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Feynman checkerboard picture and neutrino oscillations

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Abstract. In this paper we show how the Feynman checkerboard picture for the 1+1-dimensional Dirac equation can be extended to accommodate the neutrino mixing and ensuing neutrino oscillations.

1. Introduction

The relativistic checkerboard (also chessboard) picture [1, 2, 3] was an attempt by R.P. Feynman to generalize his space-time view of quantum mechanics to include special relativity. In particular, Feynman hoped that he could explain particle's spin as a result of the space-time structure alone. He abandoned this program in 60's partially because he felt dissatisfaction with not being able to extend the picture to higher dimensions and partially because Grassmann integral (invented in 60's be Berezin) became just the right tool to describe fermions in line with his path-integral philosophy. In spite of (or perhaps because of) Feynman's abandonment of the checkerboard paradigm, there has been over the years a growing number of works indicating that the checkerboard picture is not merely an interesting mathematical curiosity but it may have more substance than originally thought [4, 5, 6, 7].

In its essence the FCP represents a relativistic quantum random walk in $1 + 1$ dimensions. The underlying (Euclidean-space) stochastic process for the FCP represents a specific form of a Poisson process in which the direction of motion and helicity are simultaneously changed at random Poisson-distributed times. Here we adopt the FCP paradigm as a conceptual playground that will allow us to treat two-fermion oscillations from a rather unusual, and very instructive, point of view.

The structure of the paper is as follows. To set the stage we recall in the next section some fundamentals of the Feynman checkerboard picture (FCP) in the Euclidean regime. This will be needed in the main body of the text. In Section 3 the "checkerboard prescription" for the sum over histories will be applied to two-fermion mixing. From this, the phenomenologically observed neutrino oscillations will emerge as an important byproduct. The Euclidean formulation of the FCP will prove instrumental in guiding our intuition about the underlying inner workings of neutrino mixing. In particular, we will see that the mixing can be envisaged as a Poisson process with two types of events with mutually complementary probabilities. Various remarks and generalizations are proposed in the concluding section.

2. Feynman checkerboard picture

The original formulation of the Feynman checkerboard picture was done by Feynman in $1+1$ dimensions [1]. Though the higher-dimension generalizations are also available [6, 7], they loose the intuitive appeal of the original formulation. In order to be as close as possible to the original Feynman's intuitive view, we will confine ourselves to 1+1 dimensions.

2.1. Dirac equation in 1+1 dimensions

In ordinary 3+1 dimensions the Dirac equation can be written in a Schrödinger-like form:

$$i\hbar\frac{\partial\psi}{\partial t} \;=\; mc^2\gamma^0\psi - i\hbar c\gamma^0\gamma^i\frac{\partial\psi}{\partial x^i} \;=\; mc^2\beta\psi - i\hbar c\alpha^i\frac{\partial\psi}{\partial x^i}\,, \tag{1}$$

with the γ-matrices fulfilling the Clifford algebra $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$. In passing to 1+1 dimensions the Clifford algebra keeps its structure, but μ and ν can be only 0 or 1. The representation of the two γ-matrices is well known. In particular, one can chose $\gamma^0 = \gamma_0 = \sigma_1$ and $\gamma_1 = -\gamma^1 = i\sigma_2$. With this the Dirac equation in 1+1 dimensions can be cast in the form

$$i\hbar\frac{\partial\psi}{\partial t} \;=\; mc^2\sigma_1\psi - i\hbar c\sigma_1(-i\sigma_2)\frac{\partial\psi}{\partial x} \;=\; mc^2\sigma_1\psi - i\hbar c\sigma_3\frac{\partial\psi}{\partial x} \tag{2}$$

we have used the identity $\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$. The propagator, say $G(x,t_x;y,t_y)$, for ψ is a 2×2 matrix function of space and time and it fulfills the defining equation

$$\psi(x,t_x) \;=\; \int_{\mathbf{R}} dy\, G(x,t_x;y,t_y)\,\psi_0(y,t_y)\,. \tag{3}$$

Euclidean version of (2) is easily obtained when we analytically continue the time t to imaginary times and assume the Clifford algebra in the form $\{\gamma_E^\mu,\gamma_E^\nu\} = 2\delta^{\mu\nu}$. By setting $t = -i\tau$ ($\tau\in\mathbf{R}$) and using $\gamma_E^0 = \sigma_1$ and $\gamma_E^1 = -\sigma_2$ we obtain

$$\frac{\partial\psi_E}{\partial\tau} \;=\; \frac{mc^2}{\hbar}\,\sigma_1\psi_E - c\sigma_3\frac{\partial\psi_E}{\partial x}\,. \tag{4}$$

Here $\psi_E(x,\tau) = \psi(x,t=-i\tau)$. Among others, the equation (4) implies the correct Euclidean Klein–Gordon equation in 1+1 dimensions. The Euclidean Green function $G(x,-i\tau_x,y,-i\tau_y) \equiv P(x,\tau_x|y,\tau_y)$ satisfies the Fokker–Planck-like equation:

$$(\partial_\tau + c\sigma_3\partial_x - \sigma_1 mc^2/\hbar)\mathcal{P}(x,\tau|x',\tau') \;=\; \delta(x-x')\delta(\tau-\tau')\,. \tag{5}$$

Note also that $mc^2/\hbar = 1/\tau_C$ where τ_C is the Compton time, i.e., time in which light crosses a distance equal to particle's Compton wave length λ_C.

2.2. Euclidean checkerboard picture

Here we start by introducing the so-called *Euclidean checkerboard picture*. To this end we consider a particle with a fixed speed v moving on a line. We suppose, further, that from time to time the particle suffers a complete reversal of the direction (and hence also $v \leftrightarrow -v$). Let these reversals be random with a fixed rate, say a, of the reversal and with the probability for the reversal in a time interval $d\tau$ being $ad\tau$. Such a process represents the Poisson stochastic.

Let $p_+(x,\tau)$ and $p_-(x,\tau)$ be probability density functions (PDF's) for the particle being at the positions x at time τ and moving to the *right* and *left*, respectively. By looking at Fig. 1 we

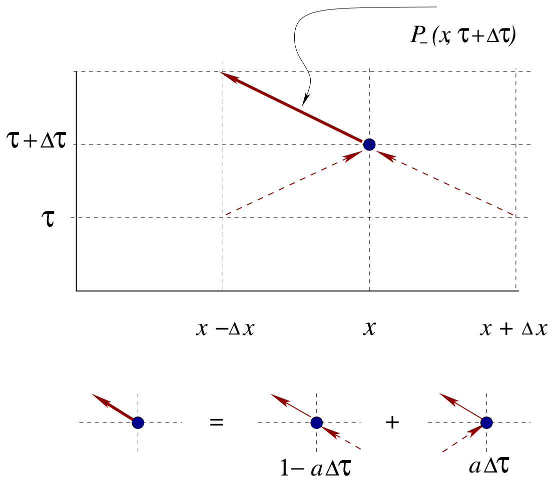

Figure 1. Graphical representation of the master equation (6).

can easily persuade ourselves that the following difference master equations hold

$$p_{\pm}(x, \tau + \Delta\tau) \;=\; p_{\pm}(x \mp \Delta x, \tau)(1 - a\Delta\tau) \;+\; p_{\mp}(x \pm \Delta x, \tau)a\Delta\tau\,. \tag{6}$$

In the continuum limit, we receive two interlocked differential equations

$$\frac{\partial p_{\pm}(x,\tau)}{\partial \tau} \;=\; -a\left[p_{\pm}(x,\tau) - p_{\mp}(x,\tau)\right] \;\mp\; v\frac{\partial p_{\pm}(x,\tau)}{\partial x}\,, \tag{7}$$

where the identification $v = dx/d\tau$ was made. At this point we introduce the probability doublet

$$P(x,\tau) \;=\; \left(\begin{array}{c} p_{+}(x,\tau) \\ p_{-}(x,\tau) \end{array}\right). \tag{8}$$

In terms of P the equation (7) reads as

$$\frac{\partial P(x,\tau)}{\partial \tau} \;=\; -aP(x,\tau) + a\sigma_1 P(x,\tau) - v\sigma_3\frac{\partial P(x,\tau)}{\partial x}\,. \tag{9}$$

The latter implies that the underlying stochastic process is the Poisson process. Equation (9) can be brought into a simpler form by performing the substitution

$$P(x,\tau) \;=\; e^{-a\tau}\mathcal{P}(x,\tau)\,. \tag{10}$$

For $\mathcal{P}$ we then have the master equation

$$\frac{\partial \mathcal{P}(x,\tau)}{\partial \tau} \;=\; a\sigma_1\mathcal{P}(x,\tau) - v\sigma_3\frac{\partial \mathcal{P}(x,\tau)}{\partial x}\,. \tag{11}$$

Before we proceed further, two comments are in order. First, the substitution (10) represents more than just a simple mathematical trick. In fcat, back in the Minkowski picture the substitution (10) represents just a multiplication of a wave function by phase factor $\exp\left[(imc^2/\hbar)t\right]$. The latter is the usual relativistic phase factor which is responsible for a shift of the ground-state energy by mc^2 [6]. Second, by comparing Eq. (4) with Eq. (11) we see that the equations are identical provided we set $\mathcal{P}(x,\tau) = \psi_E(x,\tau)$, $a = mc^2/\hbar = 1/\tau_C$ and $v = c$.

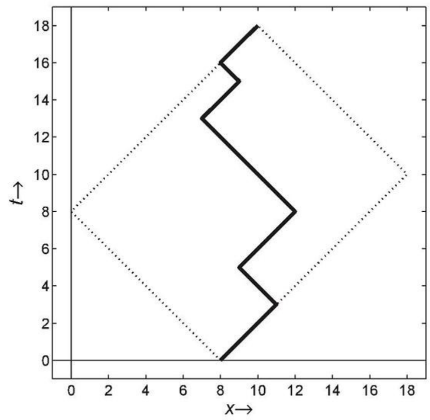

Figure 2. An example of a sample "++" trajectory that enters Feynman checkerboard picture for a spin 1/2 particle in $(1+1)$-dimensional spacetime.

The Euclidean propagator (or conditional probability) for Eq. (11) can be calculated by various means. In particular, the Feynman–Kac formula allows to phrase the propagator via path integral. There one should perform a weighted sum over *all* continuous trajectories running from the initial position (x', τ') to the finite position (x, τ) (see, e.g., Ref. [14] for a detailed exposition of this approach). It is thus interesting, that in fact not all trajectories are needed. Indeed, we have seen that (11) arises only by considering a special sub-class of zig-zag paths. From (10) follows that the propagator can be then written as

$$\mathcal{P}_{ij}(x,\tau|x',\tau') \;=\; e^{aT} P_{ij}(x,\tau|x',\tau')\,, \tag{12}$$

where $T = \tau - \tau'$ and $P(x,\tau|x',\tau')$ is the matrix-valued conditional probability density tying together the initial probabilities $P(x', \tau')$ and the final (marginal) probability $P(x, \tau)$, i.e.

$$P_i(x,\tau) \;=\; \int_{\mathbf{R}} dx' P_{ij}(x,\tau|x',\tau')\, P_j(x',\tau')\,. \tag{13}$$

Both in (12) and (13) the indices i, j run through the set $\{+,-\}$. Let us define X_{ij} as a set of all Poisson processes (with the rate a) that start at the state described by values $\{x', \tau', j\}$ and end in the state $\{x, \tau, i\}$, and let us denote $n(T)$ as the number of events that happen in our Poisson process in time T. With these we can write the transitional probability for our Poisson process by summing (marginalizing) in the the joint probability distribution $P[(n(T) = k) \cap X_{ij}]$ over all possible events k. In particular

$$P_{ij}(x,\tau|x',\tau') \;=\; \sum_k P[(n(T)=k)\cap X_{ij}] \;=\; \sum_k \mathfrak{P}[n(T)=k]P(X_{ij}|n(T)=k)\,. \tag{14}$$

The second equality in (14) is just the Bayes rule with $\mathfrak{P}[n(T) = k]$ describing the probability that in time T one observes k events. The analysis can be carried further when we discretize the time interval into N equidistant pieces. Such a regularization allows to evaluate the above conditional probability explicitly for any N. As with any regulating scheme, the limit $N \to \infty$ should be performed *after* the calculations with the fixed N are done. Consequently we can

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write

$$
\begin{aligned}
P_{ij}(x,\tau|x',\tau') &= \lim_{N\to\infty}\sum_k \mathfrak{P}[n(T)=k]\,\frac{\phi^{(N)}[(n(T)=k)\cap X_{ij}]}{\phi^{(N)}[n(T)=k]} \\
&= \lim_{N\to\infty} e^{-aT}\sum_k \left(\frac{aT}{N}\right)^k \phi^{(N)}[(n(T)=k)\cap X_{ij}]\,. \qquad (15)
\end{aligned}
$$

In the first equality we have utilized the definition of the conditional probability. The function $\phi^{(N)}[\ldots]$ denotes the *number* of distinct realizations of the random process in its argument. In the second equality we have used the fact that in the large N limit Stirling's formula gives

$$
\phi^{(N)}(n(T)=k) \;=\; \frac{N!}{k!(N-k)!} \;\cong\; \frac{N^k}{k!}\,. \qquad (16)
$$

This in turn allows to write the Euclidean propagator (12) in the form

$$
\mathcal{P}_{ij}(x,\tau|x',\tau') \;=\; \lim_{N\to\infty}\sum_k \left(\frac{mc^2T}{N\hbar}\right)^k \phi_{ij}^{(N)}(k)\,. \qquad (17)
$$

Here we have employed the simplifying notation $\phi_{ij}^{(N)}(k) \equiv \phi^{(N)}[(n(T)=k)\cap X_{ij}]$. Eq. (17) is (in spirit of Feynman's PI) a sum over alternatives, each summand being a (weighted) number of the Poisson processes (with the rate $a = mc^2/\hbar$) with exactly k reversals (events) running from the initial state $\{x',\tau',j\}$ to the final state $\{x,\tau,i\}$.

Since $\phi_{ij}^{(N)}(k)$ represents the number of paths it does not change when we pass from Euclidean to Minkowski picture. The only change is in the path weighting factor $(mc^2T/N\hbar)^k$ where the Euclidean T should change to the Minkowski iT. Consequently we can write the associated Minkowski Green function in the form

$$
G_{ij}(x,t|x',t') \;=\; \lim_{N\to\infty}\sum_k \left(i\frac{mc^2T}{N\hbar}\right)^k \phi_{ij}^{(N)}(k)\,, \qquad (18)
$$

where now $T = t - t'$. This is precisely Feynman's original result [1]. By its very construction should (18) coincide with the usual Bessel-functions based formula for the Dirac propagator. Taking into account the explicit form of $\phi_{ij}^{(N)}(k)$ this can be, indeed, explicitly checked. Following Refs. [2, 3], we can calculate, for instance, $\phi_{++}^{(N)}(k)$ as follows: first we realize that for any "++" (and also "−−") configuration k is an even number and there are precisely $k/2$ turns to the left and $k/2$ turns to the right [1]. By splitting the time axis into N equidistant pieces, we define the elementary time step $\Delta\tau \equiv T/N$. We further denote the total time (in the units of $\Delta\tau$) that corresponds to the left turns as L and the time for right turns as R. With this we can write

$$
x - x' \;=\; c(R-L)\Delta\tau \;\equiv\; c\Delta\tau M \quad \text{and} \quad R+L \;=\; N\,, \qquad (19)
$$

$$
\Rightarrow\;\; R \;=\; \frac{N+M}{2}\,, \qquad L \;=\; \frac{N-M}{2}\,. \qquad (20)
$$

On Fig. 3 we see that the total number of "++" paths starting at (x',τ') and ending at (x,τ)

[1] For "−+" and "+−" configurations k is odd. For "−+" configuration there $(k-1)/2+1$ turns to the left and $(k-1)/2$ turns to the right. For "+−" configuration is the role of *left* and *right* reversed.

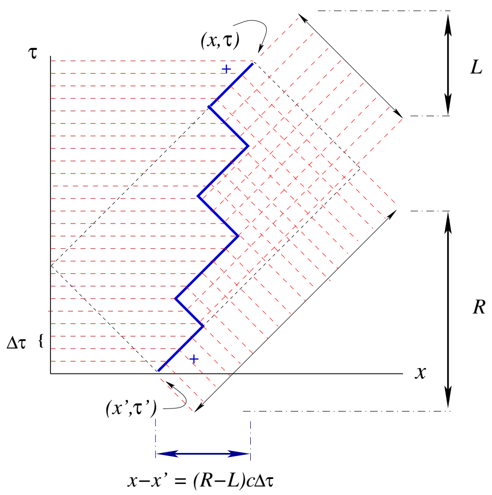

Figure 3. Conventions used in the $\phi_{++}^{(N)}(k)$ calculation.

with k turns can be written as the product of the number of different ways in which one can distribute $k/2$ right turns among $R-2$ potential time occasions for the right turn (-2 enters because the initial and final time steps are fixed by the "++" boundary conditions), times the number in which one can distribute $k/2-1$ left turns (-1 enters because one turn to the left is compulsory for $m \neq 0$) among $L-1$ potential time occasions for the left turn (-1 enters because ne turn to the left is compulsory for $m \neq 0$) , i.e.

$$\phi_{++}^{(N)}(k) \ = \ \left(\begin{array}{c} R-2 \\ k/2 \end{array} \right) \left(\begin{array}{c} L-1 \\ k/2-1 \end{array} \right) \ \cong \ \frac{L^{k/2-1}}{(k/2-1)!}\frac{R^{k/2}}{(k/2)!} \ = \ \frac{(LR)^{(k-1)/2}}{(k/2-1)!(k/2)!}\sqrt{\frac{R}{L}} \, . \tag{21}$$

Using further the fact that

$$LR \ = \ \frac{1}{4} \ = \ \left(\frac{N}{2\gamma}\right)^2 , \quad R/L \ = \ \frac{\gamma^2}{(\Delta\tau^2 N)^2}\left[T + (x-x')/c\right]^2 , \tag{22}$$

(here $v = (x-x')/T$ and γ is the Lorentz factor) and plugging (22) to (21) we receive

$$\mathcal{P}_{++}(x,t|x',t') \ \cong \ \frac{\gamma}{T}\left[T + (x-x')/c\right] \sum_{k=1}^{\infty}\left(\frac{mc^2 T}{N\hbar}\right)^{2k}\left(\frac{N}{2\gamma}\right)^{2k-1}\frac{1}{(k-1)!k!} \tag{23}$$

$$= \ \Delta\tau \, \frac{mc^2(T + (x-x')/c)}{s\hbar}\, I_1(mc^2 s/\hbar) \, , \tag{24}$$

where we have used abbreviation $s = T/\gamma = [T^2 - (x-x')^2/c^2]^{1/2}$. Similarly we would arrive at the result

$$\mathcal{P}_{-+}(x,t|x',t') \ \cong \ \Delta\tau \, \frac{mc^2}{\hbar}\, I_0(mc^2 s/\hbar) \, . \tag{25}$$

The correct normalization in the continuum limit can be obtained from the probability normalization. Going back to the Poisson process described by $P_{ij}(x,\tau|x',\tau')$ (cf. Eq. (12)) we should require the normalization

$$e^{-at}\int_{-ct}^{ct} dx \, \left[\mathcal{P}_{-+}(x,t|0,0) + \mathcal{P}_{++}(x,t|0,0)\right] \ = \ \wp(T_1 \leq t) \ = \ 1 - e^{-at} \, , \tag{26}$$

(similar normalization should hold if we have started from the "−" initial configuration). Here we have used the normalization condition for conditional distributions, i.e., $\sum_i P(A_t^i|B_0) = 1$. The summation is taken over all possible states A_t^i that the state $B_0 \equiv B(t=0)$ can reach at time t. Note that (26) does not equal "1" because for *massive* particle the future position of the particle should stay *within* the future light cone of the initial particle position. In the integral in (26) we have, however, accounted also for the boundary of the light cone. In particular, when the particle starts at $x = 0, t = 0$ the future position in the time t should be somewhere in the interval $(-ct, ct)$ (at least for times that are substantially larger than the Compton time τ_C) but not at the boundary points $x = \pm tc$. So in our case we should subtract from the *unit* probability the probability that the particle will end up at the light-cone boundary, i.e., "1" minus probability that no reversal happens (which is nothing but the probability that during t will happen at least one reversal). Explicit calculation gives, however

$$e^{-at}\int_{-ct}^{ct} dx\ [\mathcal{P}_{-+}(x,t|0,0) + \mathcal{P}_{++}(x,t|0,0)] \ = \ 2\Delta\tau(1 - e^{-at})\,. \tag{27}$$

This implies that both (24) and (25) should be divided by $2\Delta\tau$ in order to fulfill the normalization condition (26).

The passage from the Euclidean transition probability (24) to the Minkowski Green function is established by changing T to iT in the path-weighting factor. Consequently we can write

$$G_{++}(x,t|x',t') \quad \cong \quad -\frac{mc^2(T + (x-x')/c)}{2s\hbar}\, J_1(mc^2 s/\hbar)\,. \tag{28}$$

Analogous reasonings can be done also for other components. Finally we can write for the full matrix propagator

$$G_{ij}(x,t|x',t') \ = \ \frac{mc^2}{2s\hbar}\begin{pmatrix} -(T + (x-x')/c)J_1(mc^2 s/\hbar) & s\,J_0(mc^2 s/\hbar) \\ s\,J_0(mc^2 s/\hbar) & (-T + (x-x')/c)J_1(mc^2 s/\hbar) \end{pmatrix}. \tag{29}$$

This indeed coincides with the known result for the 1+1 dimensional Dirac's fermion propagator (see, e.g. Ref. [2]).

Eq. (17) offers a conceptually interesting interpretation of a representative trajectory for an Euclideanized Dirac particle in two dimensions. In particular, a massless(!) particle propagates over an average distances λ_C with velocity c before it reverts its direction. It is only on much larger spatial scales (after many directional reversals take place) where the Brownian motion with a sub-luminal average velocity emerges. So in quantum mechanics one must sum over all such zig-zag trajectories subject given Dirichlet boundary condition in order to obtain correct Feynman causal propagator for Dirac fermion.

It is quite feasible that the Minkowski metric might be just a way of interpreting our experience, while the actual underlying structure of space-time is Euclidean. This line of thoughts has been advocated, for instance, by Polyakov [12] and Hawking [13]. The original Feynman's hope was that a similar type of reasoning that we have used in this section could provide a novel explanation of the geometrical origin of spin, and potentially it could also shape a new intuition about quantum dynamics. It will be this last point that will guide us in the following section to discuss the phenomenon of two-fermion oscillation.

3. Two-fermion oscillations

Let us briefly summarize basic aspects of the traditional approach to fermion oscillations. For explicitness we will use in our exposition the phenomenologically most relevant fermion oscillations, namely neutrino oscillations. First, states in which neutrinos propagate are called "mass eigenstates", while the states that take part in interaction (and detection) are called "flavor eigenstates". Second, mass eigenstates propagate at different speeds and so the components get out of phase with each other. In particular, neutrino born as one flavor will soon start looking like another one. Third, for neutrinos there are 3 flavors $(\nu_e, \nu_\mu, \nu_\tau)$ but mixing with ν_τ is typically very weak and in the first approximation is neglected. In consequence, mass eigenstates (ν_1, ν_2) associated with (ν_e, ν_μ) have different speeds and hence different on-shell masses. Finally, because the flavor states are superpositions of mass states they must oscillate in time because the quantum mechanical phases of the involved mass states advance at slightly different rates. This is an analogue of the well known phenomenon from wave physics; when two waves with similar frequency add, they interfere and produce *beats*.

Above quantum-mechanical description is not satisfactory for various reasons and it is believed that only full quantum field theoretical approach is free from conceptual difficulties [9, 10]. We shall now demonstrate that the two-fermion oscillations can be accommodated in the FCP rather simply. In fact, it is a fortunate feature of the $1+1$-dimensional nature of the FCP that we do not need to invoke the field theory concepts (see also comment in Conclusions).

Denote a flavor doublet $\nu_f = (\nu_e, \nu_\mu)^T$ and mass doublet $\nu_m = (\nu_1, \nu_2)^T$. Then the corresponding Dirac equations read

$$\begin{aligned} i\partial_0 \nu_e &= (-i\alpha \cdot \nabla + \beta m_e)\nu_e + \beta m_{e\mu}\nu_\mu \,, \\ i\partial_0 \nu_\mu &= (-i\alpha \cdot \nabla + \beta m_\mu)\nu_\mu + \beta m_{e\mu}\nu_e \,, \end{aligned} \tag{30}$$

which can be succinctly written as

$$i\partial_0 \nu_f = (-i\alpha \cdot \nabla + \beta M_f)\nu_f \,. \tag{31}$$

Similarly for the mass doublet we have $i\partial_0 \nu_m = (-i\alpha \cdot \nabla + \beta M_d)\nu_m$. Here $M_d = \text{diag}(m_1, m_2)$ is a diagonal mass matrix. Mixing relations for two Dirac fermions are parametrized via the mixing angle θ which is conventionally defined via the implicit ralation $m_{e\mu} = m_{\mu e} = (m_\mu - m_e)\tan(2\theta)$. With this

$$\nu_e = \cos\theta \nu_1 + \sin\theta \nu_2, \quad \nu_\mu = -\sin\theta \nu_1 + \cos\theta \nu_2 \,, \tag{32}$$

which directly gives

$$\begin{aligned} m_e &= m_1 \cos^2\theta + m_2 \sin^2\theta, \quad m_\mu = m_1 \sin^2\theta + m_2 \cos^2\theta, \\ m_{e\mu} &= (m_2 - m_1)\sin\theta\cos\theta \,. \end{aligned} \tag{33}$$

To accommodate Dirac fermion oscillations in the FC picture we notice that by analogy with (6) the difference master equations for flavor states (probabilities) read (see Fig. 4)

$$\begin{aligned} p_\pm^{(e)}(x, \tau + \Delta\tau) &= p_\pm^{(e)}(x \mp \Delta x, \tau)(1 - a\Delta\tau) + p_\mp^{(e)}(x \pm \Delta x, \tau)pa\Delta\tau \\ &\quad + p_\mp^{(\mu)}(x \pm \Delta x, \tau)(1 - p)a\Delta\tau \,, \\[1ex] p_\pm^{(\mu)}(x, \tau + \Delta\tau) &= p_\pm^{(\mu)}(x \mp \Delta x, \tau)(1 - a'\Delta\tau) + p_\mp^{(\mu)}(x \pm \Delta x, \tau)p'a'\Delta\tau \\ &\quad + p_\mp^{(e)}(x \pm \Delta x, \tau)(1 - p')a'\Delta\tau \,. \end{aligned} \tag{34}$$

What this all means. Let a Poisson process representing Dirac neutrino (say ν_e) has rate a, and suppose then each time an even occurs, it can be classified into two types. First type

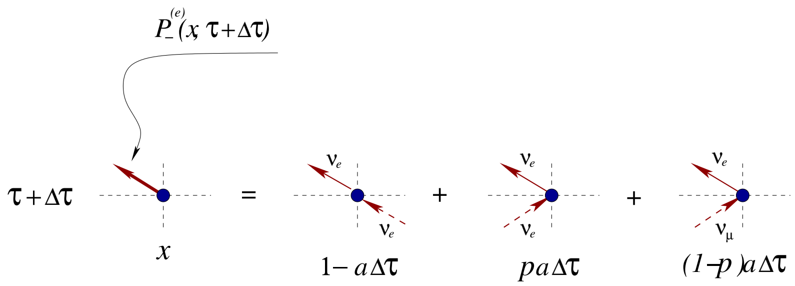

Figure 4. Graphical representation of the master equation for $p_-^{(e)}(x,\tau+\Delta\tau)$. Similar representations can be drawn also for $p_+^{(e)}$ and $p_\mp^{(\mu)}$.

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corresponding to change of direction without change of particle's identity has probability p and a second type corresponding to change of direction with the change of identity has probability $(1-p)$. In close analogy with (6) we now define the probability doublet

$$P^{(e),(\mu)}(x,\tau) \;=\; \left(\begin{array}{c} p_+^{(e),(\mu)}(x,\tau) \\ p_-^{(e),(\mu)}(x,\tau) \end{array}\right), \tag{35}$$

and set $pa = m_e c^2/\hbar$, $p'a' = m_\mu c^2/\hbar$ and $(1-p)a = (1-p')a' = m_{\mu e}c^2/\hbar$. In addition, we define $P^{(e)}(x,\tau) = e^{-a\tau}\mathcal{P}^{(e)}(x,\tau)$ and $P^{(\mu)}(x,\tau) = e^{-a'\tau}\mathcal{P}^{(\mu)}(x,\tau)$, the master equations (34) will then boil down to

$$\begin{aligned}
\frac{\partial \mathcal{P}^{(e)}(x,\tau)}{\partial \tau} &= -c\sigma_3\frac{\partial \mathcal{P}^{(e)}(x,\tau)}{\partial x} + m_e c^2/\hbar\sigma_1\mathcal{P}^{(e)}(x,\tau) + m_{\mu e}c^2/\hbar\sigma_1\mathcal{P}^{(\mu)}(x,\tau)\,,\\
\frac{\partial \mathcal{P}^{(\mu)}(x,\tau)}{\partial \tau} &= -c\sigma_3\frac{\partial \mathcal{P}^{(\mu)}(x,\tau)}{\partial x} + m_\mu c^2/\hbar\sigma_1\mathcal{P}^{(\mu)}(x,\tau) + m_{e\mu}c^2/\hbar\sigma_1\mathcal{P}^{(e)}(x,\tau)\,.
\end{aligned} \tag{36}$$

Our requirement that $m_{\mu e} = m_{e\mu}$ is a consequence of a micro-reversibility, namely that the probability for event $\nu_\mu \to \nu_e$ should be in the interval $\Delta\tau$ the same as $\nu_e \to \nu_\mu$. We have also neglected the term $e^{\pm\tau(m_e - m_\mu)c^2/\hbar}$ in front of the $m_{e\mu}$ term. This in particular shows, that the outlined Poisson process is a good description of the neutrino oscillations if $\tau(m_e - m_\mu)c^2/\hbar \ll 1$ for typical observational times τ — which indeed is the case. Equations (36) is the Euclidean version of the Cabibbo two-flavor mixing where for the Cabibbo angle one has

$$\theta \;=\; \arctan[(1-p)(1-p')/pp']\,. \tag{37}$$

In the FCP there is no need for the mass-states, since freely evolving ν_μ and ν_e propagate with speed of light, i.e., they represent particles with a sharp zero-mass value. As before, the actual inertial mass is due to a zig-zag nature of the representative trajectories in the FCP. This is akin to massless neutrinos experiencing a friction (via random noise or collisions) within the vacuum medium. The effect of this mass-giving environment is phenomenologically described via random Poisson-distributed interactions with massless neutrinos. In a sense, this is a quantum-mechanical analogue of the Higgs mechanism. In this picture oscillations are not due to different propagation speeds of mass eigenstates but they result from two different types of events that a massless neutrino in a Poisson process can experience. These two distinct events can be attributed to two different interaction channels through which massless neutrinos can interact with the mass-giving environment.

4. Conclusions

In this paper we have presented a novel heuristic picture for neutrino oscillations. A *modus operandi* we have employed is the standpoint that a stochastic picture of two-fermion oscillations in Euclidean space-time is ontologically more fundamental than the usual quantum-mechanical description. Actual inner workings of our approach were based on Feynman's checkerboard picture, i.e., a Poisson-process based random walk allowing to address quantum dynamics of Dirac particles in $1 + 1$ dimensions. We have shown here that two-fermion oscillations (e.g., $\nu_\mu \leftrightarrow \nu_e$) can be accommodated in the FCP easily by assuming that each time an event occurs, it is classified either as a type one (directional change without identity change) or a type two (directional change with identity change).

Let us add few concluding comments. First, the logical consistency of the presented picture might make one wonder how we could dispense with quantum field theory. Well, this is a byproduct of the fact that we deal with $1 + 1$ dimensional theory. Actually, because of the Mermin–Wagner–Coleman theorem the flavor-state vacuum and mass-state vacuum are unitary equivalent in $1+1$ dimensions [9] and hence the field-theoretical treatment does not produce any extra corrections as is the case in higher dimensions [9]. Extension of our approach to higher dimensions would be worth pursuing particularly in view of a naturalness with which the FCP handles fermionic particle systems. Second, checkerboard pictures are as a rule formulated only for fermions. It is, however, possible to formulate the checkerboard-like picture also for a Klein–Gordon (i.e., spinless relativistic) particle in $1 + 1$ dimensions. To this end one should use the first-order version of the Klein–Gordon equation known as the Feshbach–Villars representation of a scalar particle [6, 8]. Work along these lines will be presented elsewhere.

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