Exact Results on Moduli Spaces of Supersymmetric Gauge Theories

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1. **Chapter 2:** J. Gray, A. Hanany, Y. H. He, V. Jejjala and N. Mekareeya, "SQCD: a Geometric Aperçu," JHEP **0805** (2008) 099 [arXiv:0803.4257 [hep-th]].

2. **Chapter 3:** A. Hanany and N. Mekareeya, "Counting Gauge Invariant Operators in SQCD with Classical Gauge Groups," JHEP **0810** (2008) 012 [arXiv:0805.3728 [hep-th]].

3. **Chapter 4:** S. Benvenuti, A. Hanany and N. Mekareeya, "The Hilbert Series of the One Instanton Moduli Space," JHEP **1006** (2010) 100 [arXiv:1005.3026 [hep-th]].

4. **Chapter 5:** A. Hanany and N. Mekareeya, "Tri-Vertices and $SU(2)$'s," JHEP **1102** (2011) 069 [arXiv:1012.2119 [hep-th]].

Closely related material to this thesis is also contained in

1. A. Hanany, N. Mekareeya and G. Torri, "The Hilbert Series of Adjoint SQCD," Nucl. Phys. B **825** (2010) 52 [arXiv:0812.2315 [hep-th]].

2. A. Hanany, N. Mekareeya and A. Zaffaroni, "Partition Functions for Membrane Theories," JHEP **0809** (2008) 090 [arXiv:0806.4212 [hep-th]].

3. Y. Chen and N. Mekareeya, "The Hilbert Series of U/SU SQCD and Toeplitz Determinants," arXiv:1104.2045 [hep-th].

Summary

In this thesis, certain exact results in supersymmetric gauge theories are discussed. In these theories, holomorphic gauge invariant operators play a central role in understanding the structure of the space of solutions to vacuum equations, known as the moduli space. We focus on a technique to count such operators with various quantum numbers. The counting can be done by computing a partition function, known as the *Hilbert series*, which counts all holomorphic gauge invariant operators carrying a specified set of global $U(1)$ charges. The Hilbert series can be computed exactly for various gauge theories.

In Part I of this thesis, we compute the Hilbert series of four dimensional $\mathcal{N} = 1$ supersymmetric QCD with classical gauge groups. In part II, we count chiral operators on the one instanton moduli space on $\mathbb{R}^4$ and study the hypermultiplet moduli spaces of a large class of $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions.

We demonstrate that the Hilbert series not only contains information about the spectrum of operators in the theory, but it also carries geometrical properties of the moduli space, *e.g.* the dimension. It is also an indicator of whether the moduli space is Calabi-Yau. Moreover, Hilbert series can be used as a primary tool to test various dualities in gauge theories and in string theory.

Contents

Chapter 1

Introduction and Motivation

General relativity and quantum mechanics are two of the most successful theories of twentieth century physics. The former describes physical phenomena involving macroscopic objects, such as black holes, the expansion of the universe and the precession of the perihelion of Mercury. It has been tested on solar system and cosmological scales to a very high precision. On the other hand, quantum mechanics is a theory of microscopic objects, such as molecules, atoms and elementary particles. The spectrum of the Hydrogen atom, as an example, confirms that quantum mechanics is necessary for describing the microscopic world.

Below, we give a brief summary of the development of various fundamental aspects of theoretical physics, ranging from quantum field theory to string theory, whose ultimate goal is to seek a unified theory of all known interactions. The summary below is not intended to be comprehensive but rather, it is meant to serve as a motivation for the discussion of such topics as string theory and supersymmetric gauge theory, the main focus of interest in this thesis. For more details, the reader is referred to textbooks and review papers, such as [1–16], and references therein.

1.1 Quantum field theory

The study of the quantum theory of electromagnetic fields and their interactions with matter led to the development of quantum electrodynamics (QED). This theory unifies electromagnetism, quantum mechanics and special relativity into one framework. Particles are regarded as excitations of the corresponding field, *e.g.* the photon can be regarded as an excitation of the electromagnetic field. In this way, the discovery of QED was a starting point of the subject called *quantum field theory*. Agreement between theoretical computations and precise measurements of the Lamb shift and the anomalous magnetic moment of the electron were two of the greatest achievements of QED as a physical theory. In addition to the electromagnetic interaction, there are two other short range interactions, namely the strong and the weak nuclear interactions. The former is responsible for binding atomic nuclei together and the latter is responsible for radioactive beta decay. The theory describing the strong nuclear interaction is known as *quantum chromodynamics* (QCD), and the unified theory describing both weak nuclear interaction and electromagnetism is known as the *electroweak theory*. Each of these theories possesses a local symmetry, called *gauge symmetry*. The gauge symmetries for QCD and the electroweak theory are $SU(3)$ and $SU(2) \times U(1)$ respectively. The gauge symmetry for electromagnetism is $U(1)$ coming from the spontaneous symmetry breaking of the electroweak gauge symmetry $SU(2) \times U(1)$.

Although quantum field theory explains and predicts a number of phenomena to a satisfactory degree of accuracy, during its early development the theory encountered the problem of infinities, *i.e.* naive computations of quantum effects lead to infinite results (see *e.g.*, [1] for a review). In QED, in order to make the answers finite and agree with experimental observations, one can absorb these infinities into various parameters, namely the couplings and masses, of the theory [17]. This process is known as *renormalisation*. In this sense, quantum field theories can be divided into two classes, namely those of which all infinities can be absorbed into a *finite* number of parameters and those for which this is not possible. The former are said to be *renormalisable* and the latter are *non-renormalisable*. In the past, the problem of infinities in non-renormalisable theories was thought to be hopeless and it was therefore believed that the only sensible quantum field theories were the renormalisable ones. The requirement of renormalisability imposes strong restrictions on the theories one may consider. Indeed, physical phenomena at currently experimentally accessible energies are described accurately by the *standard model*, which is a renormalisable theory of quarks, leptons and gauge bosons transforming under the gauge symmetry $SU(3) \times SU(2) \times U(1)$ (see *e.g.*, [2, 4] for reviews).

From the modern point of view, renormalisability is no longer a fundamental physical requirement, and indeed many realistic field theories, such as Einstein's theory of gravitation, are non-renormalisable. It becomes clear that, in such theories, the divergences can also be absorbed into a redefinition of the parameters, but one needs infinitely many of them. In this sense, non-renormalisable theories are just as renormalisable as renormalisable theories (see *e.g.*, [1]). However, due to an infinite number of free parameters, non-renormalisable theories have little predictive power at a certain energy scale M. Above this energy scale, such theories are no longer well-behaved (*e.g.* the S-matrix may violate unitary bounds). This is a signal for new physics which may enter at the scale M, and in order to understand physics above this scale one must first understand the underlying fundamental theory which may not be a field theory at all. One can, however, focus on the low energy limit (far below M) of the fundamental theory – in this limit, one can consider an *effective field theory* which generally contains infinitely many non-renormalisable interactions.

Such interactions are highly suppressed by the large scale M at sufficiently low energies. Thus, from this point of view, the successful renormalisable theories, such as the standard model, the electroweak theory and QED, are simply low energy effective field theories, accompanied with non-renormalisable interactions suppressed by a large energy scale.

1.2 The hierarchy problem

At present, the standard model is regarded as a low energy approximation to an unknown fundamental theory in which gravitation is unified with the strong and electroweak interactions around 10^{16} to 10^{18} GeV.[1] This energy scale is much larger than the electroweak symmetry breaking scale of 300 GeV which characterises the standard model. Such an extraordinarily large ratio between the fundamental energy scale and the electroweak scale requires a physical explanation – this is known as the *hierarchy problem* (see *e.g.*, [3–5]). In the standard model, the invariance under the $SU(3) \times SU(2) \times U(1)$ gauge symmetry requires quarks, leptons and gauge bosons to have vanishing bare masses in the Lagrangian. Hence, the physical masses of these particles are proportional to the electroweak symmetry breaking scale, which in turn is also proportional to the mass of the scalar fields responsible for the electroweak symmetry breaking. The main issue of the hierarchy problem is that, unlike fermions and gauge bosons, scalar fields are not protected by any symmetry of the standard model from acquiring large bare masses – it is therefore not clear why these scalar fields masses, and thus all other particle masses, are not as large as 10^{16} to 10^{18} GeV. This is indeed one of the strongest motivations for introducing a new kind of symmetry, called *supersymmetry*.

1.3 Supersymmetry

Supersymmetry is a symmetry which transforms bosons into fermions and *vice-versa*. To date, the status of supersymmetry is still hypothetical, since no new particles required by supersymmetry have been experimentally detected – the search of such particles is currently a prime target of experiments at the Large Hadron Collider (LHC) at CERN. Nevertheless, supersymmetry is a valuable theoretical tool for obtaining exact results and studying various non-perturbative phenomena in quantum field theory.

An attempt to solve the hierarchy problem is to embed the standard model into a supersymmetric theory (see *e.g.*, [3, 5]). If the scalar fields along with their

[1] Here we do not consider the possibility of unification at low energies due to, for example, extra dimensions much larger than 10^{-16} GeV^{-1} or warped extra dimensions.

fermionic superpartners are in a chiral representation of some gauge group, then both scalar fields and fermions are required by supersymmetry to have vanishing bare masses. In which case, all masses in the standard model will then be related to the supersymmetry breaking scale. Thus, in principle, supersymmetry could provide a solution to the hierarchy problem. The study of supersymmetric extensions of the standard model has many open questions and is currently an active field of research.

Another motivation for supersymmetry comes from attempts in the 1960s to embed the $SU(3)$ gauge symmetry of QCD into a larger symmetry group, namely $SU(6)$, which transforms $SU(3)$ multiplets of different spins into each other (see *e.g.* [18–21] and [3] for a review). This works for the non-relativistic quark model but fails for the relativistic one. It was shown in 1967 by Coleman and Mandula (known as the *Coleman–Mandula theorem* [22]) that in a relativistic field theory it is impossible to combine space-time and internal symmetries in any other way but the trivial one. This theorem thus rules out the existence of such relativistic symmetries as the aforementioned $SU(6)$ and a large class of other continuous symmetries. Nevertheless, one can evade this theorem by invoking a symmetry which non-trivially transforms bosons and fermions into each other and whose generators satisfy anti-commutation relations (instead of commutation relations). Indeed, supersymmetry arises in this way as the only possibility (see *e.g.*, [3]).

Historically, without a reference to the Coleman-Mandula theorem, supersymmetry originated in the context of string theory, which was first formulated as a theory describing various types of hadrons as different modes of vibration of a string. The early version of string theory, known as the *bosonic string theory*, contained only bosonic degrees of freedom which is unrealistic. In order to match the particles described by string theory to those seen in the real world, fermionic degrees of freedom were introduced into the bosonic string theory. The resulting theory is known as the *Ramond-Neveu-Schwarz (RNS) model* [23–25]. Gervais and Sakita [26] then observed that in addition to the conformal invariance of the two dimensional worldsheet and the Lorentz invariance of the target space, the theory possesses a symmetry which transforms bosons into fermions and *vice-versa*. This is indeed supersymmetry in the two-dimensional worldsheet theory. Later in 1974, supersymmetry in four dimensional field theories was studied by Wess and Zumino[2] [27] – they subsequently realised that bypassing of the Coleman-Mandula theorem leads to anti-commutation relations in the supersymmetry algebra [29].

[2]The supersymmetry in four dimensions had actually been studied by Gol'fand and Likhtman [28] in 1971 before Wess and Zumino.

1.4 String theory

Although string theory was originally formulated to describe strongly interacting particles, the original theory was problematic for several reasons. For example, (i) the theory is consistent only in more than four spacetime dimensions (26 for the bosonic string theory and 10 for supersymmetric string theory), (ii) there exists a massless spin two state in string theory, but there is no such a state observed in the hadronic world, and (iii) for high energy fixed angle processes, scattering amplitudes in string theory (known as the *Veneziano amplitudes* [30]) fall off exponentially as the kinematic variable tends to infinity, whereas experimentally observed strong interaction scattering amplitudes fall off according to a power law (see *e.g.* [6]). As a result, the original motivation for string theory disappeared and QCD has since been regarded as a theory of the strong interaction.

In 1974, Scherk and Schwarz [37] came up with a proposal that completely changed the perspective on string theory – they suggested that the massless spin two particle present in the theory could be interpreted as the graviton, the quantum field of gravitation. This implies that the string tension should be related to the characteristic energy scale of gravity (also known as the *Planck scale*, which is around 10^{19} GeV). Moreover, they realised that in the low energy limit the graviton interacts according to general relativity. In this way, there was a hope that string theory might provide a unification of gravitation with all other interactions in single quantum theory.

Formulations of supersymmetry in string theory continued to develop. In 1977, Gliozzi, Scherk and Olive [31, 32] realised that by imposing suitable periodicity conditions (known as the *GSO projection*) on the spectrum of the RNS model, the theory acquires spacetime supersymmetry. Subsequently, the formalism with manifest target space supersymmetry was developed by Green and Schwarz [33–35]. As a result of their works, three different superstring theories, called the *Type I*, *Type IIA* and *Type IIB* theories, were identified and named.

In order to see these three superstring theories, one needs to consider both the left-moving modes and the right-moving modes of strings. The supersymmetries associated with the left-movers and the right-movers have either opposite chirality or the same chirality. The former gives rise to the Type IIA theory, unlike the latter which gives rise to the Type IIB theory. Moreover, taking the Type IIB strings and quotienting out by the left-right symmetry (known as the *orientifold projection* [36, 48]), one obtains a theory with unoriented strings – this is known as the Type I theory.

However, in the early 1980s, it appeared that superstrings could not describe parity-violating theories due to quantum inconsistencies called anomalies which were believed to be fatal. Gravitational anomalies in higher dimensions were first systematically investigated by Alvarez-Gaumé and Witten [38] in 1984. Based on this work, Green and Schwarz [39] showed that all gauge and gravitational anomalies in the theory could cancel provided that the gauge group was either $SO(32)$ or $E_8 \times E_8$. Shortly after the work of Green and Schwarz, Gross, Harvey, Martinec and Rohm [40–42] found the other two formulations of superstring theory in ten dimensions, namely the $SO(32)$ and the $E_8 \times E_8$ *heterotic* theories. The discovery of Green-Schwarz anomaly cancellation has led to wide acceptance of string theory – this is often referred to as *the first superstring revolution*.

1.5 String dualities, branes and M-theory

The five formulations of superstring theory, namely Type I, Type IIA, Type IIB, the heterotic $SO(32)$ and the heterotic $E_8 \times E_8$ theories, are related to each other by duality transformations; in other words, one can express the degrees of freedom of any of them in terms of the degrees of freedom of the others. We summarise these dualities below.

In the late 1980s, a duality relating two theories with different geometries for the extra dimensions, known as *T-duality*, was understood [43–45] (see also *e.g.* [46, 47] for reviews). T-duality relates Type IIA theory to Type IIB theory and the heterotic $SO(32)$ theory to the heterotic $E_8 \times E_8$ theory.

In addition to the fundamental strings, superstring theory also contains various non-perturbative object called *p-branes*, which are extended objects with $(p+1)$-dimensional world-volume. In this section, we discuss three types of extended objects in superstring theory that are very useful in applications to gauge theory, namely the D-brane, the orientifold plane and the NS5-brane.

The Type I and Type II theories contain an important class of p-branes, known as Dp-branes (or simply *D-branes* in general), whose tension is inversely proportional to the string coupling and to the $(p+1)$-th power of the string length. The subject of D-branes was originated by Dai, Leigh and Polchinski [48] and Leigh [49] in 1989. In 1995, Polchinski [50] showed that D-branes preserve half of the supersymmetries and are charged under Ramond sector gauge fields. The defining property of a D-brane is that it is an object on which fundamental strings can end. Open strings carry charges, known as *Chan–Paton* charges [51], at the end points. It was subsequently realised by Witten [56] in 1995 that the ending of the fundamental strings on N coincident D-branes gives rise to super Yang–Mills theory with the gauge group $U(N)$ and sixteen supercharges living on the world-volumes, with the Yang–Mills fields corresponding to the massless modes of the open strings attached to the D-branes.

In addition to the Dp-brane, there exists a fixed $(p+1)$-dimensional hyperplane which gives rise to a $\mathbb{Z}_2$ action on the space-time coordinates and reverses the orientation of the string [48]. This is known as an *orientifold p-plane* or an Op-plane.

It is a non-dynamical object which is charged under the same Ramond sector gauge fields and breaks the same half of the supersymmetries as the Dp-brane. On a stack of D-branes parallel to an orientifold plane, one finds a gauge theory with sixteen supercharges and a special orthogonal or a symplectic gauge group depending on the charge of the orientifold plane. This agrees with the realisation in the Type I theory due to Schwarz [53] and Marcus and Sagnotti [54] in 1982. Indeed, Type I string theory can be regarded as Type II superstring theory with orientifold planes and D-branes.

Another type of an extended object is a 5-brane, known as the *Neveu–Schwarz (NS) 5-brane* [55], which preserves half of the supersymmetry of the theory and has tension proportional to the inverse squared of the string coupling and to the sixth power of the string length. This is a solitonic object coupled magnetically to the NS-NS sector field, and hence it exists only in weakly coupled Type II and heterotic theories.

The discovery of branes has led to fascinating developments in supersymmetric gauge theory which is the main subject of this thesis (see *e.g.* [13] for a review). They are also crucial for string model-building whose ultimate goal is to reproduce and explain the structure of the standard model, such as matter content, masses, charges and interactions (see *e.g.* [15, 16] for reviews and references therein) – this is currently an active field of research.

A non-perturbative duality, called *S-duality*, which relates a strongly coupled theory to a weakly coupled theory has been studied since the late 1970s. In 1977, Montonen and Olive [57] proposed this type of duality in Yang–Mills theory. Such a proposal was sharpened to the case of $\mathcal{N} = 4$ super Yang-Mills (SYM) theory in four dimensions in 1979 by Osborn [58]. In string theory, the existence of S-duality was conjectured by Font, Ibáñez, Lüst and Quevedo [59] for the heterotic theory compactified on a six-torus. The study was then carried out further by Sen [60, 61]. It was found that both toroidally compactified heterotic string theory and $\mathcal{N} = 4$ SYM possess an exact $SL(2, \mathbb{Z})$ S-duality symmetry. In 1994, Seiberg and Witten generalised the Montonen–Olive duality to $\mathcal{N} = 2$ super Yang–Mills theory [62] and in $\mathcal{N} = 2$ supersymmetric QCD [63]. These works demonstrated that the S-duality plays a crucial role in determining the vacuum structure and the spectrum of the theories. The S-duality of the Type IIB theory was proposed by Hull and Townsend [64]. They also conjectured that the Type II superstring theory compactified on a $(d-1)$-dimensional torus T^{d-1} possesses an exact discrete symmetry $E_{d(d)}(\mathbb{Z})$, known as the *U-duality* symmetry, which contains both the T-duality symmetry $SO(d-1, d-1, \mathbb{Z})$ and the S-duality symmetry $SL(2, \mathbb{Z})$. The S-duality relation between the Type I theory and the heterotic $SO(32)$ was proposed by Polchinski and Witten [65].

S-duality allows us to describe three superstring theories, namely the Type I, the heterotic $SO(32)$ and the Type IIB theories, at strong coupling. The study of the Type IIA and the heterotic $E_8 \times E_8$ theories, on the other hand, revealed that these theories do not only exhibit an eleventh dimension at strong coupling, but also approach a common eleven dimensional limit, known as *M-theory*. At low energies it is approximated by a classical field theory called *eleven-dimensional supergravity*, which was first formulated by Cremmer, Julia and Scherk in 1978 [68]. M-theory does not contain strings, but rather a solitonic 2-brane and 5-brane, respectively known as the M2-brane and the M5-brane, which are electric and magnetic dual to each other.

In 1995, Witten [69] and Townsend [70] pointed out that the Type IIA superstring theory becomes eleven dimensional in the strongly coupled limit. Indeed, the Type IIA string and branes have natural interpretations in M-theory. For example, a fundamental Type IIA string can be regarded as an M2-brane wrapped around the compact direction. We refer the reader for more details to, *e.g.*, [13]. The 11-dimensional interpretation of the strongly coupled heterotic $E_8 \times E_8$ superstring theory is given by Hořava and Witten [72, 73]. In 1995, Aspinwall [74] and Schwarz [75, 76] proposed a way of understanding the S-duality group $SL(2, \mathbb{Z})$ of the Type IIB string theory using of the duality between M-theory on a two-torus T^2 and Type IIB superstring theory on a circle S^1. The Type IIB superstring in ten dimensions has an infinite $SL(2, \mathbb{Z})$ multiplet [77], and the interpretation as solitons and bound states is due to Witten [56] in 1995. In the toroidally compactified M-theory on T^d, the S-duality group $SL(2, Z)$, which is the symmetry group of the two-torus T^2, is a subgroup of the symmetry group $SL(d, \mathbb{Z})$ of the d-torus T^d. The groups $SL(d, \mathbb{Z})$ together with the T-duality group $SO(d-1, d-1, \mathbb{Z})$ generate the U-duality group $E_{d(d)}(\mathbb{Z})$, which is an exact symmetry of the M-theory compactified on T^d.

The relation between M-theory and the Type IIA and heterotic $E_8 \times E_8$ theories, together with S and T dualities, imply that the five superstring theories are connected by a web of dualities. The five string theories can be regarded as expansions around different corners in moduli space of M-theory (see *e.g.* [11, 12, 66, 67] for reviews). The discovery of string dualities, D-branes and M-theory is regarded as *the second superstring revolution*.

1.6 Supersymmetric gauge theory

This thesis will focus on the study of supersymmetric gauge theory. There are several reasons why supersymmetric field theories are of interest. On the phenomenological side, it provides a framework and predictions for the beyond standard model physics that the Large Hadron Collider (LHC) could discover in a few years time. On the formal side, supersymmetry is a valuable tool for rigorously studying non-perturbative phenomena in quantum field theory. Many aspects of supersymmetric gauge theories can be analysed exactly, providing a laboratory for the analysis of the dynamics of gauge theories. There is also a wide range of interesting phenomena one can explore, such as phases and dualities, which do not appear in non-supersymmetric gauge theories.

A number of interesting results about supersymmetric gauge theory and string theory can be obtained by studying the quantum field theories on the world-volumes of string theory and M-theory branes. The simplest example is super Yang–Mills theory with sixteen supercharges, arising from ground states of open strings ending on N coincident D-branes. One can also obtain gauge theories with lower supersymmetry from webs of branes. For example, by considering a stack of k Dp-branes in the background of N D$(p+4)$-branes, one can realise a supersymmetric gauge theory with eight supercharges, gauge group $U(k)$ and N flavours of fundamental matter on the world-volume of the Dp-branes. This set-up is discussed in Chapter 4 of this thesis. When the Dp-branes and the D$(p+4)$-branes are on top of each other, the Dp-branes can be viewed as k $SU(N)$ instantons on $\mathbb{R}^4$ from the perspective of an observer in the world-volume of the D$(p+4)$-branes. The Dp-D$(p+4)$ brane system can be generalised in many directions, *e.g.*, introducing branes and/or orientifold planes, rotating some of the branes relative to others, applying S and/or T duality, and considering branes ending on branes (see *e.g.* [13] for a review and references therein).

A breakthrough in the subject of gauge theories and brane dynamics was due to Hanany and Witten [78] in 1996. They investigated dynamics of three-dimensional supersymmetric gauge theories using configurations of D3-branes and NS5-branes in the ten-dimensional Type IIB superstring theory. Several phenomena, such as mirror symmetry between three-dimensional gauge theories and brane creation, can be realised in this set-up. The Hanany-Witten construction was further examined in [79, 80] and was applied to $\mathcal{N}=1$ supersymmetric gauge theories in four dimensions by [81]. Later in 1997 Witten [82] studied $\mathcal{N}=2$ supersymmetric gauge theories in four dimensions using configurations of D4-branes, NS5-branes, and D6-branes in the Type IIA string theory. These configurations were lifted to M-theory in which they consist of an M5-brane in a multi-Taub-NUT background. The dynamics of several $\mathcal{N}=2$ supersymmetric gauge theories can be interpreted geometrically and explicit solutions for the Coulomb branch of such theories can be obtained in this way. Recently, Gaiotto [83, 84] generalised this set-up to describe a large class of $\mathcal{N}=2$ superconformal field theories in four dimensions arising from M5-branes wrapping Riemann surfaces. We discuss this class of theories in Chapters 4 and 5 of this thesis.

Another major breakthrough in string theory is the discovery of a duality (known as the *AdS/CFT correspondence*) between a supersymmetric conformal field theory (CFT) and string theory/M-theory in a background of anti-de-Sitter (AdS) spacetime times a certain compact manifolds. Based on earlier works [85–87], this duality was proposed by Maldacena [88] in 1997 and was subsequently elucidated by Gubser–Klebanov–Polyakov [89] and Witten [90]. An example of this duality is the correspondence between the string states in the Type IIB superstring theory on $AdS_5 \times S^5$ and the operators of the $\mathcal{N} = 4$ super Yang–Mills theory in four dimensions, where the latter can be realised on a stack of D3-branes in the Type IIB string theory on a flat background. By taking a different compact manifold to S^5, one obtains different supersymmetric gauge theories. Finding a corresponding gauge theory to a given compact manifold (and *vice-versa*) has been a long standing problem. It was proposed that the Type IIB string theory $AdS_5 \times Y_5$, where Y_5 is a Sasaki–Einstein manifold, corresponds to a supersymmetric gauge theory on the world-volume of D3-branes at the tips of Calabi–Yau cones over the five-dimensional manifold Y_5 [91–95]. It was proposed by Hanany and Kennaway [96, 97] in 2005 that the Lagrangians of this large class of four-dimensional gauge theories are given by certain tilings on a torus, known as *brane tilings* (see *e.g.* [98, 99] for reviews). For recent applications of brane tilings to supersymmetric gauge theories on M2-branes at the tips of Calabi–Yau cones over a seven-dimensional manifold Y_7, we refer the readers to [100–104].

Given a supersymmetric gauge theory, the space of solutions of the vacuum equations, known as the *moduli space of vacua* or simply *the moduli space*, is one of the first properties one can study. The phase structure and the possible excitations at a given vacuum configuration can be investigated. In supersymmetric gauge theories, holomorphic gauge invariant operators play a central role in giving us information about the structure of the moduli space. It is therefore worthwhile to study and count these holomorphic functions. This can be done in a similar fashion to statistical mechanics, *i.e.* by constructing a partition function. Mathematically, this partition function is known as the *Hilbert series* (see *e.g.* [105, 106] for reviews of the applications of the Hilbert series to gauge and string theories). The Hilbert series not only contains information about the spectrum of the operators in the theory, it also carries geometrical properties of the moduli space (see Chapter 2 and [107, 108, 113]). It is also an indicator of whether the moduli space is Calabi-Yau (see Chapters 2, 3 and [109, 113, 114]). Moreover, Hilbert series can be used as a primary tool to test various dualities in gauge and string theories (see Chapters 4, 5 and [102, 110–112, 115]).

1.7 Outline of the thesis

In the first part of this thesis, we focus on $\mathcal{N} = 1$ supersymmetric gauge theories in $(3 + 1)$ dimensions with the $SU(N_c)$, $SO(N_c)$ and $Sp(N_c)$ gauge groups and N_f flavours of fundamental matter. Such theories are known as supersymmetric quantum chromodynamics (SQCD) with the classical gauge groups. This class of theories is one of the most extensively studied topics in quantum field theory. This is mainly due to interesting dynamics that does not appear in non-supersymmetric QCD. Moreover, the structure of such theories allows one to perform explicit and finite calculations without the necessity of numerical methods. In Chapters 2 and 3, we compute the Hilbert series of such theories and use them to study various geometrical aspects of the moduli space.

In the second part, we examine the moduli space of one instanton on $\mathbb{C}^2$ and the hypermultiplet moduli spaces of a large class of $\mathcal{N} = 2$ supersymmetric gauge theories in $(3 + 1)$ dimensions. An instanton is a stable and finite-action solution to the Euclidean field equations of a quantum field theory. The study of instantons has been a long standing classic subject since their discovery in 1975. An algebraic method to construct classical gauge group instanton solutions, known as the *ADHM construction*, can be realised on the hypermultiplet moduli space of certain $\mathcal{N} = 2$ supersymmetric gauge theories in $(3 + 1)$ dimensions.

In Chapter 4, we count the chiral operators on one instanton moduli spaces with the classical and exceptional gauge groups. The Hilbert series of the moduli spaces of one instanton for the classical gauge groups are easy to compute and turns out to take a particularly simple form. This allows for a gauge group invariant character expansion and hence is easily generalisable to exceptional gauge groups, where an ADHM construction is not known.

In Chapter 5, we study an infinite class of $\mathcal{N} = 2$ supersymmetric gauge theories in $(3+1)$ dimensions represented by graphs consisting of two building blocks, namely a tri-vertex and a line. A line represents an $SU(2)$ gauge group and a tri-vertex represents a matter field in the tri-fundamental representation of $SU(2)^3$. These graphs can be topologically classified by the genus and the number of external legs. This chapter focuses on the hypermultiplet moduli spaces of the aforementioned theories. We compute the Hilbert series which counts all chiral operators on the hypermultiplet moduli space. Several examples show that theories corresponding to different graphs with the same genus and the same number of external legs possess the same Hilbert series. This is in agreement with the conjecture that such theories are related to each other by S-duality. We also give a general expression for the Hilbert series for a graph with any genus and any number of external legs.

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Part I

Supersymmetric QCD

Chapter 2

$SU(N_c)$ SQCD with N_f flavours

2.1 Introduction and Summary

Supersymmetric Quantum Chromodynamics (SQCD) is one of the most extensively studied subjects in modern theoretical physics. Investigations within this laboratory have provided a point of contact between field theory, phenomenology, string theory, and mathematics. The moduli space of SQCD typically consists of continuous vacuum solutions of the field equations. The lifting of the classical vacuum by quantum corrections [1], the phase structure [2, 3], dualities [4], etc., have all afforded powerful insights into the theory. The reviews and lectures [5–8] collect this work and provide references to the original literature. In this chapter, we take a new perspective on this well established subject.

Observing that the vacuum moduli space of a supersymmetric gauge theory, due to its subtle structure, is best described by the language of algebraic varieties, we employ techniques from algebraic geometry to gain physical insight. This is very much in light of the recent applications of computational and algorithmic algebraic geometry to the study of field theory [9–12] as well as the discovery of the plethystic programme for systematically studying chiral gauge invariant operators using geometric methods [13–23]. Geometric quantities such as Hilbert series provide a new understanding of the theory and allow us to easily perform calculations that are cumbersome using standard methods.

Our focus in this chapter is $\mathcal{N} = 1$ SQCD with $SU(N_c)$ gauge group and N_f flavours of quarks and antiquarks that transform, respectively, in the antifundamental and fundamental representations of the gauge group. The fields are also distinguished by their transformation properties under the $SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$ global symmetry. In these initial investigations, we shall concentrate our attention on the case with a vanishing superpotential. The vacuum space is conveniently described by polynomial equations written in terms of variables which are the holomorphic gauge invariant operators (GIOs) of the theory, that is to say, the mesons, baryons, and antibaryons.

For $N_f < N_c$, the gauge group is spontaneously broken in the vacuum to $SU(N_c - N_f)$. The only GIOs are mesonic, and these parametrise a classical moduli space that is N_f^2-dimensional. However, at the quantum mechanical level, non-perturbative corrections lift the space of classical vacuum solutions completely via the dynamically generated ADS superpotential, and consequently there is no quantum moduli space for $N_f < N_c$.

For $N_f \geq N_c$, the gauge symmetry is completely broken at a generic point in the classical moduli space, which is $(2N_cN_f - N_c^2 + 1)$-dimensional. The moduli space is described by relations (syzygies) amongst mesonic operators and baryonic operators. With the incorporation of quantum corrections, the classical moduli space for the $N_f = N_c$ theories which contained the singularity at the origin is deformed to a smooth hypersurface, whereas the quantum moduli space for the $N_f > N_c$ theories is identical with the classical one. Although the precise classical relations get modified by quantum corrections for $N_f = N_c$, quantum corrections do not affect the *number* of chiral operators at each order of quarks and antiquarks. Therefore, the generating functions which count the gauge invariant operators in the $N_f \geq N_c$ theories are not changed by quantum corrections.

Algorithmic algebraic geometry, the plethystic programme, the Molien–Weyl formula, and character expansions yield a more refined understanding of textbook facts about the structure of the SQCD vacuum. In addition, the geometric invariants of the moduli space of vacua capture a vast quantity of non-trivial information about the phenomenology of the gauge theory. Algebraic geometry therefore supplies a powerful new window into the structure of SQCD.

To facilitate the reading, we have highlighted the key points in bold font as **Observations**. We collect the main results below.

Outline and Key Points:

- In Section 2.2, we stress that the vacuum moduli space of a $\mathcal{N} = 1$ gauge theory can be thought of as an affine algebraic variety and review the procedure for how to calculate this explicitly. We also discuss the importance of concepts such as primary decomposition, which breaks the moduli space up into irreducible pieces, and the Hilbert series, which enumerates the chiral GIOs of the theory.

- In Section 2.3, we examine $\mathcal{M}_{(N_f,N_c)}$, the classical moduli space of vacua of SQCD, for various values of N_c and N_f. We characterise the vacuum varieties in terms of their defining equations and find them to be affine cones over (compact) weighted projective varieties. For $N_f < N_c$, $\mathcal{M}_{(N_f,N_c)} \simeq \mathbb{C}^{N_f^2}$ (Observations 2.3.1 and 2.3.2). For $N_f = N_c$, the moduli space is a complete intersection (in fact a single hypersurface) in $\mathbb{C}^{N_f^2+2}$ with a rational function as its Hilbert series (Observations 2.3.5 and 2.3.6). For $N_f > N_c$, the moduli space is a non-complete intersection of polynomial relations (syzygies) amongst the GIOs. We also analyse the case of two colours in detail. Using characters of its global symmetry the generating function is written for arbitrary number of flavours (Observation 2.3.8 and Equation (2.3.18)).

- We find the precise weighted projective variety over which $\mathcal{M}_{(N_f,N_c)}$ is an affine cone and tabulate the first few Hilbert series for these spaces in Table 2.3. Moreover, we find in all case studies that $\mathcal{M}_{(N_f,N_c)}$ is irreducible using primary decomposition and conjecture this to hold in general (Observation 2.4.11).

- Importantly, we establish that $\mathcal{M}_{(N_f,N_c)}$ is Calabi–Yau (Observation 2.4.14).

  This follows from the fact that the Hilbert series has palindromic numerator. We outline a proof based on an independent argument.

- We discuss the quantum moduli space of SQCD in Section 2.7. For $N_f < N_c$, there is no supersymmetric vacuum. The classical vacuum geometry is an auxiliary space useful for counting gauge invariant operators. For $N_f \geq N_c$, the Hilbert series computed in the classical theory is quantum mechanically exact.

- In Section 2.5, we obtain an analytic formula for the generating function of GIOs in SQCD with fully refined fugacities corresponding to quarks and anti-quarks; this is a refined version of the Hilbert series of $\mathcal{M}_{(N_f,N_c)}$. The formula is in the form of the Molien–Weyl integral, as given in Equation (2.5.9). The results are in complete agreement with those obtained in Section 2.3 using algorithmic algebraic geometry and also affirm the fact that the generating function (Hilbert series) encodes the defining relations of the moduli space of vacua. Thus, the results of Section 2.5 verify that the geometry of the classical moduli space of $\mathcal{N} = 1$ SQCD encapsulates the structure of the chiral ring of BPS gauge invariant operators. Ours is the first systematic analysis undertaken for $(N_c \geq 2, N_f > 3)$.[1]

- In Section 2.6, we synthesise our prior results using representation theory and the character expansion.

  It proves useful to write the Hilbert series in terms of characters. This permits the generalisation of our results to an arbitrary number of colours and flavours. Subsequently, we obtain an important result, namely the full character expansion of the generating function for any values of N_f and N_c (Equations (2.6.1), (2.6.2) and (2.6.3)). We can interpret the coefficients as Young Tableaux and

[1] Earlier works [24, 25] contain some of the results for $N_c = 2$.

arrive at selection rules (Observation 5.6) for the terms appearing in the expansion.

2.1.1 Notation for representations and characters

In this thesis, we denote an irreducible representation of a group of rank r by a Dynkin label $[a_1, a_2, \ldots, a_r]$. In particular, we denote by $[1, 0, \ldots, 0]$ the fundamental representation and by $[0, \ldots, 0, 1]$ the anti-fundamental representation of $SU(N)$. We also use the subscripts $k; L$ and $k; R$ to indicate respectively the k-th postitions from the left and the right, *e.g.* $1_{k;L}$ in $[0, \ldots, 0, 1_{k;L}, 0, \ldots, 0]$ denotes the 1 in the k-th position from the left.

For representations of the product group $G_1 \times G_2$, we use the notation $[\ldots; \ldots]$ where the tuple to the left of the ; is the representation of G_1, and the tuple to the right of the ; is the representation of G_2.

Since a representation is determined by its character, we sometimes slightly abuse terminology by referring to each character by its corresponding representation. When necessary, we denote the character of the representation $[a_1, a_2, \ldots, a_r]$ of the group G of rank r, written in terms of the variables $z_1, \ldots, z_r$, by $\chi^G_{[a_1,a_2,\ldots,a_r]}(z_1, \ldots, z_r)$.

2.2 The Moduli Space of $\mathcal{N} = 1$ Gauge Theories

We begin by reviewing how to algorithmically compute the classical supersymmetric vacuum space of an $\mathcal{N} = 1$ gauge theory. Consider a general $\mathcal{N} = 1$ theory of the form

$$S = \int d^4x \, \left[\int d^4\theta \; \Phi_i^\dagger e^V \Phi_i + \left(\frac{1}{16g^2} \int d^2\theta \; \mathrm{tr} \mathcal{W}_\alpha \mathcal{W}^\alpha + \int d^2\theta \; W(\Phi_i) + \text{h.c.} \right) \right] . \tag{2.2.1}$$

The Φ_i are chiral superfields in a representation R_i of the gauge group G; V is the vector superfield in the Lie algebra $\mathfrak{g}$; $\mathcal{W}_\alpha = -\frac{1}{4}\overline{D}^2 e^{-V} D_\alpha e^V$ is the gauge field strength; and $W(\Phi_i)$ is the superpotential, which is holomorphic in Φ_i. Integrating over superspace, the scalar potential becomes

$$V(\phi_i, \overline{\phi}_i) = \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 + \frac{1}{2} \sum_a g^2 \left(\sum_i \phi_i^\dagger T^a \phi_i \right)^2 , \tag{2.2.2}$$

where ϕ_i is the lowest component of Φ_i, T^a are the generators of G, and g is the gauge coupling.[2] The potential is minimised on loci where it vanishes. The condition $V(\phi_i, \overline{\phi}_i) = 0$ yields the supersymmetry preserving D-term and F-term constraints:

$$D^a = \sum_i \phi_i^\dagger T^a \phi_i = 0 \quad \text{(D-terms)} ;$$

[2] We neglect Fayet–Iliopoulos terms associated to $U(1)$ factors in G in this discussion but these can be easily incorporated.

$$f_i = \frac{\partial W}{\partial \phi_i} = 0 \quad \text{(F-terms)} \ . \tag{2.2.3}$$

There is a D-term for each generator T^a of the gauge group and an F-term for each field. The *vacuum moduli space* $\mathcal{M}$ is the space of solutions to D- and F-flatness constraints.

The action (2.2.1) has an enormous gauge redundancy that we can most easily eliminate by working with G^C, the complexification of the gauge group.[3] The F-flatness conditions are holomorphic and invariant under G^C. The D-flatness conditions are trivial gauge fixing parameters. It is a standard fact in $\mathcal{N} = 1$ gauge theory that for any solution of the F-term equations, there exists a unique solution to the D-term equations in the completion of the orbit of the complexified gauge group. The moduli space is, therefore, the symplectic quotient

$$\mathcal{M} = \mathcal{F}//G^C \ , \tag{2.2.4}$$

where $\mathcal{F}$ is the space of F-flat field configurations. The set of holomorphic gauge invariant operators of the theory forms a basis for the D-orbits. The geometry of the vacuum is therefore an algebraic variety specified by polynomial equations in the GIOs.

2.2.1 Moduli Spaces Using Computational Algebraic Geometry

Recasting the computation of the vacuum geometry into efficient, algorithmic techniques in algebraic geometry is the subject of [9–11]. For completeness, we briefly recollect the method.

1. The F-flatness conditions are an ideal of the polynomial ring $\mathbb{C}[\phi_1, \ldots, \phi_n]$:
$$\langle f_{i=1,\ldots,n} \rangle = \langle \frac{\partial W}{\partial \phi_i} \rangle \ . \tag{2.2.5}$$

2. From the matter fields $\{\Phi_1, \ldots, \Phi_n\}$, we construct a basis of GIOs $\rho = \{\rho_1, \ldots, \rho_k\}$. The ρ_j are, by construction, uncharged under G^C. The definitions of the GIOs in terms of the fields defines a natural ring map:
$$\mathbb{C}[\phi_1, \ldots, \phi_n] \xrightarrow{\rho} \mathbb{C}[\rho_1, \ldots, \rho_k] \ . \tag{2.2.6}$$

3. The moduli space $\mathcal{M}$ is then the image of the ring map:
$$\frac{\mathbb{C}[\phi_1, \ldots, \phi_n]}{\{F = \langle f_1, \ldots, f_n \rangle\}} \xrightarrow{\rho} \mathbb{C}[\rho_1, \ldots, \rho_k] \ . \tag{2.2.7}$$

[3] We recall, for example, that the complexification of $SU(N)$ is $SL(N, \mathbb{C})$.

That is to say, $\mathcal{M} \simeq \mathrm{Im}(\rho)$ is an ideal of $\mathbb{C}[\rho_1, \ldots, \rho_k]$ which corresponds to an affine variety in $\mathbb{C}^k$. Practically, the image of the map (2.2.7), and thus the vacuum geometry $\mathcal{M}$, can be calculated using Gröbner basis methods as implemented in the algebraic geometry software packages `Macaulay 2` [26] and `Singular` [27].

2.2.2 Primary Decomposition and Hilbert Series

Having obtained the vacuum moduli space explicitly as an algebraic variety, we have many geometric tools at our disposal for analysing its structure. Two of the most fundamental concepts are the following.

Extracting Irreducible Pieces

The moduli space may not be a single irreducible piece, but rather, may be composed of various components. This is a well recognised feature in supersymmetric gauge theories. The different components are typically called **branches** of the moduli space, such as Coulomb or Higgs branches. It is an important task to identify the different components since the massless spectrum on each component has its own unique features.

We are thus naturally led to look for a process to extract the various irreducible components of the vacuum space. Such an algorithm exists and, in the mathematics literature, is called **primary decomposition** of the ideal corresponding to the moduli space. Algorithms for performing primary decomposition have been extensively studied in computational algebraic geometry (*cf.*, for example, [28] and for implementations, [26, 27]). A convenient package which calls the computational algebraic geometry programme Singular externally but which is based upon the `Mathematica` interface, which perhaps is more familiar to physicists, is `STRINGVACUA` [11]. In fact, using [11], the primary decomposition of string vacua of phenomenological significance, is one of the subjects of [12].

As a result of computations in a number of cases, we conjecture that the moduli space of $SU(N_c)$ SQCD with N_f flavours is irreducible (Conjecture 2.4.11).

The Hilbert Series

For a variety $\mathcal{M}$ in $\mathbb{C}[x_1, ..., x_k]$, the Hilbert series is the generating function for the dimension of the graded pieces:

$$H(t; \mathcal{M}) = \sum_{i=-\infty}^{\infty} (\dim_{\mathbb{C}} \mathcal{M}_i) t^i \ , \tag{2.2.8}$$

where $\mathcal{M}_i$, the i-th graded piece of $\mathcal{M}$, can be thought of as the number of independent degree i (Laurent) polynomials on the variety $\mathcal{M}$. It will be understood henceforth that we are speaking about complex dimension, and we shall simplify our notation accordingly.

As being pointed out in [13, 17, 19, 23], the Hilbert series is a key to the problem of counting GIOs in a gauge theory. Physically speaking, a Hilbert series is a function that counts chiral GIOs and has the interpretation of a partition function at zero temperature and non-zero chemical potentials for global conserved $U(1)$ charges. It can be written as a rational function whose numerator is a polynomial with integer coefficients.

One of the important expansions of the Hilbert series is a Laurent expansion about 1, and the coefficient of the leading pole can be interpreted as the volume of the dual Sasaki–Einstein manifold in the AdS/CFT context which in the case of the Calabi–Yau three-fold, this volume is related to the central charges of supersymmetric gauge theory (*cf.* [23, 29]). Although it is not clear for general SQCD what the volume means, we can nevertheless perform such an expansion. For a Hilbert series,

$$H(t;\mathcal{M}) = \frac{P(1)}{(1-t)^{\dim(\mathcal{M})}} + \dots \ , \qquad P(1) = \mathrm{degree}(\mathcal{M}) \ . \tag{2.2.9}$$

In particular, $P(1)$ always equals the degree of the variety.[4]

2.3 Supersymmetric QCD

Having set the stage with the necessary geometric background, let us specialise to the gauge theory in which we are chiefly interested. In this section, let us fix notation by introducing the content of the theory. Let there also be no superpotential, $W = 0$. Thus there will be no F-terms, and the vacuum space is determined exclusively by the D-terms and is realised as the relations among the GIOs of the theory.

We specify SQCD with gauge group $SU(N_c)$ and N_f flavours by the ordered pair (N_f, N_c). This theory has quarks Q^i_a and antiquarks $\widetilde{Q}^a_i$, with flavour indices $i = 1, \dots, N_f$ and colour indices $a = 1, \dots, N_c$. Thus, there is a total of $2N_cN_f$ chiral degrees of freedom from the quarks and antiquarks. Their quantum numbers are summarised in Table 2.1.

2.3.1 The Case of $N_f < N_c$

In this situation, at a generic point in the moduli space, the $SU(N_c)$ gauge symmetry is partially broken to $SU(N_c - N_f)$. Thus, there are

$$(N_c^2 - 1) - ((N_c - N_f)^2 - 1) = 2N_cN_f - N_f^2 \tag{2.3.1}$$

[4] We recall that when an ideal is described by a single polynomial, the degree of the variety is simply the degree of the polynomial. In the case of multiple polynomials, the degree is a generalisation of this notion. It is simply the number of points at which a generic line intersects the variety.

	Gauge symmetry	Global symmetry					
	$SU(N_c)$	$SU(N_f)_1$	$SU(N_f)_2$	$U(1)_B$	$U(1)_R$	$U(1)_Q$	$U(1)_{\widetilde{Q}}$
Q^i_a	$[0,\ldots,0,1]$	$[1,0,\ldots,0]$	$[0,\ldots,0]$	1	$\frac{N_f-N_c}{N_f}$	1	0
$\widetilde{Q}^a_i$	$[1,0,\ldots,0]$	$[0,\ldots,0]$	$[0,\ldots,0,1]$	-1	$\frac{N_f-N_c}{N_f}$	0	-1

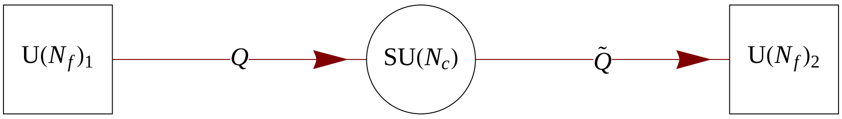

Table 2.1. The gauge and global symmetries of SQCD and the quantum numbers of the chiral supermultiplets. The quarks are Q^i_a while the antiquarks are $\widetilde{Q}^a_i$. We also draw it as a quiver theory. The circular node represents the $U(N_c)$ gauge symmetry while the two square nodes represent global $U(N_f)_1$ and $U(N_f)_2$ symmetries. Each square node gives rise to a baryonic $U(1)$ global symmetry, one of which is redundant. We thus have $U(1)_{Q,\widetilde{Q}}$ that combine into the non-anomalous $U(1)_B$ (sum) and anomalous $U(1)_A$ (difference).

broken generators. The total number of degrees of freedom of the system is, of course, unaffected by this spontaneous symmetry breaking and the massive gauge bosons each eat one degree of freedom from the chiral matter via the Higgs effect. Therefore, of the original $2N_cN_f$ chiral supermultiplets, only N_f^2 singlets are left massless. Hence, the dimension of the moduli space of vacua is

$$\dim\left(\mathcal{M}_{N_f<N_c}\right) = N_f^2 \ . \tag{2.3.2}$$

We can describe the remaining N_f^2 light degrees of freedom in a gauge invariant way by an $N_f \times N_f$ matrix field, composed of the mesons:

$$M^i_j = Q^i_a \widetilde{Q}^a_j \qquad \text{(mesons)} \ . \tag{2.3.3}$$

The M^i_j are clearly gauge invariant as the colour index on the right hand side is summed. There are no baryons since $N_f < N_c$. Thus, (2.3.3) constitute the only GIOs. Since Q and $\widetilde{Q}$ transform respectively in $[1,0,\ldots;0,\ldots,0]$ and $[0,\ldots,0;0,\ldots,1]$ of the $SU(N_f)_L \times SU(N_f)_R$ part of the global symmetry, it follows that M transforms in the bifundamental representation $[1,0,\ldots;0,\ldots,1]$ of the $SU(N_f)_L \times SU(N_f)_R$ global symmetry. We note that for the $N_f < N_c$ theory, there are no relations (constraints) between mesons. Phrasing this geometrically, and noting the dimension from (2.3.2), we have that

Observation 2.3.1. *The moduli space is freely generated: there are no relations among the generators. The space $\mathcal{M}_{N_f<N_c}$ is, in fact, nothing but $\mathbb{C}^{N_f^2}$.*

GIOs composed of k quarks and k antiquarks must be of the form: $M^{i_1}_{j_1}\ldots M^{i_k}_{j_k}$. Because of the symmetry under the interchange of any two M's, this product transforms in the representation $\mathrm{Sym}^k[1,0,\ldots,0;0,\ldots,1]$ of the $SU(N_f)_L \times SU(N_f)_R$

global symmetry. A computation of this k-th symmetric product for a bifundamental representation is rather amusing and gives

$$\text{Sym}^k[1,0,\dots,0;0,\dots,1] = \sum_{n_1,\dots,n_{N_f}\geq 0} [n_1,n_2,\dots,n_{N_f-1};n_{N_f-1},\dots,n_2,n_1]\,\delta\left(k-\sum_{j=1}^{N_f} j n_j\right) , \tag{2.3.4}$$

where[5] the only dependence on k comes from the constraint on the number of boxes in the Young diagram which is represented by the δ function. The total dimension of these representations gives

$$\frac{1}{k!}(N_f^2)(N_f^2+1)\dots(N_f^2+k-1) = \binom{N_f^2+k-1}{k} \tag{2.3.5}$$

independent components. We can sum this to give a generating function for the gauge invariants and obtain:

Observation 2.3.2. *The generating function of GIOs for SQCD with $N_f < N_c$ is*

$$g^{N_f<N_c}(t) = \sum_{k=0}^{\infty} \binom{N_f^2+k-1}{k} t^{2k} = \frac{1}{(1-t^2)^{N_f^2}} . \tag{2.3.6}$$

We note that this formula does not depend on the number of colours N_c. The expression (2.3.6) is to be expected from the plethystic programme, it is simply the Hilbert series for $\mathbb{C}^{N_f^2}$, with weight 2 for each meson.[6] We will return to this point in the following section.

We end this subsection by emphasising that what we have said so far about the $N_f < N_c$ theories is only valid in the semiclassical regime. If full quantum effects are taken into account, there will no longer be a supersymmetric vacuum. In Section 2.7, we discuss how semiclassical results are modifed in the quantum theory. Until then, let us proceed with calculations in the semiclassical limit.

2.3.2 The Case of $N_f \geq N_c$

In this case, at a generic point in the moduli space, the $SU(N_c)$ gauge symmetry is broken completely and hence the number of remaining massless chiral supermultiplets (*i.e.* the dimension of the moduli space) is given by

$$\dim\left(\mathcal{M}_{N_f\geq N_c}\right) = 2N_cN_f - (N_c^2-1) . \tag{2.3.7}$$

[5] We emphasise that in this equation, summations run over $n_1,\dots,n_{N_f}$ but only $n_1,\dots n_{N_f-1}$ appear in the representation on the right hand side.

[6] Section 2.6.2 demonstrates that the expression in (2.3.6) can be written in terms of the plethystic exponential as $g^{N_f<N_c}(t) = \text{PE}\,[\dim[1,0,\dots,0;0,\dots,1]t^2] = \text{PE}\,[N_f^2 t^2]$.

We can describe the light degrees of freedom in a gauge invariant way by the following basic generators:

$$\begin{aligned}
&M^i_j = Q^i_a \widetilde{Q}^a_j && \text{(mesons)} \ ; \\
&B^{i_1 \dots i_{N_c}} = Q^{i_1}_{a_1} \dots Q^{i_{N_c}}_{a_{N_c}} \epsilon^{a_1 \dots a_{N_c}} && \text{(baryons)} \ ; \\
&\widetilde{B}_{i_1 \dots i_{N_c}} = \widetilde{Q}^{a_1}_{i_1} \dots \widetilde{Q}^{a_{N_c}}_{i_{N_c}} \epsilon_{a_1 \dots a_{N_c}} && \text{(antibaryons)} \ .
\end{aligned} \tag{2.3.8}$$

Observation 2.3.3. *For $N_f \geq N_c$, under the global $SU(N_f)_L \times SU(N_f)_R$, the mesons M transform in the bifundamental $[1, 0, \dots; 0, \dots, 0, 1]$ representation, the baryons B and antibaryons $\widetilde{B}$ transform respectively in $[0, 0, \dots, 1_{N_c;L}, 0, \dots, 0; 0, \dots, 0]$ and $[0, \dots, 0; 0, \dots, 1_{N_c;R}, 0 \dots, 0]$.*

In the above, $1_{j;L}$ denotes a 1 in the j-th position from the left, and $1_{j;R}$ denotes a 1 in the j-th position from the right.

The total number of basic generators for the GIOs, coming from the three contributions in (2.3.8) is therefore

$$N_f^2 + \binom{N_f}{N_c} + \binom{N_f}{N_f - N_c} = N_f^2 + 2\binom{N_f}{N_c} \ . \tag{2.3.9}$$

We emphasise that the basic generators in (2.3.8) are not independent, but they are subject to the following constraints (see, *e.g.*, [5]). Since the product of two epsilon tensors can be written as the antisymmetrised sum of Kronecker deltas, it follows that

$$B^{i_1 \dots i_{N_c}} \widetilde{B}_{j_1 \dots j_{N_c}} = M^{[i_1}_{j_1} \dots M^{i_{N_c}]}_{j_{N_c}} \ . \tag{2.3.10}$$

We can rewrite this constraint more compactly as

$$(*B)\widetilde{B} = *(M^{N_c}) \ , \tag{2.3.11}$$

where $(*B)_{i_{N_c+1} \dots i_{N_f}} = \frac{1}{N_c!} \epsilon_{i_1 \dots i_{N_f}} B^{i_1 \dots i_{N_c}}$. Another constraint follows from the fact that any product of M's, B's and $\widetilde{B}$'s antisymmetrised on $N_c + 1$ (or more) upper or lower flavour indices must vanish:

$$M \cdot *B = M \cdot *\widetilde{B} = 0 \ , \tag{2.3.12}$$

where a '$\cdot$' denotes a contraction of an upper with a lower flavour index. It can be shown (see, *e.g.*, [5]) that all other constraints follow from the basic ones (2.3.11) and (2.3.12).

Counting the number of quarks and antiquarks in these basic constraints and using Observation 2.3.3, we find that

Observation 2.3.4. *For $N_f \geq N_c$, under the global $SU(N_f)_L \times SU(N_f)_R$, constraint (2.3.11) transforms as $[0, \dots, 0, 1_{N_c;L}, 0, \dots, 0; 0, \dots, 0, 1_{N_c;R}, 0, \dots, 0]$. Similarly, in (2.3.12), the first constraint transforms as $[0, \dots, 0, 1_{(N_c+1);L}, 0, \dots, 0; 0, \dots, 0, 1]$ and the second as $[1, 0, \dots, 0; 0, \dots, 0, 1_{(N_c+1);R}, 0, \dots, 0]$.*

The representation notation is as in Observation 2.3.3. Indeed, the dimension of the representation corresponding to the constraint (2.3.11) is $\binom{N_f}{N_c}^2$, and the dimension of each of the representations corresponding to the constraints (2.3.12) is $N_f\binom{N_f}{N_c+1}$. Thus, there are $\binom{N_f}{N_c}^2 + 2N_f\binom{N_f}{N_c+1}$ basic constraints.

Because of these constraints, the spaces $\mathcal{M}_{N_f \geq N_c}$ are not freely generated and provide us with interesting algebraic varieties which we will study in the ensuing section. Moreover, these constraints also prevent us from writing and summing a generating function as directly as in Observation 2.3.2. Nevertheless, we will see how the Hilbert series gives us the right answer.

The Case of $N_f = N_c$

The special case of $N_f = N_c$ deserves some special attention. From (2.3.9), the total number of basic generators for the GIOs, coming from the three contributions in (2.3.8), is $N_f^2 + 2$. From (2.3.7), the dimension of the moduli space is

$$\dim\left(\mathcal{M}_{N_f=N_c}\right) = N_f^2 + 1 \ . \tag{2.3.13}$$

There is one constraint (2.3.11), which in this case can be reduced to a single hypersurface:

$$\det(M) = (*B)(*\widetilde{B}) \ , \tag{2.3.14}$$

where we have used the identity $\det M = (1/N_c!) {*}{*}(M^{N_c})$. According to Observation 2.3.4, this constraint transforms in the trivial $[0, \ldots, 0; 0, \ldots, 0]$ representation of $SU(N_f)_L \times SU(N_f)_R$ (since the length of the weight before and after the semicolon is the rank of $SU(N_f)$, or $N_f - 1$, there are no 1's). Note that the relation (2.3.12) does not provide any additional information and (2.3.11) constitutes the only constraint. Since, in this case, the dimension of the moduli space equals the number of the basic generators minus the number of constraints, we arrive at another important conclusion:

Observation 2.3.5. *The moduli space $\mathcal{M}_{N_f=N_c}$ is a complete intersection. It is in fact a single hypersurface in $\mathbb{C}^{N_f^2+2}$.*

An interesting question to consider is to determine the number of independent GIOs that can be constructed from the basic generators (2.3.8) subject to the constraints (2.3.11) and (2.3.12). In the case $N_f = N_c$, where the only constraint is (2.3.14), the generating function can be easily computed from the knowledge that the modul space is a complete intersection (See [13] for a detailed discussion on this). There are N_c^2 mesonic generators of weight t^2 and two baryonic generators of weight t^{N_c}, subject to a relation of weight t^{2N_c}. As a result, the generating function takes the form

Observation 2.3.6. *For $N_f = N_c$ SQCD, the generating function for the GIOs is*

$$g^{N_f=N_c}(t) = \frac{1-t^{2N_c}}{(1-t^2)^{N_c^2}(1-t^{N_c})^2}. \tag{2.3.15}$$

This is indeed the Hilbert series of the hypersurface (2.3.14).

2.3.3 Special Case: $N_c = 2$

Let us illustrate this technology with the concrete example of $N_c = 2$ colours and a general number N_f of flavours. Here we can obtain nice general expressions. There are N_f quarks transforming in the fundamental representation and N_f antiquarks in the antifundamental of the $SU(2)$ gauge group. However, since both of these representations are identical for $SU(2)$, there is no distinction to be made between quarks and antiquarks. Therefore, all quark fields can be written in the form Q^i_a, with a colour (gauge) index $a = 1, 2$ and a multiplet index $i = 1, \ldots, 2N_f$. Hence, we first have:

Observation 2.3.7. *The global flavour symmetry of $(N_f, N_c = 2)$ for general N_f is $SU(2N_f)$.*

The basic generators of GIOs are mesons:

$$M^{ij} = Q^i Q^j \ , \tag{2.3.16}$$

where the contraction over the colour indices a, b by an epsilon symbol[7] has been suppressed in order to avoid the potential confusion between the gauge and global symmetries. The fundamental representation of $SU(2)$ has only two colour indices and therefore we find that any product of M's antisymmetrised on three (or more) flavour indices vanishes. This results in a simple condition for $N_f \geq 2$:

$$\epsilon_{i_1 \ldots i_{2N_f}} M^{i_1 i_2} M^{i_3 i_4} = 0 \ , \tag{2.3.17}$$

where $i_1, \ldots, i_{2N_f} = 1, \ldots, 2N_f$.

Counting the number of quarks in (2.3.16) and (2.3.17), we find that

Observation 2.3.8. *For $N_c = 2$, under the $SU(2N_f)$ global symmetry, the meson transforms in the $[0, 1, 0, \ldots, 0]$ representation, and the basic constraint (2.3.17) transforms as $[0, 0, 0, 1, 0, \ldots, 0]$. The dimension of these representations are respectively $\binom{2N_f}{2}$ and $\binom{2N_f}{4}$.*

[7] It is an epsilon contraction rather than a summation because the doublet of $SU(2)$ is a pseudoreal representation.

We see that the GIOs in the $N_c = 2$ theories must be (symmetric) products of mesons, namely M^k at the order of $2k$ quarks. Without the constraints generated by (2.3.17), we would say that M^k transforms in the representation $\mathrm{Sym}^k[0,1,0,\ldots,0]$ of $SU(2N_f)$. However, as we have just noted, any product of M's antisymmetrised on three (or more) flavour indices vanishes. It then follows that the GIOs at the order $2k$ of quarks transform in the irreducible representation $[0,k,0,\ldots,0]$. Therefore, we reach an important conclusion that

Observation 2.3.9. *The generating function for $(N_f, N_c = 2)$ theory for general $N_f \geq 1$ is*

$$g^{(N_f,N_c=2)}(t) = \sum_{k=0}^{\infty} \dim[0,k,0,\ldots,0]t^{2k} = \sum_{k=0}^{\infty} \frac{(2N_f+k-1)!(2N_f+k-2)!}{(2N_f-1)!(2N_f-2)!(k+1)!k!}t^{2k}$$
$$= {}_2F_1(2N_f-1, 2N_f; 2; t^2) \ , \tag{2.3.18}$$

where ${}_2F_1$ is the standard hypergeometric series.

It is interesting that a hypergeometric function should be the Hilbert series of an algebraic variety (for specific integer values of N_f, of course, the hypergeometric degenerates into rational functions, examples of which we will see later).

2.4 The Algebraic Geometry of SQCD Vacuum

We have now presented SQCD in some detail. Though some of the information is standard, we have also recast the vacuum structure in a geometric language and have obtained new analytic formulae for the generating functions of GIOs. In this section, let us continue along this geometric vein and use the techniques introduced in Section 2.2.1 to algorithmically find the supersymmetric vacuum space. This not only furnishes a good check of our methods but also gives us new geometric insight into SQCD.

Since there is no superpotential, the ring map (2.2.7) here becomes

$$\mathbb{C}[Q^i_a, \widetilde{Q}^a_i] \stackrel{\rho}{\longrightarrow} \mathbb{C}[M^i_j, B^{i_1\ldots i_{N_c}}, \widetilde{B}^{i_1\ldots i_{N_c}} := \rho_1,\ldots,\rho_k] \ , \qquad k = N_f^2 + 2\binom{N_f}{N_c} \ , \tag{2.4.1}$$

and the classical moduli space $\mathcal{M}$ is readily computed as the variety associated to the image ideal in the target $\mathbb{C}[M^i_j, B^{i_1\ldots i_{N_c}}, \widetilde{B}^{i_1\ldots i_{N_c}}]$. Therefore, we have that:

Observation 2.4.10. *The classical vacuum moduli space of SQCD, as an explicit affine algebraic variety, is defined by the* syzygies, *or relations amongst the mesons and baryons.*

Equations (2.3.11) and (2.3.12) are precisely these syzygies.

2.4.1 The Example of $(N_f = 4, N_c = 2)$

Let us study an example in detail. Take the non-trivial case of two colours and four flavours. Using (2.4.1) we immediately find that in full component form, it is given by 70 homogeneous quadratic equations, each containing three monomials, in 28 variables. The dimension is 13 and the degree is 132. (For brevity we do not present the lengthy polynomials here.) Therefore $\mathcal{M}_{(4,2)}$ is an affine variety realised as the non-complete intersection of dimension 13 and degree 132 in $\mathbb{C}^{28}$. We can say more since each equation is homogeneous. (This is not true in general; we will discuss shortly how using appropriate weights naturally homogenises the problem.) We can projectivise to $\mathbb{P}^{27}$ and then $\mathcal{M}_{(4,2)}$ is, by definition, an affine cone over a projective variety of dimension 12 and degree 132 in $\mathbb{P}^{27}$.

Let us adhere to the notation of [10] and let

$$(d, \delta|n|m_1^{n_1} m_2^{n_2} \ldots) := \begin{array}{l}\text{Affine variety of complex dimension } d\text{, realised as an affine cone}\\ \text{over a projective variety of dimension } d-1 \text{ and degree } \delta\text{,}\\ \text{given as the intersection of } n_i \text{ polynomials of degree } m_i \text{ in } \mathbb{P}^n.\end{array} \tag{2.4.2}$$

Then, in this notation, we can write

$$\mathcal{M}_{(N_f=4,N_c=2)} \simeq (13, 132|27|2^{70}) \ . \tag{2.4.3}$$

The dimension and degree are but two simple quantities one could ask about an algebraic variety. Another important property, as discussed in Section 2.2.2, is whether the associated ideal is primary. This can be ascertained either by direct methods or by performing a full primary decomposition which extracts the irreducible pieces. We perform this analysis and find that $\mathcal{M}_{(4,2)}$ is in fact an irreducible variety. We can find its Hilbert series, in second form, as

$$H(t;\ \mathcal{M}_{(4,2)}) = \frac{1 + 15\,t + 50\,t^2 + 50\,t^3 + 15\,t^4 + t^5}{(1-t)^{13}} \ . \tag{2.4.4}$$

Note that the weight for the meson here is t which is different than the weight t^2 given in (2.3.18). This change of variables affects the degree of embedding but not the dimension of the moduli space. Physically, this change of variables can be interpreted as a redefinition of the Boltzmann constant by a factor 2. Indeed, other than this change of $t \to t^2$, the standard definition of ${}_2F_1$ for $N_f = 4$, substituted into (2.3.18), gives precisely the above expression and we may rest assured.

Now, the exponent of the denominator encodes the dimension; the numerator, evaluated at 1, gives the degree, which is 132. Another remarkable property of the numerator is that it is palindromic, *i.e.* the coefficients a_n and a_{5-n} are the same. As we shall see below, this suggests that our affine variety $\mathcal{M}_{(4,2)}$ is in fact Calabi–Yau!

2.4.2 Other Examples

We now move on to a host of examples. We tabulate $\mathcal{M}$ for some low values of (N_f, N_c). If $\mathcal{M}$ happens to be an affine cone over a projective variety in unweighted projective space, we will use the above notation, otherwise, we will simply indicate the pair (d, δ) for dimension and degree, respectively. This information is summarised in Table 2.2.

$N_f \backslash N_c$	1	2	3	4	5
1	$(2, 2)$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$
2	$(4, 6)$	$(5, 2\|5\|2^1)$	$\mathbb{C}^4$	$\mathbb{C}^4$	$\mathbb{C}^4$
3	$(6, 20)$	$(9, 14\|14\|2^{15})$	$(10, 3)$	$\mathbb{C}^9$	$\mathbb{C}^9$
4	$(8, 70)$	$(13, 132\|27\|2^{70})$	$(16, 115)$	$(17, 4)$	$\mathbb{C}^{16}$
5	$(10, 252)$	$(17, 1430\|44\|2^{210})$	$(22, 10410)$	$(25, 744)$	$(26, 5)$

Table 2.2. The classical moduli space $\mathcal{M}$ of SQCD with N_f flavours and N_c colours, explicitly as affine algebraic varieties. The pair (d, δ) denotes dimension and degree respectively. When $\mathcal{M}$ is defined by homogeneous equations, and is thus an affine cone over a projective variety, we use the notation in (2.4.2). For $N_f < N_c$, the moduli space is freely generated and is just flat space.

2.4.3 $U(1)$-Charges and Weighted Embeddings

The forms of the moduli spaces and Hilbert series above may not look immediately enlightening. This is because we have been working in affine embeddings without taking into account the inherent weights associated with the problem. A not dissimilar situation has already been noted in [17], where it was pointed out that the del Pezzo surfaces are *much* easier to realise in weighted projective spaces than as ordinary projective varieties.

We notice that the GIOs are each composed of products of fundamental fields. In an $\mathcal{N} = 1$ supersymmetric theory, there is always a $U(1)$-charge, which could be construed as the R-charge, that we assign to the fields. For example, for the GIOs above in pure SQCD, if we normalise and assign an R-charge 1 to each fundamental quark Q^i_a and antiquark $\widetilde{Q}^j_a$, then each mesonic GIO would have R-charge of 2 and each (anti)baryonic GIO, an R-charge of N_c. We will find it useful to weight the target ring in (2.2.7) as $[2 : 2 : \ldots : 2 : N_c : N_c : \ldots : N_c]$ and thus we modify the map in (2.4.1) to

$$\mathbb{C}[Q^i_a, \widetilde{Q}^a_i] \stackrel{\rho}{\longrightarrow} \mathbb{C}[M^i_j, B^{i_1 \ldots i_{N_c}}, \widetilde{B}^{i_1 \ldots i_{N_c}} := \rho_1, \ldots, \rho_k]_{[2:\ldots:2:N_c:\ldots:N_c]} \ . \tag{2.4.5}$$

Here we have labelled the target ring with weighted variables explicitly. The equations that describe the vacuum varieties are always homogeneous in the projective spaces weighted in this manner.

In light of all of the moduli spaces being, strictly, affine cones over weighted projective varieties, we need to refine the notation in (2.4.2) to

$$(d, \delta | n[w_1 : \ldots : w_{n+1}] | m_1^{n_1} m_2^{n_2} \ldots) := \begin{array}{l}\text{Affine variety of complex dimension } d \text{, realised as}\\ \text{an affine cone over a weighted projective variety}\\ \text{of dimension } d-1 \text{ and degree } \delta \text{, given as}\\ \text{the intersection of } n_i \text{ polynomials of degree } m_i\\ \text{in weighted projective space } \mathbb{P}^n_{[w_1:\ldots:w_{n+1}]}.\end{array} \quad (2.4.6)$$

Under our weighting scheme by the R-charge given in (2.4.5), the moduli space of SQCD, for some low values, is presented in Table 2.3. There are several agreements, as can be seen from the table. The dimensions do indeed agree with (2.3.7); moreover, for $N_f = N_c$, $\mathcal{M}$ is indeed a single hypersurface as can be seen from the defining equations, in accord with (2.3.14) and (2.3.15). Next, we compute the weighted Hilbert series of the second kind and present them to the right of moduli space. The ensuing sections show how these rather complicated rational functions, here found using algorithmic algebraic geometry, can be obtained from the plethystic programme.

(N_f, N_c)	$\mathcal{M}$	Hilbert Series $H(\mathcal{M}; t)$
$(2, 2)$	$(5, 4\|5[2:2:2:2:2:2]\|4^1)$	$\frac{1+t^2}{(1-t^2)^5}$
$(3, 2)$	$(9, 896\|14[2^{15}]\|4^{15})$	$\frac{1+6\,t^2+6\,t^4+t^6}{(1-t^2)^9}$
$(4, 2)$	$(13, 4325376\|27[2^{28}]\|4^{70})$	$\frac{1+15\,t^2+50\,t^4+50\,t^6+15\,t^8+t^{10}}{(1-t^2)^{13}}$
$(5, 2)$	$(17, 383862702080\|44[2^{45}]\|4^{210})$	$\frac{1+28\,t^2+196\,t^4+490\,t^6+490\,t^8+196\,t^{10}+28\,t^{12}+t^{14}}{(1-t^2)^{17}}$
$(3, 3)$	$(10, 6\|10[2^9:3^2]\|6^1)$	$\frac{1+t^3}{(1-t^2)^9\,(1-t^3)}$
$(4, 3)$	$(16, 88128\|23[2^{16}:3^8]5^8 6^{16} 7^{12})$	$\frac{1+4\,t^2+4\,t^3+10\,t^4+8\,t^5+14\,t^6+8\,t^7+10\,t^8+4\,t^9+4\,t^{10}+t^{12}}{(1-t^2)^{12}(1-t^3)^4}$
$(4, 4)$	$(17, 8\|17[2^{16}:4^2]\|8^1)$	$\frac{1+t^4}{(1-t^2)^{16}\,(1-t^4)}$

Table 2.3. With natural weighting in (2.4.5), the vacuum moduli space $\mathcal{M}_{(N_f,N_c)}$ of SQCD are all affine cones over (compact, homogeneous) weighted projective varieties, using notation in (2.4.6). We also compute the (weighted, second form) Hilbert series. Indeed, for $N_f < N_c$, $\mathcal{M}_{(N_f,N_c)}$ is trivially $\mathbb{C}^{N_f^2}$, with Hilbert series $(1-t^2)^{-N_f^2}$.

The degrees of the varieties listed in Table 2.3 are rather large, but this is merely a vestige of the fact that we have assigned high weights to the GIOs corresponding to the number of fundamental fields contained within. Let us return to the unweighted case for a moment. Examining (2.2.9), we see that the highest power in $\frac{1}{1-t}$ is the dimension of $\mathcal{M}$ and the coefficient of that leading order term is the degree of $\mathcal{M}$. This is a fundamental property of the Hilbert series of second kind. Now, in the weighted case in Table 2.3, such a relation persists, and we see immediately that the leading coefficient in the same expansion of the Hilbert series, c, and the degree d of the variety obey the relation $c \prod_i w_i = d$. This is simply the generalisation of the $c = d$ situation of the unweighted case above.

2.4.4 Further Geometric Properties

As emphasized in the introduction, our technique allows writing down explicit equations for the moduli space. In component form, these equations can be quite complicated. For illustration, we write down $\mathcal{M}_{(N_f,N_c)}$; for some low values:

$$\begin{aligned}
M_{1,1} &= \{-y_1 + y_2 y_3\} \ ; \\
M_{2,1} &= \{-y_6 y_8 + y_4, -y_5 y_8 + y_2, -y_6 y_7 + y_3, -y_5 y_7 + y_1\} \ ; \\
M_{2,2} &= \{y_2 y_3 - y_1 y_4 + y_5 y_6\} \ ; \\
M_{3,3} &= \{y_3 y_5 y_7 - y_2 y_6 y_7 - y_3 y_4 y_8 + y_1 y_6 y_8 + y_2 y_4 y_9 - y_1 y_5 y_9 + y_{15} y_{21}\} \ .
\end{aligned} \tag{2.4.7}$$

These explicit equations allow us to do far more than merely compute the dimension, degree and Hilbert series. However complicated the equations are, computational algebraic geometry has standard algorithms for manipulating them. First, we can see whether the vacuum moduli space has reducible components by primary decomposition. For all of the cases that we have considered, we conjecture:

Conjecture 2.4.11. *The classical moduli space of $SU(N_c)$ SQCD with N_f flavours is irreducible for all value of N_f and N_c.*

(It should be noted that the algorithms we have employed check this only over the rationals and not over complex coefficient fields.) The irreducibility of moduli spaces is certainly not a feature of generic gauge theories; many reducible cases exist in the literature from very early studies of supersymmetric gauge theories (see *e.g.*, [42]). Few recent ones are presented, for example, in [23]. An argument[8] why Conjecture 2.4.11 may be true in general is that the moduli space as a symplectic quotient (2.2.4), in the absence of a superpotential is simply $\mathbb{C}^{2N_cN_f}/SL(N_c,\mathbb{C})$. Since $\mathbb{C}^{2N_cN_f}$ is irreducible and $SL(N_c,\mathbb{C})$ is a continuous group, we expect the resulting quotient to be also irreducible.

Next, we see that for $N_c = 1$ (Wess–Zumino model with no continuous gauge group and $2N_f$ chiral multiplets), the moduli space is manifestly toric (*i.e.* generated as a monomial ideal, consisting of equations of the form 'monomial = monomial'). This is no surprise, since $N_c \geq 2$ are non-Abelian actions.

Importantly, we can also calculate such familiar quantities, given the defining equation, as the Euler number χ of the compact weighted projective base over which the moduli space is an affine cone. We find that, for example, $\chi(\mathrm{Base}(\mathcal{M}_{2,2})) = 1$.

[8] We are grateful to Alberto Zaffaroni for this point.

Finding such topological invariants of the moduli space is clearly of great interest and deserves investigation in its own right; we hence leave this to subsequent work. What is perhaps a little surprising is a universal property of the SQCD vacuum: that it is, in fact, Calabi–Yau. We now delve into this fact in the next subsection.

2.4.5 The SQCD Vacuum Is Calabi–Yau

We observe that the numerators of the Hilbert series in Table 2.3 are *palindromic*, *i.e.* they have the symmetry $a_k = a_{n-k}$ where n is the degree of the numerator and a_k are the coefficients. A rigorous proof of this observation for all Hilbert series of $\mathcal{M}_{(N_f,N_c)}$ using plethystic technique will be given in Section 2.5.3.

We first argue that the coordinate rings of the moduli space of SQCD is Cohen–Macaulay. This can be done simply by invoking the following theorem:

Theorem 2.4.12. (Hochster–Roberts 1974. [30])[9] *The invariant ring of a linearly reductive group acting on a regular ring is Cohen–Macaulay.*

The following theorem tells us that the coordinate rings of the moduli space of SQCD is Gorenstein.

Theorem 2.4.13. (Stanley 1978. [31]) *The numerator to the Hilbert series of a graded Cohen–Macaulay domain R is palindromic if and only if R is Gorenstein.*

Since an affine Gorenstein variety is, by definition, affine Calabi–Yau, we reach an important conclusion that $\mathcal{M}_{(N_f,N_c)}$ is, in fact, an affine Calabi–Yau cone over a weighted projective variety. In brief,

Theorem 2.4.14. *The classical moduli space of $SU(N_c)$ SQCD with N_f flavours is Calabi–Yau*[10].

2.5 Plethystics and The Molien-Weyl Formula

Let us now move on to the problem of enumerating gauge invariants and encoding global symmetries. There have been a series of works (*e.g.*, [16, 18, 21, 22, 24, 25, 34–39]) that count the number of BPS GIOs in various gauge theories. However, for SQCD, the computations were usually limited to the case $N_c = 2$ due to technical difficulties. Recently, a **plethystic programme** has provided a general recipe for counting GIOs. In this section and below, we demonstrate that this programme provides us with not only a very systematic way of counting the GIOs, but also a deeper understanding of the moduli spaces of SQCD.

[9] We are grateful to Richard Thomas for drawing our attention to this important theorem.

[10] By Calabi–Yau conditions, we mean that the first chern class vanishes and that there exists a unique holomorphic middle dimensional form.

In SQCD the chiral GIOs are symmetric functions of quarks and antiquarks which transform respectively in the bifundamental $[1,0,\ldots,0;0,\ldots,0,1]$ of $SU(N_f)_L\times SU(N_c)$ and the bifundamental $[1,0,\ldots,0;0,\ldots,0,1]$ of $SU(N_c)\times SU(N_f)_R$. Let us denote the character of the (anti) fundamental representation of $SU(N)$, respectively, as $\chi^{SU(N)}_{[0,\ldots,1]}$, and $\chi^{SU(N)}_{[1,0,\ldots,0]}$. To write down explicit formulae and for performing computations we need to introduce weights for the different elements in the maximal torus of the different groups. We use $z_a, a=1,\ldots,N_c-1$ for colour weights and $t_i,\tilde{t}_i, i=1,\ldots,N_f$ for flavour weights. These weights have the interpretation of fugacities for the charges they count and the characters of the representations are functions of these variables. Correspondingly, the character for a quark is $\chi^{SU(N_f)_L\times SU(N_c)}_{[1,0,\ldots,0;0,\ldots,0,1]}(t_i,z_a)$ and the character for an antiquark is $\chi^{SU(N_c)\times SU(N_f)_R}_{[1,0,\ldots,0;0,\ldots,0,1]}(z_a,\tilde{t}_i)$. We further introduce two fugacities which count the number of quarks and antiquarks, t, and $\tilde{t}$, respectively. A convenient combinatorial tool which constructs symmetric products of representations is the **plethystic exponential**, which is a generator for symmetrisation [13, 17–19, 21]. To briefly remind the reader, the plethystic exponential, PE, of a function $g(t_1,\ldots,t_n)$ is defined to be $\exp\left(\sum\limits_{k=1}^{\infty}\frac{1}{k}g(t_1^k,\ldots,t_n^k)\right)$. Whence, we have that

$$\begin{aligned}&\mathrm{PE}\left[t\chi^{SU(N_f)_L\times SU(N_c)}_{[1,0,\ldots,0;0,\ldots,0,1]}(t_i,z_a)+\tilde{t}\chi^{SU(N_c)\times SU(N_f)_R}_{[1,0,\ldots,0;0,\ldots,0,1]}(z_a,\tilde{t}_i)\right]\\ \equiv\ &\exp\left[\sum_{k=0}^{\infty}\frac{1}{k}\left(t^k\chi^{SU(N_f)_L\times SU(N_c)}_{[1,0,\ldots,0;0,\ldots,0,1]}(t_i^k,z_a^k)+\tilde{t}^k\chi^{SU(N_c)\times SU(N_f)_R}_{[1,0,\ldots,0;0,\ldots,0,1]}(z_a^k,\tilde{t}_i^k)\right)\right]. \end{aligned}\qquad(2.5.1)$$

A somewhat more explicit form for the character can be

$$t\chi^{SU(N_f)_L\times SU(N_c)}_{[1,0,\ldots,0;0,\ldots,0,1]}(t_i,z_a)=\chi^{SU(N_c)}_{[0,\ldots,0,1]}(z_l)\sum_{i=1}^{N_f}t_i\ ,\qquad(2.5.2)$$

which then gives

$$\begin{aligned}&\mathrm{PE}\left[\chi^{SU(N_c)}_{[1,0,\ldots,0]}(z_l)\sum_{i=1}^{N_f}\tilde{t}_i+\chi^{SU(N_c)}_{[0,\ldots,0,1]}(z_l)\sum_{j=1}^{N_f}t_j\right]\\ &=\exp\left[\sum_{k=0}^{\infty}\frac{1}{k}\left(\chi^{SU(N_c)}_{[1,0,\ldots,0]}(z_l^k)\sum_{i=1}^{N_f}\tilde{t}_i^k+\chi^{SU(N_c)}_{[0,\ldots,0,1]}(z_l^k)\sum_{j=1}^{N_f}t_j^k\right)\right]\ .\end{aligned}\qquad(2.5.3)$$

Here, the dummy variables t_i and $\tilde{t}_j$ are the fugacities associated to quarks and antiquarks counting the $U(1)$-charges in the maximal torus of the global symmetry. Henceforth, we shall take their values to be such that $|t_i|<1$ for all i.

We emphasize that in order to obtain the generating function that counts *gauge invariant* quantities, we need to project the representations of the gauge group generated by the plethystic exponential onto the trivial subrepresentation, which consists of the quantities *invariant* under the action of the gauge group. Using knowledge from representation theory, this can be done by integrating over the whole group (see, *e.g.*, Appendix A of [38] .) Hence, the generating function for the (N_f, N_c) theory is given by

$$g^{(N_f,N_c)}(t_1,\dots t_{N_f},\tilde{t}_1,\dots,\tilde{t}_{N_f}) = \int_{SU(N_c)} \mathrm{d}\mu_{SU(N_c)} \times \\ \mathrm{PE}\left[\chi^{SU(N_c)}_{[1,0,\dots,0]}(z_l)\sum_{i=1}^{N_f}\tilde{t}_i + \chi^{SU(N_c)}_{[0,\dots,0,1]}(z_l)\sum_{j=1}^{N_f}t_j\right] . \tag{2.5.4}$$

This formula is also used in the commutative algebra literature (see, *e.g.*, [40]) and is called the **Molien–Weyl formula**. We note that the Haar measure $\mu_{SU(N_c)}$ can be written explicitly using Weyl's integration formula (see, *e.g.*, Section 26.2 of [41]):

$$\int_{SU(N_c)} \mathrm{d}\mu_{SU(N_c)} = \frac{1}{(2\pi i)^{N_c-1}N_c!}\oint_{|z_l|=1}\prod_{l=1}^{N_c-1}\frac{\mathrm{d}z_l}{z_l}\Delta(\phi)\Delta(\phi^{-1}) , \tag{2.5.5}$$

where $\{\phi_a(z_1,\dots,z_{N_c-1})\}_{a=1}^{N_c}$ are coordinates on the maximal torus of $SU(N_c)$ with $\prod_{a=1}^{N_c}\phi_a = 1$, and $\Delta(\phi) = \prod_{1\le a<b\le N_c}(\phi_a-\phi_b)$ is the Vandermonde determinant.

Let us take the weights of the fundamental representation of $SU(N_c)$ to be as follows:

$$L_1 = (1,0,\dots,0) , \quad L_k = (0,0,\dots,-1,1,\dots 0) , \quad L_{N_c} = (0,\dots,-1) , \tag{2.5.6}$$

where all L's are (N_c-1)-tuples, and for L_k (with $2 \le k \le N_c-1$), we have -1 in the $(k-1)$-th position and 1 in the k-th position. With this choice of weights, the corresponding coordinates on the maximal torus of $SU(N_c)$ are

$$\phi_1 = z_1 , \quad \phi_k = z_{k-1}^{-1}z_k , \quad \phi_{N_c} = z_{N_c-1}^{-1} , \tag{2.5.7}$$

where $2 \le k \le N_c-1$. Hence, the characters of the fundamental and antifundamental representations are respectively

$$\chi^{SU(N_c)}_{[1,0,\dots,0]}(z_1,\dots,z_{N_c-1}) = \sum_{a=1}^{N_c}\phi_a = z_1 + \sum_{k=2}^{N_c-1}\frac{z_k}{z_{k-1}} + \frac{1}{z_{N_c-1}} , \\ \chi^{SU(N_c)}_{[0,\dots,0,1]}(z_1,\dots,z_{N_c-1}) = \sum_{a=1}^{N_c}\phi_a^{-1} = \frac{1}{z_1} + \sum_{k=2}^{N_c-1}\frac{z_{k-1}}{z_k} + z_{N_c-1} . \tag{2.5.8}$$

Putting all the above together, we arrive at the Molien–Weyl formula for computing the generating function for GIOs of SQCD and which also gives an analytic way of computing the Hilbert series for the vacuum moduli space $\mathcal{M}_{(N_f,N_c)}$:

$$g^{(N_f,N_c)}(t_1,\dots t_{N_f},\tilde{t}_1,\dots,\tilde{t}_{N_f}) = \frac{1}{(2\pi i)^{N_c-1}N_c!}\oint_{|z_l|=1}\prod_{l=1}^{N_c-1}\frac{\mathrm{d}z_l}{z_l}\Delta(\phi)\Delta(\phi^{-1})\times$$
$$\mathrm{PE}\left[\left(z_1+\sum_{k=2}^{N_c-1}\frac{z_k}{z_{k-1}}+\frac{1}{z_{N_c-1}}\right)\sum_{i=1}^{N_f}\tilde{t}_i+\left(\frac{1}{z_1}+\sum_{k=2}^{N_c-1}\frac{z_{k-1}}{z_k}+z_{N_c-1}\right)\sum_{j=1}^{N_f}t_j\right] . \tag{2.5.9}$$

2.5.1 The Case of Two Colours: $N_c = 2$

Thus armed, we can compute the generating function for SQCD. Let us begin with two colours where some results are known.

The Example of $(N_f = 1, N_c = 2)$

There are two chiral multiplets (*i.e.* a quark and an antiquark being identified) in the theory, and we denote their fugacities by t_1 and t_2. From (2.5.4), the generating function $g^{(N_f=1,N_c=2)}$ is given by

$$g^{(1,2)}(t_1,t_2) = \int_{SU(2)}\mathrm{d}\mu_{SU(2)}(z)\,\mathrm{PE}[\chi_{[1]}^{SU(2)}(z)(t_1+t_2)] , \tag{2.5.10}$$

where $\chi_{[1]}^{SU(2)}(z) = z + 1/z$. Using (2.5.3), we find that

$$\mathrm{PE}\left[\left(z+\frac{1}{z}\right)(t_1+t_2)\right] = \exp\left(\sum_{l=1}^{2}\sum_{k=1}^{\infty}\frac{(zt_l)^k+(z^{-1}t_l)^k}{k}\right)$$
$$= \frac{1}{(1-t_1z)(1-t_2z)(1-\frac{t_1}{z})(1-\frac{t_2}{z})} , \tag{2.5.11}$$

where we have used the fact that $-\log(1-x) = \sum_{k=1}^{\infty} x^k/k$. Using formula (2.5.5), we can write the Haar measure in (2.5.10) as

$$\int_{SU(2)}\mathrm{d}\mu_{SU(2)}(z) \to \frac{1}{2}\frac{1}{2\pi i}\oint_{|z|=1}\frac{\mathrm{d}z}{z}(1-z^2)(1-z^{-2}) . \tag{2.5.12}$$

Therefore we can rewrite (2.5.9) in the form of Molien integral formula (see, *e.g.*, [24]):

$$g^{(1,2)}(t_1,t_2) = \frac{1}{2}\frac{1}{2\pi i}\oint_{|z|=1}\mathrm{d}z\frac{(1-z^2)(1-z^{-2})}{z(1-t_1z)(1-t_2z)(1-t_1z^{-1})(1-t_2z^{-1})} . \tag{2.5.13}$$

Recall that the fugacities t_1 and t_2 have been taken to be such that $0 < |t_1|, |t_2| < 1$. The integrand therefore has poles at $z = 0,\ t_1,\ t_2$. By the residue theorem, we find that

$$g^{(1,2)}(t_1,t_2) = \frac{1}{1-t_1t_2} = \sum_{j=0}^{\infty}(t_1t_2)^j . \tag{2.5.14}$$

The term $t_1 t_2$ in the denominator implies that there is only one basic generator of GIOs which is constructed from two chiral multiplets. Moreover, the series expansion suggests that any other operator in the chiral ring is given as a power of such a basic generator. If we set $t_1 = t_2 = t$, then

$$g^{(1,2)}(t) = \frac{1}{1-t^2} = 1 + t^2 + t^4 + t^6 + \ldots \ , \tag{2.5.15}$$

which is in agreement with the result presented in [21, 24]. Of course, this is also the result for the Hilbert series in Observation 2.3.2 at $N_f = 1$, so we have agreement with the algebro-geometric perspective as well.

$(N_f, N_c = 2)$ with Arbitrary Flavours

Let us move on to the case of arbitrary number N_f of flavours and two colours. Now, we have $2N_f$ chiral multiplets. From (2.5.4), the generating function is then given by

$$g^{(N_f,N_c=2)}(t_1, \ldots, t_{2N_f}) = \int_{SU(2)} \mathrm{d}\mu_{SU(2)}(z) \ \mathrm{PE}\left[\left(z + \frac{1}{z}\right) \sum_{i=1}^{2N_f} t_i\right] \ , \tag{2.5.16}$$

where, according to (2.5.3), the plethystic exponential can be written as

$$\exp\left(\sum_{k=1}^{\infty} \sum_{l=1}^{2N_f} \frac{(zt_l)^k + (z^{-1}t_l)^k}{k}\right) = \prod_{l=1}^{2N_f} (1 - t_l z)^{-1}(1 - t_l z^{-1})^{-1} \ , \tag{2.5.17}$$

where again we have used the log expansion. Changing the measure of integration as above, (2.5.9) becomes

$$g^{(N_f,N_c=2)}(t_1, \ldots, t_{2N_f}) = \frac{1}{2}\frac{1}{2\pi i} \oint_{|z|=1} \frac{\mathrm{d}z}{z}(1-z^2)(1-z^{-2}) \prod_{l=1}^{2N_f} (1 - t_l z)^{-1}(1 - t_l z^{-1})^{-1} \ . \tag{2.5.18}$$

The integral can again be evaluated by residues.

For example, in the case $N_f = 2$, the poles located within the unit circle are $z = 0,\ t_1,\ \ldots,\ t_4$ and we find that

$$g^{(N_f=2,N_c=2)}(t_1, \ldots, t_4) = \frac{1 - t_1 t_2 t_3 t_4}{(1-t_1t_2)(1-t_1t_3)(1-t_1t_4)(1-t_2t_3)(1-t_2t_4)(1-t_3t_4)} \ . \tag{2.5.19}$$

The expressions $t_i t_j$ (with $1 \le i < j \le 4$) in the denominator indicate that there are six basic generators of GIOs, each of which is constructed from two chiral multiplets. Explicitly, these basic generators are mesons. Moreover, the numerator suggests that there is one constraint between these generators at order four of chiral multiplets, namely

$$\mathrm{Pf}\ M = \epsilon_{i_1 \ldots i_4} M^{i_1 i_2} M^{i_3 i_4} = 0 \ , \tag{2.5.20}$$

a constraint which has already been seen in (2.3.17). The general formula for $N_f > 1$ can be written as

$$g^{(N_f>1,N_c=2)}(t_1,\dots,t_{2N_f}) = \frac{\sum_{k=1}^{2N_f}(-1)^k(t_k)^{2N_f-3}(1-t_k^2)\prod_{\substack{1\le i<j\le 2N_f\\ i,j\neq k}}(t_i-t_j)(1-t_it_j)}{2\prod_{1\le i<j\le 2N_f}(t_i-t_j)(1-t_it_j)} . \tag{2.5.21}$$

Now, if we unrefine and set $t_i = t$ for all $i = 1,\dots,2N_f$, we should reproduce the results for the Hilbert series discussed before. Let us present some results for small values of N_f:

$$\begin{aligned} g^{(1,2)}(t) &= \tfrac{1}{1-t^2} , \\ g^{(2,2)}(t) &= \tfrac{1-t^4}{(1-t^2)^6} = \tfrac{1+t^2}{(1-t^2)^5} , \\ g^{(3,2)}(t) &= \tfrac{1+6t^2+6t^4+t^6}{(1-t^2)^9} , \\ g^{(4,2)}(t) &= \tfrac{1+15t^2+50t^4+50t^6+15t^8+t^{10}}{(1-t^2)^{13}} , \\ g^{(5,2)}(t) &= \tfrac{1+28\,t^2+196\,t^4+490\,t^6+490\,t^8+196\,t^{10}+28\,t^{12}+t^{14}}{(1-t^2)^{17}} . \end{aligned} \tag{2.5.22}$$

These highly non-trivial results are in perfect agreement with the right column of Table 2.3, obtained from a completely different method. We remark that `Macaualy 2` [26] can also compute the refined (multi-variate) Hilbert series; we have performed this computation for some examples and the results are exactly as in (2.5.21). This is encouraging indeed.

In general, the formula $g^{(N_f,N_c=2)}(t)$ can be expanded in power series:

$$\begin{aligned} g^{(N_f,N_c=2)}(t) = 1 &+ \frac{2N_f(2N_f-1)}{2}t^2 + \frac{(2N_f-1)(2N_f)^2(2N_f+1)}{12}t^4 \\ &+ \frac{(2N_f-1)(2N_f)^2(2N_f+1)^2(2N_f+2)}{4(3!)^2}t^6 + \dots . \end{aligned} \tag{2.5.23}$$

We can rewrite this equation more compactly, as in (2.3.18), which in fact holds for all $N_f \ge 1$:

$$g^{(N_f,N_c=2)}(t) = \sum_{k=0}^{\infty}\frac{(2N_f+k-1)!(2N_f+k-2)!}{(2N_f-1)!(2N_f-2)!(k+1)!k!}t^{2k} = {}_2F_1(2N_f-1,2N_f;2;t^2) . \tag{2.5.24}$$

Plethystic Logarithms and $\mathcal{M}_{(N_f,N_c=2)}$

Recall that according to the plethystic programme the Hilbert series is itself the plethystic exponential of a function that encodes the defining relations. This does not contain quite as much information as the defining equations themselves, given in, *e.g.*, (2.4.7), but it does give the generators and the the relations at each degree.

We will thus use the **plethystic logarithm** to deduce the number of generators and constraints at each order of quarks and antiquarks from the generating function [17, 18]. We recall the expression for the plethystic logarithm, PL, the inverse function to PE, is

$$\mathrm{PL}[g^{(N_f,N_c)}(t)] = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log(g^{(N_f,N_c)}(t^k)) \ , \tag{2.5.25}$$

where $\mu(k)$ is the Möbius function. The significance of the series expansion of the plethystic logarithm is stated in [17, 18]: *the first terms with plus sign give the basic generators while the first terms with the minus sign give the constraints between these basic generators.* If the formula (2.5.25) is an infinite series of terms with plus and minus signs, then the moduli space is not a complete intersection and the constraints in the chiral ring are not trivially generated by relations between the basic generators, but receives stepwise corrections at higher degree. These are the so-called **higher syzygies**.

Let us calculate the plethystic logarithms for $N_f = 1, \ldots, 4$:

$$\begin{aligned}
\mathrm{PL}[g^{(1,2)}(t)] &= t^2 \ , \\
\mathrm{PL}[g^{(2,2)}(t)] &= 6t^2 - t^4 \ , \\
\mathrm{PL}[g^{(3,2)}(t)] &= 15t^2 - 15t^4 + 35t^6 - 126t^8 + 504t^{10} + \ldots \ , \\
\mathrm{PL}[g^{(4,2)}(t)] &= 28t^2 - 70t^4 + 420t^6 - 3360t^8 + 29148t^{10} + \ldots \ .
\end{aligned} \tag{2.5.26}$$

Take $\mathrm{PL}[g^{(4,2)}(t)]$ as an example: from Observation (2.3.8), we see that the coefficient 28 of t^2 are the number of mesons and the coefficient -70 indicates that there are 70 constraints among mesons according to (2.3.17).

We can conclude some properties of the moduli spaces from these results as follows. For $(N_f = 1, N_c = 2)$, there are no constraints between the generators and hence the moduli spaces are *freely generated.* For $(N_f = 2, N_c = 2)$, there are six basic generators at order two, and one constraint between these generators at order four. Since the dimension of the moduli space (which is $\dim \mathcal{M}_{(N_f=2,N_c=2)} = 2^2 + 1 = 5$) plus the number of constraints (one) is equal to the number of basic generators (six), the moduli space in this case is a *complete intersection.* These conclusions agree with Observations 2.3.1 and 2.3.5.

2.5.2 The Case of Three Colours: $N_c = 3$

Emboldened by our success with two colours, let us move on to three.

$(N_f, N_c = 3)$ with Arbitrary Flavours

There are N_f quarks transforming in the fundamental representation and N_f antiquarks transforming in the antifundamental representation. Using the notation we introduced in (2.5.4), we find that the generating function is

$$g^{(N_f,N_c=3)}(t_1,\ldots,t_{N_f},\tilde{t}_1,\ldots,\tilde{t}_{N_f}) = \int_{SU(3)} \mathrm{d}\mu_{SU(3)} \times \\ \mathrm{PE}\left[\left(\chi^{SU(3)}_{[1,0]}(z_1,z_2)\sum_{i=1}^{N_f}\tilde{t}_i + \chi^{SU(3)}_{[0,1]}(z_1,z_2)\sum_{j=1}^{N_f} t_j\right)\right] , \qquad (2.5.27)$$

with $\chi^{SU(3)}_{[1,0]}(z_1,z_2) = z_1+\frac{z_2}{z_1}+\frac{1}{z_2}$, $\chi^{SU(3)}_{[0,1]}(z_1,z_2) = \frac{1}{z_1}+\frac{z_1}{z_2}+z_2$ and the Haar measure becomes

$$\int_{SU(3)} \mathrm{d}\mu_{SU(3)} = \frac{1}{6}\frac{1}{(2\pi i)^2}\oint_{|z_1|=1}\frac{\mathrm{d}z_1}{z_1}\oint_{|z_2|=1}\frac{\mathrm{d}z_2}{z_2}\times \\ \left(1-\frac{z_1^2}{z_2}\right)\left(1-\frac{z_2^2}{z_1}\right)(1-z_1z_2)\left(1-\frac{z_2}{z_1^2}\right)\left(1-\frac{z_1}{z_2^2}\right)\left(1-\frac{1}{z_1z_2}\right) . \qquad (2.5.28)$$

The plethystic exponential in (2.5.27) can be simplified to

$$\prod_{i=1}^{N_f}\left[(1-\tilde{t}_iz_1)(1-\tilde{t}_iz_1^{-1}z_2)(1-\tilde{t}_iz_2^{-1})(1-t_iz_1^{-1})(1-t_iz_1z_2^{-1})(1-t_iz_2)\right]^{-1} . \qquad (2.5.29)$$

We note that for the z_2 integral, the poles inside the unit circle are located at $z_2 = 0,\ \tilde{t}_i,\ t_iz_1$, and for the z_1 integral, such poles are located at $z_1 = 0,\ \prod_{i<j}\tilde{t}_i\tilde{t}_j,\ t_i$. Using the residue theorem, we find that

$$g^{(1,3)}(t_1,\tilde{t}_1) = \frac{1}{1-t_1\tilde{t}_1} , \qquad (2.5.30)$$

$$g^{(2,3)}(t_1,t_2,\tilde{t}_1,\tilde{t}_2) = \frac{1}{\prod_{1\le i,j\le 2}(1-t_i\tilde{t}_j)} , \qquad (2.5.31)$$

$$g^{(3,3)}(t_1,t_2,t_3,\tilde{t}_1,\tilde{t}_2,\tilde{t}_3) = \frac{1-\prod_{i=1}^3 t_i\tilde{t}_i}{(1-\prod_{i=1}^3 t_i)(1-\prod_{j=1}^3\tilde{t}_j)\prod_{1\le i,j\le 3}(1-t_i\tilde{t}_j)} . \qquad (2.5.32)$$

Since the generating function $g^{(4,3)}$ in eight variables is very long (three pages in a `Mathematica` notebook), we shall not present its formula here. However, if we unrefine and set $t_i = t$ and $\tilde{t}_i = \tilde{t}$, the calculation is slightly easier and we obtain that

$$g^{(4,3)}(t,\tilde{t}) = \left((1-t^3)^4(1-t\tilde{t})^{16}(1-\tilde{t}^3)^4\right)^{-1}\times \\ \left[1-4\tilde{t}^4t+6\tilde{t}^8t^2-16\tilde{t}^3t^3+24\tilde{t}^6t^3-16\tilde{t}^9t^3-4\tilde{t}t^4+31\tilde{t}^4t^4- \right. \\ 20\tilde{t}^7t^4+10\tilde{t}^{10}t^4-24\tilde{t}^8t^5+24\tilde{t}^3t^6-36\tilde{t}^6t^6+24\tilde{t}^9t^6+10\tilde{t}^{12}t^6- \\ 20\tilde{t}^4t^7-16\tilde{t}^7t^7+24\tilde{t}^{10}t^7-16\tilde{t}^{13}t^7+6\tilde{t}^2t^8-24\tilde{t}^5t^8+72\tilde{t}^8t^8- \\ 24\tilde{t}^{11}t^8+6\tilde{t}^{14}t^8-16\tilde{t}^3t^9+24\tilde{t}^6t^9-16\tilde{t}^9t^9-20\tilde{t}^{12}t^9+10\tilde{t}^4t^{10}+ \\ 24\tilde{t}^7t^{10}-36\tilde{t}^{10}t^{10}+24\tilde{t}^{13}t^{10}-24\tilde{t}^8t^{11}+10\tilde{t}^6t^{12}-20\tilde{t}^9t^{12}+31\tilde{t}^{12}t^{12}- \\ \left. 4\tilde{t}^{15}t^{12}-16\tilde{t}^7t^{13}+24\tilde{t}^{10}t^{13}-16\tilde{t}^{13}t^{13}+6\tilde{t}^8t^{14}-4\tilde{t}^{12}t^{15}+\tilde{t}^{16}t^{16}\right] . \qquad (2.5.33)$$

If we completely unrefine and set $t_i = \tilde{t}_i = t$, we will have the following results which will be useful later and which again agree completely with Table 2.3:

$$
\begin{aligned}
g^{(1,3)}(t) &= \tfrac{1}{1-t^2} = 1 + t^2 + t^4 + t^6 + t^8 + t^{10} + \dots \ , \\
g^{(2,3)}(t) &= \tfrac{1}{(1-t^2)^4} = 1 + 4t^2 + 10t^4 + 20t^6 + 35t^8 + 56t^{10} + \dots \ , \\
g^{(3,3)}(t) &= \tfrac{1-t^6}{(1-t^3)^2(1-t^2)^9} = \tfrac{1+t^3}{(1-t^3)(1-t^2)^9} \\
&= 1 + 9t^2 + 2t^3 + 45t^4 + 18t^5 + 167t^6 + 90t^7 + 513t^8 + 332t^9 + \\
&\quad 1377t^{10} + 1008t^{11} + 3335t^{12} + 2664t^{13} + \dots \ , \\
g^{(4,3)}(t) &= \left((1-t^2)^{16}(1-t^3)^8\right)^{-1} \times \\
&\quad [1 - 8t^5 - 16t^6 + 31t^8 + 48t^9 + 12t^{10} - 40t^{11} - 68t^{12} - 48t^{13} + 4t^{14} + 48t^{15} + 72t^{16} + \\
&\quad 48t^{17} + 4t^{18} - 48t^{19} - 68t^{20} - 40t^{21} + 12t^{22} + 48t^{23} + 31t^{24} - 16t^{26} - 8t^{27} + t^{32}] \\
&= 1 + 16t^2 + 8t^3 + 136t^4 + 120t^5 + 836t^6 + 960t^7 + 4163t^8 + 5480t^9 + 17708t^{10} + \dots \ , \\
g^{(5,3)}(t) &= \left((1-t)^{22}(1+t)^{16}(1+t+t^2)^7\right)^{-1} \times \\
&\quad [1 + t + 10t^2 + 23t^3 + 68t^4 + 135t^5 + 281t^6 + 446t^7 + 695t^8 \\
&\quad + 895t^9 + 1090t^{10} + 1115t^{11} + 1090t^{12} + 895t^{13} + 695t^{14} \\
&\quad + 446t^{15} + 281t^{16} + 135t^{17} + 68t^{18} + 23t^{19} + 10t^{20} + t^{21} + t^{22}] \\
&= 1 + 25t^2 + 20t^3 + 325t^4 + 450t^5 + 3025t^6 + 5280t^7 + 22550t^8 + \dots \ .
\end{aligned}
\tag{2.5.34}
$$

Plethystic Logarithms and $\mathcal{M}_{(N_f, N_c=3)}$

As before, we can take the plethystic logarithms of the generating functions to find the defining equations of $\mathcal{M}_{(N_f, N_c=3)}$. For $N_f = 1, \dots, 5$ we have:

$$
\begin{aligned}
\mathrm{PL}[g^{(1,3)}(t)] &= t^2 \ , \\
\mathrm{PL}[g^{(2,3)}(t)] &= 4t^2 \ , \\
\mathrm{PL}[g^{(3,3)}(t)] &= 9t^2 + 2t^3 - t^6 \ , \\
\mathrm{PL}[g^{(4,3)}(t)] &= 16t^2 + 8t^3 - 8t^5 - 16t^6 + 31t^8 + 48t^9 - 16t^{10} + \dots \ , \\
\mathrm{PL}[g^{(5,3)}(t)] &= 25t^2 + 20t^3 - 50t^5 - 110t^6 + 30t^7 + 575t^8 + 1010t^9 - 1177t^{10} + \dots \ .
\end{aligned}
\tag{2.5.35}
$$

As an example, let us consider $\mathrm{PL}[g^{(4,3)}(t)]$: from Observation 2.3.3, the coefficient 16 of t^2 is the dimension of the bifundamental representation of $SU(4) \times SU(4)$ and hence it is the number of mesons; the coefficient 8 of t^3 is the number of baryons + antibaryons. The coefficient -8 of t^5 indicates the number of constraints at order 5 of quarks + antiquarks, namely the ones given by (2.3.12). Similarly, the coefficient -16 of t^6 indicates the number of constraints at order 6 of quarks + antiquarks, namely the ones given by (2.3.11).

We can conclude some properties of the moduli spaces from these results as follows. For $N_f = 1, 2$, there are no constraints between the generators and hence the moduli spaces are *freely generated*. For $N_f = 3$, there are nine basic generators at order two quarks and antiquarks, two basic generators at order three quarks and antiquarks, and one constraint between these generators at order six quarks and antiquarks. Since the dimension of the moduli space (which is $\dim \mathcal{M}_{(N_f=3,N_c=3)} = 3^2+1 = 10$) plus the number of constraints (one) is equal to the number of basic generators (which is $9+2=11$), the moduli space in this case is a *complete intersection*. These conclusions agree with Observations 2.3.1 and 2.3.5.

2.5.3 Palindromic Numerator: A Proof Using Plethystics

We have observed in many case studies before that the numerator of the generating function (Hilbert series) for SQCD is palindromic, *i.e.* it can be written in the form:

$$P(t) = \sum_{k=0}^{N} a_k t^k \ , \tag{2.5.36}$$

with symmetric coefficients $a_{N-k} = a_k$. This observation (cf. Section 2.4.5) would imply that the SQCD chiral ring is Gorenstein Cohen–Macaulay, and that the classical moduli space is an affine Calabi–Yau variety. In this section, as promised, we shall show that this palindromic property holds in general:

Theorem 2.5.1. *Let $P(t)$ be a numerator of the generating function (Hilbert series) $g^{(N_f,N_c)}(t)$ and suppose that $P(1)\neq 0$. Then, $P(t)$ is palindromic.*

We shall use the following lemma to prove the above theorem.

Lemma 2.5.2. *Let $d = \dim(\mathcal{M}_{(N_f,N_c)})$. Then, the generating function obeys:*

$$g^{(N_f,N_c)}(1/t) = (-1)^d t^{2N_f N_c} g^{(N_f,N_c)}(t) \ . \tag{2.5.37}$$

Proof. Let us start by writing down $g^{(N_f,N_c)}(t)$ as follows:

$$\begin{aligned} g^{(N_f,N_c)}(t) &= \int_{SU(N_c)} \mathrm{d}\mu_{SU(N_c)}\, \mathrm{PE}[N_f \left(\chi^{SU(N_c)}_{[1,0,\dots,0]} + \chi^{SU(N_c)}_{[0,\dots,0,1]}\right) t] \\ &= \int_{SU(N_c)} \frac{\mathrm{d}\mu_{SU(N_c)}}{\prod_{i=1}^{N_c}(1-t\phi_i)^{N_f}(1-t\phi_i^{-1})^{N_f}} \ , \end{aligned} \tag{2.5.38}$$

where ϕ_i are the coordinates on the maximal torus of the $SU(N_c)$ gauge group. We emphasise that, as before, the modulus of the argument of the function $g^{(N_f,N_c)}$ must be less than 1. Now consider $g^{(N_f,N_c)}(1/t)$. Under the transformation t to $1/t$, the integrand in (2.5.38) changes to

$$\frac{1}{\prod_{i=1}^{N_c}(1-t^{-1}\phi_i)^{N_f}(1-t^{-1}\phi_i^{-1})^{N_f}} = \frac{t^{2N_f N_c}}{\prod_{i=1}^{N_c}(1-t\phi_i)^{N_f}(1-t\phi_i^{-1})^{N_f}} \ . \tag{2.5.39}$$

Since $|t|<1$ implies that $|1/t|>1$ and *vice-versa*, great care must be taken when evaluating the integral in order to keep the directions of contour integrations and hence the overall sign correct. An easy way to obtain the correct overall sign is to think about the expansion of $g^{(N_f,N_c)}(t)$ as a Laurent series around $t=1$:

$$g^{(N_f,N_c)}(t) = \sum_{k=-d}^{\infty} c_k (t-1)^k \sim \frac{c_{-d}}{(t-1)^d} \ , \tag{2.5.40}$$

for $t \to 1$. (Recall that d is the dimension of the moduli space, which is equal to the order of the pole at $t = 1$.) Therefore, we see that as $t \to 1$, the signs of $g^{(N_f,N_c)}(1/t)$ and $g^{(N_f,N_c)}(t)$ differ by $(-1)^d$. Combining this result with (2.5.39), we prove the assertion (2.5.37). □

We are now ready for our claim.

Proof of Theorem 2.5.1. We note that the denominator of the generating function $g^{(N_f,N_c)}$ is in the form $\prod_k (1-t^{a_k})^{b_k}$, where a_k and b_k are non-negative integers. Observe that upon the transformation t to $1/t$, the denominator picks up the sign $(-1)^{\sum_k b_k}$. Now if the numerator $P(t)$ does not vanish at $t=1$, then $\sum_k b_k$ is exactly the order of the pole of the generating function at $t = 1$, which is equal to the dimension d of the moduli space. Since $P(t) = g^{(N_f,N_c)}(t) \prod_k (1-t^{a_k})^{b_k}$, it follows from (2.5.37) that $P(t)$ is indeed palindromic. □

Therefore, the numerator of the Hilbert series (generating function) for $\mathcal{M}_{(N_f,N_c)}$ is in general palindromic and thus $\mathcal{M}_{(N_f,N_c)}$ is Calabi–Yau.

2.6 Character Expansion and Global Symmetries

In the previous section, we have obtained the generating functions analytically for various (N_f, N_c) theories. As we mentioned earlier, the coefficients of t^k in $g^{(N_f,N_c)}(t)$ is the number of independent GIOs at the k-th order of quarks and antiquarks. We shall see in this section that this number is in fact *the dimension of some irreducible representation of the global symmetry* at that order. This is in the spirit of how plethystics of the master space encode the global symmetries of the theory [23]. Moreover, we shall see that the character expansion allows us to write down the generating function for *any* (N_f, N_c) theory in a very compact and enlightening way as follows:

$$g^{(N_f,N_c)}(t,\tilde{t}) = \sum_{n_1,n_2,\ldots,n_k,\ell,m\geq 0} [n_1,n_2,\ldots,n_k,\ell_{N_c;L},0,\ldots,0;0,\ldots,0,m_{N_c;R},n_k,\ldots,n_2,n_1] t^a \tilde{t}^b \ . \tag{2.6.1}$$

where $k = N_c - 1$, $a = \ell N_c + \sum_{j=1}^{k} j n_j$, $b = m N_c + \sum_{j=1}^{k} j n_j$ and we have again used the notation below Observation 2.3.3 for the representation. We shall discuss this important result further in Observation 2.6.5.

For $N_f = N_c$ this formula goes through and has the form

$$g^{(N_c,N_c)}(t,\tilde{t}) = \sum_{n_1,n_2,\ldots,n_k,\ell,m\geq 0} [n_1,n_2,\ldots,n_k;n_k,\ldots,n_2,n_1] \ t^a \ \tilde{t}^b \ , \tag{2.6.2}$$

whereas for $N_f < N_c$ this formula has N_f infinite sums and takes the form

$$g^{(N_c,N_c)}(t,\tilde{t}) = \sum_{n_1,n_2,\dots,n_{N_f}\geq 0} [n_1, n_2, \dots, n_{N_f-1}; n_{N_f-1}, \dots, n_2, n_1]\ t^a\, \tilde{t}^a \ , \tag{2.6.3}$$

with $a = \sum_{j=1}^{N_f} j n_j$.

2.6.1 The Case of Two Colours Revisited

Let us begin again with the simplest case of $N_c = 2$. The formula (2.3.18) suggests:

Observation 2.6.1. *For any N_f, the character expansion of the $(N_f, N_c = 2)$ generating function can be written as*

$$g^{(N_f,N_c=2)}(t) = \sum_{k=0}^{\infty} \chi^{SU(2N_f)}_{[0,k,0,\dots,0]} t^{2k} \ . \tag{2.6.4}$$

In the following subsections, we shall derive (2.6.4) for various case studies.

The Example of $(N_f = 2, N_c = 2)$

Let us first study two flavours. The generating function $g^{(N_f=2,N_c=2)}(t_1, \dots, t_4)$ was given in (2.5.19). Since the global symmetry here is $SU(4)$, we shall write this equation as a series expansion of the characters of $SU(4)$ representations. It is convenient here to take the coordinates on the maximal torus of $SU(4)$ to be[11]

$$\phi_1 = \frac{z_1 z_2}{z_3} \ , \quad \phi_2 = \frac{z_1 z_3}{z_2} \ , \quad \phi_3 = \frac{z_2 z_3}{z_1} \ , \quad \phi_4 = \frac{1}{z_1 z_2 z_3} \ . \tag{2.6.5}$$

With this choice of coordinates, the character of the fundamental representation of $SU(4)$ can be written as

$$\chi^{SU(4)}_{[1,0,0]}(z_1, z_2, z_3) = \sum_{a=1}^{4} \phi_a = \frac{z_1 z_2}{z_3} + \frac{z_1 z_3}{z_2} + \frac{z_2 z_3}{z_1} + \frac{1}{z_1 z_2 z_3} \ . \tag{2.6.6}$$

Let us write the fugacities t_i, where $i = 1, \dots, 4$, as

$$t_i = t\phi_i \ . \tag{2.6.7}$$

Substituting this in (2.5.19), we find that

$$g^{(N_f=2,N_c=2)}(t; z_1, z_2, z_3) = (1 - t^4) \left(\prod_{i=1}^{3} (1 - t^2 z_i^2)(1 - t^2/z_i^2) \right)^{-1} \ , \tag{2.6.8}$$

[11] We note that this choice of coordinates is different from those in (2.5.7). The present choice is more convenient here.

with the series expansion

$$g^{(N_f=2,N_c=2)}(t;z_1,z_2,z_3) = (1-t^4) \sum_{n_1,\dots,n_6=0}^{\infty} t^{2(n_1+\dots+n_6)} z_1^{2(n_2-n_1)} z_2^{2(n_4-n_3)} z_3^{2(n_6-n_5)} \ . \tag{2.6.9}$$

Next we shall prove that the expression in (2.6.9) is indeed the *character expansion* of the $SU(4)$ global symmetry.

We shall state and prove two lemmata that will be of use later:

Lemma 2.6.2. *Let V be the fundamental representation of $SU(4)$. Then*[12]

$$[0,m,0] \oplus \mathrm{Sym}^{m-2}(\Lambda^2 V) = \mathrm{Sym}^m(\Lambda^2 V) \ . \tag{2.6.10}$$

Proof. Consider the Plücker embedding of the Grassmannian of two-dimensional quotient spaces of V, $G = \mathrm{Grass}^2 V$, in the projective space $\mathbb{P}(\Lambda^2 V^*)$ of one-dimensional quotients of $\Lambda^2 V$. Note that G is a quadric hypersurface in $\mathbb{P}^5$, and so polynomials vanishing on G are those divisible by the quadratic polynomial that defines G. Since the space of all homogeneous polynomials of degree m on $\mathbb{P}(\Lambda^2 V^*)$ is $\mathrm{Sym}^m(\Lambda^2 V)$, we see that the subspace of those polynomials of degree m on $\mathbb{P}(\Lambda^2 V^*)$ that vanish on G is $\mathrm{Sym}^{m-2}(\Lambda^2 V)$. Then we have the exact sequence

$$0 \to \mathrm{Sym}^{m-2}(\Lambda^2 V) \to \mathrm{Sym}^m(\Lambda^2 V) \to W_m \to 0 \ , \tag{2.6.11}$$

where it can be shown [41] that W_m is an irreducible representation $[0,m,0]$. Since the exact sequence splits, the relation (2.6.10) follows. □

Lemma 2.6.3. *Let V be the fundamental representation of $SU(4)$ and let $\{\lambda_j\}_{j=1}^6$ be the eigenvalues of the action of the maximal torus on $\Lambda^2 V$. Then*

$$\chi^{SU(4)}_{\mathrm{Sym}^k(\Lambda^2 V)} = \sum_{1\le i_1 \le \dots \le i_k \le 6} \lambda_{i_1} \dots \lambda_{i_k} \ . \tag{2.6.12}$$

Proof. Let us take a basis of $\Lambda^2(V)$ to be $\{X_1,\dots,X_6\}$ for $SU(4)$. Let T be a maximal torus of $\Lambda^2(V)$ and let $D \in T$. Then D is a diagonal matrix, say, $D = \mathrm{diag}(\lambda_1,\dots,\lambda_6)$. Therefore, the eigenvalue of D corresponding to the eigenvector

[12] We shall use the notion Sym^k for symmetric powers and Λ^k for exterior powers. For the fundamental representation $V = [1,0,\dots,0]$, $\mathrm{Sym}^k V = [k,0,\dots,0]$ and $\Lambda^k V = [0,\dots,1,0,\dots,0]$ (where 1 occurs in the k-th position from the left). Their characters in the case $k=2$ are given by the formulae

$$\chi_{\mathrm{Sym}^2 V}(g) = \frac{1}{2}\left(\chi_V(g)^2 + \chi_V(g^2)\right) \ ,$$
$$\chi_{\Lambda^2 V}(g) = \frac{1}{2}\left(\chi_V(g)^2 - \chi_V(g^2)\right) \ .$$

$X_{i_1} \otimes \ldots \otimes X_{i_k}$ is $\lambda_{i_1} \ldots \lambda_{i_k}$. Since we know that the monomials of degree k in $X_1, \ldots, X_6$ form a basis of $\mathrm{Sym}^k(\Lambda^2 V)$, (2.6.12) follows. □

From (2.6.6), the character of $\Lambda^2 V$ is given by

$$\begin{aligned} \chi^{SU(4)}_{\Lambda^2 V}(z_1, z_2, z_3) &= \frac{1}{2}\left(\chi^{SU(4)}_V(z_1, z_2, z_3)^2 - \chi^{SU(4)}_V(z_1^2, z_2^2, z_3^2) \right) \\ &= \frac{1}{z_1^2} + z_1^2 + \frac{1}{z_2^2} + z_2^2 + \frac{1}{z_3^2} + z_3^2 \ . \end{aligned} \tag{2.6.13}$$

Therefore, we can take the eigenvalues λ_j of the action of the maximal torus of $\Lambda^2 V$ to be

$$\lambda_1 = z_1^2 \ , \quad \lambda_2 = z_1^{-2} \ , \quad \lambda_3 = z_2^2 \ , \quad \lambda_4 = z_2^{-2} \ , \quad \lambda_5 = z_3^2 \ , \quad \lambda_6 = z_3^{-2} \ . \tag{2.6.14}$$

Substituting these into (2.6.12), we obtain

$$\chi^{SU(4)}_{\mathrm{Sym}^k(\Lambda^2 V)} = \sum_{\substack{n_1+\ldots+n_6=k \\ n_1,\ldots,n_6 \geq 0}} z_1^{2(n_2-n_1)} z_2^{2(n_4-n_3)} z_3^{2(n_6-n_5)} \ . \tag{2.6.15}$$

Combining (2.6.9), (2.6.10), and (2.6.15), we find that the expression (2.6.9) is indeed a character expansion:

$$g^{(N_f=2, N_c=2)}(t) = \sum_{k=0}^{\infty} \left(\chi^{SU(4)}_{\mathrm{Sym}^k(\Lambda^2 V)} - \chi^{SU(4)}_{\mathrm{Sym}^{k-2}(\Lambda^2 V)} \right) t^{2k} = \sum_{k=0}^{\infty} \chi^{SU(4)}_{(0,k,0,\ldots,0)} t^{2k} \ . \tag{2.6.16}$$

This is in agreement with the formula (2.6.4).

The Example of $(N_f > 1, N_c = 2)$

Let us move on to a general number $N_f > 1$ flavours; here the global symmetry is $SU(2N_f)$. Denote coordinates on the maximal torus of $SU(2N_f)$ by $\{\phi_j\}_{j=1}^{2N_f}$ and, as before, substituting $t_j = t\phi_j$ into the generating function $g^{(N_f>1, N_c=2)}$ in (2.5.21), we obtain

$$g^{(N_f>1,N_c=2)}(t; z_i) = \frac{\sum\limits_{k=1}^{2n} (-1)^k (t\phi_k)^{2n-3} (1 - t^2\phi_k^2) \prod\limits_{\substack{1 \leq i < j \leq 2n \\ i,j \neq k}} (t\phi_i - t\phi_j)(1 - t^2 \phi_i \phi_j)}{2 \prod\limits_{1 \leq i < j \leq 2n} (t\phi_i - t\phi_j)(1 - t^2 \phi_i \phi_j)} \ . \tag{2.6.17}$$

This equation can be simplified to the formula (2.6.4) using various identities of Schur polynomials (see, *e.g.*, Appendix A of [41]) and the fact that the character $\chi_{(0,k,0,\ldots,0)}$ is given by the Weyl character formula (see, *e.g.*, Section 24.2 of [41]):

$$\chi_{[0,k,0,\ldots,0]} = \begin{vmatrix} \chi_{[k,0,\ldots,0]} & \chi_{[k+1,0,\ldots,0]} \\ \chi_{[k-1,0,\ldots,0]} & \chi_{[k,0,\ldots,0]} \end{vmatrix} \ , \tag{2.6.18}$$

where the character $\chi_{[k,0,\ldots,0]}$ is given by

$$\chi_{[k,0,\ldots,0]} = \sum_{\substack{(i_1,\ldots,i_k) \\ \sum_{1\le\alpha\le k} \alpha i_\alpha = k}} \frac{P_1^{i_1} P_2^{i_2} \ldots P_d^{i_k}}{i_1!1^{i_1} \cdot i_2!2^{i_2} \cdot \ldots \cdot i_k!k^{i_k}} \ , \tag{2.6.19}$$

where $P_j := \chi_{[1,0,\ldots,0]}(z_1^j, \ldots, z_{2N_f}^j)$. For example, $\chi_{[3,0,\ldots,0]} = \frac{1}{3!}P_1^3 + \frac{1}{2}P_1P_2 + \frac{1}{3}P_3$.

From $SU(2N_f)$ to $SU(N_f)_L \times SU(N_f)_R$

According to Observation 2.3.7, we know that the global symmetry of $(N_f, N_c = 2)$ theory is $SU(2N_f)$. However, for $N_c > 2$, we have talked mainly about the $SU(N_f)_L \times SU(N_f)_R$ global symmetry. In this section, we shall demonstrate how to decompose various representations of $SU(2N_f)$ into those of $SU(N_f)_L \times SU(N_f)_R$.

Here we shall denote the chemical potential counting the quarks in $SU(2N_f)$ by t, and the ones counting the quarks and antiquarks in $SU(N_f)_L \times SU(N_f)_R$ respectively by q and $\tilde{q}$. Therefore, we have the relation

$$q\chi^{SU(N_f)_L}_{[1,0,\ldots,0]}(x_i) + \tilde{q}\chi^{SU(N_f)_R}_{[0,\ldots,1]}(\tilde{x}_i) = t\chi^{SU(2N_f)}_{[1,0,\ldots,0]}(z_j) \ , \tag{2.6.20}$$

where $\{x_i\}_{i=1}^{N_f-1}$ and $\{\tilde{x}_i\}_{i=1}^{N_f-1}$ are respectively variables of coordinates on the maximal torus of the first and the second $SU(N_f)$, and $\{z_j\}_{j=1}^{2N_f-1}$ are variables of coordinates on the maximal torus of $SU(2N_f)$.

Let us choose the coordinates on the maximal tori according to (2.5.7) and (2.5.8). With this choice of coordinates, (2.6.20) will be satisfied, if

$$q = tb \ , \quad \tilde{q} = \frac{t}{b} \ , \quad z_i = x_i b^i \ , \quad z_{N_f} = b^{N_f} \ , \quad z_{N_f+i} = b^{N_f-i}\tilde{x}_{N_f-i} \ , \tag{2.6.21}$$

where $1 \le i \le N_f - 1$ and b is the chemical potential counting $U(1)_B$-charges. Observe that with this solution, the numbers of variables with zero degree, namely b, x's, $\tilde{x}$'s and z's, are equal on both side of (2.6.20).

As an example, the fundamental representation $[1, 0, \ldots, 0]$ of $SU(2N_f)$ can be decomposed via its character as follows:

$$\begin{aligned}
\chi^{SU(2N_f)}_{[1,0,\ldots,0]}(z_j) &= z_1 + \frac{z_2}{z_1} + \frac{z_3}{z_2} + \ldots + \frac{z_{2N_f-1}}{z_{2N_f-2}} + \frac{1}{z_{2N_f-1}} \\
&= b\left(x_1 + \frac{x_2}{x_1} + \ldots + \frac{x_{N_f-1}}{x_{N_f-2}} + \frac{1}{x_{N_f-1}}\right) + \frac{1}{b}\left(\tilde{x}_{N_f-1} + \frac{\tilde{x}_{N_f-2}}{\tilde{x}_{N_f-1}} + \ldots + \frac{\tilde{x}_1}{\tilde{x}_2} + \frac{1}{\tilde{x}_1}\right) \\
&= b\chi^{SU(N_f)_L}_{[1,0,\ldots,0]}(x_i) + \frac{1}{b}\chi^{SU(N_f)_R}_{[0,\ldots,0,1]}(\tilde{x}_i) \\
&= b\chi^{SU(N_f)_L\times SU(N_f)_R}_{[1,0,\ldots,0;0,\ldots,0]}(x_i) + \frac{1}{b}\chi^{SU(N_f)_L\times SU(N_f)_R}_{[0,\ldots,0;0,\ldots,0,1]}(x_i) \ ,
\end{aligned} \tag{2.6.22}$$

where we have used (2.6.21) to obtain the second equality. Hence, we may write

$$[1,0,\ldots,0]_{SU(2N_f)} \to [1,0,\ldots,0;0,\ldots,0]_{SU(N_f)_L\times SU(N_f)_R}\oplus[0,\ldots,0;0,\ldots,1]_{SU(N_f)_L\times SU(N_f)_R}~. \tag{2.6.23}$$

Let us now decompose the representation $[0,k,0,\ldots,0]_{SU(2N_f)}$. We write down the character $\chi^{SU(2N_f)}_{[0,1,0,\ldots,0]}$ using the formula (2.6.18). Once we substitute z's by x's and $\tilde{x}$'s according to (2.6.21), we obtain the following decomposition:

$$[0,k,0,\ldots,0]_{SU(2N_f)} \to \sum_{n_1=0}^{k} \sum_{n_1+\ell+m=k} [n_1,\ell,0,\ldots,0;0,\ldots 0,m,n_1]_{SU(N_f)_L\times SU(N_f)_R}~. \tag{2.6.24}$$

We can therefore replace t^{2k} by $q^{n_1+2\ell}\tilde{q}^{n_1+2m}$ and rewrite the character expansion in (2.6.4) as follows:

Observation 2.6.4. *The $SU(N_f)_L\times SU(N_f)_R$ character expansion of the generating function $g^{(N_f,N_c=2)}$ is given by*

$$g^{(N_f,N_c=2)}(q,\tilde{q}) = \sum_{n_1,\ell,m\geq 0} [n_1,\ell,0\ldots,0;0,\ldots,0,m,n_1]q^{n_1+2\ell}\tilde{q}^{n_1+2m}~, \tag{2.6.25}$$

where the square bracket denotes the character of the $[n_1,\ell,0\ldots,0;0,\ldots,0,m,n_1]$ representation of $SU(N_f)_L\times SU(N_f)_R$. This equation takes the form of (2.6.1), as expected.

This result is what is expected if we temporarily distinguish quarks from antiquarks in $N_c=2$ theory. The reason is as follows. A meson can either be regarded as an object transforming in the representation $[1,0,\ldots,0;0,\ldots,0,1]$ of $SU(N_f)_L\times SU(N_f)_R$ *or* as an object transforming in the representations $[0,1,0,\ldots,0;0,\ldots,0]$ or $[0,\ldots,0;0,\ldots,0,1,0]$ of $SU(N_f)_L\times SU(N_f)_R$, in which case it can respectively be regarded as a 'baryon' or an 'antibaryon'. As we mentioned in Section 2.3.3, any GIO in the $N_c=2$ theory must be a (symmetric) product of mesons. Therefore, without the constraints generated by (2.3.17), we would say that a GIO transforms in the $SU(N_f)_L\times SU(N_f)_R$ representation

$$[n_1,0,\ldots,0;0,\ldots,0,n_1]\otimes_S\mathrm{Sym}^\ell[0,1,0,\ldots,0;0,\ldots,0]\otimes_S\mathrm{Sym}^m[0,\ldots,0;0,\ldots,0,1,0],$$

for some non-negative integers n_1, ℓ, m. However, as we mentioned in a comment preceding the constraint (2.3.17) that any product of M's antisymmetrised on 3 (or more) flavour indices must vanish, it follows that the result of these symmetric tensor products is an irreducible representation with all the numbers located after the second positions from the left and right being zeros, *i.e.* $[n_1,\ell,0\ldots,0;0,\ldots,0,m,n_1]$, which is in accordance with the result in (2.6.25).

2.6.2 Character Expansion for General (N_f, N_c)

Having revisited two colours let us now study the general case. Armed with an insight from (2.6.25), we now propose the character expansion of the generating function of *any* (N_f, N_c) theory.

Observation 2.6.5. *The character expansion of the generating function of SQCD is:*

$$g^{(N_f,N_c)}(t,\tilde{t}) = \sum_{n_1,n_2,\ldots,n_k,\ell,m\geq 0} [n_1, n_2, \ldots, n_k, \ell_{N_c;L}, 0, \ldots, 0; 0, \ldots, 0, m_{N_c;R}, n_k, \ldots, n_2, n_1] t^a \tilde{t}^b \,. \tag{2.6.26}$$

In the above, $k = N_c - 1$, $a = \ell N_c + \sum_{j=1}^{k} j n_j$ is the number of boxes in the Young diagram for the representation of $SU(N_f)_L$, $b = mN_c + \sum_{j=1}^{k} j n_j$ is the number of boxes in the Young diagram for the representation of $SU(N_f)_R$, and we have again used the notation below Observation 2.3.3 for the representation.

We note that, as in (2.6.25), the asymmetry between ℓ and m arises due to the fact that the baryons and antibaryons transform respectively as $[0, \ldots, 0, 1_{N_c;L}, 0, \ldots, 0; 0, \ldots, 0]$ and $[0, \ldots, 0; 0, \ldots, 0, 1_{N_c;R}, 0, \ldots, 0]$. Moreover, since any product of M's, B's, $\widetilde{B}$'s antisymmetrised on $N_c + 1$ (or more) flavour indices must vanish, it follows that all the numbers located after the N_c-th positions from the left and right are zeros.

The character expansion of the $N_f < N_c$ theory. We mentioned earlier that the meson, which transforms in the bifundamental representation $[1, 0, \ldots, 0; 0, \ldots 1]$ of the global symmetry $SU(N_f)_L \times SU(N_f)_R$, is the only basic generator of the GIOs. It follows that the character expansion of the $N_f < N_c$ theory is encoded in the plethystic exponential:

$$\begin{aligned} g^{N_f<N_c}(t,\tilde{t}) &= \mathrm{PE}\ \left[[1,0,\ldots,0;0,\ldots,1]t\tilde{t}\right] \\ &= \sum_{k=1}^{\infty} \mathrm{Sym}^k[1,0,\ldots,0;0,\ldots,1]\left(t\tilde{t}\right)^k \\ &= \sum_{k=1}^{\infty}\sum_{i=1}^{N_f}\sum_{n_i=0}^{\infty} [n_1,n_2,\ldots,n_{N_f-1};n_{N_f-1},\ldots,n_2,n_1]\,\delta\left(k-\sum_{j=1}^{N_f} j n_j\right)\left(t\tilde{t}\right)^k\ , \\ &= \sum_{i=1}^{N_f}\sum_{n_i=0}^{\infty} [n_1,n_2,\ldots,n_{N_f-1};n_{N_f-1},\ldots,n_2,n_1]\left(t\tilde{t}\right)^{\sum_{j=1}^{N_f} j n_j}\ , \end{aligned} \tag{2.6.27}$$

where the second equality follows from the basic property of the plethystic exponential which produces all possible symmetric products of the function on which it acts, and the third equality follows from (2.3.4).

A non-trivial check of the general character expansion (2.6.26). We note that the dimension of the representation $[a_1, \ldots, a_{n-1}]$ of $SU(n)$ is given by the formula (see, *e.g.*, (15.17) of [41]):

$$\dim\,[a_1, \ldots, a_{n-1}] = \prod_{1 \le i < j \le n} \frac{(a_i + \ldots + a_{j-1}) + j - i}{j - i} \,. \tag{2.6.28}$$

Applying this dimension formula to the representations in (2.6.26) for various (N_f, N_c) and summing the series into closed forms, we obtain the expressions which are in agreement of the earlier results, *e.g.* (2.5.30)–(2.5.34).

As an example, let us consider $(N_f = 5, N_c = 3)$ theory. Using formula (2.6.28), we find that

$$\begin{aligned}
\dim\,[n_1, n_2, \ell, 0; 0, m, n_2, n_1] = {} & [(4!\ 3!\ 2!\ 1!)^{-1} \times \\
& (n_1+1)(n_1+n_2+2)(n_1+n_2+\ell+3)(n_1+n_2+\ell+4) \times \\
& (n_2+1)(n_2+\ell+2)(n_2+\ell+0+3) \times \\
& (\ell+1)(\ell+0+2) \times \\
& (0+1)] \times [\text{the same expression with } \ell \to m] \,.
\end{aligned}$$

Replacing the representation in (2.6.26) with this expression and summing over n_1, n_2, ℓ, m, upon setting $t = \tilde{t}$ we recover the expression for $g^{(5,3)}$ in (2.5.34).

Character Expansion of a Plethystic Logarithm. Using Observations 2.3.3 and 2.3.4, we can write down the character expansion of the first few terms in the plethystic logarithm of the generating function $g^{(N_f,N_c)}$ from (2.6.26) as follows:

$$\begin{aligned}
\mathrm{PL}\left[g^{(N_f,N_c)}(t,\tilde{t})\right] = {} & [1,0,0,0,0;0,0,0,0,1]t\tilde{t} + [0,\ldots,0,1,0,0;0,0,0,0,0]t^{N_c} + \\
& [0,0,0,0,0;0,0,1,0,\ldots,0]\tilde{t}^{N_c} - [0,\ldots,0,1,0;0,0,0,0,1]t^{N_c+1}\tilde{t} - \\
& [1,0,0,0,0;0,1,0,\ldots,0]t\tilde{t}^{N_c+1} - [0,\ldots,0,1,0,0;0,0,1,0,\ldots,0]t^{N_c}\tilde{t}^{N_c} + \ldots \,.
\end{aligned} \tag{2.6.29}$$

where the positions of 1's in the representations of $SU(N_f)_L$ are indicated by the powers of t, and the positions of 1's in the representations of $SU(N_f)_R$ are indicated by the powers of $\tilde{t}$. This expansion coincides precisely with the observations on the generators, relations and their transformation properties under the global symmetry.

Having seen a number of character expansions, we can establish some *selection rules* for the coefficients in the character expansion which will be extremely useful in reducing the work in calculating the character expansion:

Observation 2.6.6. (Selection rules for the coefficients in the character expansion)

1. *Each irreducible representation of the global symmetry appears at most once as a coefficient in the character expansion;*

2. *Each irreducible representation appearing as a coefficient in the character expansion corresponds to a Young tableau, with t and $\tilde{t}$ counting the number of boxes. The chemical potential t counts the number of boxes in the irreducible representation of $SU(N_f)_L$. Similarly, the chemical potential $\tilde{t}$ counts the number of boxes in the irreducible representation of $SU(N_f)_R$;*

3. *Suppose that the coefficient of the term $t^{k_1}\tilde{t}^{k_2}$ is $[a;b]$. Then, the coefficient of the term $t^{k_2}\tilde{t}^{k_1}$ is $[\bar{b};\bar{a}]$, where the bar indicates the complex conjugate representation. For $N_f \leq N_c$ this is correct modulo N_c;*

4. *If the degrees of t and $\tilde{t}$ are equal, then the coefficient of such a term is a real (self-conjugate) representation. In the square bracket notation, the numbers in the bracket are palindromic with respect to the semicolon.*

In the following subsections, we demonstrate the above observations in various examples.

The Example of $(N_f = 2, N_c = 3)$

Recall that the global symmetry of the theory is $SU(2) \times SU(2) = SO(4)$, and the generating function is given in (2.5.31). Treating the fugacities t_1, t_2 as the fugacities for the fundamental representation of the first $SU(2)$ and treating the fugacities $\tilde{t}_1$, $\tilde{t}_2$ as the fugacities for the antifundamental representation (which is identical to the fundamental representation) of the second $SU(2)$, we make the substitutions:

$$t_1 = tz \ , \quad t_2 = \frac{t}{z} \ , \quad \tilde{t}_1 = \tilde{t}w \ , \quad \tilde{t}_2 = \frac{\tilde{t}}{w} \ . \tag{2.6.30}$$

Substituting these into (2.5.31), we find that

$$\begin{aligned} g^{(2,3)}(t,\tilde{t};z,w) &= \frac{1}{\left(1-\frac{t\tilde{t}}{wz}\right)\left(1-\frac{t\tilde{t}w}{z}\right)\left(1-\frac{t\tilde{t}z}{w}\right)\left(1-t\tilde{t}wz\right)} \\ &= \mathrm{PE}\left[\,[1;1]t\tilde{t}\,\right] \end{aligned} \tag{2.6.31}$$

where the second equality follows because the character of the bifundamental representation of $SU(2)\times SU(2)$ (otherwise known as the vector representation of $SO(4)$) is $[1;1] = [1;0][0;1] = (z+1/z)(w+1/w)$. This is in agreement with (2.6.27). Therefore, the character expansion is

$$g^{(2,3)}(t,\tilde{t}) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[n;n]\left(t\tilde{t}\right)^{n+2m} \ . \tag{2.6.32}$$

The Example of $(N_f = 3, N_c = 3)$

The global symmetry of the theory is $SU(3) \times SU(3)$, and the generating function was given in (2.5.32). Treating the fugacities t_1, t_2, t_3 as the fugacities for the fundamental representation of the first $SU(3)$ and treating the fugacities $\tilde{t}_1$, $\tilde{t}_2$, $\tilde{t}_3$ as the fugacities for the antifundamental representation of the second $SU(3)$, we make the substitutions:

$$t_1 = tz_1 \;, \quad t_2 = \frac{z_2}{z_1} t \;, \quad t_3 = \frac{t}{z_2} \;, \quad \tilde{t}_1 = \frac{\tilde{t}}{w_1} \;, \quad \tilde{t}_2 = \frac{w_1}{w_2}\tilde{t} \;, \quad \tilde{t}_3 = \tilde{t} w_2 \;. \tag{2.6.33}$$

Substituting these into (2.5.32), we find that

$$g^{(3,3)}(t,\tilde{t}) = \left(1 - [0,0;0,0]t^3\tilde{t}^3\right) \mathrm{PE}\left[\, [1,0;0,1]t\tilde{t} + [0,0;0,0]t^3 + [0,0;0,0]\tilde{t}^3 \,\right] \tag{2.6.34}$$

where $[1,0;0,1] = [1,0;0,0][0,0;0,1] = \left(z_1 + \frac{z_2}{z_1} + \frac{1}{z_2}\right)\left(\frac{1}{w_1} + \frac{w_1}{w_2} + w_2\right)$ and $[0,0;0,0] = 1$. This result is what to be expected using the comment preceding Observation 2.3.6. Alternatively, we can use Equation (2.6.2) and write the character expansion of $g^{(3,3)}$ as

$$g^{(3,3)}(t,\tilde{t}) = \sum_{n_1,n_2,\ell,m \geq 0} [n_1,n_2;n_2,n_1] t^{n_1+2n_2+3\ell}\, \tilde{t}^{n_1+2n_2+3m} \;. \tag{2.6.35}$$

The Example of $(N_f = 4, N_c = 3)$

The global symmetry here is $SU(4) \times SU(4)$ and the unrefined generating function was given in (2.5.33). Using the same procedure as shown in previous examples, we obtain the generating function $g^{(4,3)}(t,\tilde{t}; z_1, \ldots, z_3, w_1, \ldots, w_3)$ as follows. The numerator can be written as

$$\begin{aligned}
&1 - [1,0,0;0,0,0]\tilde{t}^4 t + [0,1,0;0,0,0]\tilde{t}^8 t^2 - [0,0,1;1,0,0]\tilde{t}^3 t^3 + [0,0,1;0,1,0]\tilde{t}^6 t^3 - \\
&\quad [0,0,1;0,0,1]\tilde{t}^9 t^3 - [0,0,0;0,0,1]\tilde{t}t^4 - [0,0,0;0,1,1]\tilde{t}^7 t^4 + [0,0,0;0,0,2]\tilde{t}^{10} t^4 - \\
&\quad [0,1,1;0,0,0]\tilde{t}^8 t^5 + [0,1,0;1,0,0]\tilde{t}^3 t^6 + [0,1,0;0,0,1]\tilde{t}^9 t^6 + [2,0,0;0,0,0]\tilde{t}^{12} t^6 - \\
&\quad [1,1,0;0,0,0]\tilde{t}^4 t^7 + [0,0,1;1,0,0]\tilde{t}^7 t^7 + [0,0,1;0,1,0]\tilde{t}^{10} t^7 - [0,0,1;0,0,1]\tilde{t}^{13} t^7 + \\
&\quad [0,0,0;0,1,0]\tilde{t}^2 t^8 - [0,0,0;1,1,0]\tilde{t}^5 t^8 + ([0,2,0;0,0,0] + [1,0,1;0,0,0] + \\
&\quad 2[0,0,0;0,0,0] + [0,0,0;1,0,1] + [0,0,0;0,2,0])\tilde{t}^8 t^8 + \\
&\quad \text{c.c./exchange up to } \tilde{t}^{16} t^{16} \;,
\end{aligned} \tag{2.6.36}$$

where 'c.c./exchange' means that the rest of the terms can be obtained by exchanging the representations before and after the semicolon, and/or taking a complex conjugate representation, according to Observation 2.6.6. For example, the coefficient of $\tilde{t}^7 t^{10}$ is $[0,1,0;1,0,0]$, and the coefficient of $\tilde{t}^{12} t^{15}$ is the conjugate representation of that of $\tilde{t}^{16-12} t^{16-15} = \tilde{t}^4 t$: $-[0,0,1;0,0,0]$. The coefficient of $\tilde{t}^8 t^8$ is a real (self-conjugate) representation. The reciprocal of the denominator can be written as

$$\mathrm{PE}\;\left[[1,0,0;0,0,1]t\tilde{t} + [0,0,1;0,0,0]t^3 + [0,0,0;1,0,0]\tilde{t}^3\right] \;. \tag{2.6.37}$$

Alternatively, we can write the character expansion of $g^{(4,3)}$ as follows:

$$g^{(4,3)}(t,\tilde{t}) = \sum_{n_1,n_2,n_3,m_3 \geq 0} [n_1,n_2,n_3;m_3,n_2,n_1] t^{n_1+2n_2+3n_3} \tilde{t}^{n_1+2n_2+3m_3} . \qquad (2.6.38)$$

2.7 A Note on the Quantum Moduli Space of SQCD

In this section, we shall summarise the quantum effects on the vacuum moduli space of SQCD. Excellent reviews collecting this work are [1, 5–7, 33].

The $N_f < N_c$ theories

A non-perturbative Affleck–Dine–Seiberg (ADS) superpotential [1, 5–7, 33], whose form is consistent with symmetries and holomorphy, is dynamically generated:

$$W_{\text{ADS}} = C_{N_c,N_f} \left(\frac{\Lambda^{3N_c - N_f}}{\det M} \right)^{1/(N_c - N_f)} , \qquad (2.7.1)$$

where Λ is the scale of the theory and C_{N_c,N_f} is in general renormalisation scheme-dependent. Because of the dependence of W_{ADS} on meson fields with negative powers, it is never zero, but flows to zero at infinity. Consequently, at any finite values of the meson fields, W_{ADS} is non-zero, and there is no supersymmetric vacuum. Quantum corrections therefore lead to a 'runaway' vacuum.

Although this superpotential is non-polynomial in the quark and antiquark fields, we can still solve for the F-terms and examine the moduli space of solutions, a problem we have adapted to `STRINGVACUA` [11]. As expected, there is no stable vacuum. The classical vacuum is an auxiliary space that allows for the enumeration of GIOs via the Hilbert series. While the classical vacuum variety does not have a physical meaning in the full quantum theory, it nevertheless encapsulates information about the operatorial structure of SQCD for $N_f < N_c$.

The $N_f \geq N_c$ theories

The case of $N_f = N_c$. The moduli space is still parameterised by the basic generators M, B, and $\tilde{B}$. The classical constraint (2.3.14) is however modified by a one instanton effect [1, 5–7], and the quantum moduli space is described by the relation

$$\det(M) - (*B)(*\widetilde{B}) = \Lambda^{2N_c} . \qquad (2.7.2)$$

From the constraint (2.3.14), we see that the classical moduli space is singular at the origin: $M = B = \tilde{B} = 0$. This singularity does not exist in the true vacuum (2.7.2), and so the latter geometry is *everywhere smooth*. Although details of the GIOs and constraints at each order of quarks and antiquarks are modified, their *numbers* are unaffected. Thus, in spite of different geometrical properties between the classical and quantum moduli spaces, the Hilbert series is not corrected quantum mechanically.

The case of $N_f > N_c$. In this case, the quantum moduli space *coincides* with the classical moduli space [1, 7]. Thus, geometric and algebraic features of the classical vacuum variety $\mathcal{M}_{N_f>N_c}$ are also properties of the true vacuum of the theory.

A comment on Seiberg duality. In the conformal window, the convenient description of SQCD may be in terms of dual variables [4]. An early motivation in checking Seiberg duality using Hilbert series is due to Pouliot [24]. Later Römelsberger [25] showed that the Hilbert series of $SU(2)$ SQCD with three flavours and its magnetic dual match. There are, however, no further geometric checks in the literature. It is relatively easy to verify that the dimensions of the electric and magnetic theories agree. Using the Hilbert series to more carefully examine the geometric aspects of Seiberg duality is clearly an interesting problem that deserves investigation in its own right. We leave this to subsequent work.

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Chapter 3

SQCD with Classical Gauge Groups

3.1 Introduction and Summary

As pointed out in Chapter 2, the vacuum moduli space of Supersymmetric Quantum Chromodynamics (SQCD) has a very rich structure from which we can employ various algebraic and geometrical techniques to gain physical insights. The plethystic programme, Molien–Weyl formula and character expansion techniques provide a very satisfactory way in constructing generating functions (Hilbert Series) which solve the complicated problem on counting gauge invariant operators. Having studied the $SU(N_c)$ SQCD in Chapter 2, we extend our work to the other classical gauge groups, namely $SO(N_c)$ and $Sp(N_c)$. Several aspects of the SO and Sp gauge theories, *e.g.* dualities, deconfinement, s-confinement, have been extensively studied in a series of works [2–15].

In this chapter, we focus on $\mathcal{N} = 1$ SQCD with $SO(N_c)$ and $Sp(N_c)$ gauge groups with N_f flavours of quarks transforming, respectively, in the vector and fundamental representations of the gauge group. The global symmetries of the theory are respectively $SU(N_f) \times U(1)_R$ and $SU(2N_f) \times U(1)_R$. We shall concentrate our attention on the case with a vanishing superpotential. The vacuum space is conveniently described by polynomial equations written in terms of variables which are the holomorphic gauge invariant operators (GIOs) of the theory, namely the mesons and baryons for the SO theories, and the mesons for the Sp theories.

Outline and Key Points:

- In Section 3.2, we examine the classical moduli space of $SO(N_c)$ SQCD with N_f flavours. For $N_f < N_c$, the moduli space is $\mathbb{C}^{\frac{1}{2}N_f(N_f+1)}$ (Observation 3.2.1) with the Hilbert Series given by (3.2.4). For $N_f = N_c$, the moduli space is a complete intersection and is, in fact, a single hypersurface in $\mathbb{C}^{\frac{1}{2}N_f(N_f+1)+1}$ (Observation 3.2.2) with the Hilbert Series given by (3.2.10). For $N_f > N_c$, the moduli space is a non-complete intersection of polynomial relations (syzygies) amongst the GIOs. We use the plethystic exponential and Molien–Weyl formula to derive generating functions for various N_f and N_c. We also use the plethystic logarithm to count basic generators of GIOs and basic constraints between them.

- In Section 3.2.4, we synthesise our prior results using representation theory and the character expansion. It proves useful to write the Hilbert series in terms of characters. This permits the generalisation of our results to an *arbitrary number of colours and flavours*. Subsequently, we obtain an important result, namely the full character expansion of the generating function for *any* N_f in an *arbitrary* $SO(N_c)$ theory (Equations (3.2.29), (3.2.30), (3.2.31)).

- In Section 3.3, we investigate the classical moduli space of $Sp(N_c)$ SQCD with N_f flavours. For $N_f \leq N_c$, the moduli space is $\mathbb{C}^{\frac{1}{2}N_f(N_f+1)}$ (Observation 3.3.1) with the Hilbert Series given by (3.3.3). For $N_f = N_c + 1$, the moduli space is a complete intersection and is, in fact, a single hypersurface in $\mathbb{C}^{(2N_c+1)(N_c+1)}$ (Observation 3.3.2) with the Hilbert Series given by (3.3.6). For $N_f > N_c + 1$, the moduli space is a non-complete intersection of syzygies amongst the GIOs. The plethystic exponential and Molien–Weyl formula are used to derive generating functions for various N_f and N_c. We also count basic generators of GIOs and basic constraints using the plethystic logarithm.

- The full character expansion of the generating function for *any* N_f in an *arbitrary* $Sp(N_c)$ theory is given in (3.3.10).

- In Section 3.4, we study how the SO and Sp gauge theories arise from the SU gauge theory due to an orientifold $\mathbb{Z}_2$ action. Without specifying the explicit brane construction, we consider an orientifold projection on the global symmetry, the basic generators, and the basic constraints in the SU theory. We find that the projection occurs in two steps: The antifundamental index is first turned into a fundamental index, and the resulting symmetry then gets respectively symmetrised and antisymmetrised in the SO and Sp theories.

- In Section 3.5, we take a geometric aperçu of the moduli space of SQCD. We establish that the classical moduli space is an irreducible affine Calabi–Yau cone.

3.2 $SO(N_c)$ SQCD with N_f flavours

We specify SQCD with gauge group $SO(N_c)$ and N_f flavours by the ordered pair $(N_f, SO(N_c))$. This theory has quarks Q^i_a, with flavour indices $i = 1, \ldots, N_f$ and colour indices $a = 1, \ldots, N_c$. Thus, there is a total of $N_c N_f$ chiral degrees of freedom from the quarks. Their quantum numbers are summarised in Table 3.1. For reviews on this theory see, *e.g.*, [2–4]. For $N_f < N_c$, the moduli space is $\mathbb{C}^{\frac{1}{2}N_f(N_f+1)}$ (Observation 3.2.1) with the Hilbert Series given by (3.2.4).

	Gauge symmetry	Global symmetry		
	$SO(N_c)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
Q^i_a	$[1, 0, \ldots, 0]$	$[1, 0, \ldots, 0]$	1	$\frac{N_f+2-N_c}{N_f}$

Table 3.1. The gauge and global symmetries of $SO(N_c)$ SQCD with N_f flavours and the quantum numbers of the chiral supermultiplets.

For $N_f \leq N_c - 2$, at a generic point in the classical moduli space, the $SO(N_c)$ gauge symmetry is broken to $SO(N_c - N_f)$. Since the dimension of $SO(N)$ is $\frac{1}{2}N(N-1)$, there are

$$\frac{1}{2}N_c(N_c-1) - \frac{1}{2}(N_c - N_f)(N_c - N_f - 1) = N_c N_f - \frac{1}{2}N_f(N_f+1)$$

broken generators. Therefore, of the original $N_c N_f$ chiral supermultiplets, only

$$N_c N_f - \left[N_c N_f - \frac{1}{2}N_f(N_f+1)\right] = \frac{1}{2}N_f(N_f+1)$$

singlets are left massless. Hence, the dimension of the moduli space of vacua is

$$\dim\left(\mathcal{M}_{N_f \leq N_c - 2}\right) = \frac{1}{2}N_f(N_f+1)\ . \tag{3.2.1}$$

For $N_f \geq N_c - 1$, at a generic point in the moduli space, the $SO(N_c)$ gauge symmetry is broken completely and hence the number of remaining massless chiral supermultiplets (*i.e.* the dimension of the moduli space) is given by

$$\dim\left(\mathcal{M}_{N_f \geq N_c - 1}\right) = N_f N_c - \frac{1}{2}N_c(N_c-1)\ . \tag{3.2.2}$$

According to [2], we see that the 'D-flatness' constraints force us to consider matter field solutions in two cases, namely $N_f < N_c$ and $N_f \geq N_c$. We shall focus on GIOs in each of these cases below.

3.2.1 The Case of $N_f < N_c$

We can describe the $\frac{1}{2}N_f(N_f+1)$ light degrees of freedom in a gauge invariant way by the mesons:

$$M^{ij} = Q^i_a Q^j_b \delta^{ab} \qquad \text{(meson)} \ . \tag{3.2.3}$$

We emphasise that the indices i and j are symmetric. Therefore, the meson transforms in the global $SU(N_f)$ representation $\mathrm{Sym}^2[1,0,\dots,0] = [2,0,\dots,0]$. We note that for the $N_f < N_c$ theory, there are no relations (constraints) between mesons. Phrasing this geometrically, and noting the dimension from (3.2.1), we have that

Observation 3.2.1. *The moduli space $\mathcal{M}_{N_f<N_c}$ is freely generated: there are no relations among the generators. The space $\mathcal{M}_{N_f<N_c}$ is, in fact, nothing but $\mathbb{C}^{\frac{1}{2}N_f(N_f+1)}$.*

Using the plethystic programme [1, 18, 21–24], we can immediately write down the generating function of GIOs for $N_f < N_c$ as[1]

$$g^{N_f<N_c}(t) = \frac{1}{(1-t^2)^{\frac{1}{2}N_f(N_f+1)}} \ . \tag{3.2.4}$$

where t is a fugacity which can be taken to be conjugate to the R-charge. We emphasise that this formula does not depend on the number of colours N_c. This expression is simply the Hilbert series for $\mathbb{C}^{\frac{1}{2}N_f(N_f+1)}$, with weight 2 for each meson.

3.2.2 The Case of $N_f \geq N_c$

We can describe the light degrees of freedom in a gauge invariant way by the following basic generators:

$$\begin{aligned} M^{ij} &= Q^i_a Q^j_b \delta^{ab} && \text{(mesons)} \ ; \\ B^{i_1\dots i_{N_c}} &= Q^{i_1}_{a_1} \dots Q^{i_{N_c}}_{a_{N_c}} \epsilon^{a_1\dots a_{N_c}} && \text{(baryons)} \ . \end{aligned} \tag{3.2.5}$$

For $N_f \geq N_c$, under the global $SU(N_f)$ symmetry, the mesons M transform in the $[2,0,\dots,0]$ representation and the baryons B transform respectively in $[0,0,\dots,1_{N_c;L},0,\dots,0]$, where $1_{j;L}$ denotes a 1 in the j-th position from the left. The dimensions of these representations are respectively $\frac{1}{2}N_f(N_f+1)$ and $\binom{N_f}{N_c}$.

We emphasise that the basic generators in (3.2.5) are not independent, but they are subject to the following constraints. Any product of M's and B's antisymmetrised on N_c+1 (or more) upper or lower flavour indices must vanish:

$$M \cdot *B = 0 \ , \tag{3.2.6}$$

[1] This expression can be written in terms of the plethystic exponential, which is defined in (3.2.11), as $\mathrm{PE}\left[t^2 \dim\,[2,0,\dots,0]\right] = \mathrm{PE}\left[\frac{1}{2}N_f(N_f+1)t^2\right]$.

where $(*B)_{i_{N_c+1}\dots i_{N_f}} = \frac{1}{N_c!}\epsilon_{i_1\dots i_{N_f}} B^{i_1\dots i_{N_c}}$ and a '$\cdot$' denotes a contraction of an upper with a lower flavour index. We note that this constraint transform in the global $SU(N_f)$ representation

$$[1, 0, ..., 0, 1_{N_c+1;L}, 0, ..., 0] \ .$$

Another constraint follows from the facts that the rank of the meson M is N_c and that the product of two epsilon tensors can be written as the antisymmetrised sum of Kronecker deltas:

$$B^{i_1\dots i_{N_c}} B_{j_1\dots j_{N_c}} = M^{[i_1}_{j_1} \dots M^{j_{N_c}]}_{j_{N_c}} \ , \qquad (3.2.7)$$

We note that this constraint transforms in the global $SU(N_f)$ representation

$$\mathrm{Sym}^2[0, \dots, 0, 1_{N_c;L}, 0, \dots, 0] \ .$$

Because of these constraints, the spaces $\mathcal{M}_{N_f \geq N_c}$ are not freely generated. Moreover, they also prevent us from writing a generating function as directly as in (3.2.4). Nevertheless, we will see that the Molien–Weyl formula gives us the right answer.

The Case of $N_f = N_c$

The special case of $N_f = N_c$ deserves some special attention. The total number of basic generators for the GIOs, coming from the two contributions in (3.2.5), is $\frac{1}{2}N_f(N_f + 1) + 1$. From (3.2.2), the dimension of the moduli space is

$$\dim\left(\mathcal{M}_{N_f=N_c}\right) = \frac{1}{2}N_f(N_f + 1) \ . \qquad (3.2.8)$$

There is one constraint (3.2.7), which in this case can be reduced to a single hypersurface:

$$\mathcal{B}^2 = \det(M) \ . \qquad (3.2.9)$$

This constraint transforms in the trivial representation $[0, \dots, 0]$ of the global symmetry $SU(N_f)$ (as the length of the weight is the rank of $SU(N_f)$ or $N_f - 1$, there are no 1's). Note that the relation (3.2.6) does not provide any additional information and (3.2.7) constitutes the only constraint. Since, in this case, the dimension of the moduli space equals the number of the basic generators minus the number of constraints, we arrive at another important conclusion:

Observation 3.2.2. *The moduli space $\mathcal{M}_{N_f=N_c}$ is a complete intersection. It is in fact a single hypersurface in $\mathbb{C}^{\frac{1}{2}N_f(N_f+1)+1}$.*

An interesting question to consider is to determine the number of independent GIOs that can be constructed from the basic generators (3.2.5) subject to the constraints (3.2.6) and (3.2.7). In the case $N_f = N_c$, where the only constraint is (3.2.9), the generating function can be easily computed from the knowledge that the modul space is a complete intersection (See [18] for a detailed discussion on this). There are $\frac{1}{2}N_f(N_f+1) = \frac{1}{2}N_c(N_c+1)$ mesonic generators of weight t^2 and one baryonic generator of weight t^{N_c}, subject to a relation of weight t^{2N_c}. As a result, the generating function takes the form[2]

$$g^{N_f=N_c}(t) = \frac{1-t^{2N_c}}{(1-t^2)^{\frac{1}{2}N_c(N_c+1)}(1-t^{N_c})} = \frac{1+t^{N_c}}{(1-t^2)^{\frac{1}{2}N_c(N_c+1)}} \ . \qquad (3.2.10)$$

This is indeed the Hilbert series of the hypersurface (3.2.9).

3.2.3 Counting Gauge Invariants: the Plethystic Exponential and Molien–Weyl formula

Since a special orthogonal group falls into one of the two categories of the classical groups, namely $B_n = SO(2n+1)$ and $D_n = SO(2n)$, we use this notation throughout the section unless indicated otherwise. We note that the Lie algebras of B_n and D_n both have the same rank n.

To write down explicit formulae and for performing computations we need to introduce weights for the different elements in the maximal torus of the different groups. We use z_a (where a runs over $1, \ldots, n$) for colour weights and t_i (where $i = 1, \ldots, N_f$) for flavour weights. These weights have the interpretation of fugacities for the charges they count and the characters of the representations are functions of these variables. Correspondingly, the character for a quark is $\chi^{SU(N_f)\times B_n}_{[1,0,\ldots,0;1,0,\ldots,0]}(t_\alpha, z_a)$ or $\chi^{SU(N_f)\times D_n}_{[1,0,\ldots,0;1,0,\ldots,0]}(t_\alpha, z_a)$ (where $\alpha = 1, \ldots, N_f-1$) depending on which gauge group we are dealing with. We further introduce a fugacity which counts the number of quarks, t.

The method of computing the Hilbert series is similar to that discussed in Chapter 2. For the sake of completeness and easiness of reading, we go through the computation in details here. We construct chiral GIOs by first taking *symmetric* products of quarks, which transform in the bifundamental $[1,0,\ldots,0;1,0,\ldots,0]$ of $SU(N_f)\times SO(N_c)$. Recall from Chapter 2 and [1, 6, 18, 21–23, 25, 30–32] that a convenient combinatorial tool which constructs symmetric products of representations is the *plethystic exponential*, which is a generator for *symmetrisation*. To briefly remind the reader, the plethystic exponential, PE, of a function $g(t_1, \ldots, t_n)$ is defined to be

[2]This expression can be written in terms of the plethystic exponential, which is defined in (3.2.11), as $(1-t^{2N_c})\,\mathrm{PE}\left[t^2 \dim[2,0,\ldots,0] + t^{N_c} \dim[0,\ldots,0]\right] = (1-t^{2N_c})\,\mathrm{PE}\left[\frac{1}{2}N_c(N_c+1)t^2 + t^{N_c}\right]$

$\exp\left(\sum_{k=1}^{\infty}\frac{g(t_1^k,\ldots,t_n^k)}{k}\right)$. Whence, we have that

$$\mathrm{PE}\left[t\chi^{SU(N_f)\times B_n,D_n}_{[1,0,\ldots,0;1,0,\ldots,0]}(t_\alpha,z_a)\right]\equiv\exp\left[\sum_{k=0}^{\infty}\frac{1}{k}\left(t^k\chi^{SU(N_f)\times B_n,D_n}_{[1,0,\ldots,0;1,0,\ldots,0]}(t_\alpha^k,z_a^k)\right)\right]\ . \tag{3.2.11}$$

A somewhat more explicit form for the character can be

$$t\chi^{SU(N_f)\times B_n,D_n}_{[1,0,\ldots,0;1,0,\ldots,0]}(t_\alpha,z_a)=\chi^{B_n,D_n}_{[1,0,\ldots,0]}(z_a)\sum_{i=1}^{N_f}t_i\ , \tag{3.2.12}$$

where $\alpha=1,\ldots,N_f-1$ and $a=1,\ldots n$. Combining (3.2.11) with (3.2.12), we obtain

$$\mathrm{PE}\left[\chi^{B_n,D_n}_{[1,0,\ldots,0]}(z_a)\sum_{i=1}^{N_f}t_i\right]=\exp\left[\sum_{k=0}^{\infty}\frac{1}{k}\left(\chi^{B_n,D_n}_{[1,0,\ldots,0]}(z_a^k)\sum_{i=1}^{N_f}t_i^k\right)\right]\ . \tag{3.2.13}$$

Here, the dummy variables t_i are the fugacities associated to quarks counting the $U(1)$-charges in the maximal torus of the global symmetry. Henceforth, we shall take their values to be such that $|t_i|<1$ for all i.

Having obtained symmetric products of quarks using the plethystic exponential, we remind the reader that our aim is to obtain the generating function that counts *gauge invariant* quantities. Therefore, the next step is to project the representations of the gauge group generated by the plethystic exponential onto the trivial subrepresentation, which consists of the quantities *invariant* under the action of the gauge group. Using knowledge from representation theory, this can be done by integrating over the whole group (see, *e.g.*, [26–29]). Hence, the generating functions for the (N_f,B_n) and (N_f,D_n) theories are given by

$$g^{(N_f,B_n)},g^{(N_f,D_n)}=\int_{B_n,D_n}\mathrm{d}\mu_{B_n,D_n}\,\mathrm{PE}\left[\chi^{B_n,D_n}_{[1,0,\ldots,0]}(z_a)\sum_{i=1}^{N_f}t_i\right]\ . \tag{3.2.14}$$

These formulae are the *Molien–Weyl formulae*.

Let us write the above generating function in a ready-to-calculate form. For each category, we take a basis for the dual space of the Cartan subalgebra to be $\{L_a\}_{a=1}^n$. A convenient choice which we shall adopt is $L_a=(0,\ldots,0,1_{a;L},0,\ldots,0)$, where the length of the tuple is n. The weights of the fundamental representations of B_n and D_n are respectively $\{0,\pm L_a\}$ and $\{\pm L_a\}$. With this choice, we can write down the characters of the fundamental representations of B_n and D_n respectively as

$$\begin{aligned}\chi^{B_n}_{[1,0,\ldots,0]}(z_a)&=1+\sum_{a=1}^{n}\left(z_a+\frac{1}{z_a}\right)\ ,\\ \chi^{D_n}_{[1,0,\ldots,0]}(z_a)&=\sum_{a=1}^{n}\left(z_a+\frac{1}{z_a}\right)\ .\end{aligned} \tag{3.2.15}$$

Using formula (3.2.13) and the expansion $-\log(1-x) = \sum_{k=1}^{\infty} x^k/k$, we can write the plethystic exponential as

$$
\begin{aligned}
\mathrm{PE}\left[\chi^{B_n}_{[1,0,\dots,0]}(z_l)\sum_{i=1}^{N_f} t_i\right] &= \frac{1}{\prod_{i=1}^{N_f}\prod_{a=1}^{n}(1-t_i)(1-t_i z_a)\left(1-\frac{t_i}{z_a}\right)} ~, \\
\mathrm{PE}\left[\chi^{D_n}_{[1,0,\dots,0]}(z_l)\sum_{i=1}^{N_f} t_i\right] &= \frac{1}{\prod_{i=1}^{N_f}\prod_{a=1}^{n}(1-t_i z_a)\left(1-\frac{t_i}{z_a}\right)} ~. \qquad (3.2.16)
\end{aligned}
$$

The roots of the Lie algebras of B_n and D_n are respectively $\{\pm L_a \pm L_b, \pm L_a\}$ (with $a \neq b$) and $\{\pm L_a \pm L_b\}$ (with $a \neq b$). Haar measures of special orthogonal groups can be written explicitly using Weyl's integration formula (see, *e.g.*, Section 26.2 of [19]):

$$
\begin{aligned}
\int_{B_n} \mathrm{d}\mu_{B_n} &= \frac{1}{(2\pi i)^n n! 2^n} \oint_{|z_1|=1} \dots \oint_{|z_n|=1} \frac{\mathrm{d}z_1}{z_1}\dots\frac{\mathrm{d}z_n}{z_n} \prod_{\{\alpha\}}\left(1-\prod_{l=1}^{n} z_l^{\alpha_l}\right) ~, \\
\int_{D_n} \mathrm{d}\mu_{D_n} &= \frac{1}{(2\pi i)^n n! 2^{n-1}} \oint_{|z_1|=1} \dots \oint_{|z_n|=1} \frac{\mathrm{d}z_1}{z_1}\dots\frac{\mathrm{d}z_n}{z_n} \prod_{\{\beta\}}\left(1-\prod_{l=1}^{n} z_l^{\beta_l}\right) , \qquad (3.2.17)
\end{aligned}
$$

where α and β are respectively roots of B_n and D_n, and the notation α_l denotes the number in the l-th position of the root $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and similarly for β. For reference, we shall give explicit examples for small values of N_c:

$$
\begin{aligned}
\int_{SO(3)} \mathrm{d}\mu_{SO(3)} &= \frac{1}{2\times 2\pi i}\oint_{|z|=1}\frac{\mathrm{d}z}{z}\left(1-\frac{1}{z}\right)(1-z) ~, \\
\int_{SO(4)} \mathrm{d}\mu_{SO(4)} &= \frac{1}{4\times (2\pi i)^2}\oint_{|z_1|=1}\frac{\mathrm{d}z_1}{z_1}\oint_{|z_2|=1}\frac{\mathrm{d}z_2}{z_2}\left(1-\frac{z_1}{z_2}\right)\left(1-\frac{z_2}{z_1}\right)\left(1-\frac{1}{z_1 z_2}\right)(1-z_1 z_2) ~, \\
\int_{SO(5)} \mathrm{d}\mu_{SO(5)} &= \frac{1}{8\times (2\pi i)^2}\oint_{|z_1|=1}\frac{\mathrm{d}z_1}{z_1}\oint_{|z_2|=1}\frac{\mathrm{d}z_2}{z_2}\left(1-\frac{1}{z_1}\right)(1-z_1)\left(1-\frac{1}{z_2}\right)(1-z_2)\times \\
&\quad \left(1-\frac{z_1}{z_2}\right)\left(1-\frac{z_2}{z_1}\right)\left(1-\frac{1}{z_1 z_2}\right)(1-z_1 z_2) ~. \qquad (3.2.18)
\end{aligned}
$$

As an example, we shall demonstrate how to calculate the generating function $g^{(3,SO(3))}$. Putting the above together, we find that

$$
g^{(3,SO(3))}(t_1,t_2,t_3) = \frac{1}{2\times 2\pi i}\oint_{|z|=1}\frac{\mathrm{d}z}{z}\frac{\left(1-\frac{1}{z}\right)(1-z)}{\prod_{i=1}^{3}\left[(1-t_i)(1-t_i z)\left(1-\frac{t_i}{z}\right)\right]} \qquad (3.2.19)
$$

Using the residue theorem with the poles $z = 0,\ t_1,\ t_2,\ t_3$ within the unit circle, we find that

$$
g^{(3,SO(3))}(t_1,t_2,t_3) = \frac{1-t_1^2\,t_2^2\,t_3^2}{(1-t_1 t_2 t_3)\prod_{1\le i\le j\le 3}(1-t_i t_j)} ~. \qquad (3.2.20)
$$

Note that upon setting $t_1 = t_2 = t_3 = t$, we recover formula (3.2.10) with $N_c = 3$. The latter is called the *unrefined* generating function. It suffices for our purposes to compute *unrefined* generating functions (*i.e.* setting all $t_i = t$). Results are listed below:

The case of $N_c = 3$. Let us compute unrefined generating functions for the $N_c = 3$ theory. We have

$$g^{(N_f,SO(3))}(t) = \frac{1}{2\times 2\pi i}\oint_{|z|=1}\frac{\mathrm{d}z}{z}\frac{\left(1-\frac{1}{z}\right)(1-z)}{\left[(1-t)(1-tz)\left(1-\frac{t}{z}\right)\right]^{N_f}}. \tag{3.2.21}$$

Using the residue theorem with the poles at $z = 0$ and $z = t$, we find the following generating functions:

$$\begin{aligned}
g^{(1,SO(3))}(t) &= \frac{1}{1-t^2} = 1+t^2+t^4+t^6+t^8+t^{10}+\dots\ ,\\
g^{(2,SO(3))}(t) &= \frac{1}{(1-t^2)^3} = 1+3t^2+6t^4+10t^6+15t^8+21t^{10}+\dots\ ,\\
g^{(3,SO(3))}(t) &= \frac{1-t^6}{(1-t^2)^6(1-t^3)} = \frac{1+t^3}{(1-t^2)^6}\\
&= 1+6t^2+t^3+21t^4+6t^5+56t^6+21t^7+126t^8+56t^9+252t^{10}+\dots\ ,\\
g^{(4,SO(3))}(t) &= \frac{1+t^2+4t^3+t^4+t^6}{(1-t^2)^9}\\
&= 1+10t^2+4t^3+55t^4+36t^5+220t^6+180t^7+714t^8+660t^9+1992t^{10}+\dots\ ,\\
g^{(5,SO(3))}(t) &= \frac{1+3t^2+10t^3+6t^4+6t^5+10t^6+3t^7+t^9}{(1-t^2)^{12}}\\
&= 1+15t^2+10t^3+120t^4+126t^5+680t^6+855t^7+3045t^8+\dots\ ,\\
g^{(6,SO(3))}(t) &= \frac{1+6t^2+20t^3+21t^4+36t^5+56t^6+36t^7+21t^8+20t^9+6t^{10}+t^{12}}{(1-t^2)^{15}}\\
&= 1+21t^2+20t^3+231t^4+336t^5+1771t^6+2976t^7+10521t^8+\dots\ ,\\
g^{(7,SO(3))}(t) &= 1+28t^2+35t^3+406t^4+756t^5+4060t^6+8478t^7+30975t^8+\dots\ .
\end{aligned} \tag{3.2.22}$$

The case of $N_c = 4$. Let us compute generating functions for the $N_c = 4$ theory:

$$g^{(N_f,SO(4))}(t) = \frac{1}{4\times(2\pi i)^2}\oint_{|z_1|=1}\frac{\mathrm{d}z_1}{z_1}\oint_{|z_2|=1}\frac{\mathrm{d}z_2}{z_2}\frac{\left(1-\frac{z_1}{z_2}\right)\left(1-\frac{z_2}{z_1}\right)\left(1-\frac{1}{z_1z_2}\right)(1-z_1z_2)}{\left[(1-tz_1)\left(1-\frac{t}{z_1}\right)(1-tz_2)\left(1-\frac{t}{z_2}\right)\right]^{N_f}}. \tag{3.2.23}$$

Integrating along the contour $|z_2| = 1$ enclosing the poles $z_2 = 0,\ t$ and then along the contour $|z_1| = 1$ enclosing the poles $z_1 = 0,\ t$, we find that

$$\begin{aligned}
g^{(1,SO(4))}(t) &= \frac{1}{1-t^2} = 1 + t^2 + t^4 + t^6 + t^8 + t^{10} + \dots\ , \\
g^{(2,SO(4))}(t) &= \frac{1}{(1-t^2)^3} = 1 + 3t^2 + 6t^4 + 10t^6 + 15t^8 + 21t^{10} + \dots\ , \\
g^{(3,SO(4))}(t) &= \frac{1}{(1-t^2)^6} = 1 + 6t^2 + 21t^4 + 56t^6 + 126t^8 + 252t^{10} + \dots\ , \\
g^{(4,SO(4))}(t) &= \frac{1-t^8}{(1-t^2)^{10}(1-t^4)} = \frac{1+t^4}{(1-t^2)^{10}} \\
&= 1 + 10t^2 + 56t^4 + 230t^6 + 770t^8 + 2222t^{10} + \dots\ , \\
g^{(5,SO(4))}(t) &= \frac{1+t^2+6t^4+t^6+t^8}{(1-t^2)^{14}} \\
&= 1 + 15t^2 + 125t^4 + 750t^6 + 3585t^8 + 14427t^{10} + \dots\ , \\
g^{(6,SO(4))}(t) &= \frac{1+3t^2+21t^4+20t^6+21t^8+3t^{10}+t^{12}}{(1-t^2)^{18}} \\
&= 1 + 21t^2 + 246t^4 + 2051t^6 + 13377t^8 + 72030t^{10} + \dots\ , \\
g^{(7,SO(4))}(t) &= \frac{1+6t^2+56t^4+126t^6+210t^8+126t^{10}+56t^{12}+6t^{14}+t^{16}}{(1-t^2)^{22}} \\
&= 1 + 28t^2 + 441t^4 + 4900t^6 + 41944t^8 + 291648t^{10} + \dots\ , \\
g^{(8,SO(4))}(t) &= \frac{1+10t^2+125t^4+500t^6+1310t^8+1652t^{10}+1310t^{12}+500t^{14}+125t^{16}+10t^{18}+t^{20}}{(1-t^2)^{26}} \\
&= 1 + 36t^2 + 736t^4 + 10536t^6 + 114696t^8 + 1000728t^{10} + \dots\ .
\end{aligned} \tag{3.2.24}$$

The case of $N_c = 5$. Finally, let us compute generating functions for the $N_c = 5$ theory:

$$\begin{aligned}
g^{(N_f,SO(5))}(t) &= \frac{1}{8\times(2\pi i)^2}\oint_{|z_1|=1}\frac{\mathrm{d}z_1}{z_1}\oint_{|z_2|=1}\frac{\mathrm{d}z_2}{z_2}\times \\
&\frac{\left(1-\frac{1}{z_1}\right)(1-z_1)\left(1-\frac{1}{z_2}\right)(1-z_2)\left(1-\frac{z_1}{z_2}\right)\left(1-\frac{z_2}{z_1}\right)\left(1-\frac{1}{z_1 z_2}\right)(1-z_1 z_2)}{\left[(1-t)\left(1-\frac{t}{z_1}\right)(1-tz_1)\left(1-\frac{t}{z_2}\right)(1-tz_2)\right]^{N_f}}\ .
\end{aligned} \tag{3.2.25}$$

As before, we obtain generating functions as follows:

$$
\begin{aligned}
g^{(1,SO(5))}(t) &= \frac{1}{1-t^2} = 1+t^2+t^4+t^6+t^8+t^{10}+\dots\ , \\
g^{(2,SO(5))}(t) &= \frac{1}{(1-t^2)^3} = 1+3t^2+6t^4+10t^6+15t^8+21t^{10}+\dots\ , \\
g^{(3,SO(5))}(t) &= \frac{1}{(1-t^2)^6} = 1+6t^2+21t^4+56t^6+126t^8+252t^{10}+\dots\ , \\
g^{(4,SO(5))}(t) &= \frac{1}{(1-t^2)^{10}} = 1+10t^2+55t^4+220t^6+715t^8+2002t^{10}+\dots\ , \\
g^{(5,SO(5))}(t) &= \frac{1-t^{10}}{(1-t^2)^{15}(1-t^5)} = \frac{1+t^5}{(1-t^2)^{15}} \\
&= 1+15t^2+120t^4+t^5+680t^6+15t^7+3060t^8+120t^9+11628t^{10}+\dots\ , \\
g^{(6,SO(5))}(t) &= \frac{1+t^2+t^4+6t^5+t^6+t^8+t^{10}}{(1-t^2)^{20}} \\
&= 1+21t^2+231t^4+6t^5+1771t^6+120t^7+10626t^8+1260t^9+53130t^{10}+\dots\ , \\
g^{(7,SO(5))}(t) &= \frac{1+3t^2+6t^4+21t^5+10t^6+15t^7+15t^8+10t^9+21t^{10}+6t^{11}+3t^{13}+t^{15}}{(1-t^2)^{25}} \\
&= 1+28t^2+406t^4+21t^5+4060t^6+540t^7+31465t^8+7210t^9+201376t^{10}+\dots\ , \\
g^{(8,SO(5))}(t) &= 1+36t^2+666t^4+56t^5+8436t^6+1800t^7+82251t^8+29800t^9+658008t^{10}+\dots\ , \\
g^{(9,SO(5))}(t) &= 1+45t^2+1035t^4+126t^5+16215t^6+4950t^7+194580t^8+99550t^9+\dots\ .
\end{aligned}
\tag{3.2.26}
$$

From the above examples, it is amusing to observe that

Observation 3.2.3. *The generating functions for the $N_f \geq N_c$ theory can be written as*

$$
g^{N_f \geq N_c}(t) = \frac{P_k(t)}{(1-t^2)^{\dim \mathcal{M}_{(N_f, SO(N_c))}}}\ ,
$$

where $P_k(t)$ is a degree k polynomial such that $P_k(1) \neq 0$ and $\dim \mathcal{M}_{(N_f,SO(N_c))} - k$ is a constant for a given N_c. This constant, which is $\frac{1}{2}N_c(N_c-1) = \binom{N_c}{2}$, can be computed from the case of $N_f = N_c$, where the moduli space is a complete intersection, using (3.2.10).

As we have seen several examples in Chapter 2, this observation also applies for SQCD with $SU(2)$ gauge group. Later we shall establish a similar observation for SQCD with Sp gauge group.

3.2.4 Character Expansions and Global Symmetries

In the previous section, we have obtained the generating functions analytically for various $(N_f, SO(N_c))$ theories. As we mentioned earlier, the coefficients of t^k in

$g^{(N_f,SO(N_c))}(t)$ is the number of independent GIOs at the k-th order of quarks. We shall see in this section that this number is in fact *the dimension of some representation of the global symmetry* at that order. Moreover, we shall see that the character expansion allows us to write down the generating function for *any* $(N_f, SO(N_c))$ theory in a very compact and enlightening way.

The $N_f < N_c$ theories. Let us take the simplest example: $N_f < N_c$. From (3.2.4), we see that the character expansion for the case of $N_f < N_c$ is

$$g^{N_f<N_c}(t; z_1, \ldots, z_{N_f-1}) = \mathrm{PE}\left[t^2[2,0,\ldots,0]\right] = \sum_{k=0}^{\infty} \mathrm{Sym}^k[2,0,\ldots,0]t^{2k} \ , \quad (3.2.27)$$

where the second equality follows from the basic property of the plethystic exponential which produces all possible symmetric products of the function on which it acts. We emphasise that we use the fully refined generating function which is a function of N_f variables, and so this expression depends on N_f variables, not just one variable t. We note the identity (c.f. (2.3.4) in Chapter 2)

$$\mathrm{Sym}^k[2,0,\ldots,0] = \sum_{n_1,\ldots,n_{N_f}\geq 0} [2n_1, 2n_2, \ldots, 2n_{N_f-1}]\ \delta\left(k - \sum_{j=1}^{N_f} jn_j\right) \ . \quad (3.2.28)$$

Therefore, we have the character expansion

$$g^{N_f<N_c}(t; z_1, \ldots, z_{N_f-1}) = \sum_{n_1,\ldots,n_{N_f}\geq 0} [2n_1, 2n_2, \ldots, 2n_{N_f-1}]\ t^{2a} \ , \quad (3.2.29)$$

where $a = \sum_{j=1}^{N_f} jn_j$.

Character expansion of an arbitrary $(N_f, SO(N_c))$ theory. From (3.2.27), we see that the basic building block of the GIOs in the SO theory is the irreducible representation with 2 Young boxes which are symmetrised. Any other irreducible representation is obtained by repeating this basic building block and, in fact, each of such an irreducible representation appears precisely once in the character expansion. However, when baryons get involved in the theory, this observation is slightly modified. We propose selection rules for the coefficients of the character expansion of $g^{(N_f,SO(N_c))}$ as follows:

- For $N_f > N_c$, the numbers located after the N_c-th from the left are zeros, since any product of M's and B's antisymmetrised on $N_c + 1$ (or more) upper or lower avour indices must vanish;
- The numbers located in the $1^{\mathrm{st}}, 2^{\mathrm{nd}}, \ldots, (N_c - 1)$-th postitions from the left are even, whereas the number in the N_c-th position can be either even or odd. The latter is due to the fact that the baryon transforms in the representation $[0, 0, \ldots, 1_{N_c;L}, 0, \ldots, 0]$ of the $SU(N_f)$ global symmetry.

Hence, the character expansion of the generating function of any (N_f, N_c) theory is

$$g^{(N_f>N_c, SO(N_c))}(t; z_1, \ldots, z_{N_f-1}) = \sum_{n_1,\ldots,n_{N_c}\geq 0} [2n_1, 2n_2, \ldots, 2n_{N_c-1}, n_{N_c}, 0, \ldots, 0]\; t^b \;, \tag{3.2.30}$$

where $b = 2\sum_{j=1}^{N_c-1} jn_j + n_{N_c}N_c$. For $N_f = N_c$, this formula goes through and has the form

$$g^{N_f=N_c}(t; z_1, \ldots, z_{N_c-1}) = \sum_{n_1,\ldots,n_{N_c}\geq 0} [2n_1, 2n_2, \ldots, 2n_{N_c-1}]\; t^b \;. \tag{3.2.31}$$

A non-trivial check of the general character expansion (3.2.30). We note that the dimension of the representation $[a_1, \ldots, a_{n-1}]$ of $SU(n)$ is given by the formula (see, *e.g.*, (15.17) of [19]):

$$\dim\,[a_1, \ldots, a_{n-1}] = \prod_{1\leq i<j\leq n} \frac{(a_i + \ldots + a_{j-1}) + j - i}{j-i} \;. \tag{3.2.32}$$

Applying this dimension formula to the representations in (3.2.30) for various (N_f, N_c) and summing the series into closed forms, we obtain the expressions which are in agreement of the earlier results.

As an example for the B_n gauge group, let us consider $(N_f = 5, N_c = 3)$ theory. Using formula (3.2.32), we find that

$$\begin{aligned}\dim\,[2n_1, 2n_2, n_3, 0] = {} & \tfrac{1}{4!\;3!\;2!\;1!}\times \\ & (2n_1{+}1)(2n_1{+}2n_2{+}2)(2n_1{+}2n_2{+}n_3{+}3)(2n_1{+}2n_2{+}n_3{+}0{+}4)\times \\ & (2n_2{+}1)(2n_2{+}n_3{+}2)(2n_2{+}n_3{+}0{+}3)\times \\ & (n_3{+}1)(n_3{+}0{+}2)\times \\ & (0{+}1)\;.\end{aligned}$$

Replacing the representation in (3.2.30) with this expression and summing over n_1, n_2, n_3, n_4, we recover the expression

$$g^{(5,SO(3))}(t) = \frac{1 - 3t + 9t^2 - 9t^3 + 9t^4 - 3t^5 + t^6}{(1-t)^{12}(1+t)^9} \;.$$

As an example for the D_n gauge group, let us consider $(N_f = 5, N_c = 4)$ theory. We have the dimension formula

$$\begin{aligned}\dim\,[2n_1, 2n_2, 2n_3, n_4] = {} & \tfrac{1}{4!\;3!\;2!\;1!}\times \\ & (2n_1{+}1)(2n_1{+}2n_2{+}2)(2n_1{+}2n_2{+}2n_3{+}3)(2n_1{+}2n_2{+}n_3{+}n_4{+}4)\times \\ & (2n_2{+}1)(2n_2{+}2n_3{+}2)(2n_2{+}2n_3{+}n_4{+}3)\times \\ & (2n_3{+}1)(2n_3{+}n_4{+}2)\times \\ & (n_4{+}1)\;.\end{aligned}$$

Replacing the representation in (3.2.30) with this expression and summing over n_1, n_2, n_3, n_4, we recover the expression

$$g^{(5,SO(4))}(t) = \frac{1 + t^2 + 6t^4 + t^6 + t^8}{(1-t^2)^{14}} \;.$$

3.2.5 Counting Basic Generators of Gauge Invariants and Syzygies: the Plethystic Logarithm

We will use the **plethystic logarithm** to deduce the number of generators and constraints at each order of quarks and antiquarks from the generating function [21, 22]. We recall the expression for the plethystic logarithm, PL, the inverse function to PE, is

$$\mathrm{PL}[g^{(N_f,SO(N_c))}(t_1,\dots t_{N_f})] = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log(g^{(N_f,SO(N_c))}(t_1^k,\dots,t_{N_f}^k)) \ , \tag{3.2.33}$$

where $\mu(k)$ is the Möbius function. The significance of the series expansion of the plethystic logarithm is stated in [21, 22]: *the first terms with plus sign give the basic generators while the first terms with the minus sign give the constraints between these basic generators.* If the formula (3.2.33) is an infinite series of terms with plus and minus signs, then the moduli space is not a complete intersection and the constraints in the chiral ring are not trivially generated by relations between the basic generators, but receive stepwise corrections at higher degree. These are the so-called **higher syzygies**. We shall demonstrate these facts below.

The case of $N_c = 3$.

$$\begin{aligned}
\mathrm{PL}[g^{(2,SO(3))}(t)] &= 3t^2 \ , \\
\mathrm{PL}[g^{(3,SO(3))}(t)] &= 6t^2 + t^3 - t^6 \ , \\
\mathrm{PL}[g^{(4,SO(3))}(t)] &= 10t^2 + 4t^3 - 4t^5 - 10t^6 + 15t^8 + 20t^9 + \dots \ , \\
\mathrm{PL}[g^{(5,SO(3))}(t)] &= 15t^2 + 10t^3 - 24t^5 - 55t^6 + 15t^7 + 225t^8 + 330t^9 + \dots \ , \\
\mathrm{PL}[g^{(6,SO(3))}(t)] &= 21t^2 + 20t^3 - 84t^5 - 210t^6 + 120t^7 + 1575t^8 + 2604t^9 + \dots \ , \\
\mathrm{PL}[g^{(7,SO(3))}(t)] &= 28t^2 + 35t^3 - 224t^5 - 630t^6 + 540t^7 + 7350t^8 + 13720t^9 + \dots \ .
\end{aligned}$$

The case of $N_c = 4$.

$$\begin{aligned}
\mathrm{PL}[g^{(2,SO(4))}(t)] &= 3t^2 \ , \\
\mathrm{PL}[g^{(3,SO(4))}(t)] &= 6t^2 \ , \\
\mathrm{PL}[g^{(4,SO(4))}(t)] &= 10t^2 + t^4 - t^8 \ , \\
\mathrm{PL}[g^{(5,SO(4))}(t)] &= 15t^2 + 5t^4 - 5t^6 - 15t^8 + 24t^{10} + 30t^{12} + \dots \ , \\
\mathrm{PL}[g^{(6,SO(4))}(t)] &= 21t^2 + 15t^4 - 35t^6 - 99t^8 + 504t^{10} + 245t^{12} + \dots \ , \\
\mathrm{PL}[g^{(7,SO(4))}(t)] &= 28t^2 + 35t^4 - 140t^6 - 441t^8 + 4620t^{10} - 1330t^{12} + \dots \ , \\
\mathrm{PL}[g^{(8,SO(4))}(t)] &= 36t^2 + 70t^4 - 420t^6 - 1540t^8 + 27300t^{10} - 32150t^{12} + \dots \ , \\
\mathrm{PL}[g^{(9,SO(4))}(t)] &= 45t^2 + 126t^4 - 1050t^6 - 4536t^8 + 121464t^{10} - 267765t^{12} + \dots \ .
\end{aligned}$$

The case of $N_c = 5$.

$$\begin{aligned}
\mathrm{PL}[g^{(2,SO(5))}(t)] &= 3t^2 \ , \\
\mathrm{PL}[g^{(3,SO(5))}(t)] &= 6t^2 \ , \\
\mathrm{PL}[g^{(4,SO(5))}(t)] &= 10t^2 \ , \\
\mathrm{PL}[g^{(5,SO(5))}(t)] &= 15t^2 + t^5 - t^{10} \ , \\
\mathrm{PL}[g^{(6,SO(5))}(t)] &= 21t^2 + 6t^5 - 6t^7 - 21t^{10} + 35t^{12} - 15t^{14} + 70t^{15} - 210t^{17} + \dots \ , \\
\mathrm{PL}[g^{(7,SO(5))}(t)] &= 28t^2 + 21t^5 - 48t^7 + 28t^9 - 231t^{10} + 980t^{12} - 1668t^{14} + \dots \ , \\
\mathrm{PL}[g^{(8,SO(5))}(t)] &= 36t^2 + 56t^5 - 216t^7 + 280t^9 - 1596t^{10} - 120t^{11} + 11760t^{12} + \dots \ , \\
\mathrm{PL}[g^{(9,SO(5))}(t)] &= 45t^2 + 126t^5 - 720t^7 + 1540t^9 - 8001t^{10} - 1440t^{11} + \dots \ .
\end{aligned} \tag{3.2.34}$$

Character expansion of the plethystic logarithm. We emphasise that coefficients in plethystic logarithmic series are dimensions of representations of the $SU(N_f)$ global symmetry. It is therefore possible to calculate character expansions of plethystic logarithms in a similar fashion to those of generating functions. However, since we are interested in basic generators and basic constraints, only first few terms are significant for our purposes. Consider an example of $\mathrm{PL}\left[g^{(9,SO(4))}(t_i)\right]$. The character expansion is

$$\begin{aligned}
\mathrm{PL}\left[g^{(9,SO(4))}(t; z_1, \dots, z_8)\right] &= [2,0,0,0,0,0,0,0]t^2 + [0,0,0,1,0,0,0,0]t^4 \\
&\quad -[1,0,0,0,1,0,0,0]t^6 - \left(\mathrm{Sym}^2[0,0,0,1,0,0,0,0] - [2,0,0,0,0,1,0,0]\right) t^8 + \dots \ .
\end{aligned} \tag{3.2.35}$$

For the basic generators of the GIOs, the coefficient of t^2 indicates that there are mesons at order 2 and the coefficient of t^4 indicates that there are baryons at order 4. For the basic constraints, the coefficient of t^6 suggests that there is a relation between the basic generators at order 6 given by (3.2.6) and the symmetric square in the coefficient of t^8 suggests that there is also a relation at order 8 given by (3.2.7). However, we can see that this relation at order 8 receives a correction $-[2,0,0,0,0,1,0,0]$, which results from the product between the generator at order t^2 and the relation at order t^6. This correction is the first in an infinite tower of relations that will not be dealt with here. Note that for the general $SO(N_c)$ theory, such a product is at order $N_c + 4$, and so we see that such a correction at order 8 occurs *only when* $N_c = 4$ but not for other values of N_c.

3.3 $Sp(N_c)$ SQCD with N_f flavours

Let us consider an $Sp(N_c)$ gauge theory[3] with N_f flavours of matter in the fundamental $2N_c$ dimensional representation. In the same way as before, we shall specify such a theory by $(N_f, Sp(N_c))$. Since the number of fundamentals must be even according to [16], we take our matter content to be the quarks Q^i_a, with supermultiplet index $i = 1, \ldots, 2N_f$ and colour index $a = 1, \ldots, 2N_c$. Thus, there is a total of $4N_cN_f$ chiral degrees of freedom from the quarks. Their quantum numbers are summarised in Table 3.2. For reviews on this theory see, *e.g.*, [3, 4, 7].

	Gauge symmetry	Global symmetry		
	$Sp(N_c)$	$SU(2N_f)$	$U(1)_B$	$U(1)_R$
Q^i_a	$[1,0,\ldots,0]$	$[1,0,\ldots,0]$	1	$\frac{N_f-1-N_c}{N_f}$

Table 3.2. The gauge and global symmetries of $Sp(N_c)$ SQCD with N_f flavours and the quantum numbers of the chiral supermultiplets.

3.3.1 The $N_f \leq N_c$ Theories

At a generic point in the classical moduli space, the $Sp(N_c)$ gauge symmetry is broken to $Sp(N_c - N_f)$. Since the dimension of $Sp(N)$ is $N(2N+1)$, there are

$$N_c(2N_c+1) - (N_c - N_f)(2N_c - 2N_f + 1) = 4N_cN_f - 2N_f^2 + N_f$$

broken generators. Therefore, of the original $4N_cN_f$ chiral supermultiplets, only

$$4N_cN_f - \left[4N_cN_f - 2N_f^2 + N_f\right] = N_f(2N_f - 1)$$

singlets are left massless. Hence, the dimension of the moduli space of vacua is

$$\dim\left(\mathcal{M}_{N_f \leq N_c}\right) = N_f(2N_f - 1) \ . \tag{3.3.1}$$

We can describe these light degrees of freedom in a gauge invariant way by the mesons:

$$M^{ij} = Q^i_a Q^j_b J^{ab} \qquad \text{(meson)} \ , \tag{3.3.2}$$

where the matrix $J = \mathbf{1}_{N_c} \otimes i\sigma_2$ is an invariant of $Sp(N_c)$. We emphasise that the indices i and j are antisymmetric. Therefore, the meson transforms in the global $SU(N_f)$ representation $\Lambda^2\ [1,0,\ldots,0] = [0,1,\ldots,0]$. We note that for the $N_f \leq N_c$ theory, there are no relations (constraints) between mesons. Phrasing this geometrically, and noting the dimension from (3.3.1), we have that

[3] We shall use the notation where the rank of $Sp(n)$ is n and $Sp(1)$ is isomorphic to $SU(2)$. This is in agreement with the notation of [7].

Observation 3.3.1. *The classical moduli space $\mathcal{M}_{N_f \leq N_c}$ is freely generated: there are no relations among the generators. The space $\mathcal{M}_{N_f \leq N_c}$ is, in fact, nothing but $\mathbb{C}^{N_f(2N_f-1)}$.*

Using the plethystic programme, we can immediately write down the generating function of GIOs for $N_f \leq N_c$ as

$$g^{N_f \leq N_c}(t) = \mathrm{PE}\left[t^2 \dim\left[0,1,\ldots,0\right]\right] = \mathrm{PE}\left[N_f(2N_f-1)t^2\right] = \frac{1}{(1-t^2)^{N_f(2N_f-1)}} \ . \tag{3.3.3}$$

We emphasise that this formula does not depend on the number of colours N_c. This expression is simply the Hilbert series for $\mathbb{C}^{N_f(2N_f-1)}$, with weight 2 for each meson.

3.3.2 The $N_f > N_c$ Theories

At a generic point in the moduli space, the $Sp(N_c)$ gauge symmetry is broken completely and hence the number of remaining massless chiral supermultiplets (*i.e.* the dimension of the moduli space) is given by

$$\dim\left(\mathcal{M}_{N_f > N_c}\right) = 4N_f N_c - N_c(2N_c+1) \ . \tag{3.3.4}$$

These light degrees of freedom can be parametrised by the mesons given by (3.3.2). We refer to a discussion in [7] that there is no baryon, since the invariant tensor $\epsilon^{a_1 \ldots a_{2N_c}}$ decomposes into sums of products of the J^{ab} and so baryons break up into mesons. There is also a basic constraint between mesons due to the fact that any product of Ms antisymmetrised on $2N_c+1$ (or more) upper or lower flavour indices vanishes:

$$\epsilon_{i_1 \ldots i_{2N_f}} M^{i_1 i_2} M^{i_3 i_4} \ldots M^{i_{2N_c+1} i_{2N_c+2}} = 0 \ . \tag{3.3.5}$$

The meson and the constraint (3.3.5) transform respectively in the global $SU(2N_f)$ representations $[0,1,\ldots,0]$ and $[0,\ldots,0,1_{2N_c+2;L},0,\ldots,0]$. They are respectively $\binom{2N_f}{2}$ and $\binom{2N_f}{2N_c+2}$ dimensional.

Although the moduli space $\mathcal{M}_{N_f > N_c}$ is not freely generated, the special case $N_f = N_c + 1$ has a special property:

Observation 3.3.2. *The moduli space $\mathcal{M}_{N_f = N_c+1}$ is a complete intersection and is, in fact, a single hypersurface in $\mathbb{C}^{(2N_c+1)(N_c+1)}$.*

This is because the number of the basic generators (which is $\binom{2N_c+2}{2} = 2N_c^2 + 3N_c + 1 = (2N_c+1)(N_c+1)$) minus the number of the basic constraints (which is 1) is equal to the dimension of the moduli space (which is $2N_c^2 + 3N_c$). Observation 3.3.2 allows us to immediately write down the generating function for the $(N_c+1, Sp(N_c))$ theory by noting that there are $(2N_c+1)(N_c+1)$ mesonic generators of weight t^2, subject to a relation of weight t^{2Nc+2} .

$$\begin{aligned} g^{(N_c+1,Sp(N_c))}(t) &= \left(1-t^{2N_c+2}\right) \mathrm{PE}\left[t^2 \ \dim[0,1,0,\dots,0]\right] \\ &= \left(1-t^{2N_c+2}\right) \mathrm{PE}\left[(2N_c+1)(N_c+1)t^2\right] \\ &= \frac{1-t^{2N_c+2}}{(1-t^2)^{(2N_c+1)(N_c+1)}} \\ &= \frac{1+t^2+t^4+\ldots+t^{2N_c}}{(1-t^2)^{2N_c^2+3N_c}} \ . \end{aligned} \tag{3.3.6}$$

3.3.3 Character Expansions

Let us examine character expansions of generating functions of the Sp theory.

The $N_f \leq N_c$ theories. Consider the simplest example: $N_f \leq N_c$. From (3.3.3), the character expansion is

$$g^{N_f \leq N_c}(t; z_1, \ldots, z_{2N_f-1}) = \mathrm{PE}\left[t^2[0,1,\ldots,0]\right] = \sum_{k=0}^{\infty} \mathrm{Sym}^k[0,1,0,\ldots,0]t^{2k} \ , \tag{3.3.7}$$

where the second equality follows from the basic property of the plethystic exponential which produces all possible symmetric products of the function on which it acts. We emphasise that we have used the fully refined generating function which is a function of $2N_f$ variables, and so this expression depends on $2N_f$ variables, not just one variable t. We note the identity (c.f. (2.3.4) in Chapter 2)

$$\mathrm{Sym}^k[0,1,\ldots,0] = \sum_{n_1,\ldots,n_{2N_f}\geq 0} [0,n_2,0,n_4,0,\ldots,0,n_{N_f-2},0] \ \delta\left(k - \sum_{j=1}^{N_f} 2jn_{2j}\right) \ . \tag{3.3.8}$$

Therefore, we have the character expansion

$$g^{N_f \leq N_c}(t; z_1, \ldots, z_{2N_f-1}) = \sum_{n_1,\ldots,n_{2N_f}\geq 0} [0,n_2,0,n_4,0,\ldots,0,n_{N_f-2},0] \ t^{\alpha} \ , \tag{3.3.9}$$

where $\alpha = \sum_{j=1}^{N_f} 2jn_{2j}$.

Character expansion of an arbitrary $(N_f, Sp(N_c))$ theory. From (3.3.7), we see that the basic building block of the GIOs in the Sp theory is the irreducible representation with 2 Young boxes which are antisymmetrised. Any other irreducible representation is built out of this basic building block and, in fact, each of such an irreducible representation appears precisely once in the character expansion. We propose selection rules for the coefficients of the character expansion of $g^{(N_f,Sp(N_c))}$ as follows:

- Every number located in an odd position from the left is zero;
- For $N_f > N_c + 1$, the numbers located after the $2N_c$-th position from the left are zeros, since any antisymmetrisation on $2N_c + 1$ (or more) flavour indices yields a zero.

It follows that the character expansion for an arbitrary $(N_f, Sp(N_c))$ theory is

$$g^{(N_f,Sp(N_c))}(t; z_1, \dots, z_{2N_f-1}) = \sum_{n_2,n_4,\dots,n_{2N_c}\geq 0} [0, n_2, 0, n_4, 0, n_6, 0, \dots, 0, n_{2N_c}, 0, ..., 0]\, t^{\beta} \,, \tag{3.3.10}$$

where $\beta = \sum_{j=1}^{N_c} 2jn_{2j}$. We note that for $N_c = 1$, formula (3.3.10) becomes

$$g^{(N_f,Sp(1))}(t; z_1, \dots, z_{2N_f-1}) = \sum_{k=0}^{\infty}[0, k, 0 \dots, 0]\ t^{2k} \,. \tag{3.3.11}$$

Note that this is also a character expansion of the $SU(2)$ SQCD with N_f flavour (see formula (2.6.4) of Chapter 2). Such an agreement is to be expected because of an isomorphism between $Sp(1)$ and $SU(2)$.

3.3.4 Plethystic Exponentials and Molien–Weyl Formula

Let us denote a basis for the dual space of the Cartan subalgebra by $\{L_m\}_{m=1}^{N_c}$. We choose $L_m = (0, \dots, 0, 1_{m;L}, 0, \dots, 0)$, where the length of the tuple is N_c. The weights of the fundamental representation are $\{\pm L_m\}$. With this choice of L's, we find the character of the fundamental representation to be

$$\chi^{Sp(N_c)}_{[1,0,\dots,0]}(z_l) = \sum_{m=1}^{N_c} \left(z_m + \frac{1}{z_m} \right) \,. \tag{3.3.12}$$

The roots of the Lie algebra of $Sp(N_c)$ are $\omega = \pm L_m \pm L_n$. Therefore, the Haar measure is given by

$$\int_{Sp(N_c)} \mathrm{d}\mu_{Sp(N_c)} = \frac{1}{(2\pi i)^{N_c} N_c! 2^{N_c}} \oint_{|z_1|=1} \cdots \oint_{|z_{N_c}|=1} \frac{\mathrm{d}z_1}{z_1} \cdots \frac{\mathrm{d}z_{N_c}}{z_{N_c}} \prod_{\omega} \left(1 - \prod_{l=1}^{N_c} z_l^{\omega_l} \right) , \tag{3.3.13}$$

where ω_l is the number in the l-th position of the root ω. Similarly to the case of the $SO(N_c)$ gauge group, we have

$$\begin{aligned} g^{(N_f,Sp(N_c))} &= \int_{Sp(N_c)} \mathrm{d}\mu_{Sp(N_c)}\ \mathrm{PE} \left[\chi^{Sp(N_c)}_{[1,0,\dots,0]}(z_l) \sum_{i=1}^{2N_f} t_i \right] \,. \\ &= \frac{1}{(2\pi i)^{N_c} N_c! 2^{N_c}} \oint_{|z_1|=1} \cdots \oint_{|z_{N_c}|=1} \frac{\mathrm{d}z_1}{z_1} \cdots \frac{\mathrm{d}z_{N_c}}{z_{N_c}} \frac{\prod_{\omega} \left(1 - \prod_{l=1}^{N_c} z_l^{\omega_l} \right)}{\prod_{i=1}^{2N_f} \prod_{m=1}^{N_c} (1 - t_i z_m) \left(1 - \frac{t_i}{z_m} \right)} \,. \end{aligned} \tag{3.3.14}$$

For reference, we shall list a few unrefined (i.e. $t_i = t$ for all $i = 1, \dots, 2N_f$) generating functions for the $Sp(2)$ SQCD. Using the residue theorem twice with the poles at 0 and t, we find that

$$
\begin{aligned}
g^{(1,Sp(2))}(t) &= \frac{1}{1-t^2} \\
&= 1 + t^2 + t^4 + t^6 + t^8 + t^{10} + \dots \ , \\
g^{(2,Sp(2))}(t) &= \frac{1}{(1-t^2)^6} \\
&= 1 + 6t^2 + 21t^4 + 56t^6 + 126t^8 + 252t^{10} + \dots \ , \\
g^{(3,Sp(2))}(t) &= \frac{1+t^2+t^4}{(1-t^2)^{14}} \\
&= 1 + 15t^2 + 120t^4 + 679t^6 + 3045t^8 + 11508t^{10} + \dots \ , \\
g^{(4,Sp(2))}(t) &= \frac{1+6t^2+21t^4+28t^6+21t^8+6t^{10}+t^{12}}{(1-t^2)^{22}} \\
&= 1 + 28t^2 + 406t^4 + 4032t^6 + 30744t^8 + 191736t^{10} + \dots \ , \\
g^{(5,Sp(2))}(t) &= \frac{1+15t^2+120t^4+470t^6+1065t^8+1377t^{10}+1065t^{12}+470t^{14}+120t^{16}+15t^{18}+t^{20}}{(1-t^2)^{30}} \\
&= 1 + 45t^2 + 1035t^4 + 16005t^6 + 186285t^8 + 1739133t^{10} + \dots \ .
\end{aligned}
\tag{3.3.15}
$$

We note that these results are consistent with the character expansion (3.3.10). As an example, let us consider the $(2, Sp(2))$ theory:

$$
\begin{aligned}
g^{(2,Sp(2))}(t) &= \sum_{n_2,n_4 \geq 0} \dim\, [0, n_2, 0]\; t^{2n_2+4n_4} \\
&= \frac{{}_2F_1(3,4;2;t^2)}{(1-t^4)} = \frac{(1+t^2)}{(1-t^2)^5(1-t^4)} = \frac{1}{(1-t^2)^6} \ ,
\end{aligned}
\tag{3.3.16}
$$

where the second equality follows from the dimension formula (2.3.18). As before, we see that the method of summing dimensions of representations into a closed form provides a non-trivial check of formula (3.3.10).

We can also make a similar proposition to Observation 3.2.3 that

Observation 3.3.3. *The generating functions for the $N_f \geq N_c + 1$ theory can be written as*

$$
g^{N_f \geq N_c+1}(t) = \frac{P_k(t)}{(1-t^2)^{\dim \mathcal{M}_{(N_f, Sp(N_c))}}} \ ,
$$

where $P_k(t)$ is a degree k polynomial such that $P_k(1) \neq 0$ and $\dim \mathcal{M}_{(N_f,Sp(N_c))} - k$ is a constant for a given N_c. This constant, which is $N_c(2N_c + 1)$, can be computed from the case of $N_f = N_c + 1$, where the moduli space is a complete intersection, using (3.3.2).

3.3.5 Plethystic Logarithms

Recall that the plethystic logarithm of the generating function $g^{(N_f,Sp(N_c))}$ is given by

$$\mathrm{PL}[g^{(N_f,Sp(N_c))}(t)] = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log(g^{(N_f,Sp(N_c))}(t^k)) \ . \qquad (3.3.17)$$

We emphasise again that the first terms with plus sign give the numbers of basic generators, whereas the first terms with the minus sign give the numbers of constraints between these basic generators. If the formula (3.3.17) is an infinite series of terms with plus and minus signs, then the moduli space is not a complete intersection. We shall list a few results for the $Sp(2)$ SQCD:

$$\begin{aligned}
\mathrm{PL}\left[g^{(1,Sp(2))}(t)\right] &= t^2 \ , \\
\mathrm{PL}\left[g^{(2,Sp(2))}(t)\right] &= 6t^2 \ , \\
\mathrm{PL}\left[g^{(3,Sp(2))}(t)\right] &= 15t^2 - t^6 \ , \\
\mathrm{PL}\left[g^{(4,Sp(2))}(t)\right] &= 28t^2 - 28t^6 + 63t^8 - 36t^{10} - 378t^{12} + 1728t^{14} + \ldots \ , \\
\mathrm{PL}\left[g^{(5,Sp(2))}(t)\right] &= 45t^2 - 210t^6 + 1155t^8 - 2376t^{10} - 19800t^{12} + \ldots \ . \qquad (3.3.18)
\end{aligned}$$

Take an example of $\mathrm{PL}\left[g^{(4,Sp(2))}(t)\right]$. We see that the term $28t^2$ indicates that there are 28 basic generators (mesons) at the order of 2 quarks, and the term $-28t^6$ indicates that there are 28 basic constraints (given by (3.3.5)) at the order of 6 quarks.

Character expansion of the plethystic logarithm. We can make a character expansion of the plethystic logarithm in a similar fashion as for the SO theory. Consider an example of $\mathrm{PL}\left[g^{(4,Sp(2))}\right]$. The character expansion is

$$\mathrm{PL}\left[g^{(4,Sp(2))}(t; z_1, \ldots, z_7)\right] = [0,2,0,0,0,0,0]t^2 - [0,0,0,0,0,1,0]t^6 + \ldots \qquad (3.3.19)$$

The coefficient of t^2 indicates that there is one generator (meson) that transforms in the 28 dimensional $[0,2,0,0,0,0,0]$ representation, and the coefficient of t^6 suggests that there is one relation between the mesons at order 6, given by (3.3.5), that transforms in the 28 dimensional $[0,0,0,0,0,1,0]$ representation.

3.4 An Orientifold Projection

Having the character expansion of the Hilbert Series for SQCD with all classical gauge groups, we can now turn to study relations between different theories. One natural relation arises from analogy to certain string theory backgrounds [33–35] that include orientifolds [36, 37]. In such backgrounds it is rather common that the gauge group reduces by a $\mathbb{Z}_2$ projection from a unitary gauge group to a symplectic or an orthogonal gauge group. We will now study the action of this $\mathbb{Z}_2$ on the generators and relations of the Hilbert Series. Any string theory background which embeds the Sp and SO gauge groups through an orientifold projection will have to act on irreducible representations of the global symmetry in the way specified in this section. We will henceforth refer to the $\mathbb{Z}_2$ action as an orientifold action without specifying the explicit brane or other construction.

We remind the reader that the quiver diagram of SQCD with the SU gauge group (see Chapter 2) can be drawn as in Figure 3.1.

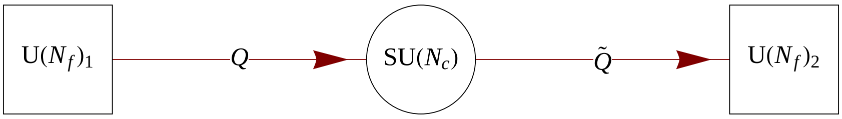

Figure 3.1. The quiver diagram of $SU(N_c)$ SQCD with N_f flavours. The red node represents the $SU(N_c)$ gauge symmetry while the two square nodes denote the global $U(N_f)$ symmetries. Each node gives rise to a $U(1)$ global symmetry, one of which is redundant.

An orientifold action on the global symmetry. The $U(N_f)_L \times U(N_f)_R$ flavour symmetry goes down to its diagonal $U(N_f)$ subgroup for the case of SO gauge group. For the $Sp(N_c)$ gauge theory, as a result of the vanishing superpotential, the global symmetry further gets enhanced to $U(2N_f)$. An antifundamental index in the $SU(N_c)$ theory becomes a fundamental index due to the orientation reversal in both $SO(N_c)$ and $Sp(N_c)$ theories.

An orientifold action on the quiver diagram. The orientifold action on the quiver diagram is to fold it along the circular $SU(N_c)$ node, together with orientation reversal of the arrow in the quiver. As a result the circular node becomes either SO or Sp, depeppppnding on the projection, and the flavour symmetry either maps to itself for the case of SO gauge group or is enhanced as stated above for the case of Sp gauge group. The resulting quiver diagrams are shown in Figure 3.2.

An orientifold action on the basic generators. A discussion on a similar problem is presented in [38]. The mesons which transform in the bifundamental representation of $SU(N_f)_L \times SU(N_f)_R$ are projected down in two steps: Firstly, the antifundamental index is turned into a fundamental index, and secondly, the resulting representation gets respectively symmetrised and antisymmetrised in the $SO(N_c)$ and $Sp(N_c)$ theories. In the $SO(N_c)$ theory, the baryon and antibaryon get identifed by the orientifold projection and inherit the irreducible representation from the embedding of $U(N_f)$ inside $U(N_f) \times U(N_f)$. On the other hand, in the $Sp(N_c)$ theory, a baryon breaks up into a product of mesons and stops being a generator

Figure 3.2. **Left diagram**: $Sp(N_c)$ SQCD with N_f flavours as a quiver theory. The circular node represents the $Sp(N_c)$ gauge symmetry, while the square node denotes the global $U(2N_f)$ symmetries. **Right diagram:** $SO(N_c)$ SQCD with N_f flavours as a quiver theory. The circular node represents the $SO(N_c)$ gauge symmetry, while the square node denotes the global $U(N_f)$ symmetries. **In each of these cases:** On the contrary to the $SU(N_c)$ SQCD, although the square node give rise to a $U(1)$ factor, the circular node does not due to the orientifold projection.

of the chiral ring. We summarise these results in Table 3.3, and note that they are consistent with the results obtained by direct computations in preceding sections.

Basic GIOs	$SU(N_c)$ SQCD	$Sp(N_c)$ SQCD	$SO(N_c)$ SQCD
	Representation of the global symmetry		
	$SU(N_f)_L \times SU(N_f)_R$	$SU(2N_f)$	$SU(N_f)$
Meson	$[1,0,\dots,0;0,\dots,0,1]$	$[0,1,0,\dots,0]$	$[2,0,\dots,0]$
Baryon	$[0,0,\dots,1_{N_c;L},0,\dots,0;0,\dots,0]$	*	$[0,0,\dots,1_{N_c;L},0,\dots,0]$
Antibaryon	$[0,\dots,0;0,\dots,1_{N_c;R},0\dots,0]$	*	**

Table 3.3. The basic generators of GIOs for *SU*, *Sp* and *SO* SQCD and how they transform under the global symmetries. In the above, * indicates that in the *Sp* theory baryons and antibaryons are simply products of mesons and stop being generators, and ** indicates that the antibaryon gets identified with the baryon such that we have one operator instead of two.

Using these observations, we can immediately write down Hilbert Series for the freely generated moduli spaces in the *Sp* and *SO* theories starting from the one for *SU* theory as follows:

$$\mathrm{PE}\left[\mathrm{Sym}^2[1,0,\dots,0]_{SU(N_f)}t^2\right] = \mathrm{PE}\left[[2,0,\dots,0]_{SU(N_f)}t^2\right]$$

$$\uparrow SO$$

$$\mathrm{PE}\left[[1,0,\dots,0;0,\dots,0,1]_{SU(N_f)_L\times SU(N_f)_R}t\,\tilde{t}\,\right]$$

$$\downarrow Sp$$

$$\mathrm{PE}\left[\Lambda^2[1,0,\dots,0]_{SU(2N_f)}t^2\right] = \mathrm{PE}\left[[0,1,0,\dots,0]_{SU(2N_f)}t^2\right]$$

An orientifold action on the basic constraints. As for the basic generators, the projection occurs in two steps: The antifundamental index is first turned into a fundamental index, and the resulting symmetry then gets respectively symmetrised and antisymmetrised in the $SO(N_c)$ and $Sp(N_c)$ theories. The results are summarised in Table 3.4, and note that they agree with the results obtained by direct computations in earlier sections

Type of relations	$SU(N_c)$ SQCD	$Sp(N_c)$ SQCD	$SO(N_c)$ SQCD
	Representation of the global symmetry		
	$SU(N_f)_L \times SU(N_f)_R$	$SU(2N_f)$	$SU(N_f)$
$BB = M \dots M$	$[\ 1_{N_c;L}\ ;\ 1_{N_c;R}\ ]$	†	$\mathrm{Sym}^2[\,1_{N_c;L}\,]$
$MB = MB$	$[\ 1_{(N_c+1);L}\ ;\ 1_{1;R}]$ $[1_{1;L}\ ;\ 1_{(N_c+1);R}\ ]$	$[\ 1_{2N_c+2;L}\ ]$	$[1_{1;L}\ \ 1_{N_c+1;L}\ ]$

Table 3.4. The basic constraints between GIOs for SU, Sp and SO SQCD and how they transform under the global symmetries. Only non-zero components of representations are presented. † indicates that in the Sp theory, the relation $BB = M \dots M$ provides us with no new information, as a baryon is simply a product of mesons.

3.5 A Geometric Aperçu

In Chapter 2 and the preceding sections, we have used the plethystic programme, the Molien–Weyl formula and the character expansion technique, to construct generating functions (Hilbert Series) which count GIOs in SQCD with *any* classical gauge group. In the following, we use Hilbert Series to extract a number of useful geometrical properties of moduli spaces. We note, *en passant*, that there have been a number of studies of moduli spaces using techniques from computational algebraic geometry [1, 39–42].

3.5.1 Palindromic Numerator

We have observed in many case studies before that the numerator of the unrefined[4] generating function (Hilbert series) for SQCD is palindromic, *i.e.* it can be written in the form of a degree k polynomial:

$$P_k(t) = \sum_{n=0}^{k} a_n t^n \ , \qquad (3.5.1)$$

[4] Although refined generating functions can be used for deducing geometrical properties of the moduli space, it is more convenient to use unrefined ones. The former are more useful for deriving of character expansions.

with symmetric coefficients $a_{k-n} = a_n$. A trivial modification of the rigorous proof given in Section 2.5.3 for the $SU(N_c)$ SQCD tells us that this palindromic property holds in general for the SO and Sp SQCD:

Theorem 3.5.1. *Let $P_k(t)$ be a numerator of the unrefined generating function (Hilbert series) $g^{(N_f,SO(N_c))}(t)$ or $g^{(N_f,Sp(N_c))}(t)$ and suppose that $P_k(1) \neq 0$. Then, $P_k(t)$ is palindromic.*

3.5.2 The SQCD Moduli Space of Vacua Is Calabi-Yau

Similar situations were encountered in Chapter 2. Due to a well-known theorem in commutative algebra called the Hochster–Roberts theorem (Theorem 2.4.12), our coordinate rings of the moduli space are Cohen–Macaulay. Therefore, as an immediate consequence of Theorem 3.5.1 and the Stanley theorem (Theorem 2.4.13), the chiral rings are also algebraically Gorenstein. Since an affine Gorenstein variety is, by definition, affine Calabi–Yau, we reach an important conclusion that $\mathcal{M}_{(N_f,N_c)}$ is, in fact, an affine Calabi–Yau cone over a weighted projective variety. In brief,

Theorem 3.5.2. *The moduli spaces of the $SO(N_c)$ and $Sp(N_c)$ theories are Calabi-Yau*[5].

3.5.3 The SQCD Moduli Space of Vacua Is Irreducible

We start this subsection by noting that the irreducibility of moduli spaces is certainly not a feature of generic gauge theories. However, similarly to Chapter 2, we find that

Conjecture 3.5.3. *The classical moduli spaces of SQCD with SO and Sp gauge groups are irreducible for all N_f and N_c.*

Similarly to Chapter 2, the moduli space (in the absense of a superpotential) of SQCD can be described by a symplectic quotient:

$$\mathbb{C}^n//G = \mathbb{C}^n/G^c \ , \tag{3.5.2}$$

where $n = N_f N_c,\ 4N_f N_c$ for $G = SO(N_c),\ Sp(N_c)$ and G^c denotes their complexifications, $G^c = SO(N_c,\mathbb{C}),\ Sp(N_c,\mathbb{C})$. Since $\mathbb{C}^n$ is irreducible and G^c is a continuous group, we expect the resulting quotient to be also irreducible.

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Part II

$\mathcal{N} = 2$ Gauge Theories in Four Dimensions

Chapter 4

The Moduli Space of One Instanton on $\mathbb{R}^4$

4.1 Introduction

Yang-Mills Instantons [1] have attracted great interest from both physicists and mathematicians since their discovery in 1975. They have served as a powerful tool in studying a number of physical and mathematical problems, ranging from the Yang-Mills vacuum structure (*e.g.*, [2–4]) to the classification of four-manifolds [5].

A method for constructing a self-dual Yang-Mills instanton solution on $\mathbb{R}^4$ is due to Atiyah, Drinfeld, Hitchin and Manin (ADHM) [6] in 1978. The ADHM construction is known for the classical gauge groups, $SU(N)$, $SO(N)$ and $Sp(N)$ (see, *e.g.*, [7–11] for explicit constructions); there is no known such construction, however, for the exceptional groups. The space of all solutions to the self-dual Yang-Mills equation modulo gauge transformations, in a given winding sector k and gauge group G is said to be the **moduli space of k G-instantons** on $\mathbb{R}^4$. In 1994-1996, Douglas and Witten [24–27] discovered that the ADHM construction can be realised in string theory. In particular, the moduli space of instantons on $\mathbb{R}^4$ is identical to the Higgs branch of supersymmetric gauge theories on a system of Dp-D$(p+4)$ branes (see, *e.g.*, [12] for a review).[1] These theories are quiver gauge theories with 8 supercharges ($\mathcal{N}=2$ supersymmetry in $(3+1)$ dimensions for $p=3$). In Section 4.3, we present the $\mathcal{N}=2$ quiver diagram of each theory as well as provide a prescription for writing down the corresponding $\mathcal{N}=1$ quiver diagram and the superpotential. The Hilbert series of the one instanton moduli space is easily computed using the ADHM construction for classical gauge groups and is written in a form that provides a natural conjectured generalization for exceptional gauge groups (even though the

[1]The Higgs branch of D3 branes near E_n type 7 branes is the moduli space of E_n instantons. Since there is no known Lagrangian for this class of theories, it is not clear how to compute the ADHM analog.

ADHM construction does not exist for the latter).

In addition to the ADHM construction, there exists an alternative description of the moduli space of instantons for simply laced (A, D and E) groups via three dimensional mirror symmetry [13]. This symmetry exchanges the Coulomb branch and the Higgs branch, and therefore maps the Coulomb branch of the ADE quiver gauge theories to moduli spaces of instantons. On the contrary to Higgs branch, one expects the Coulomb branch to receive many non-perturbative quantum corrections. As argued in [13], quantum effects correct the Coulomb branch to be the moduli space of one ADE-instanton, with the point at the origin corresponding to an instanton of zero size.[2] Nevertheless, due to such quantum corrections, this description of the instanton moduli space is not useful for exact computations using Hilbert series.

In the last section of this chapter, exceptional groups are considered, and checks that the Hilbert Series above predicts the correct dimension of the moduli space. In the case of E_n it is known [14, 15] that $\mathcal{N} = 2$ CFTs realise the moduli space of one E_n instanton. We use Argyres-Seiberg S-dualities in $\mathcal{N} = 2$ supersymmetric gauge theories [16–22] to match the Hilbert series of the theories on both sides of the duality, providing a consistency check.

4.2 Hilbert Series for One-Instanton Moduli Spaces on $\mathbb{C}^2$

We are interested in computing the partition function that counts holomorphic functions (Hilbert series) on the moduli space of k G-instantons on $\mathbb{C}^2$, were G is a gauge group of finite rank r. It is well known that this moduli space has quaternionic dimension kh_G where h_G is the dual Coxeter number of the gauge group G. In this chapter, we focus on the case of a single instanton moduli space. The moduli space is reducible into a trivial $\mathbb{C}^2$ component, physically corresponding to the position of the instanton in $\mathbb{C}^2$, and the remaining irreducible component of quaternionic dimension $h_G - 1$. Henceforth, we shall call this component the *coherent component* or the *irreducible component*. The Hilbert series for the coherent component takes the form

$$g_G^{\mathrm{Irr}}(t; x_1, \ldots, x_r) = \sum_{k=0}^{\infty} \chi[R_G(k)] t^{2k} \ , \qquad (4.2.1)$$

where $R_G(k)$ is a series of representations of G and $\chi[R]$ is the character of the representation R. The fugacities x_i (with $i = 1, \ldots, r$) are conjugate to the charges of each holomorphic function under the Cartan subalgebra of G. The moduli space of instantons is a non-compact hyperKähler space, and so there are infinitely many

[2] The Coulomb branch of the gauge theory with quiver diagram G (where G is A, D or E) and all ranks multiplied by k is $kh_G - 1$ quaternionic dimensional [13], where h_G is the dual coxeter number of G. This precisely agrees with the fact that the coherent component (eliminating the translation on $\mathbb{R}^4$) of the one G-instanton moduli space is $h_G - 1$ quaternionic dimensional.

holomorphic functions which are graded by degrees d. Setting $x_1 = \ldots = x_r = 1$, we obtain the (finite) number of holomorphic functions of degree d.

The main result of this chapter is the following:

The representation $R_G(k)$ is the irreducible representation Adj^k ,

where Adj^k denotes the irreducible representation whose Dynkin labels are $\theta_k = k\theta$, with θ the highest root of G.[3] By convention $R_G(0)$ is the trivial, one-dimensional, representation (this corresponds to the space being connected), and $R_G(1)$ is the adjoint representation.

In the case of classical gauge groups A_n, B_n, C_n, D_n it is possible to directly verify the above statement by explicit counting of the chiral operators on the Higgs branch of a certain $\mathcal{N} = 2$ supersymmetric gauge theory with a one dimensional Coulomb branch and a $A_n, \ldots, D_n$ global symmetry. The specific gauge theory can be derived in string theory by a simple system of Dp branes which probe a background of D$(p+4)$ branes in Type II theories. The moduli space of k G-instantons on $\mathbb{C}^2$ is identified with the Higgs branch of the gauge theory living on the k Dp branes. The gauge group G, which is interpreted as a global symmetry on the world volume of the Dp branes, lives on the D$(p+4)$ branes and can be chosen to be any of the classical gauge groups by an appropriate choice of a background with or without an orientifold plane. The gauge theory living on the Dp branes is a simple quiver gauge theory and is discussed in detail in Section 4.3. The F and D term equations for the Higgs branch of these theories coincides with the ADHM construction of the moduli space of instantons for classical gauge groups. Unfortunately, such a simple construction is not available for exceptional groups and other methods need to be applied. It is therefore not possible to explicitly compute the Hilbert series for exceptional groups and the main statement of this chapter is a conjecture for these cases. This conjecture is subject to a collection of tests which are presented in Section 4.5.

An example of D_4. An explicit counting of chiral operators in the well known $\mathcal{N} = 2$ supersymmetric gauge theory of $SU(2)$ with 4 flavours (see Section 4.3.3 for details), gives the Hilbert series for the coherent component of the one $D_4 = SO(8)$ instanton moduli space (omitting the trivial component $\mathbb{C}^2$) :

$$g^{\mathrm{Irr}}_{D_4}(t; x_1, x_2, x_3, x_4) = \sum_{k=0}^{\infty} [0, k, 0, 0]_{D_4} t^{2k}, \tag{4.2.2}$$

[3] For the A_n series $\theta_k = [k, 0, \ldots, 0, k]$, for the B_n and D_n series $\theta_k = [0, k, 0, \ldots, 0]$, for the C_n series $\theta_k = [2k, 0, \ldots, 0]$, for E_6 $\theta_k = [0, k, 0, 0, 0, 0]$, for G_2 $\theta_k = [0, k]$, for all other exceptional groups $\theta_k = [k, 0, \ldots, 0]$.

Setting these fugacities y_i to 1, we get the unrefined Hilbert series:

$$\begin{aligned} g^{\rm Irr}_{D_4}(t) &= \sum_{k=0}^{\infty} dim[0,k,0,0]_{D_4} t^{2k} \\ &= \frac{(1+t^2)(1+17t^2+48t^4+17t^6+t^8)}{(1-t^2)^{10}} \\ &= 1+28t^2+300t^4+\dots \ . \end{aligned} \tag{4.2.3}$$

An explicit expression for the dimension of each such representation is given by

$$\dim\ [0,k,0,0]_{D_4} = \frac{(k+1)(k+2)^3(k+3)^3(k+4)(2k+5)}{4320} \ . \tag{4.2.4}$$

Notice that summing the series we get a closed formula with a pole of order 10 at $t=1$. This means that the space is 10-complex dimensional, and is in agreement with the fact that the non-trivial component of the one-instanton moduli space for D_4 has quaternionic dimension 5 (the dual Coxeter number $h_{D_4}=6$).

In general, summing up the unrefined Hilbert series for any group G gives rational functions of the form

$$g^{\rm Irr}_{G}(t) = \frac{P_G(t^2)}{(1-t^2)^{2h-2}} \ , \tag{4.2.5}$$

where $P_G(x)$ is a palindromic polinomial of degree h_G-1.

A dimension formula for Adj^k. Formula (4.2.4) can be generalised to any classical and exceptional group. Defining

$$G_{a,b}(h,k) = \frac{\binom{(1+a)h/2-b-1+k}{k}}{\binom{(1-a)h/2+b-1+k}{k}} \ , \tag{4.2.6}$$

the dimension of the Adj^k representation is given by

$$\dim\ Adj^k = G_{1,1}(h,k) G_{a,b}(h,k) G_{1-a,1-b}(h,k) \frac{2k+h-1}{h-1} \ . \tag{4.2.7}$$

where (a,b,h) are given in Table 4.2.[4]

[4]Formula (4.2.7) generalises the Proposition 1.1 of [23]

$$\dim Adj^k = \frac{3c+2k+5}{3c+5} \frac{\binom{k+2c+3}{k}\binom{k+5c/2+3}{k}\binom{k+3c+4}{k}}{\binom{k+c/2+1}{k}\binom{k+c+1}{k}} \ ,$$

which gives the results for A_1, A_2, G_2, D_4, F_4, E_6, E_7 and E_8 if we use $c=\frac{1}{3}h_G-2$.

Lie group	Dynkin label of Adj^k	Dual coxeter number	(a,b)	$\mathcal{N}=2$ gauge theory
$A_n = SU(n+1)$	$[k,0,\ldots,0,k]$	$n+1$	$(1,1)$	Quiver diagram 4.6
$B_{n\geq 3} = SO(2n+1)$	$[0,k,0,\ldots,0]$	$2n-1$	$(1,2)$	Quiver diagram 4.8
$C_{n\geq 2} = Sp(n)$	$[2k,0,\ldots,0]$	$n+1$	$(1,1/2)$	Quiver diagram 4.10
$D_{n\geq 4} = SO(2n)$	$[0,k,0,\ldots,0]$	$2n-2$	$(1,2)$	Quiver diagram 4.8
E_6	$[0,k,0,0,0,0]$	12	$(1/3,0)$	3 M5s on 3-punctured sphere
$E_{7,8}$	$[k,0,\ldots,0]$	$18,30$	$(1/3,0)$	$4,6$ M5s on 3-punctured sphere
F_4	$[k,0,0,0]$	9	$(1/3,0)$	
G_2	$[0,k]$	4	$(1/3,0)$	

Table 4.1. Useful information on classical and exceptional groups. The last column indicates the $\mathcal{N}=2$ gauge theories, for which the Higgs branch is identified with the corresponding moduli space of instantons on $\mathbb{R}^4$.

4.3 Gauge Theories on Dp-D$(p+4)$ Brane Systems

The moduli space of instantons is known to be the Higgs branch of certain supersymmetric gauge theories [25–27]. For classical gauge groups there is an explicit construction, while for exceptional gauge groups there is a puzzle on how to explicitly write it down. Below we recall the string theory embedding of the gauge theories for classical gauge groups as worldvolume theories of Dp branes in backgrounds of D$(p+4)$ branes and summarize the gauge theory data for these theories. It is perhaps convenient to take $p=3$, so that the worldvolume theories have $\mathcal{N}=2$ supersymmetry in $(3+1)$ dimensions. The presence of these branes breaks space-time into $\mathbb{R}^{1,3}\times\mathbb{C}^2\times\mathbb{C}$. There is a $U(2)$ symmetry that acts on the $\mathbb{C}^2$ and acts as an R symmetry on the different supermultiplets in the theory. This symmetry is used below to distinguish some of the gauge invariant operators.

The gauge theory on the D3 branes is most conveniently written in terms of $\mathcal{N}=2$ quiver diagrams but for the purpose of computing the Hilbert series, it is more convenient to work using an $\mathcal{N}=1$ notation. Section 4.3.1 summarizes the basic rules of translating an $\mathcal{N}=2$ quiver diagram to an $\mathcal{N}=1$ quiver diagram with a superpotential.

4.3.1 Quiver diagrams

To write down a Lagrangian for a gauge theory with $\mathcal{N}=2$ supersymmetry it is enough to specify the gauge group, transforming in a vector multiplet, and the matter fields, transforming in hyper multiplets. This can be simply summarized by a quiver with 2 objects - nodes and lines but nevertheless has a two-fold ambiguity on how to assign the objects. A traditional mathematical approach, first introduced to the string theory literature in [28], is to assign nodes to vector multiplets and lines to hyper multiplets. This is the so called quiver diagram used below. The more physically inspired approach [29], is to assign lines to vector multiplets and nodes to hyper multiplets. This notation turns out to be more useful when the hyper multiplets carry more than two charges. On the other hand, to write down the Lagrangian for a gauge theory with $\mathcal{N}=1$ supersymmetry the data which is needed consists of 3 objects: the gauge group, the matter fields, and the interaction terms written in the form of a superpotential. This can be summarized by an oriented quiver, namely it has arrows which are absent in the $\mathcal{N}=2$ quiver, and is supplemented by a superpotential W. A simple dictionary exists between the two formulations. It goes as follows:

- A node in the $\mathcal{N}=2$ quiver diagram becomes a node with an adjoint chiral multiplet in the $\mathcal{N}=1$ quiver diagram. This adjoint chiral multiplet comes from the $\mathcal{N}=2$ vector multiplet which decomposes as a $\mathcal{N}=1$ vector multiplet and a $\mathcal{N}=1$ chiral multiplet. The map is shown in Figure 4.1.

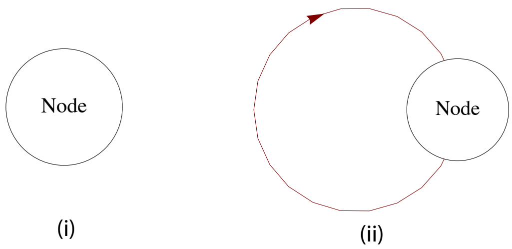

Figure 4.1. A node in the $\mathcal{N}=2$ quiver diagram (labelled (i)) becomes a node with an adjoint chiral multiplet in the $\mathcal{N}=1$ quiver diagram (labelled (ii)).

- A line in the $\mathcal{N}=2$ quiver diagram becomes a bi-directional line in the $\mathcal{N}=1$ quiver diagram. This is shown in Figure 4.2.

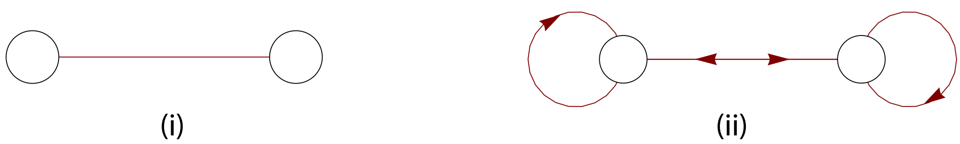

Figure 4.2. A line in the $\mathcal{N}=2$ quiver diagram (labelled (i)) becomes a bi-directional line in the $\mathcal{N}=1$ quiver diagram (labelled (ii)).

- The superpotential is given by the sum of contributions from all lines in the $\mathcal{N}=2$ quiver diagram. Each line stretched between two nodes in the $\mathcal{N}=2$ quiver diagram contributes two cubic superpotential terms. Let the two nodes be labeled by 1 and 2. Associated with each node, there is an adjoint field denoted respectively by Φ_1 and Φ_2. A line connecting between two nodes contains two $\mathcal{N}=1$ bi-fundamental chiral multiplets X_{12} and X_{21}. (The $\mathcal{N}=1$ quiver diagram is drawn in Figure 4.3.) The corresponding superpotential term is written as an adjoint valued mass term for the X fields:

$$X_{21} \cdot \Phi_1 \cdot X_{12} - X_{12} \cdot \Phi_2 \cdot X_{21} \ , \tag{4.3.1}$$

This notation means as follows. Denote the rank of nodes 1 and 2 by r_1 and r_2 respectively. then $\Phi_1, \Phi_2, X_{12}, X_{21}$ can be chosen to be $r_1 \times r_1, r_2 \times r_2, r_1 \times r_2, r_2 \times r_1$ matrices, respectively. The $\cdot$ corresponds to matrix multiplication and an impiicit trace is assumed. Note that this is a schematic notation which does not specify the index contraction whose details depend on the gauge and flavour groups. As a special case, a line from one node to itself would naturally produce a commutator.

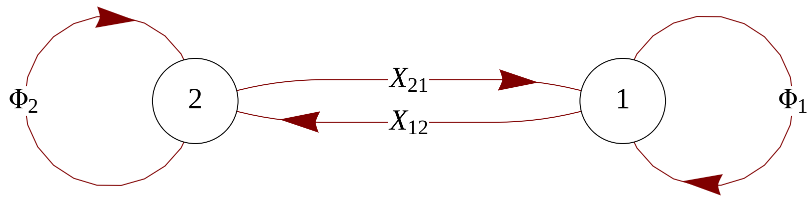

Figure 4.3. An $\mathcal{N}=1$ quiver diagram with the superpotential : $X_{21}\cdot\Phi_1\cdot X_{12} - X_{12}\cdot\Phi_2\cdot X_{21}$.

As an example, we give the $\mathcal{N}=2$ and $\mathcal{N}=1$ quiver diagrams for the $U(N)$ $\mathcal{N}=4$ super Yang-Mills (SYM) respectively in Figure 4.4 and Figure 4.5.

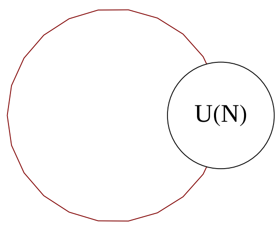

Figure 4.4. The $\mathcal{N}=2$ quiver diagram for the $\mathcal{N}=4$ SYM theory with gauge group $U(N)$. The loop around the $U(N)$ gauge group denotes an adjoint hypermultiplet.

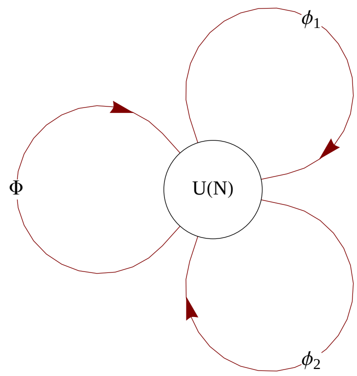

Figure 4.5. The $\mathcal{N}=1$ quiver diagram of the $\mathcal{N}=4$ SYM theory. The adjoint field Φ comes from the $\mathcal{N}=2$ vector multiplet, whereas the adjoint fields ϕ_1, ϕ_2 come from the $\mathcal{N}=2$ adjoint hypermultiplet. The superpotential is $W = \mathrm{Tr}(\phi_1 \cdot \Phi \cdot \phi_2 - \phi_2 \cdot \Phi \cdot \phi_1) = \mathrm{Tr}\,(\Phi \cdot [\phi_1, \phi_2])$.

4.3.2 k $SU(N)$ instantons on $\mathbb{C}^2$

With this quiver notation it is now very simple to write down the gauge theory living on the world volume of k D3 branes in the background of N D7 branes. In fact, the brane system very naturally forms a quiver and we can just write down a dictionary between the branes and the objects in the quiver. We will write down the theory using $\mathcal{N}=2$ quivers and then translate it to $\mathcal{N}=1$ quivers. First, the gauge theory on k D3 branes is the well known $\mathcal{N}=4$ supersymmetric theory with gauge group $U(k)$ depicted in Figure 4.4. The D7 branes are heavier and therefore give rise to a global $U(N)$ symmetry on the worldvolume of the D3 branes. As discussed below, the global $U(1)$ of $U(N)$ may be absorbed into the local $U(1)$ of $U(k)$; therefore global $SU(N)$ symmetry is represented by a square node with index N. Finally strings stretched between the D3 branes and the D7 branes are represented by a line connecting the circular node to the square node. The resulting quiver is depicted in Figure 4.6.

It is now straightforward to apply the rules of Section 4.3.1 to write down the $\mathcal{N}=1$ quiver diagram which is depicted in Figure 4.7 and its corresponding superpotential. To write down the superpotential we need explicit notation for the quiver fields and the line between the circular node and the square node corresponds to two chiral fields denoted by Q and $\widetilde{Q}$. Putting this together, W takes the form

$$\begin{aligned} W &= X_{21} \cdot \Phi \cdot X_{12} + \left(\phi^{(1)} \cdot \Phi \cdot \phi^{(2)} - \phi^{(2)} \cdot \Phi \cdot \phi^{(1)}\right) \\ &= X_{21} \cdot \Phi \cdot X_{12} + \epsilon_{\alpha\beta}\phi^{(\alpha)} \cdot \Phi \cdot \phi^{(\beta)} \ . \end{aligned} \tag{4.3.2}$$

Note that the rules for writing the quiver imply the existence of another term coming from the adjoint in the vector multiplet of the D7 branes. This term corresponds to an adjoint $U(N)$ valued mass term for the bifundamental fields X_{12}, X_{21}. In this chapter we will not treat this mass term and set it to 0, even though it is interesting to consider the effects of such a term. The adjoint fields are parametrizing the position of the D3 branes in $\mathbb{C}^2$. Since there is a natural $U(2)_g = SU(2)_g \times U(1)_g$ symmetry that acts on $\mathbb{C}^2$, the fields ϕ_1 and ϕ_2 transform as a doublet of $SU(2)_g$ symmetry and with charge 1 under $U(1)_g$. The superpotential should therefore be invariant under $SU(2)_g$ and carry charge 2 under $U(1)_g$.

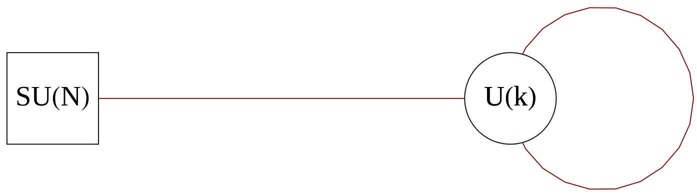

Figure 4.6. The $\mathcal{N}=2$ quiver diagram for k $SU(N)$ instantons on $\mathbb{C}^2$. The circular node represents the $U(k)$ gauge symmetry and the square node represents the $SU(N)$ flavour symmetry. The line connecting the $SU(N)$ and $U(k)$ groups denotes kN bi-fundamental hypermultiplets, and the loop around the $U(k)$ group denotes the adjoint hypermultiplet.

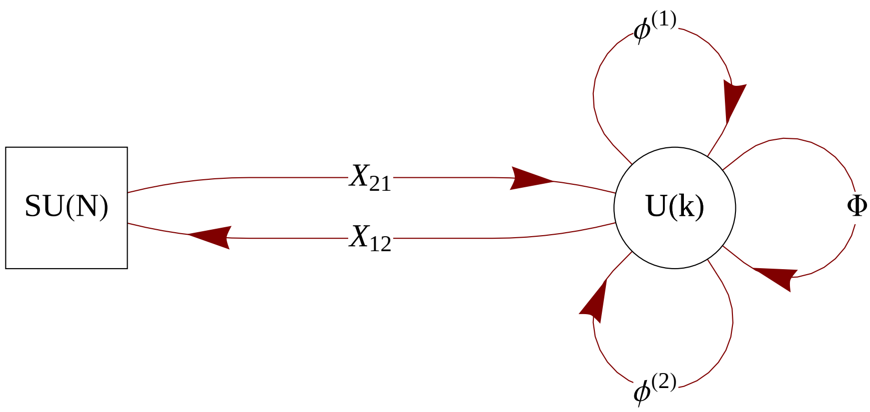

Figure 4.7. Flower quiver; The $\mathcal{N}=1$ quiver diagram for k $SU(N)$ instantons on $\mathbb{C}^2$ with the corresponding superpotential, $W = X_{21} \cdot \Phi \cdot X_{12} + \epsilon_{\alpha\beta}\phi^{(\alpha)} \cdot \Phi \cdot \phi^{(\beta)}$.

We list the charges and the representations under which the fields transform in Table 4.2.

From Table 4.2, it can be seen that the $U(1)$ of $U(N)$ can be absorbed into the local $U(1)$ (*e.g.* by means of redefining the fugacity z/q). From the brane perspective, the vector multiplet of the local $U(1)$ contains a scalar which parametrises the position of the D3-brane in the directions transverse to the D7 branes. One can

Field	$U(k)$		$U(N)$		$SU(2)_g$	$U(1)_g$
	$SU(k)$	$U(1)$	$SU(N)$	$U(1)$	global	global
Fugacity:	$z_1, \ldots, z_{k-1}$	z	$x_1, \ldots, x_{N-1}$	q	x	t
Φ	$[1, 0, \ldots, 0, 1]$	0	$[0, \ldots, 0]$	0	$[0]$	0
$\phi^{(1)}, \phi^{(2)}$	$[1, 0, \ldots, 0, 1]$	0	$[0, \ldots, 0]$	0	$[1]$	1
X_{12}	$[1, 0, \ldots, 0]$	1	$[0, \ldots, 0, 1]$	-1	$[0]$	1
X_{21}	$[0, 0, \ldots, 0, 1]$	-1	$[1, 0 \ldots, 0]$	1	$[0]$	1
$\mathrm{Tr}\,\Phi$	$[0,\ldots,0]$	0	$[0,\ldots,0]$	0	$[0]$	0
$\mathrm{Tr}\,\phi^{(1)}, \mathrm{Tr}\,\phi^{(2)}$	$[0,\ldots,0]$	0	$[0,\ldots,0]$	0	$[1]$	1

Table 4.2. The charges and the representations under which various fields transform. The fugacites of each field are assigned according to this table. The $U(2)_g$ global symmetry acts on $\phi^{(1)}$ and $\phi^{(2)}$. It is the symmetry group of $\mathbb{C}^2$, the trivial component in the moduli space.

set the origin of these directions to be at the CoM of the D7-branes and thereby eliminate the corresponding background $U(1)$ vector multiplet.

Let us compute the quaternionic dimension of the Higgs branch. From the $\mathcal{N} = 2$ quiver diagram, the line connecting the $SU(N)$ and $U(k)$ groups denotes kN hypermultiplets, and the loop around the $U(k)$ group denotes k^2 hypermultiplets. Hence, we have in total $kN + k^2$ quarternionic degrees of freedom. On a generic point on the Higgs branch, the gauge group $U(k)$ is completely broken and hence there are k^2 broken generators. As a result of the Higgs mechanism, the vector multiplet gains k^2 degrees of freedom and becomes massive. Hence, the $(kN + k^2) - k^2 = kN$ quarternionic degrees of freedom are left massless. Thus, the Higgs branch is kN quaternionic dimensional or $2kN$ complex dimensional:

$$\dim_{\mathbb{C}} \mathcal{M}^{\text{Higgs}}_{k,N} = 2kN = 2kh \; . \tag{4.3.3}$$

This agrees with the dual coxeter number of $SU(N)$ which is $h_{SU(N)} = N$.

From the brane perspective, the VEV of the scalar Φ correspond to the position of the D3-branes along the directions transverse to the D7-branes. On the Higgs branch, the gauge fields become massive freezing the whole vector multiplet and hence $\langle \Phi \rangle = 0$, setting the D3 branes to lie within the D7 branes and possibly form bound states. The hypermultiplets acquire non-zero VEVs at a generic point on the Higgs branch that parametrize all possible bound states of D3 and D7 branes. From the point of view of the D7 brane gauge theory, the D3 branes are interpreted as instantons and hence, the moduli space of classical instantons on $\mathbb{C}^2$ is identified with the Higgs branch of the quiver theory [25].

One $SU(N)$ instanton: $k = 1$

The gauge theory for 1 $SU(N)$ instanton on $\mathbb{C}^2$ is particularly simple and lives on the world volume of 1 D3 brane, $k = 1$. The gauge group is $U(1)$ and the adjoints Φ, ϕ_1, ϕ_2 are simply complex numbers, and hence the second term of (4.3.2) vanishes,

$$W = X_{21} \cdot \Phi \cdot X_{12}. \tag{4.3.4}$$

The Higgs branch. On the Higgs branch, $\Phi = 0$ and $X_{12} \cdot X_{21} = 0$. The space of F-term solutions (which we will call the F-flat space and denote by $\mathcal{F}^\flat$) is obviously a complete intersection. Using (4.3.3) the dimension of the moduli space is $2N$. On the other hand there are $2N$ bifundamental fields X_{12}, X_{21} and 2 ϕ's which are subject to 1 relation. This gives an F-flat moduli space which is $2N + 1$ dimensional and after imposing the D-term equations we get a $2N$ dimensional moduli space, as expected. The F-flat Hilbert series can be written down according to Table 4.3 as[5]

$$\begin{aligned} g^{\mathcal{F}^\flat}_{k=1,N}(t, x_1, \ldots, x_{N-1}, x, q, z) = (1-t^2)\mathrm{PE}\Big[&[1]_{SU(2)_g} t + [1,0,\ldots,0]_{SU(N)} \frac{tz}{q} \\ &+ [0,0,\ldots,0,1]_{SU(N)} \frac{tq}{z}\Big] \ . \end{aligned} \tag{4.3.5}$$

Note that the first term in the square bracket corresponds to $\phi^{(1)}$ and $\phi^{(2)}$, the second term corresponds to X_{12} and the third term correspond to X_{21}, and the factor in front of the PE corresponds to the relation.

Notice from (4.3.5) that the $U(1)$ of $U(N)$ can in fact be absorbed into the local $U(1)$. This can be seen by redefining the fugacity for the local $U(1)$ as

$$w = \frac{z}{q} \ , \tag{4.3.6}$$

and rewrite

$$\begin{aligned} g^{\mathcal{F}^\flat}_{k=1,N}(t, x_1, \ldots, x_{N-1}, x, w) = (1-t^2)\mathrm{PE}\Big[&[1]_{SU(2)_g} t + [1,0,\ldots,0]_{SU(N)} tw \\ &+ [0,0,\ldots,0,1]_{SU(N)} \frac{t}{w}\Big] \ . \end{aligned} \tag{4.3.7}$$

The right hand side can explicitly be written as a rational function:

$$\begin{aligned} &(1-t^2) \times \frac{1}{(1-tx)(1-\frac{t}{x})} \times \frac{1}{(1-twx_1)\left(1-\frac{tw}{x_{N-1}}\right)\prod_{k=2}^{N-1}(1-tw\frac{x_k}{x_{k-1}})} \\ &\times \frac{1}{\left(1-\frac{t}{w}\frac{1}{x_1}\right)\left(1-\frac{t}{w}x_{N-1}\right)\prod_{k=2}^{N-1}(1-\frac{t}{w}\frac{x_{k-1}}{x_k})} \ . \end{aligned} \tag{4.3.8}$$

[5]The *plethystic exponential (PE)* of a multi-variable function $g(t_1, \ldots, t_n)$ that vanishes at the origin, $g(0, \ldots, 0) = 0$, is defined to be $\mathrm{PE}[g(t_1, \ldots, t_n)] := \exp\left(\sum_{r=1}^{\infty} \frac{g(t_1^r, \ldots, t_n^r)}{r}\right)$. The reader is referred to Chapters 2, 3 and [30–32] for more details.

The Hilbert series. Now we project (4.3.8) onto the gauge invariant subrepresentation by performing an integration over the $U(1)$ gauge group[6]. The Hilbert series of the Higgs branch is therefore given by

$$g^{\text{Higgs}}_{k=1,N}(t, x_1, \ldots, x_{N-1}, x) = \frac{1}{2\pi i} \oint_{|w|=1} \frac{\mathrm{d}w}{w} g^{\mathcal{F}^\flat}_{k=1,N}(t, x_1, \ldots, x_{N-1}, x, w) \ . \quad (4.3.9)$$

Using the residue theorem on (4.3.8), where the poles are located at[7]

$$w = t\frac{1}{x_1},\ t\frac{x_1}{x_2},\ \ldots,\ t\frac{x_{N-2}}{x_{N-1}},\ tx_{N-1}\ , \quad (4.3.10)$$

we can write the Hilbert series in terms of representations as

$$g^{\text{Higgs}}_{k=1,N}(t, x_1, \ldots, x_{N-1}, x) = \frac{1}{(1-tx)\left(1-\frac{t}{x}\right)} \sum_{k=0}^{\infty} [k, 0, \ldots, 0, k]_{SU(N)} t^{2k} \quad (4.3.11)$$

The factor $\frac{1}{(1-tx)(1-\frac{t}{x})}$ indicates the Hilbert series for the complex plane $\mathbb{C}^2$, whose symmetry is $U(2)_g$ (with the fugacities t, x). This space $\mathbb{C}^2$ is parametrised by $\phi^{(1)}$ and $\phi^{(2)}$ and corresponds to the position of the D3-brane inside the D7-branes. The second factor corresponds to the coherent component of the one $SU(N)$ instanton moduli space. Unrefining by setting $x_1 = \ldots = x_{N-1} = x = 1$, we obtain

$$g^{\text{Higgs}}_{k=1,N}(t, 1, \ldots, 1) = \frac{1}{(1-t)^2} \times \frac{\sum_{k=0}^{N-1} \binom{N-1}{k}^2 t^{2k}}{(1-t^2)^{2(N-1)}} \ . \quad (4.3.12)$$

The order of the pole $t = 1$ is $2N$, and hence the dimension of the Higgs branch is $2N$, in accordance with (4.3.3). Note that (4.3.12) can also be derived directly from (4.3.9) as follows. Setting $x_1 = \ldots = x_{N-1} = x = 1$ in (4.3.9), we obtain

$$g^{\text{Higgs}}_{k=1,N}(t, 1, \ldots, 1) = \frac{(1-t^2)}{(1-t)^2} \frac{1}{2\pi i} \oint_{|w|=1} \frac{\mathrm{d}w}{w} \frac{1}{(1-tw)^N (1-\frac{t}{w})^N} \ . \quad (4.3.13)$$

The contribution to the integral comes from the pole at $w = t$, which is of order N. Using the residue theorem, we find that

$$g^{\text{Higgs}}_{k=1,N}(t, 1, \ldots, 1) = \frac{(1-t^2)}{(1-t)^2} \times \frac{1}{(N-1)!} \frac{\mathrm{d}^{N-1}}{\mathrm{d}w^{N-1}} \left[\frac{w^{N-1}}{(1-tw)^N}\right]_{w=t} \quad (4.3.14)$$

Using Leibniz's rule for differentiation, we thus arrive at (4.3.12).

The plethystic logarithm can be written as

$$\begin{aligned}\mathrm{PL}[g^{\text{Higgs}}_{k=1,N}(t, x_1, \ldots, x_{N-1}, x)] = {} & [1]_{SU(2)_g} t + [1, 0, \ldots, 0, 1]_{SU(N)} t^2 - ([0, 1, 0, \ldots, 0, 1, 0] + \\ & [1, 0, \ldots, 0, 1] + [0, \ldots, 0])_{SU(N)}\, t^4 + \ldots \ . \end{aligned} \quad (4.3.15)$$

Hence, the generators are $\mathrm{Tr}\,\phi^{(1)}, \mathrm{Tr}\,\phi^{(2)}$ at order t and the adjoints [1,0,...,0,1] of $SU(N)$ at the order t^2. The basic relations transform in the $SU(N)$ representation $[0, 1, 0, \ldots, 0, 1, 0] + [1, 0, \ldots, 0, 1] + [0, \ldots, 0]$.

[6]This is called the *Molien-Weyl integral formula* (see *e.g.*, Chapters 2, 3 and [31, 32]).

[7]Note that $|t| < 1$ and only poles located inside the unit circle $|w| = 1$ are included.

4.3.3 k $SO(N)$ instantons on $\mathbb{C}^2$

As pointed out in [27], the moduli space of k $SO(N)$ instantons can be realised on a system of k D3-branes with N half D7-branes on top of an $O7^-$ orientifold plane. (If the number of branes is odd, the combination of half D7 brane stuck on the $O7^-$ plane form an orientifold plane which is called $\widetilde{O7}^-$ plane.) The brane picture is similar to the one described in the previous subsection and therefore the quiver looks the same. We only need to figure out the action of the orientifold plane on the different objects in the quiver. All together, there are 4 objects in Figure 4.6.

- The gauge group on the D7 branes is projected to $SO(N)$. This is a global symmetry for the gauge theory on the D3 branes. $\mathcal{N} = 2$ supersymmetry restricts the gauge theory on the D3 branes to be $Sp(k)$. Hence,
- The gauge group on the D3 branes is projected down to $Sp(k)$.
- The bi-fundamental fields become bi-fundamentals of $SO(N) \times Sp(k)$.
- The loop around the $U(k)$ gauge group undergoes a $\mathbb{Z}_2$ projection which leaves two options - the second rank symmetric or antisymmetric representation of $Sp(k)$. To find which one, we notice that only the anti-symmetric representation is reducible into a singlet plus the rest. Since the center of mass of the instanton is physically decoupled from the rest of the moduli space, we conclude that the projection is to the antisymmetric representation.

The resulting $\mathcal{N} = 2$ quiver diagram is depicted in Figure 4.8.

Using the rules of Section 4.3.1 it is easy to find the $\mathcal{N} = 1$ quiver diagram given in Figure 4.9 and the superpotential,

$$\begin{aligned} W &= Q \cdot S \cdot Q + (A_1 \cdot S \cdot A_2 - A_2 \cdot S \cdot A_1) \\ &= Q \cdot S \cdot Q + \epsilon_{\alpha\beta} A_\alpha \cdot S \cdot A_\beta \ , \end{aligned} \tag{4.3.16}$$

where we have suppressed the contractions over the gauge indices by the tensor J^{ab} (an invariant tensor of $Sp(k)$) and the contractions over the flavour indices by δ_{ij} (an invariant tensor of $SO(N)$). The epsilon tensor $\epsilon_{\alpha\beta}$ in the second line is an invariant tensor of the global $SU(2)$ symmetry which interchanges A_1 and A_2. The mass term for Q coming from the adjoint of $SO(N)$ is set to 0.

Let us compute the quaternionic dimension of the Higgs branch. From the $\mathcal{N} = 2$ quiver diagram, the lines connecting the $SO(N)$ and $Sp(k)$ groups denotes $2kN$ half-hypermultiplets (equivalently, kN hypermultiplets), and the loop around the $Sp(k)$ group gives $k(2k-1)$ hypermultiplets. Hence, we have in total $kN + k(2k-1)$ quarternionic degrees of freedom. On the Higgs branch, $Sp(k)$ is completely broken and hence there are $k(2k+1)$ broken generators. As a result of the Higgs mechanism,

Figure 4.8. The $\mathcal{N} = 2$ quiver diagram for k $SO(N)$ instantons on $\mathbb{C}^2$. The circular node represents the $Sp(k)$ gauge symmetry and the square node represents the $SO(N)$ flavour symmetry. The line connecting the $SO(N)$ and $Sp(k)$ groups denotes $2kN$ half-hypermultiplets, and the loop around the $Sp(k)$ gauge group denotes a hypermultiplet transforming in the (reducible) second rank antisymmetric tensor.

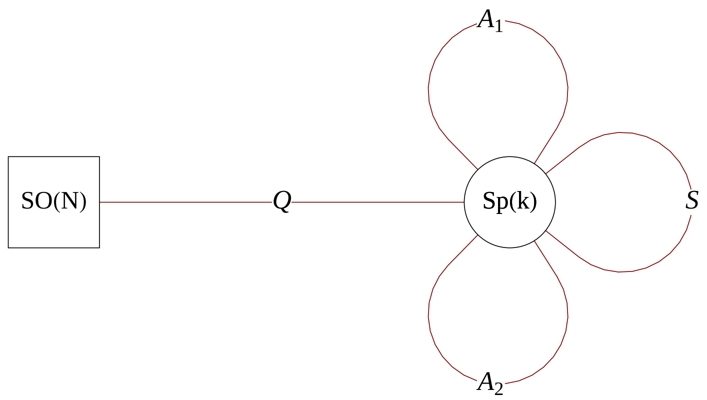

Figure 4.9. The $\mathcal{N} = 1$ quiver diagram for k $SO(N)$ instantons on $\mathbb{C}^2$. The chiral multiplet transforming in the second rank symmetric tensor (adjoint field) of $Sp(k)$ is denoted by S and the second rank antisymmetric tensors are denoted by A_1, A_2. The superpotential is given by $W = Q \cdot S \cdot Q + \epsilon_{\alpha\beta} A_\alpha \cdot S \cdot A_\beta$.

the vector multiplet gains $k(2k+1)$ degrees of freedom and becomes massive. Hence, the $kN + k(2k-1) - k(2k+1) = k(N-2)$ degrees of freedom are left massless. Thus, the Higgs branch is $k(N-2)$ quaternionic dimensional or $2k(N-2)$ complex dimensional:

$$\dim_{\mathbb{C}} \mathcal{M}^{\text{Higgs}}_{k,N} = 2k(N-2) = 2k h_{SO(N)} \ . \qquad (4.3.17)$$

Note that $h_{SO(N)} = N - 2$ is the dual coxeter number of the $SO(N)$ group.

The charges and the representations under which the fields transform are given in Table 4.3 [38].

Field	$Sp(k)$	$SO(N)$	$SU(2)_g$	$U(1)_g$
Fugacity:	$z_1,\dots,z_k$	$x_1,\dots,x_{\lfloor N/2\rfloor}$	x	t
S	$[2,0,\dots,0]$	$[0,\dots,0]$	$[0]$	0
A_1, A_2	$[0,1,0,\dots,0]+[0,\dots,0]$	$[0,\dots,0]$	$[1]$	1
Q	$[1,0,\dots,0]$	$[1,0,\dots,0]$	$[0]$	1

Table 4.3. The charges and the representations under which various fields transform. The fugacites of each field are assigned according to this table.

One $SO(N)$ instanton on $\mathbb{C}^2$: $k=1$

In the special case $k=1$, the gauge group is $Sp(1)=SU(2)$ and the superpotential (4.3.16) becomes

$$W_{k=1} = \epsilon^{ab}\epsilon^{cd} Q^i_a S_{bc} Q^i_d \ . \tag{4.3.18}$$

The Higgs branch. The Higgs branch is given by the F-term conditions: $S=0$ and $Q^i_a Q^i_b + Q^i_b Q^i_a = 0$, and the D-term condition. The Hilbert series of the F-flat moduli space is

$$\begin{aligned} g^{\mathcal{F}^\flat}(t,z,x_1,\dots,x_{\lfloor N/2\rfloor},x) &= (1-t^2)\left(1-\frac{t^2}{z^2}\right)(1-t^2z^2)\,\mathrm{PE}\Big[[1]_{SU(2)_g} t \\ &\quad + [1,0,\dots,0]_{SO(N)} t\left(z+\frac{1}{z}\right)\Big] \ . \end{aligned} \tag{4.3.19}$$

We note that the relation transforms in the representation $[2]$ of $Sp(1)$ and that the F-flat moduli space is a complete intersection of dimension $2+2N-3=2N-1$. Noting that the characters of the fundamental representations of $B_n=SO(2n+1)$ and $D_n=SO(2n)$ respectively are

$$\begin{aligned} [1,0,\dots,0]_{B_n}(x_a) &= 1+\sum_{a=1}^{n}\left(x_a+\frac{1}{x_a}\right) \ , \\ [1,0,\dots,0]_{D_n}(x_a) &= \sum_{a=1}^{n}\left(x_a+\frac{1}{x_a}\right) \ , \end{aligned} \tag{4.3.20}$$

we can write down (4.3.19) as a rational functional function

$$g^{\mathcal{F}^\flat}(t,z,x_1,\dots,x_n,x)_{B_n,D_n} = \frac{(1-t^2)}{(1-tx)(1-t/x)} \times \frac{\left(1-\frac{t^2}{z^2}\right)(1-t^2z^2)}{(1-t)^\delta\prod_{a=1}^{n}(1-tzx_a)(1-\frac{tz}{x_a})(1-\frac{t}{z}x_a)(1-\frac{t}{zx_a})} \ , \tag{4.3.21}$$

where $\delta = 1$ for B_n and $\delta = 0$ for D_n.

Performing the Molien-Weyl integral over the gauge group $Sp(1)$, we obtain the Higgs branch Hilbert series as

$$\begin{aligned} g^{\text{Higgs}}(t, x_1, \ldots, x_n, x)_{B_n, D_n} &= \frac{1}{2\pi i} \oint_{|z|=1} \mathrm{d}z \left(\frac{1-z^2}{z} \right) g^{\mathcal{F}^\flat}(t, z, x_1, \ldots, x_n, x)_{B_n, D_n} \\ &= \frac{1}{(1-tx)(1-t/x)} \times \sum_{k=0}^{\infty} [0, k, 0, \ldots, 0]_{B_n, D_n} t^{2k}, \end{aligned} \quad (4.3.22)$$

where the contributions to the integral come from the poles:

$$z = tx_1, \ldots, tx_n, \frac{t}{x_1}, \ldots, \frac{t}{x_n} \ . \quad (4.3.23)$$

The factor $\frac{1}{(1-tx)(1-t/x)}$ is the Hilbert series for $\mathbb{C}^2$ (whose symmetry is $U(2)_g$) and is parametrised by the singlets in A_1, A_2; this corresponds to the position of the D3-brane inside the D7-branes. The second factor corresponds to the coherent component of the one $SO(N)$ instanton moduli space.

Example: $N = 8$**.** The expression (4.3.19) can be written as a rational function:

$$\frac{(1-t^2)}{(1-tx)(1-t/x)} \times \frac{\left(1 - \frac{t^2}{z^2}\right)(1-t^2 z^2)}{\prod_{a=1}^{4} (1 - tzx_a)(1 - \frac{tz}{x_a})(1 - \frac{tx_a}{z})(1 - \frac{t}{zx_a})} \ . \quad (4.3.24)$$

The poles which contribute to the Molien-Weyl integral (4.3.22) are

$$z = tx_1, \ldots, tx_4, \frac{t}{x_1}, \ldots, \frac{t}{x_4} \ . \quad (4.3.25)$$

The integral (4.3.22) gives

$$g^{\text{Higgs}}(t, x_1, \ldots, x_4, x) = \frac{1}{(1-tx)(1-t/x)} \times \sum_{k=0}^{\infty} [0, k, 0, 0]_{SO(8)} t^{2k} \ . \quad (4.3.26)$$

Unrefining by setting $x_1 = \ldots = x_4 = x = 1$, we obtain

$$g^{\text{Higgs}}(t, 1, 1, 1, 1, 1) = \frac{1}{(1-t)^2} \times \frac{(1+t^2)(1 + 17t^2 + 48t^4 + 17t^6 + t^8)}{(1-t^2)^{10}} \ . \quad (4.3.27)$$

Observe that the pole at $t = 1$ is of order 12, and so the Higgs branch is indeed 12 dimensional, in agreement with (4.3.17). The plethystic logarithm is

$$\begin{aligned} \mathrm{PL}\left[g^{\text{Higgs}}(t, x_1, x_2, x_3, x_4, x) \right] = &\ [1]_{SU(2)_g} t + [0, 1, 0, 0]_{SO(8)} t^2 - ([2, 0, 0, 0] + [0, 0, 2, 0] \\ &+ [0, 0, 0, 2] + [0, 0, 0, 0])_{SO(8)} t^4 + \ldots \ , \end{aligned} \quad (4.3.28)$$

indicating that the relations are invariant under the triality of $SO(8)$.

4.3.4 k $Sp(N)$ instantons on $\mathbb{C}^2$

As pointed out in [25], the moduli space of k $Sp(N)$ instantons can be realised on a system of k D3-branes with N D7-branes on top of an $O7^+$ orientifold plane. As a result, the gauge group is projected to $SO(k)$, [8] and the scalar in the vector multiplet becomes an antisymmetric tensor, denoted by A_{ab} (where the $SO(k)$ gauge indices take values $a, b = 1, \ldots, k$). The adjoint hypermultiplet becomes a symmetric tensor, as it is the reducible second rank tensor of $SO(k)$, and is denoted by two chiral multiplets S_1 and S_2. Since representations of the $SO(k)$ group are real, the flavour symmetry is $Sp(N)$ and we have $2kN$ half-hypermultiplets. We denote the complex scalar in each half-hypermultiplet as Q^i_a (where the $Sp(N)$ flavour indices take values $i, j = 1, \ldots, 2N$).

The $\mathcal{N} = 2$ and $\mathcal{N} = 1$ quiver diagrams are given respectively in Figure 4.10 and Figure 4.11. The $\mathcal{N} = 1$ superpotential is

$$\begin{aligned} W &= Q \cdot A \cdot Q + (S_1 \cdot A \cdot S_2 - S_2 \cdot A \cdot S_1) \\ &= Q \cdot A \cdot Q + \epsilon_{\alpha\beta} S_\alpha \cdot A \cdot S_\beta \ , \end{aligned} \tag{4.3.29}$$

where we have suppressed the contractions over the flavour indices by the tensor J_{ij} (an invariant tensor of $Sp(N)$) and the contractions over the gauge indices by δ^{ab} (an invariant tensor of $SO(k)$). The epsilon tensor $\epsilon_{\alpha\beta}$ in the second line is an invariant tensor of the global $SU(2)$ symmetry which interchanges S_1 and S_2. The mass term transforming in the adjoint of $Sp(N)$ is set to 0.

Figure 4.10. The $\mathcal{N} = 2$ quiver diagram for k $Sp(N)$ instantons on $\mathbb{C}^2$. The circular node represents the $O(k)$ gauge symmetry and the square node represents the $Sp(N)$ flavour symmetry. The line connecting the $Sp(N)$ and $O(k)$ groups denotes $2kN$ half-hypermultiplets, and the loop around the $O(k)$ group denotes the second rank (reducible) symmetric tensor.

[8] For $k = 1$ we take the convention that $SO(1)$ is $\mathbb{Z}_2$. For higher values of k, the computations in this chapter do not distinguish between a gauge group $O(k)$ and a gauge group $SO(k)$ and hence this $\mathbb{Z}_2$ ambiguity is ignored.

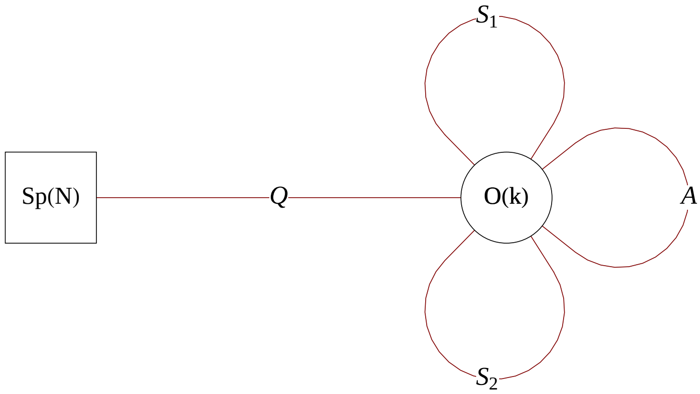

Figure 4.11. The $\mathcal{N} = 1$ quiver (flower) diagram for k $Sp(N)$ instantons on $\mathbb{C}^2$, with A being an antisymmetric tensor (adjoint field) and S_1, S_2 being symmetric tensors of $Sp(k)$. The superpotential is $W = Q \cdot A \cdot Q + \epsilon_{\alpha\beta} S_\alpha \cdot A \cdot S_\beta$.

Let us compute the quaternionic dimension of the Higgs branch. From the $\mathcal{N} = 2$ quiver diagram, the lines connecting the $Sp(N)$ and $O(k)$ groups denotes $2kN$ half-hypermultiplets (equivalently, kN hypermultiplets), and the loop around the $O(k)$ group gives $\frac{1}{2}k(k+1)$ hypermultiplets. Hence, we have in total $kN + \frac{1}{2}k(k+1)$ degrees of freedom. On the Higgs branch, we assume that $O(k)$ is completely broken and hence there are $\frac{1}{2}k(k-1)$ broken generators. As a result of the Higgs mechanism, the vector multiplet gains $\frac{1}{2}k(k-1)$ degrees of freedom and becomes massive. Hence, the $\left[kN + \frac{1}{2}k(k+1)\right] - \frac{1}{2}k(k-1) = k(N+1)$ degrees of freedom are left massless. Thus, the Higgs branch is $k(N+1)$ quaternionic dimensional or $2k(N+1)$ complex dimensional:

$$\dim_{\mathbb{C}} \mathcal{M}^{\text{Higgs}}_{k,N} = 2k(N+1) = 2k h_{Sp(N)} \ , \tag{4.3.30}$$

where $h_{Sp(N)} = N + 1$ is the dual coxeter number of the $Sp(N)$ gauge group.

We list the charges and the representations under which the fields transform in Table 4.4.

One $Sp(N)$ instanton on $\mathbb{C}^2$: $k = 1$

For $k = 1$, the gauge group becomes $O(1) \cong \mathbb{Z}_2$. Recall that we have $2N$ hypermultiplets Q^i and two gauge singlets S_1 and S_2. It is then easy to see that the moduli space in this case is

$$\mathcal{M}^{\text{Higgs}}_{k=1,N} = \mathbb{C}^{2N}/\mathbb{Z}_2 \times \mathbb{C}^2 \ , \tag{4.3.31}$$

where the factor $\mathbb{C}^2$ is parametrised by S_1 and S_2, the $\mathbb{C}^{2N}$ is parametrised by Q^i, and the orbifold action $\mathbb{Z}_2$ is -1 on each coordinate of $\mathbb{C}^{2N}$. Observe that $\mathcal{M}^{\text{Higgs}}_{k=1,N}$

Field	$SO(k)$	$Sp(N)$	$SU(2)_g$ global	$U(1)$ global
Fugacity:	$z_1, \ldots, z_k$	$x_1, \ldots, x_{\lfloor N/2 \rfloor}$	x	t
A	$[0, 1, \ldots, 0]$	$[0, \ldots, 0]$	$[0]$	0
S_1, S_2	$[2, 0, \ldots, 0] + [0, \ldots, 0]$	$[0, \ldots, 0]$	$[1]$	1
Q	$[1, 0, \ldots, 0]$	$[1, 0, \ldots, 0]$	$[0]$	1

Table 4.4. The charges and the representations under which various fields transform. The fugacites of each field are assigned according to this table.

is $2(N+1)$ complex dimensional, in accordance with (4.3.30). Physically, the $\mathbb{C}^2$ corresponds to the position (4 real coordinates) of the instanton. The coherent component of the one $Sp(N)$ instanton moduli space is therefore $\mathbb{C}^{2N}/\mathbb{Z}_2$.

One can see the last statement clearly from the Hilbert series. The Hilbert series of $\mathbb{C}^{2N}/\mathbb{Z}_2$ is given by the discrete Molien formula (see, *e.g.*, [30]):

$$\begin{aligned} g(t, x_1, \ldots, x_N; \mathbb{C}^{2N}/\mathbb{Z}_2) &= \frac{1}{2}\left(\mathrm{PE}\left[[1, 0, \ldots, 0]_{Sp(N)} t\right] + \mathrm{PE}\left[[1, 0, \ldots, 0]_{Sp(N)}(-t)\right]\right) \\ &= \sum_{k=0}^{\infty} [2k, 0, \ldots, 0] t^{2k} \ , \end{aligned} \tag{4.3.32}$$

where the plethystic exponential can be written explicitly as

$$\mathrm{PE}\left[[1, 0, \ldots, 0]_{Sp(N)} t\right] = \frac{1}{\prod_{a=1}^{N}(1 - t x_a)(1 - t/x_a)} = \sum_{n=0}^{\infty} [n, 0, \ldots, 0]_{Sp(N)} \, t^n,$$

and the $\mathbb{Z}_2$ acts on t by projecting to even powers. The last equality of (4.3.32) follows from the fact that the plethystic exponential generates symmetrisation. This is indeed the Hilbert series for the coherent component of the one $Sp(N)$ instanton moduli space. The choice of x_a in this formula is not the natural choice of weights in the representation but rather a linear combination of weights which is convenient for writing this particular formula.

4.4 $\mathcal{N} = 2$ Supersymmetric $SU(N_c)$ Gauge Theory with N_f Flavours

This section deals with the computation of the Hilbert series for the Higgs branch of the $\mathcal{N} = 2$ $SU(N_c)$ supersymmetric gauge theory with N_f flavours. It serves as a preparation for the discussion in Section 4.5, were the results will be used in checking Argyres-Seiberg duality. The global symmetry of this theory is $U(N_f) = U(1)_B \times SU(N_f)$ and since it plays a crucial role on the Higgs branch this theory will sometimes be called the $U(N_f)$ theory. The special case of $N_c = 2$ and $N_f = 4$

is discussed in Section 4.3.3 and is revisited below. The $\mathcal{N} = 2$ quiver diagram for this theory is depicted in Figure 4.12.

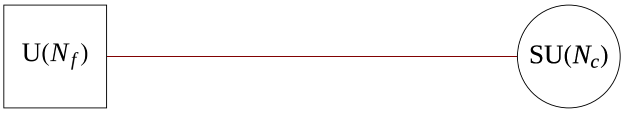

Figure 4.12. $\mathcal{N} = 2$ quiver diagram for $SU(N_c)$ gauge theory with N_f flavours.

The $\mathcal{N} = 1$ quiver diagram is depicted in Figure 4.13 and the superpotential after setting the masses to 0 is given by

$$W = \widetilde{Q} \cdot \phi \cdot Q, \tag{4.4.1}$$

giving the F-term equations on the Higgs branch, $\phi = 0$ and $Q\widetilde{Q} = 0$, where the last equation has only $N_c^2 - 1$ equations and not N_c^2. The trace meson $\widetilde{Q} \cdot Q$ need not vanish.

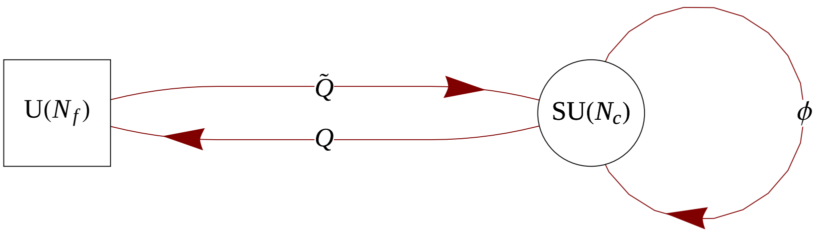

Figure 4.13. $\mathcal{N} = 1$ quiver diagram for $SU(N_c)$ gauge theory with N_f flavours. The superpotential is $W = \widetilde{Q} \cdot \phi \cdot Q$.

The Higgs branch of this theory has a Hilbert series which is easy to write down as an integral over the Haar measure of $SU(N_c)$. The reason for this lies partly with supersymmetry and partly with the simplicity of the gauge and matter content. We first argue that the F-flat moduli space is a complete intersection. Since the quaternionic dimension of the Higgs branch is $N_c N_f - (N_c^2 - 1)$, the complex dimension of the F-flat moduli space is expected to be $N_c^2 - 1$ higher than this one. Adding these together, we get that the complex dimension of the F-flat moduli space is $2N_c N_f - (N_c^2 - 1)$. On the other hand, these are precisely the number of degrees of freedom. There are $2N_c N_f$ complex variables which are subject to $N_c^2 - 1$ equations on the Higgs branch. We therefore conclude that the F-flat moduli space is a complete intersection and its Hilbert series can be written as a ratio of two plethystic exponentials,

$$g^{\mathcal{F}^\flat}_{N_c,N_f} = \frac{\text{PE}\left[[1,0,\dots,0]_{SU(N_c)}[0,\dots,0,1]_{SU(N_f)}t_1 + [0,\dots,0,1]_{SU(N_c)}[1,0,\dots,0]_{SU(N_f)}t_2\right]}{\text{PE}\left[[1,0,\dots,0,1]_{SU(N_c)}t^2\right]} , \tag{4.4.2}$$

where $t_1 = tb$ and $t_2 = t/b$ are respectively the global $U(1)$ fugacities for Q and $\widetilde{Q}$ and b is the fugacity for the baryonic symmetry $U(1)_B$. The Higgs branch is given by integrating this Hilbert series using the $SU(N_c)$ Haar measue,

$$g^{\text{Higgs}}_{N_c,N_f} = \int d\mu_{SU(N_c)} g^{\mathcal{F}^\flat}_{N_c,N_f} . \tag{4.4.3}$$

4.4.1 The case of $N_c = 3$ and $N_f = 6$

In this subsection, we focus on the $\mathcal{N} = 2$ supersymmetric $SU(3)$ gauge theory with 6 flavours.

From (4.4.2), the F-flat Hilbert series after setting all $U(6)$ fugacities to 1 can be written as

$$g^{\mathcal{F}^\flat}_{N_c=3,N_f=6} = \frac{(1-t^2)^2\left(1-\frac{t^2 z_1}{z_2^2}\right)\left(1-\frac{t^2}{z_1 z_2}\right)\left(1-\frac{t^2 z_1^2}{z_2}\right)\left(1-\frac{t^2 z_2}{z_1^2}\right)(1-t^2 z_1 z_2)\left(1-\frac{t^2 z_2^2}{z_1}\right)}{(1-tz_1)^6(1-tz_2)^6\left(1-\frac{t}{z_1}\right)^6\left(1-\frac{t}{z_2}\right)^6\left(1-\frac{tz_1}{z_2}\right)^6\left(1-\frac{tz_2}{z_1}\right)^6} , \tag{4.4.4}$$

where z_1 and z_2 are the $SU(3)$ fugacities. The Haar measure for $SU(3)$ is

$$\int \mathrm{d}\mu_{SU(3)} = \frac{1}{(2\pi i)^2}\oint_{|z_1|=1}\frac{dz_1}{z_1}\oint_{|z_2|=1}\frac{dz_2}{z_2}(1-z_1 z_2)\left(1-\frac{z_1^2}{z_2}\right)\left(1-\frac{z_2^2}{z_1}\right) , \tag{4.4.5}$$

After integrating over z_1 and z_2, we obtain the Hilbert series:[9]

$$g^{\text{Higgs}}_{N_c=3,N_f=6}(t) = \frac{P(t)}{(1-t)^{20}(1+t)^{16}(1+t+t^2)^{10}} , \tag{4.4.6}$$

where the numerator $P(t)$ is a palindromic polynomial of degree 36:

$$\begin{aligned} P(t) = {} & 1 + 6t + 41t^2 + 206t^3 + 900t^4 + 3326t^5 + 10846t^6 + 31100t^7 + 79677t^8 + \\ & +183232t^9 + 381347t^{10} + 720592t^{11} + 1242416t^{12} + 1959850t^{13} + \\ & +2837034t^{14} + 3774494t^{15} + 4624009t^{16} + 5220406t^{17} + 5435982t^{18} \\ & + \dots \text{ (palindrome) } \dots + t^{36} . \end{aligned} \tag{4.4.7}$$

[9] In using the residue theorem, the non-trivial contributions to the first integral over z_1 come from the poles $z_1 = t,\ tz_2$, and the non-trivial contributions to the second integral over z_2 come from the poles $z_2 = t,\ t^2$.

Note that the space is $20 = 2(3 \cdot 6 - 8)$ complex-dimensional, as expected. The first few orders of the power expansion of (4.4.6) reads

$$g^{\text{Higgs}}_{N_c=3,N_f=6}(t) = 1 + 36t^2 + 40t^3 + 630t^4 + 1120t^5 + \dots \ . \tag{4.4.8}$$

The plethystic logarithm is

$$PL\left[g^{\text{Higgs}}_{N_c=3,N_f=6}(t)\right] = 36t^2 + 40t^3 - 36t^4 - 320t^5 - 435t^6 + \dots \ . \tag{4.4.9}$$

The fully refined Hilbert series. In fact, one can obtain the fully refined Hilbert series directly from (4.4.2) and (4.4.3). The result can be written as a power series

$$\begin{aligned} &g^{\text{Higgs}}_{N_c=3,N_f=6}(t_1, t_2; x_1, x_2, x_3, x_4, x_5) = \\ &\frac{1}{1 - t_1 t_2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} [n_1, n_2, n_3 + n_4, n_2, n_1]_{SU(6)} {t_1}^{n_1+2n_2+3n_3} {t_2}^{n_1+2n_2+3n_4} \end{aligned} \tag{4.4.10}$$

where $x_1, \dots, x_5$ are the $SU(6)$ fugacities.

The plethystic logarithm of (4.4.10) is

$$\begin{aligned} \text{PL}\left[g^{\text{Higgs}}_{N_c=3,N_f=6}(t_1, t_2; x_1, x_2, x_3, x_4, x_5)\right] &= ([0,0,0,0,0] + [1,0,0,0,1])t_1 t_2 + \\ [0,0,1,0,0](t_1^3 + t_2^3) &- ([0,0,0,0,0] + [1,0,0,0,1])t_1^2 t_2^2 + \dots \ , \end{aligned} \tag{4.4.11}$$

where the gauge invariant operators in the representation $[0,0,0,0,0] + [1,0,0,0,1]$ of $SU(6)$ can be identified as *mesons* (see (4.4.17)) and the operators in the representation $[0,0,1,0,0]$ of $SU(6)$ can be identified as *baryons* and *antibaryons* (see (4.4.18)).

4.4.2 Generalisation to the case $N_f = 2N_c$

The formula (4.4.16) can be generalised to the case $N_f = 2N_c$. Let us first consider the simplest case of: $N_f = 2N_c = 4$, discussed in Section 4.3.3.

The $N_c = 2$ and $N_f = 4$ case. From (4.3.26), the Hilbert series of the coherent component of the Higgs branch is

$$g^{\text{Higgs}}_{N_c=2,N_f=4}(t; x_1, x_2, x_3, x_4) = \sum_{k=0}^{\infty} [0,k,0,0]_{SO(8)} t^{2k} \ , \tag{4.4.12}$$

The branching rule of the representation $[0,k,0,0]$ of $SO(8)$ to the subgroup $SU(4) \times U(1)_B$ is given by

$$[0,k,0,0]_{SO(8)} = \sum_{n_1,\dots,n_4 \geq 0} [n_1, n_2 + n_3, n_1]_{SU(4)} b^{2n_2 - 2n_3} \delta(k - n_1 - n_2 - n_3 - n_4) \ , \tag{4.4.13}$$

or equivalently the decomposition identity

$$\sum_{k=0}^{\infty}[0,k,0,0]_{SO(8)}t^{2k} = \frac{1}{1-t^2}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}[n_1,n_2+n_3,n_1]_{SU(4)}b^{2n_2-2n_3}t^{2n_1+2n_2+2n_3}\ , \tag{4.4.14}$$

where b is the fugacity of $U(1)_B$. Substituting (4.4.13) into (4.4.12), we obtain

$$\begin{aligned} g^{\text{Higgs}}_{N_c=2,N_f=4}(t;x_1,x_2,x_3;b) &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}[n_1,n_2+n_3,n_1]t^{2n_1+2n_2+2n_3+2n_4}b^{2n_2-2n_3} \\ &= \frac{1}{1-t_1t_2}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}[n_1,n_2+n_3,n_1]t_1^{n_1+2n_2}t_2^{n_1+2n_3}, \end{aligned} \tag{4.4.15}$$

where in the last line we take $t_1 = tb$ and $t_2 = tb^{-1}$.

Generalisation. From (4.4.10) and (4.4.15), we conjecture that the Hilbert series for the Higgs branch of the $SU(N_c)$ gauge theory with $N_f = 2N_c$ flavours can be written in terms of $SU(2N_c)$ representations as

$$\begin{aligned} &g^{\text{Higgs}}_{N_f=2N_c}(t_1,t_2;x_1,\ldots,x_{2N_c-1}) = \frac{1}{1-t_1t_2}\times \\ &\sum_{n_1,\ldots,n_{N_c+1}\geq 0}[n_1,n_2,\ldots,n_{N_c-1},n_{N_c}+n_{N_c+1},n_{N_c-1},\ldots,n_2,n_1]t_1^{d+N_cn_{N_c}}t_2^{d+N_cn_{N_c+1}}\ , \end{aligned} \tag{4.4.16}$$

where $d = \sum_{k=1}^{N_c-1} kn_k$. This formula can be checked by plugging in the dimensions of the representations, one finds that the Higgs branch is $2(N_c^2+1)$ complex dimensional, as expected. Note the similarity between (4.4.16) and the Hilbert series of $\mathcal{N}=1$ SQCD (see (2.6.1) of Chapter 2); however, they are not identical – the moduli space of $\mathcal{N}=1$ SQCD with $N_f \geq N_c$ is $2N_cN_f-(N_c^2-1)$ complex dimensional, whereas the moduli space of the $\mathcal{N}=2$ gauge theory is $2N_cN_f-2(N_c^2-1)$ complex dimensional.

The plethystic logarithm of (4.4.16) indicates that:

- At the order t_1t_2, there are gauge invariants transforming in the representation $[0,\ldots,0]+[1,0,\ldots,0,1]$ of $SU(N_f)$ and carrying $U(1)_B$ charge 0 These operators are *mesons*:

$$M^i_j = Q^i_a\widetilde{Q}^a_j\ , \tag{4.4.17}$$

where $a = 1,\ldots,N_c$ and $i,j = 1,\ldots,N_f$.

- At the order $t_1^{N_c}$ and $t_2^{N_c}$, there are gauge invariants transforming in the representation $[0, \ldots, 0, 1, 0, \ldots 0]$ of $SU(N_f)$ and carrying $U(1)_B$ charges N_c and $-N_c$. These operators are respectively *baryons* and *antibaryons*:

$$\begin{aligned} B^{i_1, \ldots, i_{N_c}} &= \epsilon^{a_1 \ldots a_{N_c}} Q^{i_1}_{a_1} \ldots Q^{i_{N_c}}_{a_{N_c}} \ , \\ \widetilde{B}_{i_1, \ldots, i_{N_c}} &= \epsilon_{a_1 \ldots a_{N_c}} \widetilde{Q}^{a_1}_{i_1} \ldots \widetilde{Q}^{a_{N_c}}_{i_{N_c}} \ . \end{aligned} \tag{4.4.18}$$

These generators are indeed identical to those of the $\mathcal{N} = 1$ SQCD. Hence, they satisfy the relations given by (2.3.11) and (2.3.12) of Chapter 2:

$$\begin{aligned} (*B)\widetilde{B} &= *(M^{N_c}) \ , \\ M \cdot *B &= M \cdot *\widetilde{B} = 0 \ . \end{aligned} \tag{4.4.19}$$

where $(*B)_{i_{N_c+1} \ldots i_{N_f}} = \frac{1}{N_c!} \epsilon_{i_1 \ldots i_{N_f}} B^{i_1 \ldots i_{N_c}}$ and a '$\cdot$' denotes a contraction of an upper with a lower flavour index. In addition, the F-terms impose further relations. These are given by (2.23) and (2.24) of [33]:

$$\begin{aligned} M' \cdot B = \widetilde{B} \cdot M' &= 0 \ , \\ M \cdot M' &= 0 \ , \end{aligned} \tag{4.4.20}$$

where

$$(M')^i_j = M^i_j - \frac{1}{N_c} (\mathrm{Tr}\, M) \delta^i_j \ . \tag{4.4.21}$$

4.5 Exceptional Groups and Argyres-Seiberg Dualities

In this section, we consider the Hilbert series of a single G instanton on $\mathbb{R}^4$ where G is one of the 5 exceptional groups. It is shown that the conjecture is consistent with the dimension of the instanton moduli space, by explicitly summing the unrefined Hilbert series. In the cases of E_6 and E_7, we also check that the proposed Hilbert Series are consistent with Argyres-Seiberg dualities found in [16–22]. Only for the case of E_6, we are able to carry out a full all-order check. In the case of E_7, we just match the lower dimension BPS operators. Notice that the check for BPS operators of scaling dimension 2 is equivalent to the check that the symmetries on both sides of the duality are the same. This is because BPS operators of scaling dimension 2 are in the same super multiplet of the flavour currents.

Notation: In this section, when there is no ambiguity, we denote special unitary (SU) groups in the quiver diagrams by their ranks. Each $U(1)$ global symmetry is associated with a hypermultiplet and hence each solid line connecting two nodes represents a $U(1)$ global symmetry. The dashed lines are not associated with bi-fundamental hypermultiplets and do not correspond to $U(1)$ global symmetries. Square nodes with an index 1 do not count as a $U(1)$ global symmetry.

4.5.1 E_6

The Hilbert series of one E_6-instanton on $\mathbb{R}^4$ is given by (4.2.1):

$$g^{\text{Irr}}_{E_6}(t; x_1, \ldots, x_6) = \sum_{k=0}^{\infty} [0, k, 0, 0, 0, 0] t^{2k}. \tag{4.5.1}$$

By setting the E_6 fugacities to 1, this equation can be resumed and written in the form of (4.2.5):

$$g^{\text{Irr}}_{E_6}(t; 1, \ldots, 1) = \frac{P_{E_6}(t)}{(1-t^2)^{22}}, \tag{4.5.2}$$

where

$$P_{E_6}(t) = (1+t^2)(1 + 55t^2 + 890t^4 + 5886t^6 + 17929t^8 + 26060t^{10} + \ldots \text{ (palindrome) } \ldots + t^{20}) \; . \tag{4.5.3}$$

This confirms that the complex dimension of the moduli space is $2h_{E_6} - 2 = 22$, where $h_{E_6} = 12$ is the dual Coxeter number of E_6.

Duality between the $6 - \bullet - 2 - 1$ quiver theory and the $SU(3)$ gauge theory with 6 flavours

As discussed in [19], the strongly interacting SCFT with E_6 flavour symmetry can be realised as 3 M5-branes wrapping a sphere with 3 punctures. These punctures are of the maximal type, each one is associated to $SU(3)$ global symmetry. The global symmetry $SU(3)^3$ enhances to E_6. This theory is also known as the T_3 theory [14, 15, 19, 20] and is denoted by the left picture of Figure 4.14. There is no known Lagrangian description for this theory.

The E_6 theory is denoted by a 'quiver diagram' which is analogous to those in previous sections. This is given in the right picture of Figure 4.14. The red blob denotes a theory with an unknown Lagrangian. The E_6 global symmetry is indicated in the square node. Below it is demonstrated that even though the Lagrangian is not known, it is still possible to make statements about the spectrum of operators for this theory.

The E_6 theory can be used to construct a quiver gauge theory called the $6 - \bullet - 2 - 1$ theory, depicted in Figure 4.18. This theory is proposed by Argyres and Seiberg [16] to be dual to an $SU(3)$ gauge theory with 6 flavours, whose quiver diagram is shown in Figure 4.16. The appearance of the tail in Figure 4.15 seems to be a generic feature of these dualities and follows from the splitting of branes when ending on the same brane - see Figure 20 of [29].

Let us summarise a construction of the $6 - \bullet - 2 - 1$ quiver theory. The global symmetry E_6 can be decomposed into the subgroup $SU(2) \times SU(6)$. The $SU(2)$

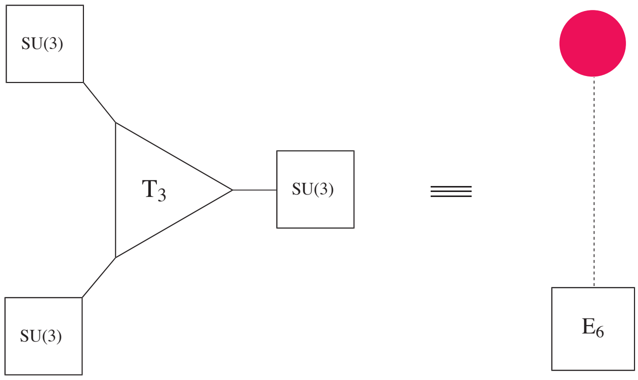

Figure 4.14. **Left:** The E_6 theory arising from 3 M5-branes wrapping a sphere with 3 maximal punctures, each is associated to $SU(3)$ global symmetry. The $SU(3)^3$ symmetry enhances to E_6. **Right:** The quiver diagram representing the E_6 theory. The red blob denotes a theory with an unknown Lagrangian description. The E_6 global symmetry is indicated in the square node.

symmetry is gauged and is coupled to the $2-1$ tail, as depicted in Figure 4.15. The resulting theory is the **the** $6-\bullet-2-1$ **quiver theory**. The $U(1)$ global symmetry is associated with the solid line in the quiver diagram. The global symmetry is thus $SU(6) \times U(1) \cong U(6)$.

Note that a necessary condition for two theories to be dual is that they have the same global symmetry. Indeed, both of the $6-\bullet-2-1$ quiver theory and the $SU(3)$ gauge theory with 6 flavours have the same global symmetry $U(6)$, even though these symmetries arise from different sources in each case.

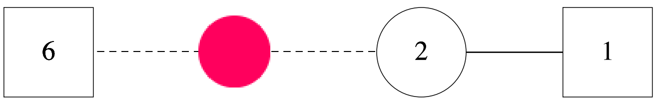

Figure 4.15. **The** $6-\bullet-2-1$ **quiver theory:** From the E_6 theory, the global symmetry E_6 is decomposed into the subgroup $SU(2) \times SU(6)$. The $SU(2)$ symmetry is gauged and is coupled to the $2-1$ tail. The $U(1)$ global symmetry is associated with the solid line in the quiver diagram. The flavour symmetry is $SU(6) \times U(1)$.

A branching rule for E_6 to $SU(2) \times SU(6)$. To proceed, we first decompose the E_6 representations into representations of $SU(2) \times SU(6)$. For this it is useful to introduce the fugacity map. The fugacities $u_1, u_2, \ldots, u_6$ of E_6 can be mapped to the fugacities x of $SU(2)$ and $y_1, \ldots, y_5$ of $SU(6)$ as follows:

$$u_1 = xy_5, \quad u_2 = y_1 y_5, \quad u_3 = y_5^2, \quad u_4 = y_2 y_5^2, \quad u_5 = y_3 y_5, \quad u_6 = y_4 \ . \quad (4.5.4)$$

Figure 4.16. The $SU(3)$ gauge theory with 6 flavours. This theory is conjectured by Argyres-Seiberg to be dual to the $6 - \bullet - 2 - 1$ quiver theory.

Using this map, one can decompose the character of an E_6 representation into the characters of $SU(2) \times SU(6)$ representations. For example, if we denote a representation of $SU(2) \times SU(6)$ of highest weight m for $SU(2)$ and highest weights $n_1, \ldots, n_5$ for $SU(6)$ by $[m; n_1, \ldots, n_5]$, then one finds that

$$\begin{aligned} [0,1,0,0,0,0]_{E_6} &= [0;1,0,0,0,1] + [1;0,0,1,0,0] + [2;0,0,0,0,0], \\ [0,2,0,0,0,0]_{E_6} &= [0;0,0,0,0,0] + [0;0,1,0,1,0] + [0;2,0,0,0,2] + \\ &\quad [1;0,0,1,0,0] + [1;1,0,1,0,1] + [2;1,0,0,0,1] + \\ &\quad [2;0,0,2,0,0] + [3;0,0,1,0,0] + [4;0,0,0,0,0] \ . \end{aligned} \tag{4.5.5}$$

These equalities can be checked by matching the characters of the representations on both sides. The general formula for the decompositions of Adj^k for any k is given in (4.5.11).

The decompositions (4.5.5) can be written in terms of dimensions as

$$\begin{aligned} 78 &\to (1,35) \oplus (2,20) \oplus (3,1) \\ 2430 &\to (1,1) \oplus (1,189) \oplus (1,405) \oplus (2,20) \oplus (2,540) \oplus (3,35) \oplus \\ &\quad (3,175) \oplus (4,20) \oplus (5,1) \ . \end{aligned} \tag{4.5.6}$$

Counting BPS operators of the $SU(3)$ gauge theory with 6 flavours. In what follows, starting from (4.5.1), we count BPS operators in the $SU(3)$ gauge theory with 6 flavours by computing the $SU(2)$ gauge invariant spectrum. For now, let us first do this order by order for the operators of small scaling dimensions. In the later subsections, we present a method to count the operators to all orders.

- At level t^2, we expect the 35 to survive, as it is an $SU(2)$ singlet. Denote the $2-1$ hypermultiplet in Figure 4.15 by q and $\tilde{q}$. Set q to have fugacity tb^3 and $\tilde{q}$ to have fugacity t/b^3, where the normalization 3 is chosen for matching with the $U(6)$ baryons. One can construct another $SU(2)$ invariant which is a singlet under $SU(6)$, by forming $q\tilde{q}$. We therefore expect the SU(3) theory with 6 flavours to have $35_0 + 1_0$ at order t^2, where the subscript 0 refers to the $U(1)_B$ baryonic charge. Indeed, in the $SU(3)$ theory of Figure 4.16 these are formed by the $SU(3)$ mesons $\tilde{Q}Q$ that decompose as $35_0 + 1_0$.

- At level t^3, the $(2, 20)$ coupled to q or to $\tilde{q}$, leads to the $SU(2)$ invariant operators which transform as $20_3 \oplus 20_{-3}$. This contributes the term $20(b^3 + 1/b^3)t^3$ to the $U(6)$ Hilbert series.
- At level t^4 we have the singlets $1+189+405$, and the 35 from order t^2 multiplied by the $SU(6)$-singlet $q\tilde{q}$, for a total of 630 operators.

These are precisely the first few terms of the Hilbert series (4.4.8) of the Higgs Branch of $SU(3)$ theory with 6 flavours:

$$g^{\text{Higgs}}_{N_c=3,N_f=6}(t) = 1 + 36t^2 + 20(b^3 + b^{-3})t^3 + 630t^4 + \dots \; . \tag{4.5.7}$$

Branching formula for Adj^k of E_6 to $SU(2) \times SU(6)$

In this subsection, we carry out the decomposition of the Adj^k-irreducible representations of E_6 into $SU(2) \times SU(6)$ to all order in k. This gives a useful check of the Argyres-Seiberg duality to all orders. The general form of the decomposition is as follows:

$$[0, k, 0, 0, 0, 0]_{E_6} = \sum_{m=0}^{2k} [m]_{SU(2)} C^k_m \tag{4.5.8}$$

where C^k_m is a reducible representation of $SU(6)$. The sets of irreps of $SU(6)$ entering in C^k_m is constructed starting by the representation R^L_p, defined by:

$$\begin{aligned} R^L_{p>0} &= \sum_{n=0}^{L} \sum_{i+2j+3/2k=n} [i, j, k+p, j, i] \\ R^{2L}_{p=0} &= \sum_{n=0}^{L} \sum_{i+2j+3/2k=2n} [i, j, k, j, i] \\ R^{2L+1}_{p=0} &= \sum_{n=0}^{L} \sum_{i+2j+3/2k=2n+1} [i, j, k, j, i] \end{aligned} \tag{4.5.9}$$

Notice that only $SU(6)$-irreps whose Dynkin labels are symmetric enter the sum, and that R^k_n contains an irreducible representation at most one time. The C^k are given in terms of the R^L_p by

$$\begin{aligned} C^k_{2m} &= \sum_{j=0}^{m} R^{k-m-j}_j \\ C^k_{2m+1} &= \sum_{j=0}^{m} R^{k-m-1-j}_j \end{aligned} \tag{4.5.10}$$

In C^k_m the same irreducible representation can appear multiple times. Summing these together we find the decomposition identity

$$\begin{aligned} &\left(1-t^4\right)\sum_{k=0}^{\infty}[0,k,0,0,0,0]_{E_6}t^{2k} \\ &= \sum_{n_1,\ldots,n_5\geq 0}[n_1+2n_2]_{SU(2)}[n_3,n_4,n_1+2n_5,n_4,n_3]_{SU(6)}t^{2n_1+2n_2+2n_3+4n_4+6n_5} \\ &+ \sum_{n_1,\ldots,n_5\geq 0}[n_1+2n_2+1]_{SU(2)}[n_3,n_4,n_1+2n_5+1,n_4,n_3]_{SU(6)}t^{2n_1+2n_2+2n_3+4n_4+6n_5+4}. \end{aligned} \tag{4.5.11}$$

Using these all order results, we can proceed to refine $g^{\mathrm{Irr}}_{E_6}(t)$ in (4.5.2) to a function of z and t (denoted as $g^{\mathrm{Irr}}_{E_6}(z,t)$), where z is the $SU(2)$ fugacity.

The Hilbert series of the $6-\bullet-2-1$ quiver theory

As discussed earlier, the $6-\bullet-2-1$ quiver theory can be obtained by first decomposing the E_6 into $SU(2)\times SU(6)$, the $SU(2)$ group is then gauged and is coupled as in the $2-1$ quiver. This process can also be described as a 'sewing' of two Riemann surfaces - one with 3 maximal punctures (corresponding to E_6) and the other with two simple puctures (corresponding to $U(2)\times U(1)$). The Hilbert series can be computed in analogy to the AGT relation [34, 35] as follows:

$$g_{6-\bullet-2-1}(t) = \int \mathrm{d}\mu_{SU(2)}(z)\; g^{\mathrm{Irr}}_{E_6}(t,z)\; g_{\mathrm{glue}}(t,z)\; g_{2-1}(t,b,z)\;, \tag{4.5.12}$$

where the Haar measure for $SU(2)$ is given by

$$\int \mathrm{d}\mu_{SU(2)} = \frac{1}{2\pi i}\oint dz\frac{1-z^2}{z}\;, \tag{4.5.13}$$

the Hilbert series for the bi-fundmentals connecting the $SU(2)$ and $U(1)$ nodes is

$$\begin{aligned} g_{2-1}(t,b,z) &= \mathrm{PE}\left[[1]_{SU(2)}\left(b^3+b^{-3}\right)t\right] \\ &= \frac{1}{(1-tzb^3)(1-t\frac{z}{b^3})\left(1-\frac{tb^3}{z}\right)\left(1-\frac{t}{zb^3}\right)}\;, \end{aligned} \tag{4.5.14}$$

and the 'gluing factor' which keeps track of the 3 F-term relations that comes from differentiating the superpotential by the adjoint chiral field of $SU(2)$ is

$$g_{\mathrm{glue}}(t,z) = \frac{1}{\mathrm{PE}\left[[2]_{SU(2)}t^2\right]} = \left(1-t^2z^2\right)\left(1-t^2\right)\left(1-\frac{t^2}{z^2}\right)\;. \tag{4.5.15}$$

The product of $g_{\mathrm{fund}}(z,t)$ and $g_{\mathrm{glue}}(z,t)$ can be written for $b=1$ as

$$\begin{aligned} g_{\mathrm{glue}}(t,z)g_{2-1}(t,1,z) &= \frac{\left(1-t^2z^2\right)\left(1-t^2\right)\left(1-\frac{t^2}{z^2}\right)}{(1-tz)^2\left(1-\frac{t}{z}\right)^2} \\ &= \sum_{n=0}^{\infty}[n]t^n + \sum_{n=0}^{\infty}[n+1]t^{n+1} + t^2 - 2\sum_{n=0}^{\infty}[n]t^{n+4}\;. \end{aligned} \tag{4.5.16}$$

If we restore the b dependence, this sum takes the form

$$\begin{aligned} & g_{\text{glue}}(t,z) g_{2-1}(t,b,z) = \\ = & \sum_{n=0}^{\infty} [n](tb^3)^n + \sum_{n=0}^{\infty} [n+1] \left(\frac{t}{b^3}\right)^{n+1} + t^2 - \sum_{n=0}^{\infty} [n] t^{n+4} (b^{3n+6} + b^{-3n-6}) \, . \end{aligned} \quad (4.5.17)$$

From (4.5.12), one sees that the integral is computed by summing over two residues, one at $z = t$ and one at $z = t^2$. For $z = t$, the residue is a rational function with denominator $(1-t)^{21}(1+t+t^2)^{21}$. For $z = t^2$, the residue is a rational function with denominator $(1-t)^{21}(1+t)^{16}(1+t^2)^{37}(1+t+t^2)^{21}$. Summing these two residues gives precisely the unrefined Hilbert series $g^{\text{Higgs}}_{N_c=3,N_f=6}(t)$ of (4.4.6).

For the refined Hilbert series, it is better to exchange the integral in (4.5.12) with the sums and use the orthonormality relation

$$\oint_{|z|=1} \frac{dz(1-z^2)}{2\pi i z} [n][m] = \delta_{n,m} \quad (4.5.18)$$

to confirm that the fully refined Hilbert series coincides with (4.4.10).

4.5.2 E_7

The Hilbert series of one E_7-instanton on $\mathbb{R}^4$ is given by (4.2.1):

$$g^{\text{Irr}}_{E_7}(t; x_1, \ldots, x_6, x_7) = \sum_{k=0}^{\infty} [k,0,0,0,0,0,0] t^{2k}. \quad (4.5.19)$$

By setting the E_7 fugacities to 1, this equation can be resumed and written in the form of (4.2.5):

$$g^{\text{Irr}}_{E_7}(t; 1, \ldots, 1) = \frac{P_{E_7}(t)}{(1-t^2)^{34}} \, , \quad (4.5.20)$$

where the numerator is a palindromic polynomial of degree 17 in t^2,

$$\begin{aligned} P_{E_7}(t) = & \ 1 + 99t^2 + 3410t^4 + 56617t^6 + 521917t^8 + 2889898t^{10} + 10086066t^{12} + \\ & 22867856t^{14} + 34289476t^{16} + \ldots \ (\text{palindrome}) \ \ldots + t^{34} \, . \end{aligned} \quad (4.5.21)$$

This is consistent with the fact that the Higgs branch is $2h_{E_7} - 2 = 34$ complex dimensional, where $h_{E_7} = 18$ is the dual Coxeter number of E_7.

Duality between the $6 - \bullet - 3 - 2 - 1$ quiver theory and the $2 - 4 - 6$ quiver theory

In [22], it was realised that the E_7 theory can be realised as 4 M5-branes wrapped over a sphere with 3 punctures. The punctures are of the type $SU(4)$, $SU(4)$, $SU(2)$.

This theory is depicted in the left picture of Figure 4.17. The Lagrangian description of this theory is unknown.

We denote the E_7 theory by a 'quiver diagram' analogue to those in previous sections. This is given in the right picture of Figure 4.17. The green blob denotes the theory with unknown Lagrangian description. The E_7 global symmetry is indicated in the square node.

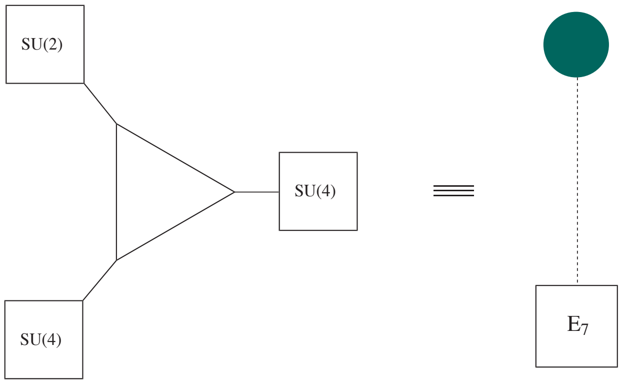

Figure 4.17. **Left:** The E_7 theory arising from 4 M5-branes wrapped over a sphere with 3 punctures of the type $SU(4)$, $SU(4)$, $SU(2)$. **Right:** The quiver diagram representing the E_7 theory. The green blob denotes a theory with an unknown Lagrangian description. The E_7 global symmetry is indicated by the square node.

The E_7 theory can be used to construct a quiver gauge theory called the $6 - \bullet - 3 - 2 - 1$ theory, depicted in Figure 4.18. The duality between this theory and the $2 - 4 - 6$ quiver theory (depicted in Figure 4.19) is proposed by [22]. Our purpose of this section is to construct and match the Hilbert series of both sides of the duality.

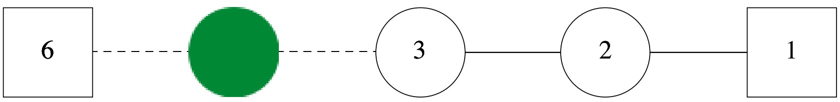

Figure 4.18. **The $6 - \bullet - 3 - 2 - 1$ quiver theory:** The global symmetry E_7 can be decomposed into the subgroup $SU(3) \times SU(6)$. The $SU(3)$ symmetry is gauged and is coupled to the $3 - 2 - 1$ tail. The $U(1)$ global symmetries are associated with the solid lines in the quiver diagram. The global symmetry is thus $SU(6) \times U(1) \times U(1)$.

Let us summarise a construction of the $6 - \bullet - 3 - 2 - 1$ quiver theory. The global symmetry E_7 can be decomposed into the subgroup $SU(3)\times SU(6)$. The $SU(3)$ symmetry is gauged and is coupled to the $3 - 2 - 1$ tail, depicted in Figure 4.18. The $U(1)$ global symmetries are associated with the hypermultiplets and hence the solid lines in the quiver diagram. The global symmetry is thus $SU(6) \times U(1) \times U(1)$.

A trick to obtain the $3-2-1$ tail is to consider the $SU(2)$ theory with 4 flavours, whose flavour symmetry of is $SO(8)$. The group $SO(8)$ contains $SU(4) \times U(1) \supset SU(3) \times U(1) \times U(1)$ as subgroups. Gauging the $SU(3)$ group in $SO(8)$ and gluing it to the $SU(3)$ group in E_7, we obtain the $6 - \bullet - 3 - 2 - 1$ quiver theory.

On the other side of the duality, we have the $2 - 4 - 6$ quiver theory, depicted in Figure 4.19. The $U(1)$ global symmetries are associated with the hypermultiplets and hence the solid lines in the quiver diagram. Therefore, the flavour symmetry is $U(6) \times U(1) \cong SU(6) \times U(1) \times U(1)$, in agreement with that of the $6 - \bullet - 3 - 2 - 1$ quiver theory. From the quiver diagram, it is clear that the $2 - 4 - 6$ quiver theory can also be obtained by gauging the $SU(2)$ subgroup of the $U(8)$ flavour group of the $SU(4)$ gauge theory with 8 flavours.

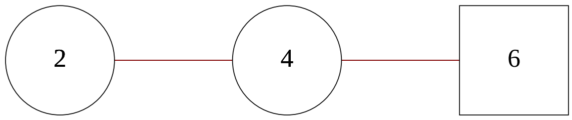

Figure 4.19. **The $2 - 4 - 6$ quiver theory.** This theory is dual to the $6 - \bullet - 3 - 2 - 1$ quiver theory.

The Hilbert series of the $2 - 4 - 6$ quiver theory

In this subsection, the refined and unrefined Hilbert series are computed. The former contains information about the global symmetries and how the gauge invariants transform under such symmetries, whereas the latter contains information about the dimension of the moduli space and the number of operators in the spectrum. In order to compute an exact form of the refined Hilbert series, general formulas involving branching rules need to be determined. However, such formulas can sometimes be very cumbersome and difficult to compute; in which case, what one can do is to compute the first few orders of the refined Hilbert series. Nevertheless, it may be possible that the unrefined Hilbert series can be computed exactly. We give an example below.

The $2 - 4 - 6$ quiver theory can be obtained by gauging the $SU(2)$ subgroup of the $U(8)$ flavour group of the $SU(4)$ gauge theory with 8 flavours. The Hilbert series written in terms of $SU(8)$ representations is given by (4.4.16). We first discuss a branching rule for $SU(8)$ to $U(1) \times SU(2) \times SU(6)$.

A branching rule for $SU(8)$ to $U(1) \times SU(2) \times SU(6)$. A map from the $SU(8)$ fugacities $x_1, \ldots, x_7$ to the $U(1)$ fugacity q, the $SU(2)$ fugacity z and the $SU(6)$

fugacities $y_1, \ldots, y_5$ can be

$$\begin{array}{llll} x_1 = q y_1, & x_2 = q^2 y_2, & x_3 = q^3 y_3, & x_4 = q^4 y_4, \\ x_5 = q^5 y_5, & x_6 = q^6, & x_7 = q^3 z \ . & \end{array}$$

For example, we have

$$\begin{aligned} [1,0,0,0,0,0,0] &= [0;1,0,0,0,0]q + [1;0,0,0,0,0]q^{-3} \\ [1,0,0,0,0,0,1] &= [0;0,0,0,0,0] + [2;0,0,0,0,0] + [1;0,0,0,0,1]q^{-4} \\ &\quad +[1;1,0,0,0,0]q^4 + [0;1,0,0,0,1] \ . \end{aligned} \tag{4.5.22}$$

Using this decomposition, the Hilbert series of the $SU(4)$ theory with 8 flavours can be written as

$$\begin{aligned} g^{\text{Higgs}}_{N_c=4,N_f=8} &= 1 + (2 + [2;0,0,0,0,0] + [1;0,0,0,0,1]\frac{1}{q^4} + [1;1,0,0,0,0]q^4 \\ &+[0;1,0,0,0,1])t^2 + \Big(4 + 2[2;0,0,0,0,0] + [4;0,0,0,0,0] + \frac{3[1;0,0,0,0,1]}{q^4} \\ &+\frac{[3;0,0,0,0,1]}{q^4} + \frac{[2;0,0,0,0,2]}{q^8} + \frac{[0;0,0,0,1,0]}{q^8} + \frac{q^4[0;0,0,0,1,0]}{b^2} \\ &+b^2 q^4[0;0,0,0,1,0] + \frac{[1;0,0,1,0,0]}{b^2} + b^2[1;0,0,1,0,0] + \frac{[0;0,1,0,0,0]}{b^2 q^4} \\ &+\frac{b^2[0;0,1,0,0,0]}{q^4} + q^8[0;0,1,0,0,0] + q^4[1;0,1,0,0,1] + [0;0,1,0,1,0] \\ &+3q^4[1;1,0,0,0,0] + q^4[3;1,0,0,0,0] + 3[0;1,0,0,0,1] + 2[2;1,0,0,0,1] \\ &+\frac{[1;1,0,0,0,2]}{q^4} + \frac{[1;1,0,0,1,0]}{q^4} + q^8[2;2,0,0,0,0] + q^4[1;2,0,0,0,1] \\ &+[0;2,0,0,0,2]\Big)t^4 + \ldots \ . \end{aligned} \tag{4.5.23}$$

The refined Hilbert series of the $2-4-6$ theory. This can be computed by gauging the $SU(2)$ symmetry. The gauging is done by integrating over the $SU(2)$ Haar measure and Supersymmetry imposes additional adjoint valued F terms, which are written below as the glue factor,

$$g_{2-4-6}(t;q;b;y_1,\ldots,y_5) = \int \mathrm{d}\mu_{SU(2)} \; g_{\text{glue}} \; g^{\text{Higgs}}_{N_c=4,N_f=8} \ , \tag{4.5.24}$$

where the gluing factor is given by

$$g_{\text{glue}}(t;z) = \frac{1}{\text{PE}\left[[2]_{SU(2)}t^2\right]} = 1 - [2]t^2 + [2]t^4 - t^6 \ . \tag{4.5.25}$$

The integral in (4.5.24) projects out the $SU(2)$ singlets. This gives

$$g_{2-4-6}(t;q,b;y_1,\ldots,y_5) = 1 + (2 + [1,0,0,0,1])t^2 + \Big(3 + \frac{1}{q^4}[0,0,0,1,0] + \frac{q^2}{b^2}[0,0,0,1,0] + b^2q^2[0,0,0,1,0] + \frac{1}{b^2q^2}[0,1,0,0,0] + \frac{b^2}{q^2}[0,1,0,0,0] + q^4[0,1,0,0,0] + [0,1,0,1,0] + 3[1,0,0,0,1] + [2,0,0,0,2]\Big)t^4 + \ldots \tag{4.5.26}$$

The unrefined Hilbert series. The unrefined Hilbert series can be computed exactly. Setting $q = b = y_1 = \ldots = y_5 = 1$ in (4.5.24), it can be easily seen that the integrand is simply a rational function of t and z. Evaluating the integral, one obtains the closed form

$$\begin{aligned} g_{2-4-6}(t) &= \frac{P(t)}{(1-t^2)^{28}(1+t^2)^{14}} \\ &= 1 + 37t^2 + 792t^4 + 12180t^6 + 145838t^8 + 1422490t^{10} + \ldots \end{aligned} \tag{4.5.27}$$

where

$$\begin{aligned} P(t) = {} & 1 + 23t^2 + 351t^4 + 3773t^6 + 29904t^8 + 180648t^{10} + 855350t^{12} + \\ & 3243202t^{14} + 10014534t^{16} + 25512281t^{18} + 54163863t^{20} + \\ & 96566265t^{22} + 145392195t^{24} + 185575556t^{26} + 201252816t^{28} \\ & + \ldots \text{ (palindrome) } \ldots + t^{56} . \end{aligned} \tag{4.5.28}$$

The plethystic logarithm of this Hilbert series is

$$\mathrm{PL}[g_{2-4-6}(t)] = 37t^2 + 89t^4 - 252t^6 - 2800t^8 + 14720t^{10} + 124524t^{12} + \ldots . \tag{4.5.29}$$

The Hilbert series of the $6 - \bullet - 3 - 2 - 1$ quiver theory

As described in Section 4.5.2, the $6 - \bullet - 3 - 2 - 1$ quiver theory can be obtained by 'gluing' the $SU(3)$ subgroup of the E_7 theory with the $SU(3)$ subgroup of the $SO(8)$ flavor symmetry for $SU(2)$ with 4 flavors. The Hilbert series of the latter, written in terms of $U(4)$ representations, is given in Equation (4.4.15). In order to gauge the $SU(3)$ subgroup, one needs to find a branching rule for $SU(4)$ to $U(1) \times SU(3)$.

A branching rule for $SU(4)$ to $U(1) \times SU(3)$. A map from the $SU(4)$ fugacities $x_1, \ldots, x_3$ to the $U(1)$ fugacity q and the $SU(3)$ fugacities z_1, z_2 can be

$$x_1 = \frac{z_1}{q}, \quad x_2 = \frac{z_2}{q^2}, \quad x_3 = \frac{1}{q^3} . \tag{4.5.30}$$

With this map, one can rewrite (4.4.15) in terms of $SU(3)$ representations as

$$
\begin{aligned}
g^{\text{Higgs}}_{3-2-1} &= \frac{1}{1-t^2}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}[n_1, n_2+n_3, n_1]_{SU(4)}t^{2n_1+2n_2+2n_3}b^{2n_2-2n_3} \\
&= \frac{1}{(1-t^2)^2}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} q^{2n_1-2n_2}\frac{b^{-2(n_1+n_2)}(1-b^{4(1+n_1+n_2)})}{(1-b^4)} \times \\
&\quad \Big[[n_1+n_3, n_2+n_3] + \sum_{n_4=0}^{n_3-1}(q^{-4n_3+4n_4}[n_1+n_3, n_2+n_4] \\
&\quad +q^{4n_3-4n_4}[n_1+n_4, n_2+n_3])\Big]t^{2(n_1+n_2+n_3)} \ . \qquad (4.5.31)
\end{aligned}
$$

Since we need to gauge $SU(3) \subset E_7$, we also need to obtain the branching rule of E_7 representations to the subgroup $SU(3) \times SU(6)$.

Branching rule for E_7 to $SU(3) \times SU(6)$. The branching rules can be obtained by matching the characters on both sides. A map of the E_7 fugacities $u_1, \ldots, u_7$ to the $SU(3)$ fugacities z_1, z_2 and the $SU(6)$ fugacities $y_1, \ldots, y_5$ can be

$$
u_1 = z_1 y_2, \quad u_2 = y_1 y_2, \quad u_3 = z_2 y_2^2, \quad u_4 = y_2^3, \quad u_5 = \frac{y_2^2 y_3}{y_4}, \quad u_6 = \frac{y_2^2}{y_4}, \quad u_7 = \frac{y_2 y_5}{y_4} \ . \qquad (4.5.32)
$$

For example, the decompositions of Adj^1 and Adj^2 of E_7 are given below. We use the notation $[a_1, a_2; b_1, \ldots, b_5]$ to denote the representations of $SU(3) \times SU(6)$.

$$
\begin{aligned}
Adj^1 &= [1,1;0,0,0,0,0] + [1,0;0,1,0,0,0] + [0,1;0,0,0,1,0] + [0,0;1,0,0,0,1] \\
Adj^2 &= [2,2;0,0,0,0,0] + [2,0;0,2,0,0,0] + [0,2;0,0,0,2,0] + [0,0;2,0,0,0,2] \\
&+ [2,1;0,1,0,0,0] + [1,1;0,1,0,1,0] + [0,1;1,0,0,1,1] + [2,0;0,0,0,1,0] \\
&+ [1,2;0,0,0,1,0] + [1,0;1,1,0,0,1] + [1,1;0,0,0,0,0] + [0,2;0,1,0,0,0] \\
&+ [1,1;1,0,0,0,1] + [1,0;0,1,0,0,0] + [1,0;0,0,1,0,1] + [0,0;1,0,0,0,1] \\
&+ [0,0;0,1,0,1,0] + [0,1;0,0,0,1,0] + [0,1;1,0,1,0,0] + [0,0;0,0,0,0,0] \ .
\end{aligned}
\qquad (4.5.33)
$$

The Hilbert series of the coherent component of the one E_7 instanton moduli space on $\mathbb{R}^4$ after using the fugacity map Equation (4.5.32) is

$$
g^{\text{Irr}}_{E_7}(t; z_1, z_2; y_1, \ldots, y_5) = \sum_{k=0}^{\infty} Adj^k(z_1, z_2; y_1, \ldots, y_5)t^{2k} \ . \qquad (4.5.34)
$$

Gluing process. We obtain the Hilbert series of the $6-\bullet-3-2-1$ quiver theory by using a similar 'gluing technique' to Equation (4.5.12):

$$
g_{6-\bullet-3-2-1}(t; q, b; y_1, \ldots, y_5) = \int \mathrm{d}\mu_{SU(3)}\ g^{\text{Irr}}_{E_7}\ g_{\text{glue}}\ g^{\text{Higgs}}_{3-2-1} \ , \qquad (4.5.35)
$$

where the gluing factor is given by the adjoint valued F terms,

$$g_{\text{glue}}(t; z_1, z_2) = \frac{1}{\text{PE}\left[[1,1]_{SU(3)} t^2\right]} \ . \tag{4.5.36}$$

Therefore, we obtain

$$\begin{aligned} g_{6-\bullet-3-2-1}(t; q, b; y_1, \ldots, y_5) &= 1 + (2 + [1,0,0,0,1])t^2 + \Big(3 + \frac{1}{q^8}[0,0,0,1,0] \\ &+ \frac{q^4}{b^2}[0,0,0,1,0] + b^2 q^4 [0,0,0,1,0] + \frac{1}{b^2 q^4}[0,1,0,0,0] + \frac{b^2}{q^4}[0,1,0,0,0] \\ &+ q^8 [0,1,0,0,0] + [0,1,0,1,0] + 3[1,0,0,0,1] + [2,0,0,0,2]\Big) t^4 + \ldots \ , \end{aligned} \tag{4.5.37}$$

in accordance with (4.5.26), up to a rescaling of q (which means simply that we use different units in counting charges):

$$g_{6-\bullet-3-2-1}(t; q, b; y_1, \ldots, y_5) = g_{2-4-6}(t; q^2, b; y_1, \ldots, y_5) \ . \tag{4.5.38}$$

Unrefining $b = q = y_1 = \ldots = y_5 = 1$, we obtain the unrefined Hilbert series up to the order t^8 as

$$g_{6-\bullet-3-2-1}(t) = 1 + 37t^2 + 792t^4 + 12180t^6 + 145838t^8 + \ldots \ . \tag{4.5.39}$$

This is in agreement with (4.5.28).

4.5.3 E_8

The resummed Hilbert series for the coherent branch of one E_8 instanton is

$$g^{\text{Irr}}_{E_8}(t; 1, \ldots, 1) = \frac{P_{E_8}(t)}{(1-t^2)^{58}} \ , \tag{4.5.40}$$

where the numerator is a palindromic polynomial of degree 58:

$$\begin{aligned} P_{E_8}(t) = \ & 1 + 190t^2 + 14269t^4 + 576213t^6 + 14284732t^8 + 234453749t^{10} + \\ & 2675683550t^{12} + 21972715186t^{14} + 133126452657t^{16} + 606326972328t^{18} + \\ & 2105555153625t^{20} + 5634990969615t^{22} + 11714759112330t^{24} + \\ & 19025183027595t^{26} + 24223919026560t^{28} + \ldots \ \text{(palindrome)} \ \ldots + t^{58} \ . \end{aligned} \tag{4.5.41}$$

This is consistent with the fact that the Higgs branch is $2h_{E_8} - 2 = 58$ complex dimensional, where $h_{E_8} = 30$ is the dual Coxeter number of E_8.

The E_8 theory arises from 6 M5-branes wrapping a sphere with 3 punctures. The 3 punctures are of the type $SU(6)$, $SU(3)$, $SU(2)$. The quiver diagram is depicted in the left picture of Figure 4.20. The Lagrangian description of this theory is unknown.

We denote the E_8 theory by a 'quiver diagram' analogue to those in previous sections. This is given in the right picture of Figure 4.20. The blue blob denotes a theory with an unknown Lagrangian description. The E_8 global symmetry is indicated in the square node.

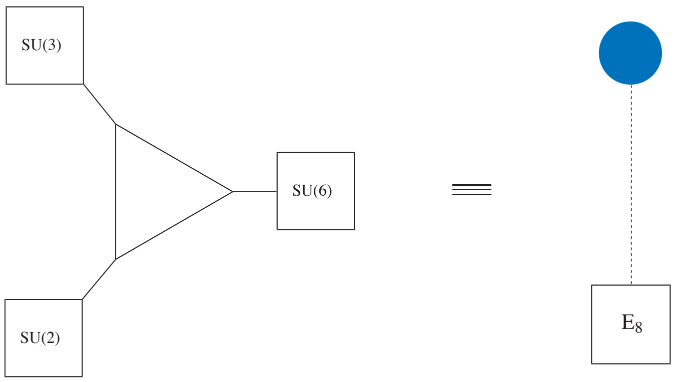

Figure 4.20. **Left:** The E_8 theory arises from 6 M5-branes wrapping a sphere with 3 punctures. The 3 punctures are of the type $SU(6)$, $SU(3)$, $SU(2)$. **Right:** The quiver diagram representing the E_8 theory. The blue blob denotes a theory with an unknown Lagrangian description. The E_8 global symmetry is indicated in the square node.

The E_8 theory can be used to construct a quiver gauge theory called the $5 - \bullet - 5 - 4 - 3 - 2 - 1$ theory, depicted in Figure 4.21. The duality between this theory and the $3 - 6_{[5]} - 4 - 2$ quiver theory (depicted in Figure 4.22) is proposed by [22].

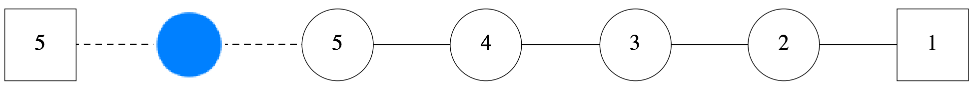

Figure 4.21. The $5 - \bullet - 5 - 4 - 3 - 2 - 1$ quiver theory. The $U(1)$ global symmetries are associated with the solid lines in the quiver diagram. The flavour symmetry is expected to be $SU(5) \times U(1)^4$.

The $5 - \bullet - 5 - 4 - 3 - 2 - 1$ theory can be constructed as follows. The global symmetry E_8 can be decomposed into $SU(5) \times SU(5)$. One of the $SU(5)$ is gauged and is coupled to the $5-4-3-2-1$ tail. The $U(1)$ global symmetries are associated with the solid lines in the quiver diagram. Hence, the flavour symmetry is expected to be $SU(5) \times U(1)^4$.

On the other side of the duality, we have the $3 - 6_{[5]} - 4 - 2$ quiver theory depicted in Figure 4.22. As in all previous quivers, the $U(1)$ global symmetries are

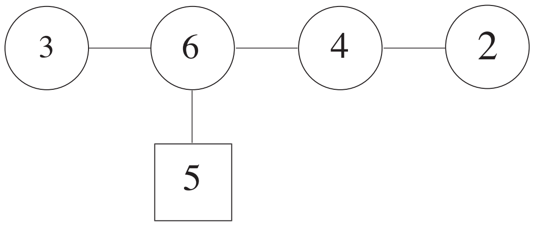

Figure 4.22. The 3−$6_{[5]}$−4−2 quiver theory. This theory is dual to the 5−•−5−4−3−2−1 theory.

associated with the solid lines in the quiver diagram, and the flavour symmetry is expected to be $SU(5) \times U(1)^4$, in agreement with that of the $5 - \bullet - 5 - 4 - 3 - 2 - 1$ quiver theory.

The computations of Hilbert series of these theories are rather involved and technical. We leave such computations for future work.

4.5.4 One F_4 instanton on $\mathbb{C}^2$

There is no simple analog of the ADHM construction. Instead the conjecture of this chapter is that the Hilbert series for the one instanton moduli space on $\mathbb{C}^2$ is a sum over symmetric adjoint representations. Explicitly, denote the adjoint representation of F_4 by $[1,0,0,0]$, and the symmetric adjoints by $[k,0,0,0]$, then the dimension of each representation is

$$\begin{aligned}&\dim\,[k,0,0,0] = \qquad (4.5.42)\\ &\frac{(k+1)(k+2)(k+3)^2(k+4)^3(k+5)^2(k+6)(k+7)(2k+5)(2k+7)(2k+9)(2k+11)}{4191264000},\end{aligned}$$

and the Hilbert series for the moduli space takes the form

$$g_{F_4}(t; x_1, x_2, x_3, x_4, x) = \frac{1}{(1-tx)(1-t/x)} \sum_{k=0}^{\infty} [k,0,0,0] t^{2k} \ , \qquad (4.5.43)$$

Where as usual, the first term is the Hilbert series for $\mathbb{C}^2$, physically interpreted as the position of the instanton and the remaining function is the Hilbert series for the coherent component of the moduli space. By setting the F_4 fugacities to 1 one can get an explicit palindromic rational function for the coherent component of the moduli space,

$$g^{\text{Irr}}_{F_4}(t) = \frac{1+36t^2+341t^4+1208t^6+1820t^8+1208t^{10}+341t^{12}+36t^{14}+t^{16}}{(1-t^2)^{16}} \qquad (4.5.44)$$

giving a non-trivial check that the dimension of this moduli space is $2(h-1) = 16$, where $h = 9$ is the dual Coxeter number of F_4.

4.5.5 One G_2 instanton on $\mathbb{C}^2$

This case also has no known simple ADHM construction. Denote the character of the adjoint representation by $[0,1]$ and the character for the k-th symmetric adjoint by $[0,k]$, with dimension

$$\dim\,[0,k] = \frac{(k+1)(k+2)(2k+3)(3k+4)(3k+5)}{120}\ . \tag{4.5.45}$$

The Hilbert series takes the form

$$g_{G_2}(t;x_1,x_2,x) = \frac{1}{(1-tx)(1-t/x)}\sum_{k=0}^{\infty}[0,k]t^{2k}, \tag{4.5.46}$$

and setting the fugacities to 1 gives

$$g_{G_2}(t;1,1,1) = \frac{1}{(1-t)^2}\frac{1+8t^2+8t^4+t^6}{(1-t^2)^6}\ , \tag{4.5.47}$$

giving a non-trivial check that the dimension of this moduli space is $2(h_{G_2}-1)=6$, where $h_{G_2}=4$ is the dual Coxeter number of G_2. Since the rank of this gauge group is 2, it is possible to compute the sum explicitly and write the Hilbert series as a rational function with characters of G_2. Omitting the trivial $\mathbb{C}^2$ part we get

$$g^{\rm Irr}_{G_2}(t;x_1,x_2) = P_{G_2}(t;x_1,x_2)\,\mathrm{PE}\left[[0,1]t^2\right]\ , \tag{4.5.48}$$

where P_{G_2} is a palindromic polynomial of degree 11 in t^2 and has the form

$$\begin{aligned} P_{G_2}(t;x_1,x_2) =\ & 1-([2,0]+1)t^4+([1,1]+[2,0]+[0,1])t^6-([3,0]+[1,1]+[0,1]+[1,0])t^8\\ &+([3,0]+[1,0])t^{10}+([3,0]+[1,0])t^{12}-([3,0]+[1,1]+[0,1]+[1,0])t^{14}\\ &+([1,1]+[2,0]+[0,1])t^8-([2,0]+1)t^{18}+t^{22}\ . \end{aligned} \tag{4.5.49}$$

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Chapter 5

Tri-vertices and $SU(2)$'s

5.1 Introduction

Recently, a new class of $\mathcal{N} = 2$ superconformal field theories in (3+1) dimensions has been explored [1]. These theories are proposed to be the worldvolume theories of M5 branes wrapping Riemann surfaces. In this chapter, we focus on the case in which the number of M5 branes is two, so that the gauge groups involved are $SU(2)$'s. These theories can be represented by graphs, called **skeleton diagram**[1], consisting of lines and trivalent vertices, where a line represents an $SU(2)$ gauge group and a trivalent vertex represents a matter field in the tri-fundamental representation of $SU(2)^3$ (see Section 5.2 for more details). Such a graph defines a unique $\mathcal{N} = 2$ Langrangian in $(3 + 1)$ dimensions. These graphs can be topologically classified by the genus g and the number of external legs e.

In this chapter, we focus on the branch of the moduli space parametrised by the vacuum expectation values (VEVs) of the hypermultiplets (called the hypermultiplet moduli space or the **Kibble branch**). Certain quantities of the Kibble branch, such as dimension and some operators, of these theories or related ones have been discussed in, for example, [2, 3, 5, 6]. In this chapter, we compute the Hilbert series for the Kibble branch of various skeleton diagrams and show that *it is possible to count all chiral operators for any genus g and any number of external legs e of the skeleton diagram*. This key result is explicitly stated in (5.7.1).

The Hilbert series is a partition function for the chiral operators in the chiral ring of supersymmetric gauge theories.[2] It can also be used as a primary tool to

[1] The skeleton diagrams give rise to an $SU(2)$ subclass of theories which are referred to in the literature as (i) generalised quiver gauge theories [1], (ii) Sicilian gauge theories [3], and (iii) tinkertoys [4]. See more details below.

[2] There are also other similar quantities such as the superconformal index [5, 9–11], which is specific to superconformal field theories. It would be interesting to find the relation between the Hilbert series and these quantities.

test various dualities in gauge theories, for example, in Chapter 4 and [7] the Hilbert series is used in the context of the Argyres-Seiberg duality [8]. In this chapter, several examples demonstrate that theories corresponding to different graphs with the same g and e possess the same Hilbert series. This is in agreement with the conjecture that such theories are related to each other by S-duality [1].

The outline and key results of this chapter are as follows. In Section 5.2, we summarise details of the skeleton diagram and give various simple examples. In Section 5.3, we introduce the notion of the Kibble branch of the moduli space and compute the dimension. It is found that the dimension of the Kibble branch only depends on the external legs e and not the genus g. In §§5.4, 5.5, 5.6, we compute Hilbert series for various examples. The main results of this chapter are collected in Section 5.7. These include the general formulae (5.7.1), (5.7.18) and (5.7.19), which are a summary of all the results in this chapter.

5.2 Skeleton diagrams of $\mathcal{N} = 2$ gauge theories

To write down a Lagrangian for a gauge theory with $\mathcal{N} = 2$ supersymmetry it is sufficient to specify the gauge group, under which vector multiplets transform in the adjoint representation, and the representations under which the hypermultiplets transform. In the case that hypermultiplets carry no more than two charges, it is convenient to represent the theory by a **quiver diagram**, whose nodes and lines represent respectively vector multiplets and hypermultiplets. Readers who are not familiar with $\mathcal{N} = 2$ quiver diagrams may wish to consult Chapter 4 for further details. However, when hypermultiplets carrying more than two charges, quiver diagrams are not good representatives of such theories. Nevertheless, some of these theories can be represented graphically by **skeleton diagrams**.

This chapter deals with an infinite class of $\mathcal{N} = 2$ supersymmetric gauge theories that are constructed by skeleton diagrams with the following simple rules: The graphs are made out of lines and trivalent vertices. Each line (——) represents an $SU(2)$ gauge group, with its length L inversely proportional to its gauge coupling g^2, *i.e.* $L \sim 1/g^2$. (Therefore, a line with infinite length has zero coupling and therefore corresponds to a global $SU(2)$ symmetry.) Each tri-valent vertex (⅄) represents a half-hypermultiplet $\mathcal{Q}_{\alpha\beta\gamma}$ transforming in the $[1;1;1]$ representation of $SU(2)^3$, where the indices $\alpha,\ \beta,\ \gamma = 1, 2$ corresponds to three different $SU(2)$ groups. A skeleton diagram defines a unique $\mathcal{N} = 2$ Langrangian in $(3+1)$ dimensions.

Let us briefly comment on the nomenclature. We emphasise that skeleton diagrams only represents $SU(2)$ gauge groups (or $SU(2)$ global symmetries in the cases of lines with infinite length) and matters in the tri-fundamental representation of $SU(2)^3$. Thus, the skeleton diagrams give rise to an $SU(2)$ subclass of theories which are referred to in the literature as (i) **generalised quiver gauge theories**

[1], (ii) **Sicilian gauge theories** [3], and (iii) **tinkertoys** [4]. Note that, for the name "generalised quiver gauge theories", the reader should not be confused with the notion of the quiver described above.

In the $\mathcal{N}=1$ language, each $\mathcal{N}=2$ vector multiplet decomposes into an $\mathcal{N}=1$ vector multiplet and an $\mathcal{N}=1$ chiral multiplet. Each $\mathcal{N}=2$ half-hypermultiplet decomposes into an $\mathcal{N}=1$ chiral multiplet. Finally, the superpotential takes the form of a sum over all nodes with a contribution of each node is

$$\mathcal{Q}_{\alpha\beta\gamma}\mathcal{Q}_{\alpha'\beta'\gamma'}\left(\phi_1^{\alpha\alpha'}\epsilon^{\beta\beta'}\epsilon^{\gamma\gamma'}+\epsilon^{\alpha\alpha'}\phi_2^{\beta\beta'}\epsilon^{\gamma\gamma'}+\epsilon^{\alpha\alpha'}\epsilon^{\beta\beta'}\phi_3^{\gamma\gamma'}\right)\ , \tag{5.2.1}$$

where the three sets of indices $\{\alpha,\alpha'=1,2\},\{\beta,\beta'=1,2\},\{\gamma,\gamma'=1,2\}$ correspond to the three different $SU(2)$ groups, and the adjoint chiral multiplets ϕ_1,ϕ_2,ϕ_3 come from the three different $SU(2)$ $\mathcal{N}=2$ vector multiplets. By convention, infinite lines give rise to adjoint valued mass terms. Note that the superpotential (5.2.1) is defined up to a constant which is determined by $\mathcal{N}=2$ supersymmetry

A motivation of skeleton diagrams comes from the study of $\mathcal{N}=2$ supersymmetric gauge theories living on M5-branes wrapping Riemann surfaces [1, 12]. In this chapter, we focus on the theories with $SU(2)$ symmetries, and so the number of M5-branes involved is two. The topology of the skeleton diagram is the same as that of the corresponding Riemann surface, namely the number of loops of the skeleton diagram is the genus of the Riemann surface and the number of external legs of the skeleton diagram is the number of punctures on the Riemann surface.

Below we give a few examples of the $\mathcal{N}=2$ theories with their skeleton diagrams.

5.2.1 The theory with a free trifundamental of $SU(2)^3$

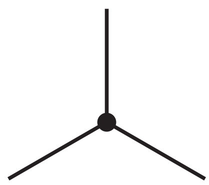

Figure 5.1. The theory with a free trifundamental field of $SU(2)^3$.

Let us consider the theory with a tri-vertex and three external legs (Figure 5.1). Each of the three legs corresponds to an $SU(2)$ global symmetry. The vertex corresponds to 8 free, possibly massive, half-hypermultiplets $\mathcal{Q}_{ijk}$ transforming in the trifundamental $[1;1;1]$ representation of the $SU(2)^3$ global symmetry. The possible mass terms are

$$W=\mathcal{Q}_{ijk}\mathcal{Q}_{i'j'k'}\left(m_1^{ii'}\epsilon^{jj'}\epsilon^{kk'}+\epsilon^{ii'}m_2^{jj'}\epsilon^{kk'}+\epsilon^{ii'}\epsilon^{jj'}m_3^{kk'}\right)\ , \tag{5.2.2}$$

where $i,\ j,\ k,\ i',\ j',\ k' = 1, 2$. This theory is also known in the literature as **the T_2 theory**, or the **triskelion** [3]. Subsequently, we use this theory as a building block to construct a number of other theories by means of 'gluing'.

5.2.2 The $SU(2)$ $\mathcal{N} = 4$ gauge theory with two free singlets

Let us consider the tadpole diagram in Figure 5.2. This diagram can be obtained by gluing together two external legs in a T_2 theory. As shown in the diagram, this theory has an $SU(2)$ gauge group (corresponding to the loop) and an $SU(2)$ global symmetry (corresponding to the external leg).

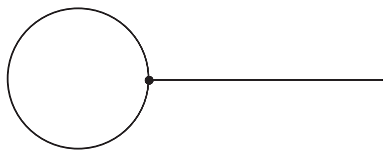

Figure 5.2. (The tadpole) The $SU(2)$ $\mathcal{N} = 4$ gauge theory with two singlets.

The vertex corresponds to a half-hypermultiplet $\mathcal{Q}_{abi}$, where $a, b = 1, 2$ are $SU(2)$ gauge indices and $i = 1, 2$ is an $SU(2)$ global index. Let us define the trace of $\mathcal{Q}$ and the traceless part of $\mathcal{Q}$ as

$$\begin{aligned} X_i &\equiv \epsilon^{ab} \mathcal{Q}_{abi} \ , \\ \varphi_{abi} &\equiv \mathcal{Q}_{abi} - \frac{1}{2} X_i \epsilon_{ab} \ . \end{aligned} \tag{5.2.3}$$

Note that, by definition, the half-hypermultiplets φ_i are traceless, *i.e.* $\epsilon^{ab}\varphi_{abi} = 0$. Hence, φ is an $SU(2)$ adjoint hypermultiplet. The vector multiplet of the $SU(2)$ gauge group and the adjoint hypermultiplet φ give rise to an $\mathcal{N} = 4$ gauge theory with an $SU(2)$ gauge group.

On the other hand, the gauge singlet X is a free hypermultiplet which is more conveniently written as two half-hypermultiplets X_1, X_2 transforming in the fundamental representation of the $SU(2)$ global symmetry.

There is also a global $U(1)$ R-symmetry under which the half-hypermultiplets $\mathcal{Q}_{ab,\alpha}$ carries the charge 1 (which is also the scaling dimension).

The representations in which X and φ transform are summarised in Table 5.1.

Let ϕ be the scalar field in the $\mathcal{N} = 2$ $SU(2)$ vector multiplet. In an $\mathcal{N} = 1$ supersymmetric notation, one can write down the superpotential (5.2.1), including a mass term, as

$$W = \mathcal{Q}_{abi}\mathcal{Q}_{a'b'j} \left(\phi^{aa'}\epsilon^{bb'}\epsilon^{ij} + \epsilon^{aa'}\phi^{bb'}\epsilon^{ij} + \epsilon^{aa'}\epsilon^{bb'}m^{ij} \right) \ .$$

Field	Gauge $SU(2)$	Global $SU(2)$	Global $U(1)$
Fugacity:	z	x	t
φ	[2]	[1]	1
X	[0]	[1]	1

Table 5.1. The hypermultiplets in the tadpole theory.

For simplicity, we set the mass term to zero and obtain

$$W = \mathcal{Q}_{abi}\mathcal{Q}_{a'b'j}\left(\phi^{aa'}\epsilon^{bb'}\epsilon^{ij} + \epsilon^{aa'}\phi^{bb'}\epsilon^{ij}\right) \ . \quad (5.2.4)$$

Observe that, by symmetry, the trace $\epsilon^{ab}\mathcal{Q}_{abi} = X_i$ does not contribute to the superpotential. Indeed, X is a free hypermultiplet. One can therefore write down the above superpotential using the traceless part of $\mathcal{Q}$ as

$$W = 2\phi^{aa'}\epsilon^{bb'}\epsilon^{ij}\varphi_{abi}\varphi_{a'b'j} \ .$$

Note that the factor in front of the superpotential is determined by supersymmetry but is not relevant to the computations done in this chapter. We shall henceforth drop this factor and take

$$W = \phi^{aa'}\epsilon^{bb'}\epsilon^{ij}\varphi_{abi}\varphi_{a'b'j} \ . \quad (5.2.5)$$

5.2.3 $SU(2)$ gauge theory with 4 flavours

Consider the skeleton diagram in Figure 5.3. This diagram can be obtained by 'gluing' two T_2 theories along one of the external legs in each diagram.

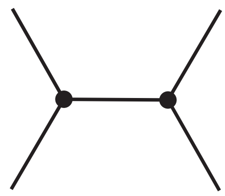

Figure 5.3. The skeleton diagram of the $SU(2)$ gauge theory with 4 flavours.

The internal line corresponds to the $SU(2)$ gauge group. Each of the four external legs corresponds to an $SU(2)$ global symmetry. The two nodes represent two trifundamental fields $\mathcal{Q}_{i_1 i_2 a}$ and $\widetilde{\mathcal{Q}}_{i_3 i_4 a}$ of $SU(2)^3$, where a is an $SU(2)$ gauge index and i_1, i_2, i_3, i_4 are the indices for the four different $SU(2)$ flavour symmetries.

Let ϕ be a scalar field in the $\mathcal{N} = 2$ $SU(2)$ vector multiplet. In an $\mathcal{N} = 1$ supersymmetric language, the superpotential (with mass terms) can be written as

$$\begin{aligned} W = {} & \mathcal{Q}_{i_1 i_2 a}\mathcal{Q}_{i'_1 i'_2 a'}\left(m_1^{i_1 i'_1}\epsilon^{i_2 i'_2}\epsilon^{aa'} + \epsilon^{i_1 i'_1}m_2^{i_2 i'_2}\epsilon^{aa'} + \epsilon^{i_1 i'_1}\epsilon^{i_2 i'_2}\phi^{aa'}\right) \\ & + \widetilde{\mathcal{Q}}_{i_3 i_4 a}\widetilde{\mathcal{Q}}_{i'_3 i'_4 a'}\left(m_3^{i_3 i'_3}\epsilon^{i_4 i'_4}\epsilon^{aa'} + \epsilon^{i_3 i'_3}m_4^{i_4 i'_4}\epsilon^{aa'} + \epsilon^{i_3 i'_3}\epsilon^{i_4 i'_4}\phi^{aa'}\right) \ . \end{aligned} \quad (5.2.6)$$

From the following decompositions of $SO(8)$ into $SU(2)^4$:

$$[1,0,0,0]_{SO(8)} = [1;1;0;0] + [0;0;1;1] \ , \tag{5.2.7}$$

one can combine $(i_1, i_2), (i_3, i_4)$ into one $SO(8)$ index I, and hence the $SU(2)^4$ global symmetry enhances to $SO(8)$. The 16 half-hypermultiplets $\mathcal{Q}_{i_1 i_2 a}$ and $\widetilde{\mathcal{Q}}_{i_3 i_4 a}$ can then be combined into Q_{aI}, which are indeed the quarks in an $SU(2)$ gauge theory with 4 flavours. The quiver diagram of this theory is depicted in Figure 5.4.

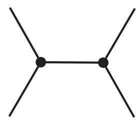

Figure 5.4. The quiver diagram of the $SU(2)$ gauge theory with 4 flavours.

In $\mathcal{N} = 1$ supersymmetric notation, one can rewrite the superpotential as

$$W = Q_{aI} Q_{bI} \phi^{ab} + \mu^{IJ} Q_{aI} Q_{bJ} \epsilon^{ab} \ . \tag{5.2.8}$$

Let us compare (5.2.6) with (5.2.8). The mass parameters μ_{IJ} transform in the adjoint representation $[0,1,0,0]$ of $SO(8)$. This can be decomposed into $SU(2)^4$ representations as

$$[0,1,0,0]_{SO(8)} = [1;1;1;1] + [2;0;0;0] + [0;2;0;0] + [0;0;2;0] + [0;0;0;2] \tag{5.2.9}$$

We see from (5.2.6) that the mass parameters $m_1^{i_1 i'_1}$, $m_2^{i_2 i'_2}$, $m_3^{i_3 i'_3}$ and $m_4^{i_4 i'_4}$ transform respectively in the $SU(2)^4$ representations $[2;0;0;0]$, $[0;2;0;0]$, $[0;0;2;0]$ and $[0;0;0;2]$. Therefore, we have the following tensor decomposition:

$$\mu^{IJ} \ \rightarrow \ m^{i_1 i_2 i_3 i_4} + m_1^{i_1 i'_1} + m_2^{i_2 i'_2} + m_3^{i_3 i'_3} + m_4^{i_4 i'_4} \ , \tag{5.2.10}$$

where the mass parameters $m^{i_1 i_2 i_3 i_4}$ transform in $[1;1;1;1]$ of $SU(2)^4$. Observe that we can set $m^{i_1 i_2 i_3 i_4}$ to zero by an $SO(8)$ transformation.

5.3 The Kibble Branch of the Moduli Space

Topology of the skeleton diagram. One can classify the skeleton diagrams according to their topological properties, namely the genus g and the number of external legs e. Henceforth, we collect these numbers in an ordered pair (g, e). Given g and e, the number of internal lines is $3g - 3 + e$ and the number of nodes (and also the number of T_2 building blocks) is the Euler characteristic $\chi = 2g - 2 + e$. Recall that an internal line corresponds to a gauge group and each node corresponds to a trifundamental matter field. Therefore,

$$\begin{aligned} \text{The number of } SU(2) \text{ gauge groups} = \mathcal{G}(g,e) &= 3g-3+e \ , \\ \text{The number of matter fields} = \chi(g,e) &= 2g-2+e \ , \\ \text{The number of } SU(2) \text{ global symmetries} &= e \ . \end{aligned} \tag{5.3.1}$$

In $\mathcal{N}=2$ supersymmetric gauge theories with one gauge group, one typically refers to two branches of the moduli space, namely the Higgs branch and the Coulomb branch. The Higgs branch is the branch on which the gauge group is *completely* broken and the vector multiplet becomes massive via the Higgs mechanism; this branch is parametrised by the massless gauge singlets of hypermultiplets. The Coulomb branch is, on the other hand, the branch on which the gauge group is broken to a collection of $U(1)$'s and the hypermultiplets generically become massive; this branch is parametrised by complex scalars in the vector multiplet.

However, for the theories with genus $g \geq 1$, the gauge group is not completely broken on the branch which is parametrised by VEVs of hypermultiplets. We conjecture that at a generic point in this branch the $SU(2)^{\mathcal{G}}$ gauge symmetry is broken to $U(1)^g$ (see Appendix A.1). In order to avoid a potential confusion with the notion of Higgs branch, we refer to this branch of the moduli space as the **Kibble branch**[3], denoted by $\mathcal{K}$. Note however that for theories with zero genus $g=0$, the Kibble branch coincides with the Higgs branch.

Let us compute the dimension of the Kibble branch for theories with genus g and e external legs. Since each T_2 building block contains 8 half-hypermultiplets (or equivalently 4 hypermultiplets) and there are χ of such building blocks, the hypermultiplets have 4χ quaternionic degrees of freedom in total. At a generic point on the Kibble branch the $SU(2)^{\mathcal{G}}$ gauge symmetry is broken to $U(1)^g$, and hence there are $3\mathcal{G}-g$ broken generators. As a result of the Higgs mechanism, the vector multiplet gains $3\mathcal{G}-g$ quarternionic degrees of freedom and become a massive $\mathcal{N}=2$ vector multiplet. Thus, from (5.3.1), the $4\chi-(3\mathcal{G}-g)=e+1$ quarternionic degrees of freedom are left massless. Thus, the quarternionic dimension of the Kibble branch is

$$\dim_{\mathbb{H}} \mathcal{K} = e+1 \ . \tag{5.3.2}$$

This is in agreement with [2]. Note that the dimension of the Kibble branch does not depend on the genus, but depends only on the number of external legs.

[3]In honour of Professor Tom Kibble's contribution to the theory of spontaneous symmetry breaking.

5.4 Theories with Genus Zero

In this section, we focus on the Hilbert series of theories with genus zero. Below the Hilbert series of these theories are studied in detail.

5.4.1 The T_2 theory $(g = 0, e = 3)$

It is clear that the moduli space of the T_2 theory is generated by the trifundamental field. Hence, the operators transform in the symmetric powers of $[1; 1; 1]$ of $SU(2)^3$. Thus, the Hilbert series of this theory can be written in an elegant way using the plethystic exponential (PE)

$$g_{T_2}(t; x_1, x_2, x_3) = \mathrm{PE}\left[[1; 1; 1]t\right] = \prod_{\epsilon_i=\pm1} \frac{1}{1 - t x_1^{\epsilon_1} x_2^{\epsilon_2} x_3^{\epsilon_3}} , \qquad (5.4.1)$$

where x_1, x_2 and x_3 are the fugacities of $SU(2) \times SU(2) \times SU(2)$, and the plethystic exponential PE of a multi-variable function $f(t_1, ..., t_n)$ that vanishes at the origin, $f(0, ..., 0) = 0$, is defined as

$$\mathrm{PE}\left[f(t_1, t_2, \ldots, t_n)\right] = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} f(t_1^k, \ldots, t_n^k)\right) . \qquad (5.4.2)$$

This expression (5.4.1) is manifestly symmetric under any permutation of the 3 external legs. The permutation group S_3 acts on exchanging the legs and the Hilbert series on the Kibble branch is an invariant function of this S_3. This point is used below to demonstrate the invariance of the Hilbert series on the Kibble branch.

One can rewrite (5.4.1) in terms of infinite sums of the irreducible representations of $SU(2)^3$ as

$$\begin{aligned} g_{T_2}(t; x_1, x_2, x_3) = \frac{1}{1-t^4} \sum_{n_1,n_2,n_3,m=0}^{\infty} \big(&[2n_1 + m; 2n_2 + m; 2n_3 + m] t^{2n_1+2n_2+2n_3+m} + \\ &[2n_1 + m + 1; 2n_2 + m + 1; 2n_3 + m + 1] t^{2n_1+2n_2+2n_3+m+3} \big) \end{aligned} \qquad (5.4.3)$$

As is shown below, this infinite sum turns out to be more useful for generalisation to any pair (g, e). In this expression, there are 4 sums, one for each external leg, and one that 'glues' all expressions together (without it, the sums would simply factorise).

5.4.2 $SU(2)$ gauge theory with 4 flavours $(g = 0, e = 4)$

The Hilbert series of this theory is computed in (5.4.7) of Chapter 4. In terms of $SO(8)$ representations, this can be written as

$$g_{N_c=2,N_f=4}(t, z_1, z_2, z_3, z_4) = \sum_{k=0}^{\infty} [0, k, 0, 0]_{SO(8)} t^{2k} , \qquad (5.4.4)$$

where z_1, z_2, z_3, z_4 are the $SO(8)$ fugacities. The moduli space of this theory is 10 complex dimensional (see *e.g.*, Section 4.4). This is in agreement with (5.3.2).

A branching rule of $SO(8)$ to $SU(2)^4$. Let us decompose these $SO(8)$ representations into $SU(2)^4$ representations. A map from the $SO(8)$ fugacities to the $SU(2)^4$ can be chosen to be

$$z_1 = x_1 x_2, \quad z_2 = x_2^2, \quad z_3 = x_3 x_2, \quad z_4 = x_4 x_2 \ , \tag{5.4.5}$$

where x_1, x_2, x_3, x_4 are the four $SU(2)$ fugacities. With such a map, one obtains, *e.g.*

$$\begin{aligned}
[1,0,0,0]_{SO(8)} &= [1;1;0;0] + [0;0;1;1] \ , \\
[0,1,0,0]_{SO(8)} &= [1;1;1;1] + [2;0;0;0] + [0;2;0;0] + [0;0;2;0] + [0;0;0;2] \ ,
\end{aligned} \tag{5.4.6}$$

etc. The formula (5.4.4) can be rewritten in terms of $SU(2)$ representations as

$$\begin{aligned}
g_{N_c=2,N_f=4} = \frac{1}{1-t^4} \sum_{n_1,\ldots,n_4,m=0}^{\infty} \big(& [2n_1+m; 2n_2+m; 2n_3+m; 2n_4+m] t^{2n_1+2n_2+2n_3+2n_4+2m} + \\
& [2n_1+m+1; 2n_2+m+1; 2n_3+m+1; 2n_4+m+1] t^{2n_1+2n_2+2n_3+2n_4+2m+4} \big) \ .
\end{aligned} \tag{5.4.7}$$

This is a form, which as in (5.4.3), turns out to be the right form to generalise to any pair (g, e). This expression is invariant under any permutation of the external legs. The permutation group S_4 acts on exchanging the legs and the Hilbert series on the Kibble branch is an invariant function of this S_4.

Gluing two T_2 theories.

One can also obtain the $SU(2)$ gauge theory with 4 flavours by gluing two T_2 theories along the external legs. Before obtaining the Hilbert series, let us briefly summarise the gluing technique.

A summary of the gluing technique

In Chapter 4, we derive Hilbert series when two Riemann surfaces are glued together along the punctures. Let us briefly summarise the gluing procedure. Suppose that the maximal punctures along which we glue possess the symmetry of a group G, whose fugacites are denoted collectively by z_k. Let the Hilbert series of the theory corresponding to the first Riemann surface be $g_1(t, x_i, z_k)$ and let the one corresponding to the second Riemann surface be $g_2(t, y_j, z_k)$, where x_i, y_j represent a dependence on additional fugacities. The Hilbert series when two Riemann surfaces are glued together is given by

$$g(t, x_i, y_j) = \int \mathrm{d}\mu_G(z_k) \; g_1(t, x_i, z_k) \; g_{\text{glue}}(t, z_k) \; g_2(t, y_j, z_k) \ , \tag{5.4.8}$$

where the gluing factor (when there is no 'self-gluing' involved) is

$$g_{\text{glue}}(t, z_k) = \frac{1}{\mathrm{PE}\left[Adj(z_k) t^2\right]} \ . \tag{5.4.9}$$

In particular, for the $SU(2)$ group, the gluing factor is given by

$$g_{\text{glue}}(t,z) = \frac{1}{\text{PE}\left[[2]_z t^2\right]} = 1 - t^2[2]_z + t^4[2]_z - t^6 \ , \tag{5.4.10}$$

where $[2]_z = z^2 + 1 + \frac{1}{z^2}$.

Note however that, when the gluing involves self-gluing of the Riemann surface, the gluing does not take the form (5.4.9). We demonstrate this point in Section 5.5.1.

The $SU(2)$ theory with 4 flavours - revisited

Let us suppose that the legs 3 (associated with the fugacity z) of the two T_2 are glued together. To obtain the Hilbert series, we apply the gluing formula (5.4.8) to the Hilbert series (5.4.3) of T_2:

$$\int \mathrm{d}\mu_{SU(2)}(z)\ g_{T_2}(t;x_1,x_2,z)\ g_{\text{glue}}(t,z)\ g_{T_2}(t;x_3,x_4,z)\ . \tag{5.4.11}$$

The gluing factor and the integration impose a 'selection rule' on m. In order to determine which m survive, we use the following identities:

$$\begin{aligned}
\int \mathrm{d}\mu_{SU(2)}(z) \sum_{n_3,m,n_3',m'} g_{\text{glue}}(t,z)[2n_3+m]_z[2n_3'+m']_z\ t^{2n_3+m+2n_3'+m'} &= 1+t^2-t^4 \\
\int \mathrm{d}\mu_{SU(2)}(z) \sum_{n_3,m,n_3',m'} g_{\text{glue}}(t,z)[2n_3+m]_z[2n_3'+m'+1]_z t^{2n_3+m+2n_3'+m'+3} &= t^4 \\
\int \mathrm{d}\mu_{SU(2)}(z) \sum_{n_3,m,n_3',m'} g_{\text{glue}}(t,z)[2n_3+m+1]_z[2n_3'+m']_z t^{2n_3+m+2n_3'+m'+3} &= t^4 \\
\int \mathrm{d}\mu_{SU(2)}(z) \sum_{n_3,m,n_3',m'} g_{\text{glue}}(t,z)[2n_3+m+1]_z[2n_3'+m'+1]_z t^{2n_3+m+2n_3'+m'+6} &= t^6\ ,
\end{aligned} \tag{5.4.12}$$

where the summations are over n_3, m, n_3', m' from 0 to ∞ and the subscripts z indicates that the characters depend on z. The first and the second identities contribute to the first term in (5.4.7):

$$\frac{1+t^2-t^4}{1-t^4} + \frac{t^4}{1-t^4} = \frac{1+t^2}{1-t^4} = \frac{1}{1-t^2} = \sum_{m=0}^{\infty} t^{2m}\ . \tag{5.4.13}$$

The third and the fourth identities contribute to the second term in (5.4.7):

$$\frac{t^4}{1-t^4} + \frac{t^6}{1-t^4} = \frac{t^4}{1-t^2} = \sum_{m=0}^{\infty} t^{2m+4}\ . \tag{5.4.14}$$

Hence, we arrive at (5.4.7), as expected.

Derivation of the identities

(The reader may skip this topic without the loss of continuity.)

We discuss the derivation of the first identity in (5.4.12); the others can be derived in a similar fashion. Let us first focus on the following expression:

$$\begin{aligned}\mathcal{A} &\equiv \int \mathrm{d}\mu_{SU(2)}(z) \sum_{n_3,m,n_3',m'} [2n_3+m]_z[2n_3'+m']_z t^{2n_3+m+2n_3'+m'} \\ &= \sum_{n_3,m,n_3',m'} \delta_{2n_3+m,2n_3'+m'} t^{2n_3+m+2n_3'+m'} \;. \end{aligned} \tag{5.4.15}$$

For a given value of $2n_3+m$, there are $\lfloor (2n_3+m)/2+1 \rfloor = n_3+1+\lfloor m/2 \rfloor$ pairs of (n_3', m') which give non-zero delta functions. Therefore, we have

$$\begin{aligned}\mathcal{A} = \sum_{n_3,m,n_3',m'} \delta_{2n_3+m,2n_3'+m'} t^{2n_3+m+2n_3'+m'} &= \sum_{n_3=0}^{\infty}\sum_{k=0}^{\infty} (n_3+1+\lfloor m/2 \rfloor) t^{4n_3+2m} \\ &= \sum_{n_3=0}^{\infty}\sum_{k=0}^{\infty} (n_3+1+k) t^{4n_3+4k}(1+t^2) \\ &= \frac{1+t^4}{(1-t^4)^2(1-t^2)} \;, \end{aligned} \tag{5.4.16}$$

where, in the second line, we considered the two separated cases, $m=2k$ and $m=2k+1$. Next, we consider the expression

$$\begin{aligned}\mathcal{B} &\equiv \int \mathrm{d}\mu_{SU(2)}(z) \sum_{n_3,m,n_3',m'} [2]_z[2n_3+m]_z[2n_3'+m']_z t^{2n_3+m+2n_3'+m'} \\ &= -1 + \sum_{n_3,m,n_3',m'} (\delta_{2n_3+m,2n_3'+m'} + \delta_{2n_3+m+2,2n_3'+m'} + \delta_{2n_3+m,2n_3'+m'+2}) t^{2n_3+m+2n_3'+m'} \\ &= -1 + \frac{1+t^4}{(1-t^4)^2(1-t^2)} + 2 \times \frac{2t^2}{(1-t^2)^3(1+t^2)^2} \\ &= \frac{t^2(5+3t^2-2t^4-t^6+t^8)}{(1-t^2)^3(1+t^2)^2} \;, \end{aligned} \tag{5.4.17}$$

where -1 in the second line compensate the case in which $2n_3+m = 2n_3'+m' = 0$. Using the gluing factor (5.4.9) (with $Adj = [2]$), we find that

$$\begin{aligned}&\int \mathrm{d}\mu_{SU(2)}(z) \sum_{n_3,m,n_3',m'} g_{\text{glue}}(t,z)[2n_3+m]_z[2n_3'+m']_z \; t^{2n_3+m+2n_3'+m'} \\ &= (1-t^6)\mathcal{A} - t^2(1-t^2)\mathcal{B} = 1+t^2-t^4 \;. \end{aligned} \tag{5.4.18}$$

Three phases of $SU(2)$ theory with 4 flavours

As pointed out in [1], there are 3 weak coupling limits of an $SU(2)$ gauge theory with 4 flavours. These corresponds to the permutations of the labels of the external legs (depicted in Figure 5.5). They have different origins from the perspective of theories on M5-branes wrapping Riemann surfaces. For example, the theory at the centre of Figure 5.5 can be obtained from the gluing of two Riemann surfaces; one contains punctures 1 and 3 and the other contains puctures 2 and 4. All of these phases are conjectured to be related to each other by S-duality [1] which states that the IR dynamics of these theories are identical. Indeed, it can easily be seen from (5.4.7) that the Hilbert series of these three phases are identical, since the permutations of the labels correspond to the permutations of $n_1, \ldots, n_4$, the dummy variables in the summations.

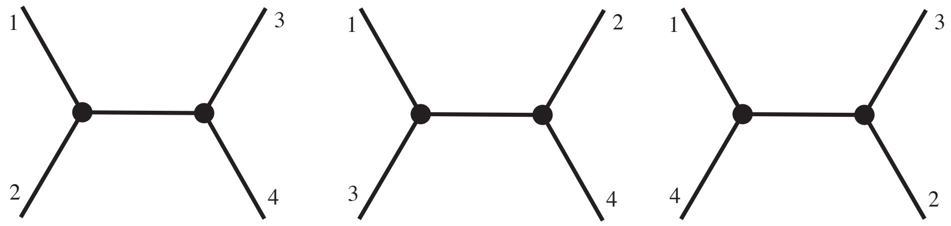

Figure 5.5. The three weak coupling limits of an $SU(2)$ gauge theory with 4 flavours.

In fact, a stronger version of this duality is that the IR dynamics depends on the pair (g, e) only and not on the specific choice of the Lagrangian. A consistency check of this duality is the set of computations below which demonstrate that the Hilbert series on the Kibble branch is an invariant of S-duality, or alternatively, depends on the choice of the pair (g, e) and not on other details of the skeleton diagram.

5.5 Theories with Genus One

Below the Hilbert series of theories with genus one are studied in detail.

5.5.1 The tadpole theory $(g = 1, e = 1)$

In this subsection, we compute the Hilbert series of the Kibble branch of the tadpole theory (Figure 5.2). We translate the $\mathcal{N} = 2$ data into the $\mathcal{N} = 1$ language. Let us denote the scalar in the vector multiplet by ϕ. In the $\mathcal{N} = 1$ language, the superpotential can be written as

$$W = \mathcal{Q}_{abi}\mathcal{Q}_{a'b'j}\left(\phi^{aa'}\epsilon^{bb'}\epsilon^{ij} + \epsilon^{aa'}\phi^{bb'}\epsilon^{ij}\right) \ . \tag{5.5.1}$$

On the Kibble branch, the field ϕ becomes massive and hence $\langle\phi\rangle = 0$. Therefore, the non-trivial F-terms are

$$(\mathcal{Q}_{abi}\mathcal{Q}_{a'b'j} + \mathcal{Q}_{bai}\mathcal{Q}_{b'a'j})\epsilon^{bb'}\epsilon^{ij} = 0 \ . \tag{5.5.2}$$

Using the fugacities according to Table 5.1, the Hilbert series of the two commuting adjoint fields is

$$\begin{aligned}&(1-t^2[2;0]+t^3[0;1])\mathrm{PE}\left[[2;1]t\right] \\ &= \frac{1-t^2(1+z^2+\frac{1}{z^2})+t^3(x+\frac{1}{x})}{\left(1-\frac{t}{x}\right)(1-tx)\left(1-\frac{t}{xz^2}\right)\left(1-\frac{tx}{z^2}\right)\left(1-\frac{tz^2}{x}\right)(1-txz^2)}\ ,\end{aligned} \tag{5.5.3}$$

where $[a;b]$ denotes the product of the characters $[a]_z[b]_x$. The F-flat Hilbert series is then given by

$$\begin{aligned}\mathcal{F}^\flat(t,z,x) &= (1-t^2[2;0]+t^3[0;1])\mathrm{PE}\left[[2;1]t+[0;1]t\right] \\ &= \frac{1-t^2(1+z^2+\frac{1}{z^2})+t^3(x+\frac{1}{x})}{\left(1-\frac{t}{x}\right)^2(1-tx)^2\left(1-\frac{t}{xz^2}\right)\left(1-\frac{tx}{z^2}\right)\left(1-\frac{tz^2}{x}\right)(1-txz^2)}\ .\end{aligned} \tag{5.5.4}$$

Integrating over the $SU(2)$ gauge group, one obtains the Kibble branch Hilbert series

$$\begin{aligned}g_{\mathrm{tadpole}}(t,x) &= \frac{1-t^4}{(1-tx)(1-\frac{t}{x})(1-t^2)(1-t^2x^2)(1-\frac{t^2}{x^2})} \\ &= (1-t^4)\mathrm{PE}\left[[1]t+[2]t^2\right] \\ &= \frac{1}{1-t^4}\sum_{n_1,n_2,m=0}^{\infty}\left([2n_1+m]t^{2n_1+m}+[2n_1+m+1]t^{2n_1+2n_2+m+3}\right)\ .\end{aligned} \tag{5.5.5}$$

The Kibble branch is therefore a 4 complex dimensional complete intersection. The generators are X at order t and

$$M_{ij} = \epsilon^{aa'}\epsilon^{bb'}Q_{abi}Q_{a'b'j} \tag{5.5.6}$$

at order t^2. The relation at order t^4 is

$$\det M = 0\ . \tag{5.5.7}$$

Note that the Kibble branch is actually $\mathbb{C}^2/\mathbb{Z}_2\times\mathbb{C}^2$, where $\mathbb{C}^2/\mathbb{Z}_2$ is generated by $M_{\alpha\beta}$ and $\mathbb{C}^2$ is generated by the two gauge singlet X_α. This can also be seen from the fact that the Hilbert series of $\mathbb{C}^2/\mathbb{Z}_2\times\mathbb{C}^2$ given by the discrete Molien formula (see *e.g.* [13]):

$$\begin{aligned}g_{\mathbb{C}^2/\mathbb{Z}_2\times\mathbb{C}^2}(t,x) &= \frac{1}{2}\left[\frac{1}{\left(1-\frac{t}{x}\right)(1-tx)}+\frac{1}{\left(1+\frac{t}{x}\right)(1+tx)}\right]\times\frac{1}{\left(1-\frac{t}{x}\right)(1-tx)} \\ &= (1-t^4)\mathrm{PE}\left[[1]t+[2]t^2\right]\ .\end{aligned} \tag{5.5.8}$$

is equal to the Hilbert series (5.5.5).

The tadpole from gluing two legs in the T_2 theory

It is clear from the skeleton diagram that the tadpole comes from gluing two legs of the T_2 theory. Let us derive the corresponding gluing factor. Starting from (5.4.1), we glue the legs 1 and 2 together (*i.e.* set $x_1 = x_2 = z$ and take $x_3 = x$); we then obtain

$$[1;1;1] \equiv ([1]_z[1]_z)[1]_x = ([2]_z + [0]_z)[1]_x \equiv [2;1] + [0;1] \ , \tag{5.5.9}$$

where $[a, b] = [a]_z[b]_x$. Observe that this is actually the representation in the plethystic exponential (5.5.4). Hence, from (5.5.4), it is immediate that the gluing factor is

$$g_{\rm glue}(t, z, x) = 1 - t^2[2;0] + t^3[0;1] \ . \tag{5.5.10}$$

Let us comment on the gluing factor as follows:

- This process involves self-gluing. The gluing factor is different from (5.4.9).
- Whenever the self-gluing gets involved, the gluing factor is no longer local. As can be seen from (5.5.10), the gluing factor does not depend only on z, the variable associated with the two legs we glue, but it depends also on x, the variable associated with the third leg which is not involved in the gluing.
- When there is no self-gluing involved, the gluing is a local process and the gluing factor is given by (5.4.9).

5.5.2 The theories with genus one and two external legs $(g = 1, e = 2)$

Below the Hilbert series of theories with genus one and two external legs are studied in detail.

The A_1 theory

In this subsection, we focus on the theory with the A_1 quiver, whose skeleton diagram is depicted in Figure 5.6. The two $SU(2)$ gauge groups are represented by the upper and lower arcs. The two external legs represent the two $SU(2)$ baryonic symmetries, $SU(2)_{B_1}$ and $SU(2)_{B_2}$. The quiver diagram of the A_1 theory is given by Figure 5.7.

Let ϕ_1 and ϕ_2 be the scalar fields in the two $\mathcal{N} = 2$ $SU(2)$ vector multiplets. In an $\mathcal{N} = 1$ notation, the superpotential can be written according to (5.2.1) as

$$\begin{aligned} W \ = \ & \mathcal{Q}_{a_1 a_2 i_1} \mathcal{Q}_{a'_1 a'_2 i'_1} (\phi_1^{a_1 a'_1} \epsilon^{a_2 a'_2} \epsilon^{i_1 i'_1} + \epsilon^{a_1 a'_1} \phi_2^{a_2 a'_2} \epsilon^{i_1 i'_1}) \\ & + \widetilde{\mathcal{Q}}_{a_1 a_2 i_2} \widetilde{\mathcal{Q}}_{a'_1 a'_2 i'_2} (\phi_1^{a_1 a'_1} \epsilon^{a_2 a'_2} \epsilon^{i_2 i'_2} + \epsilon^{a_1 a'_1} \phi_2^{a_2 a'_2} \epsilon^{i_2 i'_2}) \ , \end{aligned} \tag{5.5.11}$$

where for simplicity the mass terms of $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ are set to zero.

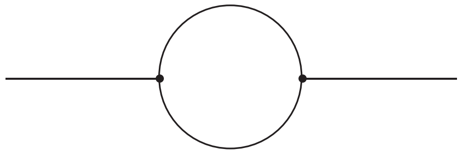

Figure 5.6. The skeleton diagram of the A_1 theory.

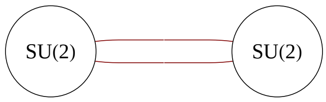

Figure 5.7. The $\mathcal{N}=2$ quiver diagram of the A_1 theory.

The F-terms. We first start from the F-terms of the A_1 theory. Since we focus on the Kibble branch, the vacuum expectation values of ϕ_1 and ϕ_2 are zero. Therefore, the non-trivial F-terms associated with the Kibble branch are the derivatives of the superpotential with respect to ϕ_1 and ϕ_2. The F-terms can be written as

$$\begin{aligned} \mathcal{F}^1_{a_1 a_1'} &= (\mathcal{Q}_{a_1 a_2 i_1} \mathcal{Q}_{a_1' a_2' i_1'} \epsilon^{i_1 i_1'} + \widetilde{\mathcal{Q}}_{a_1 a_2 i_2} \widetilde{\mathcal{Q}}_{a_1' a_2' i_2'} \epsilon^{i_2 i_2'}) \epsilon^{a_2 a_2'} = 0 \ , \\ \mathcal{F}^2_{a_2 a_2'} &= (\mathcal{Q}_{a_1 a_2 i_1} \mathcal{Q}_{a_1' a_2' i_1'} \epsilon^{i_1 i_1'} + \widetilde{\mathcal{Q}}_{a_1 a_2 i_2} \widetilde{\mathcal{Q}}_{a_1' a_2' i_2'} \epsilon^{i_2 i_2'}) \epsilon^{a_1 a_1'} = 0 \ . \end{aligned} \tag{5.5.12}$$

Dimension. Now let us compute the dimension of the F-flat space (*i.e.* the space of the F-term solutions). Since there are two nodes in the skeleton diagrams, there are $8+8=16$ half-hypermultiplets, corresponding to 16 complex dimensional space. The F-terms impose 5 complex relations. Hence, the F-flat space is $16-5=11$ complex dimensional. Due to the $\mathcal{N}=2$ supersymmetry, the D-terms also impose 5 complex relations. Hence, the Kibble branch is $11-5=6$ complex dimensional, in agreement with (5.3.2).

The Hilbert series of the F-flat space. This is given by

$$\mathcal{F}^\flat(t, z_1, z_2, x_1, x_2) = \mathcal{C}(t, z_1, z_2, x_1, x_2) \mathrm{PE}\left[[1;1;1;0]t + [1;1;0;1]t\right] \ , \tag{5.5.13}$$

where $[a;b;c;d] = [a]_{z_1}[b]_{z_2}[c]_{x_1}[d]_{x_2}$ and

$$\begin{aligned} \mathcal{C} = {} & 1 - t^2\left([2;0;0;0] + [0;2;0;0]\right) + t^4\left([2;2;0;0] + [2;0;0;0] + [0;2;0;0] + [0;0;1;1]\right) \\ & - t^5\left([1;1;1;0] + [1;1;0;1]\right) - t^6\left([2;2;0;0] + 1\right) + t^7\left([1;1;1;0] + [1;1;0;1]\right) \\ & - t^8\left([0;0;1;1] + 1\right) \ . \end{aligned} \tag{5.5.14}$$

Setting $z_1 = z_2 = x_1 = x_2 = 1$, we obtain the unrefined Hilbert series:

$$\mathcal{F}^\flat(t, 1, 1, 1, 1) = \frac{(1+t)\,(1+4t+5t^2)}{(1-t)^{11}} \ . \tag{5.5.15}$$

The pole at $t=1$ is at order 11, so the F-flat space is 11 dimensional as expected.

The Kibble branch Hilbert series. This can be obtained by integrating over the gauge fugacities:

$$g_{A_1}(t,x_1,x_2) = \int \mathrm{d}\mu_{SU(2)}(z_1)\mathrm{d}\mu_{SU(2)}(z_2)\mathcal{F}^\flat(t,z_1,z_2,x_1,x_2) \ , \tag{5.5.16}$$

where the Haar measure of $SU(2)$ is

$$\int \mathrm{d}\mu_{SU(2)}(z) = \oint_{|z|=1} \frac{1-z^2}{z}\mathrm{d}z \ . \tag{5.5.17}$$

Evaluating this integral, one obtain a rational function of t, x_1, x_2 whose power series is given by

$$\begin{aligned} g_{A_1}(t,x_1,x_2) = \frac{1}{1-t^4}\sum_{n_1,n_2,m=0}^{\infty} & [2n_1+m;2n_2+m]t^{2n_1+2n_2+2m} \\ & +[2n_1+m+1;2n_2+m+1]t^{2n_1+2n_2+2m+4} \ . \end{aligned} \tag{5.5.18}$$

Note that this expression is invariant under a permutation of the two external legs. The permutation group S_2 acts on exchanging the legs and the Hilbert series on the Kibble branch is an invariant function of this S_2.

The unrefined Hilbert series is

$$g_{A_1}(t,1,1) = \frac{(1+t^2)\,(1+3t^2+t^4)}{(1-t^2)^6} \ . \tag{5.5.19}$$

Note that the Kibble branch is indeed 6 complex dimensional, as expected. The plethystic logarithm of (5.5.18) is given by

$$\mathrm{PL}\left[g_{A_1}(t,x_1,x_2)\right] = t^2\left([2;0]+[1;1]+[0;2]\right) - t^4\left([1;1]+2[0;0]\right) + \dots \ . \tag{5.5.20}$$

The generators are listed in Table 5.2.

Since $SU(2)\times SU(2) \cong SO(4)$, it can be seen the generators transform in the 10 dimensional second rank symmetric representation[4] (*i.e.* $[2,2]+[0,0]$) of $SO(4)$.

Note that the A_1 theory with the $U(2)\times U(2)$ gauge group is considered in §4.2 of [14], where two generators, namely $M_{12}^{[1;1]}$ and $M_{21}^{[1;1]}$, are set to be equal due to imposing the F-term relation for the $U(1)$ part. However, such a relation is not imposed in our analysis.

The stickman model

The skeleton diagram of the stickman model is depicted in Figure 5.8. This model can be obtained from gluing the tadpole theory with the T_2 theory along the external legs.

[4]We denote the $SO(4)$ 2-dimensional spinor representation and its conjugate respectively by $[1,0]$ and $[0,1]$. Therefore, the $SO(4)$ vector representation is $[1,1]$, and the second rank symmetric traceless representation is $[2,2]$.

Representation of the global $SU(2) \times SU(2)$	Generators
$[2;0]$	$M^{[2;0]}_{i_1 i'_1} = \epsilon^{a_1 a'_1} \epsilon^{a_2 a'_2} \mathcal{Q}_{a_1 a_2 i_1} \mathcal{Q}_{a'_1 a'_2 i'_1}$
$[1;1]$	$M^{[1;1]}_{i_1 i_2} = \epsilon^{a_1 a'_1} \epsilon^{a_2 a'_2} \mathcal{Q}_{a_1 a_2 i_1} \widetilde{\mathcal{Q}}_{a'_1 a'_2 i_2}$
$[0;2]$	$M^{[0;2]}_{i_2 i'_2} = \epsilon^{a_1 a'_1} \epsilon^{a_2 a'_2} \widetilde{\mathcal{Q}}_{a_1 a_2 i_2} \widetilde{\mathcal{Q}}_{a'_1 a'_2 i'_2}$

Table 5.2. The generators of the A_1 theory and the representations in which they transform.

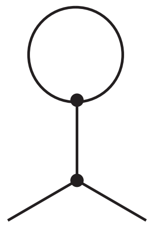

Figure 5.8. The stickman model

The Hilbert series can be obtained as follows:

$$g_{\text{man}}(t, x_1, x_2) = \int \mathrm{d}\mu_{SU(2)}(z) g_{T_2}(t, x_1, x_2, z) g_{\text{glue}}(t, z) g_{\text{tadpole}}(t, z) \ , \quad (5.5.21)$$

where g_{T_2} is given by (5.4.3), g_{tadpole} is given by (5.5.5), and the gluing factor g_{glue} is given by (5.4.9). In order to evaluate this integral, we use the the identities (5.4.12) and follow (5.4.13) and (5.4.14). The result is

$$\begin{aligned} g_{\text{man}}(t, x_1, x_2) = \frac{1}{1-t^4} \sum_{n_1, n_2, m=0}^{\infty} & [2n_1 + m; 2n_2 + m] t^{2n_1+2n_2+2m} \\ & + [2n_1 + m + 1; 2n_2 + m + 1] t^{2n_1+2n_2+2m+4} \ . \end{aligned} \quad (5.5.22)$$

Note that (5.5.22) is equal to (5.5.18). This is a consistency check of the duality conjecture.

5.6 Theories with Zero External Legs

In this section, we focus on the theories with no external legs. This class of theories has a number of interesting features. Let us mention one of them as follows. From (5.3.2), the Kibble branch of these theories is 2 complex dimensional, or equivalently 1 quarternionic dimensional. Note that a non-compact hyperKähler manifold with 1 quaternionic dimension is also known as the asymptoptic locally Euclidean (ALE) space. Hence, we expect the Kibble branch of the theories with no external leg to be $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $SU(2)$. Later we show that $\Gamma = \hat{D}_{g+1}$ for g the genus of the skeleton diagram.

In the subsequent subsections, we discuss this class of theories in detail.

5.6.1 The theories with genus two $(g = 2, e = 0)$

There are two skeleton diagrams corresponding to $(g = 2, e = 0)$. The first one, which we will refer to as the **Yin-Yang diagram**[5], is depicted in Figure 5.9(i). The corresponding quiver diagram is given in Figure 5.9(ii). The $SO(4) = SU(2) \times SU(2)$ gauge group comes from the two $SU(2)$ gauge groups corresponding to the left and the right arcs in the skeleton diagram. The two lines correspond to the 8 half-hypermultiplets (*i.e.* two nodes in the skeleton diagram). The second skeleton diagram, which we will refer to as the **dumbbell diagram**, is depicted in Figure 5.10.

We subsequently compute the Hilbert series of the Yin-Yang model and the dumbbell model and show that they are equal. This again demonstrates that the Kibble branch depends only on the topology of the skeleton diagram, but not on other details of the diagram.

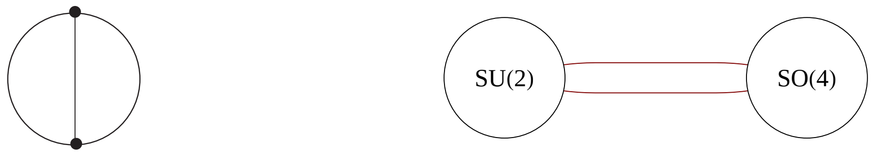

Figure 5.9. (i) A skeleton diagram with $(g = 2, e = 0)$. We refer to this diagram as the **Yin-Yang diagram**. (ii) The corresponding quiver digram. The $SO(4) = SU(2) \times SU(2)$ gauge group arises from the two $SU(2)$ gauge groups corresponding to the left and the right arcs in the skeleton diagram. The two lines corresponds to the 8 half-hypermultiplets (two nodes in the skeleton diagram).

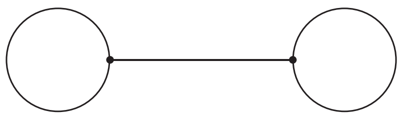

Figure 5.10. (Dumbbell) Another skeleton diagram with $(g = 2, e = 0)$.

The dumbbell model

The skeleton of the dumbbell model is depicted in Figure 5.10. This model can be obtained by gluing the tails of two tadpoles. Therefore, the Hilbert series of the

[5]The name comes from the Yin-Yang symbol .

Kibble branch of this model is

$$\begin{aligned} g_D(t) &= \int \mathrm{d}\mu_{SU(2)}(z) g_{\text{tadpole}}(t,z) g_{\text{glue}}(t,z) g_{\text{tadpole}}(t,z) \\ &= \oint_{|z|=1} \frac{\mathrm{d}z}{z}(1-z^2) \frac{(1-t^4)^2 \mathrm{PE}\left[2[1]_z t + 2[2]_z t^2\right]}{\mathrm{PE}\left[[2]_z t^2\right]} \\ &= \frac{1-t^8}{(1-t^2)(1-t^4)^2} \ . \end{aligned} \tag{5.6.1}$$

where the gluing factor $g_{\text{glue}}(t,z)$ is given by (5.4.10) and $g_{\text{tadpole}}(t,z)$ is given by (5.5.5). Observe that the Kibble branch is a two complex dimensional complete intersection. There is one generator at order t^2, two generators at order t^4, and one relation at order t^8. Note that this is the Hilbert series of $\mathbb{C}^2/\hat{D}_3$ [13].

The generators of the moduli space

The dumbbell model has three $SU(2)$ gauge groups: one corresponds to the left loop (denoted by $SU(2)_1$), one corresponds to the line (denoted by $SU(2)_2$), and one corresponds the right loop (denoted by $SU(2)_3$). In an $\mathcal{N}=1$ notation, the matter content is tabulated in Table 5.3. The superpotential, according to (5.2.1), is

$$\begin{aligned} W = &\ \mathcal{Q}_{a_1 b_1 a_2} \mathcal{Q}_{a'_1 b'_1 a'_2} (\phi_1^{a_1 a'_1} \epsilon^{b_1 b'_1} \epsilon^{a_2 a'_2} + \epsilon^{a_1 a'_1} \phi_1^{b_1 b'_1} \epsilon^{a_2 a'_2} + \epsilon^{a_1 a'_1} \epsilon^{b_1 b'_1} \phi_2^{a_2 a'_2}) \\ &+ \widetilde{\mathcal{Q}}_{a_3 b_3 a_2} \widetilde{\mathcal{Q}}_{a'_3 b'_3 a'_2} (\phi_3^{a_3 a'_3} \epsilon^{b_3 b'_3} \epsilon^{a_2 a'_2} + \epsilon^{a_3 a'_3} \phi_3^{b_3 b'_3} \epsilon^{a_2 a'_2} + \epsilon^{a_3 a'_3} \epsilon^{b_3 b'_3} \phi_2^{a_2 a'_2}) \ . \end{aligned} \tag{5.6.2}$$

Field	Gauge $SU(2)_1$	Gauge $SU(2)_2$	Gauge $SU(2)_3$
$\phi_1^{a_1 a'_1}$	[2]	[0]	[0]
$\mathcal{Q}_{a_1 b_1 a_2}$	[2] + [0]	[1]	[0]
$\phi_2^{a_2 a'_2}$	[0]	[2]	[0]
$\phi_3^{a_3 a'_3}$	[0]	[0]	[2]
$\widetilde{\mathcal{Q}}_{a_3 b_3 a_2}$	[0]	[1]	[2] + [0]

Table 5.3. The matter content in the $\mathcal{N}=1$ language of the dumbbell theory.

Note that on the Kibble branch the VEVs of ϕ_1, ϕ_2, ϕ_3 are zero. Hence, the non-trivial F-terms come from the derivatives of ϕ_1, ϕ_2, ϕ_3:

$$\begin{aligned} \epsilon^{b_1 b'_1} \epsilon^{a_2 a'_2} (\mathcal{Q}_{a_1 b_1 a_2} \mathcal{Q}_{a'_1 b'_1 a'_2} + \mathcal{Q}_{b_1 a_1 a_2} \mathcal{Q}_{b'_1 a'_1 a'_2}) &= 0 \ , \\ \epsilon^{a_1 a'_1} \epsilon^{b_1 b'_1} \mathcal{Q}_{a_1 b_1 a_2} \mathcal{Q}_{a'_1 b'_1 a'_2} + \epsilon^{a_3 a'_3} \epsilon^{b_3 b'_3} \widetilde{\mathcal{Q}}_{a_3 b_3 a_2} \widetilde{\mathcal{Q}}_{a'_3 b'_3 a'_2} &= 0 \ , \\ \epsilon^{b_3 b'_3} \epsilon^{a_2 a'_2} (\widetilde{\mathcal{Q}}_{a_3 b_3 a_2} \widetilde{\mathcal{Q}}_{a'_3 b'_3 a'_2} + \widetilde{\mathcal{Q}}_{b_3 a_3 a_2} \widetilde{\mathcal{Q}}_{b'_3 a'_3 a'_2}) &= 0 \ . \end{aligned} \tag{5.6.3}$$

The generator at order t^2 is

$$M = \epsilon^{a_1 b_1} \epsilon^{a_2 a'_2} \epsilon^{a_3 b_3} \mathcal{Q}_{a_1 b_1 a_2} \widetilde{\mathcal{Q}}_{a_3 b_3 a'_2} \ . \tag{5.6.4}$$

The generators at order t^4 are

$$\begin{aligned} \mathcal{B}_1 &= \epsilon^{a_1 b_1} \epsilon^{a'_1 b'_1} \epsilon^{a_2 b_2} \epsilon^{a'_2 b'_2} \mathcal{Q}_{a_1 b_1 a_2} U_{b_2 b'_2} \mathcal{Q}_{a'_1 b'_1 a'_2} \ , \\ \mathcal{B}_2 &= \epsilon^{a_1 b_1} \epsilon^{a'_3 b'_3} \epsilon^{a_2 b_2} \epsilon^{a'_2 b'_2} \mathcal{Q}_{a_1 b_1 a_2} U_{b_2 b'_2} \widetilde{\mathcal{Q}}_{a'_3 b'_3 a'_2} \ . \end{aligned} \tag{5.6.5}$$

where

$$U_{a_2 a'_2} = \epsilon^{a_1 a'_1} \epsilon^{b_1 b'_1} \mathcal{Q}_{a_1 b_1 a_2} \mathcal{Q}_{a'_1 b'_1 a'_2} \ . \tag{5.6.6}$$

Note that by the second F-terms in (5.6.3), it follows that

$$\epsilon^{a_3 a'_3} \epsilon^{b_3 b'_3} \widetilde{\mathcal{Q}}_{a_3 b_3 a_2} \widetilde{\mathcal{Q}}_{a'_3 b'_3 a'_2} = -U_{a_2 a'_2} \ . \tag{5.6.7}$$

Other order 4 operators can be expressed in terms of M, $\mathcal{B}_1$ and $\mathcal{B}_2$ as follows:

- Using (5.6.6) and the first F-terms in (5.6.3), we obtain

$$\det U = \frac{1}{2} \epsilon^{a_2 b_2} \epsilon^{a'_2 b'_2} U_{a_2 a'_2} U_{b_2 b'_2} = \frac{1}{2} \mathcal{B}_1 . \tag{5.6.8}$$

- Consider the operator

$$\widetilde{\mathcal{B}}_1 = \epsilon^{a_3 b_3} \epsilon^{a'_3 b'_3} \epsilon^{a_2 b_2} \epsilon^{a'_2 b'_2} \widetilde{\mathcal{Q}}_{a_3 b_3 a_2} U_{b_2 b'_2} \widetilde{\mathcal{Q}}_{a'_3 b'_3 a'_2} \ . \tag{5.6.9}$$

  From (5.6.7) and (5.6.8), we have

$$\widetilde{\mathcal{B}}_1 = -\epsilon^{a_2 b_2} \epsilon^{a'_2 b'_2} U_{a_2 a'_2} U_{b_2 b'_2} = -\mathcal{B}_1 \ . \tag{5.6.10}$$

The relation between the generators

In order to obtain the relation, we start from the following identity which is true for any symmetric matrix U_{ab}.

$$(\epsilon^{ab} A_a B_b)^2 \det U + (\epsilon^{aa'} \epsilon^{bb'} A_a U_{a'b'} B_b)^2 - (\epsilon^{aa'} \epsilon^{bb'} A_a U_{a'b'} A_b)(\epsilon^{cc'} \epsilon^{dd'} B_c U_{c'd'} B_d) = 0 \ . \tag{5.6.11}$$

Taking U to be as in (5.6.6) and taking

$$A_{a_2} = \epsilon^{a_1 b_1} \mathcal{Q}_{a_1 b_1 a_2}, \qquad B_{a_2} = \epsilon^{a_3 b_3} \mathcal{Q}_{a_3 b_3 a_2} \ , \tag{5.6.12}$$

we obtain

$$M^2 \det U + \mathcal{B}_2^2 - \mathcal{B}_1 \widetilde{\mathcal{B}}_1 = 0 \ . \tag{5.6.13}$$

Substituting in it the identities (5.6.8) and (5.6.10), we obtain

$$2M^2 \mathcal{B}_1 + \mathcal{B}_1^2 + \mathcal{B}_2^2 = 0 \ . \tag{5.6.14}$$

Note that this is indeed the relation of $\mathbb{C}^2/\hat{D}_3$.[6]

[6] Note that the relation for $\mathbb{C}^2/\hat{D}_3$ can be written as $u^2 + v^2 w = w^2$ (see *e.g.* [13]), where in this case $u = i\mathcal{B}_2, v = \sqrt{2}M, w = -\mathcal{B}_1$.

The Yin-Yang model

In this subsection, we compute the Hilbert series of the Yin-Yang model from the $\mathcal{N}=1$ quiver diagram depicted in Figure 5.11. The bi-fundamental hypermultiplets are denoted by Q_i^a and q_i^a, where we use $a,b,c=1,2$ to denote the $SU(2)$ indices and $i,j,k=1,\ldots,4$ to denote the $SO(4)$ indices. The adjoint fields in $SU(2)$ and $SO(4)$ are denoted respectively by φ and ψ. The superpotential is

$$W = \left(\epsilon_{ab}Q_i^a\psi^{ij}Q_j^b - \delta^{ij}Q_i^a\varphi_{ab}Q_j^b\right) + \left(\epsilon_{ab}q_i^a\psi^{ij}q_j^b - \delta^{ij}q_i^a\varphi_{ab}q_j^b\right) \ , \tag{5.6.15}$$

where the $SU(2)$ indices are raised and lowered using the epsilon symbol and the $SO(4)$ indices are raised and lowered using Kronecker's delta.

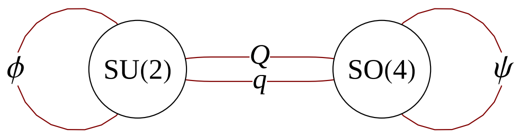

Figure 5.11. The $\mathcal{N}=1$ quiver digram of the Yin-Yang model. The superpotential is given by $W = (Q\cdot\psi\cdot Q - Q\cdot\varphi\cdot Q) + (q\cdot\psi\cdot q - q\cdot\varphi\cdot q)$.

The F-flat space. Since we focus on the Kibble branch, the vacuum expectation values of φ and ψ are zero. Therefore, the non-trivial F-terms associated with the Kibble branch are the derivatives of the superpotential with respect to φ and ψ:

$$\begin{aligned}(\mathcal{F}_\varphi)^{ab} &= \delta^{ij}(Q_i^aQ_j^b + q_i^aq_j^b) = 0 \ , \\ (\mathcal{F}_\psi)_{ij} &= \epsilon_{ab}(Q_i^aQ_j^b + q_i^aq_j^b) = 0 \ .\end{aligned} \tag{5.6.16}$$

The F-flat space is 9 complex dimensional. The fully refined Hilbert series (with the $SU(2)$ gauge fugacity z and $SO(4)$ gauge fugacities w_1, w_2) is too long to be reported here. Setting all of the gauge fugacities to unity, we obtain the unrefined Hilbert series

$$\mathcal{F}^\flat(t, z=1, w_1=w_2=1) = \frac{1+7t+19t^2+21t^3+7t^4+t^5}{(1-t)^9} \ . \tag{5.6.17}$$

The Kibble branch Hilbert series. This can be obtained as follows.

$$g_{YY}(t) = \int \mathrm{d}\mu_{SU(2)}(z)\mathrm{d}\mu_{SO(4)}(w_1,w_2)\ \mathcal{F}^\flat(t,z,w_1,w_2) \ , \tag{5.6.18}$$

where

$$\int \mu_{SO(4)}(w_1,w_2) = \oint_{|w_1|=1}\frac{\mathrm{d}w_1}{w_1}\oint_{|w_2|=1}\frac{\mathrm{d}w_2}{w_2}\left(1-\frac{w_1}{w_2}\right)(1-w_1w_2) \ . \tag{5.6.19}$$

The result of the integrations is

$$g_{YY}(t) = \frac{1-t^8}{(1-t^2)(1-t^4)^2} = \frac{1+t^4}{\left(1-t^2\right)^2\left(1+t^2\right)} \ . \tag{5.6.20}$$

Observe that this Hilbert series is identical to that of the dumbbell model (5.6.1).

The Ying-Yang model from gluing the two legs of the A_1 theory

The skeleton diagram in Figure 5.9 of the Yin-Yang model can be obtained by gluing the two external legs of the A_1 theory, whose skeleton diagram is depicted in Figure 5.6. Note that since this gluing process involves a self-gluing, the gluing factor does not take its canonical form (5.4.9). Rather, we propose that for the equation

$$g_{YY}(t) = \int \mathrm{d}\mu_{SU(2)}(z) g_{\text{glue}}(t,z) g_{A_1}(t,z,z) \tag{5.6.21}$$

with $g_{YY}(t)$ and $g_{A_1}(t,x,y)$ given respectively by (5.6.20) and (5.5.18), a solution for $g_{\text{glue}}(t,z)$ is

$$g_{\text{glue}}(t,z) = 1 - [2]_z t^2 + \frac{2t^4(1+t^2)}{1+t^4} \ . \tag{5.6.22}$$

This solution can be verified using the following identities:

$$\begin{aligned}
\int \mathrm{d}\mu(z) \sum_{n_1,n_2,m} [2n_1+m]_z[2n_2+m]_z \ t^{2n_1+2n_2+\chi m} &= \frac{1}{\left(1-t^4\right)\left(1-t^\chi\right)} \\
\int \mathrm{d}\mu(z) \sum_{n_1,n_2,m} [2n_1+m+1]_z[2n_2+m+1]_z \ t^{2n_1+2n_2+\chi m+\chi+2} &= \frac{t^{2+\chi}}{\left(1-t^4\right)\left(1-t^\chi\right)} \\
\int \mathrm{d}\mu(z) \sum_{n_1,n_2,m} [2]_z[2n_1+m]_z[2n_2+m]_z \ t^{2n_1+2n_2+\chi m} &= \frac{2t^2+t^4+t^\chi-t^{4+\chi}}{\left(1-t^4\right)\left(1-t^\chi\right)} \\
\int \mathrm{d}\mu(z) \sum_{n_1,n_2,m} [2]_z[2n_1+m+1]_z[2n_2+m+1]_z \ t^{2n_1+2n_2+\chi m+\chi+2} &= \frac{t^{2+\chi}\left(1+2t^2\right)}{\left(1-t^4\right)\left(1-t^\chi\right)} \ ,
\end{aligned} \tag{5.6.23}$$

Indeed, for $\chi = 2$, we obtain the Hilbert series for the Ying-Yang model,

$$\begin{aligned}
g_{YY}(t) &= \frac{1}{1-t^4}\left[\left(1+\frac{2t^4(1+t^2)}{1+t^4}\right)\frac{1+t^4}{\left(1-t^2\right)\left(1-t^4\right)}\right. \\
&\qquad \left. -t^2\left(\frac{t^4\left(1+2t^2\right)+3t^2+t^4-t^6}{\left(1-t^2\right)\left(1-t^4\right)}\right)\right] \\
&= \frac{1+t^4}{\left(1-t^2\right)^2\left(1+t^2\right)} \ ,
\end{aligned} \tag{5.6.24}$$

as expected.

5.6.2 The theories with genus three $(g = 3, e = 0)$

There are three phases of theories with genus 3 and zero external legs. Their skeleton diagrams are depicted in Figure 5.12, Figure 5.13 and Figure 5.14.

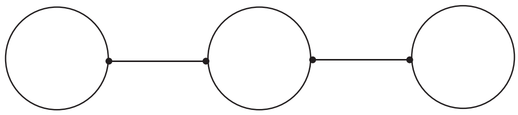

Figure 5.12. The three-loop linear model with zero external legs (TLMZ)

Figure 5.13. The tablet model. **Left:** The skeleton diagram. **Right:** The $\mathcal{N} = 2$ quiver diagram. Note that each $SO(4) \cong SU(2) \times SU(2)$ gauge symmetry arises from the $SU(2)$ corresponding to the chord and the $SU(2)$ corresponding to the arc sharing the same endpoints with the chord.

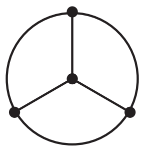

Figure 5.14. The Mercedes-Benz model

The three-loop linear model

In this subsection, we derive the Hilbert series of the TLMZ using the gluing technique.

Let us first consider the Hilbert series of the two loop linear model with one external leg depicted in Figure 5.15. This is given by

$$g_{(g=2,e=1)}(t) = \int \mathrm{d}\mu_{SU(2)}(z) g_{A_1}(t, z, x) g_{\text{glue}}(t, z) g_{\text{tadpole}}(t, z) \ , \qquad (5.6.25)$$

where g_{tadpole} is given by (5.5.5) and g_{A_1} is given by (5.5.18).

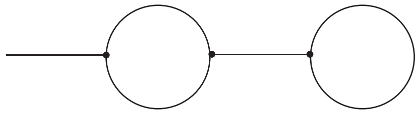

Figure 5.15. The two loop linear model with one external leg.

We can evaluate this integral by using the following identities (which are a generalisation of (5.4.12)):

$$
\begin{aligned}
&\int \mathrm{d}\mu(z) \sum_{n_3,m,n_3',m'} g_{\rm glue}(t,z)[2n_3+m]_z[2n_3'+m']_z\ t^{2n_3+\chi_1 m+2n_3'+\chi_2 m'} = \frac{(1-t^2)\left(1+t^2-t^{2+\chi_1+\chi_2}\right)}{1-t^{\chi_1+\chi_2}}\\
&\int \mathrm{d}\mu(z) \sum_{n_3,m,n_3',m'} g_{\rm glue}(t,z)[2n_3+m]_z[2n_3'+m'+1]_z t^{2n_3+\chi_1 m+2n_3'+\chi_2 m'} = \frac{t^{\chi_1}(1-t^2)}{1-t^{\chi_1+\chi_2}}\\
&\int \mathrm{d}\mu(z) \sum_{n_3,m,n_3',m'} g_{\rm glue}(t,z)[2n_3+m+1]_z[2n_3'+m']_z t^{2n_3+\chi_1 m+2n_3'+\chi_2 m'} = \frac{t^{\chi_2}(1-t^2)}{1-t^{\chi_1+\chi_2}}\\
&\int \mathrm{d}\mu(z) \sum_{n_3,m,n_3',m'} g_{\rm glue}(t,z)[2n_3+m+1]_z[2n_3'+m'+1]_z t^{2n_3+\chi_1 m+2n_3'+\chi_2 m'} = \frac{1-t^2}{1-t^{\chi_1+\chi_2}}\ .
\end{aligned}
\tag{5.6.26}
$$

Observe that for $\chi_1 = \chi_2 = 1$, we obtain the identities (5.4.12).

For our problem, the A_1 theory has $\chi_1 = 2$ and the tadpole theory has $\chi_2 = 1$. The first and the second identities of (5.6.26) contribute to

$$
\begin{aligned}
&\frac{1}{1-t^4}\left[\frac{(1-t^2)\left(1+t^2-t^{2+\chi_1+\chi_2}\right)}{1-t^{\chi_1+\chi_2}} + \frac{t^{\chi_2+2}\cdot t^{\chi_1}(1-t^2)}{1-t^{\chi_1+\chi_2}}\right]\\
&= \frac{1}{1-t^{\chi_1+\chi_2}} = \sum_{m=0}^{\infty} t^{(\chi_1+\chi_2)m}\ .
\end{aligned}
\tag{5.6.27}
$$

The third and the fourth identities of (5.6.26) contribute to

$$
\begin{aligned}
&\frac{1}{1-t^4}\left[\frac{t^{\chi_1+2}\cdot t^{\chi_2}(1-t^2)}{1-t^{\chi_1+\chi_2}} + \frac{t^{\chi_1+\chi_2+4}\cdot(1-t^2)}{1-t^{\chi_1+\chi_2}}\right]\\
&= \frac{t^{\chi_1+\chi_2+2}}{1-t^{\chi_1+\chi_2}} = \sum_{m=0}^{\infty} t^{(\chi_1+\chi_2)m+(\chi_1+\chi_2+2)}\ .
\end{aligned}
\tag{5.6.28}
$$

Putting $\chi_1 = 2, \chi_2 = 1$, we obtain the Hilbert series for the two loop linear model with one external leg as

$$
g_{(g=2,e=1)}(t,x) = \frac{1}{1-t^4}\sum_{n_1,m=0}^{\infty} [2n_1+m]t^{2n_1+3m} + [2n_1+m+1]t^{2n_1+3m+5} \tag{5.6.29}
$$

Now we compute the Hilbert series of the TLMZ (Figure 5.12). This is given by

$$
g_{\rm TLM}(t) = \int \mathrm{d}\mu_{SU(2)}(z) g_{(g=2,e=1)}(t,z) g_{\rm glue}(t,z) g_{\rm tadpole}(t,z)\ . \tag{5.6.30}
$$

We use the identities (5.6.26) with $\chi_1 = 3$ for the $(g = 2, e = 1)$ theory and $\chi_2 = 1$ for the tadpole theory. Following (5.6.27) and (5.6.28), we obtain the Hilbert series of the TLM as

$$g_{\text{TLM}}(t) = \frac{1}{1-t^4}\left(\frac{1}{1-t^4} + \frac{t^6}{1-t^4}\right) = \frac{1+t^6}{(1-t^4)^2} = \frac{1-t^{12}}{(1-t^4)^2(1-t^6)} \ . \quad (5.6.31)$$

Observe that the Kibble branch is a two complex dimensional complete intersection. There is one generator at order t^6, two generators at order t^4, and one relation at order t^{12}. Note that this is the Hilbert series of $\mathbb{C}^2/\hat{D}_4$ [13].

The generators of the moduli space

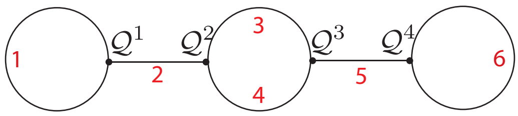

Figure 5.16. The three-loop linear model. The nodes are labelled by $\mathcal{Q}^1, \ldots, \mathcal{Q}^4$ and the gauge groups are labelled in red.

Let us label the nodes and the gauge groups according to Figure 5.16. The first generator at order 4 is

$$M = \epsilon^{a_1 b_1}\epsilon^{a_2 b_2}\epsilon^{a_3 b_3}\epsilon^{a_4 b_4}\epsilon^{a_5 b_5}\epsilon^{a_6 b_6}\,\mathcal{Q}^1_{a_1 b_1 a_2}\,\mathcal{Q}^2_{b_2 a_3 a_4}\,\mathcal{Q}^3_{b_3 b_4 a_5}\,\mathcal{Q}^4_{b_5 a_6 b_6} \ . \quad (5.6.32)$$

Another generator at order 4 is

$$\mathcal{B}_1 = \epsilon^{a_1 b_1}\epsilon^{a'_1 b'_1}\epsilon^{a_2 b_2}\epsilon^{a'_2 b'_2}\,\mathcal{Q}^1_{a_1 b_1 a_2}\,U_{b_2 b'_2}\,\mathcal{Q}^1_{a'_1 b'_1 a'_2} \ , \quad (5.6.33)$$

where

$$U_{a_2 a'_2} = \epsilon^{a_1 a'_1}\epsilon^{b_1 b'_1}\,\mathcal{Q}^1_{a_1 b_1 a_2}\,\mathcal{Q}^1_{a'_1 b'_1 a'_2} \ . \quad (5.6.34)$$

The generator at order 6 is

$$\mathcal{B}_2 = \epsilon^{a_1 b_1}\epsilon^{a_2 b_2}\epsilon^{a'_2 b'_2}\epsilon^{a_3 b_3}\epsilon^{a_4 b_4}\epsilon^{a_5 b_5}\epsilon^{a_6 b_6}\,\mathcal{Q}^1_{a_1 b_1 a_2}\,U_{b_2 b'_2}\,\mathcal{Q}^2_{a'_2 a_3 a_4}\,\mathcal{Q}^3_{b_3 b_4 a_5}\,\mathcal{Q}^4_{b_5 a_6 b_6} \ . \quad (5.6.35)$$

5.7 The General Formula for Any Genus and Any External Leg

As we have seen from several example above, we claim that the Hilbert series for a theory with genus g and e external legs is

$$g_{(g,e)}(t, x_1, \ldots, x_e) = \frac{1}{1-t^4}\sum_{n_1=0}^{\infty}\cdots\sum_{n_e=0}^{\infty}\sum_{m=0}^{\infty}\Big([2n_1+m, \ldots, 2n_e+m]\, t^{2n_1+\ldots+2n_e+\chi m} \\ + [2n_1+m+1, \ldots, 2n_e+m+1]\, t^{2n_1+\ldots+2n_e+\chi m+\chi+2}\Big) \ , \quad (5.7.1)$$

where $\chi = 2g - 2 + e$.

Observe that this expression is invariant under any permutation of the e external legs. The permutation group S_e acts on exchanging the legs and the Hilbert series on the Kibble branch is an invariant function of this S_e.

We prove this formula by induction in Section 5.7.4. Below we discuss interesting special cases of $e = 0$ and $e = 1$.

5.7.1 Special case: $e = 0$

The formula (5.7.1) reduces to

$$g_{(g,e=0)}(t) = \frac{1 - t^{4g}}{(1 - t^4)(1 - t^{2g-2})(1 - t^{2g})} = \frac{1 + t^{2g}}{(1 - t^4)(1 - t^{2g-2})} \ . \qquad (5.7.2)$$

This Hilbert series indicates that the Kibble branch of a theory with genus g and zero external legs is a two complex dimensional complete intersection. This space is isomorphic to $\mathbb{C}^2/\hat{D}_{g+1}$ [13]. There are generators at orders 4, $2g - 2$ and $2g$ subject to one relation at order $4g$.

One can write down the generators explicitly as follows. Consider the g-loop linear model with no legs depicted in Figure 5.17. Let us denote the nodes by $\mathcal{Q}^1, \ldots, \mathcal{Q}^{2g-2}$ from left to right.

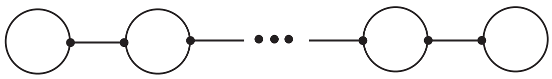

Figure 5.17. The g-loop linear model with no legs

The generator at order t^{2g-2} can be written as

$$M = \epsilon^{a_1 b_1} \epsilon^{a_2 b_2} \epsilon^{a_3 b_3} \ldots \epsilon^{a_{3g-3} b_{3g-3}} \mathcal{Q}^1_{a_1 b_1 a_2} \mathcal{Q}^2_{b_2 a_3 a_4} \ldots \mathcal{Q}^{2g-2}_{b_{3g-4} a_{3g-3} b_{3g-3}} \ . \qquad (5.7.3)$$

Observe that M is a product of all $\mathcal{Q}$'s.

The generator at order t^4 can be written as

$$\mathcal{B}_1 = \epsilon^{a_1 b_1} \epsilon^{a'_1 b'_1} \epsilon^{a_2 b_2} \epsilon^{a'_2 b'_2} \mathcal{Q}^1_{a_1 b_1 a_2} U_{b_2 b'_2} \mathcal{Q}^1_{a'_1 b'_1 a'_2} \ , \qquad (5.7.4)$$

where

$$U_{a_2 a'_2} = \epsilon^{a_1 a'_1} \epsilon^{b_1 b'_1} \mathcal{Q}^1_{a_1 b_1 a_2} \mathcal{Q}^1_{a'_1 b'_1 a'_2} \ . \qquad (5.7.5)$$

Observe that $\mathcal{B}_1$ involves in only the leftmost node $\mathcal{Q}_1$.

The generator at order t^{2g} can be written as

$$\mathcal{B}_2 = \epsilon^{a_1 b_1} \epsilon^{a_2 b_2} \epsilon^{a'_2 b'_2} \epsilon^{a_3 b_3} \ldots \epsilon^{a_{3g-3} b_{3g-3}} \mathcal{Q}^1_{a_1 b_1 a_2} U_{b_2 b'_2} \mathcal{Q}^2_{a'_2 a_3 a_4} \ldots \mathcal{Q}^{2g-2}_{b_{3g-4} a_{3g-3} b_{3g-3}} \qquad (5.7.6)$$

The relation at order t^{4g} is given by

$$2M^2 \mathcal{B}_1 + \mathcal{B}_1^g + \mathcal{B}_2^2 = 0 \ . \qquad (5.7.7)$$

5.7.2 Special case: $e = 1$

The formula (5.7.1) reduces to

$$
\begin{aligned}
g_{(g,e=1)}(t,x) &= \frac{1}{1-t^4} \sum_{n,m=0}^{\infty} \left([2n+m]\, t^{2n+\chi m} + [2n+m+1]\, t^{2n+\chi m+\chi+2} \right) \\
&= (1-t^{2\chi+2}) \mathrm{PE}\left[[2]t^2 + [1]t^{\chi} \right] \ , \qquad (5.7.8)
\end{aligned}
$$

where in this case $\chi = 2g-1$. Setting $x = 1$, the unrefined Hilbert series is

$$
g_{(g,e=1)}(t,1) = \frac{1-t^{4g}}{(1-t^2)^3(1-t^{2g-1})^2} \ . \qquad (5.7.9)
$$

The Hilbert series indicates that the Kibble branch is a four complex dimensional complete intersection. The generators at order 2 transform in the $SU(2)$ representation [2] and the generators at order $\chi = 2g-1$ transform in the representation [1]. There is one relation at order $2\chi + 2 = 4g$.

One can write down the generators explicitly as follows. Consider the g-loop linear model with one leg depicted in Figure 5.18. Let us denote the nodes by $\mathcal{Q}^1, \ldots, \mathcal{Q}^{2g-1}$ from left to right.

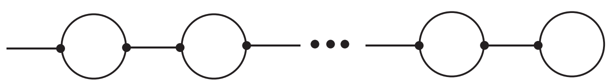

Figure 5.18. The g-loop linear model with one external leg

The generators at order t^2 can be written as

$$
M_{i_1 i_1'} = \epsilon^{aa'} \epsilon^{bb'} \mathcal{Q}^1_{abi_1} \mathcal{Q}^1_{a'b'i_1'} \ . \qquad (5.7.10)
$$

Observe that M involves in only the leftmost node $\mathcal{Q}_1$.

The generators at order t^{2g-1} can be written as

$$
B_i = \epsilon^{a_1 b_1} \epsilon^{a_2 b_2} \epsilon^{a_3 b_3} \ldots \epsilon^{a_{3g-2} b_{3g-2}} \mathcal{Q}^1_{a_1 b_1 i} \mathcal{Q}^2_{a_2 b_2 a_3} \mathcal{Q}^3_{b_3 a_4 b_4} \ldots \mathcal{Q}^{2g-2}_{a_{3g-4} b_{3g-4} a_{3g-3}} \mathcal{Q}^{2g-1}_{b_{3g-3} a_{3g-2} b_{3g-2}} \ . \qquad (5.7.11)
$$

Observe that B is a product of all $\mathcal{Q}$'s.

The relation at order t^{4g} is of the form

$$
(\det M)^g = f(g) M_{i_1 i_2} B_{j_1} B_{j_2} \epsilon^{i_1 j_1} \epsilon^{i_2 j_2} \ , \qquad (5.7.12)
$$

where $f(g)$ is some function of g. As an example, for $g = 1$, we have $f(g) = 0$ and so the relation is $\det M = 0$ (c.f. (5.5.7) of the tadpole model).

5.7.3 The total number of generators for any g and e.

In this subsection, we count the generators in a theory with any given g and e.

Let us focus on the case in which $\chi = 2g-2+e \geq 2$. The plethystic logarithm of (5.7.1) is

$$\begin{aligned} \mathrm{PL}\left[g_{(g,e)}(t,x_1,\ldots,x_e)\right] &= \left([2;0;\ldots;0]+[0;2;\ldots;0]+\ldots+[0;0;\ldots;2]\right)t^2 \\ &\quad +[1;1;\ldots;1]t^{\chi}+\ldots\ . \end{aligned} \tag{5.7.13}$$

This indicates that the generators at order t^2 transform in the representation $[2;0;\ldots;0]+[0;2;\ldots;0]+\ldots+[0;0;\ldots;2]$ of $SU(2)^e$ and the generators at order t^{χ} transform in the representation $[1;1;\ldots;1]$ of $SU(2)^e$. Hence, there are $3e$ generators at order t^2 and 2^e generators at order t^{χ}.

For $\chi = 1$, there are only two theories, namely the T_2 theory and the tadpole. In the former, there is precisely one generator $\mathcal{Q}_{ijk}$. In the latter, there is also precisely one generator M_{ij} given by (5.5.6).

5.7.4 The inductive proof of the general formula

In this subsection, we prove the formula (5.7.1) by induction. A key assumption we make here is that theories corresponding to different graphs with the same genus and the same number of external legs possess the same Hilbert series. This assumption is based on the conjecture that such theories are related to each other by S-duality and has been demonstrated by several examples so far.

We are arguing that

1. Eq. (5.7.1) is true for $(g=0, e=3)$ and $(g=1, e=1)$.
2. Eq. (5.7.1) for $(g, e=1)$ implies Eq. (5.7.1) for $(g+1, e=1)$.
3. Eq. (5.7.1) for (g, e) implies Eq. (5.7.1) for $(g, e+1)$.

After proving these steps, we establish the formula (5.7.1) for non-trivial cases. The special case $(g, e=0)$ follows immediately as discussed above.

Step 1. This can easily be done. For $(g=0, e=3)$, we consider the T_2 theory (5.4.3). For $(g=1, e=1)$, we consider the tadpole theory (5.5.5).

Step 2. Assume that Eq. (5.7.1) is true for $(g, e=1)$. We glue a theory with $(g=1, e=2)$ to the $(g, e=1)$ theory along the external legs. Hence,

$$g_{(g+1,e=1)}(t,x) = \int \mathrm{d}\mu_{SU(2)}(z)\ g_{(g,e=1)}(t,z)g_{\mathrm{glue}}(t,z)g_{(g=1,e=2)}(t,z,x) \tag{5.7.14}$$

To evaluate this, we use the identites (5.6.26) with $\chi_1 = 2g-2+1 = 2g-1$ and $\chi_2 = 2$, and follow (5.6.27) and (5.6.28). We then obtain the expression for $(g+1, e=1)$ as

$$
\begin{aligned}
g_{(g+1,e=1)}(t,x) = \frac{1}{1-t^4} \sum_{n_1=0}^{\infty} \sum_{m=0}^{\infty} & [2n_1+m]\, t^{2n_1+(2g+1)m} \\
& + [2n_1+m+1]\, t^{2n_1+(2g+1)m+(2g+3)} \ .
\end{aligned} \tag{5.7.15}
$$

This is in agreement with (5.7.1) for $(g+1, e=1)$.

Step 3. Now assume that Eq. (5.7.1) is true for (g,e). We glue the T_2 theory to the the (g,e) theory along the external legs. Hence,

$$
g_{(g,e+1)}(t, x_1, \dots x_{e+1}) = \int \mathrm{d}\mu(z)\; g_{(g,e)}(t, x_1, \dots, x_{e-1}, z) g_{\text{glue}}(t,z) g_{T_2}(t, z, x_e, x_{e+1}) \ . \tag{5.7.16}
$$

To evaluate this, we use the identites (5.6.26) with $\chi_1 = 2g-2+e$ and $\chi_2 = 1$, and follow (5.6.27) and (5.6.28). We then obtain

$$
\begin{aligned}
g_{(g,e+1)}(t, x_1, \dots, x_{e+1}) = \frac{1}{1-t^4} \sum_{n_1=0}^{\infty} \cdots \sum_{n_{e+1}=0}^{\infty} \sum_{m=0}^{\infty} & \Big([2n_1+m, \dots, 2n_{e+1}+m]\, t^{2n_1+\dots+2n_{e+1}+\tilde{\chi}m} \\
& + [2n_1+m+1, \dots, 2n_{e+1}+m+1]\, t^{2n_1+\dots+2n_{e+1}+\tilde{\chi}m+\tilde{\chi}+2} \Big) \ ,
\end{aligned} \tag{5.7.17}
$$

where $\tilde{\chi} = \chi_1 + \chi_2 = 2g-2+(e+1)$. This is in agreement with (5.7.1) for $(g, e+1)$.

5.7.5 The general formula in terms of products

The general formula (5.7.1) can actually be rewritten in another form involving products. In order to do so, we use the identity

$$
f_m(t,x) \equiv \sum_{n=0}^{\infty} [2n+m]_x t^{2n} = (1-t^2)\left([m]_x - [m-2]_x t^2\right) \mathrm{PE}\left[[2]t^2\right] \ . \tag{5.7.18}
$$

Then, it is immediate that

$$
g_{(g,e)}(t, x_1, \dots, x_e) = \frac{1}{1-t^4} \sum_{m=0}^{\infty} \left(t^{\chi m} \prod_{i=1}^{e} f_m(t, x_i) + t^{\chi m + \chi + 2} \prod_{i=1}^{e} f_{m+1}(t, x_i) \right) \ . \tag{5.7.19}
$$

Appendix A

Appendix to Chapter 5

A.1 The unbroken $U(1)^g$ gauge symmetry on the Kibble branch of the theory with genus g

A.1.1 A theory with genus one

As we state in Section 5.3, at a generic point on the Kibble branch the $SU(2)$ gauge symmetry is broken to $U(1)$, corresponding to the genus of the skeleton diagram. In this subsection, we prove to this statement by showing that two of the three components of the scalar field ϕ in the $SU(2)$ vector multiplet become massive and the other component remains massless.

In this subsection, it is convenient to work with $SU(2)$ adjoint indices $A, B, C = 1, 2, 3$. We take the generators of the $SU(2)$ group to be $T^A = \sigma^A/2$, where σ^A are the Pauli matrices. We note the identity

$$(T^A)^a{}_b (T^B)^b{}_c = \frac{1}{4}\delta^{AB}\delta^a{}_c + \frac{1}{2} i\epsilon^{ABC}(T^C)^a{}_c \ . \tag{A.1.1}$$

The adjoint fields can be written as

$$\phi^a{}_{a'} = \phi^A (T^A)^a{}_{a'} \ , \qquad \varphi^a{}_{b1} = \varphi_1^A (T^A)^a{}_b \ , \qquad \varphi^a{}_{b2} = \varphi_2^A (T^A)^a{}_b \ , \tag{A.1.2}$$

where $\Phi^A, \varphi_1, \varphi_2$ are complex numbers. We emphasise that $SU(2)$ fundamental indices a, b, a', b' are raised and lowered using the epsilon symbol. The superpotential (5.2.5) can then be rewritten as

$$W = \frac{i}{2}\epsilon^{ABC}\phi^A \varphi_1^B \varphi_2^C \ . \tag{A.1.3}$$

The equation of motion of the F auxiliary field corresponding to φ_1^C and φ_2^C is the (minus) derivative of W with respect to φ_1^C and φ_2^C.

$$
\begin{aligned}
-F_1^C &= \frac{\partial W}{\partial \varphi_1^C} = -\frac{i}{2}\epsilon^{ABC}\phi^A\varphi_2^B \ , \\
-F_2^C &= \frac{\partial W}{\partial \varphi_2^C} = \frac{i}{2}\epsilon^{ABC}\phi^A\varphi_1^B \ .
\end{aligned} \tag{A.1.4}
$$

The potential V contains the terms $(F_1^C)^*F_1^C + (F_2^C)^*F_2^C$, where

$$
\begin{aligned}
(F_1^C)^*F_1^C &= \frac{1}{4}(\phi^A)^*\phi^{A'}(\varphi_2^B)^*\varphi_2^{B'}\epsilon^{ABC}\epsilon^{A'B'C} \\
&= \frac{1}{4}(\phi^A)^*\phi^A(\varphi_2^B)^*\varphi_2^B - \frac{1}{4}(\phi^A)^*\phi^B\varphi_2^A(\varphi_2^B)^* \ , \\
(F_2^C)^*F_2^C &= \frac{1}{4}(\phi^A)^*\phi^A(\varphi_1^B)^*\varphi_1^B - \frac{1}{4}(\phi^A)^*\phi^B\varphi_1^A(\varphi_1^B)^* \ .
\end{aligned} \tag{A.1.5}
$$

These terms in the potential give rise top the mass terms of ϕ:

$$
(F_1^C)^*F_1^C + (F_2^C)^*F_2^C = m^{AB}\phi^A(\phi^B)^* \ , \tag{A.1.6}
$$

where the mass matrix m^{AB} can be determined by the second order derivative

$$
\begin{aligned}
m^{AB} &= \frac{\partial^2}{\partial\phi^A\partial(\phi^B)^*}\left[(F_1^C)^*F_1^C + (F_2^C)^*F_2^C\right] \\
&= \frac{1}{4}\left[\delta^{AB}(\varphi_1^C)^*\varphi_1^C - (\varphi_1^A)^*\varphi_1^B\right] + (1 \to 2) \ .
\end{aligned} \tag{A.1.7}
$$

The three eigenvalues of the mass matrix m^{AB} are

$$
m_1 = \frac{1}{4}U, \quad m_2 = \frac{1}{8}\left(U + \sqrt{U^2 - 16(f^A)^*f^A}\right), \quad m_3 = \frac{1}{8}\left(U - \sqrt{U^2 - 16(f^A)^*f^A}\right) \ .
$$

where U is the sums of the quadratic casimirs

$$
U = (\varphi_1^C)^*\varphi_1^C + (\varphi_2^C)^*\varphi_2^C \ , \tag{A.1.8}
$$

and f^A is the derivative of W with respect to ϕ^A (which is zero because of F-terms):

$$
f^A = \frac{\partial W}{\partial \phi^A} = \frac{i}{2}\epsilon^{ABC}\varphi_1^B\varphi_2^C = 0 \ . \tag{A.1.9}
$$

Thus, the mass eigenvalues are

$$
m_1 = \frac{1}{4}U, \quad m_2 = \frac{1}{4}U, \quad m_3 = 0 \ .
$$

Indeed, two components of ϕ are massive (and each of them has mass $U/4$) and the other component is massless.

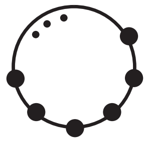

Figure A.1. For a loop, there are nodes on the loop. Each node is attached to two lines.

A.1.2 A theory with genus g

In this subsection, we give an argument that, for a theory with genus g, there is an unbroken $U(1)^g$ gauge symmetry at a generic point on the Kibble branch. As a special case, in Appendix A.1.1, we show that for the tadpole theory ($g = 1$), the unbroken gauge symmetry is $U(1)$.

For a given loop, there are nodes and legs that go around it, as depicted in Figure A.1. Consider a node and two lines which are attached to it. These two lines give rise to $V = 6$ gauge fields in the vector multiplets (3 from each line). The node itself gives rise to $H = 4$ hypermultiplets.

After the Higgs mechanism, three gauge fields become massive and hence we are left with $V = 3$, $H = 1$. If the line is external, this $H = 1$ hypermultiplet contributes to the dimension of the Kibble branch. Therefore, this effective process replaces a node with the two lines by a single line. Thus, one can keep eliminating nodes in such a way until the end result is a loop. For such a loop, there is an unbroken $U(1)$ symmetry (from Appendix A.1.1). One can proceed in this way for all loops, and concludes that for a theory with genus g, there is an unbroken $U(1)^g$ gauge symmetry at a generic point on the Kibble branch.

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