ if possible,} \\ 0 \text{ else,} \end{cases}$$

$$\tilde{f}_n \centerdot w = \begin{cases} \text{change } n \text{to } -n \text{ if possible,} \\ 0 \text{ else,} \end{cases}$$

where “if possible” means that the result is a standard column Young diagram. Similarly for $\tilde{e}_i$, where $i \in \{1, \dots, n-1\}$, and $\tilde{e}_n$. ◇

Lemma 4C.3 *The crystal $\mathcal{CT}_{\varpi_k}$ is a highest weight crystal for the $\mathfrak{sp}_{2n}$-modules $L(\varpi_k)$. In particular, the crystal graph is connected.*

Proof Well-known, see e.g. [13, Chapter 6]. □

Lemma 4C.4 *Let $S = \{s_1 < \cdots < s_k\}$ be such that $|\pm L(S)| = |S_0|$. Then $(s_1, \dots, s_k) \in \mathcal{CT}_{\varpi_k}$.*

Proof View $S \subset [1, -1]$ as a dot diagram with, as before, columns enumerated $1, 2, \dots, n$ when reading left to right. The condition that $|\pm L(S)| = |S_0|$ is equivalent to the condition that for all $x \in S_0$, $x > 0$, $|S_{\geq x}| \leq n - x + 1$, i.e. the number of dots to the right of any fully dotted column is less than or equal to the index of the fully dotted column, ensuring the presence of an empty column to match the x column with. If $s_p = x$ and $s_q = -x$, then $q - p + 1 = |S_{\geq x}| \leq n - x + 1$. Thus, $(s_1, \dots, s_k) \in \mathcal{CT}_{\varpi_k}$. □

The following is an alternative model of the $\mathfrak{sp}_{2n}$ crystal with highest weight ϖ_k, and closer to our previous notation:

Definition 4C.5 We define the $\mathfrak{sp}_{2n}$ crystal with highest weight ϖ_k to be the set:

$$\mathcal{C}_{\varpi_k} := \{S \subset [1, -1] | \, |S| = k, |\pm L(S)| = |S_0|\}.$$

The weight function we use is:

$$\mathrm{wt}\, S = \sum_{s \in S} \mathrm{wt}\, s, \quad \text{where } \mathrm{wt}\, i = \epsilon_i \text{ and } \mathrm{wt}\, {-i} = -\epsilon_i,$$

similarly as in Definition 2A.7.

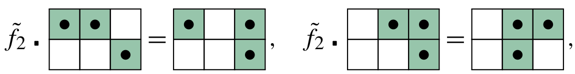
We also equip this set with the crystal operators $\tilde{f}_i$ and $\tilde{e}_i$, for $i \in \{1, \dots, n\}$, that act as follows. Let $i \in \{1, \dots, n-1\}$, then:

$$\tilde{f}_i \centerdot S = \begin{cases} (S \setminus \{i\}) \cup \{i+1\} & \text{if } S_{i,i+1} = \{i\}, \\ (S \setminus \{-(i+1)\}) \cup \{-i\} & \text{if } S_{i,i+1} = \{-(i+1)\}, \\ (S \setminus \{-(i+1)\}) \cup \{-i\} & \text{if } S_{i,i+1} = \{i, -(i+1)\}, \\ (S \setminus \{i\}) \cup \{i+1\} & \text{if } S_{i,i+1} = \{i, -i\}, \\ (S \setminus \{-(i+1)\}) \cup \{-i\} & \text{if } S_{i,i+1} = \{i+1, -(i+1)\}, \\ (S \setminus \{i\}) \cup \{i+1\} & \text{if } S_{i,i+1} = \{i, -(i+1), -i\}, \\ (S \setminus \{-(i+1)\}) \cup \{-i\} & \text{if } S_{i,i+1} = \{i, (i+1), -(i+1)\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{f}_n \centerdot S = \begin{cases} (S \setminus \{n\}) \cup \{-n\} & \text{if } S_n = \{n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly for $\tilde{e}_i$, where $i \in \{1, \dots, n-1\}$, and $\tilde{e}_n$. ◇

Example 4C.6 The action of the crystal operators on dot diagrams is quite nice: the operator $\tilde{f}_i$, say for $i \in \{1, \dots, n-1\}$, shifts a dot in the ith column rightwards, if possible, a dot in the $(i+1)$th column leftwards, if possible, or annihilates the dot diagram. This happens in the indicated order, i.e. one first looks for dots in the ith column etc. For example:



are two examples for $n = k = 3$. ◇

Lemma 4C.7 *The map* $\Phi\colon \mathcal{C}_{\varpi_k} \to \mathcal{CT}_{\varpi_k}$ *sending* $S = \{s_1 < \dots < s_k\}$ *to* $(s_1, \dots, s_k)$ *is an isomorphism of crystals.*

Proof That Φ is a well-defined bijection follows from Lemma 4C.4. Comparing the formulas for actions of the crystal operators above it is easy to see that, setting $\Phi(0) := 0$, Φ intertwines the action of the crystal operators. □

The following two lemmas follow from the theory of crystals via Lemma 4C.7. We, for completeness, included two short arguments.

Lemma 4C.8 *For each* $S \in \mathcal{C}_{\varpi_k}$*, there exists* $r \in \mathbb{Z}_{\geq 0}$ *and a sequence of elements* $j_1, \dots, j_r$ *in* $\{1, \dots, n\}$*, such that*

$$\tilde{f}_{j_1} \dots \tilde{f}_{j_r} \centerdot \{1, \dots, k\} = S.$$

Proof Thanks to Lemma 4C.7, the existence of the sequence $j_1, \dots, j_r$ is a consequence of the crystal graph of $\mathcal{CT}_{\varpi_k}$ being connected, which we established in Lemma 4C.3. □

Definition 4C.9 Let S be as in Lemma 4C.8. Then we say the *length* $\ell(S)$ of S is r. ◇

Lemma 4C.10 *The length of S is well-defined.*

Proof Let $l_1, \dots, l_s$ be a sequence in $\{1, \dots, n\}$ such that

$$\tilde{f}_{l_1} \dots \tilde{f}_{l_s} \centerdot \{1, \dots, k\} = S = \tilde{f}_{j_1} \dots \tilde{f}_{j_r} \centerdot \{1, \dots, k\}.$$

Applying the function wt we find that

$$-\sum_{i=1}^{r} \alpha_{j_i} + \epsilon_1 + \dots + \epsilon_k = \mathrm{wt}\, S = -\sum_{i=1}^{s} \alpha_{l_i} + \epsilon_1 + \dots + \epsilon_k \Rightarrow \sum_{i=1}^{r} \alpha_{j_i} = \sum_{i=1}^{s} \alpha_{l_i}.$$

Since the simple roots form a basis, it follows that there is a bijection $\{j_1, \dots, j_r\} \to \{l_1, \dots, l_s\}$. In particular, $r = s$. □

4D Filtrations for $\mathfrak{sl}_2$

Recall from the $\mathbb{A}$-version of Sect. 2E that $\Lambda_{\mathbb{A}}$ is, as an $U_{\mathbb{A}}(\mathfrak{sl}_2)$-module, isomorphic to $X_{\mathbb{A}}^{\otimes n}$ where $X_{\mathbb{A}} = {}^{1}\mathbb{A}_* \oplus {}^{-1}W_{\mathbb{A}} \oplus {}^{1}\mathbb{A}_*$. In particular, the operators $E^{(k)}$ and $F^{(k)}$ preserve the $\mathbb{A}$ span of the vectors v_S.

Lemma 4D.1 *Let $S \subset [1, -1]$ and recall* $\mathbf{w}$ *from Notation 2E.12. We have:*

$$E^{(k)} \centerdot v_S = q^{\binom{k}{2}} \sum_{\substack{\{\pm i_1, \dots, \pm i_k\} \subset S_0^c \\ i_1 < \dots < i_k}} (-q)^{\sum_{t=1}^{k} \mathbf{w}_{>i_t}(S)} \cdot v_{S \cup \{\pm i_1, \dots, \pm i_k\}},$$

$$F^{(k)} \centerdot v_S = q^{\binom{k}{2}} \sum_{\substack{\{\pm i_1, \dots, \pm i_k\} \subset S_0 \\ i_1 < \dots < i_k}} (-q)^{\sum_{t=1}^{k} \mathbf{w}_{<i_t}(S)} \cdot v_{S \setminus \{\pm i_1, \dots, \pm i_k\}}.$$

Proof We prove the claim for $E^{(k)}$. A similar argument works for $F^{(k)}$.

Let $\{\pm i_1, \dots, \pm i_k\} \subset S_0^c$, with $0 \leq i_1 < \dots < i_k$. Note that

$$\mathbf{w}_{>j}(S \cup \{\pm i_1, \dots, \pm i_k\}) = \mathbf{w}_{>j}(S) + 2 \cdot |\{i_p \mid i_p > j\}|.$$

For σ in the symmetric group $\mathfrak{S}_k$ on $\{1, \dots, k\}$, define

$$\mathbf{w}_{>\sigma}(S) := \mathbf{w}_{>i_{\sigma(1)}}(S) + \mathbf{w}_{>i_{\sigma(2)}}(S \cup \{\pm i_{\sigma(1)}\} + \dots + \mathbf{w}_{>i_{\sigma(k)}}(S \cup \{\pm i_{\sigma(1)}, \dots, \pm i_{\sigma(k-1)}\}).$$

Then, if we write $\ell(\sigma)$ for the length of the permutation σ, i.e. the number of inversions in σ, then $\mathbf{w}_{>\sigma}(S) = \mathbf{w}_{>\mathrm{id}}(S) + 2 \cdot \ell(\sigma)$.

Repeated application of Lemma 2E.16 shows that we have

$$\sum_{\sigma \in \mathfrak{S}_k} (-q)^{\mathbf{w}_{>\sigma}(S)} = (-q)^{\mathbf{w}_{>\mathrm{id}}(S)} \sum_{\sigma \in \mathfrak{S}_k} (-q)^{2\ell(\sigma)} = (-q)^{\mathbf{w}_{>\mathrm{id}}(S)} q^{\binom{k}{2}} [k]_{\mathbb{A}}!$$

as the the coefficient of $v_{S \cup \{\pm i_1, \dots, \pm i_k\}}$ in $E^k \centerdot v_S$. This implies:

$$\begin{aligned} E^{(k)} \centerdot v_S &= q^{\binom{k}{2}} \sum_{\substack{\{\pm i_1, \dots, \pm i_k\} \subset S_0^c \\ i_1 < \dots < i_k}} (-q)^{\sum_{t=1}^{k} \mathrm{wt}_{>i_t}(S \cup \{\pm i_1, \dots, \pm i_{t-1}\})} \cdot v_{S \cup \{\pm i_1, \dots, \pm i_k\}} \\ &= q^{\binom{k}{2}} \sum_{\substack{\{\pm i_1, \dots, \pm i_k\} \subset S_0^c \\ i_1 < \dots < i_k}} (-q)^{\sum_{t=1}^{k} \mathrm{wt}_{>i_t}(S)} \cdot v_{S \cup \{\pm i_1, \dots, \pm i_k\}}. \end{aligned}$$

This proves the statement. □

The following can be, for example, found in [27, Section 5.2]. With a bit of care, cf. [30, Chapter 5], these operators can be defined and used for type-1 and type-(-1) $U_q(\mathfrak{sl}_2)$-modules.

Definition 4D.2 Let M be a finite dimensional type-ϵ $U_q(\mathfrak{sl}_2)$-module. We define *Lusztig's quantum Weyl group element* T as the operator $T\colon M \to M$, which acts on $v \in M[k] = \{v \in M \mid K \centerdot v = \epsilon q^k \cdot v\}$ as

$$T \centerdot v = \sum_{\substack{a,b,c\geq 0\\ -a+b-c=k}} (-1)^b q^{b-ac} \cdot E^{(a)}F^{(b)}E^{(c)} \centerdot v.$$

(Note that these sums contain only finitely many terms, since M is finite dimensional so E and F act nilpotently.) ◇

Lemma 4D.3 *Let M be a finite dimensional type-ϵ $U_q(\mathfrak{sl}_2)$-module. Then*

$$T|_{M[k]}\colon M[k] \xrightarrow{\cong} M[-k]$$

is an $\mathbb{F}$-linear isomorphism. The inverse of T is the operator $T^{-1}\colon M \to M$, which acts on $w \in M[k]$ as

$$T^{-1} \centerdot w = \sum_{\substack{a,b,c\geq 0\\ a-b+c=m}} (-1)^b q^{ac-b} \cdot F^{(a)}E^{(b)}F^{(c)} \centerdot w.$$

Proof See [27, Section 8.4] or [30, Proposition 5.2.7]. □

Lemma 4D.4 *Suppose that M is a finite dimensional type-ϵ $U_q(\mathfrak{sl}_2)$-module and that $M_{\mathbb{A}} \subset M$ is preserved by $U_{\mathbb{A}}(\mathfrak{sl}_2)$ and satisfies $M_{\mathbb{A}}[k] = M_{\mathbb{A}} \cap M[k]$ for all $k \in \mathbb{Z}$. Then*

$$T|_{M_{\mathbb{A}}[k]}\colon M_{\mathbb{A}}[k] \xrightarrow{\cong} M_{\mathbb{A}}[-k]$$

is an $\mathbb{A}$-linear isomorphism.

Proof Since $E^{(i)}$ and $F^{(i)}$ preserve $M_{\mathbb{A}}$, it follows that T preserves $M_{\mathbb{A}}$. Then, noting that T maps $M[k]$ to $M[-k]$ and $M_{\mathbb{A}}[\pm k] = M_{\mathbb{A}} \cap M[\pm k]$, we see T maps $M_{\mathbb{A}}[k]$ to $M_{\mathbb{A}}[-k]$. Similarly, we find that T^{-1} maps $M_{\mathbb{A}}[-k]$ to $M_{\mathbb{A}}[k]$. □

Proposition 4D.5 *The $U_{\mathbb{A}}(\mathfrak{sl}_2)$-module $\Lambda_{\mathbb{A}}$ is a tilting module. In particular, $\Lambda_{\mathbb{A}}$ has a Weyl module filtration with Weyl character*

$$[\Lambda_{\mathbb{A}}] = \sum_{k=0}^{n} \mathrm{rk}_{\mathbb{A}} \Delta_{\mathbb{A}}(\varpi_k)[^{\epsilon}\Delta_{\mathbb{A}}(n-k)],$$

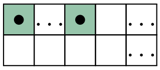
where $\epsilon = 1$, if k is even, and $\epsilon = -1$, if k is odd.

Proof Because we have that ${}^{1}\mathbb{A}_{*}$ and ${}^{-1}W_{\mathbb{A}}$ are tilting $U_{\mathbb{A}}(\mathfrak{sl}_2)$-modules (this is well-known and easy to check by hand), the claim that $\Lambda_{\mathbb{A}}$ is a tilting module, such that Λ^{even} is a type-1 module and Λ^{odd} is a type-(-1) module, follows directly from $\Lambda_{\mathbb{A}} \cong X_{\mathbb{A}}^{\otimes n}$ and the fact that tensor products of tilting modules are tilting, see Lemma 4A.6. To compute the Weyl character of $\Lambda_{\mathbb{A}}$ it suffices to determine the character of Λ_q, which is known by Theorem 3B.4. □

4E Embedding fundamental Weyl modules in exterior powers

Notation 4E.1 Let $B^k_{\Delta(\varpi_k)} = \{b_S \mid |S| = k \text{ and } |\pm L(S)| = |S_0|\}$. ◇

Lemma 4E.2 *For each $b_S \in B^k_{\Delta(\varpi_k)}$ there is an element $u_S \in U_{\mathbb{A}}(\mathfrak{sp}_{2n})$ such that $u_S \centerdot v_{\{1,\dots,k\}} = b_S$, that is*

$u_S \centerdot$  $= b_S,$

where the rightmost dot is in the kth column.

Proof Let $b_S \in B^k_{\Delta(\varpi_k)}$, i.e. $S \in \mathcal{C}_{\varpi_k}$, such that $\ell(S) = r$.

If $r = 0$, then $S = \{1, \dots, k\}$, $1 \in U_{\mathbb{A}}(\mathfrak{sp}_{2n})$, and $1 \centerdot v_{\{1,\dots,k\}} = b_{\{1,\dots,k\}}$. Suppose inductively that the result is true for all $b_T \in B^k_{\Delta(\varpi_k)}$ such that $\ell(T) < r$.

Let $r > 0$. There is a sequence of elements $j_1, \dots, j_r$ in $\{1, \dots, n\}$ such that

$$\tilde{f}_{j_1} \dots \tilde{f}_{j_r} \centerdot \{1, \dots, k\} = S.$$

Note that from Definition 4C.5 we have two options. The first option is that $j_1 \in \{1, \dots, n-1\}$ and $(\tilde{f}_{j_2} \dots \tilde{f}_{j_r} \centerdot \{1, \dots, k\})_{j_1, j_1+1}$ is one of the following:

$$\{j_1\}, \{-(j_1+1)\}, \{j_1, -(j_1+1)\}, \{\pm j_1\}, \{\pm(j_1+1)\}, \{\pm j_1, -(j_1+1)\}, \{j_1, \pm(j_1+1)\}.$$

Or, as a second option, $j_1 = n$ and $(\tilde{f}_{j_2} \dots \tilde{f}_{j_r} \centerdot \{1, \dots, k\})_{j_1} = \{n\}$.

By induction, there is an element $u_{\tilde{f}_{j_2} \dots \tilde{f}_{j_r} \centerdot \{1,\dots,k\}} \in U_{\mathbb{A}}(\mathfrak{sp}_{2n})$ such that

$$u_{\tilde{f}_{j_2} \dots \tilde{f}_{j_r} \centerdot \{1,\dots,k\}} \centerdot v_{\{1,\dots,k\}} = b_{\tilde{f}_{j_2} \dots \tilde{f}_{j_N} \centerdot \{1,\dots,k\}}.$$

Excluding the $\{\pm j_1\}$ case, we can compare the formulas in Definition 4C.5 and Lemma 4B.8 to deduce that

$$F_{j_1} b_{\tilde{f}_{j_2} \dots \tilde{f}_{j_r} \centerdot \{1,\dots,k\}} = b_{\tilde{f}_{j_1} \tilde{f}_{j_2} \dots \tilde{f}_{j_r} \centerdot \{1,\dots,k\}}.$$

Thus, we have $F_{j_1} u_{\tilde{f}_{j_2} \dots \tilde{f}_{j_r} \centerdot \{1,\dots,k\}} \in U_{\mathbb{A}}(\mathfrak{sp}_{2n})$ such that

$$F_{j_1} u_{\tilde{f}_{j_2} \dots \tilde{f}_{j_r} \centerdot \{1,\dots,k\}} \centerdot v_{\{1,\dots,k\}} = b_s.$$

It remains to consider the case where $j_1 \in \{1, \dots, n-1\}$ and $(\tilde{f}_{j_2} \dots \tilde{f}_{j_r} \centerdot \{1, \dots, k\})_{j_1, j_1+1} = \{\pm j_1\}$. It follows that

$$(\tilde{f}_{j_2} \dots \tilde{f}_{j_r} \centerdot \{1, \dots, k\}) \setminus \{-j_1\} \cup \{-(j_1+1)\} \in \mathcal{C}_{\varpi_k},$$

and therefore there is a sequence $l_1, \dots, l_s$ in $\{1, \dots, n\}$ such that

$$\tilde{f}_{l_1} \dots \tilde{f}_{l_s} \centerdot \{1, \dots, k\} = \big((\tilde{f}_{j_2} \dots \tilde{f}_{j_r} \centerdot \{1, \dots, k\}) \setminus \{-j_1\}\big) \cup \{-(j_1+1)\}.$$

Then from Definition 4C.5 we find

$$\tilde{f}_{j_1} \tilde{f}_{j_1} \tilde{f}_{l_1} \dots \tilde{f}_{l_s} \centerdot \{1, \dots, k\} = S.$$

In particular, we have $2 + s = \ell(S) = r$, so $s < r$. By induction, there is an element $u_{\tilde{f}_{l_1} \dots \tilde{f}_{l_s} \centerdot \{1,\dots,k\}} \in U_{\mathbb{A}}(\mathfrak{sp}_{2n})$ such that $u_{\tilde{f}_{l_1} \dots \tilde{f}_{l_s} \centerdot \{1,\dots,k\}} \centerdot v_{\{1,\dots,k\}} = b_{\tilde{f}_{l_1} \dots \tilde{f}_{l_s} \centerdot \{1,\dots,k\}}$. Comparing the formulas in Definition 4C.5 and Lemma 4B.8, we find

$$f_{j_1} f_{j_1} u_{\tilde{f}_{l_1} \dots \tilde{f}_{l_s} \centerdot \{1,\dots,k\}} \centerdot v_{\{1,\dots,k\}} = [2]_{\mathbb{A}} b_{\tilde{f}_{j_1} \tilde{f}_{j_1} \tilde{f}_{l_1} \dots \tilde{f}_{l_s} \centerdot \{1,\dots,k\}}.$$

Thus, we have $f_{j1}^{(2)} u_{\tilde{f}_{l_1}\dots\tilde{f}_{l_s}\centerdot\{1,\dots,k\}} \in U_{\mathbb{A}}(\mathfrak{sp}_{2n})$ and

$$f_{j1}^{(2)} u_{\tilde{f}_{l_1}\dots\tilde{f}_{l_s}\centerdot\{1,\dots,k\}} \centerdot v_{\{1,\dots,k\}} = b_s.$$

This completes the proof. □

Recall that F denotes the Chevalley generator from $U_{\mathbb{A}}(\mathfrak{sl}_2)$.

Lemma 4E.3 *If $b_S \in B^k_{\Delta(\varpi_k)}$, then $F \centerdot b_S = 0$.*

Proof From Lemma 2E.16, we have $F \centerdot v_{\{1,\dots,k\}} = 0$. Let $b_S \in B^k_{\Delta(\varpi_k)}$. By Lemma 4E.2 there is u_S such that $b_S = u_S \centerdot v_{\{1,\dots,k\}}$. It follows that

$$F \centerdot b_S = F \centerdot (u_S \centerdot v_{\{1,\dots,k\}}) = u_S \centerdot (F \centerdot v_{\{1,\dots,k\}}) = u_S \centerdot 0 = 0,$$

which uses Proposition 4A.13.(c). □

Lemma 4E.4 *We have $\mathbb{F}B^k_{\Delta(\varpi_k)} = \ker F|_{\Lambda^k_q}$ as $U_q(\mathfrak{sp}_{2n})$-modules.*

Proof Lemma 4E.3 and Corollary 3B.8 implies that

$$\mathbb{F}B^k_{\Delta(\varpi_k)} \subset \ker F|_{\Lambda^k_q} \xrightarrow{\cong} L_q(\varpi_k).$$

Moreover, from Definition 4C.5 and Lemma 4C.7, one can observe that the character of $\mathbb{F}B^k_{\Delta(\varpi_k)}$ is the same as the character of $L_q(\varpi_k)$. Hence, the inclusion $\mathbb{F}B^k_{\Delta(\varpi_k)} \to L_q(\varpi_k)$ is an isomorphism. □

Let $X = X(v \to b)$ be the matrix with rows indexed by $\{T \subset [1,-1] \mid |T| = k\}$ and columns indexed by $\{S \subset [1,-1] \mid |S| = k, |R(S)| = |S_0|\}$ such that the (T,S) entry is the coefficient of v_T in b_S. The matrix X is a type of *base change matrix* between the standard and the canonical basis.

Lemma 4E.5 *There is an $\mathbb{A}$-matrix $Y = Y(b \to v)$ with the opposite indexing when compared to X, such that $YX = \mathrm{id}$.*

Proof If we partially order the subsets of $[1,-1]$ by how far fully dotted columns are to the right, then X is a unitriangular matrix with values in $\mathbb{A}$. The lemma follows. □

Remark 4E.6 The base change argument in Lemma 4E.5 is often used in the theory of crystals, see for example [34, Proof of Lemma 4.5] for an early reference from where we got the idea from. ◇

Lemma 4E.7 *We have $\mathbb{A}B^k_{\Delta(\varpi_k)} = \ker F|_{\Lambda^k_q}$ as $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-modules.*

Proof Because of Lemma 4E.3, it suffices to show that $\ker F|_{\Lambda^k_{\mathbb{A}}}$ is contained in $\mathbb{A}B^k_{\Delta(\varpi_k)}$. Let $x \in \ker F|_{\Lambda^k_{\mathbb{A}}}$. Then in particular,

$$x = \sum_{T \subset [1,-1], |T| = k} a_T \cdot v_T, \quad a_T \in \mathbb{A}.$$

Viewing $x \in \ker F|_{\Lambda^k_q} = \mathbb{F}B_{\Delta(\varpi_k)}$, with the equality coming from Lemma 4E.4, we can write

$$x = \sum_{\substack{S \subset [1,-1], |S| = k \\ |L(S)| = |S_0|}} k_S \cdot b_S, \quad k_S \in \mathbb{F}.$$

It follows from Lemma 4E.5 that

$$k_S = \sum_{T \subset [1,-1], |T|=k} Y_{S,T} a_T \in \mathbb{A},$$

where Y is the matrix from Lemma 4E.5. Hence, we get $x \in \mathbb{A}B^k_{\Delta(\varpi_k)}$. □

Let $v^+_{\varpi_k}$ denote the highest weight vector of $\Delta_{\mathbb{A}}(\varpi_k)$ (this is unique up to some choice of scalars in $\mathbb{A}$; the choice does not play any role for us)

Proposition 4E.8 *There is an isomorphism of $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-modules*

$$\Delta_{\mathbb{A}}(\varpi_k) \to \ker F|_{\Lambda^k_{\mathbb{A}}}$$

such that $v^+_{\varpi_k} \mapsto v_{\{1,\dots,k\}}$.

Proof Write $L_q(\varpi_k) = \mathbb{F} \otimes \Delta_{\mathbb{A}}(\varpi_k)$, and continue to denote the highest weight vector by $v^+_{\varpi_k}$. By Corollary 3B.8 we have

$$v_{\{1,\dots,k\}} \in \ker F|_{\Lambda^k_{\mathbb{A}}} \subset \ker F|_{\Lambda^k_q} \xrightarrow{\cong} L_q(\varpi_k)$$

such that $v_{\{1,\dots,k\}} \mapsto v^+_{\varpi_k}$.

Thanks to Proposition 4A.13, Lemma 4A.1 implies that $\ker F|_{\Lambda^k_{\mathbb{A}}}$ is preserved by $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$. Lemma 4E.7 and Lemma 4E.2 imply that $\ker F|_{\Lambda^k_{\mathbb{A}}}$ is generated over $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$ by $v_{\{1,\dots,k\}}$. The result follows from the description of Weyl modules in Lemma 4A.3. □

4F Filtrations for $\mathfrak{sp}_{2n}$

We now aim to show that the exterior algebra $\Lambda_{\mathbb{A}}$ is a tilting $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-module.

Lemma 4F.1 *Let $0 \leq k \leq n$ and let $i \geq 0$ be minimal such that $k - 2i < 0$. Then $\Lambda^k_{\mathbb{A}} \cong \ker F^{(i)}|_{\Lambda^k_{\mathbb{A}}}$ as $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-modules.*

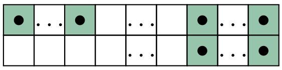
Proof This follows since F removes a fully dotted column from dot diagrams, see Definition 6.6. □

Note that Lemma 4F.1 implies that $\Lambda^k_{\mathbb{A}}$ has the following filtration by $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-submodules:

$$0 = \ker F^{(0)}|_{\Lambda^k_{\mathbb{A}}} \subset \ker F^{(1)}|_{\Lambda^k_{\mathbb{A}}} \subset \cdots \subset \ker F^{(i-1)}|_{\Lambda^k_{\mathbb{A}}} \subset \ker F^{(i)} = \Lambda^k_{\mathbb{A}}.$$

Our goal is to show that this is a Weyl filtration.

Notation 4F.2 Let $0 \leq k \leq n$ and let $i \geq 0$ such that $k - 2i \geq 0$. Then we write

$v_{k,k-2i} \leftrightsquigarrow$  ,

with the single dots in columns 1 to $k - 2i$ and the fully dotted columns $(n - i + 1)$ to n. Or in formulas, $v_{k,k-2i} := v_{\{1,\dots,k-2i\} \cup \{\pm(n-i+1),\dots,\pm n\}}$. ◇

Remark 4F.3 A consequence of Proposition 4E.8 is that $v_{k,k} = v_{\{1,\dots,k\}}$ is a highest weight vector generating a $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-submodule isomorphic to $\Delta_{\mathbb{A}}(\varpi_k)$. However, for $i > 0$, the vector $v_{k,k-2i}$ is not a singular vector. ◇

Remark 4F.4 If $x \in \ker F^{(i+1)}|_{\Lambda^k_{\mathbb{A}}}$, then $F^{(i)} \centerdot x \in \Lambda^{k-2i}_{\mathbb{A}}$ and $F \centerdot (F^{(i)} \centerdot x) = [i+1]_{\mathbb{A}} \cdot F^{(i+1)} \centerdot x = 0$. Combining this observation with Proposition 4A.13, we see that $F^{(i)}$ induces a map $F^{(i)} \centerdot \_ \in \mathrm{Hom}_{U_{\mathbb{A}}(\mathfrak{sp}_{2n})}(\ker F^{(i+1)}|_{\Lambda^k_{\mathbb{A}}}, \ker F|_{\Lambda^{k-2i}_{\mathbb{A}}})$. ◇

Lemma 4F.5 *Let $0 \leq k \leq n$ and let $i \geq 0$ such that $k - 2i \geq 0$. Then*

$$F^{(i)} \centerdot v_{k,k-2i} = \xi \cdot v_{k-2i,k-2i},$$

where $\xi = \pm q^j$ for some $j \in \mathbb{Z}$. In particular, $v_{k,k-2i} \in \ker F^{(i+1)}|_{\Lambda^k_{\mathbb{A}}}$.

Proof It follows from Lemma 4D.1 that $F^{(i)}(v_{k,k-2i}) = q^{\binom{i}{2}}(-q)^{i(n-k+i)+\binom{i}{2}} \cdot v_{k-2i,k-2i}$. □

Lemma 4F.6 *There is a surjection*

$$F^{(i)} \centerdot \_ \colon \ \ker F^{(i+1)}|_{\Lambda^k_{\mathbb{A}}} \twoheadrightarrow \ker F|_{\Lambda^{k-2i}_{\mathbb{A}}}.$$

Proof By Lemma 4F.5 the vector $v_{k,k-2i} \in \ker F^{(i+1)}|_{\Lambda^k_{\mathbb{A}}}$ and $F^{(i)} \centerdot v_{k,k-2i} = \xi \cdot v_{k-2i,k-2i}$ for $\xi = \pm q^j$ for some $j \in \mathbb{Z}$. Let $x \in \ker F|_{\Lambda^{k-2i}_{\mathbb{A}}}$. It follows from Proposition 4E.8 that there is $u \in U_{\mathbb{A}}(\mathfrak{sp}_{2n})$ such that $u \centerdot v_{k-2i,k-2i} = x$. Noting that $\xi^{-1} \in \mathbb{A}$, we can apply Lemma 4A.1 to see that $\xi^{-1} \cdot u \centerdot v_{k,k-2i} \in \ker F^{(i+1)}|_{\Lambda^k_{\mathbb{A}}}$. Since

$$F^{(i)} \centerdot (\xi^{-1} \cdot u \centerdot v_{k,k-2i}) = \xi^{-1} \cdot u \centerdot (F^{(i)} \centerdot v_{k,k-2i}) = x,$$

the lemma follows. □

We now have the desired Weyl filtration.

Proposition 4F.7 *Let $k \in \{1, \ldots, n\}$. We have an isomorphism of $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-modules*

$$\ker F^{(i+1)}|_{\Lambda^k_{\mathbb{A}}} / \ker F^{(i)}|_{\Lambda^k_{\mathbb{A}}} \cong \Delta_{\mathbb{A}}(\varpi_{k-2i}).$$

In particular, the $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-module $\Lambda^k_{\mathbb{A}}$ has a filtration by Weyl modules with Weyl character

$$[\Lambda^k_{\mathbb{A}}] = \sum_{k-2i \geq 0} [\Delta_{\mathbb{A}}(\varpi_{k-2i})].$$

(The final equation holds in the Grothendieck ring.)

Proof By Lemma 4F.6 and Proposition 4E.8, we have

$$\ker F^{(i+1)}|_{\Lambda^k_{\mathbb{A}}} / \ker F^{(i)}|_{\Lambda^k_{\mathbb{A}}} \cong \ker F|_{\Lambda^{k-2i}_{\mathbb{A}}} \cong \Delta_{\mathbb{A}}(\varpi_{k-2i}).$$

Suppose j is such that $k - 2j < 0$ and $k - 2j + 2 \geq 0$. Then $\Lambda^k \subset \ker F^{(j)}$ and the filtration

$$0 = \ker F^{(0)}|_{\Lambda^k_{\mathbb{A}}} \subset \ker F^{(1)}|_{\Lambda^k_{\mathbb{A}}} \subset \cdots \subset \ker F^{(j-1)}|_{\Lambda^k_{\mathbb{A}}} \subset \ker F^{(j)} = \Lambda^k_{\mathbb{A}}$$

has subquotients isomorphic to

$$\Delta_{\mathbb{A}}(\varpi_k), \Delta_{\mathbb{A}}(\varpi_{k-2}), \ldots, \Delta_{\mathbb{A}}(\varpi_{k-2j+2}).$$

Thus, $\Lambda^k_{\mathbb{A}}$ has Weyl character as claimed in the proposition. □

Proposition 4F.8 *The $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-module $\Lambda_{\mathbb{A}}$ is a tilting module.*

Proof Let $0 \leq k \leq n$. Proposition 4F.7 implies that $\Lambda^k_{\mathbb{A}}$ has a Weyl filtration. Thanks to Lemma 2E.6, Remark 2B.13, and Proposition 2D.4, we find that $\Lambda^k_{\mathbb{A}} \cong ({}^{\omega}\Lambda^k_{\mathbb{A}})^*$, so Λ^k also has a dual Weyl filtration.

Since $U_{\mathbb{A}}(\mathfrak{sl}_2)$ acts on $\Lambda_{\mathbb{A}}$, with weight spaces $\Lambda^i_{\mathbb{A}}$, such that $0 \leq i \leq 2n$, Lemma 4D.4 implies that $T|_{\Lambda^k_{\mathbb{A}}}$ is an isomorphism $\Lambda^k_{\mathbb{A}} \xrightarrow{\cong} \Lambda^{2n-k}_{\mathbb{A}}$. Since T is an $\mathbb{A}$-linear combination of elements in $U_{\mathbb{A}}(\mathfrak{sl}_2)$, Proposition 4A.13 implies that

$$T \in \mathrm{End}_{U_{\mathbb{A}}(\mathfrak{sp}_{2n})}(\Lambda_{\mathbb{A}}).$$

Thus, $\Lambda^i_{\mathbb{A}}$ is a tilting module for $0 \leq i \leq 2n$, and since a direct sum of tilting modules is a tilting module $\Lambda_{\mathbb{A}} = \bigoplus_{i=0}^{2n} \Lambda^i_{\mathbb{A}}$ is a tilting module. □

5 Howe duality: part II

We now prove Theorem 1B.1. We start below by collecting well-known material which we then apply to get new results.

5A Some generalities, e.g. based modules and Weyl filtrations

We fix a finite dimensional simple Lie algebra $\mathfrak{g}$. There are associated weights X, roots Φ, positive roots Φ_+, dominant weights X_+, and the Weyl group W which acts on X and has longest element w_0.

[30, Chapter 24] defines a non-unital $\mathbb{F}$-algebra, the *idempotented form* $\dot{U}_q(\mathfrak{g})$, with mutually orthogonal idempotents $\mathbf{1}_\lambda$ for $\lambda \in X$, such that

$$\dot{U}_q(\mathfrak{g}) = \bigoplus_{\lambda,\mu \in X} \mathbf{1}_\mu \, \dot{U}_q(\mathfrak{g}) \, \mathbf{1}_\lambda,$$

and a divided powers form $\dot{U}_{\mathbb{A}}(\mathfrak{g})$ which is generated as an $\mathbb{A}$-algebra by $E^{(m)}_\alpha \mathbf{1}_\lambda$ and $F^{(m)}_\alpha \mathbf{1}_\lambda$.

[30, Chapter 25] describes a way to extend the canonical basis for the positive part $U^+_q(\mathfrak{g})$ to a basis $\dot{\mathbf{B}}$ of $\dot{U}_q(\mathfrak{g})$ such that

$$\mathbb{A}\dot{\mathbf{B}} = \dot{U}_{\mathbb{A}}(\mathfrak{g}).$$

Moreover, if $b \in \dot{\mathbf{B}}$, then $b \in \dot{U}_q(\mathfrak{g}) \, \mathbf{1}_\lambda$ for a unique $\lambda \in X$.

The following can be found in [30], and is nicely summarized in [28, Section 1] (we however drop the condition on being finite dimensional from those papers):

Definition 5A.1 Let (M, B, ϕ_M) be a triple consisting of a (finite dimensional) type-1 $\dot{U}_q(\mathfrak{g})$-module M, a $\mathbb{K}$-basis B of M and an involution $\phi_M \colon M \to M$. The triple is a *based* $\dot{U}_{\mathbb{A}}(\mathfrak{g})$*-module* if $B = \bigcup_{v\in X}(B \cap M_v)$ (weight spaces), $M_{\mathbb{A}} = \mathbb{A}B$ is $\dot{U}_{\mathbb{A}}(\mathfrak{g})$-stable, $\phi_M(b) = b$ for $b \in B$, $\phi_M(u \centerdot m) = \phi(u) \centerdot \phi_M(m)$ with ϕ swapping K_i and K_i^{-1}, and the pair $(L(M) = \mathbb{Q}(q)_{(q)}B, \overline{B})$ is a crystal base of M for $\overline{B}$ the image of B in $L(M)/qL(M)$. ◇

Example 5A.2 The left regular $\dot{U}_q(\mathfrak{g})$-module $\dot{U}_q(\mathfrak{g})$ is a based module with basis $\dot{\mathbf{B}}$. The involution used in this example is as in [27, Section 11.9] (essentially, send e_i to e_i, f_i to f_i, and invert k_i and q). ◇

Example 5A.3 Weyl modules are prototypical examples of based modules. That is, each Weyl module $\Delta_{\mathbb{A}}(\lambda)$, where $\lambda \in X_+$, comes with a distinguished highest weight vector v_λ^+ and canonical basis $\mathbf{B}_{\Delta(\lambda)}$. The involution is again as in [27, Section 11.9]. There is a unique canonical basis element $v_\lambda^- \in \Delta(\lambda)$ which lies in the $w_0(\lambda)$ weight space. ◇

For based modules we have the notion of tensor product (that works similarly as for crystals) [30, Section 27.3]. In particular, for each $\lambda, \mu \in X_+$, there is a canonical basis $\mathbf{B}_{\Delta(\lambda)\otimes\Delta(\mu)}$ of $\Delta_{\mathbb{A}}(\lambda) \otimes \Delta_{\mathbb{A}}(\mu)$. The involution is described in [30, Theorem 24.3.3].

Notation 5A.4 For $\lambda \in X_+$, we write $M[\lambda]$ to denote the sum of submodules of M isomorphic to $\Delta(\lambda)$, and write $M[\geq \lambda] := \oplus_{\mu \geq \lambda} M[\mu]$. ◇

Definition 5A.5 For $b \in B$, there is a unique $\mu \in X_+$ which is maximal (with respect to the dominance order on X_+) such that $b \in M[\geq \mu]$ [30, Section 27.2.1]. This defines a map $B \to X_+$. Denote by $B[\mu]$ the fiber of this map over $\mu \in X_+$. ◇

Recall the following from [30, Section 29.1.1]: Let $b \in \dot{\mathbf{B}} \cap \dot{U}_q(\mathfrak{g})\,\mathbf{1}_\zeta$. Choose $\lambda \in X_+$ such that $(\alpha_i^\vee, \lambda)$ is "large enough", see [30, Section 25.2], for all i. Then $b \cdot v^-_{-w_0(\lambda)} \otimes v^+_{\lambda+\xi}$ is in the canonical basis $\mathbf{B}_{\Delta(-w_0(\lambda))\otimes\Delta(\lambda+\xi)}$, and by [30, Section 27.2.1] there is a unique $\mu \in X_+$ such that

$$b \cdot v^-_{-w_0(\lambda)} \otimes v^+_{\lambda+\xi} \in \mathbf{B}_{\Delta(-w_0(\lambda))\otimes\Delta(\lambda+\xi)}[\mu].$$

The weight $\mu \in X_+$ does not depend on the choice of λ and we say $b \in \dot{\mathbf{B}}[\mu]$. We have $\dot{\mathbf{B}} = \coprod_{\lambda \in X_+} \dot{\mathbf{B}}[\lambda]$.

Notation 5A.6 If $S \subset X_+$, then write $\dot{\mathbf{B}}[S] = \coprod_{\lambda \in S} \dot{\mathbf{B}}[\lambda]$ and $\dot{U}_{\mathbb{A}}(\mathfrak{g})[S] := \mathbb{A}\dot{\mathbf{B}}[S]$. ◇

Lemma 5A.7 *Suppose that $S \subset X_+$ is such that:*

(i) *If $\lambda \in S$ and $\lambda' \in X_+$ is such that $\lambda' > \lambda$, then $\lambda \in S$.*
(ii) *$X_+ \setminus S$ is finite.*

Then $\dot{U}_{\mathbb{A}}(\mathfrak{g})[S] \subset \dot{U}_{\mathbb{A}}(\mathfrak{g})$ is a two-sided ideal.

Proof This is [30, Section 29.2.1]. □

Recall the notion of a homomorphism of based modules from [30, Section 27.1.3]: such a homomorphism is a $U_q(\mathfrak{g})$-equivariant map $f : M \to M'$ such that $B \subset B' \cup \{0\}$.

Lemma 5A.8 *Suppose that $S \subset X_+$ satisfies the hypothesis of Lemma 5A.7. Then the quotient map*

$$\dot{U}_{\mathbb{A}}(\mathfrak{g}) \to \dot{U}_{\mathbb{A}}(\mathfrak{g})/\dot{U}_{\mathbb{A}}(\mathfrak{g})[S]$$

is a homomorphism of based modules.

Proof Since $\dot{U}_{\mathbb{A}}(\mathfrak{g})[S]$ is a based ideal, the quotient, equipped with basis $\dot{\mathbf{B}}[X_+ \setminus S]$, is a based module by [30, Section 27.1.4]. □

We recall the following key observation which is the main point in [28, Corollary 1.5]:

Lemma 5A.9 *Suppose that M is a (finite dimensional) based $\dot{U}_{\mathbb{A}}(\mathfrak{g})$-module. Then M has a filtration by Weyl modules.*

Proof This is a consequence of repeatedly applying [30, Proposition 27.1.7]. □

5B Quantum Schur algebras

We now recall the setting in [19], and explain how we can use this to quantize an important argument from [1].

Definition 5B.1 Let $\pi \subset X_+$ be a set of dominant weights such that if $\lambda \in \pi$ and $\mu \leq \lambda$, then $\mu \in \pi$. We say that π is a *saturated* set of weights and write

$$W \centerdot \pi := \{w \centerdot \lambda | \lambda \in \pi\}.$$

Write $\pi^c := X_+ \setminus \pi$. ◇

Note that π^c satisfies the hypothesis in Lemma 5A.7.

Definition 5B.2 We define the *(generalized quantum) Schur algebra* $S^{\pi}_{\mathbb{A}}(\mathfrak{g})$ as the quotient of $\dot{U}_{\mathbb{A}}(\mathfrak{g})$ by the ideal generated by $\mathbf{1}_{\chi}$ such that $\chi \notin W\centerdot\pi$. ◇

Definition 5B.2 appeared in [19, Definition 1.5]. It is easy to see that the $\mathbb{A}$-algebra $S^{\pi}_{\mathbb{A}}(\mathfrak{g})$ is unital with unit $\mathbf{1}_{\pi} := \sum_{\lambda \in W\centerdot\pi} \mathbf{1}_{\lambda}$.

Proposition 5B.3 *There is an explicit isomorphism* $\dot{U}_{\mathbb{A}}(\mathfrak{g})/\dot{U}_{\mathbb{A}}(\mathfrak{g})[\pi^c] \to S^{\pi}_{\mathbb{A}}(\mathfrak{g})$ *and* $\dot{\mathbf{B}}[\pi]$ *descends to an* $\mathbb{A}$*-basis of* $S^{\pi}_{\mathbb{A}}(\mathfrak{g})$.

Proof See [19, Theorem 4.2]. □

Remark 5B.4 Let $\lambda \in \pi$. Then $\mathbf{1}_{\pi}$ acts as the identity on $\Delta_{\mathbb{A}}(\lambda)$ and $\nabla_{\mathbb{A}}(\lambda)$. Moreover, if $\mathbb{K}$ is a field over $\mathbb{A}$, then $\mathbf{1}_{\pi} \centerdot T_{\mathbb{K}}(\lambda) = T_{\mathbb{K}}(\lambda)$. ◇

Lemma 5B.5 *Let* $\mathbb{K}$ *be a field over* $\mathbb{A}$*. The category* $S^{\pi}_{\mathbb{K}}(\mathfrak{g})$*-mod is a highest weight category, with poset* π*, and with standard modules, costandard modules, and tilting modules given by* $\Delta_{\mathbb{K}}(\lambda)$*,* $\nabla_{\mathbb{K}}(\lambda)$ *and* $T_{\mathbb{K}}(\lambda)$ *with indexing poset* π*.*

Proof See [19, Theorem 5.4] □

The following is a standard consequence of Lemma 5B.5:

Lemma 5B.6 *If* T *and* T' *are tilting* $S^{\pi}_{\mathbb{K}}(\mathfrak{g})$*-modules with Weyl characters*

$$[T] = \sum_{\lambda \in \pi} m_{\lambda} \cdot [\Delta_{\mathbb{K}}(\lambda)], \quad [T'] = \sum_{\lambda \in \pi} m'_{\lambda} \cdot [\Delta_{\mathbb{K}}(\lambda)],$$

in the Grothendieck ring, then we have

$$\dim_{\mathbb{K}} \mathrm{Hom}_{S^{\pi}_{\mathbb{K}}(\mathfrak{g})}(T, T') = \sum_{\lambda \in \pi} m_{\lambda} m'_{\lambda}.$$

Proof The dual Weyl character is the same as the Weyl character, so

$$[T'] = \sum_{\lambda \in \pi} m'_{\lambda} [\nabla_{\mathbb{K}}(\lambda)].$$

Lemma 5B.5 implies the usual Ext-vanishing, cf. Lemma 4A.3.(c), and the result then follows from a standard long exact sequence argument. □

Lemma 5B.7 *The Schur algebra* $S^{\pi}_{\mathbb{A}}(\mathfrak{g})$ *is free over* $\mathbb{A}$ *of rank*

$$\mathrm{rk}_{\mathbb{A}} S^{\pi}_{\mathbb{A}}(\mathfrak{g}) = \sum_{\lambda \in \pi} \left(\dim L_q(\lambda)\right)^2.$$

Proof By [30, Section 29.2.1] we have

$$\dim_{\mathbb{F}} \dot{U}_q(\mathfrak{g})/\dot{U}_q(\mathfrak{g})[\pi^c] = \sum_{\lambda\in\pi} \big(\dim_{\mathbb{F}} L_q(\lambda)\big)^2,$$

so $|\dot{\mathbf{B}}[\pi]| = \sum_{\lambda\in\pi} \big(\dim L_q(\lambda)\big)^2$. Proposition 5B.3 implies that $S^{\pi}_{\mathbb{A}}(\mathfrak{g})$ has a basis naturally in bijection with $\dot{\mathbf{B}}[\pi]$. □

Lemma 5B.8 *If $\mathbb{K}$ is a field over $\mathbb{A}$, then*

$$\dim_{\mathbb{K}} S^{\pi}_{\mathbb{K}}(\mathfrak{g}) = \sum_{\lambda\in\pi} \big(\dim \Delta_{\mathbb{K}}(\lambda)\big)^2.$$

Proof Since Weyl modules are free over $\mathbb{A}$, it follows that

$$\dim_{\mathbb{K}} \Delta_{\mathbb{K}}(\lambda) = \dim_{\mathbb{F}} \Delta_q(\lambda) = \dim_{\mathbb{F}} L_q(\lambda).$$

Therefore, the claim follows from Lemma 5B.7. □

Lemma 5B.9 *The left regular $S^{\pi}_{\mathbb{A}}(\mathfrak{g})$-module $S^{\pi}_{\mathbb{A}}(\mathfrak{g})$ has a filtration by standard modules.*

Proof Lemmas 5A.8 and 5A.9 imply that $\dot{U}_{\mathbb{A}}(\mathfrak{g})/\dot{U}_{\mathbb{A}}(\mathfrak{g})[\pi^c]$ is Weyl filtered as a module over $\dot{U}_{\mathbb{A}}(\mathfrak{g})$. The claim then follows from Proposition 5B.3. □

Lemma 5B.10 *The left regular $S^{\pi}_{\mathbb{K}}(\mathfrak{g})$-module $S^{\pi}_{\mathbb{K}}(\mathfrak{g})$ has a filtration by standard modules.*

Proof Directly from Lemma 5B.9. □

Lemma 5B.11 *Let X be an $S^{\pi}_{\mathbb{K}}(\mathfrak{g})$-module. If X has a finite standard filtration, then X embeds in a tilting module.*

Proof Each standard module Δ embeds in an indecomposable tilting module T_{Δ}. If

$$0 \to K \xrightarrow{i} X \xrightarrow{p} \Delta \to 0$$

and K has a standard filtration, then we can assume inductively that there are embeddings into tilting modules:

$$0 \to K \xrightarrow{a} T_K \quad \text{and} \quad 0 \to \Delta \xrightarrow{b} T_{\Delta}.$$

Since $\mathrm{Ext}^1(\Delta, T_K) = 0$, there is a surjection

$$\mathrm{Hom}(X, T_K) \xrightarrow{i^*} \mathrm{Hom}(K, T_K) \to 0.$$

Let $a' \colon X \to T_K$ be such that $a' \circ i = a$ and define

$$X \to T_K \oplus T_{\Delta}, \quad x \mapsto \big(a'(x), b \circ p(x)\big).$$

If $x \mapsto 0$, then injectivity of b implies $x \in \ker p = i(K)$. Injectivity of a then implies $x = i(k) = 0$. Therefore, X embeds in the tilting module $T_K \oplus T_{\Delta}$. □

Lemma 5B.12 *The left regular $S^{\pi}_{\mathbb{K}}(\mathfrak{g})$-module $S^{\pi}_{\mathbb{K}}(\mathfrak{g})$ embeds in a tilting module.*

Proof Combing Lemma 5B.10 with Lemma 5B.11. □

The following is crucial:

Proposition 5B.13 *If $T \in S^{\pi}_{\mathbb{K}}(\mathfrak{g})$-mod is a full tilting module, then T is a faithful module.*

Proof Let T be a full tilting module. The left regular module $S^{\pi}_{\mathbb{K}}(\mathfrak{g})$ is faithful and embeds in a tilting module. Hence, it embeds in a full tilting module T' and it follows that T' is faithful. Since

$$T = \bigoplus_{\lambda \in \pi} T^{\oplus m_{\lambda}}(\lambda) \quad \text{and} \quad T' = \bigoplus_{\lambda \in \pi} T^{\oplus m'_{\lambda}}(\lambda)$$

such that $m_{\lambda}, m'_{\lambda} \in \mathbb{Z}_{\geq 0}$, it follows that there is some $n \in \mathbb{Z}_{\geq 0}$ such that T' is a summand of $(T)^{\oplus n}$. Since T' is faithful, $T^{\oplus n}$ is also faithful. A module M is faithful if and only if $M^{\oplus n}$ is faithful for some n, so T is faithful as well. □

5C Integral Howe duality

Recall from Proposition 4A.13 that we have commuting actions of $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$ and $U_{\mathbb{A}}(\mathfrak{sl}_2)$ on $\Lambda_{\mathbb{A}}$. We will use these actions for the remainder of this section.

Lemma 5C.1 *An $\mathbb{A}$-linear map between two finitely generated free $\mathbb{A}$-modules is an isomorphism if it is an isomorphism for all specializations.*

Proof Let $f : \mathbb{A}^m \to \mathbb{A}^n$ be an $\mathbb{A}$ module map. Since $\mathbb{F} \otimes f : \mathbb{F}^m \to \mathbb{F}^n$ is an isomorphism, it follows that $m = n$. In particular, f can be represented as a square matrix.

Suppose now for the sake of contradiction that det f is not invertible in $\mathbb{A}$. Therefore, the ideal $(\det f) \subset \mathbb{A}$ is contained in some maximal ideal M and $\mathbb{A}/M \otimes \det f = 0$. Since f is an isomorphism for all specializations, it follows that $\mathbb{A}/M \otimes f$ is an isomorphism. Therefore $\mathbb{A}/M \otimes \det f = \det(\mathbb{A}/M \otimes f) \neq 0$, a contradiction.

Since det f is therefore invertible in $\mathbb{A}$, it follows that f is invertible. □

Proposition 5C.2 *The homomorphisms induced from the actions*

$$\phi_{\mathbb{A}} : U_{\mathbb{A}}(\mathfrak{sp}_{2n}) \to \mathrm{End}_{U_{\mathbb{A}}(\mathfrak{sl}_2)}(\Lambda_{\mathbb{A}}), \quad \psi_{\mathbb{A}} : U_{\mathbb{A}}(\mathfrak{sl}_2) \to \mathrm{End}_{U_{\mathbb{A}}(\mathfrak{sp}_{2n})}(\Lambda_{\mathbb{A}})$$

are surjective.

Proof The set $\pi := \{\varpi_0, \varpi_1, \ldots, \varpi_n\} \subset X_{C_n}$ is a saturated set of dominant weights, and the set of weights occurring in $\Lambda_{\mathbb{A}}$ is equal to $W_{C_n} \cdot \pi$. Therefore, the map $\phi_{\mathbb{A}}$ induces an $\mathbb{A}$-algebra homomorphism $\overline{\phi}_{\mathbb{A}} : S^{\pi}_{\mathbb{A}}(\mathfrak{sp}_{2n}) \to \mathrm{End}_{U_{\mathbb{A}}(\mathfrak{sl}_2)}(\Lambda_{\mathbb{A}})$.

Since $\Lambda_{\mathbb{K}} = \oplus_{k=0}^{2n} \Lambda^k_{\mathbb{K}}$ and $T_{\mathbb{K}}(\varpi_k)$ is a summand of $\Lambda_{\mathbb{K}}(\varpi_k)$ for $k \in \{0, 1, \ldots, n\}$, it follows that $\Lambda_{\mathbb{K}}$ is a full tilting module for $S^{\pi}_{\mathbb{K}}(\mathfrak{sp}_{2n})$. By Lemma 5B.13 $\overline{\phi}_{\mathbb{K}}$ is injective.

Using Corollary 5B.6 and Proposition 4D.5 we find

$$\dim_{\mathbb{K}} \mathrm{Hom}_{U_{\mathbb{K}}(\mathfrak{sl}_2)}(\Lambda_{\mathbb{K}}, \Lambda_{\mathbb{K}}) = \sum_{k=0}^{n} \big(\dim_{\mathbb{K}} \Delta_{\mathbb{K}}(\varpi_k)\big)^2.$$

It then follows from Lemma 5B.7 that

$$\dim S^{\pi}_{\mathbb{K}}(\mathfrak{sp}_{2n}) = \dim_{\mathbb{K}} \mathrm{End}_{U_{\mathbb{K}}(\mathfrak{sl}_2)}(\Lambda_{\mathbb{K}}),$$

so $\overline{\phi}_{\mathbb{K}}$ is surjective. Hence, $\overline{\phi}_{\mathbb{A}}$ is surjective by Lemma 5C.1, so $\phi_{\mathbb{A}}$ is surjective.

The set of $\mathfrak{sl}_2$ weights $\tau := \{0, 1, \ldots, n\} \subset X_{A_1}$ is a saturated set of dominant weights, and the set of $\mathfrak{sl}_2$ weights occurring in $\Lambda_{\mathbb{A}}$ is equal to $W_{A_1} \cdot \tau$. So there is an induced map $\overline{\psi}_{\mathbb{A}} : S^{\tau}_{\mathbb{A}}(\mathfrak{sl}_2) \to \mathrm{End}_{U_{\mathbb{A}}(\mathfrak{sp}_{2n})}(\Lambda_{\mathbb{A}})$. Also, $\Lambda_{\mathbb{K}}$ is a full tilting module for $S^{\tau}_{\mathbb{K}}(\mathfrak{sl}_2)$, so $\overline{\psi}_{\mathbb{K}}$ is injective. Proposition 4F.8 implies $\Lambda_{\mathbb{K}}$ is a $U_{\mathbb{K}}(\mathfrak{sp}_{2n})$ tilting module, with Δ character given by Proposition 4F.7, so a similar argument to the one above shows that

$$\dim_{\mathbb{K}} S^{\tau}_{\mathbb{K}}(\mathfrak{sl}_2) = \sum_{k=0}^{n} \big(\dim_{\mathbb{K}} \Delta_{\mathbb{K}}(k)\big)^2 = \dim_{\mathbb{K}} \mathrm{End}_{U_{\mathbb{K}}(\mathfrak{sp}_{2n})}(\Lambda_{\mathbb{K}}),$$

and therefore that $\overline{\psi}_{\mathbb{K}}$ is surjective. This implies that $\overline{\psi}_{\mathbb{A}}$ is surjective, again by Lemma 5C.1. Thus, $\psi_{\mathbb{A}}$ is surjective. □

Proposition 5C.3 *The $U_{\mathbb{A}}$-module $\Lambda_{\mathbb{A}}$ is a Howe tilting module satisfying* (1B.2) *and* (1B.3).

Proof Note that (1B.2) implies (1B.3), and recall that the tilting module properties of $\Lambda_{\mathbb{A}}$ were shown in Proposition 4D.5, Proposition 4F.8 and Sect. 5B. Thus, it remains to show (1B.2). This is however a formal consequence of Ringel duality, see [17, Appendix], and that Ringel duality can be used in our setting is Proposition 5C.2. □

Proof of Theorem 1B.1 This theorem follows now from Proposition 4A.13, Proposition 5C.2 and Proposition 5C.3. (Note that we formulated Proposition 4A.13 in terms of a left and a right action, hence the appearance of the opposite algebra when one compares that theorem to Proposition 5C.3, see also Remark 1B.5.(a).) □

Remark 5C.4 For the reader wondering whether, as $S^{\pi}_{\mathbb{A}}(\mathfrak{sp}(V))$-modules, we have $\mathbf{1}_{\pi} \cdot V^{\otimes k} \cong \Lambda^k$: this is not the case. For example, $V^{\otimes 3}$ has Weyl filtration containing three copies of V and one copy of $\Delta(\varpi_3)$, while Λ^3 has Weyl filtration $\Delta(\varpi_3)$ and one copy of V. ◇

6 Further results I: the dual action via differential operators

In the final three sections, including this one, we collect results that are not relevant for Theorem 1B.1 but of independent interest.

Our construction of the $\mathfrak{sl}_2$ in Sect. 2E is quite different from Howe's approach in [26] who uses a form of harmonic analysis, that is, differential operators. We now present the quantum version of this approach, following ideas in [43].

Remark 6.1 The results in this section, up to some details regarding idempotented forms, give the same results as Sect. 2E. We therefore decided to leave all proofs to the reader. ◇

Using Proposition 2D.4, we will write $\Lambda^*_{\mathbb{A}}$ for $\Lambda_{\mathbb{A}}$ with the generating set being $\{d_i | i \in [1, -1]\}$. In the following definition, which defines an $\mathbb{A}$-algebra containing $\Lambda_{\mathbb{A}}$ and $\Lambda^*_{\mathbb{A}}$, we write $\mathrm{X}_i \rightsquigarrow v_i$ for the elements of $\Lambda_{\mathbb{A}}$ and $\partial_j \rightsquigarrow d_j$ for the elements of $\Lambda^*_{\mathbb{A}}$ (we decided to change notation to stress the differential operator calculus).

Definition 6.2 Let $\mathcal{D}_{\mathbb{A}}$ be the $\mathbb{A}$-algebra generated by elements

$$\mathrm{X}_1, \dots, \mathrm{X}_n, \mathrm{X}_{-n}, \dots, \mathrm{X}_{-1}, \partial_1, \dots, \partial_n, \partial_{-n}, \dots, \partial_{-1},$$

subject only to the following relations. The relations of $\Lambda_{\mathbb{A}}$ for the X_i, the relations of $\Lambda_{\mathbb{A}}$ for the ∂_j and:

$$\partial_i \mathrm{X}_{i'} = -q^2 \cdot \mathrm{X}_{i'} \partial_i, i \in [1, -1], \partial_i \mathrm{X}_i = -\mathrm{X}_i \partial_i + q(q - q^{-1}) \sum_{k \in [1, i-1]} \mathrm{X}_k \partial_k + 1, i \in [1, n], \partial_{-i} \mathrm{X}_{-i}$$

$$= -\mathrm{X}_{-i}\,\partial_{-i} + q^{2(n-i+1)+1}(q-q^{-1})\cdot \mathrm{X}_i\,\partial_i + 1 + q(q-q^{-1}) \sum_{k\in[1,-(i+1)]} \partial_k\,\mathrm{X}_k,\ i\in[1,n],$$

and for the final three relations we always have $i, j \in [1,-1], i \notin \{j, j'\}$:

$$\partial_i\,\mathrm{X}_j = -q\cdot \mathrm{X}_j\,\partial_i,\ i' > j,\ \partial_i\,\mathrm{X}_j = -q\cdot \mathrm{X}_j\,\partial_i + q(-q)^{|j|-|i'|}(q-q^{-1})\cdot \mathrm{X}_{i'}\,\partial_{j'},\ i' < j,\ \mathrm{sgn}(i)\neq \mathrm{sgn}(j),$$

$$\partial_i\,\mathrm{X}_j = -q\cdot \mathrm{X}_j\,\partial_i + q^2(-q)^{|j|-|i'|}(q-q^{-1})\cdot \mathrm{X}_{i'}\,\partial_{j'},\ i' < j,\ \mathrm{sgn}(i) = \mathrm{sgn}(j).$$

Here, for $i\in[1,-1]$, let $i' = 2n+1-i$. Moreover, $\mathrm{sgn}(i) = 1$ and $|i| = i$ for $i \in [1,n]$, and $\mathrm{sgn}(i) = -1$ and $|i| = 2n+i+1$ for $i\in[-n,-1]$. ◇

The relation $\partial_i\,\mathrm{X}_i = -\mathrm{X}_i\,\partial_i + q(q-q^{-1})\sum_{k\in[1,i-1]}\mathrm{X}_k\,\partial_k + 1$ and its variations are sometimes called *q-Heisenberg commutation relations*. The name comes from the classical version of this relation being $\partial_i\,\mathrm{X}_i = -\mathrm{X}_i\,\partial_i + 1$.

Remark 6.3 The $\mathbb{A}$-algebra $\mathcal{D}_{\mathbb{A}}$ from Definition 6.2 is the symplectic analog of [43, (2.26)], i.e. the *differential operator algebra* on $\Lambda_{\mathbb{A}}$. The $\mathbb{A}$-algebra $\mathcal{D}_{\mathbb{A}}$ has similar properties as the orthogonal analog in [43]. For example, using the same notation as for $\Lambda_{\mathbb{A}}$, the first result one can show is that

$$\{\partial_T\,\mathrm{X}_S\,|\,S, T\subset[1,-1]\}$$

is an $\mathbb{A}$-basis of $\mathcal{D}_{\mathbb{A}}$ so that $\mathcal{D}_{\mathbb{A}} \cong \Lambda_{\mathbb{A}}\otimes_{\mathbb{A}}\Lambda_{\mathbb{A}}^*$ as free $\mathbb{A}$-modules. For brevity, we decided to not include any further discussion of $\mathcal{D}_{\mathbb{A}}$ and its properties. ◇

Similarly as in [43] one can show that $\mathcal{D}_{\mathbb{A}}$ gives rise to an analog of the oscillator $U_q(\mathfrak{sl}_2)$-action on $\Lambda_{\mathbb{A}}$. The rest of this section is an algebraic description of this action.

To this end, we will work with the idempotented quantum group for $\mathfrak{sl}_2$ which need the following weight operators:

Definition 6.4 For $i\in[1,n]$ define the $(\mathfrak{sl}_2)$ *weight operators* $\mathbf{1}_i$ on Λ_q by $\mathbf{1}_i \centerdot v_S := \delta_{i,-n+|S|}\cdot v_S$. ◇

Note that $\mathbf{1}_i\,\mathbf{1}_j = \delta_{ij}\,\mathbf{1}_i$.

Remark 6.5 The nondegenerate alternating two form from Sect. 3A, determines an nonzero vector $\omega\in\Lambda_1^2$. The operator E is then multiplication by ω, while F is contraction with ω.

The elements

$$\omega_q := \sum_{i=1}^{n}(-q)^i\,\mathrm{X}_i\,\mathrm{X}_{-i},\quad \omega_q^\vee := \sum_{i=1}^{n}(-q)^{-i}\,\partial_i\,\partial_{-i}\,.$$

both span trivial $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$-submodules of $\mathcal{D}_{\mathbb{A}}$. If we view ω_q as an element in $\Lambda_{\mathbb{A}}$, then this is the q-analog of ω. There is a natural action of $\mathcal{D}_{\mathbb{A}}$ on $\Lambda_{\mathbb{A}}$, inducing operators in $\mathrm{End}_{U_{\mathbb{A}}(\mathfrak{sp}_{2n})}(\Lambda_{\mathbb{A}})$: multiplication by ω_q and contraction with $\omega_q^\vee$. The following definition records the action of these operators explicitly. ◇

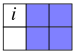
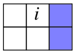
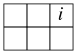
Definition 6.6 For $S\subset[1,-1]$ and $i\in[1,n]$ we write $S_{i\to} := S\cap[i+1,-(i+1)]$ and $S^c = [1,-1]\setminus S$. Define operators e and f on Λ_q by

$$e\centerdot v_S := \sum_{\{i,-i\}\subset S^c}(-q)^i(-q)^{|S_{i\to}|}\cdot v_{S\cup\{i,-i\}},\quad f\centerdot v_S := \sum_{\{i,-i\}\subset S}(-q)^i(-q)^{|S_{i\to}|}\cdot v_{S\setminus\{i,-i\}}$$

We call e and f the $(\mathfrak{sl}_2)$ *Chevalley operators*. ◇

Let us explain the action of the Chevalley generators on dot diagrams in words:

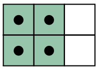
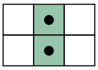
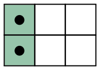
(i) The only nonzero action of f is on fully dotted columns, and f acts summing over all such columns.
(ii) f removes the two dots in fully dotted columns.
(iii) The q-factor that one gets is obtained from the column number i, giving $(-q)^i$, and the number of dots to the right, contributing $(-q)^{|S_{i\to}|}$. Here the set $S_{i\to}$ counts nodes to the right, for example, as follows:

$S_{1\to} = \{2, 3, -3, -2\} \leftrightsquigarrow$ , $S_{2\to} = \{3, -3\} \leftrightsquigarrow$ , $S_{3\to} = \emptyset \leftrightsquigarrow$ ,

where the shaded nodes are in $S_{i\to}$.
(iv) The action of e is similar, but e acts on completely empty columns and adds dots instead of removing them.

Here is an explicit example:

Example 6.7 The action of the operator f on dot diagrams is exemplified by:

$f\,.$  $= (-q)^1(-q)^2 \cdot$  $+ (-q)^2(-q)^0 \cdot$ .

Note that this is different from the action of the Chevalley generators f_i of the symplectic side which fix the number of dots. ◇

Lemma 6.8 *The operators e and f commute with the operators e_i and f_i for* $\{1, \dots, n\}$.

Proof Omitted. □

Following [6], the *idempotented quantum group* $\dot{U}_q(\mathfrak{sl}_2)$ is obtained by the $\mathbb{F}$-algebra extension of $U_q(\mathfrak{sl}_2)$ by orthogonal idempotents $\mathbf{1}_k$ for $k \in \mathbb{Z}$, the *(dot $\mathfrak{sl}_2$) weight generators*, such that,

$$\mathbf{1}_{k+2}\, E = E\, \mathbf{1}_k, \quad \mathbf{1}_{k-2}\, F = F\, \mathbf{1}_k, \quad \mathbf{1}_k\, K = K\, \mathbf{1}_k = q^k\, \mathbf{1}_k, \tag{6.9}$$

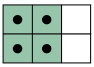
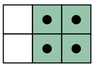
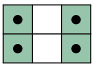
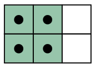
and then idempotent truncation, namely $\dot{U}_q(\mathfrak{sl}_2) = \bigoplus_{k,j\in\mathbb{Z}} \mathbf{1}_k\, U_q(\mathfrak{sl}_2)\, \mathbf{1}_j$.

In $\dot{U}_q(\mathfrak{sl}_2)$ we can use (6.9) to write elements of $\dot{U}_q(\mathfrak{sl}_2)$ as strings of E and F with one weight generator on the right, and we will do this in the following. Note that in $\dot{U}_q(\mathfrak{sl}_2)$ we have that $\mathbf{1}_{k+2}\, E = E\, \mathbf{1}_k$ replaces $KE = q^2 EK$, $\mathbf{1}_{k-2}\, F = F\, \mathbf{1}_k$ replaces $KF = q^{-2} FK$ and and $EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$ gets replaced by

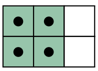
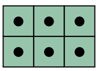
$$EF\, \mathbf{1}_k - FE\, \mathbf{1}_k = [k]_q\, \mathbf{1}_k\,.$$

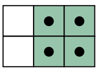
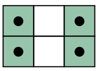
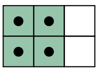
We almost have an action of $\dot{U}_q(\mathfrak{sl}_2)$ on Λ_q:

Example 6.10 Continuing Example 6.7 by applying e gives:

$e\,.\Big(f\,.$  $\Big) = (-q)^6 \cdot$  $+ (-q)^5 \cdot$  $+ \big((-q)^6 + (-q)^4\big) \cdot$ .

On the other hand, we also have

$f\,.\Big(e\,.$  $\Big) = (-q)^3 \cdot f\,.$ 

$= (-q)^8 \cdot$  $+ (-q)^7 \cdot$  $+ (-q)^6 \cdot$ .

In other words, the action of $(ef - fe)$ is a bit off compared to what we would want for quantum $\mathfrak{sl}_2$ (namely a scalar multiple of the dot diagram one starts with), which is the reason for the rescaling in Definition 6.11 below. ◇

Definition 6.11 Define the *dot* $(\mathfrak{sl}_2)$ *Chevalley operators* E and F

$$E\,\mathbf{1}_{-n+|S|} := q^{-|S|-1} \cdot e\,\mathbf{1}_{-n+|S|}, \quad F\,\mathbf{1}_{-n+|S|} := q^{-n} \cdot f\,\mathbf{1}_{-n+|S|},$$

with e and f being as in Definition 6.6, and with $\mathbf{1}_i$ as in Definition 6.4. ◇

Next, the "correct" action:

Proposition 6.12 *There is an $\dot{U}_q(\mathfrak{sl}_2)$-action on Λ_q by assigning the generators of $\dot{U}_q(\mathfrak{sl}_2)$ to the operators with the same name in Definitions 6.11 and 6.4.*

In particular, we have

$$(EF - FE)\,\mathbf{1}_{-n+|S|} \centerdot v_S = [-n + |S|]_q \centerdot v_S.$$

Proof Omitted. □

Proposition 6.13 *The two actions, the $U_q(\mathfrak{sp}_{2n})$-action and the $\dot{U}_q(\mathfrak{sl}_2)$-action, on Λ_q commute.*

Proof Omitted. □

Let $\dot{U}_q = U_q(\mathfrak{sp}_{2n}) \otimes \dot{U}_q(\mathfrak{sl}_2)$ and note that Proposition 6.13 gives a $\dot{U}_q$-action on Λ_q.

Proposition 6.14 *The $\dot{U}_q$-action and the U_q-action on Λ_q, the one in this section and the one from Sect. 2E, give the same image.*

Proof Omitted. □

Remark 6.15 After defining the appropriate integral form of the action of $\dot{U}_q$, there is an analogous statement for the integral actions. Then Proposition 6.14 says that, for the purpose of this paper, both actions do the same job, and it depends on the problem at hand which action is the more suitable one. ◇

As explained in [43] for the orthogonal case, pushing this further leads to the symplectic version of harmonic analysis on quantum spheres.

7 Further results II: on fundamental tilting modules

For $U_{\mathbb{A}}(\mathfrak{sl}_2)$ it is well-known when the Weyl modules are simple, see for example [17, Section 3.4] or [44, Proposition 3.3]. We now partially address the case of $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$.

Recall from Propositions 4F.7 and 4F.8 that

$$\mathrm{rk}_{\mathbb{A}}\, \mathrm{Hom}_{U_{\mathbb{A}}(\mathfrak{sp}_{2n})}\big(\Delta_{\mathbb{A}}(\varpi_k), \Lambda^k_{\mathbb{A}}\big) = 1 = \mathrm{rk}_{\mathbb{A}}\, \mathrm{Hom}_{U_{\mathbb{A}}(\mathfrak{sp}_{2n})}\big(\Lambda^k_{\mathbb{A}}, \nabla_{\mathbb{A}}(\varpi_{\mathbb{A}})\big).$$

We fix bases for these homomorphism spaces as follows: Fix highest weight vectors $v^+_{\varpi_k} \in \Delta_{\mathbb{A}}(\varpi_k)$ and $v^-_{\varpi_k} \in \nabla_{\mathbb{A}}(\varpi_k)$. Let $\iota_k \colon \Delta_{\mathbb{A}}(\varpi_k) \to \Lambda^k_{\mathbb{A}}$ be the unique homomorphism such that $v^+_{\varpi_k} \mapsto v_{\{1,\dots,k\}}$ and let $\pi_k \colon \Lambda^k_{\mathbb{A}} \to \nabla_{\mathbb{A}}(\varpi_k)$ be the unique homomorphism such that $v_{\{1,\dots,k\}} \mapsto v^-_{\varpi_k}$. Note that the usual Ext-vanishing Lemma 4A.3.(c) gives

$$\mathrm{rk}_{\mathbb{A}}\, \mathrm{Hom}_{U_{\mathbb{A}}(\mathfrak{sp}_{2n})}\big(\Delta_{\mathbb{A}}(\varpi_k), \nabla_{\mathbb{A}}(\varpi_k)\big) = 1$$

and a basis for this homomorphism space is given by $hs_k := \pi_k \circ \iota_k$.

Remark 7.1 The construction above is standard, see e.g. [5, Section 3] for essentially the same type of basis. ◇

Lemma 7.2 *We have*

$$\mathrm{Hom}_{U_{\mathbb{A}}(\mathfrak{sp}_{2n})}(\Lambda^k_{\mathbb{A}}, \Lambda^{k+2i}_{\mathbb{A}}) \circ \iota_k = \mathbb{A}\{(E^{(i)} \centerdot \_) \circ \iota_k\},$$
$$\pi_k \circ \mathrm{Hom}_{U_{\mathbb{A}}(\mathfrak{sp}_{2n})}(\Lambda^{k+2i}_{\mathbb{A}}, \Lambda^k_{\mathbb{A}}) = \mathbb{A}\{\pi_k \circ (F^{(i)} \centerdot \_)\}.$$

Proof By Propositions 4F.7, 4F.8 and Lemma 4A.3.(c) we know the $\mathbb{A}$-rank on both sides match, hence, for $F^{(i)}$ is this Lemma 4F.5. The first claim follows similarly. □

Definition 7.3 Consider the pairing

$$\beta_k^{k+2i}\colon \mathrm{Hom}_{U_{\mathbb{A}}(\mathfrak{sp}_{2n})}(\Lambda^{k+2i}_q, \Lambda^k) \times \mathrm{Hom}_{U_{\mathbb{A}}(\mathfrak{sp}_{2n})}(\Lambda^k_q, \Lambda^{k+2i}_q) \to \mathrm{Hom}_{U_{\mathbb{A}}(\mathfrak{sp}_{2n})}\big(\Delta_{\mathbb{A}}(\varpi_k), \nabla_{\mathbb{A}}(\varpi_k)\big)$$

defined by $(x, y) \mapsto \pi_k \circ (x \circ y) \circ \iota_k$. ◇

Recall that $\mathbb{K}$ denotes a specialization of $\mathbb{A}$.

Proposition 7.4 *The rank of the pairing $(\beta_k^{k+2i})_{\mathbb{K}}$ is equal to the multiplicity of $T_{\mathbb{K}}(\varpi_k)$ as a summand of $\Lambda^{k+2i}_{\mathbb{K}}$. Moreover,*

$$\mathrm{rk}_{\mathbb{K}}(\beta_k^{k+2i})_{\mathbb{K}} = \begin{cases} 1 & \text{if } \left[{n-k \atop i}\right]_{\mathbb{K}} \neq 0, \\ 0 & \text{else.} \end{cases}$$

Proof We have the well-known relation

$$F^{(a)} E^{(b)} \mathbf{1}_c = \sum_{d=0}^{\min\{a,b\}} \begin{bmatrix} a-b-c \\ d \end{bmatrix}_{\mathbb{A}} E^{(b-d)} F^{(a-d)} \mathbf{1}_c$$

that holds in the idempotented form $\dot{U}_q(\mathfrak{sl}_2)$ which we will use for $a = b = i$ and $j = n-k$. That is, by Lemma 7.2 and the weight space decomposition for the $U_q(\mathfrak{sl}_2)$-action on $\Lambda_{\mathbb{A}}$, we need to know the value of $F^{(i)} E^{(i)} \mathbf{1}_{n-k}$. This value is $\left[{n-k \atop i}\right]_{\mathbb{A}}$ by the above formula as only the summands for $d = \min\{a, b\}$ is nonzero. This shows that $\mathrm{rk}_{\mathbb{K}}(\beta_k^{k+2i})_{\mathbb{K}}$ is as claimed. The first part of the statement follows from general theory similarly to [5, Section 4C]. □

Let $p = \mathrm{char}(\mathbb{K})$ (with, as usual, $\mathrm{char}(\mathbb{K}) = 0$ interpreted as $p = \infty$) and let $\ell \in \mathbb{Z}_{\geq 1}$ be minimal such that $[\ell]_{\mathbb{K}} = 0$ with $\ell = \infty$ if no such minimal number exists. Write $p^{(i)} = p^{i-1}\ell$ with $p^{(0)} = 1$. The (p, ℓ)-*adic expansion* of $m \in \mathbb{Z}_{\geq 0}$ is the tuple $[\dots, a_k, \dots, a_0]_{p,\ell}$ with $a_i \in \{0, \dots, p-1\}$ for $i \neq 0$ and $a_0 \in \{0, \dots, \ell - 1\}$ such that $m = \sum_{k \in \mathbb{Z}_{\geq 0}} a_k p^{(k)}$ (we often omit the infinitely many zeros for $k \gg 0$). We compare (p, ℓ)-adic expansions componentwise.

As a consequence, we get:

Proposition 7.5 *Let $n - k = [\dots, a_0]_{p,\ell}$ and $i = [\dots, b_0]_{p,\ell}$ be the (p, ℓ)-adic expansions.*

(a) *$T_{\mathbb{K}}(\varpi_k)$ appears as direct summands of $\Lambda^{k+2i}_{\mathbb{K}}$ if and only if $[\dots, a_0]_{p,\ell} \geq [\dots, b_0]_{p,\ell}$.*
(b) *If $\ell \geq n+1$ ($n+1$ is the dual Coxeter number of $\mathfrak{sp}_{2n}$), then $T_{\mathbb{K}}(\varpi_k)$ is always a direct summand of $\Lambda^{k+2i}_{\mathbb{K}}$.*

Proof (a). By Proposition 7.4 and the famous quantum Lucas' theorem. For the latter see e.g. [36, (1.2.4)] for an early account.

(b). In this case all involved (p, ℓ)-adic expansion have only the zeroth digit, and this then follows from (a). □

Example 7.6 If $\mathbb{K} = \mathbb{F}_7$ and $\xi = 2 \in \mathbb{K}$, then $(p = 7, \ell = 3)$. Let us take $74 = [3, 3, 2]_{7,3}$, $28 = [1, 2, 1]_{7,3}$ and $25 = [1, 1, 1]_{7,3}$. We get $\left[\begin{smallmatrix}74\\28\end{smallmatrix}\right]_{\mathbb{K}} = 0$ and $\left[\begin{smallmatrix}74\\25\end{smallmatrix}\right]_{\mathbb{K}} \neq 0$. Thus, if $n = 78$ and $k = 5$ so that $78 - 5 + 1 = 74$, then $T_{\mathbb{K}}(\varpi_5)$ is a direct summand of $\Lambda_{\mathbb{K}}^{55=5+2\cdot 25}$ but not of $\Lambda_{\mathbb{K}}^{61=5+2\cdot 28}$, which we can directly read of from $[3, 3, 2]_{7,3} \not\geq [1, 2, 1]_{7,3}$ but $[3, 3, 2]_{7,3} \geq [1, 1, 1]_{7,3}$. ◇

Theorem 7.7 *The following equations inductively determine the Weyl characters of each fundamental tilting $U_{\mathbb{K}}(\mathfrak{sp}_{2n})$-module. First, the initial conditions are $[T_{\mathbb{K}}(\varpi_0)] = [\Delta_{\mathbb{K}}(\varpi_0)]$ and $[T_{\mathbb{K}}(\varpi_1)] = [\Delta_{\mathbb{K}}(\varpi_1)]$. Then, for $k > 1$:*

$$[T_{\mathbb{K}}(\varpi_k)] = \sum_{i \geq 0, k-2i \geq 0} [\Delta_{\mathbb{K}}(\varpi_{k-2i})] - \sum_{i \geq 1, k-2i \geq 0 \left[\begin{smallmatrix}n-k\\i\end{smallmatrix}\right]_{\mathbb{K}} \neq 0} [T_{\mathbb{K}}(\varpi_{k-2i})].$$

Proof For all $k \in \{0, \dots, n\}$, Proposition 7.5 implies that

$$\Lambda_{\mathbb{K}}^k \cong \bigoplus_{i \geq 0, k-2i \geq 0 \left[\begin{smallmatrix}n-k\\i\end{smallmatrix}\right]_{\mathbb{K}} \neq 0} T_{\mathbb{K}}(\varpi_{k-2i}) = T_{\mathbb{K}}(\varpi_k) \oplus \bigoplus_{\substack{i \geq 1, k-2i \geq 0 \\ \left[\begin{smallmatrix}n-k\\i\end{smallmatrix}\right]_{\mathbb{K}} \neq 0}} T_{\mathbb{K}}(\varpi_{k-2i}).$$

In particular,

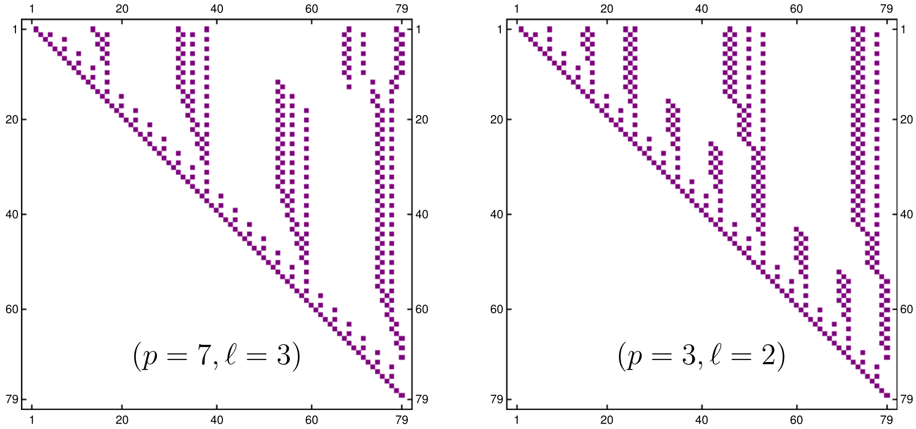
$$\Delta(\varpi_0) \cong \Lambda_{\mathbb{K}}^0 \cong T_{\mathbb{K}}(\varpi_0), \quad \Delta(\varpi_1) \cong \Lambda_{\mathbb{K}}^1 \cong T_{\mathbb{K}}(\varpi_1).$$

The desired result follows by looking at the equations determined by these isomorphisms in the Grothendieck ring, noting that the Weyl character of $\Lambda_{\mathbb{K}}^k$ is determined by the results in Proposition 4F.7. □

Example 7.8 In the setting of Example 7.6, let $n = 78$. The matrix

	$T_{\mathbb{K}}(\varpi_0)$	$T_{\mathbb{K}}(\varpi_1)$	$T_{\mathbb{K}}(\varpi_2)$	$T_{\mathbb{K}}(\varpi_3)$	$T_{\mathbb{K}}(\varpi_4)$	$T_{\mathbb{K}}(\varpi_5)$	$T_{\mathbb{K}}(\varpi_6)$	$T_{\mathbb{K}}(\varpi_7)$	$T_{\mathbb{K}}(\varpi_8)$	$T_{\mathbb{K}}(\varpi_9)$
$\Delta_{\mathbb{K}}(\varpi_0)$	1									
$\Delta_{\mathbb{K}}(\varpi_1)$		1		1						
$\Delta_{\mathbb{K}}(\varpi_2)$			1				1			
$\Delta_{\mathbb{K}}(\varpi_3)$				1						
$\Delta_{\mathbb{K}}(\varpi_4)$					1		1			
$\Delta_{\mathbb{K}}(\varpi_5)$						1				1
$\Delta_{\mathbb{K}}(\varpi_6)$							1			
$\Delta_{\mathbb{K}}(\varpi_7)$								1		1
$\Delta_{\mathbb{K}}(\varpi_8)$									1	
$\Delta_{\mathbb{K}}(\varpi_9)$										1

is the matrix of tilting:Weyl character for $k \in \{0, \dots, 9\}$. The whole matrix (left $(p = 7, \ell = 3)$ and on the right $(3, 2)$ for comparison) is



where entries 1 are shaded boxes. Not displayed entries are zero in both cases. ◇

Remark 7.9 Since $[T_{\mathbb{K}}(\varpi_k)] = [\Delta_{\mathbb{K}}(\varpi_k)]$ if and only if $\Delta_{\mathbb{K}}(\varpi_k) \cong L_{\mathbb{K}}(\varpi_k)$ by Lemma 4A.9, Theorem 7.7 also completely determines whether or not a Weyl module with fundamental highest weight is simple. When $q = 1$ and the characteristic of $\mathbb{K}$ is positive, this question has been answered in [38]. It is however not immediate how to recover their result from Theorem 7.7, or how to deduce the q analog of their result. ◇

Proposition 7.10 *If $\ell \geq n+1$, then each fundamental Weyl $U_{\mathbb{K}}(\mathfrak{sp}_{2n})$-module is simple.*

Proof By Theorem 7.7 using Lemma 4A.9. □

The following describes tilting:Weyl multiplicities on the symplectic side in terms of dual Weyl:simple on the side of $\mathfrak{sl}_2$, and also vice versa.

Proposition 7.11 *We have* $\big(T(\varpi_k) : \Delta(\varpi_l)\big) = [\nabla(n-l) : L(n-k)]$ *and* $\big(T(n-k) : \Delta(n-l)\big) = [\nabla(\varpi_l) : L(\varpi_k)]$.

The point is that the numbers $[\nabla(n-l) : L(n-k)]$ and $\big(T(n-k) : \Delta(n-l)\big)$ are well-known, see e.g. [44, Proposition 3.3], so that we can get also the tilting:Weyl multiplicities.

Proof The classical version of this is explained in [33, Proposition 6.1.3] (which is a consequence of abstract Ringel duality similarly as (1B.2)). Using Theorem 1B.1, the arguments given therein can be copied without problems. □

Example 7.12 Continuing Example 7.6, let $n = 78$ and $k = 5$. Then $78-5+1 = [3,3,2]_{7,3}$ and $\big(T(n-k) : \Delta(n-l)\big) = 0$ or $\big(T(n-k) : \Delta(n-l)\big) = 1$. The latter happens if and only if $n-l+1 \in \{[3,3,2]_{7,3}, [3,3,-2]_{7,3}, [3,-3,2]_{7,3}, [3,-3,-2]_{7,3}\} = \{74, 70, 56, 52\}$. This follows for example from [44, Proposition 3.3]. Thus, we get $[\nabla(\varpi_l) : L(\varpi_k)] = 1$ for $l \in \{5, 9, 23, 27\}$. ◇

Lastly, we record a question we arrived at computing examples using Theorem 7.7. In this question $q\dim_{\mathbb{K}}$ denotes the *quantum dimension*.

Question 7.13 Let $n = [\dots, a_0]_{p,\ell}$ and $i = [\dots, b_0]_{p,\ell}$. Suppose that $\ell > 2$. Do we have

$$\big(q\dim_{\mathbb{K}} T_{\mathbb{K}}(\varpi_i) \neq 0\big) \Leftrightarrow \big([\dots, a_0]_{p,\ell} \geq [\dots, b_0]_{p,\ell}\big)?$$

8 Further results III: canonical bases

In this final section we justify that we called the b_S from Definition 4B.6 canonical vectors.

Lemma 8.1 *The vectors* $\{v_{k,k-2i} | 0 \leq k \leq 2n, k-2i \in \mathbb{Z}_{\geq 0}\}$*, described in Notation 4F.2, generate* $\Lambda_{\mathbb{A}}$ *as a* $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$*-module. Similarly for fixed* k*.*

Proof We prove the claim for each fixed k, form which the general claim follows. The case $k = 0, 1$ is immediate. Fix $k \geq 2$ and suppose the claim is true for $k-2$.

Let $v \in \Lambda^k_{\mathbb{A}}$. Using Lemma 4F.5, we find for each $i \geq 1$ such that $k - 2i \geq 0$, ξ_i, a unit in $\mathbb{A}$, such that $F(v_{k,k-2i}) = \xi_i \cdot v_{k-2,k-2i}$. Since $F(v) \in \Lambda^{k-2}_{\mathbb{A}}$, our hypothesis implies there are $u_{k-2,k-2i} \in U_{\mathbb{A}}(\mathfrak{sp}_{2n})$ such that $F(v) = \sum_i u_{k-2,k-2i} \centerdot v_{k-2,k-2i}$. Since F commutes with $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$, the vector

$$v - \sum_i \xi_i^{-1} \cdot u_{k-2,k-2i} \centerdot v_{k,k-2i} \in \Lambda^k_{\mathbb{A}}$$

is in the kernel of F. Proposition 4E.8 implies that $F|_{\Lambda^k_{\mathbb{A}}}$ is generated over $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$ by $v_{k,k}$. The result follows. □

The following is our version of [12, Corollary 2.13]:

Lemma 8.2 *Let* $\theta \colon \Lambda_{\mathbb{A}} \to \Lambda_{\mathbb{A}}$ *be* $U_{\mathbb{A}}(\mathfrak{sp}_{2n})$*-equivariant. Suppose that* $\theta(v_{k,k-2i}) = v_{k,k-2i}$ *for all* $k \in \{0, \ldots, 2n\}$ *and* $i \in \mathbb{Z}_{\geq 0}$ *such that* $k - 2i \geq 0$*. Then* $\theta = \mathrm{id}$*.*

Proof Let $x \in \Lambda^k_{\mathbb{A}}$. By Lemma 8.1 there are $u_{k,k-2i} \in U_{\mathbb{A}}(\mathfrak{sp}_{2n})$ such that $x = \sum_i u_{k,k-2i} \cdot v_{k,k-2i}$ with $u_{k,k-2i} \in \mathbb{A}$. Thus,

$$\theta(x) = \sum_i \theta(u_{k,k-2i} \cdot v_{k,k-2i}) = \sum_i u_{k,k-2i} \cdot \theta(v_{k,k-2i}) = \sum_i u_{k,k-2i} \cdot v_{k,k-2i} = x,$$

where we used that $\theta(v_{k,k-2i}) = v_{k,k-2i}$. □

Definition 8.3 Let M, N be $\mathbb{A}$-modules. A $\mathbb{Z}$-module homomorphism $a \colon M \to N$ such that $a(q \cdot m) = q^{-1} \cdot a(m)$ will be referred to as $\mathbb{A}$-*antilinear*. ◇

Lemma 8.4 *Suppose that* a *and* b *are two* $\mathbb{A}$*-antilinear maps* $a, b \colon \Lambda_{\mathbb{A}} \to \Lambda_{\mathbb{A}}$ *such that* $\alpha \in \{a, b\}$*:*

(i) *Commutes with the Chevalley generators* e_i *and* f_i*.*
(ii) *Satisfies* $\alpha(k_i \centerdot x) = k_i^{-1} \centerdot \alpha(x)$ *for all* $x \in \Lambda_{\mathbb{A}}$*.*
(iii) *Fixes the vectors* $\{v_{k,k-2i} | 0 \leq k \leq 2n, k - 2i \geq 0\}$*.*

Then $a = b$ *and* $a^2 = \mathrm{id}$*.*

Proof The composition of two $\mathbb{A}$-antilinear maps is $\mathbb{A}$-linear. Thus, Lemma 8.2 applies to a^2, showing that a is an involution, and $a \circ b$, showing that $a \circ b = \mathrm{id}$. Algebra autopilot gives

$$b = \mathrm{id} \circ b = a \circ a \circ b = a \circ \mathrm{id} = a.$$

and the statement is proven. □

Definition 8.5 Let $S = \{x_1, x_2, \ldots, x_{|S|}\} \subset [1, -1]$ and suppose $x_1 < x_2 < \cdots < x_{|S|}$. We will write

$$v_S^{rev} := v_{x_{|S|}} \cdots v_{x_2} v_{x_1}.$$

for the *reversed standard basis vectors*. ◇

Lemma 8.6 *For all* $k \in \{0, 1, \ldots, 2n\}$*, the sets*

$$\{v_S^{rev} | S \subset [1, -1]\}, \quad \{v_S^{rev} | S \subset [1, -1], |S| = k\}$$

form an $\mathbb{A}$*-basis for* $\Lambda_{\mathbb{A}}$ *and* $\Lambda_{\mathbb{A}}^k$*, respectively.*

Proof Similarly as the proof of Proposition 2A.9 and omitted. □

Lemma 8.7 *For* $i \in \{1, \ldots, n-1\}$ *and* $i = n$ *we have:*

$$f_i \centerdot v_S^{rev} = \begin{cases} v^{rev}_{(S\setminus\{i\})\cup\{i+1\}} & \text{if } S_{i,i+1} = \{i\}, \\ v^{rev}_{(S\setminus\{-(i+1)\})\cup\{-i\}} & \text{if } S_{i,i+1} = \{-(i+1)\}, \\ q^{-1} \cdot v^{rev}_{(S\setminus\{-(i+1)\}\cup)\{-i\}} + v^{rev}_{(S\setminus\{i\}\cup)\{i+1\}} & \text{if } S_{i,i+1} = \{i, -(i+1)\}, \\ v^{rev}_{(S\setminus\{i\}\cup)\{i+1\}} & \text{if } S_{i,i+1} = \{i, -i\}, \\ q \cdot v^{rev}_{(S\setminus\{-(i+1)\}\cup)\{-i\}} & \text{if } S_{i,i+1} = \{i+1, -(i+1)\}, \\ v^{rev}_{(S\setminus\{i\}\cup)\{i+1\}} & \text{if } S_{i,i+1} = \{i, -(i+1), -i\}, \\ v^{rev}_{(S\setminus\{-(i+1)\}\cup)\{-i\}} & \text{if } S_{i,i+1} = \{i, (i+1), -(i+1)\}, \\ 0 & \text{otherwise}. \end{cases}$$

$$f_n \centerdot v_S^{rev} = \begin{cases} v^{rev}_{(S\setminus\{n\}\cup)\{-n\}} & \text{if } S_n = \{n\}, \\ 0 & \text{otherwise}. \end{cases}$$

Similarly, for $e_i \centerdot v_S^{rev}$ *and* $e_n \centerdot v_S^{rev}$*.*

Proof As for Lemma 2B.10. □

Definition 8.8 We call the $\mathbb{A}$-antilinear map defined by

$$v_S \mapsto \overline{v_S} := (-q^{-1})^{\binom{|S|}{2}} (q^{-1})^{\frac{|S\cap -S|}{2}} \cdot v_S^{rev}$$

the *bar involution*. ◇

Note that for $g(q) \in \mathbb{A}$, we have $\overline{g(q) \cdot v_S} = g(q^{-1}) \cdot \overline{v_S}$. We are about to show that this map is an involution, justifying the choice of terminology.

Lemma 8.9 *We have* $\overline{e_i \centerdot v_S} = e_i \centerdot \overline{v_S}$*,* $\overline{f_i \centerdot v_S} = f_i \centerdot \overline{v_S}$ *and* $\overline{k_i \centerdot v_S} = k_i^{-1} \centerdot \overline{v_S}$ *for all* $i \in \{1, \ldots, n\}$.

Proof The proof is a direct calculation using Lemma 2B.10, Definition 8.8, and Lemma 8.7. We give an example of one of the calculations, leaving the rest to the reader. Suppose that $S_{i,i+1} = \{i, -(i+1)\}$ and write $T := (S \setminus \{-(i+1)\}) \cup \{-i\}$ and $U := (S \setminus \{i\}) \cup \{i+1\}$. Then

$$\begin{aligned} \overline{f_i \centerdot v_S} &= \overline{v_T + q^{-1} \cdot v_U} \\ &= (-q^{-1})^{\binom{|T|}{2}} (q^{-1})^{\frac{|T\cap -T|}{2}} \cdot v_T^{rev} + q \cdot (-q^{-1})^{\binom{|U|}{2}} (q^{-1})^{\frac{|U\cap -U|}{2}} \cdot v_U^{rev} \end{aligned}$$

$$= (-q^{-1})^{\binom{|S|}{2}} \cdot q^{-1} \cdot (q^{-1})^{\frac{|S\cap -S|}{2}} \cdot v_T^{rev} + q \cdot (-q^{-1})^{\binom{|S|}{2}} \cdot q^{-1} \cdot (q^{-1})^{\frac{|S\cap -S|}{2}} \cdot v_U^{rev}$$
$$= (-q^{-1})^{\binom{|S|}{2}} (q^{-1})^{\frac{|S\cap -S|}{2}} \cdot \left(q^{-1} \cdot v_T^{rev} + v_U^{rev}\right)$$
$$= (-q^{-1})^{\binom{|S|}{2}} (q^{-1})^{\frac{|S\cap -S|}{2}} \cdot f_i \centerdot v_S^{rev}$$
$$= f_i \centerdot \overline{v_S}.$$

All other calculations are similar. □

Remark 8.10 First, note that the defining relations for $\Lambda_{\mathbb{A}}$ imply that for all $k > 0$,

$$v_{-(x+k)} v_{-(x+1)} \dots v_{-n} v_n \dots v_{x+1} = 0,$$

and therefore $v_x v_{-x} v_{-(x+1)} \dots v_{-n} v_n \dots v_{x+1} = -q^2 \cdot v_{-x} v_x v_{-(x+1)} \dots v_{-n} v_n \dots v_{x+1}$. ◇

Lemma 8.11 *We have $\overline{v_{k,k-2i}} = v_{k,k-2i}$ for all $0 \leq k \leq 2n$ and $i \in \mathbb{Z}_{\geq 0}$ with $k - 2i \in \mathbb{Z}_{\geq 0}$.*

Proof The relations for $\Lambda_{\mathbb{A}}$ and Remark 8.10 imply that

$$v_{k,k-2i}^{rev} = (-q)^{(k-1)} v_1 v_{-(n-i+1)} \dots v_{-n} v_n \dots v_{n-i+1} v_{k-2i} \dots v_2$$
$$= (-q)^{(k-1)+(k-2)} v_1 v_2 v_{-(n-i+1)} \dots v_{-n} v_n \dots v_{n-i+1} v_{k-2i} \dots v_3$$
$$= \dots$$
$$= (-q)^{(k-1)+(k-2)+\dots+(k-(k-2i))} \cdot v_1 \dots v_{k-2i} v_{-(n-i+1)} \dots v_{-n} v_n \dots v_{n-i+1}$$
$$= (-q)^{(k-1)+(k-2)+\dots+(2i)+(2i-1)} q \cdot v_1 \dots v_{k-2i} v_{n-i+1} v_{-(n-i+1)} v_{-(n-i+2)} \dots v_{-n} v_n \dots v_{n-i+2}$$
$$= (-q)^{(k-1)+\dots+(2i-2)} q^2 \cdot v_1 \dots v_{k-2i} v_{n-i+1} v_{n-i+2} v_{-(n-i+1)} v_{-(n-i+2)} \dots v_{-n} v_n \dots v_{n-i+3}$$
$$= \dots$$
$$= (-q)^{(k-1)+\dots+(2i-i)} q^i \cdot v_1 \dots v_{k-2i} v_{n-i+1} \dots v_n v_{-(n-i+1)} \dots v_{-n}$$
$$= (-q)^{(k-1)+\dots+(i)+(i-1)} q^i \cdot v_1 \dots v_{k-2i} v_{n-i+1} \dots v_n v_{-n} v_{-(n-i+1)} \dots v_{-(n-1)}$$
$$= (-q)^{(k-1)+\dots+(i-2)} q^i \cdot v_1 \dots v_{k-2i} v_{n-i+1} \dots v_n v_{-n} v_{-(n-1)} v_{-(n-i+1)} \dots v_{-(n-2)}$$
$$= \dots$$
$$= (-q)^{(k-1)+\dots+(1)} q^i \cdot v_1 \dots v_{k-2i} v_{n-i+1} \dots v_n v_{-n} \dots v_{-(n-i+1)}$$
$$= (-q)^{\binom{k}{2}} q^i \cdot v_{k,k-2i}.$$

By the definition of the bar involution we find

$$\overline{v_{k,k-2i}} = (-q^{-1})^{\binom{k}{2}} (q^{-1})^i \cdot v_{k,k-2i}^{rev} = v_{k,k-2i}.$$

The proof completes. □

Proposition 8.12 *The bar involution is an $\mathbb{A}$-antilinear involution.*

Proof Combining Lemmas 8.9 and 8.11, this follows from Lemma 8.4. □

Lemma 8.13 *The elements b_S are invariant under the bar involution.*

Proof For brevity, we only sketch a proof.

Let $S \subset [1, -1]$. Suppose that $|S| = k$ and $|\pm L(S)| + 2i = |S_0|$. We will show that there is $u_S \in U_{\mathbb{A}}(\mathfrak{sp}_{2n})$ of the form $u_S = f_{j_1} \dots f_{j_t}$, for some $j_p \in \{1, \dots, n\}$, such that $u_S \centerdot v_{k,k-2i} = b_S$. It will then follow from Lemmas 8.9 and 8.11 that

$$\overline{b_S} = \overline{u_S \centerdot v_{k,k-2i}} = u_S \centerdot \overline{v_{k,k-2i}} = u_S \centerdot v_{k,k-2i} = b_S.$$

To this end, let

$$\mathcal{C} := \{S \subset [1, -1]\},$$
$$\mathcal{C}^k := \{S \subset [1, -1] \,|\, |S| = k\}, \mathcal{C}^k_{k-2i} := \{S \subset [1, -1] \,|\, |S| = k \text{ and } |\pm L(S)| + 2i = |S_0|\}.$$

Note that

$$\mathcal{C} = \coprod_{0 \le k \le 2n} \mathcal{C}^k = \coprod_{\substack{0 \le k \le 2n \\ k-2i \ge 0}} \mathcal{C}^k_{k-2i}.$$

Define operators $\tilde{E}, \tilde{F} : \mathcal{C} \to \mathcal{C} \cup \{0\}$ as follows:

$$\tilde{E} \centerdot S := \begin{cases} S \cup \{\pm i_k\} & S_0 \setminus \pm L(S) = \{\pm i_1, \dots, \pm i_r\} \text{ and } i_1 < \dots < i_r, \\ 0 & S_0 = \pm L(S), \end{cases}$$
$$\tilde{F} \centerdot S := \begin{cases} S \setminus \{\pm i_1\} & S_0 \setminus \pm L(S) = \{\pm i_1, \dots, \pm i_r\} \text{ and } i_1 < \dots < i_r, \\ 0 & S_0 = \pm L(S). \end{cases}$$

We write $\mathcal{C}(m) := \{-m, -m+2, \dots, m-2, m\}$ to denote the $\mathfrak{sl}_2$ crystal of highest weight m, with the evident action of $\tilde{E}$ an $\tilde{F}$. One can then use Definition 2E.8 to determine a bijection

$$\Big(\mathcal{C}(0) \coprod \mathcal{C}(1) \coprod \mathcal{C}(0)\Big)^{\otimes n} \to \mathcal{C},$$

which is easily seen to be an isomorphism of $\mathfrak{sl}_2$ crystals.

For $S \in \mathcal{C}$ such that $S_0 \setminus \pm L(S) = \{\pm i_1, \dots, \pm i_r\}$, we have $S \setminus \{\pm i_1, \dots, \pm i_r\} \in \mathcal{C}_{\varpi_k}$. Moreover, the assignments

$$S \mapsto (S \setminus \{\pm i_1, \dots, \pm i_r\}, -n + |S|) \in \mathcal{C}_{\varpi_{|S|-2r}} \otimes \mathcal{C}(n - |S| + 2r)$$

induce a bijection

$$\mathcal{C} \to \coprod_{0 \le k \le n} \mathcal{C}_{\varpi_k} \otimes \mathcal{C}(n-k).$$

The formulas in Definition 4C.5 define operators $\tilde{e}_j, \tilde{f}_j : \mathcal{C} \to \mathcal{C} \cup \{0\}$, and we can then view the above bijection as an isomorphism of $\mathfrak{sp}_{2n} \times \mathfrak{sl}_2$ crystals.

For $S \in \mathcal{C}$ such that $|S| = k$ and $|S_0 \setminus \pm L(S)| = r$, then from Lemma 4E.2 we know there is $u_{\tilde{F}^r \centerdot S} = f_{j_1} \cdots f_{j_t}$ such that $u_{\tilde{F}^r \centerdot S} \centerdot v_{k-2r,k-2r} = b_{\tilde{F}^r \centerdot S}$. Now, similar to the proof of Lemma 4E.2, one compares the descriptions of $f_j \centerdot b_S$ and $\tilde{f}_j \centerdot S$, but now for $S \in \mathcal{C}$, to deduce that $u_{\tilde{F}^r \centerdot S} \centerdot v_{k,k-2r} = b_S$, as desired. □

Given two subsets $X, Y \subset \{1, \dots, n\}$, we write $X \le Y$ if $X = \{x_1 < x_2 < \dots < x_m\}$, $Y = \{y_1 < y_2 < \dots < y_m\}$, and $x_k \le y_k$ for $k \in \{1, \dots, m\}$.

Definition 8.14 We define a partial order on the standard basis $\{v_S | S \subset [1, -1]\}$ by $v_S \le v_T$ if and only if:

(i) $S \setminus S_0 = T \setminus T_0$.
(ii) $S_0 \leq T_0$.

(Note that $v_S \leq v_T$ is equivalent to $\mathrm{wt}\, v_S = \mathrm{wt}\, v_T$ and $S_0 \leq T_0$.) ◇

Lemma 8.15 *There is a unique symmetric bilinear form*

$$(\_,\_)\colon \Lambda_{\mathbb{A}} \times \Lambda_{\mathbb{A}} \to \mathbb{A}$$

determined by $(v_S, v_T) = \delta_{S,T}$.

Proof Clear. □

Using Lemma 8.15 we define:

Definition 8.16 We define the sesquilinear form

$$\langle\_,\_\rangle\colon \Lambda_{\mathbb{A}} \times \Lambda_{\mathbb{A}} \to \mathbb{A}, \quad \langle x, y\rangle := (x, \overline{y}).$$

(*Sesquilinear* means q-linear in first argument and q-antilinear in the second argument). ◇

Note that $\langle \overline{y}, \overline{x}\rangle = \langle x, y\rangle$ and $\langle v_S, \overline{v_T}\rangle = \delta_{S,T}$.

Lemma 8.17 *Let* $S \subset [1, -1]$. *We have*

$$\overline{v_S} \in v_S + \sum_{v_T \geq v_S} \mathbb{A} \cdot v_T.$$

In particular, if S does not have any fully dotted columns, then $\overline{v_S} = v_S$.

Proof The proof is similar to the proof of Lemma 8.11. That is, going from $\overline{v_S}$ to v_S can be done using the defining relations of $\Lambda_{\mathbb{A}}$ to reverse the order. The only relation one needs to be careful in the process is

$$v_{-i} v_i = -q^2 \cdot v_i v_{-i} + (q - q^{-1}) \sum_{k \in [1, n-i]} (-q)^{k+1} \cdot v_{i+k} v_{-(i+k)}.$$

The term in this relation that is multiplied by $(q - q^{-1})$ produces fully dotted columns further to the right. Applying this observation repeatedly shows the statement by additionally observing that the scalar in front of the leading term comes out as claimed. □

Lemma 8.18 *The free $\mathbb{A}$-module $\Lambda_{\mathbb{A}}$ equipped with standard basis $\{v_S | S \subset [1, -1]\}$, partial order $\leq$ from Definition 8.14, and bar involution $v_S \mapsto \overline{v_S}$ equip $\Lambda_{\mathbb{A}}$ satisfies the conditions of a* balanced pre-canonical structure *in the terminology of [47, Definitions 1.1 and 1.5]. (The balanced part comes from $\langle v_S, \overline{v_T}\rangle = \delta_{S,T}$.)*

Proof Directly from the definitions. □

By *canonical* we mean [47, Definition 1.7].

Theorem 8.19 *The basis $\{b_S | S \subset [1, -1]\}$ is canonical.*

Proof Because of [47, Lemma 1.8], this follows from

$$b_S \in v_S + \sum_{T \geq S} q^{-1}\mathbb{Z}[q^{-1}] \cdot v_T.$$

and bar invariance of b_S as in Lemma 8.13. □

Remark 8.20 Using [47, Theorem 1.9] one can now show that $\{b_S | S \subset [1, -1]\}$ is *the* canonical basis (up to signs). Details are omitted. ◇

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