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Path integral formulation for position –time-dependent mass systems

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Abstract. The problem of position-time dependent mass with general time-dependent potential is treated in the framework of the phase space path integral approach. By means of an appropriate point canonical transformation, the problem is converted into that relative to a constant mass with stationary potential. Particular examples are also considered.

1. Introduction

In recent years, the study of quantum mechanical systems with position-dependent effective masses has received considerable attention[1-8]. They constitute interesting and useful models for the description of several physical problems in different areas of the material sciences and condensed matter physics, especially in the case of many- body problems[9], electronic properties of semi-conductors[10], quantum dots[11], quantum liquids[12] and metal clusters[13],...etc. This wide range of applications has led to the development of methods and techniques for studding such systems. Among them, we can cite the point-canonical transformation method[6,7,8], the algebraic methods[12-16] and the supersymmetric quantum mechanics[17]. Note that in all of theses methods, the common procedure is to convert the position-dependent mass problem into that of constant mass and the main aim is to get energy spectra and/or the wave functions for theses systems once the position-dependent mass is given. The problem of variable mass can also be formulated by the path integral approach. Some examples have been treated in configuration space [18,19,20] where in [16] the Green's function of position-dependent mass has been related to that of constant mass according to a direct calculation. In this work we are interested in developing a systematic procedure to study one-dimensional path integral in phase space for a class of position-time dependent masses and time-dependent potentials. This later can provide not only many exact results known in the literature but also a various new ones.

By using an explicitly time-dependent canonical transformation as well as a time transformation, we were able to absorb the time dependence of the path integral. Then by shifting the momentum and performing an other judicious time transformation, we reduced the problem with position-time dependent mass to that related to a constant mass and stationary potential.

As applications, we have considered two different mass distributions each of which being relative to a chosen potential so that the corresponding path integral can be exactly resolved.

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2. The Hamiltonian

There are several forms for the hermitian Hamiltonian with a position dependent mass. The most general one that can save the hermiticity of Hamiltonian is

$$\hat{H} = \frac{1}{4}(m^{\alpha}\hat{p}m^{\beta}\hat{p}m^{\gamma} + m^{\gamma}\hat{p}m^{\beta}\hat{p}m^{\alpha} + \hat{V}(\hat{x},t), \quad \text{with} \;\; \alpha+\beta+\gamma=-1. \tag{1}$$

In our case we choose:
1-The Hamiltonian with $\alpha = 1$ and $\beta = \gamma = 0$
2- The time-dependent potential of the form $V(x,t) = f(t)^2 V(f(t)x)$, where $f(t)$ is a time-dependent function. 3-The mass has the form $m(x,t) = m(f(t)x)$.
By reordering the Hamiltonian this will produces an effective potential term (See Ref. [21])

$$\hat{H} = \frac{1}{4}(\frac{1}{m}\hat{p}^2 + \hat{p}^2\frac{1}{m}) - f^2\frac{m''}{8m^2} + \frac{9f^2}{32}\frac{m'^2}{m^3} + f^2\hat{V}(f\hat{x}), \tag{2}$$

Here the primes denote the derivatives with respect to the coordinate x.
The propagator related to the system Eq.(2) can be given in phase space by the following relation

$$\mathrm{K}(x'',t'';x',t') = (m(f''x'')m(f'x'))^{-1/4}\int D[x(t)]D[p(t)]$$
$$\times\, exp(i\int dt(p\dot{x} - \frac{1}{2}\frac{p^2}{m} + f^2\frac{m''}{8m^2} - \frac{9f^2}{32}\frac{m'^2}{m^3} - f^2V(fx)). \tag{3}$$

To absorb the time-dependency of the system, we perform the following time-dependent canonical transformation:

$$x = g(t)Q, \quad p = \frac{P}{g(t)},$$

with the generating function

$$F(x,P,t) = \frac{xP}{g(t)}, \tag{4}$$

where $g(t)$ is a real function.
The propagator is transformed as follows

$$K(Q'',P'',t'';Q',P',t') = \frac{(m(f''g''Q'')m(f'g'Q'))^{-1/4}}{\sqrt{g(t'')g(t')}}\int\frac{D[Q(t)]D[P(t)]}{2\pi}$$
$$\times\, exp(i\int_{t'}^{t''} dt(P\dot{Q} - \frac{1}{2}\frac{P^2}{mg^2} + f^2\frac{m''}{8m^2} - \frac{9f^2}{32}\frac{m'^2}{m^3} +$$
$$-f^2V(fgQ) + \frac{\dot{g}}{g}PQ)). \tag{5}$$

Since $g(t)$ is an arbitrary function, we will choose it such that $g(t)f(t) = 1$, to make the potential $V(x,t)$ and the mass functions $m(x,t)$ time-independent. Moreover by using the time rescaling $dt = f^{-2}(t)d\tau$, this will let Eq.(5) takes the following expression

$$K(Q'',P'',\tau'';Q',P',\tau') = \frac{(m(f''g''Q'')m(f'g'Q'))^{-1/4}}{\sqrt{g(t'')g(t'')}}\int\frac{D[Q(\tau)]D[P(\tau)]}{2\pi}$$

$$\times exp(i\int_{\tau'}^{\tau''} d\tau(P\dot{Q} - \frac{1}{2}\frac{P^2}{m} + \frac{m''}{8m^2} - \frac{9}{32}\frac{m'^2}{m^3} - V(Q) - g\dot{g}PQ)). \tag{6}$$

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In order to make the propagator completely time-independent, the function f(t) is chosen to satisfy the condition $\dot{f}f^{-3}(t) = g\dot{g} = \kappa$. Then shifting the momentum by $m\kappa Q$, or $P = P - m\kappa Q$, will cancel the PQ's term in the action

$$K(Q'', P'', t''; Q', P', t') = \frac{(m(Q'')m(Q'))^{-1/4}}{\sqrt{g(t'')g(t')}} e^{-\kappa i(G(Q'')-G(Q'))} \int \frac{D[Q(\tau)]D[P(\tau)]}{2\pi}$$
$$\times\, exp(i \int_{\tau'}^{\tau''} d\tau (P\dot{Q} - \frac{1}{2}\frac{P^2}{m} + \frac{m''}{8m^2} - \frac{9}{32}\frac{m'^2}{m^3} - V(Q)$$
$$-\frac{\kappa^2}{2} m Q^2)), \tag{7}$$

where the function $G(z)$ is

$$G_z(z) = \frac{\partial G(z)}{\partial z} = m(z)z. \tag{8}$$

Since the mass is position-dependent function, the kinetic term $\frac{P^2}{2m}$ has not a familiar form. To achieve a more convenient form of the path integral, we will perform the following time-transformation

$$\xi(Q(\tau))d\tau = ds. \tag{9}$$

This will lead to a new expression of the propagator

$$K(Q'', P'', s''; Q', P', s') = \frac{(m(Q'')\xi(Q'')^2 m(Q')\xi(Q')^2)^{-1/4}}{\sqrt{g(t'')g(t')}} e^{-\kappa i(G(Q'')-G(Q'))}$$
$$\times \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iET} \int_0^{\infty} dS \int D[Q(s)]D[P(s)]$$
$$\times exp(i \int ds (P\dot{Q} - \frac{1}{2}\frac{P^2}{m\xi} - \frac{\kappa^2}{2}\frac{m}{\xi}Q^2 - \frac{\xi'^2}{8m\xi^3} +$$
$$-\frac{m'\xi'}{4m^2\xi^2} - \frac{1}{\xi}(-\frac{m''}{8m^2} + \frac{9}{32}\frac{m'^2}{m^3} + V(Q)) + \frac{E}{\xi}). \tag{10}$$

Since ξ is an arbitrary function, we will choose it such that $\xi m = 1$. To obtain the following standard expression for the propagator

$$K(Q'', P'', s''; Q', P', s') = \frac{(m(Q'')m(Q'))^{1/4}}{\sqrt{g(t'')g(t'')}} e^{-\kappa i(G(Q'')-G(Q'))} \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iET}$$
$$\times \int_0^{\infty} dS \int D[Q(s)]D[P(s)] exp(i \int ds (P\dot{Q} - \frac{1}{2}P^2 +$$
$$-\frac{\kappa^2}{2} m^2 Q^2 - \frac{5}{32}\frac{m'^2}{m^2} + \frac{1}{8}\frac{m''}{m} - mV(Q) + mE). \tag{11}$$

Then the problem is transformed to that of a constant mass but with different potential. There are other choices of ξ such that the problem can be evaluated, for example $\xi m = \frac{\alpha}{Q}$ and $\xi m = \frac{\alpha}{Q^2}$. The choice depends on the nature of the system.

3. Applications

3.1. Example 1

We will be interested to the system of the mass $m(x) = x^\sigma$ and the potential $V(x) = V_0 + \frac{\beta}{x^\sigma} + \frac{\gamma}{x^{2\sigma}}$, $\sigma = \pm 2$.

This system is stationary and is the case where $\kappa = 0$. The function $f(t)$ is chosen to be 1. Then after performing the transformations given above, one can find that the propagator related to this problem is[22]

$$K(Q'', P'', s''; Q', P', s') = (x'^\sigma x''^\sigma)^{1/4} \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iET} \int_0^\infty dS e^{i\beta S} \int D[x(s)]D[p(s)]$$
$$\times exp(i \int ds(p\dot{x} - \frac{1}{2}p^2 - \frac{1}{32}\frac{\sigma(\sigma+4)}{x^2} + (E - V_0)x^\sigma - \frac{\gamma}{x^\sigma})$$
$$= \frac{\omega}{i}(x'^{\sigma+2} x''^{\sigma+2})^{1/4} \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iET} \int_0^\infty dS \frac{e^{i\beta S}}{sin(\omega S)}$$
$$\times e^{i\frac{\omega}{2}(x''^2 + x'^2)cot(\omega S)} I_{2\nu}\left(\omega \frac{x'' x'}{isin(\omega S)}\right) \tag{12}$$

with $\nu = \sqrt{\frac{1}{8}\sigma(\sigma+4) + \frac{\gamma}{2} + \frac{1}{16}}$ and $\omega^2 = -2(E - V_0)$ for $\sigma = 2$, and $\nu = \sqrt{\frac{1}{8}\sigma(\sigma+4) - \frac{E-V_0}{2} + \frac{1}{16}}$ and $\omega^2 = 2\gamma$, for $\sigma = -2$.

The integral over S in (12) can be performed using the formula

$$\int_0^\infty dx\ exp(2px - \beta coth(x))cosech(x) J_{2\gamma}(\alpha cosech(x))$$
$$= \alpha^{-1}\frac{\Gamma(1/2 - p + \gamma)}{\Gamma(2\gamma+1)} M_{-p,\gamma}(\sqrt{\alpha^2+\beta^2} - \beta) W_{p,\gamma}(\sqrt{\alpha^2+\beta^2} - \beta) \tag{13}$$

Thus (12) becomes

$$K(Q'', P'', s''; Q', P', s') = (-1)^{\nu+1}(x'^\sigma x''^\sigma)^{1/4} \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iET} \frac{\Gamma(1/2 - p + \nu)}{\Gamma(2\nu+1)\omega} M_{-p,\nu}(\frac{\omega}{2}x'^2) W_{p,\nu}(\frac{\omega}{2}x''^2), \tag{14}$$

with $p = \frac{\beta}{2\omega}$. Consequently, the energy levels related to the bound state of this system are given by the relation

$$211/2 - p + \nu = -n, \quad n \in N. \tag{15}$$

By inserting the value of p in (15) we will find that the energies are

- For $\sigma = 2$

$$E_n = -\frac{\beta^2/8}{(1/2 + n + \nu)^2} + V_0, \tag{16}$$

- For $\sigma = -2$

$$E_n = -2(-n + \frac{\beta}{4\gamma} - 1/2)^2 - 7/8 + V_0. \tag{17}$$

3.2. Example 2

We will be interested to the system of the mass $m(x) = e^{\lambda x}$ and the potential $V(x) = V_0 + \beta e^{-\lambda x} + \gamma e^{\lambda x}$. For this system, the mass grows exponentially $m = e^{\lambda x}$, where λ is a real constant.

Following the steps given above one can find that

$$K(Q'', P'', s''; Q', P', s') = e^{\frac{\lambda}{4}(x''+x')} \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iET} \int_0^{\infty} dS e^{-i\frac{\lambda^2+32\beta}{32}S} \times \int D[x(s)]D[p(s)]exp(i\int ds(p\dot{x} - \frac{1}{2}p^2 + (E - V_0)e^{\lambda x} - \gamma e^{2\lambda x}). \tag{18}$$

This expression is recognized as the path integral of the Morse potential [22]. Following [23,24], one can find

$$K(Q'', P'', s''; Q', P', s') = e^{\frac{\lambda}{4}(x''+x')} \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iET} \int_0^{\infty} dS e^{-i\frac{\lambda^2+32\beta}{32}S} \int \frac{dE_M}{2\pi} e^{-iE_M S} \frac{(-1)^{\lambda'+1}}{i\lambda\omega exp(\frac{\lambda}{2}(x''+x'))} \frac{\Gamma(\lambda' - p + 1/2)}{\Gamma(2\lambda' + 1)} \times M_{-p,\lambda'}(-\omega e^{\frac{\lambda}{2}x'}) W_{-p,\lambda'}(-\omega e^{\frac{\lambda}{2}x''}), \tag{19}$$

where $p = 2\frac{E-V_0}{\lambda^2\omega}$, $\omega = \sqrt{8\gamma/\lambda^2}$ and $\lambda' = \sqrt{-2E_M/\lambda^2}$.

Performing the S and E_M integrals we find that $E_M = -\frac{\lambda^2+32\beta}{32}$, so that the propagator is written as

$$K(Q'', P'', s''; Q', P', s') = \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iET} \frac{(-1)^{\lambda'+1}}{i\lambda\omega exp(\frac{\lambda}{4}(x''+x'))} \frac{\Gamma(\lambda' - p + 1/2)}{\Gamma(2\lambda' + 1)} \times M_{-p,\lambda'}(-\omega e^{\frac{\lambda}{2}x'}) W_{-p,\lambda'}(-\omega e^{\frac{\lambda}{2}x''}). \tag{20}$$

The discrete energy levels can be found from the poles of the Γ function in the numerator. Replacing λ', and p by their values, one can find that

$$E_n = \frac{\lambda^2\omega}{2}(\sqrt{1/16 + 2\beta/\lambda^2} + 1/2 + n) + V_0. \tag{21}$$

We have presented two examples here, but using the method developed above, one can find many systems that can have exact solutions, like for example the system with the mass $m(x) = m_0/x$, associated to the potential $V(x) = \alpha/x + \beta x$ will be reduced to the one dimensional system of a free particle in a Coulomb potential and an inverse quadratic potential.

4. Conclusion

In the present work, we reduced the phase space path integral with position-dependent mass and time-dependent potential to that with constant mass and stationary potential, simply by using explicitly time-dependent canonical transformation and appropriate time transformations. The general form of the propagator is given and closed expressions are deduced for tow specific mass functions particles moving in familiar physical potentials, together with their energy spectra and corresponding wave functions.

We should point out that our result can provide solutions for systems with different mass functions and typical potentials frequently used in the literature and can also be extended to get solutions for systems with more complicated time-dependent mass distributions combined with other potentials to model interesting physical phenomena.

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