EXTENSIONS OF THE CLASSICAL DOUBLE COPY

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ABSTRACT

EXTENSIONS OF THE CLASSICAL DOUBLE COPY

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The classical double copy is a map between certain exact solutions of general relativity and Maxwell's theory with two different formulations: the Kerr-Schild double copy (KSDC) and the Weyl double copy (WDC). This thesis extends both versions in various directions. In three spacetime dimensions, matter fields should be coupled to gravity to match the vacuum solution of Maxwell's theory, i.e., the Coulomb solution. In addition to presenting a novel matter coupling with improved behaviour at infinity, some generalizations that admit the study of the regularity of the solutions on both sides are also considered. Also, a formulation of the KSDC with a general background metric in generic D-dimensions is given. For the WDC, a regularization scheme that makes it compatible with the much more general results in the KSDC is proposed.

Keywords: Classical Double Copy, Kerr-Schild Double Copy, Weyl Double Copy

ÖZ

KLASİK ÇİFT KOPYANIN GENİŞLETİLMESİ

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Klasik çift kopya, genel göreliliğin belirli tam çözümleri ile Maxwell teorisi arasında Kerr-Schild çift kopya (KSÇK) ve Weyl çift kopya (WÇK) şeklinde iki farklı formülasyonu olan bir ilişkilendirmedir. Bu tez her iki versiyonun da çeşitli yönlerde genişletilmesini içermektedir. Bu kapsamda üç uzay-zaman boyutunda maddesel alanların, Maxwell teorisinin vakum çözümüne, yani Coulomb çözümüne uyacak şekilde yerçekimiyle etkileştirilmesi gerektiği gösterilmiştir. Sonsuzdaki davranışı iyileştirilmiş yeni bir madde konfigürasyonu sunmanın yanı sıra, her iki taraftaki çözümlerin düzenliliğinin (*regularization*) incelenmesini sağlayan bazı genellemeler de ele alınmıştır. Ayrıca, D boyutlu genel bir artalan metriği için KSÇK'nin geneleştirilmiş bir formülasyonu sunulmuştur. WÇK için de sonuçlarını KSÇK'deki genel sonuçlarla uyumlu hale getiren bir regülarizasyon şeması önerilmiştir.

Anahtar Kelimeler: Klasik Çift Kopya, Kerr-Schild Çift Kopyası, Weyl Çift Kopyası

"I claded in the four elements as a niqab.
I emerged from a point and got purified.
Seen by those who see with the eyes of their soul,
A vista I am, in the depth of a vista."
Bektashi Nefes by Gencî, 19th century

"All science would be superfluous if the outward appearance and the essence of things directly coincided." **Karl Marx, 19th century**

To all those seeking the truth in all times and all geographies...

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LIST OF ABBREVIATIONS

ABBREVIATIONS

2D	2 Dimensional
3D	3 Dimensional
4D	4 Dimensional
GR	General Relativity
(A)dS	(Anti) de Sitter
KS	Kerr-Schild
KSDC	Kerr-Schild Double Copy
WDC	Weyl Double Copy
SWDC	Sourced Weyl Double Copy
RWDC	Regularized Weyl Double Copy
BTZ	Bañados-Teitelboim-Zanelli
BCJ	Bern-Carrasco-Johansson

NOTATION: Latin indices $(i, j, k, \cdots)$ run from 1 to (D-1) and Grek indices $(\mu, \nu, \rho, \cdots)$ run from 0 to D. We follow Einstein's summation convention, *i.e.* $v_\mu v^\mu = \sum_{\mu=0}^{D} u_\mu v^\mu$ and use units $c = \hbar = 1$ throughout the thesis.

CHAPTER 1

INTRODUCTION

The idea of the double copy, also named BCJ double copy after its discoverers Bern, Carrasco and Johansson, emerged as a relation between the scattering amplitudes in gauge and gravity theories [1]. A gluon amplitude in the gauge theory is expressed as a sum of cubic graphs which schematically takes the form

$$A_{\text{Gluon}} = \sum_i \frac{n_i c_i}{d_i}, \tag{1.1}$$

where c_i, n_i and d_i represent the color factors, the kinematic factors and the propagators of each graph respectively. This sum can be written in different ways by making use of generalized gauge transformations, i.e., gauge transformations and field redefinitions. When it is expressed in the so-called color-dual gauge, where the kinematic factors obey the same algebra as the color factors, the graviton amplitude in the gravity theory can be obtained by replacing the color factors c_i by another set of kinematic operators $\tilde{n}_i$ as follows

$$A_{\text{Graviton}} = \sum_i \frac{n_i \tilde{n}_i}{d_i}. \tag{1.2}$$

The new kinematic factors $\tilde{n}_i$ should be also in the color-dual gauge but, in general, they may be taken from a different gauge theory. Therefore, one obtains a gravity = (Yang-Mills)^2-type relationship where the gravity theory is called the "double copy" of the two Yang-Mills (YM) theories, while each YM theory is referred to as a "single copy" of the gravity theory.

Since it enables the "squaring" of the Yang-Mills (YM) ones to yield gravitational amplitudes, the double copy has evolved into a crucial tool in the study of scattering amplitudes [2, 3]. From semi-classical tree-level amplitudes, perturbative classical solutions can also be produced [4–41]. Although the nonlinearity of the Yang-Mills and Einstein equations makes it appear impossible to establish a map between exact classical solutions, it is possible to obtain a linear structure for some classes of spacetimes on the gravity side and then map them to the linearized solutions of YM theory, or solutions of Maxwell's theory. Two different but related procedures can be used to achieve the classical double copy, a non-perturbative realization of the double copy idea. The Ricci tensor with mixed indices is linear in the perturbation for spacetimes admitting Kerr-Schild (KS) coordinates, which is useful in the KS double copy [42, 43]. As an alternative, one can work with the Weyl spinor, which is the spinor representation of the Weyl tensor. Similar to the KS construction, one can identify a spinor basis that linearizes the Weyl spinor for certain algebraically special spacetimes and relate it to a field strength spinor corresponding to a solution of Maxwell's equations [44]. There is a growing body of literature on the topic that offers numerous examples and generalizations [45–70] (see [71–73] for reviews). Thanks to the results from Twistor theory. [74–76], we now have a pretty good understanding of its roots and boundaries.

Having a new theoretical tool to investigate the relation between two seemingly different theories, it is important to consider situations where it might break, which not only enables one to understand the limitations if there are any but also opens doors for extensions of the idea to more general cases. In this thesis, following this logic in various cases, some extensions of the classical double copy will be presented. Our focus will be the following shortages and limitations of the original constructions:

1. In the original formulation of the KSDC [42], the Newtonian potential seems to play an important role. While it is possible to obtain black hole solutions with similar physical properties to higher-dimensional examples, it is also well-known that GR in 3D has no well-defined Newtonian limit. Therefore, it is crucial to understand the validity of the KSDC in 3D.

2. Also in the original construction [42], the spacetimes under consideration are written in the KS coordinates with a flat background metric. However; for many solutions of GR with physical importance, this is not possible or the choice of the background spacetime is not unique. Most significantly, this is the case for th AdS and the Lifshitz black holes, our main probes to study strongly

coupled gauge theories through holography [77–80]. In order to understand whether there exists any relation between our main tools to relate gauge and gravity theories, the double copy and the holography, this should be overcome by finding a formulation of the KSDC with an arbitrary background metric.

3. In the Weyl Souble Copy (WDC), while it was shown to be valid for asymptotically flat vacuum spacetimes [62], a proposal to deal with sources was later made by considering solutions of Einstein-Maxwell theory [40]. An implicit assumption in the Sourced WDC (SWDC) is that the contribution of each term in the metric function to the Weyl spinor is non-zero, this is not the case for some AdS and Lifshitz black hole solutions. Therefore, it is not certain whether the KSDC and the WDC are consistent when more general spacetimes are considered. Especially, the status of holographically relevant spacetimes should be understood.

The outline of the thesis is as follows: After a review of some background material in Chapter 2, we work with KSDC in 3D in Chapters 3,4,5. In these chapters, we will discuss the validity of the KSDC in 3D for rotating solutions, the effect of the cosmological constant, alternatives for the matter couplings to ensure the validity of the construction and their possible generalizations. In Chapter 6, we will give a formulation of the KSDC with a generic, curved background metric. The solutions to the issues related to the SWDC and its relation to the KSDC will be presented in Chapter 7. We will end the thesis with a summary and discussions in Chapter 8.

CHAPTER 2

BACKGROUND MATERIAL

In this chapter, we present some background material that might be useful to understand the discussion in the rest of the thesis. For the first two sections we closely follow the references [81] and [82].

2.1 Degree of Freedom Counting and the Newtonian Limit in Three-dimensional Gravity

General relativity is based on the well-known Einstein-Hilbert action for d dimensional spacetime

$$S_{EH} = \int \mathrm{d}^d x \sqrt{-g} \left[\frac{1}{\kappa^2} (R - 2\Lambda) + \mathcal{L}_{matter} \right] . \tag{2.1}$$

Here, $g = \det(g_{\mu\nu})$, R is the scalar curvature, Λ is the cosmological constant, and $\mathcal{L}_{matter}$ represents the Lagrangian for the matter fields. Varying the action with respect to the metric gives the Einstein equations

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = \frac{1}{\kappa^2} T_{\mu\nu} , \tag{2.2}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor and $R_{\mu\nu}$ is the Ricci tensor. The stress-energy tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_{matter})}{\delta g^{\mu\nu}} . \tag{2.3}$$

The Riemann curvature tensor $R_{\mu\nu\sigma\rho}$ contains the all information about how the spacetime is curved and it can be written in terms of trace and traceless parts as follows

$$R_{\mu\nu\sigma\rho} = W_{\mu\nu\sigma\rho} + \left(g_{\mu\nu}R_{\sigma\rho} - g_{\mu\rho}R_{\nu\sigma} - g_{\nu\sigma}R_{\mu\rho} + g_{\nu\rho}R_{\mu\sigma}\right) - \frac{1}{2}R\left(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}\right). \tag{2.4}$$

The traceless part of the Riemann tensor is represented by the Weyl tensor $W_{\mu\nu\sigma\rho}$. For $d = 3$, the Weyl tensor equals to zero and the Riemann tensor has only trace part. Taking the trace of Einstein's equations in $d = 3$, we get

$$R = -\kappa^2 T, \tag{2.5}$$

from which the trace-reversed Einstein's equations follows as

$$R_{\mu\nu} = \frac{\kappa^2}{2}\left(T_{\mu\nu} - T g_{\mu\nu}\right). \tag{2.6}$$

Using (2.5) and (2.6) in the decomposition of the Riemann tensor in (2.4), one finds

$$R_{\mu\nu\sigma\rho} = \frac{\kappa^2}{2}\left(g_{\mu\nu}T_{\sigma\rho} - g_{\mu\rho}T_{\nu\sigma} - g_{\nu\sigma}T_{\mu\rho} + g_{\nu\rho}T_{\mu\sigma}\right) - \frac{\kappa^2}{2}T\left(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}\right). \tag{2.7}$$

This means that the Riemann tensor, namely the curvature of the spacetime, is localy determined by the matter distribution. Additionally, for empty spacetime, where $T_{\mu\nu} = 0$, the spacetime is locally flat:

$$R_{\mu\nu\sigma\rho} = 0\,. \tag{2.8}$$

Therefore, we can say that three-dimensional gravity can exist only in the presence of matter in spacetime. The vacuum solutions of the Einstein equations are flat in three spacetime dimensions.

We can also get this result in an algebraic way. In d spacetime dimensions, the Riemann tensor has

$$N_R = \frac{1}{12}d^2(d^2 - 1) \tag{2.9}$$

independent components. For $d = 4$, $N_R = 20$, and for $d = 3$, $N_R = 6$

We also know that the Einstein tensor is a symmetric tensor and it has

$$N_E = \frac{1}{2}d(d + 1) \tag{2.10}$$

independent components. This means that $N_E = 10$ for $d = 4$, and $N_E = 6$ for $d = 3$. Considering this, we can easily see that in $d \geq 4$ the Riemann tensor contains more information than the Einstein tensor. However, for $d = 3$, both the Riemann and the Einstein tensors contains the same amount of information. At the perturbative level, it is easy to show that this is because the gravitation is over-constrained in $d = 3$. We can write the metric as flat metric $\eta_{\mu\nu}$ and the perturbative part $h_{\mu\nu}$:

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu} \,. \tag{2.11}$$

$h_{\mu\nu}$ has $\frac{1}{2}d(d{+}1)$ independent components. However, the physical degrees of freedom of the graviton can be obtained from the transverse and traceless parts of its spatial components h_{ij}. In d-dimensional spacetime these both amount to $(d-1)$ seperate constraints. These constraints cannot completely determine the $\frac{1}{2}d(d{-}1)$ independent components of h_{ij} for $d \geq 4$. On the other hand, for $d = 3$, $2(d-1) = 4$ seperate constraints exist for the $\frac{1}{2}d(d-1) = 3$ independent components of h_{ij}. This means that the graviton does not propagate in $d = 3$.

2.1.1 Newtonian Limit of 2+1 dimensional gravity

Let us continue the discussion by taking the Newtonian limit of pure gravity in d dimensions. Hereby, we will prove that the Newtonian potential decouples from any point sources in three dimensional spacetime.

We will start by considering the weak fields. Namely, fields live in a spacetime whose metric is quasi-stationary in a specific coordinate system ($\partial_0 g_{\mu\nu} \approx 0$), and weakly deviates from flat space $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$. We will consider a stationary weak source i.e. a stress-energy tensor has only one non-zero component $T_{00} = -\rho$ and it is small enough.

Now we can examine the non-relativistic trajectories, i.e. geodesics $x^\mu(\tau)$ for which $|\dot{x}^i(\tau)| \ll 1$, in this coordinate system by neglecting all terms beyond linear order in ρ and $h_{\mu\nu}$. These reduce the spatial geodesic equation to

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{1}{2}\vec{\nabla} h_{00} \equiv -\nabla\Phi \,, \tag{2.12}$$

where we can identify

$$\Phi \equiv -\frac{1}{2}h_{00} = -\frac{1}{2}(1 + g_{00}), \tag{2.13}$$

as the Newtonian potential. Furthermore, in d dimensions, the 00-component of the trace reversed Einstein equations reads

$$\vec{\nabla}^2 \Phi = \frac{\kappa^2}{2}\frac{d-3}{d-2}\rho \,. \tag{2.14}$$

At this point, we should note that the Newtonian and Poisson equations are gauge dependent in the Newtonian limit

If we specialise the equations for the case of the point mass M which is sitting at the origin, the Newtonian gravitational field $\vec{g} = -\vec{\nabla}\Phi$, at a distance r from the orign becomes

$$\vec{g} = -\frac{\kappa^2}{2}\left[\frac{(d-3)\Gamma\left(\frac{d+1}{2}\right)}{(d-1)(d-2)\pi^{(d-1)/2}}\right]\frac{M}{r^{d-2}}\,\hat{r}\,. \tag{2.15}$$

This equation tells us that the Newton's constant for the d dimensional spacetime G_d is

$$G_d = \frac{\kappa^2}{2}\left[\frac{(d-3)\Gamma\left(\frac{d+1}{2}\right)}{(d-1)(d-2)\pi^{(d-1)/2}}\right]\,. \tag{2.16}$$

Finally, by considering the last equation, we can easily read that for $d = 4$, the Newton's constant $G_{d=4} = \frac{1}{16\pi}\kappa^2$. In contrast to $d \geq 4$ case, for $d = 3$, $G_{d=3} = 0$. Therefore, we can conclude the discussion by noting that, in $d = 3$, the Newtonian gravitational field $\vec{g}$ vanishes identically at any distance in the presence of a point mass M at the origin. The Newtonian potential Φ is no longer coupled via Poisson equation to the matter distribution ρ and this means that the trajectories are straight lines in the Newtonian limit.

2.2 Some Important Solutions in 3D Gravity

2.2.1 Conical Defects

In three dimensions, at the points where the particles are located, there arises conical defects (or conical singularities) characterized by the deficit angle $\theta_{def} \leq 2\pi$. This parameter produces an upper bound for the mass of the particle. We can show this by studying point particle solution in three dimensional general relativity.

A general three dimensional black hole solution can be written in polar coordinates as

$$\mathrm{d}s^2 = -A^2(r)\mathrm{d}t^2 + B^2(r)dr^2 + r^2\mathrm{d}\phi^2\,. \tag{2.17}$$

Here, $0 \leq r < \infty$ and $0 \leq \phi < 2\pi$.

In order to determine the $A(r)$ and $B(r)$ for vacuum case, the gravity equations should be taken as $R_{\mu\nu} = 0$, and these lead to

$$A(r) = A_0, \quad B(r) = B_0\,. \tag{2.18}$$

Here, A_0 and B_0 are constants. Now, let us make some coordinate redefinitions as

$$\tau \equiv A_0 t, \quad \rho \equiv B_0 r, \text{ and } \quad \phi \equiv B_0\theta\,. \tag{2.19}$$

In these coordinates the line element becomes

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + d\rho^2 + \rho^2\mathrm{d}\theta^2\,. \tag{2.20}$$

According to (2.19) we can write that $0 \leq \rho < \infty$ and $0 \leq \theta < \dfrac{2\pi}{B_0}$. We know that when the angular coordinate ϕ has period 2π, the new angular coordinate θ has $\dfrac{2\pi}{B_0}$. In other words, the solution is singular in the interval of $\dfrac{2\pi}{B_0} \leq \theta < 2\pi$. This angular interval $2\pi\left(1 - \dfrac{1}{B_0}\right)$ can be defined as defict angle θ_{def}.

The solution in (2.20) can be embeded as a cone into the $1{+}3$ dimensional Minkowski spacetime in cylindrical coordinates

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 \tag{2.21}$$

with a cone equation

$$z^2 = (1 - B_0^2)r^2\,. \tag{2.22}$$

On the other hand, we can define a energy distribution for a point particle at origin as

$$\rho = M\,\delta^{(2)}(\vec{r})\,. \tag{2.23}$$

For this source, the solution of gravity equations gives

$$B_0 = \left[1 - \frac{M}{2\pi}\right]^{1/2}. \tag{2.24}$$

Thus, the deficit angle for point particle solution can be written as

$$\theta_{def} = 2\pi \left[1 - \left(1 - \frac{M}{2\pi}\right)^{1/2}\right] , \tag{2.25}$$

and the angle parameter is valid at the interval

$$0 \leq \theta < 2\pi\sqrt{1 - \frac{M}{2\pi}} . \tag{2.26}$$

We can easily see that there is an upper bound for mass parameter since, $\frac{M}{2\pi}$ can be at most one in (2.26).

At this point, we should note that there is no parallelism with Schwarzschild solution in $d \geq 4$. We have $G_{tt} \sim M\,\delta^{(d-1)}(\vec{r})$. However, the remaining diagonal components are also proportinonal to δ-function.

2.2.2 AdS_3 spacetime

One of the most important solutions in 3D gravity is the AdS_3 spacetime. Let us first show how we can obtain it by starting from a four dimensional flat space with $SO(2,2)$ symmetry

$$dS^2_{(4)} = -\mathrm{d}T^2 - \mathrm{d}U^2 + \mathrm{d}X^2 + \mathrm{d}Y^2 . \tag{2.27}$$

In this space, we can define a hypersurface which carries the rotational symmetries of the ambient spacetime as

$$-T^2 - U^2 + X^2 + Y^2 = \ell^2 . \tag{2.28}$$

By using the following reparametrizations

$$\begin{aligned} T &= \ell \cosh\rho \cos t , \qquad & X &= \ell \sinh\rho \cos\theta , \\ U &= \ell \cosh\rho \sin t , \qquad & Y &= \ell \sinh\rho \sin\theta , \end{aligned} \tag{2.29}$$

the line element in (2.27) can be written as

$$\mathrm{d}s^2 = \ell^2 \left[-\cosh^2\rho\, \mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\rho\, \mathrm{d}\theta^2\right] . \tag{2.30}$$

This metric is the induced metric on the surface in (2.28). It is the metric of AdS_3 spacetime in global coordinates. Here the coordinate ρ is defined on the $0 \leq \rho < \infty$

and the θ is periodic with 2π. In (2.30) we can do another parametrization as

$$r = \text{sinh}\rho \,, \tag{2.31}$$

and we obtain AdS_3 metric in static coordinates as

$$\mathrm{d}s^2 = \ell^2 \left[-(1+r^2)\,\mathrm{d}t^2 + \frac{1}{1+r^2}\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 \right] . \tag{2.32}$$

Another useful coordinate system for the AdS_3 spacetime is the Poincaré coordinates where the line element is given by

$$\mathrm{d}s^2 = -\frac{r^2}{\ell^2}\mathrm{d}t^2 + \frac{\ell^2}{r^2}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 \,. \tag{2.33}$$

For the AdS_3 spacetime, the Ricci tensor satisfies

$$R_{\mu\nu} = 2\Lambda g_{\mu\nu} \quad \text{with} \quad \Lambda = -\frac{1}{\ell^2} \,. \tag{2.34}$$

When we introduce a cosmological constant in the Einstein-Hilbert action as

$$S_{EH} = \frac{1}{\kappa^2}\int \mathrm{d}^3x\sqrt{-g}\,[R - 2\Lambda] \,, \tag{2.35}$$

the Einstein's equations and its trace-reversed version becomes

$$\begin{aligned} G_{\mu\nu} + \Lambda g_{\mu\nu} &= 0 \,, \\ R_{\mu\nu} &= 2\Lambda g_{\mu\nu} \,, \end{aligned} \tag{2.36}$$

from which we see that the AdS_3 spacetime is a solution of 3D GR with a negative cosmological constant $\Lambda = -\frac{1}{\ell^2} < 0$.

2.2.3 Bañados-Teitelboim-Zanelli Black Hole

Due to the lack of a well-defined Newtonian limit and propagating degrees of freedom, it was thought for a very long time that one cannot have black hole solutions in $d = 3$. However, Banados-Teitelboim and Zanelli (BTZ) showed that one can obtain a black hole solution with the addition of a negative cosmological constant [83].

The line element of the BTZ black hole can be written as

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\mathrm{d}\theta - \Omega(r)\mathrm{d}t\right)^2 \,, \tag{2.37}$$

where

$$f(r)=\frac{r^2}{\ell^2}-M+\frac{J}{4r^2}\quad\text{and}\quad\Omega(r)=-\frac{J}{2r^2}\,,\qquad |J|\leq M\ell\,,\ M>0\,. \tag{2.38}$$

Here, the parameters M and J are the mass and the angular momentum respectively.

There are two horizons located at

$$r_{\pm}^2=\frac{M\ell^2}{2}\left\{1\pm\left[1-\left(1-\frac{J}{M\ell}\right)^2\right]^{1/2}\right\}\,, \tag{2.39}$$

and if rotation parameter is set to zero ($J=0$), there is only one horizon

$$r_h=\sqrt{\frac{M}{2}}\ell\,, \tag{2.40}$$

completely analogous to the higher-dimensional black holes.

An important fact about the BTZ black hole is that it is locally isomorphic to the AdS_3 spacetime and the difference is in the global topology. To understand this, let us study the embedding of the static BTZ black hole ($J=0$). We again consider the line element in (2.27) and hypersurface in (2.28). Let us do a reparametrization for coordinates in (2.28) similar to (2.29)

$$\begin{aligned} T&=\sqrt{\frac{r^2}{M}-\ell^2}\sinh\left(\frac{\sqrt{M}}{\ell}t\right)\,,\qquad X=\sqrt{\frac{r^2}{M}-\ell^2}\cosh\left(\frac{\sqrt{M}}{\ell}t\right)\\ U&=\frac{r}{\sqrt{M}}\cosh\left(\sqrt{M}\phi\right)\,,\qquad Y=\frac{r}{\sqrt{M}}\sinh\left(\sqrt{M}\phi\right)\,, \end{aligned} \tag{2.41}$$

with which the induced metric becomes

$$\mathrm{d}s^2=-f(r)\mathrm{d}t^2+\frac{\mathrm{d}r^2}{f(r)}+r^2\mathrm{d}\phi^2\,,\qquad f(r)=\frac{r^2}{\ell^2}-M\,. \tag{2.42}$$

This is just the metric of the static BTZ black hole. For this identification to be globally valid, we need to identify ϕ with $\phi+2\pi$, namely ϕ has a period with 2π.

The topology of this solution is sensitive to the mass parameter M. When we try to remove it ($M=1$), we need to rescale the coordinates t and r, then this will change the period of the coordinate ϕ. This shows that BTZ and AdS_3 are topologically distinct.

2.3 Maximally Symmetric Spacetimes and the Deviation Tensor

AdS_3 spacetime is just an example of a maximally symmetric spacetime, which is a spacetime of constant curvature that possesses the maximum number of symmetries (same as the Minkowski spacetime). In this section, we review some important properties of maximally symmetric spacetimes by following [84], which will lead us to the definition of the deviation tensor that, as shown in Chapter 6, plays a major role in the formulation of the KSDC with a general background metric. For a maximally symmetric spacetime, the Riemann tensor is given by[1]

$$\bar{R}_{\mu\alpha\nu\beta} = \frac{\epsilon}{L^2}\left(\bar{g}_{\mu\nu}\bar{g}_{\alpha\beta} - \bar{g}_{\mu\beta}\bar{g}_{\nu\alpha}\right) \tag{2.43}$$

where $\epsilon = +1, 0, -1$ correspond to de Sitter (dS), Minkowski and Anti-de Sitter (AdS) spacetimes and L is the dS/AdS radius when $\epsilon \neq 0$. Taking the trace yield the Ricci tensor and the Ricci scalar as

$$\bar{R}_{\mu\nu} = \epsilon\,\frac{d-1}{L^2}\,\bar{g}_{\mu\nu}\,, \tag{2.44}$$

$$\bar{R} = \epsilon\,\frac{d(d-1)}{L^2}\,. \tag{2.45}$$

Using (2.44), one can show that the spacetime is a solution of vacuum Einstein equations if the cosmological constant is chosen as

$$\Lambda = \epsilon\,\frac{(d-1)(d-2)}{2\,L^2} \tag{2.46}$$

and, therefore, the Ricci tensor becomes

$$\bar{R}_{\mu\nu} = \frac{2\,\Lambda}{d-2}\,\bar{g}_{\mu\nu}. \tag{2.47}$$

This motivates us to define the deviation tensor which characterizes the deviation from a maximally symmetric spacetime as

$$\Delta_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{2\,\Lambda}{d-2}\,\bar{g}_{\mu\nu}, \tag{2.48}$$

which vanishes for maximally symmetric spacetimes provided that the cosmological constant is given by (2.46). When the background is Minkowski spacetime, one has $\bar{R}_{\mu\nu} = 0$ and,

$$\Delta(\text{Minkowski})_{\mu\nu} = -\frac{2\,\Lambda}{d-2}\,\bar{g}_{\mu\nu}. \tag{2.49}$$

[1] We use barred quantities since, in this work, we consider the possibility of a background metric being that of a maximally symmetric spacetime.

2.4 Spinor formalism and the Petrov classification in $d = 4$

Here, by following the appendix of [85], we review spinor formalism for a 4D curved spacetime endowed with the metric $g_{\mu\nu}$. In our notation, spacetime indices are given by $\{\mu, \nu, \dots\}$, frame indices by $\{a, b, \dots\}$, while the spinor indices are $\{A, B, \dots\}$ and their conjugates $\{A', B', \dots\}$.

2.4.1 Tetrads and Spinor Basis

We introduce a complex null tetrad $\{l, n, m, \bar{m}\}$ which is used to construct the metric $g^{(0)}_{\mu\nu}$ and satisfies

$$g^{(0)}_{\mu\nu} = -2\, l_{(\mu}n_{\nu)} + 2m_{(\mu}\bar{m}_{\nu)} \tag{2.50}$$

where

$$\begin{aligned} l^\mu l_\mu = n^\mu n_\mu = m^\mu m_\mu = \bar{m}^\mu \bar{m}_\mu = 0 \\ l^\mu n_\mu = -1 = -m^\mu \bar{m}_\mu \,. \end{aligned} \tag{2.51}$$

and, l and n are real null vectors and m is typically complex with $\bar{m}$ as its conjugate. The vierbein can be used as, for example, $l^a = e^{(0),a}{}_\mu l^\mu$, to obtain the frame tetrad that corresponds to the tetrad vectors. Specifically, we use the tetrad set

$$\begin{aligned} l^a &= \frac{1}{\sqrt{2}}\,(1, -1, 0, 0) \qquad & n^a &= \frac{1}{\sqrt{2}}\,(1, 1, 0, 0) \\ m^a &= \frac{1}{\sqrt{2}}\,(0, 0, i, 1) \qquad & \bar{m}^a &= \frac{1}{\sqrt{2}}\,(0, 0, -i, 1)\ , \end{aligned} \tag{2.52}$$

which generates Minkowski space by representing (2.50) as $\eta_{ab} = -2\, l_{(a}n_{b)} + 2m_{(a}\bar{m}_{b)}$ and, all frame indices are raised by η^{ab}.

We can use the Pauli four-vectors for the shift between tensors and spinors and they are given by

$$\sigma^a_{AA'} = \frac{1}{\sqrt{2}}(1, \vec{\sigma})_{AA'}\,. \tag{2.53}$$

where the σ_i's are well known SU(2) generators

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.54}$$

The Pauli four-vectors obey the following normalization rules

$$\sigma^a_{AA'}\sigma_a^{BB'} = \delta^B_A\delta^{B'}_{A'}\,, \qquad \sigma^a_{AA'}\sigma_b^{AA'} = \delta^a_b\,. \tag{2.55}$$

Then, any spacetime or frame vector has a spinor counterpart as

$$V_a \longrightarrow V_{AA'} = V_a\sigma^a_{AA'} = V_\mu e^{(0),\mu}{}_a\sigma^a_{AA'}\,. \tag{2.56}$$

Next, we can define a spinor basis $\{o_A, \iota_A\}$ and its conjugate basis $\{\bar{o}_{A'}, \bar{\iota}_{A'}\}$, whose indices are raised and lowered by the two-dimensional Levi-Civita symbol

$$\epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{AB}\,, \tag{2.57}$$

and its conjugate $\epsilon^{A'B'}$. The frame tetrad and the basis spinors are related by

$$\begin{aligned} o_A\bar{o}_{A'} &= l_a\sigma^a_{AA'}, \qquad \iota_A\bar{\iota}_{A'} = n_a\sigma^a_{AA'}\,, \\ o_A\bar{\iota}_{A'} &= \bar{m}_a\sigma^a_{AA'}, \qquad \iota_A\bar{o}_{A'} = m_a\sigma^a_{AA'}\,. \end{aligned} \tag{2.58}$$

The spinor basis satisfies

$$\begin{aligned} \epsilon^{AB}o_Ao_B =& 0 = \epsilon^{AB}\iota_A\iota_B, \\ \epsilon^{AB}o_A\iota_B =& -1 = -\epsilon^{AB}\iota_Ao_B. \end{aligned} \tag{2.59}$$

We can determine the (normalized) spinor basis vectors by using (2.59) and the tetrads which we chose as in (2.52) as follows

$$o_A = \frac{1}{\sqrt{2}}(1,1)\,, \qquad \iota_A = \frac{1}{\sqrt{2}}(1,-1)\,. \tag{2.60}$$

Next, we introduce the Infeld-van der Waerden symbols

$$\sigma^{ab}_{AB} = \sigma^{[a}_{AA'}\,\bar{\sigma}^{b]A'C}\epsilon_{CB}\,, \tag{2.61}$$

which, along with the spacetime vierbeins, allow us to obtain the spinorial counterparts of any even ranked tensors. For example, defining

$$\sigma^{\mu\nu}_{AB} = e^{(0),\mu}{}_a\,e^{(0),\nu}{}_b\,\sigma^{ab}_{AB}, \tag{2.62}$$

the Weyl spinor and spinor field strength spinor read

$$\Psi_{ABCD} = \frac{1}{4}W_{\mu\nu\alpha\beta}\sigma^{\mu\nu}_{AB}\sigma^{\alpha\beta}_{CD}\,, \tag{2.63}$$

$$f_{AB} = \frac{1}{2} F_{\mu\nu} \sigma^{\mu\nu}_{AB} \,, \tag{2.64}$$

where $W_{\mu\nu\alpha\beta}$ is the Weyl tensor and $F_{\mu\nu}$ is the standard field strength tensor. Both Ψ_{ABCD} and f_{AB} are completely symmetric in their indices.

2.4.2 Field Strength Spinor

The spinor field strength in the Weyl double copy resides in "an appropriate flat limit" of the entire spacetime metric, which we associate with the Minkowski vierbeins $g^{(0)}_{\mu\nu} = e^{(0),a}_{\mu} e^{(0),b}_{\nu} \eta_{ab}$. Then the frame field strength F_{ab} in relation to the spacetime field strength can be defined as

$$F_{\mu\nu} = e^{(0),a}_{\mu} e^{(0),b}_{\nu} F_{ab} \,. \tag{2.65}$$

Importantly, F_{ab} lives on Minkowski space while $F_{\mu\nu}$ lives on a different form of flat space described by $g^{(0)}_{\mu\nu}$. The spinor field strength associated with the frame field strength F_{ab} is analogous to (2.64).

$$f_{AB} = \frac{1}{2} F_{ab} \sigma^{ab}_{AB} \,. \tag{2.66}$$

We should note that for each of the type D metrics, f_{AB} is in the form of σ_3 as

$$f_{AB} = Z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{AB} \tag{2.67}$$

where Z is a general complex function. The nonzero components of F_{ab} are related to the real and imaginary parts of Z respectively as

$$F_{01} = -\frac{1}{2} Re(Z) \,, \qquad F_{23} = -\frac{1}{2} Im(Z) \,. \tag{2.68}$$

Finally, we can construct $F_{\mu\nu}$ by using flat space vierbeins as

$$\begin{aligned} F_{\mu\nu} &= F_{ab} e^{(0),a}{}_{\mu} e^{(0),b}{}_{\nu} \\ &= -\frac{1}{2} \left(Re(Z)\, e^{(0),a}{}_{[\mu} e^{(0),b}{}_{\nu]} + Im(Z)\, e^{(0),a}{}_{[\mu} e^{(0),b}{}_{\nu]} \right) , \end{aligned} \tag{2.69}$$

We can note that, from (2.69), if Z has an imaginary component, then $F_{\mu\nu}$ has a magnetic field component when vierbeins are diagonal (as would be the case if $g_{\mu\nu}$ is the metric in spherical polar or oblate spheroidal coordinates).

2.4.3 Weyl Spinor

The Weyl spinor Ψ_{ABCD} may be expanded using the spinor basis $\{o_A, \iota_A\}$ as

$$\begin{aligned}\Psi_{ABCD} =& \Psi_0\, \iota_{(A}\iota_B\iota_C\iota_{D)} - 4\,\Psi_1\, o_{(A}\iota_B\iota_C\iota_{D)} + 6\,\Psi_2\, o_{(A}o_B\iota_C\iota_{D)} \\ & - 4\,\Psi_3\, o_{(A}o_Bo_C\iota_{D)} + \Psi_4\, o_{(A}o_Bo_Co_{D)}\,, \end{aligned} \tag{2.70}$$

where parentheses represent symmeterization, with the convention $k_{(AB)} = \frac{1}{2}\left(k_{AB} + k_{BA}\right)$. The components $\Psi_I \in \mathbb{C}$ are Weyl scalars are related to the Petrov classification of the spacetime and, they are defined by using null tetrads as

$$\begin{aligned}\Psi_0 &= W_{\mu\nu\alpha\beta}\, n^\mu m^\nu n^\alpha m^\beta\,, \\ \Psi_1 &= W_{\mu\nu\alpha\beta}\, n^\mu l^\nu n^\alpha m^\beta\,, \\ \Psi_2 &= W_{\mu\nu\alpha\beta}\, n^\mu m^\nu \bar{m}^\alpha l^\beta\,, \\ \Psi_3 &= W_{\mu\nu\alpha\beta}\, n^\mu l^\nu \bar{m}^\alpha l^\beta\,, \\ \Psi_4 &= W_{\mu\nu\alpha\beta}\, \bar{m}^\mu l^\nu \bar{m}^\alpha l^\beta\,. \end{aligned} \tag{2.71}$$

For any Petrov type D spacetime, one can always choose an appropriate coordinate system such that $\Psi_2 \neq 0$ while all other Ψ_I's are equal to zero, and the Weyl spinor can be written as

$$\Psi_{ABCD} = 6\,\Psi_2\, o_{(A}o_B\iota_C\iota_{D)}. \tag{2.72}$$

2.4.4 Petrov Classification of Field Strength and Weyl Spinors

Now we can briefly introduce the spinor classification of field strength $F_{\mu\nu}$ and Weyl $W_{\mu\nu\sigma\rho}$ tensors by following the discussion in relevant parts in [86, 87]. We can start the discusson by considering field strength tensor.

Let ξ^A be an arbitary spinor with components ξ^0 and ξ^1. We can expand the field strength spinor by using these spinor basis as a quadratic polinomial in ξ^0 and ξ^1

$$f_{AB}\xi^A\xi^B = f_{00}\xi^0\xi^0 + 2f_{01}\xi^0\xi^1 + f_{11}\xi^1\xi^1\,, \tag{2.73}$$

If we define a complex number

$$\zeta = \frac{\xi^0}{\xi^1}, \tag{2.74}$$

we can write equation (2.73) as

$$f_{AB} = (\xi^1)^2 \left[f_{00}\zeta^2 + 2f_{01}\zeta + f_{11}\right] . \tag{2.75}$$

whose factorization reads

$$\begin{aligned} f_{AB} =&(\xi^1)^2(\alpha_0\zeta + \alpha_1)(\beta_0\zeta + \beta_1) \\ =&(\alpha_A\xi^A)(\beta_B\xi^B) . \end{aligned} \tag{2.76}$$

Thus we can write the field strength spinor as

$$f_{AB} = \alpha_{(A}\beta_{B)} . \tag{2.77}$$

This equation is called canonical decomposition of f_{AB} and the spinors α_A, β_B are principal spinors of the field strength spinor. Each of these spinors determines a principal null direction. By using these principal spinors we classify the field strength spinor as described in Table 2.1.

Table 2.1: Classification of the field stength spinor

Type	Partition	f_{AB}	f_{AB} satisfies
I	{11}	$\alpha_{(A}\beta_{B)}$	$f_{AB}\xi^A\xi^B = 0$
N	{2}	$\alpha_{(A}\alpha_{B)}$	$f_{AB}\xi^A = 0$
O	{-}	0	$f_{AB} = 0$

Here we should note that, in fact, the "partition" column shows the two possible coincidences for the principal null directions. The last row is included for completeness.

Now, we can turn our attention to the Weyl spinor, for which we can follow a similar procedure. Again, the Weyl tensor can be decomposed as

$$\begin{aligned} \Psi_{ABCD}\xi^A\xi^B\xi^C\xi^D =&\Psi_{0000}\xi^0\xi^0\xi^0\xi^0 + 4\Psi_{1000}\xi^1\xi^0\xi^0\xi^0 + 6\Psi_{1100}\xi^1\xi^1\xi^0\xi^0 \\ &+ 4\Psi_{0111}\xi^0\xi^1\xi^1\xi^1 + \Psi_{1111}\xi^1\xi^1\xi^1\xi^1 . \end{aligned} \tag{2.78}$$

and if the definition of ζ in (2.74) is used, one can write equation (2.78) as

$$\Psi_{ABCD} = (\xi^1)^4 \left[\Psi_{0000}\zeta^4 + 4\Psi_{1000}\zeta^3 + 6\Psi_{1100}\zeta^2 + 4\Psi_{0111}\zeta + \Psi_{1111}\right] . \tag{2.79}$$

Its factorization gives

$$\Psi_{ABCD} =(\xi^1)^4(\alpha_0\zeta+\alpha_1)(\beta_0\zeta+\beta_1)(\gamma_0\zeta+\gamma_1)(\delta_0\zeta+\delta_1)$$
$$=(\alpha_A\xi^A)(\beta_B\xi^B)(\gamma_G\xi^G)(\delta_D\xi^D) \tag{2.80}$$

Thus, we get

$$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}. \tag{2.81}$$

Analogous to the field strength spinor f_{AB}, the Ψ_{ABCD} shown in (2.81) is canonical decomposition of Weyl spinor and the spinors α_A, β_B, γ_C and δ_D are principal spinors that determine the principal null directions. In Table 2.2 the classification scheme can be found.

Table 2.2: Classification of the Weyl Spinor

Petrov Type	Partition	Ψ_{ABCD}	Ψ_{ABCD} satisfies
I	{1111}	$\alpha_{(A}\beta_B\gamma_C\delta_{D)}$	$\Psi_{ABCD}\xi^A\xi^B\xi^C\xi^D=0$
II	{211}	$\alpha_{(A}\alpha_B\gamma_C\delta_{D)}$	$\Psi_{ABCD}\xi^A\xi^B\xi^C=0$
D	{22}	$\alpha_{(A}\alpha_B\beta_C\beta_{D)}$	$\Psi_{ABCD}\xi^A\xi^B\xi^C=0$
III	{31}	$\alpha_{(A}\alpha_B\alpha_C\beta_{D)}$	$\Psi_{ABCD}\xi^A\xi^B=0$
N	{4}	$\alpha_{(A}\alpha_B\alpha_C\alpha_{D)}$	$\Psi_{ABCD}\xi^A=0$
O	{-}	0	$\Psi_{ABCD}=0$

The symbols D and N in the "Petrov type" column means degenerate (or double) and null types respectively. All types except I are algebraically special while type I is algebraically general.

We can end our discussion by choosing ξ^0 and ξ^1 as o and ι respectively by using the arbitrariness of ξ^A. In this case the equation (2.81) turns into the equation (2.70)

$$\Psi_{ABCD} = \Psi_0\,\iota_{(A}\iota_B\iota_C\iota_{D)} - 4\,\Psi_1\,o_{(A}\iota_B\iota_C\iota_{D)} + 6\,\Psi_2\,o_{(A}o_B\iota_C\iota_{D)}$$
$$- 4\,\Psi_3\,o_{(A}o_Bo_C\iota_{D)} + \Psi_4\,o_{(A}o_Bo_Co_{D)}\,. \tag{2.82}$$

In the same way the field strength spinor reads

$$f_{AB} = f_0\,o_{(A}o_{B)} + 2f_1\,o_{(A}\iota_{B)} + f_2\,\iota_{(A}\iota_{B)}. \tag{2.83}$$

Using Table 2.2, for D type metrics, we can easily say that the only non-zero coefficient in (2.70) is Ψ_2. In this case we can write

$$\Psi_{ABCD} = 6\,\Psi_2\, o_{(A}o_B\iota_C\iota_{D)}. \tag{2.84}$$

At this point, if we consider the (2.77) and (2.67), for D type metrics, we can expect a relation of the following form

$$\begin{aligned}\Psi_{ABCD} &= 6\,\Psi_2\, o_{(A}o_B\iota_C\iota_{D)} \sim \frac{1}{S} Z_1 Z_2 o_{(A}\iota_B o_C \iota_{D)} \\ &= \frac{1}{S} Z_1 o_{(A}o_{B)} Z_2 \iota_{(C}\iota_{D)} = \frac{1}{S} f^{(1)}_{(AB} f^{(2)}_{CD)}.\end{aligned} \tag{2.85}$$

Here, the functions Z_1 and Z_2 can be corresponded to f_0 and f_2, in (2.83), respectively. They are related to each other as $Z_1 = -Z_2 = Z$, and S is the linear combination of the real and imaginary parts of Z. Eventually, the last simple but inspiring equation (2.85) is the essence of the Weyl double copy.

CHAPTER 3

THE KERR-SCHILD DOUBLE COPY IN THREE DIMENSIONS

3.1 Kerr-Schild Double Copy with a Flat Background Metric and the Coulomb Solution

The main idea of the Kerr-Schild double copy is to consider stationary metrics of the Kerr-Schild (KS) form

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}, \qquad \partial_0 g_{\mu\nu} = 0 \tag{3.1}$$

where the deviation $h_{\mu\nu}$ from the background Minkowski space $\eta_{\mu\nu}$ is constructed from a scalar ϕ and a vector k_μ as

$$h_{\mu\nu} = \phi\, k_\mu k_\nu. \tag{3.2}$$

Here, the vector k_μ is null and geodesic with respect to both the background metric $\eta_{\mu\nu}$ and the full metric $g_{\mu\nu}$. When the metric is in the KS form (see [98] for a review), the Ricci tensor with mixed indices becomes linear in the deviation $h_{\mu\nu}$ and the trace-reversed Einstein equations take the form

$$\begin{aligned} R^\mu{}_\nu &= \frac{\kappa}{2}\left(\partial^\alpha\partial^\mu h_{\nu\alpha} + \partial^\alpha\partial_\nu h^\mu{}_\alpha - \partial^\alpha\partial_\alpha h^\mu{}_\nu\right). \\ &= \frac{\kappa^2}{2}\left[T^\mu{}_\nu - \frac{1}{d-2}\delta^\mu{}_\nu T\right], \end{aligned} \tag{3.3}$$

where $\kappa^2 = 8\pi G$. Choosing $k^0 = -1$, the $\mu 0$ components of the Ricci tensor with mixed indices become

$$\partial_\nu\left[\partial^\mu\left(\phi k^\nu\right) - \partial^\nu\left(\phi k^\mu\right)\right] = \kappa\left[T^\mu_0 - \frac{1}{d-2}\delta^\mu_0 T\right]. \tag{3.4}$$

It is easy to see that if one makes the following identifications [42]

$$A_\mu \equiv \phi\, k_\mu, \qquad g \equiv \frac{\kappa}{2}, \tag{3.5}$$

(3.4) becomes the Abelian Yang-Mills equations

$$\partial_\nu F^{\nu\mu} = gJ^\mu, \tag{3.6}$$

where $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ is the field strength, g is the gauge coupling and the source is given by[1]

$$J^\mu = 2\left[T^\mu_0 - \frac{1}{d-2}\delta^\mu_0 T\right]. \tag{3.7}$$

The time component of (3.6) is

$$-\partial^2\phi = -\vec{\nabla}^2\phi = gJ^0. \tag{3.8}$$

Due to the linearization of the Ricci tensor for the metrics of the KS form, one obtains the linearized equations of YM theory and biadjoint scalar theory, namely, Maxwell's equations (3.6) and Poisson's equation (3.8). While the gauge field A_μ is called the single copy of the KS graviton $h_{\mu\nu}$, the scalar field ϕ is interpreted as the zeroth copy of the gauge field A_μ.

While the construction was also extended to metrics with multi KS forms [45], time dependence [49] and backgrounds more general than Minkowski [47], the only known way to study metrics with no KS form is to employ perturbation theory [4–9, 11, 26, 33, 188, 189]. Other developments in the classical double copy include the relation between the sources in the gravity and the gauge theory sides [190], and nonperturbative [50, 141–143] and global [140] aspects. However, the restrictive nature of the KS double copy seems to be an obstacle to a better understanding. In [67], the authors used 3D physics as a testing ground for the classical double copy since, at first sight, it is not obvious how it works. General relativity in 3D has no propogating degrees of freedom and therefore the first question that needs to be answered is how the degree of freedom of the photon, which is one in 3D, is matched in the gravity side. The second question is related to the nature of Newtonian and Coulomb potentials in 3D. An application of Gauss' law to a point particle of charge Q and mass M suggests a logarithmic form for both as follows

$$\oint \mathbf{E}\cdot d\mathbf{A} \propto Q, \qquad E \propto \frac{1}{r}, \qquad \phi \propto \log r, \tag{3.9}$$

$$\oint \mathbf{g}\cdot d\mathbf{A} \propto M, \qquad g \propto \frac{1}{r}, \qquad \Phi \propto \log r. \tag{3.10}$$

[1] See [47] for a covariant version of the KS double copy where no particular time coordinate is chosen.

Whereas, in 3D, the Coulomb potential given in (3.9) is a consequence of Maxwell's equations, general relativity has no Newtonian limit giving rise to (3.10). In taking the Newtonian limit, one considers a weak deviation from the flat space ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$) and a stationary weak source ($T^0_0 = -\rho$, $T^i_0 = 0 = T^i_j$) and the geodesic equation reduces to

$$\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = \frac{1}{2}\vec{\nabla}h_{00}. \tag{3.11}$$

Using

$$\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = \vec{g} = -\vec{\nabla}\Phi, \tag{3.12}$$

one finds the Newtonian potential as

$$\Phi = -\frac{1}{2}h_{00}\,. \tag{3.13}$$

However, when $d = 3$, the 00 component of Einstein equations becomes

$$\vec{\nabla}^2\Phi = 0, \tag{3.14}$$

resulting in a trivial Newtonian potential $\Phi = 0$. If the double copy construction is possible, this problem should be automatically solved since using the 00 component of the KS graviton $h_{00} = \kappa\phi$ in (3.13) yields a nontrivial Newtonian potential as

$$\Phi = -\frac{\kappa}{2}\phi. \tag{3.15}$$

When one starts from the Coulomb solution and achieves a double copy, the logarithmic form of the Coulomb potential is naturally moved into the metric and one obtains a nontrivial solution.

It turns out that, in 3D, the construction possesses a unique feature with no higher-dimensional counterpart. Although one starts with a vacuum solution in the gauge theory side, one obtains a nonvacuum gravity solution with a nontrivial energy-momentum tensor. In [67], the source was interpreted as a dilaton, which also seems to solve the degree of freedom problem. It is also in agreement with the fact that the double copy of the pure YM theory is gravity coupled to a two-form field and a dilaton, and the absence of the two-form field can be explained by the symmetric nature of the KS ansatz (3.1).

This chapter aims to study the construction of [67] and further examine the nature of the source by considering a stationary solution, which is a natural generalization of the static solution. Additionally, we introduce a cosmological constant, find solutions in the KS form and present the corresponding gauge theory single copies. The outline of this chapter is as follows: In Sec. 3.2, we review the main findings of [67] and give an alternative way to obtain the static solution. In Sec. 3.3, we find the stationary version of the solution and show that it is sourced by not a dilaton but a spacelike perfect fluid. Then, we give its gauge theory single copy and discuss some properties of the solution briefly. We end this section by discussing the addition of the cosmological constant. In Sec. 3.4, we conclude with comments on the validity of the classical double copy based on our results.

3.2 Static Solution

Our starting point is a point charge in 3D Maxwell's theory. In polar coordinates (t, r, θ), the flat space metric takes the form

$$\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2, \tag{3.16}$$

and the current vector is given by

$$J^\mu \partial_\mu = gQ\delta^{(2)}(\vec{r})\partial_t \tag{3.17}$$

where Q is the charge of the particle. The Coulomb solution is obtained as

$$A_\mu = \phi k_\mu, \qquad k_\mu \mathrm{d}x^\mu = -\mathrm{d}t, \qquad \phi = -\frac{gQ}{2\pi}\log r, \tag{3.18}$$

due to the relation $\vec{\nabla}^2 \log r = 2\pi\delta^{(2)}(\vec{r})$ in two spatial dimensions. In order to obtain a metric in the KS form, we first write the solution in a gauge where the vector k_μ is null as follows

$$A_\mu = \phi k_\mu, \qquad -k_\mu \mathrm{d}x^\mu = \mathrm{d}t + \mathrm{d}r, \qquad \phi = -\frac{gQ}{2\pi}\log r. \tag{3.19}$$

Identifying the charge with the black hole mass parameter, $Q \to M$, the double copy is given by the metric

$$\begin{aligned} \mathrm{d}s^2 &= \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \kappa\,\phi\,(k_\mu \mathrm{d}x^\mu)^2\,, \\ &= -(1+2GM\log r)\mathrm{d}t^2 + (1-2GM\log r)\mathrm{d}r^2 - 4GM\log r\,\mathrm{d}t\mathrm{d}r + r^2\mathrm{d}\theta^2. \end{aligned} \tag{3.20}$$

Note that the vector k_μ is null with respect to both metrics $\eta_{\mu\nu}$ and $g_{\mu\nu}$. The Ricci tensor of the metric (3.20) reads

$$R_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2GM \end{pmatrix}. \tag{3.21}$$

The $\theta\theta$ component of the Ricci tensor is nonzero everywhere and the same should be true for the energy-momentum tensor $T_{\mu\nu}$, implying a nonlocal source. The approach of [67] is to consider the coupling of gravity to a free scalar field as

$$S = \int \mathrm{d}^3x\sqrt{-g}\left[\frac{\epsilon_1}{\kappa^2}R - \frac{\epsilon_2}{2}\left(\partial\varphi\right)^2\right], \tag{3.22}$$

where $\epsilon_i = \pm 1$ $(i = 1, 2)$ control the sign of the kinetic terms and take a negative value for a ghost graviton or a dilaton. The trace-reversed field equations which follow from the action (3.22) are

$$R_{\mu\nu} = \frac{\epsilon}{2}\partial_\mu\varphi\partial_\nu\varphi, \tag{3.23}$$

where $\epsilon = \epsilon_1\epsilon_2$. The metric in (3.20) is a solution if $\epsilon = -1$ and the gradient of the field is

$$\partial_\mu\varphi = \sqrt{\frac{M}{2\pi}}(0, 0, 1), \tag{3.24}$$

which implies that the dilaton is linear in the azimuthal angle. The matter field equation is also satisfied as

$$\Box\varphi \equiv \nabla_\mu\nabla^\mu\varphi = 0. \tag{3.25}$$

The analysis of [67] proceeds by taking $\epsilon_1 = +1$ and $\epsilon_2 = -1$, i.e., the dilaton should be a ghost to support the metric given in (3.20). It was also shown that, in a proper generalized gauge, the part of the Lagrangian which is quadratic in the fields can be put in a form where the graviton and the dilaton kinetic terms have nonghost signs, exhibiting a parallelism with the double copy construction in scattering amplitudes. The existence of the scalar hair was attributed to two facts: the scalar field is a ghost and it does not respect the symmetries of the spacetime, i.e., $\partial_\mu\varphi \neq 0$. However, the same solution can be obtained by taking $\epsilon_1 = -1$ and $\epsilon_2 = +1$, and therefore, the former does not play a role here (see the Appendix A for a discussion of the no-hair theorem for free scalar fields).

This choice of the signs has the advantage that the dilaton is not a ghost any more and the "wrong" sign for the graviton kinetic term has no physical importance since it does not propagate any dynamical degree of freedom. This is, indeed, an approach which is employed to preserve the unitarity of modified gravity theories such as topologically massive gravity [88, 89] and new massive gravity [90]. In this case, the quadratic part of the Lagrangian contains no dynamical ghost, and therefore there is no need for performing a generalized gauge transformation. Hence, one might speculate that it might have also interesting consequences for the double copy in 3D scattering amplitudes, which, we believe, deserves further study.

Motivated by this possibility of obtaining the solution with different sign choices, one might also ask whether there is any other freedom in the construction of the solution, which is answered in [67] to a certain extent. It was shown that, while it is not possible to see the source as a timelike fluid (perfect or viscous), the solution can also be obtained by coupling to a spacelike perfect fluid, whose energy-momentum tensor reads

$$T_{\mu\nu} = (\rho + P)\, u_\mu u_\nu + P g_{\mu\nu}, \qquad u^2 = +1, \tag{3.26}$$

where u^μ is the velocity of the fluid. The Einstein equations in this case become

$$R_{\mu\nu} = \frac{\kappa^2}{2}\left[(\rho + P)\, u_\mu u_\nu - (\rho + 3P)\, g_{\mu\nu}\right]. \tag{3.27}$$

Comparing this with (3.23) and removing the metric term in (3.27) by choosing $\rho = -3P$ gives

$$R_{\mu\nu} = -\kappa^2 P\, u_\mu u_\nu. \tag{3.28}$$

Therefore, it yields the same solution if the pressure is chosen to be the norm of the gradient of the field φ as

$$P = \frac{1}{2}(\partial\varphi)^2 = \frac{M}{4\pi r^2} = -\frac{1}{3}\rho \tag{3.29}$$

and the fluid velocity is given by

$$u_\mu = (0, 0, r)\,. \tag{3.30}$$

This alternative reflects the correspondence between scalar fields and perfect fluids [91–93]. However, as we will show in the next section by studying a stationary solution in the KS form, the correspondence does not always hold and one is forced to choose the spacelike fluid interpretation.

3.3 Stationary Solution

In this section, we will put different interpretations of the source on a test by studying a more nontrivial solution. To introduce rotation, we write the flat metric in spheroidal coordinates (t, r, θ) as

$$\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}t^2 + \frac{r^2}{r^2+a^2}\mathrm{d}r^2 + (r^2+a^2)\mathrm{d}\theta^2, \tag{3.31}$$

where a will be the rotation parameter. The null vector k_μ is parametrized as

$$-k_\mu\mathrm{d}x^\mu = \mathrm{d}t + \frac{r^2}{r^2+a^2}\mathrm{d}r + a\mathrm{d}\theta. \tag{3.32}$$

For a metric in KS form with $\phi = \phi(r)$, the metric becomes

$$\begin{aligned} ds^2 &= \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \kappa\,\phi\,(k_\mu\mathrm{d}x^\mu)^2\,, \\ &= -(1-\kappa\phi(r))\mathrm{d}t^2 + \frac{r^2\left[a^2+r^2(1+\kappa\phi(r))\right]}{(a^2+r^2)^2}\mathrm{d}r^2 + \left[a^2(1+\kappa\phi(r))+r^2\right]\mathrm{d}\theta^2 \\ &\quad + \frac{2\kappa a r^2\phi(r)}{a^2+r^2}\,\mathrm{d}r\mathrm{d}\theta + \frac{2\kappa r^2\phi(r)}{a^2+r^2}\,\mathrm{d}r\mathrm{d}t + 2\kappa a\phi(r)\,\mathrm{d}t\mathrm{d}\theta, \end{aligned} \tag{3.33}$$

and the independent components of the Ricci tensor read

$$\begin{aligned} R_{00} &= \kappa\frac{\left(a^2-r^2+\kappa r^2\phi(r)\right)\phi'(r) + r\left(-a^2-r^2+\kappa r^2\phi(r)\right)\phi''(r)}{2r^3}, \\ R_{11} &= \kappa\frac{r\left(a^2+r^2+\kappa r^2\phi(r)\right)\left(\phi'(r)+r\phi''(r)\right)}{2\left(a^2+r^2\right)^2}, \\ R_{22} &= \kappa\frac{\left(a^4+3r^2a^2+\kappa r^2\phi(r)a^2+2r^4\right)\phi'(r) - a^2r\left(a^2+r^2-\kappa r^2\phi(r)\right)\phi''(r)}{2r^3}, \\ R_{01} &= \kappa^2\frac{r\phi(r)\left(\phi'(r)+r\phi''(r)\right)}{2\left(a^2+r^2\right)}, \\ R_{02} &= \kappa\frac{a\left(\left(a^2+r^2+\kappa r^2\phi(r)\right)\phi'(r) + r\left(-a^2-r^2+\kappa r^2\phi(r)\right)\phi''(r)\right)}{2r^3}, \\ R_{12} &= \kappa^2\frac{ar\phi(r)\left(\phi'(r)+r\phi''(r)\right)}{2\left(a^2+r^2\right)}, \end{aligned} \tag{3.34}$$

where primes denote the derivative with respect to r. The trace-reversed Einstein equations when the source is the dilaton (3.23) or the spacelike fluid (3.28) are of the form $R_{\mu\nu} \propto V_\mu V_\nu$, where V_μ is a three-vector and one can use this to constrain the function $\phi(r)$. From, for example, $(R_{01})^2 = R_{00}R_{11}$, it is easy to see that the only consistent solution takes the form $\phi(r) \propto \log r$ and the proportionality constant is determined by requiring one to get the static solution as $a \to 0$, which yields

$$\phi(r) = -\frac{\kappa M}{4\pi}\log r. \tag{3.35}$$

Using this in (3.23) with again $\epsilon = -1$, one sees that the gradient of the scalar φ should be given by

$$\partial_\mu \varphi = \sqrt{\frac{M}{2\pi}} \left(\frac{a}{r^2}, 0, \frac{a^2 + r^2}{r^2} \right). \tag{3.36}$$

When the rotation is turned on, $a \neq 0$, this introduces an r dependence in the t and θ components, which conflicts with the fact that the r component is zero, hence no r dependence. Therefore, unlike the case for the static metric, there is no consistent solution for the function φ.

However, one can still use the spacelike fluid as the source and the metric with the scalar ϕ given in (3.35), is a solution of Einstein equations (3.28) when

$$P = \frac{1}{2}(\partial\varphi)^2 = \frac{M}{4\pi r^2} = -\frac{1}{3}\rho, \qquad u_\mu = \left(\frac{a}{r}, 0, \frac{a^2 + r^2}{r} \right). \tag{3.37}$$

Therefore, the stationary solution cannot be sourced by a dilaton and the spacelike fluid becomes compulsory to obtain a stationary solution in the KS form.

The gauge theory single copy can easily be obtained as

$$A_\mu \mathrm{d}x^\mu = \phi k_\mu \mathrm{d}x^\mu = \frac{gQ}{2\pi} \log r \left(\mathrm{d}t + \frac{r^2}{r^2 + a^2} \mathrm{d}r + a\mathrm{d}\theta \right), \tag{3.38}$$

which is a solution of Maxwell's equations (3.6) with the current vector

$$J^\mu = \rho v^\mu, \qquad \rho = \frac{Qa^2}{\pi r^4}, \qquad v^\mu = \left(1, 0, -\frac{1}{a} \right), \tag{3.39}$$

which describes a rotating nonlocal charge distribution with angular velocity $\omega = -\frac{1}{a}$ with respect to the origin. Checking the nonzero components of the field strength tensor,

$$F_{rt} = \frac{gQ}{2\pi r}, \qquad F_{r\theta} = \frac{agQ}{2\pi r}, \tag{3.40}$$

one sees that the magnetic field is created due to the rotation in the gravity side.

In order to see some main properties of the metric (3.33) with the scalar ϕ given in (3.35), it is useful to write it down in Boyer-Lindquist coordinates, which is achieved by the transformations [94]

$$\begin{aligned} \mathrm{d}\theta &\mapsto \mathrm{d}\Theta + h_1 \mathrm{d}r, \\ \mathrm{d}t &\mapsto \mathrm{d}T + h_2 \mathrm{d}r, \end{aligned} \tag{3.41}$$

where

$$h_1 = -\frac{\kappa a r^2 \phi(r)}{(a^2+r^2)\left(a^2-\kappa r^2\phi(r)+r^2\right)},$$
$$h_2 = \frac{\kappa r^2\phi(r)}{a^2-\kappa r^2\phi(r)+r^2}. \tag{3.42}$$

In these coordinates, the metric is given by

$$\mathrm{d}s^2 = -\left(1-\kappa\phi(r)\right)\mathrm{d}T^2 + \frac{r^2\mathrm{d}r^2}{a^2-\kappa r^2\phi(r)+r^2} + \left(r^2+a^2(1+\kappa\phi(r))\right)\mathrm{d}\Theta^2 + 2\kappa a\phi(r)\,\mathrm{d}\Theta\mathrm{d}T \tag{3.43}$$

When the explicit form of the scalar $\phi(r)$ given in (3.35) is used, it becomes

$$\mathrm{d}s^2 = -\left(1+2GM\log r\right)\mathrm{d}T^2 + \frac{r^2\mathrm{d}r^2}{a^2+r^2\left(1+2GM\log r\right)} + \left(r^2+a^2(1-2GM\log r)\right)\mathrm{d}\Theta^2 - 4aGM\log r\,\mathrm{d}\Theta\mathrm{d}T. \tag{3.44}$$

From the curvature invariants

$$R^{\mu\nu}R_{\mu\nu} = \frac{4G^2M^2}{r^4},$$
$$R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho} = 4R^{\mu\nu}R_{\mu\nu} - R^2 = \frac{12G^2M^2}{r^4}, \tag{3.45}$$

one sees that the metric has a real singularity at $r=0$. For appropriately chosen parameters, it has an event horizon enclosed by an *ergocircle*, which might be thought of a 3D analog of the Kerr black hole sourced by a spacelike fluid. The metric asymptotically takes the form

$$\mathrm{d}s^2\Big|_{r\to\infty} = -2GM\log r\,\mathrm{d}T^2 + \frac{\mathrm{d}r^2}{2GM\log r} + r^2\mathrm{d}\Theta^2, \tag{3.46}$$

and therefore, it is not asymptotically flat. However, it is asymptotically locally flat[2] as can be seen by the vanishing of the curvature invariants (3.45) as $r\to\infty$.

When a cosmological constant is introduced, the trace-reversed Einstein equations become

$$R_{\mu\nu} - 2\Lambda g_{\mu\nu} = -\kappa^2 P\,u_\mu u_\nu. \tag{3.47}$$

This time, the metric given in (3.33) is a solution when

$$\phi(r) = -\frac{\kappa M}{4\pi}\log r + \frac{\Lambda}{\kappa}r^2, \tag{3.48}$$

[2] To our knowledge, the black hole solution discovered in [95] is the only other known solution of this type in 3D.

with the fluid properties given in (3.37). The gauge theory single copy is given by

$$A_\mu \mathrm{d}x^\mu = \phi k_\mu \mathrm{d}x^\mu = \left(\frac{gQ}{2\pi}\log r - \frac{\Lambda}{2g}r^2\right)\left(\mathrm{d}t + \frac{r^2}{r^2+a^2}\mathrm{d}r + a\mathrm{d}\theta\right), \tag{3.49}$$

and the nonzero components of the field strength tensor read

$$F_{rt} = \frac{gQ}{2\pi r} - \frac{\Lambda r}{g}, \qquad F_{r\theta} = a\left(\frac{gQ}{2\pi r} - \frac{\Lambda r}{g}\right), \tag{3.50}$$

with the magnetic field again created by the rotation. Maxwell's equations are now given by

$$\partial_\nu F^{\nu\mu} = g\left(J^\mu_{(\Lambda=0)} + \bar{J}^\mu\right), \tag{3.51}$$

where the first current vector $J^\mu_{(\Lambda=0)}$ describes the source in the absence of the cosmological constant and is given by (3.39). The second current vector describes a constant charge density filling all space as follows

$$\bar{J}^\mu = \rho_0 \bar{v}^\mu, \qquad \rho_0 = \frac{2\Lambda}{g^2}, \qquad \bar{v}^\mu = (1,0,0), \tag{3.52}$$

which is the expected effect of adding a cosmological constant in the gravity side.

When written in Boyer-Lindsquit coordinates using (3.43), the metric becomes

$$\begin{aligned} ds^2 =& -(1+2GM\log r - \Lambda r^2)\mathrm{d}T^2 + \frac{r^2}{a^2 + r^2(1+2GM\log r - \Lambda r^2)}\mathrm{d}r^2 \\ &+ \left(r^2 + a^2(1-2GM\log r + \Lambda r^2)\right)\mathrm{d}\Theta^2 + 2a(-2GM\log r + \Lambda r^2)\,\mathrm{d}\Theta\mathrm{d}T. \end{aligned} \tag{3.53}$$

From the curvature invariants

$$\begin{aligned} R^{\mu\nu}R_{\mu\nu} &= \frac{4G^2M^2}{r^4} - \frac{8G\Lambda M}{r^2} + 12\Lambda^2, \\ R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho} = 4R^{\mu\nu}R_{\mu\nu} - R^2 &= \frac{12G^2M^2}{r^4} - \frac{8G\Lambda M}{r^2} + 12\Lambda^2, \end{aligned} \tag{3.54}$$

the singularity at $r=0$ is again apparent and an event horizon enclosed by an ergocircle can be identified by a certain choice of the parameters. Taking $\Lambda = -\frac{1}{\ell^2}$, the asymptotic form of the metric is

$$\mathrm{d}s^2\big|_{r\to\infty} = -\left(\frac{r}{\ell}\right)^2\mathrm{d}T^2 + \left(\frac{\ell}{r}\right)^2\mathrm{d}r^2 + r^2\mathrm{d}\Theta^2, \tag{3.55}$$

which is the anti-de Sitter (AdS) spacetime with radius ℓ. Therefore, the metric given in (3.53) serves as an interesting alternative to the well-known BTZ black hole [83], which is, of course, more physical since it is a solution of the cosmological Einstein theory with no matter field like our spacelike fluid.

3.4 Summary

In this chapter, we revisited the static solution constructed in [67]. Being able to obtain the static solution with different sign choices for the kinetic terms of the graviton and the dilaton, yielding a quadratic Lagrangian with no propagating ghost field, we claim that the study of scattering amplitudes in 3D might offer an interesting insight into the double copy because the change of sign of the graviton kinetic term is problematic in higher dimensions.

Turning our attention to a stationary version of the solution in the KS form, we showed that it cannot be sourced by a free scalar field and the source should be a spacelike fluid. Even in this form, it presents itself as an interesting example of the classical double copy where the gauge theory source is a nonlocal rotating charge distribution. By introducing a cosmological constant, we obtained a rotating, asymptotically AdS solution whose single copy gauge field describes an electric field and a magnetic field, which is proportional to the rotation parameter in the gravity side, and the effect of the cosmological constant shows itself in the gauge theory as a constant charge distribution filling all space. Based on the expectation from the scattering amplitudes that the double copy should be given by gravity coupled to a dilaton, obtaining a stationary solution sourced by a dilaton or understanding why it is not possible remains an open problem, whose solution might give a better understanding of the classical double copy.

CHAPTER 4

THE KERR-SCHILD DOUBLE COPY OF COULOMB SOLUTIONS IN THREE DIMENSIONS

As we have seen in the previous chapter, due to the lack of degrees of freedom and a Newtonian limit in 3D GR, it is not obvious, at first sight, how the procedure works.[1] For the Coulomb solution, which is the simplest nontrivial solution in the gauge theory side, the gauge boson degrees of freedom is mapped to the graviton and the KS scalar characterizing the spacetime metric is directly linked to the Newtonian potential as given in (4.13). However, without matter coupling, this cannot be realized since the Newtonian potential vanishes identically. When the problem was tackled by coupling a free scalar field, a hairy black hole with the desired properties can be obtained if the scalar is a ghost, which is equivalent to coupling a spacelike fluid [67]. In order to support the black hole solution, the scalar should be linear in the azimuthal angle, and therefore, does not vanish at infinity as suggested by the no-hair theorem. In the linearized theory, the ghost sign can be removed by a certain generalized gauge transformation; however, it is still somewhat unsatisfactory to not have a reasonable behavior of the matter field at infinity. When investigated further in [68], it was proposed to take the Einstein-Hilbert term with a ghost sign, which does not introduce a dynamical ghost in the theory and removes the need for a generalized gauge transformation to get rid of the ghost in the linearized theory.

In this chapter, we aim to present an alternative for the matter coupling with a better behavior at infinity, which, as we will see, also provides a beautiful connection to a well-known solution of 3D black hole physics. In Sec. 4.1, we will make a review

[1] See [96, 97] for the study of 3D amplitudes where a degree of freedom is introduced in the gravity side by adding a Chern-Simons term in the action. As a result, one obtains the amplitudes in topologically massive gravity as the double copy of the amplitudes of topologically massive electrodynamics.

of the construction of [67] by emphasizing the points that will be relevant to our later discussion. In Sec. 4.2, considering the on-shell duality of a scalar and a gauge vector together with the KS ansatz for the metric, we will find that the same type of solution can be obtained in Einstein-Maxwell theory. In Sec. 4.3, we will obtain the most general static solution of the KS form by introducing a cosmological constant. When the cosmological constant is zero, we will show that a charged black hole solution with the correct Newtonian limit, which also gives rise to the Coulomb solution as its single copy, can be obtained when a ghost sign is used in the action. The electric field corresponding to the gauge field in the gravity side vanishes at infinity, and therefore, provides the promised improvement. For a negative cosmological constant, the charged Bañados-Teitelboim-Zanelli black hole [83] follows from the most general solution without taking any ghost sign in the action. We end this section by studying the gauge theory single copy of the solution. Finally, we present our conclusions in Sec. 4.4.

4.1 The Coulomb Solution from the Free Scalar

4.1.1 Solution with the correct Newtonian potential

In [67], the Coulomb solution was obtained as the single copy of the static black hole solution of GR coupled to a free scalar with the following action:

$$S = \int \mathrm{d}^3x\sqrt{-g}\left[\frac{\zeta_1}{\kappa^2}R - \frac{\zeta_2}{2}\left(\partial\varphi\right)^2\right], \qquad \kappa^2 = 8\pi G, \tag{4.1}$$

where $\zeta_i = \pm 1$ $(i = 1, 2)$ control the sign of the kinetic terms and take a negative value for a ghost graviton or a dilaton [68]. The field equations which follow from the action (4.1) are

$$R_{\mu\nu} = \zeta\frac{\kappa^2}{2}\partial_\mu\varphi\partial_\nu\varphi, \tag{4.2}$$

$$\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\varphi) = 0, \tag{4.3}$$

where $\zeta = \zeta_1\zeta_2$. Let us consider the following static KS metric around the Minkowski space

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi(r)\, k_\mu k_\nu\,, \tag{4.4}$$

where the vector k_μ is null and geodesic with respect to both the background metric $\eta_{\mu\nu}$ and the full metric $g_{\mu\nu}$ (see chap. 32 of [98] for a detailed discussion of important properties). Writing the background line element in polar coordinates

$$\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2, \tag{4.5}$$

the vector k_μ can be written as follows

$$-k_\mu \mathrm{d}x^\mu = \mathrm{d}t + \mathrm{d}r, \tag{4.6}$$

and the line element in the KS coordinates becomes

$$\begin{aligned} \mathrm{d}s^2 &= \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \phi(r)(k_\mu \mathrm{d}x^\mu)^2 \\ &= -\left[1-\phi(r)\right]\mathrm{d}t^2 + \left[1+\phi(r)\right]\mathrm{d}r^2 + 2\phi(r)\mathrm{d}t\mathrm{d}r + r^2\mathrm{d}\theta^2. \end{aligned} \tag{4.7}$$

If one assumes[2]

$$\partial_\mu \varphi = (0, 0, c), \qquad\qquad c = \text{constant}, \tag{4.8}$$

the equation for the scalar field (4.3) is satisfied independent of the KS scalar ϕ as follows:

$$\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\varphi) = \partial_\mu(\sqrt{-\eta}\, \eta^{\mu\nu}\partial_\nu\varphi) = 0, \tag{4.9}$$

where we have used $\det g = \det \eta$, $g^{\theta\theta} = \eta^{\theta\theta}$ and $g^{t\theta} = g^{r\theta} = 0$.

With this in hand, one can now check the gravity equations (4.2). The independent nonzero components of the Ricci tensor read

$$\begin{aligned} R_{tt} &= \frac{[\phi(r)-1]\,[r\,\phi''(r)+\phi'(r)]}{2r}, \\ R_{tr} &= \frac{\phi(r)\,[r\,\phi''(r)+\phi'(r)]}{2r}, \\ R_{rr} &= \frac{[\phi(r)+1]\,[r\,\phi''(r)+\phi'(r)]}{2r}, \\ R_{\theta\theta} &= r\,\phi'(r), \end{aligned} \tag{4.10}$$

and the only nonzero component of the right-hand-side of the equations is

$$(\text{RHS})_{\theta\theta} = 4\pi\zeta G c^2. \tag{4.11}$$

[2] In [67], the authors directly use the KS scalar yielding the correct Newtonian potential and conclude that the solution should be sourced by a free scalar. Here, we give the derivation in a way that will be useful in our later discussion.

From the $\theta\theta$ component, one finds the solution for the KS scalar as

$$\phi = b + 4\pi\zeta G c^2 \log(r), \qquad b = \text{constant}, \tag{4.12}$$

which also satisfies the remaining components. The constant c and the parameter ζ can be fixed by considering the Newtonian limit. The Newtonian potential is given by

$$\Phi = -\frac{1}{2}(1 + g_{00}) = -\frac{\phi}{2}, \tag{4.13}$$

and in order to mimic the Newtonian gravity

$$\vec{g} = -\vec{\nabla}\Phi = -\frac{GM}{r}\hat{r}, \tag{4.14}$$

the KS scalar should be in the following form:

$$\phi = -2GM \log(r) + \text{constant}. \tag{4.15}$$

Therefore, we need to fix the parameters as follows:

$$c = \sqrt{\frac{M}{2\pi}}, \qquad \zeta = -1, \tag{4.16}$$

with which, the KS scalar becomes

$$\phi = -2GM \log(r) + b. \tag{4.17}$$

Since $\zeta = -1$, one should choose the "wrong sign" for one of the kinetic terms in the action (4.1). While the scalar was chosen to be a ghost in [67], introducing the Einstein-Hilbert (EH) term with the negative sign has the advantage that it does not propagate any physical degree of freedom [68].

In order to fix the integration constant b, we write the metric (4.4) in the Boyer-Lindsquit (BL) coordinates by the following coordinate transformation:

$$\mathrm{d}t \to \mathrm{d}t + \frac{\phi(r)}{1 - \phi(r)}\,\mathrm{d}r, \tag{4.18}$$

which leads to the line element

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\theta^2, \qquad f(r) = 1 - \phi(r), \tag{4.19}$$

for a generic KS scalar. In our case, the metric function becomes

$$f(r) = 1 - b + 2GM \log(r). \tag{4.20}$$

In order to recover the Minkowski spacetime when the black hole mass vanishes ($M = 0$), one should take $b = 0$. Therefore, the KS scalar and the metric function are given by

$$\phi(r) = -2GM \log(r), \tag{4.21}$$

$$f(r) = 1 + 2GM \log(r). \tag{4.22}$$

We refer the reader to [67] for an analysis of the motion of massive particles where the authors show that stable orbits exist for a certain range of parameters. In Sec. 4.3, we will show that it is true for the solution of Einstein-Maxwell theory.

4.1.2 Gauge theory single copy

The gauge theory single copy for a generic matter coupling can be obtained by considering the trace-reversed gravity equations

$$R^{\mu}{}_{\nu} = \frac{\kappa^2}{2}\left[T^{\mu}{}_{\nu} - \delta^{\mu}{}_{\nu} T\right], \qquad T = T^{\mu}{}_{\mu}, \tag{4.23}$$

where $T_{\mu\nu}$ is the energy-momentum tensor. For a KS metric (4.4), the Ricci tensor with mixed indices reads

$$R^{\mu}{}_{\nu} = \frac{1}{2}\left[\partial^{\alpha}\partial^{\mu}(\phi\, k_{\nu} k_{\alpha}) + \partial^{\alpha}\partial_{\nu}(\phi\, k^{\mu} k_{\alpha}) - \partial^{\alpha}\partial_{\alpha}(\phi\, k^{\mu} k_{\nu})\right], \tag{4.24}$$

which is linear in the perturbation. If $k^0 = +1$ and one identifies $A_{\mu} = \phi\, k_{\mu}$, the $\mu 0$ component can be written as

$$R^{\mu}{}_{0} = \frac{1}{2}\partial_{\nu} F^{\nu\mu}, \tag{4.25}$$

where $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$ is the field strength tensor of the gauge field A_{μ}. Therefore, the $\mu 0$ component of the gravity equations can be mapped to Maxwell's equations as follows:

$$\partial_{\nu} F^{\nu\mu} = g J^{\mu}, \tag{4.26}$$

where the source is given by

$$J^{\mu} = 2\left[T^{\mu}{}_{0} - \delta^{\mu}{}_{0} T\right], \tag{4.27}$$

and the gauge coupling is obtained by $\frac{\kappa^2}{2} \to g$ [42].

Application of this procedure to our solution gives the single copy gauge field as[3]

$$A_\mu \mathrm{d}x^\mu = \phi k_\mu \mathrm{d}x^\mu = Q \log r \, (\mathrm{d}t + \mathrm{d}r), \tag{4.28}$$

where we have made the replacement $2GM \to Q$. This is just the Coulomb solution in a gauge where $A_\mu A^\mu = 0$, and the source is

$$J^\mu \partial_\mu = Q \, \delta^2(\vec{r}) \, \partial_t, \tag{4.29}$$

which corresponds to a charged particle in the flat spacetime.

Although we achieved the Coulomb solution as the single copy, the construction has the undesired feature that the scalar field in the gravity side is linear in the azimuthal angle. This is not unexpected due to the existence of the scalar hair (see Appendix A. for a detailed discussion of no-hair theorem for free scalar fields). One way to obtain a matter configuration which is well behaved at infinity is to consider the coupling of a gauge vector since it will yield a global charge, i.e., the electric charge, and therefore, produces no hair. In $d \geq 4$, this is just the Reissner-Nordström black hole solution of Einstein-Maxwell theory, whose metric can be also written in the KS form around Minkowski background.

4.2 Scalar-Vector Duality and Its Consequences

In this section, we will present the duality between a free scalar and a gauge vector in three dimensions by following Sec. 7.8 of [99], and then, discuss its consequences for the classical double copy. In d-dimensions, the number of on-shell degrees of freedom of a p-form gauge field is $C(d-2,p) = \frac{(d-2)!}{p!(d-p-2)!}$. Due to the identity $C(d-2,p) = C(d-2,d-p-2)$, a p-form and a $(d-p-2)$-form in d-dimensions have the same number of degrees of freedom. In $d = 3$, this implies that a scalar ($p = 0$) and a vector gauge field ($p = 1$) have the same number of degrees of freedom, which is one. Indeed, one can also show that the free field equations are equivalent and the solutions are in one-to-one correspondence. In order to see that, let us consider the

[3] We write the solution with a different normalization than [67, 68] to simplify the solution of Einstein-Maxwell theory that we will give in Sec. 4.3. The Maxwell action is taken as (4.35) and we formulate the scalar-vector duality in Appendix B accordingly.

following flat space action

$$S = \int \mathrm{d}^3x\sqrt{-\eta}\left[\frac{1}{8\pi}f_{\mu\nu}f^{\mu\nu} + \frac{1}{2\sqrt{2\pi}}\epsilon^{\mu\nu\rho}f_{\mu\nu}\partial_\rho\varphi\right], \tag{4.30}$$

where $\epsilon^{\mu\nu\rho} = \frac{1}{\sqrt{-\eta}}\varepsilon^{\mu\nu\rho}$ is the Levi-Civita tensor and we take the Minkowski spacetime in polar coordinates (4.5) for later convenience. The equation for φ gives

$$\epsilon^{\mu\nu\rho}\partial_\mu f_{\nu\rho} = 0, \tag{4.31}$$

which implies $f_{\mu\nu} = 2\partial_{[\mu}a_{\nu]}$, i.e., $f_{\mu\nu}$ is the field strength tensor of a gauge field a_μ. Checking the equation for $f_{\mu\nu}$ gives how it is related to the scalar φ as follows:

$$f_{\mu\nu} = -\sqrt{2\pi}\,\epsilon_{\mu\nu\rho}\partial^\rho\varphi \implies \partial_\mu\varphi = \frac{1}{2\sqrt{2\pi}}\,\epsilon_{\mu\nu\rho}f^{\nu\rho}. \tag{4.32}$$

Inserting the expression for $f_{\mu\nu}$ into the action (4.30) yields the action for a free scalar

$$S_{\text{scalar}} = \int \mathrm{d}^3x\sqrt{-\eta}\,\left[\frac{1}{2}\left(\partial\varphi\right)^2\right], \tag{4.33}$$

with the field equation

$$\partial_\mu(\sqrt{-\eta}\,\eta^{\mu\nu}\partial_\nu\varphi) = 0. \tag{4.34}$$

In the same way, one can also eliminate φ from the action (4.30) by using the expression in (4.32), which yields the action for a free gauge field

$$S_{\text{vector}} = \int \mathrm{d}^3x\sqrt{-\eta}\,\left[-\frac{1}{8\pi}f_{\mu\nu}f^{\mu\nu}\right], \tag{4.35}$$

where the field equation is given by

$$\partial_\nu(\sqrt{-\eta}\,\eta^{\nu\alpha}\eta^{\mu\beta}f_{\alpha\beta}) = 0. \tag{4.36}$$

We are now in a position to discuss the implications of the duality. As we have seen in (4.9), the scalar field configuration given in (4.8) is a solution when the spacetime is curved and endowed with the metric (4.4), or equivalently, flat and endowed with the Minkowski metric in polar coordinates (4.5). Our analysis shows that the actions for the free scalar (4.33) and the free vector (4.35) are equivalent and the solutions to free field equations (4.33,4.35) are in one-to-one correspondence where the relation between the solutions is given in (4.32). For the solution of the scalar field given in (4.8), this implies that, for the corresponding vector solution, the only nonzero component of the field strength tensor is

$$f_{tr} \propto \frac{1}{r}, \tag{4.37}$$

which is the electric part. Introducing the electric charge as the proportionality constant, the gauge field and the field strength tensor can be written as

$$a_\mu \mathrm{d}x^\mu = q\log(r)\mathrm{d}t,$$
$$\frac{1}{2}\, f_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{q}{r}\, \mathrm{d}r \wedge \mathrm{d}t. \tag{4.38}$$

Similar to the scalar case, this field configuration is also a solution when the spacetime is curved and endowed with the metric (4.4) as follows:

$$\partial_\nu(\sqrt{-\eta}\,\eta^{\nu\alpha}\eta^{\mu\beta} f_{\alpha\beta}) = \partial_\nu(\sqrt{-g}\, g^{\nu\alpha} g^{\mu\beta} f_{\alpha\beta}) = 0. \tag{4.39}$$

As emphasized in [99], although the equivalence is true for the simplest kinetic actions, it is not guaranteed to hold in a more general setup. We have proven that the scalar solution in curved spacetime with a KS metric implies that the electric field configuration (4.38) is also a solution in such a spacetime. However, one should keep in mind that this statement is independent of the KS scalar $\phi(r)$. Therefore, when coupled to gravity, the matter equations will definitely be satisfied if the metric can be put in the KS form (4.4); however, the KS scalars in these two cases might differ. Indeed, one immediately sees that the static black hole solution obtained by coupling the vector field to gravity should be a charged black hole solution. In the next section, we will study the black hole solution with this new matter coupling.

4.3 Einstein-Maxwell Theory in Three Dimensions

4.3.1 The most general static solution of Kerr-Schild form

Motivated by the results of the previous section, we consider Einstein-Maxwell theory with a cosmological constant described by the following action:

$$S = \int \mathrm{d}^3x\sqrt{-g}\left[\frac{\zeta_1}{\kappa^2}\,(R-2\Lambda) - \frac{\zeta_2}{8\pi} f_{\mu\nu} f^{\mu\nu}\right], \qquad \kappa^2 = 8\pi G, \tag{4.40}$$

where, similar to the scalar case, $\zeta_i = \pm 1$ ($i = 1, 2$) control the sign of the kinetic terms and take a negative value for a ghost graviton or a vector. The field equations arising from the action (4.40) are given by

$$R_{\mu\nu} - 2\Lambda g_{\mu\nu} = \zeta\frac{\kappa^2}{8\pi}\left(2 f_\mu{}^\alpha f_{\nu\alpha} - g_{\mu\nu} f_{\alpha\beta} f^{\alpha\beta}\right), \tag{4.41}$$
$$\partial_\nu(\sqrt{-g}\, f^{\nu\mu}) = 0, \tag{4.42}$$

where $\zeta = \zeta_1\zeta_2$. As discussed in the previous section, for a static metric in the KS form (4.4), the solution for the vector field is given in (4.38). With this at hand, one can solve the gravitational field equations (4.41). Introducing the cosmological constant modifies the left-hand side as follows:

$$\begin{aligned}
(\text{LHS})_{tt} &= \frac{[\phi(r)-1]\,[r\,\phi''(r)+\phi'(r)-4\Lambda r]}{2r},\\
(\text{LHS})_{tr} &= \frac{\phi(r)\,[r\,\phi''(r)+\phi'(r)-4\Lambda r]}{2r},\\
(\text{LHS})_{rr} &= \frac{[\phi(r)+1]\,[r\,\phi''(r)+\phi'(r)-4\Lambda r]}{2r},\\
(\text{LHS})_{\theta\theta} &= r\,\phi'(r)-2\Lambda r^2,
\end{aligned} \tag{4.43}$$

and the only nonzero component of the right-hand-side of the equations is

$$(\text{RHS})_{\theta\theta} = 2\zeta G q^2. \tag{4.44}$$

From the $\theta\theta$ component, the KS scalar can be solved as

$$\phi(r) = C + \Lambda r^2 + 2\zeta G q^2 \log(r), \qquad C = \text{constant} \tag{4.45}$$

which also solves the other components of the field equations. In order to give a physical meaning to the integration constant C, we again write the metric in the BL coordinates (4.19) via the transformation (4.18), which leads to the following metric function:

$$f(r) = 1 - C - \Lambda r^2 - 2\zeta G q^2 \log(r). \tag{4.46}$$

Having found the most general solution, we are now ready to investigate physically interesting possibilities.

4.3.2 Solutions with the correct Newtonian potential ($\Lambda = 0$)

In order to obtain solutions with the correct Newtonian potential, we take $\Lambda = 0$ since only the logarithmic term is needed. In the Newtonian limit, the gravitational field in terms of the KS scalar can be obtained from Eqs. (4.13)-(4.14), which yields

$$\vec{g} = \frac{1}{2}\vec{\nabla}\phi. \tag{4.47}$$

For the KS scalar given in (4.45) with $\Lambda = 0$, we obtain

$$\vec{g} = \frac{\zeta G q^2}{r}\hat{r}, \tag{4.48}$$

which shows that in order to preserve the attractive nature of the gravitational force, one should have $\zeta = -1$, i.e., either the EH term or the vector kinetic term in the action (4.40) should carry a ghost sign (see the Appendix B for more details).

Note that our solution should be a charged black hole and the gravitational attraction is provided by the electric charge. Therefore, the integration constant should be a function of the mass of the black hole $C = C(M)$. In order to fix the constant, we again need to check the metric function in the BL coordinates given in (4.46) and demand that the metric reduces to the Minkowski metric $[f(r) = 1]$ when the mass and the charge are set to zero ($M \to 0$, $q \to 0$). This constraint can be satisfied by taking the mass term with different signs as follows:

$$f^{\pm}(r) = 1 \pm 8GM + 2Gq^2 \log(r), \tag{4.49}$$

where for both choices $f(r)$ has a single zero, and therefore, admits one event horizon.

Although we have already ensured the correct Newtonian limit, it is interesting to have a closer look at the properties of the metric as done in [67] for the scalar case. For this purpose, we check the geodesic motion of a timelike particle described by the equation

$$\frac{1}{2}E^2 = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + V^{\pm}_{\text{eff}}, \qquad V^{\pm}_{\text{eff}} = \frac{1}{2}\left(\frac{L^2}{r^2} + 1\right) f^{\pm}(r), \tag{4.50}$$

where E and L are the energy and the angular momentum of the particle which are defined through the timelike and the angular Killing vectors $\xi^{\mu}_{(t,\theta)}$ as follows:

$$E = -g_{\mu\nu}\, \xi^{\mu}_{(t)} u^{\nu}, \qquad L = g_{\mu\nu}\, \xi^{\mu}_{(\theta)} u^{\nu}, \tag{4.51}$$

where u^{μ} is the velocity the particle. The Newtonian limit of the effective potential $V^{\pm}_{\text{eff}}$ can be obtained by neglecting the L^2G terms as follows[4]:

$$V^{\pm}_{\text{N}} = \frac{1}{2} \pm 4GM + \frac{L^2}{2r^2} + Gq^2 \log(r). \tag{4.52}$$

The metric function obtained by coupling a free scalar (4.22) and the ones we obtained by coupling a gauge vector (4.49) have the same functional form $[f(r) = A + B\log(r)$, A, B: constant]. Therefore, all the physically important properties of the solution that is discussed in [67] are also valid for our solutions. They can be summarized as follows:

[4] For a general analysis, one should write the logarithmic term in both V_{eff} and V_{N} by introducing a length scale as $\log(\frac{r}{r_0})$. We set $r_0 = 1$ for simplicity.

1. The Newtonian potential V_{N} has an infinite barrier at short distances and matches with the effective potential V_{eff} at long distances.

2. A timelike particle cannot escape to infinity due to the logarithmic divergence of the potential as $r \to \infty$.

3. The effective potential V_{eff} develops a local maximum ($V_{\text{eff}}^{\text{max}}$) and a local minimum ($V_{\text{eff}}^{\text{min}}$) when the angular momentum of the particle L is larger than a certain value (L_{min}). A timelike particle moves along a stable orbit provided that $V_{\text{eff}}^{\text{min}} \leq E < V_{\text{eff}}^{\text{max}}$. On the other hand, the Newtonian potential V_{N} always admit stable orbits.

4. When $E = V_{\text{eff}}^{\text{min}}$ and $L > L_{\text{min}}$, timelike geodesics form circular orbits, i.e., one has $\frac{\mathrm{d}r}{\mathrm{d}t} = 0$ in (4.50).

5. Since the central potential is not that of an inverse-square central force [$V(r) = -\frac{k}{r}$, k: constant] or a radial harmonic oscillator [$V(r) = \frac{1}{2}kr^2$, k: constant], Bertrand's theorem assures that there will be precession for orbits with $E > V_{\text{eff}}^{\text{min}}$.

In Fig. 4.1, we show that properties 1-4 hold for the metric functions $f^{\pm}(r)$ given in (4.49) by tuning the parameters such that $L_{\text{min}} = 1$. Having shown that the qualitative properties of the metric is the same, we refer the reader to [67] where the authors present timelike geodesics, and also, show that more precession is observed in the relativistic orbits when compared to the Newtonian orbits.

The KS scalars corresponding to the metric functions (4.49) are given by

$$\phi^{\pm}(r) = \mp 8GM - 2Gq^2 \log(r), \tag{4.53}$$

and lead to the following single copy gauge field:

$$A_\mu \mathrm{d}x^\mu = \phi k_\mu \mathrm{d}x^\mu = (\pm 8GM + 2Gq^2 \log r)\,(\mathrm{d}t + \mathrm{d}r). \tag{4.54}$$

The constant factor does not play a role and this is just the Coulomb solution (4.28) with the identification $2Gq^2 \to Q$, i.e., the electric charge in the gravity side q yields a positively charged point particle in the gauge theory. This is a remarkable difference

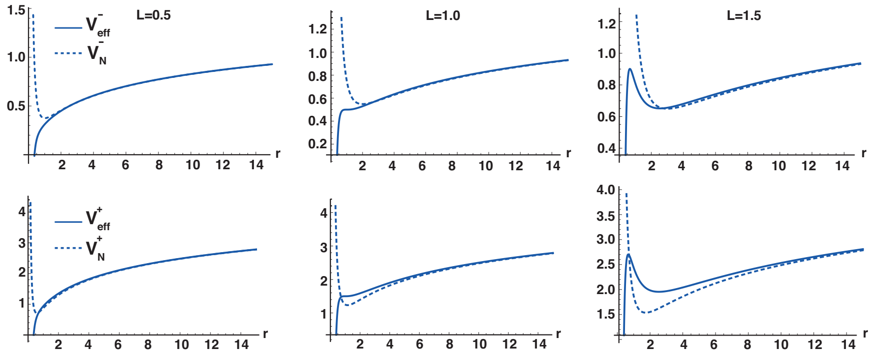

Figure 4.1: First row shows the effective potential V_{eff}^- and the Newtonian potential V_{N}^- for the metric function $f^-(r)$ with $GM = \frac{1}{16}$ and $Gq^2 = \frac{1}{4}$. The second row shows the effective potential V_{eff}^+ and the Newtonian potential V_{N}^+ for the metric function $f^+(r)$ with $GM = \frac{1}{16}$ and $Gq^2 = \frac{3}{4}$. For both cases, timelike geodesics are stable orbits when $L > L_{\text{min}} = 1$.

compared to the higher dimensional cases.[5] In dimensions higher than three ($d \geq 4$), the static solution of the Einstein-Maxwell theory, the Reissner-Nordström black hole, has the following KS scalar:

$$\phi(r) = \frac{2GM}{r^{d-3}} - \frac{Gq^2}{r^{2(d-3)}}, \tag{4.55}$$

where M and q are the mass and the electric charge of the black hole respectively. The gauge field in the gravity side and the corresponding field strength tensor are given by

$$a_\mu \mathrm{d}x^\mu = -\frac{q}{r^{(d-3)}}\mathrm{d}t, \tag{4.56}$$

$$\frac{1}{2} f_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = (d-3)\frac{q}{r^{d-2}}\, \mathrm{d}r \wedge \mathrm{d}t. \tag{4.57}$$

The gauge theory source for the solution is as follows:

$$J^\mu = \rho\, \delta^\mu{}_0, \qquad \rho = 2GM\delta^{d-1}(\vec{r}) - \frac{2(d-3)^2 Gq^2}{r^{2(d-2)}}, \tag{4.58}$$

where we see that the mass M of the black hole shows itself as the charge of a point particle and the electric charge q results in a nonlocalized charge distribution which vanishes as $r \to \infty$. Obtaining the Coulomb solution in $d = 3$ is a very peculiar property, which is possible thanks to the fact that the existence of the electric charge in

[5] Various aspects of the charged black holes solutions in the context of the KS double copy are discussed in [47] and the source terms for $d = 4$ are given in [48] .

the gravity side changes the KS scalar (4.53) such that the modification has the same functional form [$\log(r)$] with the Coulomb solution in $d = 3$. In higher dimensions, as can be seen in (4.55), the mass term carries the functional form of the Coulomb solution given in (4.56), and therefore, yields a point charge in the gauge theory. The modification due to the electric charge has a different functional form and produces a nonlocalized charge density as described in (4.58).

4.3.3 The charged Bañados-Teitelboim-Zanelli (BTZ) black hole ($\Lambda < 0$)

We have shown that the Coulomb solution can be obtained as a gauge theory single copy by considering Einstein-Maxwell theory where either the EH term or the vector kinetic term carries a ghost sign. One can introduce the cosmological constant Λ such that solutions with the correct Newtonian potential are recovered when $\Lambda = 0$. Instead, we will study the charged BTZ black hole whose metric function reads [83]

$$f(r) = -8GM + \frac{r^2}{\ell^2} - 2Gq^2 \log(r). \tag{4.59}$$

Comparing this with the most general solution (4.46) gives that the parameters should be chosen as follows:

$$C = 1 + 8GM, \qquad \Lambda = -\frac{1}{\ell^2}, \qquad \zeta = 1, \tag{4.60}$$

where the last one equation shows that no ghost field is needed to obtain the solution.

Gravitational field equations with the cosmological constant (4.41) can be mapped to Maxwell's equations by again checking the $\mu 0$ component of the trace-reversed equations and using (4.25), which yield

$$\partial_\nu F^{\nu\mu} = g \left[J^\mu_{(\Lambda=0)} + \bar{J}^\mu \right]. \tag{4.61}$$

Here, $J^\mu_{(\Lambda=0)}$ is the source in the absence of the cosmological constant, whose general form is given in (4.27). $\bar{J}^\mu$ represents the effect of the cosmological constant on the source and takes the following form:

$$\bar{J}^\mu = \rho_c \bar{v}^\mu, \qquad \rho_c = \frac{4\Lambda}{g}, \qquad \bar{v}^\mu = (1, 0, 0)\,, \tag{4.62}$$

which is a constant charge density filling all space. The KS scalar corresponding to the metric function (4.59) is

$$\phi(r) = 1 + 8GM + \Lambda r^2 + 2Gq^2 \log(r), \tag{4.63}$$

with the single copy gauge field

$$A_\mu \mathrm{d}x^\mu = \phi k_\mu \mathrm{d}x^\mu = -\left[1 + 8GM + \Lambda r^2 + 2Gq^2 \log(r)\right] (\mathrm{d}t + \mathrm{d}r). \tag{4.64}$$

The field strength tensor reads

$$\frac{1}{2} F_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \left[\frac{Q}{r} - \Lambda r\right] \mathrm{d}r \wedge \mathrm{d}t, \tag{4.65}$$

where we have made the replacement $-2Gq^2 \to Q$. We see that the gauge theory single copy of the charged BTZ black hole is the Coulomb solution ($Q < 0$) modified by a term which describes an electric field linearly increasing with the radial coordinate r (since $\Lambda < 0$) and the source is a point charge located in a medium of constant charge density as follows:

$$J^\mu \partial_\mu = \left[Q\, \delta^2(\vec{r}) + \frac{4\Lambda}{g}\right] \partial_t. \tag{4.66}$$

It is important to note that this is the usual behavior of the Schwarzschild-AdS black hole in higher dimensions when written around a flat background metric [48].

As we see, the Coulomb solution ($Q < 0$) modified by the cosmological constant can be obtained from the well-known charged BTZ black hole without any need for introducing a ghost. From (4.47), one can calculate the gravitational field in the Newtonian limit as

$$\vec{g} = \left[\frac{\Lambda r}{2} + \frac{Gq^2}{r}\right] \hat{r}, \tag{4.67}$$

which shows that a negative cosmological constant ($\Lambda < 0$) is needed for an attractive force, which is possible when $r > \sqrt{\frac{2Gq}{-\Lambda}}$. The geodesics of the charged BTZ black hole exhibit a very rich structure and the details can be found in [100].

4.4 Conclusions

In this chapter, we have studied the KS double copy of the Coulomb solution in 3D, which is an important consistency check for the classical double copy due to the lack of degrees of freedom and a Newtonian limit in GR. The double copy solution should have the correct Newtonian limit, and in 3D, this can only be achieved by matter coupling. In [67], the solution was constructed by coupling to a scalar but has some undesired features. It is a hairy black hole which requires that either the EH term or the scalar kinetic term carry a ghost sign, and the scalar field does not vanish at infinity. Making use of the on-shell duality of a free scalar and a gauge vector, we have shown that a solution with the correct Newtonian limit can also be obtained as a solution of Einstein-Maxwell theory such that the single copy is again the Coulomb solution. While at least one ghost sign is still needed, the electric field in the gravity side vanishes at infinity, which is an improvement compared to the scalar case.

When a negative cosmological constant is introduced, the charged BTZ black hole is a solution without any need for a ghost field, and we have shown that the single copy gauge field is the Coulomb solution ($Q < 0$) modified by a term describing an electric field whose magnitude linearly increases with the distance to the point charge. The source is a point particle sitting in a medium of constant charge density, which is the usual effect of the cosmological constant. At the expense of this modification, this remarkably establishes a connection to the well-known black hole solutions in 3D gravity, which, we believe, shows the potential of 3D KS double copy to have many other interesting features.

CHAPTER 5

GENERALIZED BLACK HOLES IN THREE DIMENSIONS

As we have seen in the Chapters 3 and 4, the double copy of the Coulomb solution in three dimensions is a non-vacuum solution that can be obtained through different matter couplings. It is the static black hole solution of Einstein-Maxwell theory or general relativity minimally coupled to a free scalar field (with one ghost sign in the action in both cases). In this chapter, we consider generalizations of these matter couplings by paying particular attention to the regularity of the static black solution on the gravity side and the corresponding single copy electric field in the gauge theory. We show that i) Einstein-Born-Infeld theory yields a singular double copy, which admits stable orbits for certain choices of parameters, with a regular single copy electric field. ii) Black hole solutions constructed by [101] by coupling to the scalar field exemplify mostly regular double copies with regular single copy electric fields and also admit stable orbits. Additionally, we use these solutions to investigate the connection between horizons on the gravity side and electric fields on the gauge theory side, which was previously observed in four dimensions.

The outline of this chapter is as follows: In Section 5.1, we review the basics of the KS double copy for a static spacetime and discuss the regularity of the solutions. After an investigation of Einstein-Born-Infeld theory as a generalization of Einstein-Maxwell theory in Section 5.2, we move on to generalizations of the scalar coupling in Section 5.3. We end this chapter with conclusions and discussions in Section 5.4.

5.1 Basics of the 3D Kerr-Schild Double Copy and Regularity of the Solutions

For a spacetime admitting KS coordinates, it is possible to write down the components of metric tensor in the following form [102]

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu \,, \tag{5.1}$$

where ϕ is a scalar field and the vector k_μ is null and geodesic with respect to the full metric $g_{\mu\nu}$ and the flat background metric $\eta_{\mu\nu}$ (see chap. 32 of [98] for a summary of important properties). In these coordinates, the Ricci tensor with mixed indices becomes linear in the perturbation as follows

$$R^\mu{}_\nu = \frac{1}{2}\left[\partial^\alpha\partial^\mu(\phi k_\nu k_\alpha) + \partial^\alpha\partial_\nu(\phi k^\mu k_\alpha) - \partial^\alpha\partial_\alpha(\phi k^\mu k_\nu)\right] \,. \tag{5.2}$$

If one writes down the background line element in polar coordinates

$$\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 \,, \tag{5.3}$$

and parametrizes the null vector as

$$-k_\mu\mathrm{d}x^\mu = \mathrm{d}t + \mathrm{d}r \,, \tag{5.4}$$

the $\mu 0$-components become

$$R^\mu{}_0 = \frac{1}{2}\partial_\nu F^{\nu\mu} \,, \quad F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} \,, \quad A_\mu \equiv \phi k_\mu . \tag{5.5}$$

With minimal matter coupling, the trace-reversed gravitational field equations are given by

$$R^\mu{}_\nu = \frac{\kappa^2}{2}\left(T^\mu{}_\nu - \delta^\mu{}_\nu T\right) \,, \qquad \kappa^2 = 8\pi G \,, \tag{5.6}$$

where $T_{\mu\nu}$ is the energy-momentum tensor and T is its trace. Checking the $\mu 0$-components, one obtains the Maxwell equations

$$\partial_\nu F^{\nu\mu} = g J^\mu , \tag{5.7}$$

where the source is given by

$$J^\mu = 4\left(T^\mu{}_0 - \delta^\mu{}_0 T\right) \,, \tag{5.8}$$

and the gauge coupling is obtained by the identification[1] $\kappa^2 \to 4g$. Therefore, for each solution of the gravitational field equations that admit KS coordinates (5.1), the double copy, one can obtain a single copy solution of Maxwell's equations.

[1] We choose our conventions such that when $G = 1$, which is used in our numerical calculations, one has $\partial_\nu F^{\nu\mu} = 2\pi J^\mu$.

Unlike higher dimensions, in order to obtain the Coulomb's solution as the single copy, one needs matter coupling in 3D. One possibility is Einstein-Maxwell theory described by the action

$$S = \int \mathrm{d}^3x\sqrt{-g}\left[\frac{1}{\kappa^2}R + \frac{1}{8\pi}f_{\mu\nu}f^{\mu\nu}\right] , \tag{5.9}$$

where $f_{\mu\nu} = 2\partial_{[\mu}a_{\nu]}$. As shown in [69], one needs to introduce the Maxwell term with a ghost sign in order to obtain the correct Newtonian limit $[\phi = c\log(r)\,, c > 0]$[2]. Taking $\phi = \phi(r)$, $a_\mu \mathrm{d}x^\mu = a_t(r)\mathrm{d}t$, one obtains the following static solution

$$\phi(r) = -8GM - 2Gq^2\log(r)\,, \quad a_\mu\mathrm{d}x^\mu = -q\log(r)\mathrm{d}t\,, \tag{5.10}$$

where q is the charge and M is the mass parameter of the black hole. The corresponding single copy solution is

$$A_\mu\mathrm{d}x^\mu = \phi(r)k_\mu\mathrm{d}x^\mu = (8GM + 2Gq^2\log(r))\mathrm{d}t\,, \tag{5.11}$$

which is just the Coulomb's solution with the identification $2Gq^2 \to -Q$ where Q is the charge of the point particle in Maxwell's theory.

Alternatively, one can consider the coupling to a free scalar as

$$S = \int \mathrm{d}^3x\sqrt{-g}\left[\frac{1}{\kappa^2}R + \frac{1}{2}(\partial\varphi)^2\right] , \tag{5.12}$$

with again a ghost sign for the matter term. Taking $\phi = \phi(r)$, a static black hole solution is obtained provided that $\varphi = p\,\theta$ (p: constant). The solution is given by

$$\phi(r) = -2GM\log(r)\,, \quad \varphi = \sqrt{\frac{M}{2\pi}}\theta\,. \tag{5.13}$$

The single copy gauge field

$$A_\mu\mathrm{d}x^\mu = 2GM\log(r)\mathrm{d}t, \tag{5.14}$$

is again the Coulomb's solution this time with the identification $2GM \to -Q$.

The line element for the metric given in KS coordinates (5.1)

$$\begin{aligned}\mathrm{d}s^2 &= \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \phi(r)(k_\mu\mathrm{d}x^\mu)^2\\ &= -\left[1-\phi(r)\right]\mathrm{d}t^2 + \left[1+\phi(r)\right]\mathrm{d}r^2 + 2\phi(r)\mathrm{d}t\mathrm{d}r + r^2\mathrm{d}\theta^2,\end{aligned} \tag{5.15}$$

[2] One can also take the Einstein - Hilbert term with a ghost sign but throughout this paper, we will use the matter terms with a ghost sign.

can be written in the Boyer-Lindsquit (BL) coordinates by the following coordinate transformation

$$\mathrm{d}t \to \mathrm{d}t + \frac{\phi(r)}{1-\phi(r)}\mathrm{d}r, \tag{5.16}$$

as follows

$$\mathrm{d}s^2 = -h(r)\mathrm{d}t^2 + \frac{1}{h(r)}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2\,, \quad h(r) = 1-\phi(r)\,. \tag{5.17}$$

In BL coordinates, the existence of stable orbits can be easily studied. For timelike particles, the geodesic motion is governed by the equation

$$\frac{1}{2}\mathcal{E}^2 = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + V_{\text{eff}}\,, \tag{5.18}$$

where the effective potential is given by

$$V_{\text{eff}} = \frac{1}{2}\left(\frac{L^2}{r^2}+1\right)h(r)\,. \tag{5.19}$$

The energy and the angular momentum of the particle are expressed in terms of the timelike and angular Killing vectors $\xi_{(t,\theta)}$ as

$$\mathcal{E} = -g_{\mu\nu}\xi^{\mu}_{(t)}u^{\nu}\,, \qquad\qquad L = g_{\mu\nu}\xi^{\mu}_{(\theta)}u^{\nu}\,, \tag{5.20}$$

where u_μ is the velocity of the particle. The Newtonian potential V_{Newton} can be obtained by neglecting GL^2 terms in the effective potential V_{eff}. The stable orbits were shown to exist in [69] for the vector coupling and in [67] for the scalar coupling.

When the single copy is the Coulomb's solution, both the double copy and the single copy have a singularity at $r = 0$. In 3D, there are only three independent curvature invariants that can be constructed from contractions of the metric and the Riemann tensor [103–105]. For a general static solution with the line element (5.15), they are given by

$$R = \frac{\phi'}{r} + \phi'', \tag{5.21}$$

$$R^{\mu}{}_{\nu}R^{\nu}{}_{\mu} = \left(\frac{\phi'}{r}\right)^2 + \frac{1}{2}R^2, \tag{5.22}$$

$$R^{\mu}{}_{\nu}R^{\nu}{}_{\rho}R^{\rho}{}_{\mu} = \left(\frac{\phi'}{r}\right)^3 + \frac{1}{4}R^3, \tag{5.23}$$

and the electric field corresponding to single copy is

$$E(r) \equiv F_{rt} = -\phi'(r) = h'(r). \tag{5.24}$$

We see that if $\frac{\phi'}{r}$ and ϕ'' are regular, then the double copy solution is regular and it can be checked by looking at the curvature scalar R alone. However; it is not guaranteed by the regularity of the single copy electric field. For example, if one takes the scalar potential corresponding to a point charge in Born-Infeld electromagnetism

$$\phi(r) = -Q \log \left[\frac{r + \sqrt{r^2 + \frac{Q^2}{b^2}}}{2} \right], \tag{5.25}$$

and studies a generalized static solution by assuming that the KS ansatz (5.1) gets no correction, as suggested by [66], the curvature scalar and the single copy electric field read

$$\begin{aligned} R(r) &= -\frac{Q^3}{r\left(r^2 + \frac{Q^2}{r^2}\right)^{3/2}} \\ E(r) &= \frac{Q}{\sqrt{r^2 + \frac{Q^2}{b^2}}}\,. \end{aligned} \tag{5.26}$$

Around $r = 0$, one has

$$\begin{aligned} R(r) &= -\frac{b}{r} + \frac{3b^3 r}{2Q^2} - \frac{15 b^5 r^3}{8Q^4} + \mathcal{O}(r^5)\,, \\ E(r) &= b - \frac{b^3 r^2}{2Q^2} + \frac{3b^5 r^4}{8Q^4} + \mathcal{O}(r^5)\,. \end{aligned} \tag{5.27}$$

This is a simple example where we explicitly see that one might have regular single copy electric fields despite having a singularity on the gravity side. If we start from a general KS scalar $\phi(r)$ that is regular at $r = 0$ as follows

$$\phi(r) = a_0 + a_1 r + a_2 r^2 + \mathcal{O}(r^3), \tag{5.28}$$

where a_i's ($i = 0, 1, 2$) are arbitrary constants, we obtain

$$R(r) = \frac{a_1}{r} + 4a_2 + 9a_3 r + \mathcal{O}(r^2), \tag{5.29}$$

$$E(r) = a_1 + 2a_2 r + \mathcal{O}(r^2). \tag{5.30}$$

Therefore; the necessary and sufficient condition for the regularity of both single and double copy solutions is $a_1 = 0$, which is the vanishing of the single copy electric field at the origin[3], i.e., $E(r = 0) = 0$. In the next sections, by considering more generalized matter couplings, we will provide examples of different possibilities, in the usual context of KS double copy without making any assumptions such as made in [66].

5.2 Einstein-Born-Infeld Theory

One of the simplest and most natural generalizations of Einstein-Maxwell theory is Einstein-Born-Infeld theory described by the action

$$S = \int \mathrm{d}^3x\sqrt{-g}\left[\frac{\zeta_1}{\kappa^2}R + \zeta_2\mathcal{L}_{\mathrm{BI}}(f)\right], \quad \kappa^2 = 8\pi G\,, \tag{5.31}$$

where we have introduced $\zeta_i = \pm 1$ $(i = 1, 2)$ to control the sign of the kinetic terms (-1: ghost, $+1$: not ghost) and the Lagrangian of the Born-Infeld electrodynamics is given by [106]

$$\mathcal{L}_{\mathrm{BI}}(f) = \frac{b^2}{2\pi}\left(1 - \sqrt{1 + \frac{f^2}{2b^2}}\right), \qquad f_{\mu\nu} = 2\partial_{[\mu}a_{\nu]}, \tag{5.32}$$

which reduces to that of Maxwell theory as $b \to \infty$. Assuming a static line element of the KS form (5.15) and $a_\mu \mathrm{d}x^\mu = a_t(r)\mathrm{d}t$, the matter equations

$$\partial_\mu\left(\frac{\sqrt{-g}f^{\mu\nu}}{\sqrt{1 + \dfrac{f^2}{2b^2}}}\right) = 0\,, \tag{5.33}$$

where $f^2 = f_{\mu\nu}f^{\mu\nu}$, are solved by the following scalar potential and the corresponding independent nonzero component of the field strength tensor

$$a_t(r) = -q\log\left[\frac{r + \psi(r)}{2}\right], \qquad f_{rt} = \frac{q}{\psi(r)}, \tag{5.34}$$

where

$$\psi(r) = \sqrt{r^2 + \frac{q^2}{b^2}}. \tag{5.35}$$

With the energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{2\pi}\frac{f_\mu{}^\alpha f_{\nu\alpha}}{\sqrt{1 + \dfrac{f^2}{2b^2}}} + g_{\mu\nu}\mathcal{L}_{\mathrm{BI}}(f), \tag{5.36}$$

[3] In 4D, one also need to have $a_0 = 0$, which can be achieved by changing a different integration constant in obtaining the solution. However; this chances the asymptotic behaviour of the metric (see [66] for details).

the trace-reversed Einstein equations read

$$R_{\mu\nu} - 2\Lambda g_{\mu\nu} = \zeta \frac{\kappa^2}{4\pi} \left(\frac{2 f_\mu{}^\alpha f_{v\alpha} - g_{\mu\nu} f^2}{\sqrt{1 + \frac{f^2}{2b^2}}} + 2\, g_{\mu\nu} \mathcal{L}_{\rm BI}(f) \right), \tag{5.37}$$

where $\zeta = \zeta_1 \zeta_2$. The independent components of the left-hand side of the equations are

$$\begin{aligned}
(\mathrm{LHS})_{tt} &= \frac{[\phi(r) - 1]\,[r\phi''(r) + \phi'(r) - 4\Lambda r]}{2r}, \\
(\mathrm{LHS})_{tr} &= \frac{\phi(r)\,[r\phi''(r) + \phi'(r) - 4\Lambda r]}{2r}, \\
(\mathrm{LHS})_{rr} &= \frac{[\phi(r) + 1]\,[r\phi''(r) + \phi'(r) - 4\Lambda r]}{2r}, \\
(\mathrm{LHS})_{\theta\theta} &= r\phi'(r) - 2\Lambda r^2 \,,
\end{aligned} \tag{5.38}$$

with the following components at the right-hand side

$$\begin{aligned}
(\mathrm{RHS})_{tt} &= \frac{2b^2 \zeta G\, [\phi(r) - 1]\, [r - \psi(r)]^2}{r\psi(r)}, \\
(\mathrm{RHS})_{tr} &= -\frac{2b^2 \zeta G \phi(r)\, [r - \psi(r)]^2}{r\psi(r)}, \\
(\mathrm{RHS})_{rr} &= \frac{2b^2 \zeta G\, [1 + \phi(r)]\, [r - \psi(r)]^2}{r\psi(r)}, \\
(\mathrm{RHS})_{\theta\theta} &= -4b^2 \zeta G r\, [r - \psi(r)] \,.
\end{aligned} \tag{5.39}$$

Similar to the Einstein-Maxwell case, the $\theta\theta$-component is the easiest one to solve and it yields

$$\phi(r) = -8GM + \zeta G q^2 + 2\zeta G b^2 \left[r^2 - r\psi(r) - \frac{q^2}{b^2} \log\left(\frac{r + \psi(r)}{2} \right) \right], \tag{5.40}$$

which also solves the other components[4]. Note that we have chosen the integration constant such that the expansion around $b \to \infty$

$$\phi(r) = -8GM + 2\zeta G q^2 \log(r) + \frac{\zeta G q^4}{4b^2 r^2} + \mathcal{O}\left[\frac{1}{b^3} \right], \tag{5.41}$$

[4] The static solution for a nonzero cosmological constant and no ghost sign in the action was given in [107, 108].

gives the result for $\zeta = -1$ in the EM case (5.10) at the leading order. The gravitational field in the Newtonian limit is given by

$$\begin{aligned} \vec{g} &= \frac{1}{2}\vec{\nabla}\phi = -\frac{1}{2}\vec{\nabla}h, \\ &= \frac{2\zeta G\left[q^2 + b^2 r\left(r - \sqrt{r^2 - \frac{q^2}{b^2}}\right)\right]}{\psi(r)}\hat{r}, \end{aligned} \tag{5.42}$$

which is attractive everywhere when $\zeta = -1$. Therefore, we again need to choose one ghost sign in the action. For $\zeta = -1$, the Ricci scalar and the electric field corresponding to the single copy $A_\mu = \phi k_\mu$ are given by

$$\begin{aligned} R(r) &= -\frac{4G\left[q^2 + 2b^2 r\left(r - \psi(r)\right)\right]}{r\psi(r)}, \\ E(r) &= \frac{4G\left[q^2 + b^2 r\left(r - \psi(r)\right)\right]}{\psi(r)}. \end{aligned} \tag{5.43}$$

Checking their behavior as $r \to 0$,

$$\begin{aligned} R(r) &= -\frac{4Gbq}{r} + 8Gb^2 - \frac{6Gb^3 r}{q} + \mathcal{O}(r^3), \\ E(r) &= 4Gbq - 4Gb^2 r + \frac{2Gb^3 r^2}{q} + \mathcal{O}(r^3), \end{aligned} \tag{5.44}$$

one sees that, while the single copy electric field is regular around the origin, we have a singularity on the gravity side.

For particle orbits, one can ensure to preserve the following main properties of the static solution of Einstein-Maxwell theory by choosing an appropriate set of parameters: i) The Newtonian potential V_{Newton} possesses an infinite barrier at short distances and become equal to the effective potential V_{eff} at large distances. ii) Timelike particles are forbidden to reach the infinity due to the logaritmic behaviour of the potential as $r \to \infty$. iii) There is a critical value L_c of the angular momentum of the particle such that, when $L > L_c$, the effective potential V_{eff} develops a local minimum and a local maximum, making stable orbits possible. We refer the reader to Figure 5.1 for an explicit demonstration of these properties and the regularity of the solutions, together with the charge density in the gauge theory.

Before proceeding further, we would like to note that, similar to the Einstein-Maxwell theory, one can again make use of the duality of scalars and gauge vectors to realize a

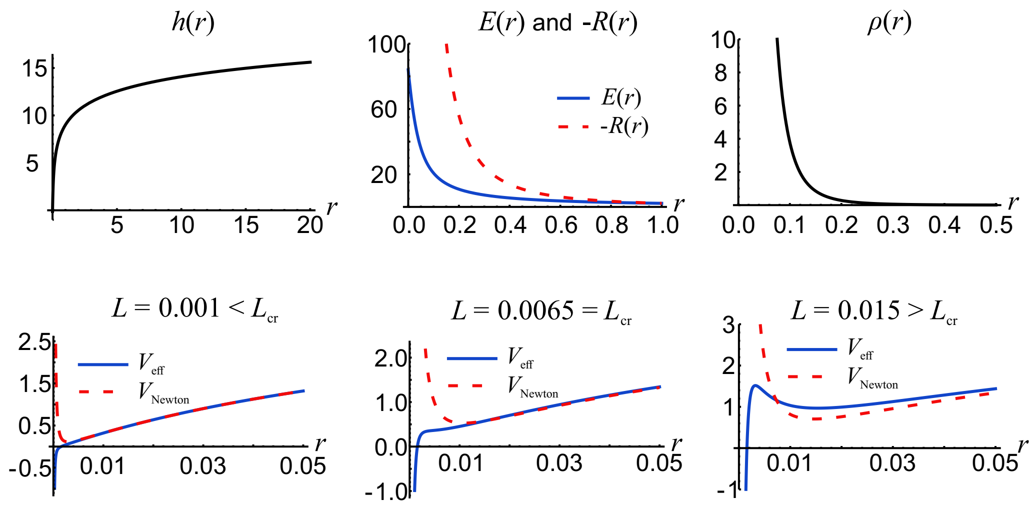

Figure 5.1: Details of the static solution of Einstein-Born-Infeld theory for $(G = 1, M = 1, q = 0.525, b = 10)$.

solution with the same physical properties. The matter part of the Lagrangian is given by

$$\mathcal{L}_{\text{scalar}}(\varphi) = -\frac{b^2}{2\pi}\left(1 - \sqrt{1 + \frac{2\pi}{b^2}(\partial\varphi)^2}\right), \tag{5.45}$$

which reduces to the free scalar Lagrangian as $b \to \infty$ and the scalar field should read

$$\varphi = \sqrt{\frac{M}{2\pi}}\theta, \qquad M = q^2, \tag{5.46}$$

in order to obtain the same solution with that of Einstien-Born-Infeld theory up to an integration constant.

5.3 Generalization of the Scalar Coupling

In this section, we will study a generalization of the scalar coupling which not only allows different possibilities regarding the regularity of the single and double copies, but also demonstrates a simple relation between horizons on the gravity side and the corresponding electric field in Maxwell's theory, which is based on the following observation [65]: Since the single copy electric field is equal to the derivative of the metric function in BL coordinates ($E(r) = h'(r)$), it becomes zero at a maximum, a minimum or a saddle point. Since the horizons are located at the zeros of the metric function ($h(r_i) = 0$), there should exist at least one point between two adjacent horizons where the electric field is zero, corresponding to a minimum or maximum. The theory that we will consider is described by the action [101]

$$S = \int \mathrm{d}^3x\sqrt{-g}\left[\frac{1}{\kappa^2}(R - 2\Lambda) - Q\right], \tag{5.47}$$

where the matter term is given by

$$\begin{aligned} Q =& \sum_{n=1} \alpha_n \ell^{2(n-1)} (\partial\varphi)^{2n} \\ & - \sum_{m=0} \beta_m \ell^{2(m+1)} (\partial\varphi)^{2m} \left[(3+2m)R^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - R(\partial\varphi)^2\right]. \end{aligned} \tag{5.48}$$

Here, α_n and β_m are arbitrary dimensionless constants. The trace-reversed Einstein equations are in the following form

$$R_{\mu\nu} = \frac{\kappa^2}{2}\Theta_{\mu\nu}, \qquad \Theta_{\mu\nu} = \frac{1}{\Gamma(\varphi)}\left[\widetilde{T}_{\mu\nu} + \frac{4\Lambda}{\kappa^2}g_{\mu\nu}\right]. \tag{5.49}$$

We give the expressions for $\Gamma(\varphi), \widetilde{T}_{\mu\nu}$ and the field equation for the scalar field in Appendix Csince they are quite cumbersome and do not play a direct role in our discussion. Applying the usual prescription, we again obtain Maxwell's equations with a source defined in terms of the Θ-tensor as follows

$$\partial_\nu F^{\nu\mu} = gJ^\mu, \qquad J^\mu = 4\,\Theta^\mu_{\ 0}. \tag{5.50}$$

This theory admits a family of black holes and horizonless spacetimes whose line elements in BL coordinates are in the form (5.17). (see [101] for the most general form of the solution). For our purposes, it is enough to take non-zero values for $(\alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1)$ and set all the other constants to zero. In this case, for $\varphi = p\,\theta$ (p:constant), all the field equations are solved if the metric function is given by

$$h(r) = \frac{(1+8GM)r^4 - \Lambda r^6 - 8\pi G\alpha_1 p^2 r^4 \log(r) + 4\pi G\alpha_2\ell^2 p^4 + 2\pi G\alpha_3\ell^4 p^6 r^2}{8\pi G\beta_0\ell^2 p^2 r^2 + 24\pi G\beta_1\ell^4 p^4 + r^4}, \tag{5.51}$$

where we have chosen the integration constant such that such that we recover the free scalar case when $p = \sqrt{\frac{M}{2\pi}}$, $\alpha_1 = -\frac{1}{2}$, $\alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \Lambda = 0$. The curvature scalar and the single copy electric field can be easily calculated from equations (5.21) and (5.24) respectively. Their expansions around $r = 0$ are given by

$$\begin{aligned} R(r) &= \frac{\alpha_3\beta_0 - 6\alpha_2\beta_1}{6\beta_1^2\ell^2} + \frac{\left[c_1 + c_2\log(r)\right]r^2}{216\pi G\beta_1^3\ell^4 p^4} + \mathcal{O}(r^3) \\ E(r) &= -\frac{r\left(\alpha_3\beta_0 - 6\alpha_2\beta_1\right)}{18\beta_1^2\ell^2} + \mathcal{O}\left(r^3\right), \end{aligned} \tag{5.52}$$

where $c_1 = c_1(G, M, p, \alpha_2, \alpha_3, \beta_0, \beta_1)$ and $c_2 = c_2(G, p, \alpha_1, \beta_1)$ are constants. From the expansions, we see that one can obtain non-singular single and double copies by taking $\beta_1 \neq 0$. When $\beta_0 = \beta_1 = 0$, the expansions around $r = 0$ become

$$\begin{aligned} R(r) &= -\frac{24\pi G\alpha_3 \ell^4 p^6}{r^6} - \frac{8\pi G\alpha_2 \ell^2 p^4}{r^4} + \frac{8\pi G\alpha_1 p^2}{r^2} + 6\Lambda + \mathcal{O}(r^3), \\ E(r) &= -\frac{8\pi G\alpha_3 \ell^4 p^6}{r^5} - \frac{8\pi G\alpha_2 \ell^2 p^4}{r^3} - \frac{8\pi G\alpha_1 p^2}{r} - 2\Lambda r + \mathcal{O}(r^3). \end{aligned} \tag{5.53}$$

The expression for the electric field shows that one has a point charge at the origin with $Q = -8\pi G\alpha_1 p^2$.

We present four cases by using different set of parameters, which are given in Table 5.1:

- Case I: Taking a nonzero value for β_1, we obtain regular single and double copies. This is a horizonless geometry and the electric field $E(r)$ becomes zero at two points, which are maximum and minimum of the metric function $h(r)$.
- Case II: Again taking $\beta_1 \neq 0$ guarantees the regularity of the single and double copies. There is one event horizon and the electric field $E(r)$ is zero only at the origin, where the minimum of the metric function $h(r)$ occurs. Stable orbits exist when $L > L_{\text{cr}}$.
- Case III: $\beta_1 \neq 0$ yields regular single and double copies. We have two event horizons and the electic field becomes zero at two points: A local maximum ($r = 0$) and a global minimum located between two horizons.
- Case IV: $\beta_1 = 0$ gives single and double copies which are singular at the origin. There are three event horizons and the electric field is zero at the following points: a local maximum between the first and the second horizons, and a local miminum between the second and the third horizons.

All the details can be seen in Figures 5.2, 5.3, 5.4 and 5.5. Although, stable orbits exist for all values of the angular momentum L, in cases I, III and IV; there is a critical value L_{cr}, beyond which, there arises a second region where a particle in a stable orbit can be present. We do not show them explicitly since it is sufficient to

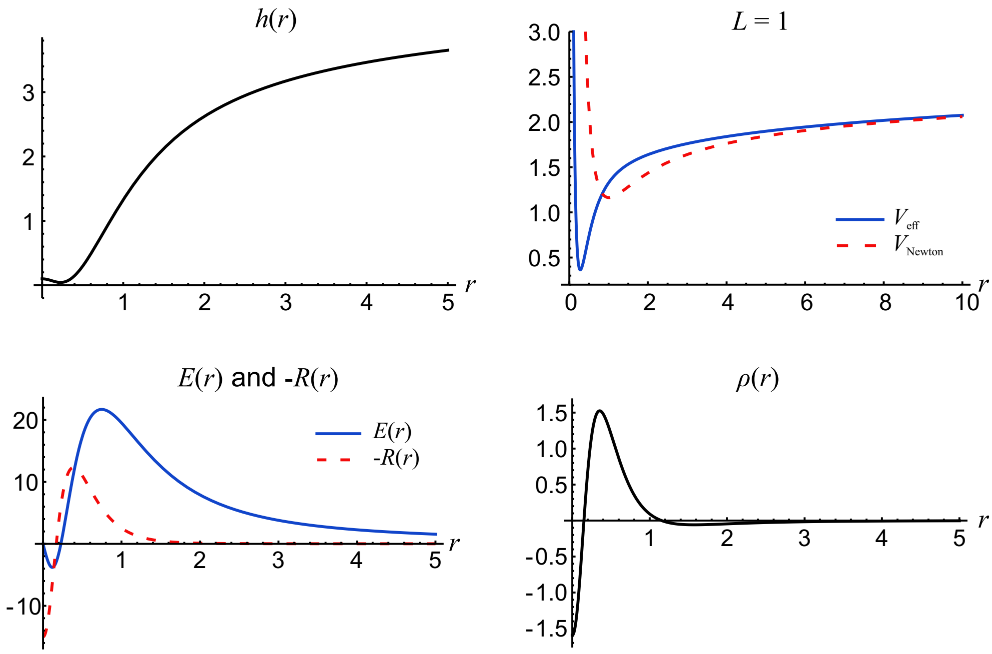

Figure 5.2: Details of Case I presented in Section 5.3 for parameters given in Table 5.1.

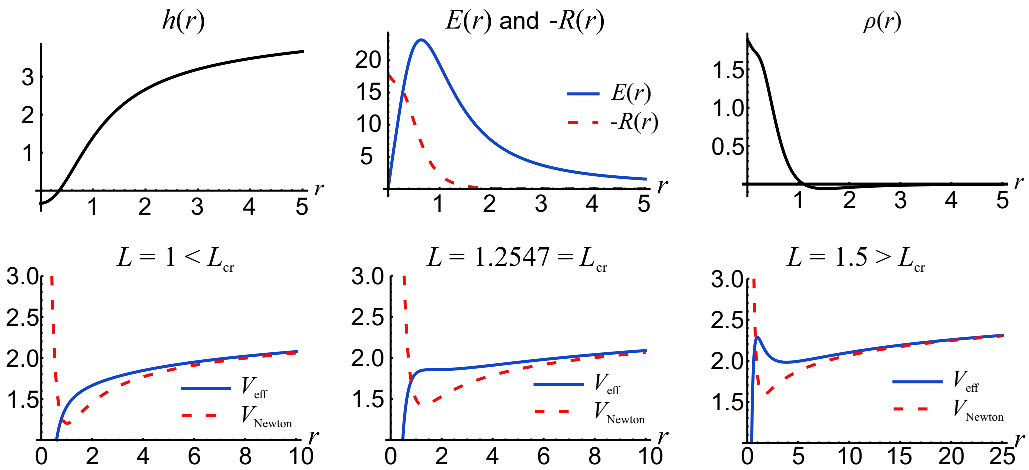

Figure 5.3: Details of Case II presented in Section 5.3 for parameters given in Table 5.1.

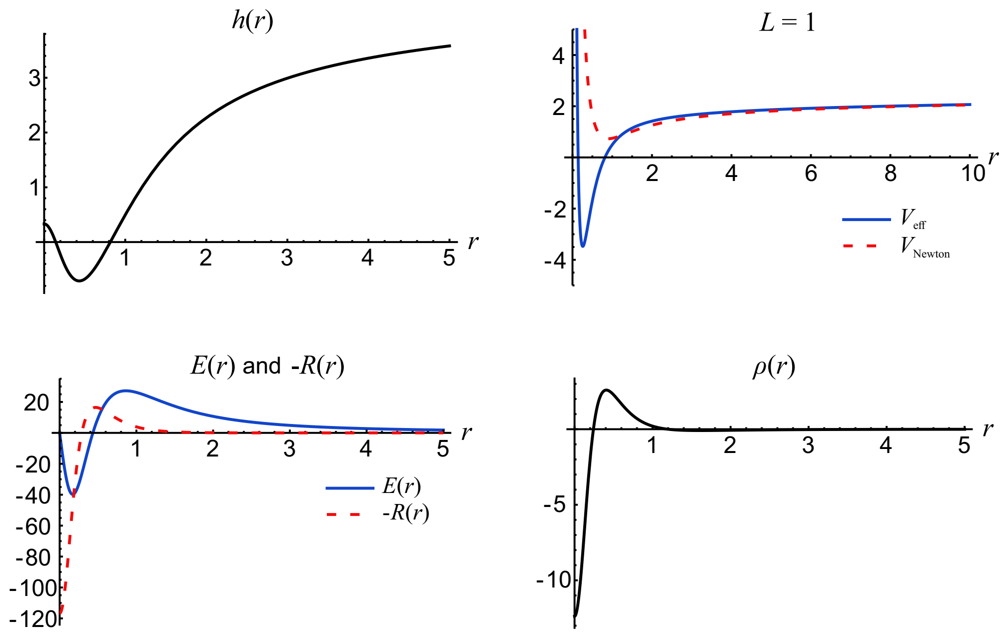

Figure 5.4: Details of Case III presented in Section 5.3 for parameters given in Table 5.1.

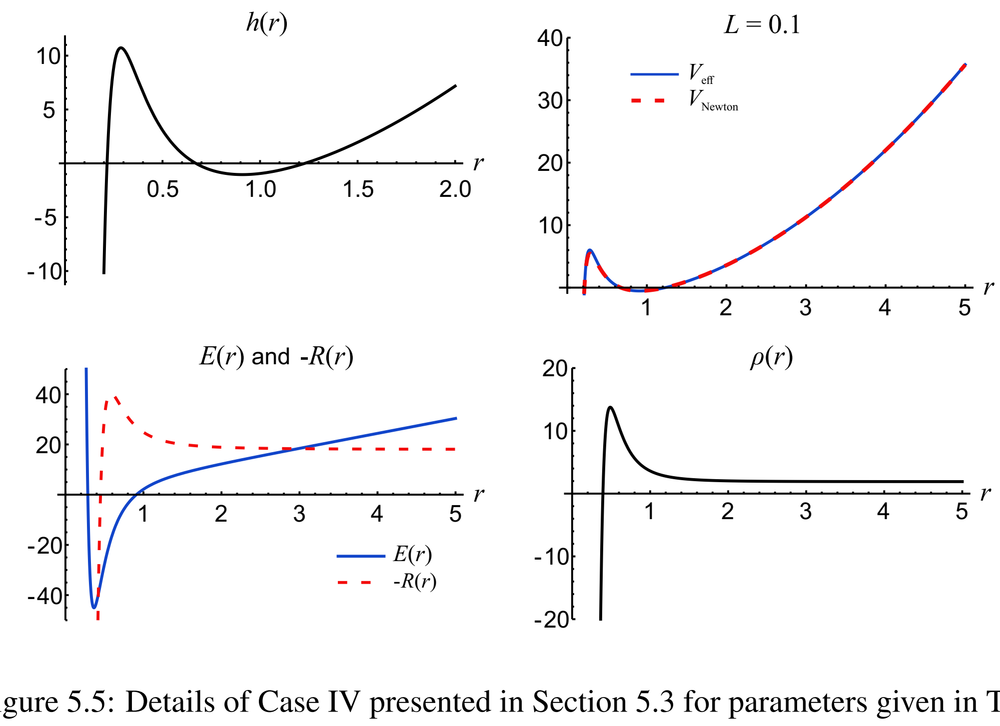

Figure 5.5: Details of Case IV presented in Section 5.3 for parameters given in Table 5.1. Note that in addition to the nonlocal charge distribution shown in the figure, one has a point charge with $Q = 2$ at the origin.

Table 5.1: Choice of parameters for different solutions of the scalar theory. We always take $p = \sqrt{\frac{M}{2\pi}}$, $\alpha_1 = -\frac{1}{2}$, as in the free scalar case, and $G = 1$, which leads to $\partial_\nu F^{\nu\mu} = 2\pi J^\mu$.

	G	M	p	ℓ	Λ	α_1	α_2	α_3	β_0	β_1
Case I (No Horizon)	1	$\frac{1}{4}$	$\sqrt{\frac{M}{2\pi}}$	1	0	$-\frac{1}{2}$	-10	30	1	1
Case II (Single Horizon)	1	$\frac{1}{4}$	$\sqrt{\frac{M}{2\pi}}$	1	0	$-\frac{1}{2}$	1	-5	1	1
Case III (Two Horizons)	1	$\frac{1}{4}$	$\sqrt{\frac{M}{2\pi}}$	1	0	$-\frac{1}{2}$	-100	100	1	1
Case IV (Three Horizons)	1	1	$\sqrt{\frac{M}{2\pi}}$	1	-3	$-\frac{1}{2}$	10	-5	0	0

show the existence of one such region for our purposes. As we have shown at the end of Section 5.1 on general grounds, the necessary and sufficient condition for the regularity of the solutions on both gravity and gauge theory sides is the vanishing of the electric field at the origin, which is realized in Cases II and III and can be explicitly seen in Figures 5.3,5.4.

5.4 Summary and Discussions

In this paper, we have studied generalizations of the matter couplings in 3D admitting a static black hole solution which gives rise to the Coulomb's solution as its single copy. For these matter couplings, Einstein-Maxwell theory or GR minimally coupled to a free scalar field, both the double copy and the single copy solution has a singularity at the origin. As a generalization of the former, we studied Einstein-Born-Infeld theory and showed that although the static black hole solution, which admits stable particle orbits, is singular, the single copy electric field is regular at the origin. For the latter, we have intestigated a theory recently discovered in [101], which forms an extremely useful theoretical laboratory since the most general solution offers different possibilities regarding the regularity of the black hole solution and the number of event horizons. We have given examples where both the double and single copy are regular. Morover, starting from a horizonless geometry, we have considered spacetimes with increasing number of event horizons and exhibited the relation between the event horizons on the gravity side and the corresponding electric field in Maxwell's theory.

All these examples show that many physically important properties of the KS double copy can also be realized in 3D with the most notable exception that, in the simplest case where the single copy is the Coulomb solution, the double copy is a non-vacuum solution which can be obtained by taking the Einstein-Hilbert term or the matter term with a ghost sign in the action. In the generalizations that we have considered in this paper, we introduced the couplings such that one recovers the Coulomb case in an appropriate limit. However; a wide range of different possibilities exist without this requirement.

As a final note, we would like to mention that there exists a different interpretation of the single copy in the case of a non-minimal coupling on the gravity side. Writing the gravitational field equations (5.49) by introducing an effective Newton constant as

$$R_{\mu\nu} = \frac{\kappa_{\text{eff}}^2}{2}\Theta'_{\mu\nu}, \qquad \kappa_{\text{eff}}^2 = \frac{\kappa^2}{\Gamma(\varphi)}, \tag{5.54}$$

where

$$\Theta'_{\mu\nu} = \widetilde{T}_{\mu\nu} + \frac{4\Lambda}{\kappa^2}, \tag{5.55}$$

one can obtain solutions to Maxwell's equations with an effective gauge coupling

$$\partial_\nu F^{\nu\mu} = g_{\text{eff}} J^\mu, \qquad J^\mu = 4\Theta'^{\mu}_{0}, \tag{5.56}$$

with the identification $\kappa_{\text{eff}}^2 \to 4g_{\text{eff}}$. In this picture, the cosmological constant plays its usual role in the case of a minimal matter coupling and produces a constant charge density filling all space. However; a dynamical mechanism for the evolution of the gauge coupling seems to be missing on the gauge theory side. This might be an interesting direction for future study.

CHAPTER 6

THE KERR-SCHILD DOUBLE COPY IN CURVED SPACETIMES

In the pioneering work on the KS double copy [42], the analyses was performed by considering solutions of general relativity which can be written in the Kerr-Schild form with the flat background metric. The fact that the Ricci tensor with mixed indices becomes linear in the perturbation for such solutions provides a natural way to map them to solutions of Maxwell's theory defined on the flat spacetime. A natural extension is to consider spacetimes with non-flat background metrics, which was first studied in [45]. Later, a more systematic analysis was given in [46] and it was shown that there exist two different ways to realize the double copy structure when the background metric is curved, called Type-A and Type-B double copies. In the Type-A double copy, one maps both the background and the perturbation by using the flat metric as the base. Alternatively, in the Type-B double copy, only the perturbation is mapped by taking the base metric as that of the background spacetime, yielding solutions of Maxwell's theory defined on the curved background. A wide range of examples with constant curvature background was presented in [47] where the authors showed the crucial role played by the Killing vectors in the construction. For the stationary solutions, the contraction of the gravity equations with the time-like Killing vector was used, which is essentially checking the $\mu 0$-components of the trace-reversed equations as done previously. More non-trivial evidence was obtained from the wave solutions where the contraction with the null Killing vector yielded a reasonable single copy.

The linearity of the Ricci tensor in the perturbation, which is the crucial property that makes the whole construction work, holds in the case of a generic curved background spacetime. Motivated by this, in Section 6.1, we will give a general formulation of the KS double copy without any simplifying assumption about the background metric. With the assumption that some redundant terms vanish, one obtains Maxwell's equation defined on the curved background where the source term gets a contribution from the curvature of the background, which vanishes for a constant curvature spacetime, in addition to the energy-momentum tensor in the gravity side. In order to see the implications, we will study different solutions of general relativity with a cosmological constant. In Section 6.2, solutions with a maximally symmetric background will be examined. When the background is chosen to be of constant curvature, there is no effect on the source. However, choosing a flat background leads to a constant charge density filling all space. While it has been observed before, our formalism explicitly demonstrates that this is due to the deviation of the background from a constant curvature spacetime. In Section 6.3, in order to exhibit the effect of a curved background, we will consider the Lifshitz black hole with two different matter couplings.

6.1 General Formulation

In this section, we give a general formulation of the KS double copy in curved spacetime. For that, we will consider classical solutions of cosmological general relativity minimally coupled to matter, which is described by the action

$$S = \frac{1}{16\pi G_d} \int \mathrm{d}^d x \sqrt{-g}\, \left[R - 2\Lambda + \mathcal{L}_m\right], \tag{6.1}$$

where G_d is the $d-$dimensional Newton's constant, Λ is the cosmological constant and $\mathcal{L}_m$ is the matter part of the Lagrangian density. The field equations arising from the action (6.1) are

$$G_{\mu\nu} + \Lambda\, g_{\mu\nu} = T_{\mu\nu}. \tag{6.2}$$

For the KS double copy, one needs the trace-reversed equations with mixed indices

$$R^{\mu}_{\ \nu} - \frac{2\,\Lambda}{d-2}\,\delta^{\mu}_{\ \nu} = \widetilde{T}^{\mu}_{\ \nu}, \tag{6.3}$$

where the matter contribution is given by

$$\widetilde{T}^{\mu}_{\ \nu} = T^{\mu}_{\ \nu} - \frac{1}{d-2}\,\delta^{\mu}_{\ \nu}\, T. \tag{6.4}$$

For a metric in the KS form,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \phi\, k_{\mu} k_{\nu}, \tag{6.5}$$

where the vector k_μ is null and geodesic with respect to both the background and the full metric as

$$\bar{g}^{\mu\nu}k_\mu k_\nu = g^{\mu\nu}k_\mu k_\nu = 0, \qquad k^\nu\bar{\nabla}_\nu k^\mu = k^\nu\nabla_\nu k^\mu = 0, \tag{6.6}$$

the Ricci tensor with mixed indices becomes linear in the perturbation as follows [98]

$$R^\mu{}_\nu = \bar{R}^\mu{}_\nu - \phi\, k^\mu k^\alpha \bar{R}_{\alpha\nu} + \frac{1}{2}\left[\bar{\nabla}^\alpha\bar{\nabla}^\mu\left(\phi\, k_\alpha k_\nu\right) + \bar{\nabla}^\alpha\bar{\nabla}_\nu\left(\phi\, k^\mu k_\alpha\right) - \bar{\nabla}^2\left(\phi\, k^\mu k_\nu\right)\right]. \tag{6.7}$$

Since the aim is to obtain Maxwell's equations in the background spacetime, we rewrite the Ricci tensor in the KS coordinates (6.7) by using the gauge field $A_\mu \equiv \phi\, k_\mu$ as

$$R^\mu_\nu = \bar{R}^\mu{}_\nu - \frac{1}{2}\left[\bar{\nabla}_\alpha F^{\alpha\mu}k_\nu + E^\mu_\nu\right], \tag{6.8}$$

where $F_{\mu\nu} = 2\,\bar{\nabla}_{[\mu}A_{\nu]}$ is the field strength tensor and

$$E^\mu_\nu = X^\mu_\nu + Y^\mu_\nu - \bar{R}^\mu{}_{\alpha\beta\nu}A^\alpha k^\beta + \bar{R}_{\alpha\nu}A^\alpha k^\mu, \tag{6.9}$$

with X^μ_ν and Y^μ_ν given by

$$X^\mu_\nu = -\bar{\nabla}_\nu\left[A^\mu\left(\bar{\nabla}_\alpha k^\alpha + \frac{k^\alpha\bar{\nabla}_\alpha\phi}{\phi}\right)\right], \tag{6.10}$$

$$Y^\mu_\nu = F^{\alpha\mu}\bar{\nabla}_\alpha k_\nu - \bar{\nabla}_\alpha\left(A^\alpha\bar{\nabla}^\mu k_\nu - A^\mu\bar{\nabla}^\alpha k_\nu\right). \tag{6.11}$$

Using this form of the Ricci tensor (6.8) in the trace-reversed equations (6.3) gives,

$$\Delta^\mu{}_\nu - \frac{1}{2}\left[\bar{\nabla}_\alpha F^{\alpha\mu}k_\nu + E^\mu_\nu\right] = \widetilde{T}^\mu{}_\nu, \tag{6.12}$$

where we introduce the *deviation tensor*

$$\Delta^\mu{}_\nu = \bar{R}^\mu{}_\nu - \frac{2\,\Lambda}{d-2}\,\delta^\mu{}_\nu, \tag{6.13}$$

which vanishes for a constant curvature spacetime if the cosmological constant Λ is appropriately chosen, and therefore, characterizes the deviation of the background spacetime from a spacetime with constant curvature (see Sec. 2.3 for more explanation).

In order to solve for the field strenght term, we consider the contraction of this equation (6.12) with a Killing vector V^ν of both the background and the full metric, i.e.,

$$\nabla_{(\mu}V_{\nu)} = \bar{\nabla}_{(\mu}V_{\nu)} = 0, \tag{6.14}$$

which gives the single copy equation as

$$\bar{\nabla}_\nu F^{\nu\mu} + E^\mu = J^\mu, \tag{6.15}$$

where the extra part is

$$E^\mu = \frac{1}{V \cdot k} E^\mu_{\ \nu} V^\nu, \tag{6.16}$$

and the gauge theory source is given by

$$J^\mu = 2\left[\Delta^\mu - \widetilde{T}^\mu\right], \tag{6.17}$$

with

$$\Delta^\mu = \frac{1}{V \cdot k} \Delta^\mu_{\ \nu} V^\nu, \qquad\qquad \widetilde{T}^\mu = \frac{1}{V \cdot k} \widetilde{T}^\mu_{\ \nu} V^\nu, \tag{6.18}$$

which are the contributions from the background spacetime and the matter part of the Lagrangian respectively.

Contracting the single copy equation (6.15) with the Killing vector V^μ, one obtains the zeroth copy equation as

$$\bar{\nabla}^2 \phi + \mathcal{Z} + \mathcal{E} = j, \tag{6.19}$$

where

$$\mathcal{Z} = \frac{V \cdot Z}{V \cdot k}, \qquad\qquad \mathcal{E} = \frac{V \cdot E}{V \cdot k}, \qquad\qquad j = \frac{V \cdot J}{V \cdot k}, \tag{6.20}$$

with vectors E^μ and J^μ given in (6.16 - 6.17) and,

$$Z^\mu = \bar{\nabla}_\alpha k^\mu \, \bar{\nabla}^\alpha \phi + \bar{\nabla}_\alpha \left[2\phi \bar{\nabla}^{[\alpha} k^{\mu]} - k^\alpha \bar{\nabla}^\mu \phi\right]. \tag{6.21}$$

For *any* solution of the gravitational field equations (6.2) that can be written in the KS form (6.5), the gauge field $A_\mu = \phi \, k_\mu$ solves the single copy equation (6.15) and the scalar ϕ solves the zeroth copy equation (6.19). In this paper, we will study black hole solutions in the KS form by using the time-like Killing vector[1] $V^\mu = \delta^\mu_{\ 0}$. For the examples that we will consider in this paper, one has

$$V \cdot k = 1, \qquad E^\mu = E^\mu_{\ 0} = 0, \qquad \mathcal{E} = E^0 = 0, \qquad \Delta^\mu = \Delta^\mu_{\ 0}, \qquad \widetilde{T}^\mu = \widetilde{T}^\mu_{\ 0}, \tag{6.22}$$

[1] In [47], it was shown that the wave-type solutions with maximally symmetric background metrics can be studied by choosing a null Killing vector.

and the single copy and the zeroth copy equations becomes Maxwell's and Poisson's equations

$$\begin{aligned} \bar{\nabla}_\nu F^{\nu\mu} &= J^\mu, \\ \bar{\nabla}^2\phi + \mathcal{Z} &= j, \end{aligned} \tag{6.23}$$

where the source terms are given by

$$J^\mu = 2\left[\Delta^\mu - \widetilde{T}^\mu\right], \qquad j = J_0 = \bar{g}_{0\mu}J^\mu, \tag{6.24}$$

and

$$\mathcal{Z} = Z_0 = \bar{g}_{0\mu}Z^\mu, \tag{6.25}$$

with Z^μ given in (6.21). The $\mathcal{Z}$-term in Poisson's equation vanishes when the background metric is flat and takes a different form depending on the background spacetime.

The principal result of our analysis is that the deviation of the background metric from a constant curvature spacetime, which is characterized by the deviation tensor defined in (6.13), affects nontrivially the gauge theory source as described in (6.17) for an arbitrary Killing vector and in (6.28) for the time-like Killing vector. Previously, this has been observed as a constant charge distribution filling all space when the background is taken to be flat. In Section 6.3, we will show that this remains to be true when the background is the Lifshitz spacetime. Therefore, we write the contribution from the background spacetime as

$$\Delta^\mu = \frac{1}{2}\rho_c\,\delta^\mu_{\ 0}, \tag{6.26}$$

where ρ_c is the constant charge density. The matter contribution can also be written in the following form

$$\widetilde{T}^\mu = -\frac{1}{2}\rho_m v^\mu, \tag{6.27}$$

where ρ_m is the charge density due to the matter in the gravitational theory and v^μ is the velocity of the charge distribution. These lead to the following form of the gauge theory source

$$J^\mu = \rho_c\,\delta^\mu_{\ 0} + \rho_m v^\mu, \tag{6.28}$$

which we will use throughout this paper[2]. In the next section, we will review some previously studied examples through our general formalism.

6.2 Maximally Symmetric Background Spacetime

In this section, we focus on solutions of theories described by the action (6.1) with the corresponding field equations (6.2) which can be written in the KS form (6.5) around a maximally symmetric background spacetime. For a non-zero cosmological constant ($\Lambda \neq 0$), the background spacetime can be chosen to be Minkowski or AdS spacetimes. In the former case, the gauge theory copy is defined on Minkowski spacetime and the deviation tensor defined in (6.13) takes the form

$$\Delta^{\mu}{}_{\nu}(\text{Minkowski}) = -\frac{2\Lambda}{d-2}\delta^{\mu}{}_{\nu}, \tag{6.29}$$

since $\bar{R}^{\mu}{}_{\nu} = 0$, and the constant charge density in the gauge theory source for a timelike Killing vector (6.28) is determined by the cosmological constant as

$$\rho_c = -\frac{4\Lambda}{d-2}. \tag{6.30}$$

Since the modification to the Poisson's equation given in (6.25) vanishes when the background is Minkowski spacetime, the single copy and the zeroth copy equations become

$$\begin{aligned} \bar{\nabla}_{\nu} F^{\nu\mu} &= J^{\mu}, \\ \bar{\nabla}^2 \phi &= j, \end{aligned} \tag{6.31}$$

where the general form of the sources is given by

$$J^{\mu} = \rho_c\, \delta^{\mu}{}_{0} + \rho_m v^{\mu}, \qquad j = -(\rho_c + \rho_m). \tag{6.32}$$

Here, ρ_m is the charge density due to the matter fields and the velocity vector v^{μ} can be read from (6.27). For static solutions, one has a static charge distribution, and therefore, $v^{\mu} = \delta^{\mu}{}_{0}$. For stationary solutions, one obtains a rotating charge distribution and the velocity vector takes a form accordingly.

[2] As discussed in [47], for black hole solutions, one has localized sources describing a point charge at the origin. Since our main aim is to study the effect of the background spacetime, we will only give the non-localized part of the gauge theory source.

In the latter case, the gauge theory copy is Maxwell's theory on AdS spacetime and the deviation tensor vanishes

$$\Delta^{\mu}{}_{\nu}(\text{AdS}) = 0, \tag{6.33}$$

which implies that there is no constant charge density in the gauge theory source ($\rho_c = 0$). The Poisson's equation is modified due to the curvature of the background as described in (6.23). In what follows, we will give examples in $d = 4$, for which the equations take the following form

$$\begin{aligned} \bar{\nabla}_\nu F^{\nu\mu} &= J^\mu, \\ \bar{\nabla}^2\phi - \frac{1}{6}\bar{R}\,\phi &= j, \end{aligned} \tag{6.34}$$

and the sources are fixed by only the matter contribution as

$$J^\mu = \rho_m v^\mu, \qquad\qquad j = -\rho_m. \tag{6.35}$$

In the remainder of this section, we will elaborate on this, by applying our formalism to some examples that were investigated previously in the literature, with a special focus on the sources, and make a comparison between Minkowski and AdS backgrounds whenever possible.

6.2.1 AdS$_4$ Spacetime around Minkowski Background

As the simplest example, we consider the AdS$_4$ spacetime [45], which is a solution when $d = 4$, $\mathcal{L}_m = 0$ in (6.1). It can be written in the KS form (6.5) around the Minkowski metric

$$\bar{g}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{6.36}$$

where the null vector and the scalar function are given by

$$k_\mu\mathrm{d}x^\mu = \mathrm{d}t + \mathrm{d}r, \qquad\qquad \phi(r) = \frac{\Lambda r^2}{3}. \tag{6.37}$$

As a result, the gauge field takes the form

$$A_\mu\mathrm{d}x^\mu = \frac{\Lambda r^2}{3}\,(\mathrm{d}t + \mathrm{d}x)\,, \tag{6.38}$$

and the only non-zero component of the field strength tensor is

$$F_{rt} = \frac{2\Lambda r}{3}. \tag{6.39}$$

The effect of the cosmological constant on the sources shows itself as a constant charge density as follows

$$\rho_c = -2\Lambda. \tag{6.40}$$

6.2.2 Bañados-Teitelboim-Zanelli (BTZ) Black Hole

An interesting example in three dimensions is the BTZ black hole [47], which is a solution when $d = 3$ and $\mathcal{L}_m = 0$ in (6.1) for $\Lambda < 0$ [83]. This black hole solution can be obtained by identifyting points of AdS_3 spacetime by a discrete subgroup of SO(2,2) [109] and its gauge theory copy possesses the same characteristics with AdS_d spacetime with $d \geq 4$. Its KS form [110] is given around the Minkowski spacetime in spheroidal coordinates

$$\bar{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \frac{r^2}{r^2 + a^2}\mathrm{d}r^2 + (r^2 + a^2)\mathrm{d}\theta^2, \tag{6.41}$$

where a is the rotation parameter. The null vector k_μ is parametrized as

$$k_\mu \mathrm{d}x^\mu = \mathrm{d}t + \frac{r^2}{r^2 + a^2}\mathrm{d}r + a\mathrm{d}\theta, \tag{6.42}$$

and the scalar is given by

$$\phi(r) = 1 + 8GM + \Lambda r^2. \tag{6.43}$$

The corresponding gauge field is given by

$$A_\mu \mathrm{d}x^\mu = \left(1 + 8GM + \Lambda r^2\right)\left[\mathrm{d}t + \frac{r^2}{r^2 + a^2}\mathrm{d}r + a\mathrm{d}\theta\right]. \tag{6.44}$$

Due to the rotation, there is also a magnetic field and the independent components of the field strenght tensor are

$$F_{rt} = 2\Lambda r, \qquad F_{r\theta} = aF_{rt} = 2\Lambda a r. \tag{6.45}$$

The constant charge density corresponding to the BTZ black hole reads

$$\rho_c = -4\Lambda. \tag{6.46}$$

Here, we content ourselves with showing that the constant charge density term appears due to the general property of the deviation tensor (6.29) and refer the reader to [67] for a more detailed discussion.

6.2.3 Schwarzschild-AdS$_4$ Black Hole

Our next example is the Schwarzschild-AdS$_4$ Black Hole which is a solution with $d = 4$ and $\mathcal{L}_m = 0$ in (6.1). In [47], it was studied around AdS$_4$ spacetime whose metric in global static coordinates reads

$$\bar{g}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\left[1-\frac{\Lambda r^2}{3}\right]\mathrm{d}t^2 + \left[1-\frac{\Lambda r^2}{3}\right]^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right), \quad (6.47)$$

and the null vector and the scalar are given by

$$k_\mu\mathrm{d}x^\mu = \mathrm{d}t + \left[1-\frac{\Lambda r^2}{3}\right]^{-1}\mathrm{d}r, \qquad \phi(r) = \frac{2M}{r}. \quad (6.48)$$

The gauge field

$$A_\mu\mathrm{d}x^\mu = \frac{2M}{r}\left[\mathrm{d}t + \left[1-\frac{\Lambda r^2}{3}\right]^{-1}\mathrm{d}r\right], \quad (6.49)$$

has the field strenght tensor with the following non-zero component

$$F_{rt} = -\frac{2M}{r^2}. \quad (6.50)$$

We obtain vacuum solutions of (6.34) since the background is chosen to be of constant-curvature, which implies $\rho_c = 0$ and there is no contribution from the matter fields ($\rho_m = 0$).

The solution can also be written around the Minkowski spacetime

$$\bar{g}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right), \quad (6.51)$$

with the null vector and the scalar defined as

$$k_\mu\mathrm{d}x^\mu = dt + dr, \qquad \phi(r) = \frac{2M}{r} + \frac{\Lambda r^2}{3}. \quad (6.52)$$

The gauge field now becomes

$$A_\mu dx^\mu = \left[\frac{2M}{r} + \frac{\Lambda r^2}{3}\right](\mathrm{d}t + \mathrm{d}r), \quad (6.53)$$

with the field strenght tensor

$$F_{rt} = -\frac{2M}{r^2} + \frac{2\Lambda r}{3}. \quad (6.54)$$

This time, in the gauge theory source, the only contribution comes from the cosmological constant as

$$\rho_c = -2\Lambda. \quad (6.55)$$

6.2.4 Reissner-Nordström-AdS$_4$ Black Hole

In order to see the effect of the matter coupling, we now consider Reissner-Nordström-AdS$_4$ black hole. The matter part of the action is

$$\mathcal{L}_m = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu}, \tag{6.56}$$

with contribution to the trace-reversed equations

$$\tilde{T}_{\mu\nu} = \frac{1}{2} f_{\mu\alpha} f_\nu{}^\alpha - \frac{1}{8} g_{\mu\nu} f_{\alpha\beta} f^{\alpha\beta}. \tag{6.57}$$

When the metric is written in the KS form around AdS$_4$ spacetime (6.47) with the null vector given in (6.48), the scalar function reads [47]

$$\phi(r) = \frac{2M}{r} - \frac{Q^2}{4r^2}, \tag{6.58}$$

where M and Q are the mass and the charge of the black hole respectively. The gauge field becomes

$$A_\mu = \left[\frac{2M}{r} - \frac{Q^2}{4r^2}\right] \left[\mathrm{d}t + \left[1 - \frac{\Lambda r^2}{3}\right]^{-1} \mathrm{d}r\right], \tag{6.59}$$

which leads to the field strength tensor

$$F_{rt} = -\frac{2M}{r^2} + \frac{Q^2}{2r^3}. \tag{6.60}$$

While the constant curvature background implies no constant charge density ($\rho_c = 0$), the matter field produces the following static charge density

$$\rho_m = \frac{Q^2}{2r^4}, \qquad v^\mu = \delta^\mu{}_0. \tag{6.61}$$

One should note that our formalism gives the modification to the Poisson's equation and the source as

$$\mathcal{Z} = \Lambda \frac{Q^2 - 4Mr^2}{3r^2}, \tag{6.62}$$

$$j = \frac{Q^2 (\Lambda r^2 - 3)}{6r^2}, \tag{6.63}$$

and one obtains the standard form given in (6.34-6.35) only after simplifications.

When written around the Minkowski spacetime (6.51) with the null vector (6.52), the scalar function is given by

$$\phi(r) = \frac{2M}{r} - \frac{Q^2}{4r^2} + \frac{\Lambda r^2}{3}, \tag{6.64}$$

and the gauge field is

$$A_\mu = \left[\frac{2M}{r} - \frac{Q^2}{4r^2} + \frac{\Lambda r^2}{3}\right](\mathrm{d}t + \mathrm{d}r)\,, \tag{6.65}$$

with the field strength tensor

$$F_{rt} = -\frac{2M}{r^2} + \frac{Q^2}{2r^3} + \frac{2\Lambda r}{3}. \tag{6.66}$$

In addition to the static charge density ρ_m due to the matter part of the Lagrangian, the constant charge density is produced by the non-zero deviation of the Minkowski background (6.29), which are given by

$$\rho_c = -2\Lambda, \qquad \rho_m = \frac{Q^2}{2r^4}. \tag{6.67}$$

6.3 Lifshitz Black Holes

So far, we studied metrics that can be written in the KS form around a maximally symmetric background and presented the differences that arise due to the deviation if the Minkowski spacetime from a constant-curvature spacetime, which are a constant charge density in the source and correspondingly, electric and magnetic (if the black hole rotates) fields that linearly increase with the radial coordinate r. As an example of a solution with a curved background, in this section, we will consider the Lifshitz black hole in d-dimensions, whose metric reads

$$ds^2 = L^2\left[-r^{2z}h(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2 h(r)} + r^2\sum_{i=1}^{d-2}\mathrm{d}x_i^2\right], \tag{6.68}$$

where the function $h(r)$ has a single zero at a finite value of r and, $h(r \to \infty) = 1$. Asymptotically, the metric takes the form

$$\mathrm{d}s^2\big|_{r\to\infty} = L^2\left[-r^{2z}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + r^2\sum_{i=1}^{d-2}\mathrm{d}x_i^2\right], \tag{6.69}$$

which is the Lifshitz spacetime. In this form, it is apparent that it describes an asymptotically Lifshitz black hole with a planar horizon. With the following coordinate transformation[3],

$$\mathrm{d}t \to \mathrm{d}t + \alpha\,\mathrm{d}r, \qquad \alpha = \frac{h(r)-1}{h(r)}\,r^{-(z+1)}, \tag{6.70}$$

[3] We were informed that the KS form of the Lifshitz black hole was first obtained through this transformation in [111].

one can write the metric in the KS form where the background is the Lifshitz spacetime with the metric

$$\bar{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = L^2 \left[-r^{2z}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + r^2 \sum_{i=1}^{d-2} \mathrm{d}x_i^2 \right], \tag{6.71}$$

The null vector and the scalar are given by

$$k_\mu \mathrm{d}x^\mu = \mathrm{d}t + \frac{1}{r^{z+1}}\mathrm{d}r, \qquad\qquad \phi(r) = L^2 \left[1 - h(r)\right] r^{2z}. \tag{6.72}$$

Note that, for $z = 1$, the background metric becomes the AdS spacetime in Poincare coordinates. For $z > 1$, the background metric is not maximally symmetric and the deviation tensor will give a non-trivial contribution. The Ricci tensor for the background metric reads

$$\bar{R}^\mu{}_\nu = \mathrm{diag}\left[-\frac{z\,(z+d-2)}{L^2}, -\frac{z^2+d-2}{L^2}, -\frac{z+d-2}{L^2}, -\frac{z+d-2}{L^2} \right], \tag{6.73}$$

which reduces to that of AdS spacetime (2.47) when $z = 1$. The relevant part is still a constant given by

$$\bar{R}^\mu{}_0 = -\frac{z\,(z+d-2)}{L^2}\delta^\mu{}_0, \tag{6.74}$$

which leads to the following background contribution to the gauge theory source

$$\Delta^\mu(\text{Lifshitz}) = -\left[\frac{z\,(z+d-2)}{L^2} + \frac{2\Lambda}{d-2} \right] \delta^\mu{}_0, \tag{6.75}$$

and, as a result, the following constant charge density

$$\rho_c = -2 \left[\frac{z\,(z+d-2)}{L^2} + \frac{2\Lambda}{d-2} \right]. \tag{6.76}$$

After this general discussion, we will study two different realizations of the Lifshitz black hole with different matter couplings.

6.3.1 Lifshitz Black Hole from a Massless Scalar and a Gauge Field

The first solution that we consider is obtained by the following coupling of a massless scalar to a gauge field [112]

$$\mathcal{L}_m = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{4}e^{\lambda\varphi} f_{\mu\nu} f^{\mu\nu}, \tag{6.77}$$

whose contribution to the trace-reversed equations is

$$\widetilde{T}_{\mu\nu} = \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi + \frac{1}{2}e^{\lambda\varphi} f_{\mu\alpha} f_\nu{}^\alpha - \frac{1}{4(d-2)} g_{\mu\nu} e^{\lambda\varphi} f_{\alpha\beta} f^{\alpha\beta}. \tag{6.78}$$

The equations for the matter fields are

$$\partial_\mu \left(\sqrt{-g} e^{\lambda\varphi} f^{\mu\nu}\right) = 0, \tag{6.79}$$

$$\partial_\mu \left(\sqrt{-g}\partial^\mu \varphi\right) - \frac{\lambda}{4}\sqrt{-g} e^{\lambda\varphi} f_{\mu\nu} f^{\mu\nu} = 0. \tag{6.80}$$

The Lifshitz black hole is a solution in this theory with the following metric function [113]

$$h(r) = 1 - \frac{r_+^{z+d-2}}{r^{z+d-2}}, \qquad z \geq 1, \tag{6.81}$$

provided that the matter fields and the cosmological constant are given by

$$\begin{aligned} f_{rt} &= q\, e^{-\lambda\varphi} r^{z-d+1}, \qquad & e^{\lambda\varphi} = r^{\lambda\sqrt{2(z-1)(d-2)}}, \\ \lambda^2 &= \frac{2(d-2)}{z-1}, \qquad & q^2 = 2L^2(z-1)(z+d-2), \\ \Lambda &= -\frac{(z+d-3)(z+d-2)}{2L^2}. \end{aligned} \tag{6.82}$$

For $z = 1$, the matter fields vanish and one obtains the Schwarzschild-AdS black hole with a planar horizon. By using the coordinate transformation (6.70), the metric can be put in the KS form around the Lifshitz background (6.71) with the null vector and the scalar given in (6.72). The explicit form of the scalar for the metric function (6.81) reads

$$\phi(r) = \frac{L^2 r_+^{z+d-2}}{r^{d-z-2}}, \tag{6.83}$$

The corresponding gauge field is

$$A_\mu \mathrm{d}x^\mu = \frac{L^2 r_+^{z+d-2}}{r^{d-z-2}} \left[\mathrm{d}t + \frac{1}{r^{z+1}}\mathrm{d}r\right], \tag{6.84}$$

and with the following non-zero component of the field strength tensor

$$F_{rt} = -\frac{(d-z+2)L^2 r_+^{z+d-2}}{r^{d-z+1}}. \tag{6.85}$$

Since the matter configuration (6.82) does not change under the coordinate transformation, it can be directly used in the rest of the calculations. It turns out that the contribution from the deviation tensor and the energy-momentum tensor to the gauge theory source are equal to each other and given by

$$\Delta^\mu = \widetilde{T}^\mu = -\frac{(d-3)(z-1)(z+d-2)}{(d-2)L^2}, \tag{6.86}$$

and therefore, the single copy is

$$\bar{\nabla}_\nu F^{\nu\mu} = 0. \tag{6.87}$$

Although we started from a non-vacuum solution, the gauge field given in (6.84) is vacuum solution of the gauge theory. The modification to the Poisson's equation can again be written in terms of the background Ricci scalar and the KS scalar as

$$\mathcal{Z} = \frac{z(z-d+2)}{z^2+(d-2)z+\frac{1}{2}(d-1)(d-2)}\bar{R}\,\phi, \tag{6.88}$$

which leads to the following zeroth copy

$$\bar{\nabla}^2\phi + \frac{z(z-d+2)}{z^2+(d-2)z+\frac{1}{2}(d-1)(d-2)}\bar{R}\,\phi = 0. \tag{6.89}$$

6.3.2 Lifshitz Black Hole from a Massive Vector and a Gauge Field

The second solution that we consider is a charged Lifshitz black hole obtained through the following matter coupling [114]

$$\mathcal{L}_m = -\frac{1}{4}f_{\mu\nu}f^{\mu\nu} - \frac{1}{2}m^2 a_\mu a^\mu - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}, \tag{6.90}$$

where a_μ is a massive vector field with the field strength $f_{\mu\nu} = 2\,\partial_{[\mu}a_{\nu]}$ and $\mathcal{F}_{\mu\nu}$ is the field strength of the gauge field. The matter field equations are

$$\partial_\mu\left(\sqrt{-g}f^{\mu\nu}\right) = m^2\sqrt{-g}\,a^\nu \tag{6.91}$$

$$\partial_\mu\left(\sqrt{-g}\mathcal{F}^{\mu\nu}\right) = 0, \tag{6.92}$$

and the contribution to the trace-revesed equations is given by

$$\widetilde{T}_{\mu\nu} = \frac{1}{2}f_{\mu\alpha}f_\nu{}^\alpha - \frac{1}{4(d-2)}g_{\mu\nu}f_{\alpha\beta}f^{\alpha\beta} + \frac{1}{2}m^2 a_\mu a_\nu + \frac{1}{2}\mathcal{F}_{\mu\alpha}\mathcal{F}_\nu{}^\alpha - \frac{1}{4(d-2)}g_{\mu\nu}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}. \tag{6.93}$$

The charged Lifshitz black hole is a solution with the metric function

$$h(r) = 1 - \frac{q^2}{2(d-2)^2 r^z}, \tag{6.94}$$

for the matter configuration

$$a_t = L\sqrt{\frac{2(z-1)}{z}}h(r)r^z, \qquad \mathcal{F}_{rt} = qLr^{z-d-1}. \tag{6.95}$$

The mass of the vector field, the cosmological constant and the Lifshitz exponent should also be fixed as follows

$$m = \sqrt{\frac{(d-2)z}{L^2}}, \qquad \Lambda = -\frac{(d-3)z+(d-2)^2+z^2}{2L^2}, \qquad z = 2\,(d-2)\,. \tag{6.96}$$

The metric can be put in the KS form through the coordinate transformation (6.70) around the Lifshitz background (6.71) with the null vector and the scalar given in (6.72). The explicit form of the scalar for the metric function (6.81) reads

$$\phi(r) = \frac{L^2 q^2 r^z}{2\,(d-2)^2}. \tag{6.97}$$

The single copy gauge field and the non-zero component of the field strength tensor are

$$A_\mu \mathrm{d}x^\mu = \frac{L^2 q^2 r^z}{2\,(d-2)^2}\left[\mathrm{d}t + \frac{1}{r^{z+1}}\mathrm{d}r\right], \tag{6.98}$$

$$F_{rt} = \frac{z L^2 q^2 r^{z-1}}{2\,(d-2)^2}. \tag{6.99}$$

This time, the coordinate transformation (6.70) affects the matter configuration (6.95) non-trivially, yielding an additional radial component of the massive vector as follows

$$a_r = \alpha\, a_t. \tag{6.100}$$

The contribution from the deviation tensor and the energy-momentum tensor to the gauge theory source are this time given by

$$\Delta^\mu = -\frac{(z-1)\left[(d-3)\,z + (d-2)^2\right]}{(d-2)\,L^2}\delta^\mu_{\ 0}, \tag{6.101}$$

$$\widetilde{T}^\mu = \Delta^\mu + \frac{q^2}{2L^2 r^z}\delta^\mu_{\ 0}, \tag{6.102}$$

Similar to the previous example, the constant charge density contribution from the deviation tensor (6.101) again disappears, however, this time the contribution from the energy-momentum tensor (6.102) has an additional term, which leads to a non-vacuum solution. The single copy is

$$\bar{\nabla}_\nu F^{\nu\mu} = J^\mu, \qquad J^\mu = -\frac{q^2}{L^2 r^z}\delta^\mu_{\ 0}. \tag{6.103}$$

The modification to Poisson's equation in this case can be written as

$$\mathcal{Z} = \frac{z^2}{z^2 + (d-2)z + \frac{1}{2}(d-1)(d-2)}\bar{R}\,\phi, \tag{6.104}$$

which yields

$$\bar{\nabla}^2\phi + \frac{z^2}{z^2 + (d-2)z + \frac{1}{2}(d-1)(d-2)}\bar{R}\,\phi = j, \qquad j = q^2 r^z. \tag{6.105}$$

6.4 Summary and Discussions

In this chapter, extending the construction of [47], we gave a formulation of the classical double copy with a generic, curved background spacetime. Apart from obtaining solutions of Maxwell's theory defined on curved backgrounds, our formulation makes the effect of the background spacetime on the gauge theory source much more transparent through the deviation tensor that we defined in (6.13). For an arbitrary Killing vector of the background and the full metric, the result is given in (6.17-6.18). Choosing a flat background for a solution with a non-zero cosmological constant yields a constant charge density filling all space in the gauge theory due to the general property presented in (6.29). The effect disappears when the background is chosen to be a constant curvature spacetime, which can be explained due to the vanishing of the deviation tensor for a suitably chosen cosmological constant (6.33). Furthermore, we studied two different realizations of the Lifsthiz black hole, whose background is not maximally symmetric. While the contribution to the gauge theory source again turns out to be a constant as described in (6.75-6.76), it is removed by the matter fields in the gravity side, yielding a vacuum solution in one case.

CHAPTER 7

REGULARIZED WEYL DOUBLE COPY

An alternative version of classical double copy is the Weyl double copy (WDC) [39, 44], relying on some previous results on Petrov type D and N spacetimes admitting a Killing spinor [115–117], according to which a particular relation between the completely symmetric Weyl spinor Ψ_{ABCD} corresponding to a type D or N vacuum solution of 4d Einstein's equations (with $\Lambda = 0$) and the symmetric field strenght spinor f_{AB} corresponding to a solution of Maxwell's equation defined on the curved spacetime characterized by the Weyl spinor Ψ_{ABCD}. Furthermore, one can also obtain a scalar field that satisfies the Poisson's equation on the same curved background spacetime. When the curved spacetime metric is given in the KS coordinates, the scalar and gauge fields have the identical equations when defined on the flat background spacetime. As a result, a spinorial version of the double copy is obtained. It is possible to derive this version of the double copy from twistor theory [74] and develop a deeper understanding. For example, when the momentum space origins of the double copy are considered, it is already quite surprising to have a local relation in position space. Using twistor techniques for type D spacetimes, this was shown to be a consequence of the very special properties in the momentum space, which could be possible only for algebraically special spacetimes [118].

In a recent letter [40], although not covered in the original theorems which inspired the construction of the WDC, it was shown that, sources in the gravity side might be handled term-by-term by considering a sum of scalar-gauge theories and the sourced WDC (SWDC) takes the following form

$$\Psi_{ABCD} = \sum_{i=1}^{N} \frac{1}{S_{(i)}} f^{(i)}_{(AB} f^{(i)}_{CD)}. \tag{7.1}$$

Here, the $i = 1$ term represents the scalar-gauge theories corresponding to the vacuum solution $\bar{\nabla}_\nu F^{\nu\mu}_{(i=1)} = 0$ and other terms satisfy a sourced Maxwell's equation $\bar{\nabla}_\nu F^{\nu\mu}_{(i>1)} = J^{\mu}_{(i>1)}$. In other words, for each term in the metric addition to the vacuum part, one can find a sourced Maxwell's equation with a source proportional to the one described by the KSDC procedure. The complex scalar S satisfies the Poisson's equation and, in general, a linear combination of its real and imaginary parts corresponds to the zeroth copy ϕ in KSDC. So far, this procedure has been implemented to the Kerr-Newman black hole solution, the charged C-metric, and the most general type D solution of Einstein-Maxwell theory [85]. This proposal can be tested in a non-trivial way by solving gauge field equations term-by-term according to the KS formulation and one can also demonstrate that one loses the connection to the KSDC if a single product of spinor fields are used.

For type D metrics, in a suitable spinor basis $\{o_A, \iota_B\}$, the Weyl spinor is given by $\Psi_{ABCD} = 6 o_{(A} o_B \iota_C \iota_{D)} \sum_i \Psi_2^{(i)}$, where the only non-zero Weyl scalar is $\Psi_2^{(i)}$ which is obtained for the part of the metric function corresponding to the i-th source [1]. In this chapter, since many properties of the single copy solution take a simple form we will deal with only static black hole solutions. In our formalism, the single copy field strength spinor reads $f_{AB} = Z\, o_{(A} \iota_{B)}$ where Z is real and directly related to the radial single copy electric field $e = F_{rt}$ for these static solutions. On the other hand, the generally complex field S has only real part, and therefore, can be identified with the zeroth copy ϕ. As a result, this relatively simple structure suffices to check the following consistency condition

$$\Psi_2^{(i)} \propto \frac{Z^2_{(i)}}{\phi_{(i)}}, \tag{7.2}$$

term-by-term for the consistency of the proposal of [40] for the SWDC with the KSDC.

When this procedure is applied to a metric of the following form with a flat background metric

$$\mathrm{d}s^2 = -h(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{h(r)} + r^2 \mathrm{d}\Omega^2 \tag{7.3}$$

$$h(r) = 1 + \sum_n \frac{a_n}{r^n} \tag{7.4}$$

$$\mathrm{d}\bar{s}^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2, \tag{7.5}$$

[1] For a detailed explanation, see the Section 2.4

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$ as first done in [40], one finds

$$\phi = -\sum_n \frac{a_n}{r^n}, \qquad e = \sum_n \frac{n a_n}{r^{n+1}} \propto Z, \tag{7.6}$$

$$\Psi_2 = \sum_n \frac{(n+1)(n+2)a_n}{12\, r^{n+2}}. \tag{7.7}$$

We see that the consistency condition (7.2) is satisfied for asymptotically flat solutions ($n > 0$). One has $a_1 = -2M$, $a_2 = \frac{Q^2}{4}$ for the Reissner-Nordström (RN) solution of Einstein-Maxwell theory, where M and Q are black hole's mass and electric charge respectively, and the scalar-gauge theory equations are satisfied term-by-term exactly as described by the SWDC in (7.1).

7.1 Some Problems in Weyl Double Copy

We are now in a position to discuss some problems regarding the proposal for the SWDC. We would like to emphasize that we do not claim that there is a pathology in the procedure as given in [40]; however, there are issues that needs to be resolved for matching the SWDC with more general results in the KS side of the double copy.

First of all, since the Weyl tensor $W_{\mu\nu\rho\sigma}$ transforms homogeneously under a conformal transformation of the metric ($\tilde{g}_{\mu\nu} = e^{\psi} g_{\mu\nu}, \tilde{W}_{\mu\nu\rho\sigma} = e^{\psi} W_{\mu\nu\rho\sigma}$), the Weyl tensor and the Weyl spinor vanishes for conformally flat spacetimes. Therefore, when one applies the same prescription for the SWDC to a term in the metric function which is conformally flat when considered alone, the procedure seems to break down since the corresponding contribution to the Weyl spinor vanishes and obviously cannot match a a non-zero relevant contribution in the KSDC. Such a phenomenon cannot be observed when working with asymptotically flat spacetimes since it corresponds to $n = -1, -2$ in (7.4) for which the contribution to the Weyl scalar Ψ_2 vanishes as can be seen in (7.7). Hence, the resolution of this issue is crucial for the correct formulation of SWDC for type D solutions that are not asymptotically flat.

A related problem appears for the RN-AdS$_4$ black hole solution of EM theory with a negative cosmological constant described by the Lagrangian $\mathcal{L} = \sum_i \mathcal{L}_i$ with $\mathcal{L}_{(i=1)} = R$, $\mathcal{L}_{(i=2)} = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$, and $\mathcal{L}_{(i=3)} = -2\Lambda$. The metric function in (7.4) reads

$$h(r) = 1 - \frac{2M}{r} + \frac{Q^2}{4r^2} - \frac{\Lambda r^2}{3}. \tag{7.8}$$

According to [85], the single copy solution should be defined in a “suitable flat space limit", which is the AdS_4 spacetime in the global static coordinates with the line element

$$d\bar{s}^2 = -\left[1 - \frac{\Lambda r^2}{3}\right] dt^2 + \left[1 - \frac{\Lambda r^2}{3}\right]^{-1} dr^2 + r^2 d\Omega^2. \tag{7.9}$$

This is a natural choice that we obtain when the mass M and the charge Q are set to zero. Applying the machinery of the SWDC, we find

$$\phi = \frac{2M}{r} - \frac{Q^2}{4r^2}, \qquad e = -\frac{2M}{r^2} + \frac{Q^2}{2r^3} \propto Z, \tag{7.10}$$

$$\Psi_2^{(i=1)} = -\frac{M}{r^3}, \qquad \Psi_2^{(i=2)} = \frac{Q^2}{r^4}, \qquad \Psi_2^{(i=3)} = 0, \tag{7.11}$$

where the single copy properties (7.10) were first given in [47]. We see that although the contribution from the conformally flat part of the metric that arises due to the cosmological constant vanishes, the consistency condition (7.2) is still satisfied since the cosmological constant has no effect on the properties of the single copy solution (7.10) defined on the AdS_4 spacetime.

On the other hand, the same solution can be put into the KS form with a flat background metric given in (7.5). This time, the single copy properties get a contribution from the cosmological costant as follows [48]

$$\phi = \frac{2M}{r} - \frac{Q^2}{4r^2} + \frac{\Lambda r^2}{3}, \qquad e = -\frac{2M}{r^2} + \frac{Q^2}{2r^3} + \frac{2\Lambda r}{3} \propto Z, \tag{7.12}$$

while the Weyl spinor takes the same form characterized by the Weyl scalars given in (7.11). The consistency condition (7.2) is obviously not satisfied by the contribution to the metric function from the cosmological constant, which is just the $n = -2$ term in (7.4). For a complete equivalence of the KSDC and the SWDC, one would expect to cover both single copies in the SWDC.

One final problem is that some solutions to matter coupled GR do not have a vacuum part, i.e., when the matter coupling is turned off there remains no solution that both carries the symmetries of the ansatz that is used to derive the solutions and satisfies the field equations at the same time. It is desirable to understand whether, and if yes how, the SWDC can be realized in such cases

7.2 Resolution by Regularization

For the resolution of the first two problems, we are inspired by a simple observation for the RN-AdS$_4$ black holes: Although the standard procedure fails when the background metric is taken to be flat, the radial dependence of the Weyl scalar is still correct ($\Psi_2^{(i=3)} \propto$ constant apart from the $(n+2)$ factor which produces zero) such that the consistency condition (7.2) is satisfied with $\phi_{(i=3)} \propto r^2$ and $e_{(i=3)} \propto r$. Motivated by this, we propose a three-step regularization procedure for the SWDC such that it will still work when one has a conformally flat part in the metric function that has a non-trivial effect on the single copy solution: Let us say the conformally flat part is of the form $\frac{a_{n*}}{r^{n*}}$ such that for $n = n_*$, the contribution to the Weyl scalar Ψ_2^* vanishes. To cure this, one should proceed as follows:

1. Take the problematic term as $\frac{a_{n*}}{r^n}$, i.e., use an arbitrary exponent n instead of n_*.

2. Let $a_{n_*} \to \frac{a_{n*}}{n-n_*}$ and calculate Ψ_2^* using this coefficient.

3. Set $n = n_*$ at the end.

This way, the $(n - n_*)$ term in the Weyl scalar Ψ_2^* is removed in a controlled manner and a non-zero contribution is obtained. For a metric of the form (7.3), the consistency condition (7.2) is automatically satisfied and the problematic term in the RN-AdS$_4$ black hole with a flat background metric corresponds to $n_* = -2$. Although we are not aware of a solution of matter coupled GR with an $n_* = -1$ term, such a problem would be easily solved.

One might rightfully argue that this form of the metric ($g_{tt}g_{rr} = -1$) is too simple to check the validity of the regularized Weyl double copy (RWDC). Therefore, we put it to the test with a set of examples that requires the most general formulation of the KSDC summarized at the beginning, i.e., the Lifshitz black holes. As a by-product, they will show us how the third problem must be handled.

7.3 Lifshitz black holes

In order to study a scenario as general as possible, we will consider the following ansatz for Lifshitz black holes with different horizon topologies

$$\mathrm{d}s^2 = L^2 \left[-r^{2z} h(r) \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2 h(r)} + r^2 \mathrm{d}\Sigma_k^2 \right], \tag{7.13}$$

where $k = +1, 0, -1$ correspond to S^2, E^2 and H^2 respectively, and z is the dynamical exponent. When $z \neq 1$, one has an anisotropic scaling of the time coordinate in the boundary field theory and these solutions can be used to probe the properties of finite temperature non-relativistic systems holographically when $k = 0$ [80] ($z = 1$ corresponds to the relativistic case). When written in the KS coordinates, the background spacetime is the Lifsthiz spacetime whose line element can be found by setting $h(r) = 1$ in (7.13). For a generic metric function as in (7.4), one has

$$\phi = -L^2 \sum_n \frac{a_n}{r^{n-2z}}, \quad e = L^2 \sum_n \frac{(n-2z)a_n}{r^{n-2z+1}} \propto r^{z-1} Z, \tag{7.14}$$

$$\Psi_2 = -\frac{k}{6L^2 r^2} + \frac{z(z-1)}{6L^2} + \sum_n \frac{(n-z)(n-2z+2)\, a_n}{12 L^2 r^n}. \tag{7.15}$$

First of all, since the Lifshitz black holes cannot be obtained as vacuum solutions, all the non-zero a_n's here can only be generated by matter coupling. Therefore; there is no analogue of the mass term in the RN-AdS$_4$ black hole and no vacuum part of the metric. As a result; in the Weyl scalar Ψ_2 in (7.15), there is a term which is independent of a_n's and has no direct meaning regarding the properties of the single copy. Note that it vanishes for $z = 1$ and $k = 0$, which corresponds to the AdS$_4$ spacetime in Poincaré coordinates that is a vacuum solution. Other than this term, the consistency condition (7.2) is satisfied with the data given above provided that a regularization is employed for $n_* = z$ and $n_* = 2z - 2$. In Table 7.1, we provide various examples.

Table 7.1: Matter couplings, metric functions, k and z for three different Lifshitz black hole solutions considered in the text. φ is a scalar field. $\mathcal{F}_{\mu\nu} = 2\partial_{[\mu}a_{\nu]}$ and $\mathcal{G}_{\mu\nu}$ are two-form fields. $\mathcal{H}_{\mu\nu\tau} = \partial_{[\mu}B_{\nu\tau]}$ is a three-form field.

I	$\mathcal{L}_m = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{4}e^{\lambda\varphi}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$	$h(r) = 1 - \left(\frac{r_+}{r}\right)^{z+2}$	$k = 0,\, z > 1$
II	$\mathcal{L}_m = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2}m^2 a_\mu a^\mu - \frac{1}{4}\mathcal{G}_{\mu\nu}\mathcal{G}^{\mu\nu}$	$h(r) = 1 - \frac{q^2}{8r^z}$	$k = 0,\, z > 1$
III	$\mathcal{L}_m = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{12}\mathcal{H}_{\mu\nu\tau}\mathcal{H}^{\mu\nu\tau} - C\epsilon^{\mu\nu\alpha\beta}B_{\mu\nu}\mathcal{F}_{\alpha\beta}$	$h(r) = 1 + \frac{k}{2r^2}$	$k \neq 0,\, z = 2$

The Examples in the Table 7.1 are from [112, 114, 119] respectively. Example I is the double copy of the vacuum solution of Maxwell's equations on the Lifshitz spacetime, which is the only known example in the KSDS where a non-vacuum gravity solution is mapped to a vacuum gauge theory solution in $d > 3$ (3D KSDC is a completely different story [67–70]). In Example II, one has a sourced single copy solution and a regularization with $n_* = z$ is required for the agreement of the KSDC and the SWDC. In both of these examples, the event horizon is planar and the KS single copies were obtained in [48]. Example III is a demostration of the equivalence of two formulations of the classical double copy for a naked singularity where constant t and r surfaces are S^2, and a black hole with H^2 horizon. Again, a regularization with $n_* = z = 2$ is required [2].

7.4 Conclusions and Outlook

In this chapter, by introducing a regularization procedure in the SWDC, we have shown how it can be made consistent with the KSDC when a certain part of the metric function in static black hole solutions is conformally flat but still affects the single copy properties nontrivially. Intriguingly, this turns out to be much more than curing a mathematical difficulty in the formulation. Handling this issue is directly related to

[2]For detailed examination of matter Lagrangian, see the Appendix D.

the AdS and the Lifshitz black holes which form the basis of probing the properties of strongly coupled gauge theories at finite temperature holographically, [77–80] which is just another development that we owe to string theory. Since both the double copy and the AdS/CFT correspondence (together with its non-relativistic generalization) originate from string theory, we have a strong expectation that achieving consistency of different formulations of the classical double copy will be an important first step to understand whether the double copy ideas and holography is related in some way. While the gauge theory in the double copy lives in the same number of dimensions as the gravity theory, the gauge theory dual in holography is defined on the conformal boundary, which has one less dimension. Although one cannot point out a relation for now, the success of the WDC in capturing the asymptotic structure of spacetimes [120, 121] together with our regularization procedure might be helpful in this regard.

We would like to note that, in all the examples considered here, the constant a_{n_*} is proportional to either the cosmological constant Λ or the Newton's constant G which is hidden in the event horizon radius r_+. This makes one to think that our regularization procedure might have a field theoretical origin, which, we believe, deserves further study.

In our analysis, we have reached another important finding: When a solution does not have a vacuum part, then, at the left-hand-side of the SWDC formula (7.1), the leading term loses its meaning in the interpretation of the single copy. The Lifshitz black holes are the first kind of such examples but we expect the same behaviour in different type of solutions.

Despite the success of the RWDC that we have presented here, we would like to mention that the study of some solutions of $\mathcal{N} = 0$ supergravity, the effective field theory emerging in the low energy limit of closed string theory, by using twistor methods has recently led to a double copy formula different than the one considered here [122]. For a better understanding, more general and also different type of examples including rotating black holes and wave solutions should be studied.

CHAPTER 8

CONCLUSION

In this thesis, we have focused on extensions of the classical double copy both in the Kerr-Schild and the spinorial Weyl formulations.

Chapter 2 presents essential background material crucial for understanding the thesis. It covers topics such as degree of freedom counting and Newtonian potential in 3D gravity, important solutions in 3D gravity, maximally symmetric spacetimes, and the deviation tensor that is employed in the most general formulation in the KSDC. Additionally, it explores spinor formalism and Petrov classification in $d = 4$.

In chapter 3, we explain why the KSDC in 3D seems to be problematic and study the proposal given in [67] for the KSDC of the Coulomb solution where the authors claim that one can employ a ghost free scalar or a spacelike fluid. We show that one can get rid of the ghost sign of the scalar by taking the EH term with a wrong sign, which is not problematic since there are no propagating degrees of freedom. We also consider a rotating generalization of the solution which is naturally implied by the KS coordinates and show that it cannot be sourced by a free scalar field and the use of a spacelike fluid becomes compulsory. The effect of the cosmological constant is also discussed.

Chapter 4 provides an alternative matter coupling for the double copy of the Coulomb solution in 3D with the help of the on-shell duality of p-forms. Instead of a scalar field, which does not vanish at infinity when it sources the correct gravitational solution, one can employ a gauge field with a ghost sign which can again be moved to the EH term. In this case, the gauge field takes the usual Coulomb form and a better behaviour at infinity is achieved. This matter coupling has an additional advantage that when a cosmological constant is introduced it provides a natural connection to the charged BTZ black hole.

After obtaining the two possible matter couplings that give rise to the double copy of the Coulomb solution in 3D, we study certain generalizations of them with the aim of understanding the regularity structure of the solutions in both sides of the double copy in chapter 5. While for the vector coupling we study the well-known Born-Infeld generalization, we use the scalar-tensor theory discovered in [101] that admits regular black hole solutions for the generalization of the free scalar. We use these scenarios to investigate the connection between horizons on the gravity side and electric fields on the gauge theory side, which was previously observed in four dimensions.

Chapter 6 is devoted to the most general formulation of the KSDC where no simplifying assumption about the background metric is made. We show that the gauge theory source is affected by a curvature term that characterizes the deviation of the background spacetime from a constant curvature spacetime. We demonstrate this effect explicitly by studying gravitational solutions with non-zero cosmological constant. We show that, when the background is flat, the constant charge density filling all space in the gauge theory that has been observed in previous works is a consequence of this curvature term. As an example of a solution with a curved background, we study the Lifshitz black hole with two different matter couplings. The curvature of the background, i.e., the Lifshitz spacetime, again yields a constant charge density; however, unlike the previous examples, it is canceled by the contribution from the matter fields. For one of the matter couplings, there remains no additional non-localized source term, providing an example for a non-vacuum gravity solution corresponding to a vacuum gauge theory solution in arbitrary dimensions.

In chapter 7, we propose a regularization procedure in the sourced Weyl double copy, the spinorial version of the classical double copy, such that it matches much more general results in the Kerr-Schild version. In the regularized Weyl double copy, the AdS and the Lifshitz black holes, which form the basis of the study of strongly coupled gauge theories at finite temperature through the AdS/CFT correspondence and its non-relativistic generalization, become treatable. We speculate that this might prove useful for finding out a relation between the classical double copy and holograpy.

We would like to note that all the results presented in chapters 3,4,5,6,7 are based on [48, 68–70, 123].

In summary, the extension of the original formulation of the KSDC in [42] was extended to three-spacetime dimensions through different matter couplings, and also to spacetimes with a curved background metric. Additionally, an extension of the WDC that is applicable to the holographically viable spacetimes that is compatible with the most general formulation of the KSDC was given.

The most natural next step for all this progress is to look for a connection between holography and the classical double copy. Considering the fact that the precursor of the AdS/CFT correspondence is the result of Brown and Hanneaux [124] the asymptotic symmetry algebra of 3D GR with a negative cosmological constant is two copies of Virasoro algebra corresponding to symmetries of the 2d conformal field theories, an understanding of the asymptotic data in the context of Classical double copy should be very useful. Studies along this direction in 4D can be found in [125, 126] and the results presented in these works, when supplemented with the regularization procedure described in Chapter 7, are expected to help in establishing such a connection. In order to benefit the full power of the Brown-Hanneaux result, the spinorial version of the Classical double copy in 3D, the Cotton double copy, and its application to the asymptotic structure of 3D spacetimes is needed.

The current view in the literature is that the Cotton double copy is applicable to only type N spacetimes, excluding the black hole solutions which are of type D. However; this is due to the fact that the Cotton tensor, and therefore the Cotton spinor vanishes for AdS_3 spacetime or any spacetime which is locally isomorphic to AdS_3 such as the BTZ black hole. Therefore, an important problem for the future research is to look for a similar regularization procedure in the Cotton double copy. Another approach to search for a connection to the holography would be a formulation of the Classical double copy at the action level. Since the current formulations are based the considerations of the solutions to gravitational field equations at a particular gauge, a covariant formulation has the potential to yield many interesting results apart from relating the double copy to holography.

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APPENDICES

A. No-Hair Theorem for Free Scalar Fields

As we have seen in Sec. 3.2, the scalar hair can be obtained by a different choice of the scalar kinetic term than [67]. Here, we show explicitly why this is possible by reviewing the formulation of the no-hair theorem for free scalar fields [127–129] . Any static metric can be written as

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + h_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{A.1}$$

where $N = N(x^i)$ and $h_{ij} = h_{ij}(x^i)$. Then, assuming no time dependence, the equation for the free scalar field (3.25) becomes

$$\Box\varphi(x^i) = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\varphi\right). \tag{A.2}$$

With the help of relations

$$\begin{aligned}\sqrt{-g} &= N\sqrt{h},\\ \frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}\,h^{ij}\partial_j\varphi\right) &= \mathcal{D}^i\mathcal{D}_i\varphi,\end{aligned} \tag{A.3}$$

where $\mathcal{D}_i$ is the covariant derivative with respect to the spatial metric h_{ij}, it can be written as

$$\Box\varphi(x^i) = \frac{1}{N}\mathcal{D}^i N\mathcal{D}_i\varphi + \mathcal{D}^i\mathcal{D}_i\varphi\,. \tag{A.4}$$

Multiplying by $N\varphi$ and integrating over the spatial region Σ between the event horizon and infinity yields

$$0 = \int_\Sigma \mathrm{d}^2x\sqrt{h}\,\left[\varphi\mathcal{D}^i N\mathcal{D}_i\varphi + N\varphi\mathcal{D}^i\mathcal{D}_i\varphi\right], \tag{A.5}$$

which, after integrating the first term by parts, becomes

$$0 = \oint_{\partial\Sigma} \mathrm{d}S_i \, N\varphi\mathcal{D}^i\varphi - \int_{\Sigma} \mathrm{d}^2x\sqrt{h}\, N\mathcal{D}_i\varphi\mathcal{D}^i\varphi. \tag{A.6}$$

The surface integral consists of integrations over the event horizon $\partial\Sigma_{\text{h}}$ and the spatial infinity $\partial\Sigma_{\text{inf}}$ as

$$\oint_{\partial\Sigma} \mathrm{d}S_i \, N\varphi\mathcal{D}^i\varphi = \int_{\partial\Sigma_{\text{h}}} \mathrm{d}S_i \, N\varphi\mathcal{D}^i\varphi + \int_{\partial\Sigma_{\text{inf}}} \mathrm{d}S_i \, N\varphi\mathcal{D}^i\varphi. \tag{A.7}$$

The first term is zero since the function N, by definition, vanishes at the event horizon while the second term is zero provided that the field φ or its derivative $\mathcal{D}_i\varphi$ vanishes at infinity. With this assumption, (A.6) becomes

$$0 = \int_{\Sigma} \mathrm{d}^2x\sqrt{h}\, N\mathcal{D}_i\varphi\mathcal{D}^i\varphi. \tag{A.8}$$

Since the integrand is positive definite, one must have $\mathcal{D}_i\varphi = 0$ and therefore $\varphi =$ constant throughout the entire region Σ. In the usual formulation of the no-hair theorem, one takes $\varphi = 0$ at the spatial infinity $\partial\Sigma_{\text{inf}}$, which is a reasonable assumption for physical fields. This implies that the constant should be set to zero, and hence $\varphi = 0$, i.e., no scalar can be present in the region Σ.

For our analysis, two things are important: First, in the formulation of the no-hair theorem, there is no reference to the sign of the kinetic term of the scalar field in the action since we directly start from its field equation. Therefore, whether it is a ghost or not does not play a role. Second, it becomes possible to obtain a scalar hair because we do not demand $\mathcal{D}_i\varphi = 0$ at infinity. Indeed, one has $\mathcal{D}_i\varphi =$ constant for the static solution.

B. Scalar-Vector Duality as Described in Chapter 4

Note that in our formulation of the scalar-vector duality in Sec. 4.2, we have shown that free field equations in flat spacetime are equivalent and a scalar field which is linear in the azimutal angle ($\varphi = c\,\theta$, $c =$ constant) implies the Coulomb solution for the gauge vector. This can be generalized to curved spacetime as long as the metric is in the KS form (4.4). However; our analysis is limited to the matter equations and the gravity equations have to be checked independently. The action (4.30) that we use to establish the duality leads to a free scalar action with a ghost sign (4.33) and a free Maxwell action with a nonghost sign (4.35), which is enough to show the equivalence of the matter equations. When coupling to the gravitation is considered, in order to obtain a black hole solution with the same physical properties, one needs to have a ghost scalar and a ghost vector (we assume $\zeta_1 = 1$, i.e., the EH term has the "right sign," for simplicity).

Indeed, as shown in [101] recently, the duality can also be formulated such that the gravitational action with the correct sign for the matter coupling is obtained directly. The authors consider theories described by Lagrangians of the form $\mathcal{L}(g^{\mu\nu}, R_{\mu\nu}, \partial_\mu\varphi)$ and study solutions in the following form

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\theta^2, \qquad \varphi = c\,\theta, \qquad c = \text{constant}, \tag{B.1}$$

which includes the solution that we studied in Sec. 4.1 with the metric written in the BL coordinates. It is possible to find theories such that the scalar equations is automatically satisfied, the equation for the metric function gets a nontrivial modification and can be solved analytically. They also show that the same solution can be obtained from a dual Lagrangian of the form $\mathcal{L}_{\text{dual}}(g^{\mu\nu}, R_{\mu\nu}, f_{\mu\nu})$. When the matter fields are related as[1]

$$f_{\mu\nu} = -\sqrt{2\pi}\,\epsilon_{\mu\nu\rho}\frac{\partial\mathcal{L}}{\partial(\partial_\rho\varphi)} \tag{B.2}$$

the dual Lagrangian is given by

$$\mathcal{L}_{\text{dual}} = \mathcal{L} - \frac{1}{2\sqrt{2\pi}}\epsilon^{\mu\nu\rho} f_{\mu\nu}\partial_\rho\varphi, \tag{B.3}$$

[1] We use a different normalization than [101] to obtain the vector kinetic term with coefficient $\frac{1}{8\pi}$, as we have used throughout the text.

where the original Lagrangian $\mathcal{L}$ should also be written in terms of $f_{\mu\nu}$ by using the relation (B.2). In [101], nonminimal matter couplings are used and regular, electrically charged black hole solutions in three dimensions are obtained. In this work, we only have the kinetic terms, which are the simplest possible matter couplings. Starting from the Lagrangian

$$\mathcal{L} = \frac{1}{\kappa^2} R + \frac{1}{2} (\partial\varphi)^2 , \tag{B.4}$$

leads to the following dual Lagrangian

$$\mathcal{L}_{\text{dual}} = \frac{1}{\kappa^2} R + \frac{1}{8\pi} f_{\mu\nu} f^{\mu\nu}. \tag{B.5}$$

Both Lagrangians $\mathcal{L}$ and $\mathcal{L}_{\text{dual}}$ support the solution with the correct Newtonian potential as long as the electric charge q and the constant c are related as

$$c = \frac{q}{\sqrt{2\pi}}, \tag{B.6}$$

which is a consequence of (B.2). Using the relation between the mass parameter M in the scalar case with the constant c (4.16), this implies

$$M = q^2, \tag{B.7}$$

which shows the relation between the coefficient of the logarithmic terms in the metric functions (4.22) and (4.49).

C. Details of the Field Equations of the Scalar Theory in Chapter 5

The field equations of scalar fields φ in (5.48) gives the following energy-momentum tensor and field equations where

$$\Gamma(\varphi) = 1 - \frac{\kappa^2}{2}\sum_{m=0} \beta_m \ell^{2(m+1)} (\partial\varphi)^{2(m+1)} \tag{C.1}$$

$$\tilde{T}_{\mu\nu} = \tilde{T}^{(1)}_{\mu\nu} + \tilde{T}^{(2)}_{\mu\nu} + \tilde{T}^{(3)}_{\mu\nu}. \tag{C.2}$$

The $\tilde{T}^{(1)}_{\mu\nu}$, $\tilde{T}^{(3)}_{\mu\nu}$ and $\tilde{T}^{(3)}_{\mu\nu}$ can be given as

$$\begin{aligned}
\tilde{T}^{(1)}_{\mu\nu} &= \sum_{n=1} \alpha_n (\partial\varphi)^{2(n-1)} \ell^{2(n-1)} \left(n\partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}(n-1)(\partial\varphi)^2\right) \\
\tilde{T}^{(2)}_{\mu\nu} &= \sum_{m=0} \beta_m \ell^{2(m+1)} (\partial\varphi)^{2(m-1)} \big[2(3+2m) R_{\alpha(\mu}\partial_{\nu)}\varphi\partial^\alpha\varphi(\partial\varphi)^2 \\
&\quad + m(3+2m)\partial_\mu\varphi\partial_\nu\varphi R^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - (m+1)(\partial\varphi)^2 \\
&\quad \times \left(\partial_\mu\varphi\partial_\nu\varphi R + g_{\mu\nu} R^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - g_{\mu\nu} R (\partial\varphi)^2\right)\big], \\
\tilde{T}^{(3)}_{\mu\nu} &= \nabla^\alpha\nabla_{(\mu} E_{\nu)\alpha} - \frac{1}{2}\Box E_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\Box E,
\end{aligned} \tag{C.3}$$

with

$$E_{\mu\nu} = \sum_{m=0} \beta_m \ell^{2(m+1)} (\partial\varphi)^{2m} \left[(3+2m)\partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}(\partial\varphi)^2\right]. \tag{C.4}$$

The equation for the scalar field reads

$$\begin{aligned}
0 &= 2\nabla_\mu \sum_{n=1} \left\{ n\alpha_n \ell^{2n-1} (\partial\varphi)^{2(n-1)} \partial^\mu\varphi \right\} - 2\nabla_\mu \sum_{m=0} \big\{ \beta_m \ell^{2(m+1)} (\partial\varphi)^{2(m-1)} \\
&\quad \times \left[m(3+2m)\partial^\mu\varphi R^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi + (3+2m)(\partial\varphi)^2 R^{\mu\alpha}\partial_\alpha\varphi - (m+1) R (\partial\varphi)^2 \partial^\mu\varphi \right] \big\}.
\end{aligned} \tag{C.5}$$

D. Some Details of Example III Considered in Chapter 7

In chapter 7, we introduce a matter Lagrangian for the Lifshitz topological black hole in the table 7.1 as [119]

$$\mathcal{L}_m = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{12}\mathcal{H}_{\mu\nu\tau}\mathcal{H}^{\mu\nu\tau} - C\epsilon^{\mu\nu\alpha\beta}B_{\mu\nu}\mathcal{F}_{\alpha\beta}\,. \tag{D.1}$$

We can derive the energy-momentum tensor and the field equations of the matter fields by taking variation of the matter Lagrangian $\mathcal{L}_m$ with respect to the metric and matter fields respectively. The energy-momentum tensor reads

$$T_{\mu\nu} = \frac{1}{2}\left(\mathcal{F}_{\mu\tau}\mathcal{F}^{\tau}{}_{\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{F}^2\right) + \frac{1}{4}\left(\mathcal{H}_{\mu\sigma\tau}\mathcal{H}_{\nu}{}^{\sigma\tau} - \frac{1}{6}g_{\mu\nu}\mathcal{H}^2\right), \tag{D.2}$$

whose trace-reversed form can be obtained as

$$\begin{aligned}\widetilde{T}_{\mu\nu} &= T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\\ &= \frac{1}{2}\left(\mathcal{F}_{\mu\tau}\mathcal{F}^{\tau}{}_{\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{F}^2\right) + \frac{1}{4}\left(\mathcal{H}_{\mu\sigma\tau}\mathcal{H}_{\nu}{}^{\sigma\tau} - \frac{1}{2}g_{\mu\nu}\mathcal{H}^2\right).\end{aligned} \tag{D.3}$$

The field equations for the matter fields are found as

$$\nabla^{\nu}\mathcal{F}_{\mu\nu} = -\frac{C}{6}\epsilon_{\mu\nu\alpha\beta}\mathcal{H}^{\nu\alpha\beta}\,, \tag{D.4}$$

$$\nabla^{\tau}\mathcal{H}_{\mu\nu\tau} = \frac{C}{2}\epsilon_{\mu\nu\alpha\beta}\mathcal{F}^{\alpha\beta}\,. \tag{D.5}$$

For the solutions that we consider $z = 2$, and the constant C and the cosmological constant Λ should be fixed as

$$(C\ell)^2 = 4, \qquad \Lambda = -\frac{5}{\ell^2}\,. \tag{D.6}$$

The metric function is given by

$$h(r) = 1 + \frac{k}{2r^2}, \quad (k \neq 0). \tag{D.7}$$

When the constants t and r surfaces are parametrized as

$$d\Omega_k^2 = \begin{cases} d\theta^2 + \sin^2\theta d\phi^2, & k = +1,\\ d\theta^2 + \sinh^2\theta d\phi^2, & k = -1,\end{cases} \tag{D.8}$$

the matter configuration is as follows

$$\mathcal{F}_{rt} = -2\ell r, \qquad \mathcal{H}_{r\theta\varphi} = 2\ell^2 r \begin{cases}\sin\theta, & k = 1\\ \sinh\theta, & k = -1\end{cases} \tag{D.9}$$

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PUBLICATIONS

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