Aspects of Chiral Symmetries in Holography

By

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I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

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Prashant Gupta

Dedicated to the resilient spirit of humans

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Abstract

This thesis explores the derivation of chiral current algebras from different gravitational theories, including $\mathbb{R}^{1,3}$ gravity, AdS_4 gravity, and $3d$ conformal gravity. We propose chiral boundary conditions consistent with the variational principle for $4d$ asymptotically flat solutions yielding a chiral $\mathfrak{bms}_4$ asymptotic symmetry algebra. This symmetry algebra was earlier discovered in the context of celestial conformal field theory. It is an infinite-dimensional chiral extension of the Poincaré algebra and includes a copy of Virasoro algebra, a copy of $\mathfrak{sl}(2,\mathbb{R})$ current algebra, and a doublet of commuting currents with conformal weight $h = \frac{3}{2}$. Additionally, a novel non-chiral infinite-dimensional symmetry algebra for $\mathbb{R}^{1,3}$ gravity is introduced through alternative non-chiral boundary conditions, considering the boundary metric in conformal gauge.

Generalising the chiral boundary conditions of $\mathbb{R}^{1,3}$ gravity for AdS_4 gravity, we derive chiral locally AdS_4 solutions in the Newman-Unti gauge consistent with variational principle whose asymptotic symmetry algebra we show, to be an infinite dimensional chiral extension of $\mathfrak{so}(2,3)$. This symmetry algebra coincides with the chiral $\mathfrak{bms}_4$ algebra in the flat space limit. We posit this symmetry algebra as the chiral version of recently discovered Λ-$\mathfrak{bms}_4$ algebra. We postulate line integral charges from the bulk AdS_4 gravity corresponding to this chiral symmetry algebra and show that the charges obey the semi-classical limit of $\mathcal{W}$-algebra that includes a level k, $\mathfrak{sl}(2,\mathbb{R})$ current algebra. The complete quantum version of this algebra which we denote by $\mathcal{W}(2;(3/2)^2,1^3)$ already existed in the literature in a different context.

We also construct four inequivalent $\mathcal{W}$-algebras based on exactly four inequivalent embedding of $\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{so}(2,3)$. These algebras are denoted as $\mathcal{W}(2;2^2,1)$, $\mathcal{W}(2,4)$, $\widetilde{\mathcal{W}}(2;2^2,1)$ and $\mathcal{W}(2;(3/2)^2,1^3)$. We show the emergence of $\mathcal{W}(2;(3/2)^2,1^3)$ and $\mathcal{W}(2,4)$ algebras in the semi-classical limit from the asymptotic symmetries of $\mathfrak{so}(2,3)$ Chern-Simons theory formulation of $3d$ conformal gravity. The analogous calculation for $\mathcal{W}(2;2^2,1)$ already existed in the literature which can be extended trivially for the case of $\widetilde{\mathcal{W}}(2;2^2,1)$. The emergence of $\mathcal{W}(2;(3/2)^2,1^3)$ algebra from asymptotic symmetry analysis of three-dimensional gravitational Chern-Simons action is also demonstrated in this thesis. This action describes $3d$ conformal gravity in the second-order formulation and plays an important role in AdS_4 gravity. Charges generating this symmetry algebra are calculated using the modified Lee-Wald covariant phase space formalism.

Finally, classical theories for scalar fields in arbitrary Carroll spacetimes, invariant under Carrollian diffeomorphisms and Weyl transformations, are constructed. These theories, upon gauge fixing, become Carrollian conformal field theories, with a classification provided for scalar field theories in three dimensions up to two derivative orders. We show that only a specific case of these theories arises in the ultra-relativistic limit of a covariant parent theory.

Contents

Chapter 1

Synopsis

1.1 Introduction

Asymptotic symmetries play a crucial role in the study of holography. For instance, before the seminal work of Maldacena [1] that conjectured the AdS/CFT correspondence, it was shown by Brown and Henneaux [2] that the commutator algebra of canonical charges that generate asymptotic symmetries of asymptotically locally AdS_3 configurations with Dirichlet boundary conditions is the sum of two commuting copies of Virasaro algebra. It is also well known that these are global symmetries of $2d$ conformal field theory (CFT) defined on the time-like boundary of such spacetimes. They further calculated the central charge associated with the two copies of Virasaro algebra to be $c = \frac{3l}{2G}$, where l is the AdS radius and G is Newton constant. This work was the first validation of one of the expectations from the holographic principle, which states that the global symmetries of the holographic dual boundary theory are isomorphic to asymptotic symmetries of bulk gravitational theory.

Similarly, before the initial works on flat space holography, in the 1960s Bondi, van der Burg, Metzner [3] and Sachs [4] showed the existence of an infinite dimensional algebra of asymptotic symmetries for spacetimes that approach Minkowski spacetime near the boundary null infinity. Their analysis was done in a particular gauge now famously known as Bondi gauge . This algebra, denoted by $\mathfrak{bms}_4$, is a semi-direct sum of Lorentz algebra and supertranslations (parametrized by an arbitrary function defined on the sphere at the boundary). Though done in the context of gravitational waves, this work later became the foundation of the flat space holography program. In a series of seminal works by Strominger and collaborators [5] , it was proved that the Ward identities of charges associated with $\mathfrak{bms}_4$ algebra are equivalent to Weinberg soft theorems for gravitational theories. The connection of this algebra with the soft theorems and memory effects have far-reaching consequences on gravitational theories in asymptotically flat spacetimes, which include imposing strict constraints on the infrared structure of the gravitational S-matrix. The study of asymptotic symmetries has proved insightful in our search for dual boundary theories in the context of flat space holography as well. There are two candidates of dual holographic theories for $4d$ gravitational theories in asymptotically flat spacetimes, (i) celestial conformal field theory, a putative $2d$ theory defined on the celestial sphere at null infinity [6] and (ii) Carrollian conformal field theory, a $3d$ theory defined on null infinity akin to the traditional co-dimension one holography like AdS/CFT correspondence [7]. These theories include $\mathfrak{bms}_4$ symmetry algebra as their global symmetries.

The asymptotic symmetries are known to be sensitive to the boundary conditions imposed on the bulk configuration space. Therefore, it is important to study the interplay of these boundary conditions with the asymptotic symmetry algebra as it allows us to probe the features and properties of the boundary theory, and explore the relationship between various physical bulk configurations and the corresponding boundary theories. A prime example of such a case includes relaxing the Brown and Henneaux boundary conditions for asymptotically locally AdS_3 spacetimes, such that it breaks the symmetry between left-moving and right-moving sectors of the $2d$ CFT and gives rise to $2d$ chiral CFT or $2d$ polyakov induced gravity in chiral gauge at the boundary [8, 9, 10].

In the context of $4d$ asymptotically flat spacetime, one proposes boundary conditions on the celestial sphere at null infinity where $\mathfrak{bms}_4$ gets enhanced as follows,

- Extended $\mathfrak{bms}_4$, where Lorentz algebra is enhanced to two copies of Witt algebra [11].
- Generalised $\mathfrak{bms}_4$, where Lorentz algebra is enhanced to the algebra of diffeomorphisms on a celestial sphere (Diff (S^2)) [12].

However, there has not been an attempt to consider chiral boundary conditions for $4d$ flat space gravity along the lines of AdS_3 gravity [8, 9, 10]. This question becomes important in light of the recent results in celestial CFT, where a sector in the celestial CFT holographically computes MHV graviton scattering amplitudes [13, 14]. The complete symmetry algebra of such a sector includes one copy of the $\mathfrak{sl}(2, \mathbb{R})$ current algebra, one copy of Virasoro algebra and a pair of commuting $h = \frac{3}{2}$

conformal primaries which is also a current algebra doublet. This symmetry algebra is an infinite dimensional chiral extension of the Poincare algebra $\mathfrak{iso}(1,3)$, which will be referred to as the chiral $\mathfrak{bms}_4$ algebra. Then, a natural question, which will be answered in the thesis is whether one can obtain such an algebra as the asymptotic symmetry algebra of the $4d$ flat space $\mathbb{R}^{1,3}$ gravity.

The situation is different for AdS_4 gravity, where until recently, it was thought that the symmetry algebra of asymptotically $\text{AdS}_d + 1$ spacetimes for $d \geq 3$ is finite-dimensional $\mathfrak{so}(2,d)$ for a class of boundary conditions. Compère and collaborators in [15, 16], proposed new boundary conditions for generic asymptotically AdS_4 spacetimes in Bondi gauge, where they uncover the symmetry algebra of residual gauge transformations to be an infinite dimensional algebra, the Λ-$\mathfrak{bms}_4$. The Λ-$\mathfrak{bms}_4$ and its corresponding phase space in the flat space limit ($\Lambda \to 0$) coincide with the generalised $\mathfrak{bms}_4$ algebra and the phase space associated with it. This raises the following questions, which we address in the thesis,

1. Is there a chiral extension of $\mathfrak{so}(2,3)$ algebra that in some appropriate flat space limit reduces to the chiral $\mathfrak{bms}_4$?
2. Can one obtain such a chiral Λ-$\mathfrak{bms}_4$ algebra from the AdS_4 gravity?
3. How many chiral extensions of $\mathfrak{so}(2,3)$ are possible, and is there a systematic way to derive them?
4. Can one obtain all these chiral extensions as the asymptotic symmetry algebra of $3d$ conformal gravity?

One of the two proposed duals of flat space gravity is a Carrollian conformal field theory defined at null infinity, an example of a Carrollian manifold. Earlier efforts to construct Carrollian conformal field theories have involved taking the ultra-relativistic limit $c \to 0$ (c being the speed of light) of some parent relativistic conformal theory [17, 18]. This is because a simple way to obtain flat Carrollian geometry is to take $c \to 0$ limit of the Minkowski spacetime. Also, there have been few other attempts to construct Carrollian conformal field theories over the years [19], however, none of these attempts were general enough to demonstrate all the features of Carrollian manifolds as they did not include field theories defined on curved Carrollian manifolds. Hence, the question of constructing more general conformal field theories intrinsically on a Carrollian manifold is an important one. We will address this in the thesis by providing a systematic method to construct such field theories.

The thesis comprises two parts. In the first part, we concern ourselves with some aspects of flat space holography. The following is a summary of the main results of this part of the thesis:

- **$\widehat{\mathfrak{sl}}_2$ symmetry of $\mathbb{R}^{1,3}$ gravity**: We propose chiral boundary conditions for $4d$ asymptotically flat solutions consistent with the variational principle. The asymptotic symmetry algebra of such solutions is shown to be the chiral $\mathfrak{bms}_4$. We also construct boundary terms that lead to a well defined variational principle. Six arbitrary holomorphic functions parametrize the resultant solutions, which correspond to Goldstone modes associated with the spontaneous breaking of this symmetry algebra in the gravitational vacuum.

- **Scalar Carollian conformal Field theories**: We provide an explicit construction of classes of conformal scalar field theories on Carrollian manifolds in three dimensions using the geometrical covariant tensors of the Carroll geometry. These theories by construction have their symmetry algebra as Carrollian conformal algebra $\mathfrak{cca}_3^{(z)}$, a one parameter generalization of $\mathfrak{bms}_4$, defined for generic values of dynamical exponent z.

In the second part of the thesis we derive all the possible infinite dimensional chiral extensions of $\mathfrak{so}(2,3)$ algebra and their holographic realizations. We propose boundary conditions for AdS_4 gravity that will give rise to one of these chiral extension of $\mathfrak{so}(2,3)$ as its asymptotic symmetry algebra. Furthermore, we will derive all these chiral algebras in the semi-classical limit (large central charge c or level κ limit) from the asymptotic symmetries of $3d$ conformal gravity by providing appropriate boundary conditions. This analysis fills a gap in the literature where most of the work done on the asymptotic symmetries of $3d$ conformal gravity except the one in [20], did not include $\mathfrak{so}(2,3)$ algebra [21, 22, 23]. In the following, we provide a summary of the main results of the second part of the thesis:

- **A chiral $\mathcal{W}$-algebra extension of $\mathfrak{so}(2,3)$** : We propose chiral boundary conditions for AdS_4 gravity in Neumann-Unti gauge inspired by the chiral boundary conditions of $\mathbb{R}^{1,3}$ gravity considered in the first part of the thesis and, obtain the symmetry algebra for resultant solutions. The result is a $\mathcal{W}$-algebra, which coincides with chiral $\mathfrak{bms}_4$ algebra in the flat space limit. This algebra is the chiral version of Λ-$\mathfrak{bms}_4$ algebra. The derivation of this algebra from the AdS_4 gravity is in the semi-classical limit. The complete quantum version of this algebra is very similar to a $\mathcal{W}$-algebra that already existed in the literature [24], but, in this thesis, we re-derive for $\mathfrak{so}(2,3)$ this quantum algebra analogue of chiral $\mathfrak{bms}_4$ using the tools of $2d$ CFT and denote it by $\mathcal{W}(2;(3/2)^2,1^3)$.

- **All chiral $\mathcal{W}$-algebra extension of $\mathfrak{so}(2,3)$**: We construct all the four infinite dimensional chiral extensions of $\mathfrak{so}(2,3)$ by showing that there are exactly four inequivalent ways to embed the $\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{so}(2,3)$. Each of these embeddings gives rise to a $\mathcal{W}$-algebra. We use the associativity of operator product expansions on the postulated operator product algebra for each of the four cases to derive these algebras. The four resultant $\mathcal{W}$-algebras are denoted by, $\mathcal{W}(2;2^2,1), \widetilde{\mathcal{W}}(2;2^2,1)\,, \mathcal{W}(2,4), \mathcal{W}(2;(3/2)^2,1^3)$. The semi-classical limit of $\mathcal{W}(2;2^2,1)$ was known from the asymptotic symmetries of $3d$ conformal gravity [20]. The $\mathcal{W}(2,4), \mathcal{W}(2;(3/2)^2,1^3)$ can be identified with those that existed in the older literature in different contexts [24, 25].

- **Chiral boundary conditions of $3d$ conformal gravity**: Using the Chern Simons theory formulation of $3d$ conformal gravity in its first order formulation, we provide boundary conditions and gauge fixing of $\mathfrak{so}(2,3)$ gauge connection compatible with variational principle. We then show that the symmetry algebra of charges that generate residual gauge transformations of these gauge connections form representations of $\mathcal{W}(2,4), \mathcal{W}(2;(3/2)^2,1^3)$ algebras in the semi-classical limit. The analogous calculation for $\mathcal{W}(2;2^2,1)$ already existed in the literature [20], which can be extended trivially for the case of $\widetilde{\mathcal{W}}(2;2^2,1)$. We also demonstrate that one can obtain some of these $\mathcal{W}$ algebras as the asymptotic symmetry algebra for $3d$ conformal gravity directly in its second-order formulation.

We now provide more details about these results, highlighting some of the key points that form the crux of these results.

1.1.1 $\widehat{\mathfrak{sl}}_2$ symmetry of $\mathbb{R}^{1,3}$ gravity

As mentioned in the introduction, the chiral $\mathfrak{bms}_4$ algebra of celestial CFT introduced by the authors in [13] includes one copy of the $\mathfrak{sl}(2,\mathbb{R})$ current algebra (with generators $J_{a,n}$ for $a=0,\pm1$ and $n\in\mathbb{Z}$), one copy of Virasoro (L_n with $n\in\mathbb{Z}$) and two $h=\frac{3}{2}$ conformal primaries (with generators $P_{s,r}$ with $s=\pm\frac{1}{2}$ and $r\in\mathbb{Z}+\frac{1}{2}$). The Ward identities of these operators $P_{s,r}$ and $J_{a,n}$ are equivalent to leading and sub-leading soft graviton theorems. The algebra of these generators is,

$$
\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n}\,, \quad [J_{a,m}, J_{b,n}] = (a-b)\,J_{a+b,m+n}\,, \quad [L_m, J_{a,n}] = -n\,J_{a,m+n},\\
&[L_n, P_{s,r}] = \frac{1}{2}(n-2r)P_{s,n+r}\,, \quad [J_{a,m}, P_{s,r}] = \frac{1}{2}(a-2s)P_{a+s,m+r}\,, \quad [P_{s,r}, P_{s',r'}] = 0
\end{aligned} \tag{1.1}
$$

with possible central terms, that remains undetermined. This algebra is different from the classical $\mathfrak{bms}_4$ algebra of [3, 4, 26]. In this thesis, we present asymptotically locally flat solutions in $\mathbb{R}^{1,3}$ gravity consistent with the variational principle and derive the asymptotic symmetry algebra of such solutions to be the chiral $\mathfrak{bms}_4$ in (1.1). One implements the asymptotically locally flatness by demanding that $R^{\mu}{}_{\nu\rho\sigma} \to 0$ as $r \to \infty$ where $R^{\mu}{}_{\nu\rho\sigma}$ is the Riemann tensor of asymptotically locally flat solutions of $\mathbb{R}^{1,3}$ gravity, expanded in the radial coordinate r near the boundary. We argue that to obtain the asymptotic symmetries of these asymptotically locally flat solutions, one does not need to solve for the complete solution space. It is sufficient to find the locally flat solutions that share the same boundary conditions as these asymptotically locally flat solutions and study their asymptotic symmetries. Therefore, we analyse locally flat solutions, which are Ricci flat solutions with vanishing Riemann tensor.

We first provide all locally flat solutions in the Neumann-Unti gauge with u-independent boundary metric on spatial part (Σ_2) of null infinity, where u is the coordinate of null infinity. We then impose chiral boundary conditions on locally flat solutions by taking the metric on Σ_2 of the null infinity (celestial sphere or celestial plane) with coordinates $(z, \bar{z})$ of the form: $\Omega_0(z,\bar{z})\,dz(d\bar{z} + f(z,\bar{z})dz)$ (sometimes called Polyakov or chiral gauge) and further impose additional conditions coming from a well-defined variational principle. The topology of Σ_2 was chosen to be either $\mathbb{R}^2$ (with $\Omega_0 = 1$) or $\mathbb{S}^2$ (with $\Omega_0 = 4(1+z\bar{z})^{-2}$). To impose a consistent and correct variational principle, we also constructed appropriate set of boundary terms (Gibbon Hawking type) and add them to Einstein Hilbert action. The resultant chiral locally flat solutions are given in terms of six arbitrary holomorphic functions. After this, we computed vector fields that keep us within this class of locally flat solutions, and show that these vector fields close under the modified commutator, namely the Courant bracket. The commutators of these vector fields at the boundary obey chiral $\mathfrak{bms}_4$ algebra in eq (1.1). These vector fields result in the non-trivial transformations of background fields in the solution space, which is same for either choice of Σ_2.

1.1.2 Constructing Carrollian CFTs

Carrollian conformal field theories are defined at the boundary of asymptotically flat spacetimes which includes the null infinity, an example of a Carrollian manifold. Therefore to better understand the properties of these field theories defined on such a boundary, we provide an effective framework to construct such theories intrinsically on a Carrollian manifold.

A three-dimensional Carrollian manifold as defined in [27], is a fibre bundle with a two-dimensional base and one-dimensional fibre. We use the local coordinates $\mathbf{x}$ on the base and t on the fibre. There are three geometrical quantities that describe a Carrollian manifold, metric on the base $a_{ij}(t,\mathbf{x})$, a 1-form $b_i(t,\mathbf{x})$ (called the Ehereshmann connection) and a scalar $\omega(t,\mathbf{x})$. One defines the following Carroll diffeomorphisms, which keep this structure invariant,

$$t \to t'(t,\mathbf{x}), \quad \mathbf{x} \to \mathbf{x}'(\mathbf{x}) \tag{1.2}$$

Under these diffeomorphisms the geometrical data transform as,

$$\begin{aligned} a'_{ij}(t',\mathbf{x}') &= \left(J^{-1}\right)^k_{\ i} \left(J^{-1}\right)^l_{\ j} a_{kl}(t,\mathbf{x}), \quad \omega'(t',\mathbf{x}') = J^{-1}\omega(t,\mathbf{x}) \\ b'_i(t',\mathbf{x}') &= \left(J^{-1}\right)^k_{\ i} \left(b_k(t,\mathbf{x}) + J^{-1} J_k\, \omega(t,\mathbf{x})\right) \end{aligned} \tag{1.3}$$

where $J = \frac{dt'}{dt}$, $J_i = \frac{dt'}{d\mathbf{x}^i}$ and $J^i_{\ j} = \frac{d\mathbf{x}'^i}{d\mathbf{x}^j}$ are the Jacobian of Carroll diffeomorphisms. In addition to the Carroll diffeomorphisms, one also defines Carroll-Weyl transformations on background geometrical quantities as follows,

$$\begin{aligned} \tilde{a}_{ij}(t,\mathbf{x}) = B(t,\mathbf{x})^{-2}\, a_{ij}(t,\mathbf{x}), \quad \tilde{\omega}(t,\mathbf{x}) = B(t,\mathbf{x})^{-z}\, \omega(t,\mathbf{x}), \\ \tilde{b}_i(t,\mathbf{x}) = B(t,\mathbf{x})^{-z}\, b_i(t,\mathbf{x}). \end{aligned} \tag{1.4}$$

where $B(t,\mathbf{x})$ is an arbitrary function and z is the dynamical exponent. We work with a real scalar field $\Phi(t,\mathbf{x})$ that transforms under Caroll diffeomorphisms and Caroll-Weyl transformations as,

$$\Phi'(t',\mathbf{x}') = \Phi(t,\mathbf{x}), \quad \widetilde{\Phi}(t,\mathbf{x}) = B(t,\mathbf{x})^{\delta}\, \Phi(t,\mathbf{x}) \tag{1.5}$$

respectively, where δ is the Weyl weight of the scalar field $\Phi(t,\mathbf{x})$. We then construct the most general type of Carroll diffeomorphisms and Carroll-Weyl covariant equation of motions as well as invariant actions on generic 3-dimensional Carrollian manifolds for scalar fields up to two derivative order in both time (t) and space ($\mathbf{x}$)

coordinates for general values of z (dynamical exponent) and δ.

For this we start by listing invariants under Carroll diffeomorphisms involving the geometrical quantities (a_{ij}, b_i, ω) and the field $\Phi(t, \mathbf{x})$ and combining them into Weyl covariant blocks. We also construct various Weyl covariant time and space derivatives up to second order on the scalar fields that form building blocks of appropriate Lagrangian densities. This results in the following,

- Two classes of diffeomorphic and Weyl covariant equations of motion, one with two-time derivatives of the field (time-like), and the other with (up to) two space derivatives (space-like). These exist for general values of z and δ.
- For the special value of $z = 1$, one can combine time-like and space-like equations of motion with at least five real parameters.
- For $z = 2$, there are two classes of the equation of motion: one with second order time-derivatives and the other with first-order time as well as second-order space derivatives.
- The invariant actions exist only when $\delta = \frac{z}{2}$ for space-like action and $\delta = 1 - \frac{z}{2}$ for time-like action with both types of actions combining for $z = 1$.
- Having a stable monomial potential (Φ^{2n}) further restricts the value of z to be determined by the degree of monomials.
- When one expands the equation of motion and action of a conformally coupled scalar field in pseudo-Riemannian geometry as a polynomial in c, the speed of light, using the Randers-Pappetrou parametrization of $3d$ metric, the coefficient of order $\mathcal{O}\left(\frac{1}{c^2}\right)$ corresponds to the time-like equation/action and the coefficient $\mathcal{O}(1)$ corresponds to the space-like equation/action mentioned in the first point for $z = 1$ and $\delta = \frac{1}{2}$ case with appropriate identification of parameters.
- Gauge fixing the metric a_{ij} and ω using local symmetries leads to field theories with Carrollian conformal algebra $\mathfrak{cca}_3^{(z)}$ worth of symmetries that have an additional fluctuating two-component field b_i along with the scalar field $\Phi(t, \mathbf{x})$. For $z = 1$, this conformal algebra is isomorphic to the $\mathfrak{bms}_4$. Alternatively, if one gauge fixes b_i, ω and the determinant of a_{ij}, then the global symmetries of the field theories defined on the Carroll manifold is isomorphic to $\mathcal{R} \times \mathcal{A}$, where $\mathcal{A}$ is the algebra of volume-preserving diffeomorphisms on the sphere. This algebra is also known to arise from AdS_4 gravity [15].

1.1.3 A chiral $\mathcal{W}$- algebra extension of $\mathfrak{so}(2,3)$

The Λ-$\mathfrak{bms}_4$ algebra of [15, 16] mentioned in the introduction is a non-chiral extension of $\mathfrak{so}(2,3)$. The chiral boundary conditions that we use to obtain (1.1) in flat space gravity were a subset of $\mathfrak{gbms}_4$ boundary conditions, therefore, we consider such chiral boundary conditions suitably adapted to asymptotically locally AdS_4 solutions and examine the asymptotic symmetry algebra that one obtains after imposing these boundary conditions.

Following the same arguments as that of flat space gravity, we derive the asymptotic symmetries of locally AdS_4 solutions that share the same chiral boundary conditions as asymptotically AdS_4 solutions. To this end, we solve for locally AdS_4 equation in the Neumann-Unti gauge and obtain a solution after imposing chiral boundary conditions on the co-dimension-2 boundary of AdS_4 whose topology we take it to be $\mathbb{R}^2$. We further impose boundary conditions obtained from a well defined variational principle after adding the boundary terms we use in the case of $\mathbb{R}^{1,3}$ gravity. The boundary conditions we use are a subset of those employed in [15, 16]. The resultant chiral locally AdS_4 solutions that we obtained are in a specific form such that in the flat space limit ($l \to \infty$, where l is the AdS length), one obtains the chiral locally flat solutions that we mentioned in section (1.1.1). Our chiral locally AdS_4 solutions are given in terms of six holomorphic fields,

$$\{T(z), J_a(z), G_s(z)\} \quad \text{where } a \in 0, \pm 1\,, \quad s \in \{-\frac{1}{2}, \frac{1}{2}\}. \tag{1.6}$$

We then compute vector fields that keep us within the same class of chiral locally AdS_4 solutions and show that six holomorphic functions parametrize the vector fields. The algebra of these vector fields is a Lie algebroid as the structure constant depends on the fields J_a , which in the flat space limit reduces to (1.1). Furthermore, the variations of the fields (1.6) under these vector fields are computed.

One can write an expression for conjectured line integral charges that will generate the correct variations of the currents in (1.6) due to large gauge transformations, given the OPEs between various currents. The derivation of these charges from the bulk is an open problem that won't be addressed in this thesis. The operator product expansions between these currents and the commutation relations obtained from them obey a $\mathcal{W}$-algebra. This $\mathcal{W}$-algebra is similar to the semi-classical limit of the one already existing in the literature, which is a $N = 1$ case of an infinite series of $\mathcal{W}$-algebra defined for $2N$ bosonic spin-$\frac{3}{2}$ currents and spin-1 Kac-Moody current for $\mathfrak{sp}(2N)$ [24]. We re-derive this algebra in the context of the chiral extension of

$\mathfrak{so}(2,3)$ algebra for spin-1 $\mathfrak{sl}(2,\mathbb{R})$ Kac-Moody current by imposing Jacobi identities on the commutators of the modes of the chiral (quasi-) primaries. The resultant algebra is,

$$\begin{aligned}
&[L_m, L_n] = (m-n)\, L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}\,,\\
&[L_m, J_{a,n}] = -n\, J_{a,m+n}\,, \quad [L_m, P_{s,r}] = \frac{1}{2}(n-2r)\, P_{s,n+r}\,,\\
&[J_{a,m}, J_{b,n}] = -\frac{1}{2}\kappa\, m\, \eta_{ab}\, \delta_{m+n,0} + f_{ab}{}^{c}\, J_{c,m+n}, \quad [J_{a,n}, P_{s,r}] = P_{s',n+r}(\lambda_a)^{s'}{}_{s}\,,\\
&[P_{s,r},\, P_{s',r'}] = \epsilon_{ss'}\Big[\alpha\Big(r^2-\frac{1}{4}\Big)\delta_{r+r',0} + \beta\, L_{r+r'} + \gamma\,(J^2)_{r+r'}\Big] + \delta\,(r-r')\, J_{a,r+r'}(\lambda^a)_{ss'}
\end{aligned} \tag{1.7}$$

$$\text{with}\;\; c = -\frac{6\kappa\,(1+2\kappa)}{5+2\kappa},\;\; \alpha = -\frac{1}{4}\gamma\kappa(3+2\kappa),\;\; \beta = \frac{1}{4}\gamma(5+2\kappa),\;\; \delta = -\frac{1}{2}\gamma(3+2\kappa)$$

where $\eta_{ab} = (3a^2-1)\delta_{a+b,0}$, $f_{ab}{}^{c} = (a-b)\delta^{c}_{a+b}$, $(\lambda^a)_{ss'} = \frac{1}{2}\delta^{a}_{s+s'}$, $(\lambda_a)_{ss'} = \eta_{ab}(\lambda^b)_{ss'}$, $\kappa \neq -5/2$, and γ can be fixed to be any non-zero function of κ by rescaling the $\mathcal{P}_{s,r}$ appropriately. $(J^2)_n$ are the modes of the normal ordered quasi-primary $\eta^{ab}(J_aJ_b)(z)$. We denote this algebra by $\mathcal{W}(2;(3/2)^2,1^3)$ and show that in the semi-classical limit with proper identification of parameters, this algebra matches with the algebra obtained from bulk AdS_4 gravity. In a pure CFT calculation, we also provide the expressions of 3-point and 4-point correlation functions of the currents in (1.6) having the symmetry algebra 5.1, that obey crossing symmetry and conformal invariance.

1.1.4 All chiral $\mathcal{W}$- algebra extension of $\mathfrak{so}(2,3)$

After uncovering the algebra (5.1), we show that there are four infinite dimensional chiral extensions of $\mathfrak{so}(2,3)$ based on exactly four inequivalent embedding of $\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{so}(2,3)$. The steps that we use to construct all these chiral extensions of $\mathfrak{so}(2,3)$ are as follows,

1. To start with, we show that there are exactly four ways to embed a copy of $\mathfrak{sl}(2,\mathbb{R})$ inside $\mathfrak{so}(2,3)$
2. Then we identify in each case a maximal sub-algebra $\mathfrak{h} \subset \mathfrak{so}(2,3)$ that commutes with the $\mathfrak{sl}(2,\mathbb{R})$ of the previous step.
3. The rest of the generators arrange themselves into finite dimensional irreducible representations of the subalgebras $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{h}$. These steps facilitate writing $\mathfrak{so}(2,3)$ Lie algebra in four avatars.

4. We then postulate a quasi-primary chiral stress tensor $T(z)$ with central charge c for the $\mathfrak{sl}(2,\mathbb{R})$ and a chiral current algebra of $\mathfrak{h}$ at level κ.

5. To the generators in step three, we associate chiral fields that are both conformal primaries and the current algebra primaries of appropriate conformal weights and current algebra representations.

6. We then use the $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{h}$ symmetries to give the general ansatz of the OPEs among the additional chiral fields of the previous step with some unknown parameters that are to be determined.

7. Lastly, to implement the associativity of OPEs, we impose Jacobi identities on the commutators of the modes of the chiral (quasi-) primaries.

Going through these steps, we find that there are exactly four chiral $\mathcal{W}$- algebra extensions of $\mathfrak{so}(2,3)$ that may be denoted as follows,

$$\mathcal{W}(2;2^2,1),\widetilde{\mathcal{W}}(2;2^2,1)\,,\mathcal{W}(2,4),\mathcal{W}(2;(3/2)^2,1^3).$$

Out of these, $\mathcal{W}(2,4)$ was already well known in the literature [25]. It is associated with the principal embedding of $\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{so}(2,3)$, where $\mathfrak{h}=\emptyset$. The algebra $\mathcal{W}(2;(3/2)^2,1^3)$ given in eq.(5.1) has $\mathfrak{h}=\mathfrak{sl}(2,\mathbb{R})$. The algebra $\mathcal{W}(2;2^2,1)$ has $\mathfrak{h}=\mathfrak{so}(1,1)$ and may be called conformal $\mathfrak{bms}_3$. Its semi-classical limit was found to be the asymptotic symmetry algebra of a $3d$ conformal gravity in [20]. This algebra is given as follows

$$\begin{aligned}
&[\mathcal{L}_m,\mathcal{L}_n]=(m-n)\,\mathcal{L}_{m+n}+\frac{c}{12}m(m^2-1)\,\delta_{m+n,0}\,,\quad [L_m,P^a_n]=(m-n)\,P^a_{m+n},\\
&[L_m,H_n]=-n\,H_{m+n}\quad [H_m,H_n]=\kappa\,m\,\delta_{m+n,0},\quad [H_m,P^a_n]=P^b_{m+n}\,\epsilon_b{}^a\\
&[P^a_m,P^b_n]=\eta^{ab}\left[\frac{\alpha}{6}m(m^2-1)\delta_{m+n,0}+(m-n)\left(\beta\,L_{m+n}+\gamma\,(H^2)_{m+n}\right)\right]\\
&+\epsilon^{ab}\Big[\delta\,(m^2+n^2-mn-1)\,H_{m+n}+\sigma\,\Lambda_{m+n}+\omega\,\Sigma_{m+n}\Big]
\end{aligned}\tag{1.8}$$

where $a\in\{0,1\}$ and the non-zero components of η^{ab} are $\eta^{00}=-1,\eta^{11}=1$ and ϵ^{ab} is an antisymmetric tensor with $\epsilon^{01}=1$. The values of the coefficients are,

$$\begin{aligned}
&\alpha=\frac{3\kappa^2(\kappa-1)}{\kappa+1}\sigma,\quad \beta=-\frac{\kappa}{2}\sigma,\quad \gamma=\frac{2\kappa-1}{2(\kappa+1)}\sigma,\quad \delta=-\frac{\kappa(\kappa-1)}{2(\kappa+1)}\sigma,\\
&\omega=-\frac{2\kappa}{\kappa+1}\sigma,\quad c=-\frac{2\,(6\kappa^2-8\kappa+1)}{\kappa+1}\,.
\end{aligned}\tag{1.9}$$

Λ_m, Σ_m are modes of the $h = 3$ composite normal ordered quasi primaries,

$$\Lambda(z) := (TH)(z) - \frac{1}{2}\partial^2 H(z), \quad \Sigma(z) := \frac{1}{2\kappa}((H^2)H)(z) - \frac{1}{2}\partial^2 H(z) \tag{1.10}$$

where $T(z)$ is the field associated with Virasaro mode L_n. The algebra $\widetilde{\mathcal{W}}(2; 2^2, 1)$ has $\mathfrak{h} = \mathfrak{so}(2)$. It is closely related to $\mathcal{W}(2; 2^2, 1)$ and has the same structure except δ^{ab} replaces η^{ab} in (1.9) and other minor changes accompanied by the Euclidean signature of the Killing form of current algebra $\mathfrak{h}$.

Since each of the four chiral $\mathcal{W}$-algebras is an extension of the $3d$ conformal algebra $\mathfrak{so}(2, 3)$ one should expect to realise all of them as the asymptotic symmetry algebra of $3d$ conformal gravity. To show this, we work with $3d$ conformal gravity in the first order formalism, formulated as the Chern-Simons theory with $\mathfrak{so}(2, 3)$ gauge algebra given by the following action,

$$S = \frac{k}{4\pi} \int_M \mathrm{Tr}\Big(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\Big) \tag{1.11}$$

and go through the following standard steps,

- We start by considering one-form $\mathcal{A}$ in $3d$ written in one of the avatars of $\mathfrak{so}(2, 3)$ that matches with the global part of the $\mathcal{W}$ algebra that we are interested in deriving as the asymptotic symmetry algebra.
- We choose gauge and boundary conditions for $\mathcal{A}$ compatible with a well defined variational principle and solve its equation of motion.
- One then obtains the set of residual gauge transformations that obey gauge and boundary conditions of the connection $\mathcal{A}$. These residual gauge transformations induce the variation of fields in the connection.
- After calculating the charges that generate these residual gauge transformations, one writes down the commutators between the modes of the fields that parametrize the gauge connection $\mathcal{A}$.

Using the methodology described in the points above, we propose boundary condition of $\mathfrak{so}(2, 3)$ gauge connection and obtain $\mathcal{W}(2, 4), \mathcal{W}(2; (3/2)^2, 1^3)$ as the asymptotic symmetry algebra in the semi-classical limit. As mentioned before, the semi-classical limit of $\mathcal{W}(2; 2^2, 1)$ was shown as the asymptotic symmetry algebra of $\mathfrak{so}(2, 3)$ Chern-Simons theory previously in the literature [20]. The $\widetilde{\mathcal{W}}(2; 2^2, 1)$ can then be trivially derived from this result.

1.1.5 Chiral $\mathcal{W}$-algebra from second-order formulation of $3d$ conformal gravity

Over the years, the second-order formulation of three-dimensional conformal gravity with the following action,

$$S=\frac{k}{2}\int d^3x\,\sqrt{-g}\,\epsilon^{\lambda\mu\nu}\left(\Gamma^{\rho}_{\lambda\sigma}\,\partial_\mu\Gamma^{\sigma}_{\rho\nu}+\frac{2}{3}\,\Gamma^{\rho}_{\lambda\sigma}\,\Gamma^{\sigma}_{\mu\tau}\,\Gamma^{\tau}_{\nu\rho}\right) \tag{1.12}$$

has not garnered much attention, even though it is equivalent to $\mathfrak{so}(2,3)$ gauge theory formulation of conformal gravity. k is the dimensionless coupling constant, which is the level of the Chern-Simons action. The equation of motion for the action defined in 1.12 is Cotton tensor vanishing condition $C_{\mu\nu}=0$, where $C_{\mu\nu}=\epsilon_{\mu}{}^{\sigma\rho}\nabla_\sigma\left(R_{\rho\nu}-\frac{1}{4}R\,g_{\rho\nu}\right)$. The Lagrangian density is invariant up to a total derivative under diffeomorphisms as well as Weyl transformations. We provide chiral boundary conditions for solutions to $C_{\mu\nu}=0$ that solve variational principle and show the emergence of $\mathcal{W}(2;(3/2)^2,1^3)$- algebra as the asymptotic symmetry algebra.

This action becomes relevant when one quantizes the AdS gravity using the Neumann boundary condition where the holographic stress tensor $T_{ij}=0$, and one expects the boundary action to be an induced gravity action. One can add a bulk Pontryagin term $S_{pontryagin}=k\int d^4x\,\sqrt{-g}\epsilon^{\mu\nu\sigma\lambda}\,R^{\alpha}{}_{\beta\mu\nu}\,R^{\beta}_{\alpha\sigma\nu}$ with an arbitrary coupling constant to the bulk $4d$ action for negative cosmological constant. The Pontryagin term is a total derivative, equivalent to Chern Simons Gravity action in eq (1.12) at the boundary. In the large k limit, this action dominates the induced gravity action at the boundary and thus becomes relevant for AdS_4 holography. One solves the equation $C_{\mu\nu}=0$ after writing the metric in Fefferman Graham gauge expansion as follows,

$$ds^2=\frac{l^2}{r^2}\,dr^2+\frac{r^2}{l^2}\left(g^{(0)}_{ab}+\frac{l}{r}\,g^{(1)}_{ab}+\frac{l^2}{r^2}\,g^{(2)}_{ab}+\cdots\right)dx^a\,dx^b \tag{1.13}$$

Further, we also impose the gauge condition to fix Weyl symmetry as follows,

$$\partial_r\left(\frac{det(g)}{r^2}\right)=0 \tag{1.14}$$

where $det(g)$ is the determinant of the full metric in (1.13). The boundary coordinates x^a are light-cone coordinates with $x^a\in\{x^-,x^+\}$. We choose the following chiral boundary conditions that satisfy the variational principle,

$$g^{(0)}_{--} = 0\,,\; g^{(0)}_{+-} = -\frac{1}{2}\,,\; g^{(1)}_{--} = 0\,,\; g^{(1)}_{+-} = 0\,,\; g^{(2)}_{--} = -\frac{N^2}{4}.$$

The solution is,

$$g^{(0)}_{++} = J^-(x^+)\,e^{-\frac{i\,N\,x^-}{l}} + J^0(x^+) + J^+(x^+)\,e^{\frac{i\,N\,x^-}{l}} = J^a(x^+)\;e^{a\,\frac{i\,N\,x^-}{l}} \tag{1.15}$$

$$g^{(1)}_{++} = G^{-1/2}(x^+)\,e^{-\frac{i\,N\,x^-}{2\,l}} + G^{1/2}(x^+)\,e^{\frac{i\,N\,x^-}{2\,l}} = G^s(x^+)\;e^{s\,\frac{i\,N\,x^-}{l}} \tag{1.16}$$

$$g^{(2)}_{++} = T(x^+) + \text{Terms containing function of } J^a \text{ and } G^s \tag{1.17}$$

The complete solution is parameterized by $\{J^a, G^s, T\}$, $a \in \{0, \pm 1\}, s \in \{\pm\frac{1}{2}\}$ and subleading terms are given in terms of these chiral currents. The residual gauge transformations will induce a variation of these fields. The line integral charges that generate these transformations are calculated by modified Lee-Wald covariant phase space formalism, first proposed by Tachikawa [28] for Lagrangian densities that are not invariant under symmetry transformations. It turns out that the $\mathcal{O}(r)$ divergences in diffeomorphism charges cancel the contribution from the Weyl charges, and the resultant canonical charges are integrable. The algebra of charges is $\mathcal{W}(2;(3/2)^2,1^3)$ in the semi-classical limit.

1.1.6 Conclusions

- We provide a general prescription to construct actions and equations of motions of Carrollian conformal field theories for scalars that one can extend to other higher spin and fermionic Carrollian conformal field theories, which can potentially describe flat space gravity/string theories holographically.

- We introduced novel chiral boundary conditions for classical gravity in four dimensions without a cosmological constant ensuring asymptotic local flatness near null infinities. The resultant locally flat solutions after imposing chiral boundary conditions are complex in $\mathbb{R}^{1,3}$ gravity (real in $\mathbb{R}^{2,2}$ gravity). These complex solutions are 'condensates' of soft gravitons with definite helicity, determined by six holomorphic (anti-holomorphic) functions.

- The analysis of chiral locally AdS_4 solution and the emergence of $\mathcal{W}(2;(3/2)^2,1^3)$ algebra from the asymptotic symmetry algebra points to the fact that by imposing appropriate bulk boundary conditions, one obtains a topological sector in AdS_4 that can be holographically described by some co-dimension two hologram.

- We found exactly four chiral $\mathcal{W}$-algebra extensions of $\mathfrak{so}(2,3)$ Lie algebra and show that these algebras in the semi-classical limit emerge as the asymptotic symmetry algebras of $3d$ $\mathfrak{so}(2,3)$ Chern Simons gauge theory. Just as chiral algebras of celestial CFT play a role in $4d$ flat space gravity, the relevance of these chiral $\mathcal{W}$-algebras in probing the infrared effects of the bulk AdS_4 gravity needs to be understood better.
- We also show the emergence of $\mathcal{W}(2;(3/2)^2,1^3)$ algebra from the second order formalism of $3d$ conformal gravity, which we argue to describe holographically AdS_4 gravity for some specific bulk configurations and boundary conditions.

The proposed organization of the thesis

1. The first chapter will comprise a general introduction of asymptotic symmetries in the context of 4-dimensional asymptotically flat spacetime and 3-dimensional conformal gravity with past relevant results and important formulae reviewed.
2. The second chapter will mainly analyze chiral boundary conditions for asymptotically flat spacetime in $\mathbb{R}^{1,3}$ gravity and obtain symmetry algebra for such solutions.
3. The third chapter will deal with a general method for constructing Carrollian conformal scalar field theories in 3 dimensions on a curved Carroll manifold.
4. The fourth chapter will discuss chiral boundary conditions for asymptotically AdS_4 spacetime, obtaining symmetry algebra for such solutions and re-deriving $2d$ chiral $\mathcal{W}$- algebra that governs such asymptotic symmetry algebra.
5. The fifth chapter will discuss the derivation of all chiral $\mathcal{W}$-algebra extensions of $\mathfrak{so}(2,3)$ and further derive these $\mathcal{W}$-algebras as the asymptotic symmetry algebra of $3d$ conformal gravity in the first-order and second-order formulation.
6. The sixth chapter will be the conclusion of the results and future directions.

Chapter 2

Introduction

The starting point to answer several relevant questions in quantum gravitational theories is to study the symmetries of these theories. One such class of symmetries are called asymptotic symmetries, which become prominent near the spacetime boundaries and represent the physical symmetries of gravitational theories. They are also called 'large' gauge transformations. Because of their origin and nature, they play a crucial role in determining the gravitational dynamics in holography. One of the most successful examples of holographic duality is AdS/CFT correspondence. However, before the seminal work of Maldacena [1] that conjectured the AdS/CFT correspondence, Brown and Henneaux showed in [2] that the commutator algebra of canonical charges that generate asymptotic symmetries of asymptotically locally AdS_3 configurations with Dirichlet boundary conditions is the sum of two commuting copies of Virasaro algebra. It is also well known that these are global symmetries of $2d$ conformal field theory (CFT) defined on the time-like boundary of such spacetimes. They further calculated the central charge associated with the two copies of Virasaro algebra to be $c = \frac{3l}{2G}$, where l is the AdS radius and G is the Newton constant. This work was the first validation of one of the expectations from AdS/CFT correspondence, which states that the global symmetries of the holographic quantum field theory at the boundary are isomorphic to asymptotic symmetries of corresponding bulk gravitational theory in asymptotically locally AdS spacetimes. Understanding the symmetries on both sides of the correspondence is vital for exploring the holographic duality and gaining insights into quantum gravitational theories.

Similarly, in the 1960s, Bondi, van der Burg, Metzner [3] and Sachs [4] showed the existence of an infinite dimensional algebra of asymptotic symmetries for four-dimensional spacetimes that approach Minkowski spacetime near the boundary null infinity. Their analysis was done in a particular gauge now famously known as the Bondi gauge. This algebra, denoted by $\mathfrak{bms}_4$, is a semi-direct sum of Lorentz algebra and supertranslations (parameterised by an arbitrary function defined on the sphere at the null infinity). It is an infinite dimensional extension of Poincare algebra $\mathfrak{iso}(1,3)$ where translations enhance to supertranslations. The action of Lorentz transformations on the sphere at null infinity is isomorphic to global conformal transformations. In [29, 30, 11], Barnich and Troessaert showed that one can extend the original $\mathfrak{bms}_4$ algebra further by considering local conformal transformations on the sphere denoted by superrotations. Therefore, the resultant symmetry algebra gets enhanced to a direct sum of supertranslations and two copies of Virasaro algebra. This extension is known as extended $\mathfrak{bms}_4$ ($\mathfrak{ebms}_4$) algebra.

Due to the connection of asymptotic symmetries and soft theorems, one has been able to understand better the constraints imposed on $\mathcal{S}$-matrix in the IR regime [31, 32]. In [33], Strominger established the invariance of $\mathcal{S}$-matrix under the diagonal element of the BMS group acting on the scattering data at the future and past null infinity using the results of [34]. In the subsequent paper [35], the authors showed that the Ward identity associated with the supertranslation symmetries was equivalent to Weinberg leading soft graviton theorem [36, 37]. This leading soft graviton theorem is a universal factorisation property of scattering amplitudes at the tree level. It relates the $\mathcal{S}$-matrix element of any quantum field theory including gravity in the limit when the energy ω of one of the graviton goes to zero, to the $\mathcal{S}$-matrix element without the soft graviton multiplied by the leading soft factor that is proportional to $\mathcal{O}(\omega)$. This work [35] led to a renewed interest in the study of soft graviton theorems [38, 39, 40, 41] and a plethora of work connecting the invariance of $\mathcal{S}$-matrix under these asymptotic symmetries via corresponding soft theorems ensued [42, 43, 44, 45]. Notably, it was shown in [46] that the subleading universal soft graviton theorem at tree level corresponding to $\mathcal{O}(1)$ pole in ω [38] implies the Ward identity of $\mathcal{S}$-matrix associated with the Virasaro symmetry of $\mathfrak{ebms}_4$ algebra. However, as the superrotations include singular transformations on the sphere, one could not establish this equivalence entirely the other way. To remedy this Campliglia and Laddha in [47, 12] proposed a different extension of original $\mathfrak{bms}_4$ called generalised $\mathfrak{bms}_4$ algebra ($\mathfrak{gbms}_4$), where Lorentz algebra is enhanced to the algebra of arbitrary diffeomorphisms on the sphere $\mathfrak{diff}(\mathbb{S}^2)$ at null infinity. For smooth vector fields in $\mathfrak{diff}(\mathbb{S}^2)$, they showed that the Ward identity of $\mathcal{S}$-matrix associated with $\mathfrak{diff}(\mathbb{S}^2)$ symmetry is equivalent to Cachazo-Strominger subleading soft theorem [38]. Recently Freidel and collaborators proposed another extension of

$\mathfrak{bms}_4$ called Weyl $\mathfrak{bms}_4$ algebra which includes local Weyl rescalings of the boundary metric in addition to supertranslations and arbitrary diffeomorphisms of the sphere metric [48].

This connection of asymptotic symmetries of gravitational theories in $\mathbb{R}^{1,3}$ gravity with the soft graviton theorems of quantum field theory [36, 38, 39, 40, 41] has led to the proposal of codimension-two holography for $\mathbb{R}^{1,3}$ gravity called celestial holography. In celestial holography, one rewrites four-dimensional scattering amplitudes in boost eigenstates instead of momentum eigenstates that recast these scattering amplitudes as the correlation functions of two-dimensional putative celestial conformal field theory defined on the celestial sphere at null infinity [49, 50, 51, 52, 53]. This recasting uses the isomorphism between Lorentz algebra and $\mathfrak{sl}(2, \mathbb{C})$ to classify conformal states on the sphere [54, 55, 56, 57]. Written in the boost eigenstates, also called conformal basis, one rewrites the soft theorems in $\mathbb{R}^{1,3}$ gravity as conformal Ward identity with the insertion of soft currents on the celestial sphere. These soft currents correspond to the choice of particular conformal dimension Δ in celestial CFT [58, 52, 59]. Furthermore, one interprets the collinear limit between two operators on the amplitude side as OPEs between the corresponding conformal primaries in the celestial CFT [60, 61, 62].

The developments in celestial CFT have revealed more insights into the symmetries of gravitational theories defined in $\mathbb{R}^{1,3}$ gravity from the boundary perspective. For instance, in [13, 14], the authors showed that the leading and subleading soft graviton theorems written in a conformal basis are equivalent to conformal Ward identity of two commuting $h = \frac{3}{2}$ currents and a $\mathfrak{sl}(2, \mathbb{R})$ current algebra respectively defined on the celestial sphere. This chiral symmetry algebra, which we will refer to as chiral $\mathfrak{bms}_4$, describes a sector in celestial CFT which holographically computes MHV graviton scattering amplitude at tree level in $\mathbb{R}^{1,3}$ gravity [13, 14]. Later in [63], using the OPE between two conformal primaries graviton operators, the authors showed the existence of an infinite tower of symmetry generating currents defined on the celestial sphere for positive helicity gravitons. They further speculate that this infinite tower of symmetry-generating chiral currents with higher spin may have links to the infinite number of soft theorems defined in the bulk $\mathbb{R}^{1,3}$ gravity [64, 65]. Later, Strominger in [66] demonstrated that this infinite tower of currents form a closed algebra known as the $w_{1+\infty}$ algebra, with the chiral $\mathfrak{bms}_4$ algebra [13, 14] as the sub-algebra within this $w_{1+\infty}$ algebra.

To strengthen the case for flat space holography, one should be able to derive the symmetries revealed by celestial CFT from the bulk $\mathbb{R}^{1,3}$ gravity. As a first step, we note that a similar $\mathfrak{sl}(2,\mathbb{R})$ current algebra appeared in the context of AdS_3 gravity [10]. The authors in [10] considered boundary metric for asymptotically AdS_3 solutions in chiral Polyakov gauge and showed the emergence of a $\mathfrak{sl}(2,\mathbb{R})$ current algebra from the asymptotic symmetries with the level k given in terms of central charge c of the Virasaro algebra. This suggests that considering chiral boundary conditions in $\mathbb{R}^{1,3}$ gravity may lead to the derivation of chiral $\mathfrak{bms}_4$ algebra [13, 14]. We study this possibility with a positive conclusion in this thesis. See [67] for a related later work where the authors have derived charges that form the canonical representation of these $w_{1+\infty}$ algebra using vacuum Einstein equations.

In a parallel development, the results of [68, 69] established the isomorphism between conformal Carroll algebra $\mathfrak{cca}_3^{(z)}$ for $z = 1$ and $\mathfrak{bms}_4$ algebra that generated interest in Carrollian conformal field theories. One defines such theories on the Carroll manifolds, which are null and have richer structures [27, 70] compared to (Pseudo) Riemannian manifolds. The examples of Carroll manifolds include one of the boundaries of $\mathbb{R}^{1,3}$ gravity (null infinity) and the horizon of black holes [71]. The fact that the Carrollian conformal field theories defined at null infinity contain $\mathfrak{bms}_4$ as their global symmetry algebra led to the expectation that these theories are viable candidates for dual boundary theories to gravitational theories in $\mathbb{R}^{1,3}$ gravity [72]. One can obtain Carroll algebra from the group contraction of Poincare algebra by taking the ultra-relativistic limit ($c \to 0$, where c is the speed of light) [73, 74]. Thus, earlier attempts to obtain field theories exhibiting Carrollian conformal symmetries involved taking the ultra-relativistic limit of parent relativistic conformal field theories defined in Minkowski spacetime [72, 17]. Such theories were defined on the flat Carrollian geometries and excluded the examples of general field theories conformally coupled to the background Carroll geometries, which are not necessarily flat [27]. The developments in [19, 75, 76], which involved constructing some of the Carrollian fluid hydrodynamics equations directly on these curved Carrollian manifolds, led to the construction of conformally coupled field theories on general Carrollian manifolds [77, 78, 79, 80, 81, 82]. See [18, 83, 84, 85, 86] for different approaches to the Carrollian CFTs.

Several works have explored the connections between Carrollian CFTs and celestial CFTs [7, 87, 88, 89, 90, 91]. One of the differences between these two proposed dual theories is that celestial CFT is a proposal for codimension-two holography. In contrast, Carrollian CFT is the proposal for codimension-one holography akin to AdS/CFT correspondence, and both address different aspects of flat space holography. Celestial holography is better suited to study the scattering problem as it provides a better framework to examine the scattering amplitudes by rewriting them as correlation functions on the celestial sphere. In Carrollian CFT, due to the incorporation of an extra u coordinate of boundary, the study of the evolution of dynamical data on null infinity is natural. In celestial CFT, the momentum operators act as weight-shifting operators of conformal primaries [53], whereas, in Carrollian CFT, the action of momentum operators is much more natural as they manifestly act by shifting the u-coordinate of null infinity [13].

The boundary conditions play an integral role in the study of asymptotic symmetries. The symmetry algebra can be enhanced or restricted depending on the boundary conditions. For example, in the derivation of the classical $\mathfrak{bms}_4$ algebra and $\mathfrak{ebms}_4$, the boundary metric on a 2-dimensional celestial sphere was kept fixed [3, 4, 92]. This boundary condition was relaxed to only holding the determinant of the 2-dimensional boundary metric on the celestial sphere fixed while deriving $\mathfrak{gbms}_4$ [47, 12]. Relaxation of this condition further leads to Weyl $\mathfrak{bms}_4$ algebra [48]. The interplay between boundary conditions and asymptotic symmetries has been documented very well in the AdS/CFT correspondence, especially in AdS_3 [2, 8, 9, 10, 93, 94]. The asymptotically locally AdS solutions to the Einstein equations near the boundary are given in terms of two independent data, (i) a boundary metric and (ii) a holographic stress tensor which satisfies constraint equations [95, 96]. This behaviour is unlike the flat case where an infinite number of free data appear while solving Einstein equations [97, 98]. One can quantise gravity in the bulk AdS spacetime in different ways by imposing different boundary conditions at the AdS boundary [99, 100]. The following type of boundary conditions are the most commonly imposed in AdS/CFT correspondence

- Dirichlet boundary condition, where one fixes the boundary metric [2, 101, 96].
- Neumann boundary condition, where one fixes the holographic stress tensor to zero [99, 100].
- Mixed boundary conditions, where one fixes some components of boundary metric and of the holographic stress tensor after adding a boundary action to the bulk action that gives a finite contribution to the stress tensor [8, 99, 9, 10].

The symmetry algebra of asymptotically AdS_{d+1} spacetimes for $d \geq 3$ with Dirichlet boundary conditions is finite-dimensional $\mathfrak{so}(2, d)$, whereas, for Neumann boundary conditions all the residual diffeomorphisms turn out to be pure gauge transformations [99, 95].[1] This is because, according to [99], for Neumann boundary conditions the boundary theory is an induced gravity for which codimension-two charges from the bulk perspective and codimension-one charges from the boundary perspective go to zero. Recently, Compère and collaborators in [15, 16] proposed boundary conditions for generic asymptotically locally AdS_4 spacetimes in Bondi gauge, where they uncover the symmetry algebra of residual gauge transformations to be an infinite dimensional algebra, the Λ-$\mathfrak{bms}_4$. The Λ-$\mathfrak{bms}_4$ and its corresponding phase space in the flat space limit ($\Lambda \to 0$) coincide with the generalised $\mathfrak{bms}_4$ algebra and the phase space associated with it. To impose a well-defined variational principle, they implement mixed boundary conditions, giving rise to reduced symmetry algebra, which is a direct sum of $\mathbb{R} \oplus \mathcal{A}$, where $\mathbb{R}$ denotes the abelian time translations and $\mathcal{A}$ is the algebra of 2-dimensional area-preserving diffeomorphisms. These results in AdS_4 gravity [15, 16, 99] and the derivation of chiral $\mathfrak{bms}_4$ algebra from $\mathbb{R}^{1,3}$ gravity that we will present in this thesis raises the following questions:

- Is there a chiral extension of $\mathfrak{so}(2,3)$ algebra that in some appropriate flat space limit reduces to the chiral $\mathfrak{bms}_4$?
- Can one obtain such a chiral extension of $\mathfrak{so}(2,3)$ from the study of asymptotic symmetries in AdS_4 gravity similar to the chiral extensions of $\mathfrak{so}(2,2)$ from AdS_3 gravity [2, 10, 93]?[2]
- How many chiral extensions of $\mathfrak{so}(2,3)$ are possible, and is there a systematic way to derive them?

We address these questions in this thesis and see how Neumann boundary conditions [99] play a part in answering the first two questions. One such chiral extension of $\mathfrak{so}(2,3)$ already existed in the literature namely, conformal $\mathfrak{bms}_3$ or $W_{(2,2,2,1)}$ [20]. It was obtained from the asymptotic symmetries of three-dimensional $\mathfrak{so}(2,3)$ Chern-Simons gauge theory which is equivalent to the first-order formulation of $3d$ conformal gravity [103]. In the same way, we expect all the other chiral extensions of $\mathfrak{so}(2,3)$ that we will derive in this thesis to emerge from the asymptotic symmetries of $3d$ conformal gravity in the semi-classical limit (large central charge c or level κ limit). This analysis aims to fill the gap in the literature, where the previous works on asymptotic symmetries of $3d$ conformal gravity except in [20], did not include

[1] Brown York charges corresponding to Neumann boundary conditions evaluate to zero since for asymptotically AdS configurations, they are proportional to holographic stress tensor contracted with boundary Killing field [102, 95].

[2] See Appendix (G) for all possible chiral extensions of $\mathfrak{so}(2,2)$.

the global symmetry algebra $\mathfrak{so}(2,3)$ [21, 22, 23]. See [104, 105, 106] for the role played by $3d$ conformal gravity in AdS$_4$ gravity.[3]

The thesis comprises two parts. In the first part, we concern ourselves with some aspects of flat space holography. The following is a summary of the main results of this part of the thesis:

- **$\widehat{\mathfrak{sl}}_2$ symmetry of $\mathbb{R}^{1,3}$ gravity**: We propose chiral boundary conditions for $4d$ asymptotically flat solutions in Newman-Unti gauge consistent with the variational principle. The asymptotic symmetry algebra of such solutions is shown to be chiral $\mathfrak{bms}_4$ which was derived from the celestial CFT in [13, 14]. We also construct boundary terms which are suitable for boundaries near null infinity that lead to a well-defined variational principle. We obtain chiral currents that are predicted to exist from celestial CFT in $\mathbb{R}^{1,3}$ gravity [13, 14].

  We also show the existence of a novel non-chiral infinite dimensional symmetry algebra of $\mathbb{R}^{1,3}$ gravity by proposing different non-chiral boundary conditions where the boundary metric is considered in conformal gauge.

- **Scalar Carollian conformal Field theories**: To fully understand the properties of Carollian CFTs, one needs to construct such theories intrinsically on Carollian manifolds. We address this by providing a systematic method to construct such field theories. Specifically, we provide an explicit construction of classes of conformal scalar field theories on Carollian manifolds in three dimensions using the geometrical covariant tensors of the Carroll geometry. These theories by construction have their symmetry algebra as Carollian conformal algebra $\mathfrak{cca}_3^{(z)}$ [70], a one-parameter generalization of $\mathfrak{bms}_4$, defined for generic values of dynamical exponent z. We also show that based on the different ways of gauge fixing the background data using the local symmetries of Carroll Weyl and Carroll Diff. invariant field theories, one obtains a different set of residual symmetry algebra including generalised $\mathfrak{bms}_4$ algebra.

In the second part of the thesis, we present the results in AdS$_4$ and $3d$-conformal gravity. The summary of the main results of this part of the thesis are:

- **A chiral $\mathcal{W}$-algebra extension of $\mathfrak{so}(2,3)$ from AdS$_4$ gravity**: We propose chiral boundary conditions for AdS$_4$ gravity in Newman-Unti gauge inspired

[3] See [81, 107] for the analogous role played by $3d$ Carrollian conformal gravity for $4d$ flat space gravity.

by the chiral boundary conditions of $\mathbb{R}^{1,3}$ gravity considered in the first part of the thesis and, obtain the symmetry algebra for resultant solutions. This symmetry algebra is an infinite dimensional chiral extension of $\mathfrak{so}(2,3)$, which coincides with the chiral $\mathfrak{bms}_4$ algebra in the flat space limit. We posit this symmetry algebra as the chiral version of Λ-$\mathfrak{bms}_4$ algebra.

We propose line integral charges from the bulk AdS_4 gravity corresponding to this chiral symmetry algebra and show that the charges themselves obey a semi-classical limit of a $\mathcal{W}$-algebra that includes a level k, $\mathfrak{sl}(2,\mathbb{R})$ current algebra. We then derive the complete quantum version of this $\mathcal{W}$ algebra using the tools of $2d$ CFT such as the associativity of operator product algebra and denote it by $\mathcal{W}(2;(3/2)^2,1^3)$.

- **All chiral $\mathcal{W}$-algebra extensions of $\mathfrak{so}(2,3)$**: Using the same techniques of $2d$ CFT, we construct all four infinite dimensional chiral extensions of $\mathfrak{so}(2,3)$ by showing that there are exactly four inequivalent ways to embed the $\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{so}(2,3)$. Each of these embeddings gives rise to a $\mathcal{W}$-algebra. The four resultant $\mathcal{W}$-algebras are denoted by, $\mathcal{W}(2;2^2,1)$, $\widetilde{\mathcal{W}}(2;2^2,1)$, $\mathcal{W}(2,4)$ and previously mentioned $\mathcal{W}(2;(3/2)^2,1^3)$. The semi-classical limit of $\mathcal{W}(2;2^2,1)$ was known from the asymptotic symmetries of $3d$ conformal gravity [20]. The $\mathcal{W}(2,4)$, $\mathcal{W}(2;(3/2)^2,1^3)$ can be identified with those that existed in the older literature in different contexts [24, 25].

- **$\mathcal{W}$-algebras from $3d$ conformal gravity**: In the Chern-Simons theory formulation of $3d$ conformal gravity, we provide boundary conditions and gauge fixing of $\mathfrak{so}(2,3)$ gauge connection compatible with variational principle. We then show that the symmetry algebra of charges that generate residual gauge transformations of these gauge connections form representations of $\mathcal{W}(2,4)$, $\mathcal{W}(2;(3/2)^2,1^3)$ algebras in the semi-classical limit. The analogous calculation for $\mathcal{W}(2;2^2,1)$ already existed in the literature [20], which can be extended trivially for the case of $\widetilde{\mathcal{W}}(2;2^2,1)$.

  The gravitational Chern-Simons action equivalently describes the $3d$ conformal gravity in its second-order formulation. After making the case for the importance of this action for AdS_4 gravity, we show the emergence of $\mathcal{W}(2;(3/2)^2,1^3)$ algebra from the asymptotic symmetry analysis of the solutions to gravitational Chern-Simons term. Furthermore, we calculate charges that generate this symmetry algebra by the modified Lee-Wald covariant phase space formalism method defined for Lagrangian density like gravitational Chern-Simons which we review in the next section.

The organization of the thesis

1. We end this chapter by reviewing some of the preliminary concepts and important results used in this thesis.
2. The second chapter will mainly analyze chiral boundary conditions for asymptotically flat spacetime in $\mathbb{R}^{1,3}$ gravity and obtain symmetry algebra for such solutions.
3. The third chapter will deal with a general method for constructing Carrollian conformal scalar field theories in 3 dimensions on a curved Carroll manifold.
4. The fourth chapter will discuss chiral boundary conditions for asymptotically AdS_4 spacetime, obtaining symmetry algebra for such solutions and deriving $2d$ chiral $\mathcal{W}$-algebra that governs such asymptotic symmetry algebra.
5. The fifth chapter will discuss the derivation of all four chiral $\mathcal{W}$-algebra extensions of $\mathfrak{so}(2,3)$. In this chapter we further derive $\mathcal{W}(2;(3/2)^2,1^3)$ and $\mathcal{W}(2,4)$ as the asymptotic symmetry algebra of $3d$ $\mathfrak{so}(2,3)$ gauge theory. We also show the emergence of $\mathcal{W}(2;(3/2)^2,1^3)$ from the second-order formulation of $3d$ conformal gravity described by gravitational Chern-Simons action.
6. The sixth chapter will discuss future directions.

2.1 Preliminaries

2.1.1 Review of asymptotic symmetries

In this section, we review some of the techniques associated with the asymptotic symmetries that will be useful for our analysis later. For a comprehensive review please refer to [108, 109]. Gravity is a gauge theory with diffeomorphisms playing the role of gauge transformations. This is also evident from the fact that the charges associated with these diffeomorphisms vanish. To obtain non-trivial canonical charges associated with these theories one defines the charges on the codimension-two boundaries of bulk space-time. This is because the physical gauge transformations associated with these gravitational theories exist near the boundary and generate non-trivial boundary dynamics. These are large gauge transformations that do not vanish at the boundary and give rise to non-zero charges. To find asymptotic symmetries of these gravitational theories one starts by gauge fixing some of the components of the metric using diffeomorphisms. This removes some of the gauge redundancies but there is still some residual gauge freedom left. One then imposes non-trivial boundary conditions on the fields and metric components of the gravitational theories near the boundary of spacetimes and demands that the residual gauge transformations preserve those boundary conditions. As defined in [108], there exist two definitions of the asymptotic symmetry group. In the first definition, which is weaker, the asymptotic symmetry group of gravitational theories are set of residual gauge diffeomorphisms that preserve the boundary conditions. The second, which is a stronger condition defines the asymptotic symmetries as a set of residual gauge diffeomorphisms that preserve boundary conditions and have non-vanishing charges associated with these diffeomorphisms. Therefore it is quite evident that the boundary conditions play a huge role in determining the set of asymptotic symmetries.

We will work in four dimensions with the coordinates (r, u, x^a) $(a \in \{1, 2\})$ suitable for asymptotically flat spacetimes and gauge fix our metric in Newman-Unti gauge [110] which is characterised by the following conditions

$$g_{rr} = g_{ra} = 0, \quad g_{ur} = -1. \tag{2.1}$$

The line element after gauge fixing is given by

$$ds^2 = h_{uu}\, du^2 - 2\, du\, dr + 2\, g_{ua}\, du\, dx^a + g_{ab}\, dx^a\, dx^b, \tag{2.2}$$

where h_{uu}, g_{ua}, g_{ab} are functions of (u, r, x^a) coordinates. The boundary topology is $\Sigma_2 \times \mathbb{R}$, where codimension-two hypersurface Σ_2 can be $\mathbb{S}^2$ or $\mathbb{R}^2$. This gauge is different from the commonly used Bondi gauge which imposes the following gauge fixing conditions on the metric,

$$g_{rr} = g_{rz} = g_{r\bar{z}} = 0, \quad \partial_r \left(\frac{det\,(g_{ab})}{r^4} \right) = 0. \tag{2.3}$$

As mentioned in [92], both ways of gauge fixing differ only by choice of radial coordinates which does not impact the calculation of asymptotic symmetries and hence asymptotic symmetry algebra in both of these gauges are found to be isomorphic to each other. Apart from these gauge conditions, the following boundary conditions are imposed which are not too stringent and allow for the possibility of some physical examples [5]

$$\lim_{r\to\infty} g_{uu} = -\,1 + \mathcal{O}(r^{-1}), \quad g_{ua} = \mathcal{O}(1), \quad g_{ab} = r^2\gamma_{ab} + \mathcal{O}(r), \tag{2.4}$$

where γ_{ab} is a $2d$ fixed metric defined on the Σ_2. Under any residual diffeomorphisms the gauge conditions on metric component should be preserved, therefore to find the diffeomorphisms that leave the form of asymptotically flat metric invariant, we demand that under such transformations the Bondi gauge conditions (2.1) are obeyed. This translates to

$$\mathcal{L}_\xi g_{rr} = 0, \quad \mathcal{L}_\xi g_{ra} = 0, \quad \mathcal{L}_\xi g_{ur} = 0. \tag{2.5}$$

The Lie derivative of the metric tensor is

$$\mathcal{L}_\xi g_{\mu\nu} = \partial_\sigma \xi^\sigma g_{\mu\nu} + \partial_\mu \xi^\sigma g_{\sigma\nu} + \partial_\nu \xi^\sigma g_{\mu\sigma}, \tag{2.6}$$

and the vector field in the coordinate-dependent chart is written as follows,

$$\xi = \xi^u(u,r,x^a)\,\partial_u + \xi^r(u,r,x^a)\,\partial_r + \xi^a(u,r,x^a)\,\partial_a. \tag{2.7}$$

Therefore solving (2.5) one obtains

$$\begin{aligned} \xi^u &= \xi^u(u,x^a)\,, \quad \xi^a = Y^a(u,x^c) - \partial_b \xi^u_{(0)}(u,x^c)\int_r^\infty g^{ab} dr, \\ \xi^r &= -r\,\partial_u \xi^u + \int_r^\infty g_{ua}\,\partial_r\,\xi^a. \end{aligned} \tag{2.8}$$

The asymptotic symmetries should also preserve boundary conditions (2.4), which means that the corresponding Lie derivative of the metric component should obey the same boundary conditions and fall-offs. For example, the Lie derivative of $\mathcal{L}_\xi g_{uu}$ should fall off as $\mathcal{O}\left(\frac{1}{r}\right)$, which means all the higher order terms in r should go to zero for this Lie derivative. This allows us to solve for various functions of the vector fields and further reduce the dependence of functions on the number of coordinates

$$\xi^u(u,z,\bar{z}) = \xi^u_{(0)}(z,\bar{z}) + u\,\xi^u_{(1)}(z,\bar{z}), \quad \xi^a_{(0)} = \xi^a_{(0)}(x^c), \tag{2.9}$$

$$-\,2\,\partial_u \xi^u\,\gamma_{ab} + D_a \xi^c_{(0)}\,\gamma_{cb} + D_b \xi^c_{(0)}\,\gamma_{ab} = 0. \tag{2.10}$$

The equation (2.10) implies that $\xi^u_{(1)} = \frac{1}{2} D_a \xi^a_{(0)}$ and

$$D_a \xi_{b(0)} + D_b \xi_{a(0)} = D_c \xi^c_{(0)} \, \gamma_{ab}. \tag{2.11}$$

This is the conformal killing equation for ξ^a for metric γ_{ab} defined on Σ_2 at null infinity. D_a is the covariant derivative associated with the metric γ_{ab} . The resultant vector fields that preserve the Newman-Unti gauge (2.1) and boundary conditions (2.4) to leading order i r are,

$$\xi = \left(\xi^u_{(0)}(x^a) + \frac{u}{2} D_c \xi^c_{(0)} \right) \partial_u + \xi^a_{(0)}(x^c) \partial_a - \frac{r}{2} \, D_a \xi^a_{(0)}(x^c) \, \partial_r + \cdots , \tag{2.12}$$

with $\xi^a_{(0)}(x^a)$ satisfying CKV equation (2.11). The $\cdots$ in (2.12) denote subleading terms that are determined in terms of $\{\xi^u_{(0)}(x^a), \xi^a_{(0)}(x^c)\}$. Also if at some order in r, there exists free data in the metric component in (2.2), then the variation of it would be the coefficient of the Lie derivative of that metric component at the same order. Now that we have our full set of vector fields that enable us to move in the space of solutions, to check that they satisfy some Lie algebra, the Lie brackets between any two of these vector fields should close under the Courant bracket (see for instance [111])

$$[\, \xi_1 \, , \, \xi_2 \,]_M := \xi_1^\sigma \, \partial_\sigma \, \xi_2 - \xi_2^\sigma \, \partial_\sigma \, \xi_1 - \delta_{\xi_1} \, \xi_2 + \delta_{\xi_2} \, \xi_1. \tag{2.13}$$

The last two extra terms in (2.13) are to take into account any background field dependence of ξ. $\delta_{\xi_1} \xi_2$ means the variation of background field in ξ_2 with respect to the variation in ξ_1. Calculating the bracket as in (2.13) we find:

$$[\, \xi_1 \, , \, \xi_2 \,]_M = \left(\hat{\xi}^u_{(0)} + \frac{u}{2} \, D_c \, \hat{\xi}^c_{(0)} \right) \partial_u + \left(\hat{\xi}^a_{(0)} \right) \, \partial_a - \frac{r}{2} \, D_a \, \hat{\xi}^a_{(0)} \, \partial_r + \cdots \tag{2.14}$$

where

$$\hat{\xi}^u_{(0)} = \xi^a_{1(0)} \, \partial_a \, \xi^u_{2\,(0)} + \frac{1}{2} \, \xi^u_{1\,(0)} \, D_a \, \xi^a_{2(0)} - \xi^a_{2(0)} \, \partial_a \, \xi^u_{1\,(0)} - \frac{1}{2} \, \xi^u_{2\,(0)} \, D_a \, \xi^a_{1(0)}, \tag{2.15}$$

$$\hat{\xi}^a_{(0)} = \xi^b_{1(0)} \, \partial_b \, \xi^a_{2(0)} - \xi^b_{2(0)} \, \partial_b \, \xi^a_{1(0)}. \tag{2.16}$$

Considering the Lie bracket with $\xi^a_{1(0)} = \xi^a_{2(0)} = 0$ one can check that the resultant $\hat{\xi}^u_{(0)} = 0$ and therefore they commute just like translations in Poincare algebra $\mathfrak{iso}(1,3)$. However here $\xi^u_{(0)}$ is an arbitrary function defined on Σ_2 and is a vector field associated with *supertranslations*, which are angle-dependent translations in the null direction u at null infinity. The vector field at the boundary when one set

$\xi^u_{(0)} = 0$ is given as $\xi_{bdry} = \frac{u}{2}\, D_c \xi^c_{(0)} \partial_u + \xi^a_{(0)} \partial_a$.

Global $\mathfrak{bms}_4$ algebra: When the vector fields $\xi^a_{(0)}$ are restricted to the globally well-defined transformations on Σ_2 then their algebra obeys global conformal algebra $\mathfrak{sl}(2,\mathbb{C})/\mathbb{Z}_2$ which is isomorphic to Lorentz algebra $\mathfrak{so}(1,3)$. The $\xi^u_{(0)}(x^a)$ for example can be expanded in the natural basis of spherical harmonics Y_{lm} which are smooth functions on $\Sigma_2 = \mathbb{S}^2$. The normal translations are the special case of supertranslations for $l=0$ and $l=1$ spherical harmonics. Therefore the resulting algebra becomes the semi-direct sum of these smooth functions (supertranslations) and global conformal transformations.

Extended $\mathfrak{bms}_4$ algebra: Taking a cue from the two-dimensional conformal field theory, if the functions $\xi^a_{(0)}, \xi^u_{(0)}$ are allowed to be meromorphic functions with a finite set of poles on $\mathbb{S}^2$ or $\mathbb{R}^2$ then they can be expanded in Laurent series if one works in stereographic coordinates ($x^a \in \{z, \bar{z}\}$) where the metric on Σ_2 takes the form $\gamma_{ab}\, dx^a dx^b = \Omega(z,\bar{z}) dz\, d\bar{z}$.

The general solution of the conformal killing equation in (2.11) is $\xi^z = Y(z)$ and $\xi^{\bar{z}} = \bar{Y}(\bar{z})$. Now if we denote the vector field with components $Y(z) = -z^{n+1}, \xi^u_{(0)} = 0, \bar{Y}(\bar{z}) = 0$ by L_n and $\bar{Y}(\bar{z}) = -\bar{z}^{n+1}, \xi^u_{(0)} = 0, Y(z) = 0$ by $\bar{L}_n$ the commutation relation using (2.16) is,

$$[\, L_m, L_n\,] = (m-n) L_{m+n}, \quad [\, \bar{L}_m, \bar{L}_n\,] = (m-n) \bar{L}_{m+n}, \quad [\, L_m, \bar{L}_m\,] = 0. \qquad (2.17)$$

If one denotes supertranslation vector field with components $\bar{Y}(\bar{z}) = Y(z) = 0, \xi^u_{(0)} = z^m\, \bar{z}^n$ by $T_{m,n}$ then the other commutators are given by,

$$[\, L_n, T_{p,q}\,] = \left(\frac{n+1}{2} - p\right) T_{p+l,q}, \qquad [\, \bar{L}_n, T_{p,q}\,] = \left(\frac{n+1}{2} - q\right) T_{p,q+l} \qquad (2.18)$$

The extended $\mathfrak{bms}_4$ algebra is the direct sum of supertranslations and local conformal translations which are also called superrotations [11]. These transformations allow for singular transformations of the boundary metric.

Generalised $\mathfrak{bms}_4$ algebra: If one loosen the boundary conditions (2.4) to,

$$g_{uu} = \mathcal{O}(1)\,, \quad g_{ab} = r^2\, q_{ab} + \mathcal{O}(1) \qquad (2.19)$$

where q_{ab} is an arbitrary metric independent of u on Σ_2 with fixed determinant. The vector fields associated with the residual gauge transformations that preserve these boundary conditions (2.19) are,

$$\xi_{GBM} = \left(\xi^u_{(0)}(x^a) + \frac{u}{2} D_c \xi^c_{(0)}\right)\partial_u + \xi^a_{(0)}(x^c)\partial_a - \frac{r}{2}\, D_a \xi^a_{(0)}(x^c)\,\partial_r + \cdots \tag{2.20}$$

with CKV condition dropped on $\xi^a_{(0)}$. The symmetry algebra is then the direct sum of supertranslations and algebra of arbitrary diffeomorphisms on Σ_2 $\mathfrak{diff}(\mathbb{S}^2)$ with the following transformation of the boundary metric,

$$\delta q_{ab} = -2\, D_c\, \xi^c_{(0)}\, q_{ab} + D_a\, \xi_{b(0)} + D_b\, \xi_{a(0)}. \tag{2.21}$$

2.1.2 Current Algebra from Celestial CFT

The $\mathcal{S}$-matrix elements of $\mathbb{R}^{1,3}$ gravity can be recast as $2d$ conformal correlators, called the celestial amplitudes [50, 49]. By taking conformal soft limits one can uncover Ward identities of various $2d$ conformal currents – referred to as the conformal soft theorems. Here we review the existence of two current algebra symmetries of the $2d$ celestial amplitudes that follow from the leading and subleading conformal soft theorems for positive helicity soft graviton operators (see [13, 14] for notation and more details). One starts with the leading order conformal soft theorem for an outgoing positive helicity soft graviton operator $S_0^+(z,\bar{z})$

$$\big\langle S_0^+(z,\bar{z}) \prod_{i=1}^n \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\big\rangle = -\left(\sum_{k=1}^n \frac{\bar{z}-\bar{z}_k}{z-z_k}\epsilon_k \mathcal{P}_k\right)\big\langle \prod_{i=1}^n \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\big\rangle, \tag{2.22}$$

where $\mathcal{P}_k\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i) = \delta_{ki}\phi_{h_i+1/2,\bar{h}_i+1/2}(z_i,\bar{z}_i)$, $\epsilon_k = \pm 1$ for an outgoing (incoming) particle. Also $(h_k,\bar{h}_k) = (\frac{\Delta_k+\sigma_k}{2}, \frac{\Delta_k-\sigma_k}{2})$ where Δ_k and σ_k are the scaling dimension and helicity of the k-th particle respectively. The RHS of equation (2.22) is a polynomial in $\bar{z}$, so we can expand it around $\bar{z}=0$ and rewrite the equation as

$$\begin{aligned}&\big\langle S_0^+(z,\bar{z}) \prod_{i=1}^n \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\big\rangle \\ &= \left(\sum_{k=1}^n \frac{\bar{z}_k}{z-z_k}\epsilon_k\mathcal{P}_k\right)\big\langle \prod_{i=1}^n \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\big\rangle - \bar{z}\left(\sum_{k=1}^n \frac{1}{z-z_k}\epsilon_k\mathcal{P}_k\right)\big\langle \prod_{i=1}^n \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\big\rangle.\end{aligned} \tag{2.23}$$

Now if we define two currents $C^{\frac{1}{2}}(z)$ and $C^{-\frac{1}{2}}(z)$ in the following way,

$$S_0^+(z,\bar{z}) = C^{\frac{1}{2}}(z) - \bar{z}\, C^{-\frac{1}{2}}(z) \tag{2.24}$$

then equation (2.23) implies separate Ward identities of these two currents given by,

$$\big\langle C^{\frac{1}{2}}(z)\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\big\rangle = \left(\sum_{k=1}^{n}\frac{\bar{z}_k}{z-z_k}\epsilon_k\mathcal{P}_k\right)\big\langle\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\big\rangle, \tag{2.25}$$

and

$$\big\langle C^{-\frac{1}{2}}(z)\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\big\rangle = \left(\sum_{k=1}^{n}\frac{1}{z-z_k}\epsilon_k\mathcal{P}_k\right)\big\langle\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\big\rangle. \tag{2.26}$$

One can see from (2.25) and (2.26) that $C^{\frac{1}{2}}(z)$ and $C^{-\frac{1}{2}}(z)$ are generators of infinitesimal supertranslations acting as:

$$\delta_{P^{\frac{1}{2}}}\phi_{h,\bar{h}}(z,\bar{z}) = \bar{z}\,P^{\frac{1}{2}}(z)\,\epsilon\mathcal{P}\,\phi_{h,\bar{h}}(z,\bar{z}), \tag{2.27}$$

and

$$\delta_{P^{-\frac{1}{2}}}\phi_{h,\bar{h}}(z,\bar{z}) = P^{-\frac{1}{2}}(z)\,\epsilon\mathcal{P}\phi_{h,\bar{h}}(z,\bar{z}) \tag{2.28}$$

respectively. Next, we turn to the emergence of $\mathfrak{sl}(2,\mathbb{R})$ current algebra from the celestial CFT. For this one starts with the corresponding subleading conformal soft graviton theorem (the holographic analogue of [38]) for a positive helicity outgoing soft graviton operator $S_1^+(z,\bar{z})$

$$\big\langle S_1^+(z,\bar{z})\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\big\rangle = \sum_{k=1}^{n}\frac{(\bar{z}-\bar{z}_k)^2}{z-z_k}\left[\frac{2\bar{h}_k}{\bar{z}-\bar{z}_k}-\bar{\partial}_k\right]\big\langle\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\big\rangle, \tag{2.29}$$

where $S_1^+(z,\bar{z})$ is given by

$$S_1^+(z,\bar{z}) = \lim_{\Delta\to 0}\Delta\,G_\Delta^+(z,\bar{z}). \tag{2.30}$$

One follows the same procedure as the leading conformal soft theorem above. Expanding the RHS of (2.29) in powers of $\bar{z}$ gives

$$\begin{aligned}
&\big\langle S_1^+(z,\bar{z}) \prod_{i=1}^{n} \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\big\rangle \\
&= -\sum_{k=1}^{n} \frac{\bar{z}_k^2\bar{\partial}_k + 2\bar{h}_k\bar{z}_k}{z-z_k} \langle\prod_{i=1}^{n} \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle + 2\bar{z}\sum_{k=1}^{n} \frac{\bar{h}_k + \bar{z}_k\bar{\partial}_k}{z-z_k} \langle\prod_{i=1}^{n} \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle \\
&\quad -\bar{z}^2 \sum_{k=1}^{n} \frac{1}{z-z_k}\frac{\partial}{\partial\bar{z}_k} \langle\prod_{i=1}^{n} \phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\rangle.
\end{aligned} \tag{2.31}$$

Using this one can define three currents $J^i(z)$ where $i = 0, \pm 1$, which are the generators of $\mathfrak{sl}(2,\mathbb{R})$ current algebra. In terms of these currents, we can write the soft graviton operator $S_1^+(z,\bar{z})$ as,

$$S_1^+(z,\bar{z}) = -J^1(z) + 2\,\bar{z}\,J^0(z) - \bar{z}^2 J^{-1}(z). \tag{2.32}$$

The mode algebra of the currents $J^i(z)$ is

$$\left[J^i_m, J^j_n\right] = (i-j)J^{i+j}_{m+n}. \tag{2.33}$$

The symmetry algebra we have discussed has been obtained by analysing the correlation functions between the primary operators of the $2d$ celestial field theory and conformally soft gravitons. If a holographic duality exists between gravitational theories in four-dimensional asymptotically flat spacetime and $2d$ celestial conformal field theory then the bulk theory should also possess these symmetries. This motivates us to search for these symmetries directly in the bulk $\mathbb{R}^{1,3}$ gravity.

2.1.3 An $\mathfrak{sl}(2,\mathbb{R})$ current algebra from AdS$_3$ gravity

An $\mathfrak{sl}(2,\mathbb{R})$ current algebra can be seen as the asymptotic symmetry algebra of AdS_3 gravity by taking the boundary metric of asymptotically AdS_3 geometries in the Polyakov gauge [112]. Working in the Fefferman-Graham gauge the locally AdS_3 geometries can be written as

$$ds^2_{lAdS_3} = l^2\,\frac{dr^2}{r^2} + r^2\left(g^{(0)}_{ab} + \frac{l^2}{r^2}g^{(2)}_{ab} + \frac{l^4}{r^4}g^{(4)}_{ab}\right)dx^a dx^b \tag{2.34}$$

along with

$$g^{(4)}_{ab} = \frac{1}{4} g^{(2)}_{ac} g^{cd}_{(0)} g^{(2)}_{db}, \quad \nabla^a_{(0)} g^{(2)}_{ab} = \frac{1}{2} \partial_b R_0, \quad \text{and} \quad g^{ab}_{(0)} g^{(2)}_{ab} = \frac{1}{2} R_{(0)}. \tag{2.35}$$

Choosing the chiral gravity gauge of Polyakov

$$ds^2 = (F(z,\bar{z})\, dz + d\bar{z})\, dz \tag{2.36}$$

for the boundary metric $g^{(0)}_{ab}$ the differential conditions impose the following equation:

$$\left(\partial_z g^{(2)}_{\bar{z}\bar{z}} - F\,\partial_{\bar{z}} - 2\,\partial_{\bar{z}} F\right) g^{(2)}_{\bar{z}\bar{z}} = \frac{1}{2} \partial^3_{\bar{z}} F\,. \tag{2.37}$$

One also has

$$g^{(2)}_{zz} = \kappa(z) + F^2\, g^{(2)}_{\bar{z}\bar{z}} + \frac{1}{4}\left[(\partial_{\bar{z}} F)^2 + 2\, F\, \partial^2_{\bar{z}} F + 2\, \partial_z \partial_{\bar{z}} F\right]. \tag{2.38}$$

Finally, the bulk variational problem can be satisfied by setting $g^{(2)}_{\bar{z}\bar{z}} = 0$ as shown in [10]. This makes $F(z,\bar{z})$ a polynomial of degree 2 in $\bar{z}$. The residual bulk diffeomorphisms lead to one copy of $\mathfrak{sl}(2,\mathbb{R})$ current algebra and one copy of Virasoro algebra.[4] We will implement a similar procedure in $\mathbb{R}^{1,3}$ gravity and demonstrate that an analogous $\mathfrak{sl}(2,\mathbb{R})$ chiral current algebra emerges in this context as well.

2.1.4 Asymptotic symmetries of $\mathfrak{so}(2,3)$ gauge theory

The $3d$ conformal gravity theory was originally formulated in a first-order formalism where the basic field is the vielbein $e_\mu{}^a$ ($\mu, \nu, \cdots$ are spacetime indices and $a, b, \cdots$ are local Lorentz indices.). The torsionless condition on the manifold is given by

$$\left[\partial_\mu e^a_\nu + \omega^{ab}_\mu e^b_\nu\right] - \left[\partial_\nu e^a_\mu + \omega^{ab}_\nu e^b_\mu\right] = 0. \tag{2.39}$$

The spin connection is defined in terms of $e_\mu{}^a$ and the Christoffel symbol associated with the metric such that (2.39) is solved. In $3d$ one can construct a Lorentz vector from a spin connection,

$$\omega^a =: \frac{1}{2} \epsilon^{abc} \omega_{bc}. \tag{2.40}$$

[4] Strictly speaking the $\mathfrak{sl}(2,\mathbb{R})$ current algebra is the correct one only when one is talking about the Lorentizan AdS_3 theory. When dealing with Euclidean theory we should say the relevant algebra is $\mathfrak{sl}(2,\mathbb{C})$.

The action associated with $3d$ conformal gravity is then [113, 114, 103]

$$S = \int_M \epsilon^{\mu\nu\rho} \left[\omega_{\mu a} \left(\partial_\nu \omega_\rho{}^a - \partial_\rho \omega_\nu{}^a \right) + \frac{2}{3} \epsilon^{abc} \, \omega_{\mu a} \, \omega_{\nu b} \, \omega_{\rho c} \right] . \tag{2.41}$$

The equation of motion one gets after varying the action (2.41) with respect to $e_\mu{}^a$ is

$$\nabla_\rho W_{\mu\nu} - \nabla_\nu W_{\mu\rho} = 0, \tag{2.42}$$

where $W_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} \, R$ is a Schoutten tensor. ∇ is the covariant derivative for the Christoffel symbol associated with the metric $g_{\mu\nu}$. The equation (2.42) is a necessary and sufficient condition for the metric $g_{\mu\nu}$ to be conformally flat in three dimensions. The LHS of (2.42) is also the definition of Cotton tensor in $3d$.

Witten and Horne [103] showed that the equation of motion (2.42) associated with the action (2.41) is equivalent to flat connection of $\mathfrak{so}(2,3)$ Chern-Simons gauge theory given by the action

$$S_{CS} = \frac{k}{4\pi} \int_M \text{Tr} \left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right), \tag{2.43}$$

where $\mathcal{A}$ is a $\mathfrak{so}(2,3)$ Lie-algebra value one form gauge field given by,

$$A_\mu dx^\mu = e^a{}_\mu \, P_a + \omega_\mu^a J_a + \lambda_\mu^a K_a + \phi \, D. \tag{2.44}$$

P_a, J_a, K_a and D are generators associated with three translations, three Lorentz transformations, three special conformal transformations and a dilatation respectively and obey the following $\mathfrak{so}(2,3)$ algebra

$$\begin{aligned} &[P_a, J_b] = \epsilon_{abc} P^c \,, \quad [J_a, J_b] = \epsilon_{abc} J^c \,, \quad [K_a, J_b] = \epsilon_{abc} K^c \,, \quad [P_a, D] = P_a, \\ &[P_a, K_b] = -\epsilon_{abc} \, J^c + \eta_{ab} D \,, \quad [K_a, D] = -K_a. \end{aligned} \tag{2.45}$$

The gauge choice considered to show such an equivalence was $\phi = 0$ [103] as it ensured that the vielbein $e_\mu{}^a$ are non-degenerate which was required by the conformal gravity theory (2.41). The action (2.43) is also gauge equivalent to (2.41) up to a boundary term with this gauge choice [23]. To find the asymptotic symmetries of $\mathfrak{so}(2,3)$ gauge theory we go through the following steps,

- We start by considering one-form $\mathcal{A}$ in $3d$ which takes value in $\mathfrak{so}(2,3)$ algebra.
- Just like in Chern-Simons theory formulation of AdS_3 gravity and its higher

spin extensions [115, 116, 117], one can gauge away the radial dependence and effectively work with a $2d$ gauge connection $\mathfrak{a}$ at the boundary of the manifold. If r is the radial coordinate then for a choice of $\mathfrak{b}$, one can define

$$\mathfrak{a}(x^a) = \mathfrak{b}^{-1}\,\mathcal{A}\,\mathfrak{b} - d\mathfrak{b}^{-1}\,\mathfrak{b} \tag{2.46}$$

where x^a are the coordinates on $2d$ boundary manifold and $\mathfrak{b} = f(r)$.

- From the flat connection condition of $\mathcal{A}$, one obtain equations of motion for $\mathfrak{a}$ using $d\mathfrak{a}+\mathfrak{a}\wedge\mathfrak{a} = 0$. We further choose gauge and boundary conditions that are compatible with a well-defined variational principle. The boundary conditions should be such that the resultant background geometry should also include global AdS_3 and flat $\mathbb{R}^{1,2}$ gravity solutions as well. These conditions allow us to solve for constraints on various components of $\mathfrak{a}$ arising from equations of motion. The full solution space can then be obtained in terms of free independent data.

- Then one consider gauge transformation parameter λ which is a scalar and also takes value in $\mathfrak{so}(2,3)$ algebra and uses $\delta\mathfrak{a} = d\Lambda + [\mathfrak{a},\Lambda]$ to obtain the set of residual gauge transformations that do not spoil the gauge and boundary conditions of the connection $\mathfrak{a}$. These residual gauge transformations in turn induce variation of the free data in the connection.

- The canonical charges that generate these gauge transformations are given by [118]

$$\not{\delta} Q = -\frac{k}{2\pi}\int \mathrm{Tr}\,\left(\delta\mathcal{A}_\mu \Lambda\right). \tag{2.47}$$

- One can write down the poison brackets between various free data in $\mathfrak{a}$ such that the variation of fields can be obtained as $\delta_\Lambda f = \{Q_\Lambda, f\}$. The commutator from Poisson brackets can be obtained after expanding the fields in terms of the mode.

2.1.5 Charges using modified Lee-Wald covariant phase space formalism

Here we will review the formalism to calculate the charges for Lagrangian density that are not invariant under the symmetry transformations and will use the notation and formulae of [28, 119]. We consider Lagrangian density $L(\phi)$ (ϕ denotes the collection of various fields) as a D-form in D-dimensions and χ as the collection of symmetry parameters such that

$$\delta_\chi L(\phi) = \mathcal{L}_\chi L(\phi) + d\,\Xi_\chi, \tag{2.48}$$

where Ξ_χ is a $(D-1)$ form. In Wald's derivation of the entropy formula [120, 121], (2.48) is without the second term. The variation of L is given by,

$$\delta L = \sum_\phi E_\phi \delta\phi + d\Theta(\phi, \delta\phi). \tag{2.49}$$

E_ϕ is the equations of motion associated with ϕ and Θ is a presymplectic potential which is a $D-1$ form. The on-shell Noether current is then given by

$$J^\mu_\chi(\phi) = \Theta^\mu(\phi, \delta_\chi\phi) - \chi^\mu\, L(\phi) - \Xi^\mu_\chi(\phi), \tag{2.50}$$

which is conserved $\partial_\mu J^\mu_\chi(\phi) \simeq 0$, where $\simeq$ is the symbol for on-shell equality. One can construct $K^{\mu\nu}$ from this current such that

$$J^\mu_\chi(\phi) \simeq \partial_\nu K^{\mu\nu}_\chi(\phi). \tag{2.51}$$

Consider the variation of presymplectic potential under symmetry transformation as

$$\delta_\chi\Theta^\mu(\phi, \delta\phi) = \mathcal{L}_\chi\Theta^\mu(\phi, \delta\phi) + \Pi^\mu_\chi(\phi, \delta\phi). \tag{2.52}$$

The Lee-Wald symplectic potential is given by,

$$w^\mu(\phi, \delta\phi, \delta_\chi\phi) = \frac{1}{16\pi}\left(\delta\Theta^\mu(\phi, \delta_\chi\phi) - \delta_\chi\Theta^\mu(\phi, \delta\phi) - \Theta^\mu(\phi, \delta_{\delta_\chi}\phi)\right). \tag{2.53}$$

The presence of the last term in (2.53) is to compensate for the background dependence of the symmetry parameter χ. Using (2.50) and (2.51), the first term becomes

$$\delta\Theta^\mu(\phi, \delta_\chi\phi) \simeq \delta\partial_\nu K^{\mu\nu}_\chi(\phi) + \chi^\mu\delta L(\phi) + \delta\chi^\mu\, L(\phi) + \delta\Xi^\mu_\chi(\phi). \tag{2.54}$$

The third term is given as

$$\Theta^\mu(\phi, \delta_{\delta_\chi}\phi) \simeq \partial_\nu K^{\mu\nu}_{\delta\chi}(\phi) + \delta\chi^\mu\, L(\phi) + \Xi^\mu_{\delta\chi}(\phi). \tag{2.55}$$

Substituting (2.54), (2.55) and (2.52) in (2.53) one gets,

$$w^\mu(\phi,\delta\phi,\delta_\chi\phi) \simeq \frac{1}{16\pi}\left(\delta\partial_\nu K^{\mu\nu}_\chi(\phi) - \partial_\nu K^{\mu\nu}_{\delta\chi} + \delta\Xi^\mu_\chi(\phi) - \Xi^\mu_{\delta\chi}(\phi) - \Pi^\mu_\chi(\phi,\delta\phi)\right)$$
$$+ \chi^\mu d\Theta(\phi,\delta\phi) - \mathcal{L}_\chi\Theta^\mu(\phi,\delta\phi). \tag{2.56}$$

Using Cartan's magic formula $\mathcal{L}_\chi = i_\chi\, d + d\, i_\chi$,

$$\chi^\mu d\Theta(\phi,\delta\phi) - \mathcal{L}_\chi\Theta^\mu(\phi,\delta\phi) = i_\chi d\Theta - (i_\chi\, d + d\, i_\chi)\Theta = d\, i_\chi\Theta \tag{2.57}$$

and the theorem in [122], one can show that,

$$\delta\Xi^\mu_\chi(\phi) - \Xi^\mu_{\delta\chi}(\phi) - \Pi^\mu_\chi(\phi,\delta\phi) \simeq \partial_\nu\Sigma^{\mu\nu}_\chi. \tag{2.58}$$

Therefore the final expression of (2.53) after substituting everything is,

$$w^\mu(\phi,\delta\phi,\delta_\chi\phi) \simeq \partial_\nu Q^{\mu\nu}_\chi(\phi,\delta\phi), \tag{2.59}$$

where $Q^{\mu\nu}_\chi$ is given by

$$Q^{\mu\nu}_\chi = \frac{1}{16\pi}\left(\delta K^{\mu\nu}_\chi(\phi) - K^{\mu\nu}_{\delta\chi}(\phi) + 2\,\xi^{[\mu}\Theta^{\nu]}(\phi,\delta\phi) + \Sigma^{\mu\nu}_\chi(\phi,\delta\phi)\right). \tag{2.60}$$

(2.60) is the final expression for the surface charge for Lagrangian that is not invariant under the symmetry transformations. It has to be integrated appropriately on a codimension-two hypersurface.

Chapter 3

$\widehat{\mathfrak{sl}_2}$ symmetry of $\mathbb{R}^{1,3}$ gravity

As mentioned in the introduction, the chiral $\mathfrak{bms}_4$ algebra of celestial CFT introduced by the authors in [13, 14] includes one copy of the $\mathfrak{sl}(2,\mathbb{R})$ current algebra (with generators $J_{a,n}$ for $a = 0, \pm 1$ and $n \in \mathbb{Z}$), one copy of Virasoro (L_n with $n \in \mathbb{Z}$) and two $h = \frac{3}{2}$ conformal primaries (with generators $P_{s,r}$ with $s = \pm\frac{1}{2}$ and $r \in \mathbb{Z} + \frac{1}{2}$). The Ward identities of the currents associated with the operators $P_{s,r}$ and $J_{a,n}$ in celestial CFT are equivalent to leading and sub-leading soft graviton theorems as was reviewed in section (2.1.2). The algebra that these generators obey is,

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n}\,, \quad [J_{a,m}, J_{b,n}] = (a-b)\, J_{a+b,m+n}\,, \quad [L_m, J_{a,n}] = -n\, J_{a,m+n},\\
&[L_n, P_{s,r}] = \frac{1}{2}(n-2r)P_{s,n+r}\,, \quad [J_{a,m}, P_{s,r}] = \frac{1}{2}(a-2s)P_{a+s,m+r}\,, \quad [P_{s,r}, P_{s',r'}] = 0
\end{aligned} \tag{3.1}$$

with possible central terms, that remain undetermined.[1] This algebra is different from the classical $\mathfrak{bms}_4$ algebra of [3, 4]. In this chapter, we will derive the $4d$ bulk realisation of chiral $\mathfrak{bms}_4$ algebra and the currents associated with it. The $\mathfrak{sl}(2,\mathbb{R})$ current algebra like the one in (3.1) has already been derived from asymptotic symmetries of solutions in AdS$_3$ gravity [10]. We briefly reviewed this in (2.1.3). Taking a cue from the boundary conditions imposed in [10], we apply them to the solutions in $\mathbb{R}^{1,3}$ gravity. We present asymptotically locally flat solutions in $\mathbb{R}^{1,3}$ gravity consistent with the variational principle and derive the asymptotic symmetry algebra of such solutions to be the chiral $\mathfrak{bms}_4$ algebra. One implements the asymptotically locally flatness by demanding that $R^\mu{}_{\nu\rho\sigma} \to 0$ as $r \to \infty$ where $R^\mu{}_{\nu\rho\sigma}$ is the Rie-

[1] The computation of authors in [14] did not determine the value of central charge c of the Virasaro algebra.

mann tensor of the asymptotically locally flat solutions of $\mathbb{R}^{1,3}$ gravity, expanded in the radial coordinate r near the boundary. To obtain the asymptotic symmetries of these asymptotically locally flat solutions, one does not need to solve for the complete solution space. It is sufficient to find the locally flat solutions that share the same boundary conditions as these asymptotically locally flat solutions and study their asymptotic symmetries. Therefore, we analyse locally flat solutions, which are Ricci flat solutions with vanishing Riemann tensor.

The results of this chapter include,

- In Section (3.1), we provide all locally flat solutions (including complex ones) with u-independent boundary metric on spatial part (Σ_2) of null infinity, where u is the coordinate of null infinity.
- In Section (3.2) we impose a consistent and well-defined variational principle, and construct an appropriate set of boundary terms (Gibbon Hawking type) for the Einstein- Hilbert action.
- After choosing a chiral gauge for the metric on Σ_2, in Section (3.3), we show that the residual large diffeomorphisms generate chiral $\mathfrak{bms}_4$ algebra. In Section (3.4), we consider the boundary metric in a conformal gauge and show that the asymptotic symmetry algebra for the resultant solutions has a holomorphic and anti-holomorphic sector with each sector comprising of a Witt algebra and two $\mathfrak{u}(1)$-type current algebras.

We provide a discussion of our results in Section (3.5). In the Appendix A, we carry out the general analysis of asymptotic symmetries in the Newman-Unti gauge and we present the analysis of the $\mathfrak{iso}(1,3)$ gauge theory in the Appendix B, which is relevant for the discussion of charges in the main body of the chapter. This chapter is mostly based on the work [123].

3.1 A class of ALF spacetimes in four dimensions

We start with collecting some useful formulae towards the construction of asymptotically locally flat (ALF) solutions in $\mathbb{R}^{1,3}$ (or $\mathbb{R}^{2,2}$) gravity. We work with the coordinates $(r, u, z, \bar{z})$ suitable for the future null infinity and find it convenient to use the Newman-Unti gauge [92],

$$g_{rr} = g_{rz} = g_{r\bar{z}} = 0, \quad g_{ur} = -1. \tag{3.2}$$

Let us write the remaining metric components g_{ij} for $i, j \in \{u, z, \bar{z}\}$ in the following form:

$$g_{ij}(r, u, z, \bar{z}) = \sum_{n=0}^{\infty} r^{2-n}\, g_{ij}^{(n)}(u, z, \bar{z}) \tag{3.3}$$

We seek Ricci flat metrics ($R_{\mu\nu} = 0$) which are also asymptotically locally flat. The latter condition is implemented by demanding that $R^{\mu}{}_{\nu\sigma\lambda} \to 0$ as $r \to \infty$ [124].

The leading non-trivial conditions from solving $R_{\mu\nu} = 0$ for large r in a power series in $1/r$ require that $g_{ij}^{(0)}$ is degenerate. Since we are interested in solutions for which the metric $g_{ab}^{(0)}$ for $a, b \in \{z, \bar{z}\}$ on the spatial manifold Σ_2 is non-degenerate we solve this condition by assuming:

$$g_{uu}^{(0)} = g_{uz}^{(0)} = g_{u\bar{z}}^{(0)} = 0\,. \tag{3.4}$$

Then the next non-trivial condition implies that $g_{uu}^{(1)} = -\frac{1}{2}\partial_u(\log\det g_{ab}^{(0)})$. One also finds the condition $\det(\hat{\gamma}) = \frac{1}{4}(\mathrm{tr}\,\hat{\gamma})^2$, for the 2×2 matrix $\hat{\gamma}^a_b = g^{ac}_{(0)}\partial_u g^{(0)}_{cb}$. Any 2×2 matrix that satisfies such relation has to have both its eigenvalues the same – and therefore can only be proportional to the 2×2 identity matrix I_2. Thus we arrive at the following conditions: $\partial_u g_{ab}^{(0)} = \lambda\, g_{ab}^{(0)}\ \forall\ a, b \in \{z, \bar{z}\}$ with the same λ. This immediately leads to $g_{ab}^{(0)}(u, z, \bar{z}) = \Omega(u, z, \bar{z}) q_{ab}(z, \bar{z})$ where q_{ab} is u independent general 2×2 matrix. At $\mathcal{O}(1/r)$ from vanishing of R_{uz} and $R_{u\bar{z}}$ we find $g_{uz}^{(1)} = g_{u\bar{z}}^{(1)} = 0$. We will further restrict that the conformal factor Ω for the boundary metric $g_{ab}^{(0)}$ is independent of u which in turn implies $g_{uu}^{(1)} = 0$. Further one finds:

$$g_{uu}^{(2)} = -\frac{1}{2}R_0 - \frac{1}{2}\partial_u\left(g^{ab}_{(0)} g^{(1)}_{ab}\right)\,, \tag{3.5}$$

$$g_{ua}^{(2)} = V_a(z, \bar{z}) + \frac{1}{2}D^b g^{(1)}_{ba} - \frac{1}{2}D_a\left(g^{cd}_{(0)} g^{(1)}_{cd}\right) \tag{3.6}$$

and

$$g_{ab}^{(2)}(u, z, \bar{z}) = D_{ab}(z, \bar{z}) + \frac{1}{4} g^{(1)}_{ac}\, g^{cd}_{(0)}\, g^{(1)}_{cb}\,. \tag{3.7}$$

and so on, where $V_a(z,\bar{z})$ and $D_{ab}(z,\bar{z})$ are unconstrained u-independent $2d$ vector and rank–2 symmetric tensor respectively. D_a and R_0 are the covariant derivative and Ricci scalar associated with the boundary metric $g^{(0)}_{ab}$ respectively. It turns out that imposing vanishing of $R^{\mu}{}_{\nu\sigma\lambda}$ at order r implies

$$\partial_u^2\left[g^{(1)}_{ab}-\frac{1}{2}\left(g^{cd}_{(0)}g^{(1)}_{cd}\right)\,g^{(0)}_{ab}\right]=0\,. \tag{3.8}$$

This means that one can determine the u-dependence of the trace-free part of $g^{(1)}_{ab}$ completely and it is at most linear in u. Any such solution can be written as

$$g^{(1)}_{ab}-\frac{1}{2}\left(g^{cd}_{(0)}g^{(1)}_{cd}\right)\,g^{(0)}_{ab}=g^{(1,0)}_{ab}(z,\bar{z})+(u-u_0)\,g^{(1,1)}_{ab}(z,\bar{z}) \tag{3.9}$$

where $\{g^{(1,0)}_{ab}(z,\bar{z}),g^{(1,1)}_{ab}(z,\bar{z})\}$ are traceless and symmetric. The remaining data in $(g^{(0)}_{ab},g^{(1)}_{ab})$ has to satisfy some further differential conditions. In particular:

$$D^b\left(\partial_u\left[g^{(1)}_{ba}-\frac{1}{2}\left(g^{cd}_{(0)}g^{(1)}_{cd}\right)\,g^{(0)}_{ba}\right]+\frac{1}{2}R_0\,g^{(0)}_{ba}\right)=0. \tag{3.10}$$

The equation (3.10) is similar to (2.35) of AdS_3 gravity.

Further constraints arising from asymptotic flatness are: $V_a(z,\bar{z})=0$, $\partial_u g^{(3)}_{uu}(u,z,\bar{z})=0$. Along with $\partial_a g^{(3)}_{uu}$ determined in terms of the data at order r^2 and order r and differential conditions on D_{ab}. More unconstrained data also appears at $\mathcal{O}(r^{-1})$ and Ricci flat solutions can be systematically constructed order by order in powers of r^{-1}. It turns out that $g^{ab}_{(0)}g^{(1)}_{ab}$ remains unconstrained and we set it to zero.[2]

We now turn to analyse locally flat solutions which share boundary conditions with our asymptotically locally flat solutions and find the asymptotic symmetries associated with them.

3.1.1 Classes of locally flat geometries

These form a subset of Ricci flat geometries discussed above which have vanishing Riemann tensors. With our NU gauge and boundary conditions we find that any

[2] In [92] this condition follows from the constraints on the affine parameter r for the null geodesic generator of $u=$ const. null hypersurface.

such locally flat solution can be written as polynomials in r:

$$ds^2_{LF} = g^{(2)}_{uu}\, du^2 - 2\, du\, dr + \left(r^2\, g^{(0)}_{ab} + r g^{(1)}_{ab} + g^{(2)}_{ab}\right)\, dx^a\, dx^b + 2\, g^{(2)}_{ua}\, dx^a\, du, \quad (3.11)$$

where

$$g^{(2)}_{uu} = -\frac{1}{2}R_0, \quad g^{(2)}_{ua} = \frac{1}{2}D^b g^{(1)}_{ba}, \quad g^{(2)}_{ab} = \frac{1}{4}g^{(1)}_{ac}\, g^{cd}_{(0)} g^{(1)}_{db}, \quad g^{ab}_{(0)}\, g^{(1)}_{ab} = 0, \quad (3.12)$$

$$g^{(1)}_{ab} = g^{(1,0)}_{ab}(z,\bar{z}) + u\, g^{(1,1)}_{ab}(z,\bar{z}), \quad (3.13)$$

along with the differential conditions:

$$D^b \partial_u g^{(1)}_{ba} + \frac{1}{2}\partial_a R_0 = 0, \quad (3.14)$$

$$D_a g^{(2)}_{ub} - D_b g^{(2)}_{ua} + \frac{1}{4}\left(g^{(1)}_{ac}\, g^{cd}_{(0)}\, \partial_u g^{(1)}_{db} - g^{(1)}_{bc}\, g^{cd}_{(0)}\, \partial_u g^{(1)}_{da}\right) = 0. \quad (3.15)$$

We consider the boundary metric $g^{(0)}_{ab}$ to be u- independent. The equation (3.14) is one of the constraints on generalised $\mathfrak{bms}_4$ phase space, as derived in [125, 126]. When these differential conditions (3.14, 3.15) are solved for $g^{(0)}_{ab}$ and $g^{(1)}_{ab}$ then the rest of the 4d metric components ($g^{(2)}_{uu}$, $g^{(2)}_{ua}$, $g^{(2)}_{ab}$) can be found using (3.12, 3.13). The u-dependent part of the equation (3.15) is satisfied identically when we use (3.14) and thus can be simplified to:

$$\left[D_a D^c g^{(1,0)}_{bc} - D_b D^c g^{(1,0)}_{ac} + \frac{1}{2}\, g^{cd}_{(0)}\, \left(g^{(1,0)}_{ac}\, g^{(1,1)}_{db} - g^{(1,0)}_{bc}\, g^{(1,1)}_{da}\right)\right] = 0\,. \quad (3.16)$$

(3.16) are the generalisation of Christodolou Klainermann (CK) conditions for boundary metric that is allowed by generalised $\mathfrak{bms}_4$ boundary conditions [127, 125, 126].

3.1.2 Complete set of solutions

It turns out that the equations (3.14, 3.15) can be solved completely which we turn to now. To solve these equations first we take the boundary metric $g^{(0)}_{ab}$ in the form

$$g^{(0)}_{ab} = \Omega \begin{pmatrix} f & \frac{1}{2}(1+f\,\bar{f}) \\ \frac{1}{2}(1+f\,\bar{f}) & \bar{f} \end{pmatrix} \quad (3.17)$$

with determinant: $-\frac{\Omega^2}{4}(1-f\bar{f})^2$ where $(f,\ \bar{f},\ \Omega)$ are arbitrary functions of $(z,\bar{z})$. Then we further parameterise $f(z,\bar{z})$ and $\bar{f}(z,\bar{z})$ as follows in terms of two other independent functions $\zeta(z,\bar{z})$ and $\bar{\zeta}(z,\bar{z})$:

$$f(z,\bar{z}) = \frac{\partial_z\bar{\zeta}(z,\bar{z})}{\partial_{\bar{z}}\bar{\zeta}(z,\bar{z})}, \quad \bar{f}(z,\bar{z}) = \frac{\partial_{\bar{z}}\zeta(z,\bar{z})}{\partial_z\zeta(z,\bar{z})}. \tag{3.18}$$

Notice that this parameterisation is not one-to-one: suppose $\zeta(z,\bar{z})$ ($\bar{\zeta}(z,\bar{z})$) gives rise to a given $f(z,\bar{z})$ ($\bar{f}(z,\bar{z})$) then so does $\psi(\zeta(z,\bar{z}))$ ($\bar{\psi}(\bar{\zeta}(z,\bar{z}))$). Then the solutions for $g^{(1)}_{ab}$ components can be given explicitly in terms of the functions $(f,\bar{f},\Omega)$ and $\rho = \left(\partial\zeta\bar{\partial}\bar{\zeta}\right)^{-1/2}$ and their derivatives.

To keep the expressions simple we drop all the coordinate dependencies $(z,\bar{z})$ of various functions and use $\partial_z f(z,\bar{z}) \to \partial f$, $\partial_{\bar{z}} f(z,\bar{z}) \to \bar{\partial} f$ etc. We define:

$$\{\chi(z,\bar{z}),z\} = \frac{\partial^3\chi}{\partial\chi} - \frac{3}{2}\frac{(\partial^2\chi)^2}{(\partial\chi)^2}, \quad \{\chi(z,\bar{z}),\bar{z}\} = \frac{\bar{\partial}^3\chi}{\bar{\partial}\chi} - \frac{3}{2}\frac{(\bar{\partial}^2\chi)^2}{(\bar{\partial}\chi)^2},$$

$$\tau_{ab} = \frac{\partial_a\partial_b\Omega}{\Omega} - \frac{3\,\partial_a\Omega\,\partial_b\Omega}{2\,\Omega^2}, \tag{3.19}$$

where $a \in \{z,\bar{z}\}$. Then components $g^{(1,0)}_{zz}$ and $g^{(1,0)}_{\bar{z}\bar{z}}$ are given by:

$$\begin{aligned} g^{(1,0)}_{zz}/\sqrt{\Omega} = &-\frac{\rho}{(1-f\bar{f})^2}\left[2(1+f^2\bar{f}^2)\partial^2 v + 4f^2\,\bar{\partial}^2 v\right] \\ &+\partial v\Big[\frac{2\,\rho}{(1-f\bar{f})^3}\left(-\bar{f}(1+f^2\bar{f}^2)\,\partial f - (1-2f\bar{f}-f^2\bar{f}^2)\,\bar{\partial} f - f(1+f^2\bar{f}^2)\,\partial\bar{f} + 2f^2\,\bar{\partial}\bar{f}\right) \\ &\qquad - \frac{4}{(1-f\bar{f})^2}\left((1+f^2\bar{f}^2)\partial\rho - f(1+f\bar{f})\,\bar{\partial}\rho\right)\Big] \\ &+\bar{\partial} v\Big[\frac{2\rho}{(1-f\bar{f})^3}\left((1+f^2\bar{f}^2)\,\partial f - 2f^2\bar{f}\,\bar{\partial} f + 2f^3\bar{f}\partial\bar{f} - 2f^3\,\bar{\partial}\bar{f}\right) \\ &\qquad + \frac{4f}{(1-f\bar{f})^2}\left((1+f\bar{f})\,\partial\rho - 2f\,\bar{\partial}\rho\right)\Big] + \frac{4f(1+f\bar{f})}{(1-f\bar{f})^2}\rho\,\partial\bar{\partial} v, \end{aligned} \tag{3.20}$$

where $\upsilon = \upsilon(z,\bar z)$ is also arbitrary. Next, the $g_{zz}^{(1,1)}$ is given by

$$\begin{aligned}
g_{zz}^{(1,1)} &= \frac{\partial\Omega}{\Omega(1-f\bar f)^3}\Big[-2f^2\,\bar\partial\bar f + (1+f^2\bar f^2)f\,\partial\bar f + (1+f^2\bar f^2)\bar f\,\partial f + (1-2f\bar f - f^2\bar f^2)\,\bar\partial f\Big]\\
&+ \frac{\bar\partial\Omega}{\Omega(1-f\bar f)^3}\Big[-2\bar f\,f^3\,\partial\bar f + 2f^3\,\bar\partial\bar f + 2f^2\bar f\,\bar\partial f - (1+f^2\bar f^2)\,\partial f\Big]\\
&+ \frac{1}{(1-f\bar f)^2}\Big[(1+f^2\bar f^2)\,\tau_{zz} - 2f\,(1+f\bar f)\tau_{z\bar z} + 2f^2\tau_{\bar z\bar z}\Big]\\
&- \frac{1}{(1-f\bar f)^2}\Big[(1-f\,\bar f)^2\{\zeta,z\} + f^2\,(1-f\bar f)^2\,\{\bar\zeta,\bar z\}\Big]\\
&+ \frac{1}{(1-f\bar f)^2}\Big[2f\,\partial^2\bar f - 2f^2\,\partial\bar\partial\bar f + f(1+2f\bar f - f^2\bar f^2)\,\bar\partial^2 f - (1+f^2\bar f^2)\,\partial\bar\partial f\Big]\\
&+ \frac{1}{(1-f\bar f)^3}\Big[-2f^3\,\partial\bar f\bar\partial\bar f + f^2\,(1+f\bar f)\,(\partial\bar f)^2 - \frac{1}{2}(1-3f\bar f - 3f^2\bar f^2 + f^3\bar f^3)\,(\bar\partial f)^2\\
&\qquad - \bar f(1+f^2\bar f^2)\,\partial f\,\bar\partial f + (1+f^2\bar f^2)\,\partial f\,\partial\bar f - 2f\,(1+f\bar f)\,\bar\partial f\,\partial\bar f + 2f^2\,\bar\partial f\,\bar\partial\bar f\Big].
\end{aligned} \tag{3.21}$$

The components $g_{\bar z\bar z}^{(1,0)}$ and $g_{\bar z\bar z}^{(1,1)}$ can be obtained from $g_{zz}^{(1,0)}$ and $g_{zz}^{(1,1)}$ respectively by $f \leftrightarrow \bar f$ and $\partial \to \bar\partial$ (with ρ, υ and Ω unchanged).[3]

To summarise, a general locally flat metric solving equations (3.14, 3.15) is parameterised by the following arbitrary set of functions: $(\Omega(z,\bar z), \zeta(z,\bar z), \bar\zeta(z,\bar z))$ in $g_{ab}^{(0)}$ and ρ, and $\upsilon(z,\bar z)$ in $g_{ab}^{(1)}$. The remaining components $g_{z\bar z}^{(1,1)}$ and $g_{z\bar z}^{(1,0)}$ can be obtained from the above ones using the tracelessness condition of $g_{ab}^{(1)}$:

$$g_{z\bar z}^{(1)} = \frac{1}{1+f\bar f}\Big[g_{zz}^{(1)}\,\bar f + g_{\bar z\bar z}^{(1)}\,f\Big]\,. \tag{3.23}$$

We have generated this solution by starting with flat spacetime with metric $ds^2_{\mathbb{R}^{1,3}} = -2\,du\,dr + r^2\,dz\,d\bar z$ and making a finite coordinate transformation that keeps us in the chosen gauge. A similar exercise was done in [128, 129] with a different parameterisation of the boundary metric. We find our parameterisation better suited for the problem at hand. The solution in [129] matches with our LF solution after the following identifications, $\Phi = \frac{\log\rho(z,\bar z)^{-2}}{\Omega(z,\bar z)}$ and $G = \big(\zeta(z,\bar z), \bar\zeta(z,\bar z)\big)$.

[3] As commented above under $(\zeta,\bar\zeta) \to (\psi(\zeta), \bar\psi(\bar\zeta))$ the functions $(f,\bar f)$ in (3.18) remain unaffected, but we have

$$\{\zeta,z\} \to (\partial_z\zeta)^2\,\{\psi(\zeta),\zeta\} + \{\zeta,z\}, \quad \{\bar\zeta,\bar z\} \to (\partial_{\bar z}\bar\zeta)^2\,\{\bar\psi(\bar\zeta),\bar\zeta\} + \{\bar\zeta,\bar z\} \tag{3.22}$$

Under this change, the resulting configurations remain LF solutions.

3.1.3 Gauge fixing the boundary metric

As shown in the appendix (A), the residual diffeomorphisms that preserve the NU gauge and the boundary conditions act as diffeomorphisms and Weyl transformations on $g^{(0)}_{ab}$. These boundary diffeomorphisms can be used to gauge fix the $2d$ metric $g^{(0)}_{ab}$ down to one independent function. The most commonly used gauge choice for 2-dimensional metrics is the conformal gauge. In our language, this amounts to setting $f(z,\bar{z}) = \bar{f}(z,\bar{z}) = 0$ and letting Ω fluctuate. However, we are interested in getting a symmetry algebra that includes $\mathfrak{sl}(2,\mathbb{R})$ current algebra, so we choose the boundary metric in the Polyakov gauge [112]:

$$\bar{f}(z,\bar{z}) = 0 \text{ or } f(z,\bar{z}) = 0 \quad \text{with} \quad \Omega \quad \text{fixed to a given function.} \tag{3.24}$$

Below we provide the solutions for this gauge.

Locally flat solutions in Polyakov gauge

To be specific, we choose $\bar{f} = 0$. Then the components of $g^{(0)}_{ab}$ reduce to

$$g^{(0)}_{zz} = \Omega(z,\bar{z})\,\frac{\partial_z\bar{\zeta}_((z,\bar{z})}{\partial_{\bar{z}}\bar{\zeta}_((z,\bar{z})} := \Omega(z,\bar{z})\,f(z,\bar{z}), \quad g^{(0)}_{z\bar{z}} = \frac{1}{2}\Omega(z,\bar{z}), \quad g^{(0)}_{\bar{z}\bar{z}} = 0. \tag{3.25}$$

Similarly the components $g^{(1,0)}_{ab}$ and $g^{(1,1)}_{ab}$ in (3.20, 3.21) reduce to:

$$g^{(1,1)}_{\bar{z}\bar{z}} = -\{\bar{\zeta}(z,\bar{z}),\bar{z}\} + \tau_{\bar{z}\bar{z}}(\Omega),$$

$$g^{(1,0)}_{\bar{z}\bar{z}}/\sqrt{\Omega} = -2\sqrt{\frac{\partial_{\bar{z}}\bar{\zeta}(z,\bar{z})}{\partial_z\zeta(z)}}\;\partial_{\bar{z}}\left[\frac{\partial_{\bar{z}}\upsilon(z,\bar{z})}{\partial_{\bar{z}}\bar{\zeta}(z,\bar{z})}\right], \tag{3.26}$$

$$\begin{aligned} g^{(1,1)}_{zz} = {} & f^2\,g^{(1,1)}_{\bar{z}\bar{z}} - \{\zeta(z),z\} + f\,\partial^2_{\bar{z}}f - \frac{1}{2}\,(\partial_{\bar{z}}f)^2 - \partial_z\partial_{\bar{z}}f \\ & + \frac{\partial_z\Omega(z,\bar{z})}{\Omega(z,\bar{z})}\partial_{\bar{z}}f(z,\bar{z}) - \frac{\partial_{\bar{z}}\Omega(z,\bar{z})}{\Omega(z,\bar{z})}\partial_z f(z,\bar{z}) \\ & + \tau_{zz}(\Omega) + f(z,\bar{z})^2\tau_{\bar{z}\bar{z}}(\Omega) - 2\,f(z,\bar{z})\,\tau_{z\bar{z}}(\Omega), \end{aligned} \tag{3.27}$$

$$g^{(1,0)}_{zz}/\sqrt{\Omega(z,\bar z)} = -2\,\rho\,\left[\partial_z^2 \upsilon + 2\,f^2\,\partial_{\bar z}^2\upsilon - 2\,f\,\partial_z\partial_{\bar z}\upsilon\right]$$
$$+ 2\,\rho\,\frac{\partial_z^2\zeta}{\partial_z\zeta}\,\left[\partial_z\upsilon - 2\,f\,\partial_{\bar z}\upsilon\right] + 2\,\left[\rho\,\left(\partial_z f - 2\,f\,\partial_{\bar z} f\right) - 2\,f\,\partial_z\rho\right]\partial_{\bar z}\upsilon, \qquad (3.28)$$

where $\rho = \frac{1}{\sqrt{\partial_z\zeta(z)\,\partial_{\bar z}\bar\zeta(z,\bar z)}}$. We also have to take

$$g^{(1)}_{z\bar z} = f(z,\bar z)\,g^{(1)}_{\bar z\bar z}. \qquad (3.29)$$

Now that we know $g^{(0)}_{ab}$ and $g^{(1)}_{ab}$ the rest of the components can be found using the relations (3.12, 3.13). We will not be interested in general conformal factor $\Omega(z,\bar z)$ when the boundary metric is in the Polyakov gauge. We will be interested in only two choices $\Omega = 1$ for $\Sigma_2 = \mathbb{R}^2$ and $\Omega = 4\,(1+z\bar z)^{-2}$ for $\Sigma_2 = \mathbb{S}^2$. These choices mean that the quantities τ_{zz}, $\tau_{\bar z\bar z}$ vanish.

Restriction to $g^{(1)}_{\bar z\bar z} = 0$

In the next section, we will show that the solutions with vanishing $g^{(1,1)}_{\bar z\bar z}$ and $g^{(1,0)}_{\bar z\bar z}$ also satisfy the variational principle. Anticipating that result we write down the solutions with $g^{(1)}_{\bar z\bar z} = 0$. In the cases for Ω's with $\tau_{zz} = \tau_{\bar z\bar z} = 0$, we see from (3.26) that

$$\bar\zeta(z,\bar z) = \frac{g_1(z)\,\bar z + g_2(z)}{g_3(z)\,\bar z + g_4(z)}, \quad \upsilon(z,\bar z) = \upsilon_{(1)}(z) + \bar\zeta(z,\bar z)\,\upsilon_{(2)}(z). \qquad (3.30)$$

Then $f(z,\bar z) = \frac{\partial_z\bar\zeta(z,\bar z)}{\partial_{\bar z}\bar\zeta(z,\bar z)}$ turns out to be a polynomial of degree two in $\bar z$:

$$\begin{aligned}(\det\mathfrak{g})\,f(z,\bar z) &= [g_2'(z)g_4(z) - g_2(z)g_4'(z)] \\ &\quad + [g_1'(z)g_4(z) - g_1(z)g_4'(z) + g_2'(z)g_3(z) - g_2(z)g_3'(z)]\ \bar z \\ &\quad + [g_1'(z)g_3(z) - g_1(z)g_3'(z)]\ \bar z^2 \\ &:= -(\det\mathfrak{g})\sum_{a=0}^{2} J_{a-1}(z)\,\bar z^a, \end{aligned} \qquad (3.31)$$

where $\det\mathfrak{g} := g_1(z)\,g_4(z) - g_2(z)\,g_3(z)$. For (3.31) to make sense we have to assume that $\det \mathfrak{g} \neq 0$.[4] Furthermore,

$$g_{zz}^{(1,0)} = -2\sqrt{\frac{\Omega}{\zeta'(z)\partial_{\bar z}\bar\zeta(z,\bar z)}}\left[\upsilon''_{(1)}(z) + \bar\zeta(z,\bar z)\,\upsilon''_{(2)}(z) - \frac{\zeta''(z)}{\zeta'(z)}\left(\upsilon'_{(1)}(z) + \bar\zeta(z,\bar z)\,\upsilon'_{(2)}(z)\right)\right]. \tag{3.34}$$

Since $\partial_{\bar z}\bar\zeta(z,\bar z) = (\det\mathfrak{g})/(g_3(z)\,\bar z + g_4(z))^2$ we see that $g_{zz}^{(1,0)}/\sqrt{\Omega}$ is linear in $\bar z$. Therefore we write $g_{zz}^{(1,0)}$ as

$$g_{zz}^{(1,0)} = \sqrt{\Omega}\,\left[C_{-1/2}(z) + \bar z\, C_{1/2}(z)\right], \tag{3.35}$$

where

$$\begin{aligned} &C_{-1/2}(z) = \\ &-\frac{2}{\det\mathfrak{g}\,\sqrt{\zeta'(z)}}\left[g_4(z)\left(\upsilon''_{(1)}(z) - \frac{\zeta''(z)}{\zeta'(z)}\upsilon'_{(1)}(z)\right) + g_2(z)\left(\upsilon''_{(2)}(z) - \frac{\zeta''(z)}{\zeta'(z)}\upsilon'_{(2)}(z)\right)\right], \\ &C_{1/2}(z) = \\ &-\frac{2}{\det\mathfrak{g}\,\sqrt{\zeta'(z)}}\left[g_3(z)\left(\upsilon''_{(1)}(z) - \frac{\zeta''(z)}{\zeta'(z)}\upsilon'_{(1)}(z)\right) + g_1(z)\left(\upsilon''_{(2)}(z) - \frac{\zeta''(z)}{\zeta'(z)}\upsilon'_{(2)}(z)\right)\right]. \end{aligned} \tag{3.36}$$

We can equivalently write

$$g_{zz}^{(1,0)}(z,\bar z) = \sqrt{\Omega(z,\bar z)}\sum_{r,s\in\{-1/2,1/2\}}\epsilon_{rs}\bar z^{r+1/2}\,C^s \tag{3.37}$$

[4]There is another interesting interpretation of the coefficients J_a of the powers of $\bar z$ in (3.31) as follows. First, consider the 2×2 matrix

$$\mathfrak{g} = \begin{pmatrix} g_1(z) & g_2(z) \\ g_3(z) & g_4(z) \end{pmatrix} \tag{3.32}$$

with $\det\mathfrak{g} := g_1(z)\,g_4(z) - g_2(z)\,g_3(z) \neq 0$ making $\mathfrak{g} \in GL_2(\mathbb{C})$. Then define $A(z) = \mathfrak{g}^{-1}(z)\,\partial_z\mathfrak{g}(z)$ which is an element of the algebra $\mathfrak{gl}_2$. Consider a 2×2 matrix representation of $\mathfrak{sl}_2$ algebra $[t^a, t^b] = (a-b)\,t^{a+b}$ – for instance, in terms of the Pauli matrices, we can take $t^{(0)} = \frac{1}{2}\sigma_3$, $t^{(\pm1)} = \frac{i}{2}(\sigma_2 \pm i\,\sigma_1)$. Then it turns out that

$$J_a(z) = 2\,\eta_{ab}\,\mathrm{Tr}\left[t^b\,A(z)\right] := \eta_{ab}\,J^b(z) \tag{3.33}$$

where the non-zero components of η_{ab} are $\eta_{+1\,-1} = \eta_{-1\,+1} = 1/2$ and $\eta_{00} = -1$. One also has $\mathrm{Tr}\,[A(z)] = \frac{1}{2\det\mathfrak{g}}\partial_z\det\mathfrak{g}$ and without loss of generality we may choose $\det\mathfrak{g} = 1$. This makes $A(z) \in \mathfrak{sl}_2$ and $A(z) = J_a(z)\,t^a$.

with $\epsilon_{-\frac{1}{2}\frac{1}{2}} = -\epsilon_{\frac{1}{2}-\frac{1}{2}} = 1$ and $C_r(z) = \epsilon_{rs}\, C^s(z)$. Finally, we have

$$g^{(1,1)}_{zz} = -\kappa(z) + \frac{1}{2}\,\eta_{ij} J^i(z)\, J^j(z) - \partial_z J^{(0)}(z) + \bar{z}\,\partial_z J^{(-1)}(z) \\ + \frac{\partial_z \Omega(z,\bar{z})}{\Omega(z,\bar{z})} \partial_{\bar{z}} f(z,\bar{z}) - \frac{\partial_{\bar{z}} \Omega(z,\bar{z})}{\Omega(z,\bar{z})}\, \partial_z f(z,\bar{z}) - 2\, f(z,\bar{z})\, \tau_{z\bar{z}}(\Omega) \qquad (3.38)$$

where $\kappa(z) = \{\zeta(z), z\}$.

Please note that we will be using $a, b \cdots$ as the coordinate indices on Σ_2 where the coordinates are $x^a \in \{z, \bar{z}\}$, as well as component indices of the currents $J^a(z)$ and η_{ab} with $a \in \{0, \pm 1\}$. Now we will leave the Polyakov gauge solutions found in this section in store until we learn to impose the variational problem – which will be done in the next section.

3.2 Boundary terms and variational principle

We now turn to the variational problem among the configurations we have considered so far. To be precise we consider our configuration space to be all the four-dimensional metrics $g_{\mu\nu}$ that are in the NU gauge which asymptotically approach the locally flat solutions. Then we will define our solution space to be a subset of Ricci flat configurations, that also satisfy a variational principle $\delta S = 0$.

The usual prescription of boundary terms consists of adding a Gibbons-Hawking term and a set of possible counter terms to the standard Einstein Hilbert action S_{EH} [102, 101]. There are two essential aspects required of such boundary terms:

1. The boundary action S_{bdy} has to be consistent with at least the symmetries that leave the gauge choice and the boundary surface invariant.

2. The variation of $S_{EH} + S_{bdy}$ should be proportional to variations of metric data on the boundary - but not its derivatives.

The standard Gibbons-Hawking term is invariant under the full set of $3d$ diffeomorphisms of the boundary [101]. In the context of AdS_{n+1} gravity in the Fefferman-Graham gauge residual symmetries are the diffeomorphisms of the n-dimensional subspace $r = r_0$ and thus adding the Gibbons-Hawking term and other counter terms that are also invariant under the n-dimensional diffeomorphisms is justified.

In Appendix (A), we find $4d$ vector fields which leave our boundary conditions and NU gauge choice invariant. Now we pose this question in our context: what are the subset of asymptotic symmetries of the class of geometries that leave our 3-dimensional $r = r_0$ hypersurface fixed? Generators of any such coordinate transformations should have $\xi^r = 0$. The vector fields should continue to solve (A.1, A.3, A.4).

The first consequence of the condition $\partial_r \xi^r = 0$, is that $(\partial_u - g_{ua} g^{ab} \partial_b)\, \xi^u = 0$ from (A.1). Since $\xi^r_{(0)} = \xi^u_{(1)} = \partial_u \xi^u$ from (A.3), we also have $\partial_u \xi^u = 0$ implying $g_{ua} g^{ab} \partial_b\, \xi^u = 0$. From (A.4) working in the gauge $g^{(0)}_{ab}\, g^{ab}_{(1)} = 0$ we see that $\xi^r_{(1)} = 0$ requires ξ^u to be a harmonic function in $2d$. Since the harmonic equation is Weyl invariant and in $2d$ every metric is Weyl equivalent to flat space the solutions to $\Box \xi^u = 0$ are $\xi^u(z, \bar{z}) = \xi^u_{(z)}(z) + \xi^u_{(\bar{z})}(\bar{z})$. Substituting this expression of ξ^u into ${g_u}^a \partial_a\, \xi^u = 0$ implies ${g_u}^z\, \partial_z \xi^u_{(z)}(z) + {g_u}^{\bar{z}}\, \partial_{\bar{z}} \xi^u_{(\bar{z})}(\bar{z}) = 0$. This equation is background dependent and a linear combination of holomorphic and anti-holomorphic functions $(\partial_z \xi^u_{(z)}(z), \partial_{\bar{z}} \xi^u_{(\bar{z})}(\bar{z}))$ and unless the coefficients $({g_u}^z, {g_u}^{\bar{z}})$ are zero (and non-generic) the only solutions are $(\partial_z \xi^u_{(z)}(z) = 0, \partial_{\bar{z}} \xi^u_{(\bar{z})}(\bar{z}) = 0)$ which in turn implies $\partial_a \xi^u = 0$ and this is what we work with. To summarise the subset of symmetries of our class of geometries that leave $r = r_0$ invariant are: $(\xi^r = 0, \xi^u = \xi^u_{(0)}, \xi^a = \xi^a_{(0)}(z, \bar{z}))$ – that is, rigid translations in u and arbitrary diffeomorphisms of $(z, \bar{z})$.[5] Now we seek the boundary terms that respect at least these symmetries.

The bulk four-dimensional metrics in the NU gauge are of the form:

$$g_{\mu\nu} = \begin{pmatrix} g_{uu} & -1 & g_{ua} \\ -1 & 0 & 0 \\ g_{au} & 0 & g_{ab} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & g_{ua}{g_u}^a - g_{uu} & {g_u}^b \\ 0 & {g^a}_u & g^{ab} \end{pmatrix} \tag{3.39}$$

where g^{ab} is the inverse of g_{ab}, ${g_u}^a = g_{ub} g^{ba}$, etc. We take the unit normal to the $r = r_0$ surface as: $n_\mu = \frac{1}{\sqrt{g_{ua}{g_u}^a - g_{uu}}} \delta^r_\mu$ and $n^\mu = \frac{1}{\sqrt{g_{ua}{g_u}^a - g_{uu}}} g^{r\mu}$ such that $n_\mu n^\mu = 1$. The induced metric on $r = r_0$ surface is:

$$\gamma_{ij} = g_{\mu\nu}\, e^\mu_i e^\nu_j = \begin{pmatrix} g_{uu} & g_{ua} \\ g_{au} & g_{ab} \end{pmatrix} \tag{3.40}$$

[5] Another way to arrive at the same conclusion is the following. We should have expected that the residual transformations do not mix different orders of powers of r. This in turn implies that ξ^a is independent of r (ξ^u is already independent of r). Then the last of (A.1) implies $\partial_a \xi^u$ is zero, and the fact that ξ^r should also vanish requires $\partial_u \xi^u = 0$, thus making ξ^u a constant and $\xi^a = \xi^a_{(0)}(z, \bar{z})$. These are the generators of the boundary coordinate transformations: $(u \to u' = u + u_0, x^a \to x'^a = x'^a(x))$. The Jacobian of such transformations is: $\frac{\partial u'}{\partial u} = 1$, $\frac{\partial u'}{\partial x^a} = \frac{\partial x'^a}{\partial u} = 0$ with the only non-trivial part $J^a_b = \frac{\partial x'^a}{\partial x^b}$.

with its inverse

$$\gamma^{ij} = \frac{1}{N^2} \begin{pmatrix} -1 & g_u^a \\ g_u^a & N^2\, g^{ab} - g_u^a g_u^b \end{pmatrix}, \tag{3.41}$$

with $N^2 = -g_{uu} + g_{au}\, g_u^a$ and the usual completeness relation $g^{\mu\nu} = n^\mu n^\nu + \gamma^{ij} e_i^\mu e_j^\nu$. Thus the metric data in the line element

$$ds_{bdy}^2 = g_{uu}\, du^2 + 2\, g_{ua} du\, dx^a + g_{ab}\, dx^a\, dx^b \tag{3.42}$$

on the boundary is (g_{uu}, g_{ua}, g_{ab}). The expected set of symmetries of the boundary ($u \to u' = u + u_0, x^a \to x'^a = x'^a(x)$) is much smaller than the $3d$ diffeos in (u, x^a). Therefore, we expect much more freedom in the choice of the boundary terms. The transformations of the components of the induced metric under the reduced symmetry are:

$$\begin{aligned} g_{u'u'}(u', x'^a) = g_{uu}(u, x^a), \quad g_{u'x'^a}(u', x') &= \frac{\partial x^b}{\partial x'^a} g_{ux^b}(u, x) \\ g_{x'^a x'^b}(u', x') &= \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^a} g_{x^c x^d}(u, x). \end{aligned} \tag{3.43}$$

This enables us to classify the basic scalars under the boundary symmetries and some of them are:

$$\begin{aligned} &\text{no derivatives}: g_{uu}, \quad g^{ab} g_{ua} g_{ub}, \\ &\text{one derivative}: \partial_u g_{uu}, g^{ab} g_{ua} \partial_u g_{ub}, \ g_{ua} g_{ub} \partial_u g^{ab} \ g^{ab} \partial_u g_{ab}, \ D^a g_{ua}, \\ &\text{two derivative}: R[g_{ab}], \ \Box g_{uu}, \ \partial_u^2 g_{uu}, \partial_u D^a g_{ua}, \ \partial_u g^{ab} \partial_u g_{ab}, \ g^{ab}\, \partial_u g_{ua} \partial_u g_{ub} \ etc. \end{aligned} \tag{3.44}$$

The integration measure invariant under our boundary symmetry $(u, x^a) \to (u + u_0, x'^a(x))$ is $\int du \int d^2x \sqrt{\sigma}$ where $\sigma = \det g_{ab}$, which can be used to integrate any function of the scalars listed above to obtain a potential boundary term.

3.2.1 The boundary terms

Now we look for the potential Gibbons-Hawking-type terms we can construct consistent with our boundary symmetries. The bulk action is the standard Einstein-

Hilbert action

$$S_{EH} = \frac{1}{16\,\pi\,G}\int d^4x\,\sqrt{-g}\,R\,. \tag{3.45}$$

Then the variation of the action within our configuration space around a configuration satisfying the Einstein equation $R_{\mu\nu} = 0$ is

$$\delta S_{EH} = \frac{1}{16\,\pi\,G}\int d^4x\,\sqrt{-g}\,g^{\mu\nu}\left(\nabla_\kappa \delta\Gamma^\kappa_{\mu\nu} - \nabla_\nu\,\delta\Gamma^\kappa_{\mu\kappa}\right) = \frac{1}{16\,\pi\,G}\int_\Sigma d^3x\,\sqrt{-\gamma}\,n_\mu J^\mu, \tag{3.46}$$

where $J^\kappa = g^{\mu\nu}\,\delta\Gamma^\kappa_{\mu\nu} - g^{\mu\kappa}\,\delta\Gamma^\nu_{\mu\nu}$. $\delta\Gamma^\kappa_{\mu\nu} = \frac{1}{2}g^{\kappa\omega}\left(\nabla_\mu\delta g_{\omega\nu} + \nabla_\nu\delta g_{\mu\omega} - \nabla_\omega\delta g_{\mu\nu}\right)$ which leads to $J^\mu = g^{\mu\nu}g^{\sigma\lambda}\left(\nabla_\sigma\delta g_{\nu\lambda} - \nabla_\nu\delta g_{\sigma\lambda}\right)$. We will have to manipulate this term carefully. Since our surface is defined by $r = r_0$ the unit normal is $n_\mu = N\delta^r_\mu$ where $N = \frac{1}{\sqrt{g^{rr}}}$. But we also have $g^{rr} = \frac{\gamma}{g}$ where $\gamma = \det(\gamma_{ij})$ and $g = \det g_{\mu\nu}$ $\implies \sqrt{-\gamma} = N\,\sqrt{-g}$. This boundary term can be written explicitly for geometries in the NU gauge with $n_\mu = \frac{1}{\sqrt{g^{rr}}}\delta^r_\mu$ and we find:

$$\begin{aligned}
\sqrt{-\gamma}\,n_\mu J^\mu = {} & \sqrt{\sigma}\,\partial_r\delta\omega + \frac{1}{2}\sqrt{\sigma}\,\sigma^{ab}\partial_r\sigma_{ab}\,\delta\omega + \frac{1}{2}\sqrt{\sigma}\,\omega\,\partial_r\sigma^{ab}\delta\sigma_{ab} + \sqrt{\sigma}\,\omega\,\sigma^{ab}\partial_r\delta\sigma_{ab} \\
& - \sqrt{\sigma}\,v_a\partial_r v_b\,\delta\sigma^{ab} + \sqrt{\sigma}\,v^a v^b\,\partial_r\delta\sigma_{ab} + \partial_r(\sqrt{\sigma})\,v^a v^b\delta\sigma_{ab} + \sqrt{\sigma}\,v^a v_b\,\partial_r\sigma^{cb}\,\delta\sigma_{ac} \\
& - \frac{1}{2}\sqrt{\sigma}\,v_a v^a\,\partial_r\sigma_{ab}\delta\sigma^{ab} - \sqrt{\sigma}\,v^c v_c\,\sigma^{ab}\,\partial_r\delta\sigma_{ab} \\
& - \sqrt{\sigma}\,\sigma^{ab}\,\partial_r v_a\,\delta v_b - \sqrt{\sigma}\,v_a\partial_r\sigma^{ab}\,\delta v_b - 2\,\partial_r(\sqrt{\sigma})\,v^a\,\delta v_a - 2\sqrt{\sigma}\,v^a\partial_r\delta v_a \\
& + \frac{1}{2}\sqrt{\sigma}\,\partial_u\sigma_{ab}\,\delta\sigma^{ab} + \sqrt{\sigma}\,\sigma^{ab}\,\partial_u\delta\sigma_{ab} \\
& + \sqrt{\sigma}\,(D_a\delta\sigma_{bc})\left(\sigma^{ab}\,v^c - v^a\,\sigma^{bc}\right) - \partial_a\left(\sqrt{\sigma}\,\sigma^{ab}\delta v_b\right),
\end{aligned} \tag{3.47}$$

where we introduced the notation: $\omega = g_{uu}$, $v_a = g_{ua}$ and $\sigma_{ab} = g_{ab}$ along with σ^{ab} being the inverse of σ_{ab} and $v^a = \sigma^{ab}v_b$. In this, we seek that those terms with tangential derivatives (that is, derivatives w.r.t $(u, z, \bar{z})$) on variations of the boundary data $(\delta\omega, \delta v_a, \delta\sigma_{ab})$ have to be cancelled. The usual strategy involves completing such terms into either total variations or total derivatives so that the variations (derivatives) are moved away from the derivatives (variations) [130, 131].

To do this one may use the following identities:

$$\sqrt{\sigma}\,\partial_r\delta\omega = \delta\left(\sqrt{\sigma}\,\partial_r\omega\right) - \partial_r\omega\,\delta\left(\sqrt{\sigma}\right), \tag{3.48}$$

$$\begin{aligned}&\sqrt{\sigma}\,\left(\omega\,\sigma^{ab} + v^a v^b - v^c v_c\,\sigma^{ab}\right)\,\partial_r\delta\sigma_{ab}\\ &= \delta\,\left[\sqrt{\sigma}\,\left(\omega\,\sigma^{ab} + v^a v^b - v^c v_c\,\sigma^{ab}\right)\,\partial_r\sigma_{ab}\right] - \partial_r\sigma_{ab}\;\delta\left(\sqrt{\sigma}\,\left(\omega\,\sigma^{ab} + v^a v^b - v^c v_c\,\sigma^{ab}\right)\right),\end{aligned} \tag{3.49}$$

$$-\,2\sqrt{\sigma}\,v^a\partial_r\delta v_a = \delta\left[-2\sqrt{\sigma}\,v^a\partial_r v_a\right] + 2\,\partial_r v_a\,\delta\left(\sqrt{\sigma}\,v^a\right), \tag{3.50}$$

$$\sqrt{\sigma}\,\sigma^{ab}\,\partial_u\delta\sigma_{ab} = \delta\left[\sqrt{\sigma}\,\sigma^{ab}\,\partial_u\sigma_{ab}\right] - \partial_u\sigma_{ab}\,\delta\left[\sqrt{\sigma}\,\sigma^{ab}\right], \tag{3.51}$$

$$\begin{aligned}&\sqrt{\sigma}\,(D_a\delta\sigma_{bc})\,\left(\sigma^{ab}\,v^c - v^a\,\sigma^{bc}\right)\\ &= \partial_a\left[\sqrt{\sigma}\,\left(\sigma^{ab}\,v^c - v^a\,\sigma^{bc}\right)\,\delta\sigma_{bc}\right] - \sqrt{\sigma}\,\delta\sigma_{bc}\,\left[\sigma^{ab}\,D_a v^c - \sigma^{bc}D_a v^a\right].\end{aligned} \tag{3.52}$$

Using these and moving the total variations and total derivatives (in $(z,\bar{z})$ coordinates) to the lhs we find:

$$\begin{aligned}&\sqrt{-\gamma}\,n_\mu J^\mu - \delta\left(\sqrt{\sigma}\,\partial_r\omega\right) - \delta\,\left[\sqrt{\sigma}\,\left(\omega\,\sigma^{ab} + v^a v^b - v^c v_c\,\sigma^{ab}\right)\,\partial_r\sigma_{ab}\right]\\ &+\delta\left[2\sqrt{\sigma}\,v^a\partial_r v_a\right] - \delta\left[\sqrt{\sigma}\,\sigma^{ab}\,\partial_u\sigma_{ab}\right] - \partial_a\left[\sqrt{\sigma}\,\left(\sigma^{ab}\,v^c - v^a\,\sigma^{bc}\right)\,\delta\sigma_{bc}\right] + \partial_a\left(\sqrt{\sigma}\,\sigma^{ab}\delta v_b\right)\end{aligned} \tag{3.53}$$

$$\begin{aligned}= &-\partial_r\omega\,\delta\left[\sqrt{\sigma}\right] - \partial_r\sigma_{ab}\;\delta\left[\sqrt{\sigma}\,\left(\omega\,\sigma^{ab} + v^a v^b - v^c v_c\,\sigma^{ab}\right)\right]\\ &+\frac{1}{2}\sqrt{\sigma}\,\sigma^{ab}\partial_r\sigma_{ab}\,\delta\omega + \frac{1}{2}\sqrt{\sigma}\,\omega\,\partial_r\sigma^{ab}\delta\sigma_{ab}\\ &-\sqrt{\sigma}\,v_a\partial_r v_b\,\delta\sigma^{ab} + \partial_r(\sqrt{\sigma})\,v^a v^b\delta\sigma_{ab} + \sqrt{\sigma}\,v^a v_b\,\partial_r\sigma^{cb}\,\delta\sigma_{ac} - \frac{1}{2}\sqrt{\sigma}\,v_c v^c\,\partial_r\sigma_{ab}\delta\sigma^{ab}\\ &-\sqrt{\sigma}\,\sigma^{ab}\,\partial_r v_a\,\delta v_b - \sqrt{\sigma}\,v_a\partial_r\sigma^{ab}\,\delta v_b - 2\,\partial_r(\sqrt{\sigma})\,v^a\,\delta v_a + 2\,\partial_r v_a\,\delta\left[\sqrt{\sigma}\,v^a\right]\\ &+\frac{1}{2}\sqrt{\sigma}\,\partial_u\sigma_{ab}\,\delta\sigma^{ab} - \partial_u\sigma_{ab}\,\delta\left[\sqrt{\sigma}\,\sigma^{ab}\right]\\ &-\sqrt{\sigma}\,\delta\sigma_{bc}\,\left[\sigma^{ab}\,D_a v^c - \sigma^{bc}D_a v^a\right].\end{aligned} \tag{3.54}$$

Note that there are no terms involving a tangential derivative of $(\delta\omega, \delta v_a, \delta\sigma_{ab})$ on the rhs. Thus the boundary action required to be added to the Einstein-Hilbert action (3.45) in our context is,

$$\begin{aligned}(16\pi G)\,S_{bdy.} &= \int d^3x\sqrt{\sigma}\left[2\;v^a\partial_r v_a - \partial_r\omega - \left(\omega\,\sigma^{ab} + v^a v^b - v^c v_c\,\sigma^{ab}\right)\,\partial_r\sigma_{ab} - \sigma^{ab}\,\partial_u\sigma_{ab}\right]\\ &= \int d^3x\sqrt{\sigma}\,\partial_r(v^a v_a - \omega) + 2\,\left[(v^a v_a - \omega)\,\partial_r - \partial_u\right]\,\sqrt{\sigma}\end{aligned} \tag{3.55}$$

and the total divergence term in δS_{EH} is $(16\pi G)^{-1}$ times

$$\partial_a \left[\sqrt{\sigma}\ \left(\delta v^a - v^a\, \sigma_{bc}\, \delta\sigma^{bc}\right)\right] \tag{3.56}$$

that we ignore (this amounts to assuming that the geometry of Σ_2 with coordinates $(z, \bar{z})$ is either compact or, when it is not, the integrand falls off fast enough near its asymptotes).

In summary, we have managed to find the boundary action (3.55) we sought such that the variation of the total (bulk plus boundary) action is a linear combination of $(\delta\omega, \delta v_a, \delta\sigma_{ab})$ but not their derivatives:

$$\begin{aligned}(16\pi G)\, \delta S_{EH+bdy.} &= \int_{r=r_0} du\, d^2z\, \sqrt{\sigma}\ \left[\mathfrak{t}_{ab}\, \delta\sigma^{ab} + \mathfrak{j}_a\, \delta v^a + \mathfrak{o}\, \delta\omega\right] \\ &= \int_{r=r_0} du\, d^2z\, \sqrt{\sigma}\ \left[-(\mathfrak{t}^{ab} + \mathfrak{j}^{(a}v^{b)})\, \delta\sigma_{ab} + \mathfrak{j}^a \delta v_a + \mathfrak{o}\, \delta\omega\right], \end{aligned} \tag{3.57}$$

where $\mathfrak{t}^{ab} = \sigma^{ac}\sigma^{bd}\mathfrak{t}_{cd}$, $\mathfrak{j}^a = \sigma^{ab}\mathfrak{j}_b$. The coefficients of $(\delta\omega, \delta v^a)$ are:

$$\begin{aligned}\delta\omega : \mathfrak{o} &= -\frac{1}{2}\, \sigma^{ab}\, \partial_r\sigma_{ab}, \\ \delta v^a : \mathfrak{j}_{\mathfrak{a}} &= -\partial_r\sigma_{ab}\, v^b + \partial_r v_a + \sigma^{cd}\, \partial_r\sigma_{cd}\, v_a,\end{aligned} \tag{3.58}$$

and the coefficient of $\delta\sigma^{ab}$ is :

$$\begin{aligned}\mathfrak{t}_{ab} :=& \frac{1}{2}\partial_r\omega\, \sigma_{ab} + \frac{1}{2}(v^2 - \omega)\ \left(\partial_r\sigma_{ab} - \sigma^{cd}\, \partial_r\sigma_{cd}\, \sigma_{ab}\right) - \frac{1}{2}\left(\partial_u\sigma_{ab} - \sigma^{cd}\partial_u\sigma_{cd}\, \sigma_{ab}\right) \\ &+ \left(D_{(a}v_{b)} - \sigma_{ab}\, D_c v^c\right) - \frac{1}{2}\sigma^{cd}\partial_r\sigma_{cd}\, v_a v_b + \frac{1}{2}v^c v^d\, \partial_r\sigma_{cd}\, \sigma_{ab} - v^c\partial_r v_c\, \sigma_{ab},\end{aligned} \tag{3.59}$$

where $v^2 = v_a v^a$ and $D_{(a}v_{b)} = \frac{1}{2}(D_a v_b + D_b v_a)$ and so on. Finally, we can now substitute the boundary conditions we have and find the expressions in powers of

$1/r$.

$$\mathfrak{o} = -\frac{1}{r}2 + \frac{1}{r^2}\left(g^{ab}_{(0)}g^{(1)}_{ab}\right) + \frac{1}{r^3}\left[\left(g^{ab}_{(0)}g^{(2)}_{ab}\right) - \frac{1}{2}\left(g^{ab}_{(0)}g^{(1)}_{bc}\,g^{cd}_{(0)}g^{(1)}_{da}\right)\right] + \cdots, \tag{3.60}$$

$$\mathfrak{j}_a = \frac{1}{r}2\,g^{(2)}_{ua} + \frac{1}{r^2}\left[g^{(3)}_{ua} - \left(g^{ab}_{(0)}g^{(1)}_{ab}\right)\,g^{(2)}_{ua} + g^{(1)}_{ab}\,g^{bc}_{(0)}g^{(2)}_{uc}\right] + \cdots, \tag{3.61}$$

$$\begin{aligned}\mathfrak{t}_{ab} = &-r\,\frac{1}{2}\left[\partial_u g^{(1)}_{ab} - \left(g^{cd}_{(0)}g^{(1)}_{cd}\right)\,g^{(0)}_{ab} - 2\,g^{(2)}_{uu}\,g^{(0)}_{ab}\right]\\
&+\frac{1}{2}g^{(3)}_{uu}g^{(0)}_{ab} + \left(\frac{3}{2}g^{(1)}_{ab} - \frac{1}{2}\left(g^{cd}_{(0)}g^{(1)}_{cd}\right)\,g^{(0)}_{ab}\right)\,g^{(2)}_{uu} + D_{(a}g^{(2)}_{b)u} - D^c g^{(2)}_{cu}\,g^{(0)}_{ab}\\
&+\frac{1}{2}\partial_u\left(g^{cd}_{(0)}g^{(2)}_{cd}\right)\,g^{(0)}_{ab} - \frac{1}{2}\partial_u g^{(2)}_{ab} + \partial_u\left(g^{cd}_{(0)}g^{(1)}_{cd}\right)\,g^{(1)}_{ab} - \frac{1}{2}\left(g^{(1)}_{ac}g^{cd}_{(0)}\,\partial_u g^{(1)}_{db} + \partial_u g^{(1)}_{ac}g^{cd}_{(0)}\,g^{(1)}_{db}\right)\\
&+\frac{1}{2}\left(g^{cd}_{(0)}g^{(1)}_{cd}\right)\,\partial_u g^{(1)}_{ab} - \frac{1}{4}\partial_u\left(g^{cd}_{(0)}g^{(1)}_{cd}\right)^2\,g^{(0)}_{ab}\\
&+\mathcal{O}(1/r).\end{aligned} \tag{3.62}$$

Substituting these expressions in (3.57), we find in large $r \to \infty$ limit the variation of the Lagrangian density is given by

$$\begin{aligned}\delta\mathcal{L}_{EH+bdy.} = &-2\,r^3\sqrt{g^{(0)}}\,\delta g^{(0)}_{uu} - r^2\sqrt{g^{(0)}}\,\left[\frac{1}{2}\left(g^{cd}_{(0)}g^{(1)}_{cd}\right)\,\delta g^{(0)}_{uu} + 2\,\delta g^{(1)}_{uu}\right]\\
&-r\sqrt{g^{(0)}}\left[2\,\delta g^{(2)}_{uu} + g^{(2)}_{uu}\,g^{ab}_{(0)}\,\delta g^{(0)}_{ab} + \frac{1}{2}\left(g^{cd}_{(0)}g^{(1)}_{cd}\right)\,\delta g^{(1)}_{uu} - 2\,g^{(2)}_{ua}\,g^{ab}_{(0)}\delta g^{(0)}_{ub}\right.\\
&\left.-\frac{1}{2}\left(g^{ab}_{(0)}\partial_u g^{(1)}_{bc}\,g^{cd}_{(0)}\,\delta g^{(0)}_{da} - \left(g^{cd}_{(0)}\partial_u g^{(1)}_{cd}\right)\left(g^{ab}_{(0)}\,\delta g^{(0)}_{ab}\right)\right)\right] - \sqrt{g^{(0)}}\,\delta\mathcal{L}_0 + \mathcal{O}(1/r),\end{aligned} \tag{3.63}$$

where:

$$\begin{aligned}\delta\mathcal{L}_0 = &\,2\,\delta g^{(3)}_{uu} + \frac{1}{2}g^{(3)}_{uu}\,\left(g^{cd}_{(0)}\delta g^{(0)}_{cd}\right) + \frac{1}{2}\left(g^{cd}_{(0)}g^{(1)}_{cd}\right)\,\delta g^{(2)}_{uu} + \frac{1}{2}g^{(2)}_{uu}\,\left(g^{ab}_{(1)}\delta g^{(0)}_{ab}\right)\\
&-\frac{1}{2}\left[\,\mathrm{Tr}\,g^{(3)} + \frac{1}{2}\,\mathrm{Tr}\,g^{(1)}\;\mathrm{Tr}\,g^{(2)} - \mathrm{Tr}\left(g^{(1)}g^{-1}_{(0)}g^{(2)}g^{-1}_{(0)}\right)\right.\\
&\left.-\frac{1}{8}\,\mathrm{Tr}\,g^{(1)}\,\mathrm{Tr}\left(g^{(1)}g^{-1}_{(0)}g^{(1)}g^{-1}_{(0)}\right) + \frac{1}{4}\,\mathrm{Tr}\left(g^{(1)}g^{-1}_{(0)}\right)^3\,\right]\delta g^{(0)}_{uu} - \left(D_{(a}g^{(2)}_{b)u} - D^c g^{(2)}_{uc}\,g^{(0)}_{ab}\right)\,\delta g^{ab}_{(0)}\\
&-\frac{1}{4}\,\mathrm{Tr}\,g^{(1)}\,\left(\partial_u g^{(1)}_{ab} - \partial_u(\mathrm{Tr}\,g^{(1)})g^{(0)}_{ab}\right)\,\delta g^{ab}_{(0)} + \frac{1}{2}\left(\partial_u g^{(2)}_{ab}\delta g^{ab}_{(0)} - \mathrm{Tr}(\partial_u g^{(2)})\,\left(g^{(0)}_{ab}\,\delta g^{ab}_{(0)}\right)\right)\\
&+g^{(2)}_{ua}g^{ab}_{(1)}\delta g^{(0)}_{bu} - g^{(3)}_{au}\,g^{ab}_{(0)}\,\delta g^{(0)}_{bu} - 2g^{(2)}_{au}g^{ab}_{(0)}\,\delta g^{(1)}_{bu} + g^{(2)}_{uu}\,g^{ab}_{(0)}\delta g^{(1)}_{ab}\\
&-\frac{1}{2}\left(\mathrm{Tr}\left(g^{-1}_{(0)}\partial_u g^{(1)}g^{-1}_{(0)}\delta g^{(1)}\right) - \mathrm{Tr}\left(g^{-1}_{(0)}\partial_u g^{(1)}\right)\mathrm{Tr}\left(g^{-1}_{(0)}\delta g^{(1)}\right)\right),\end{aligned} \tag{3.64}$$

with $\mathrm{Tr}g^{(2)} = g^{ab}_{(0)}g^{(2)}_{ab}$ and so on. As we will be taking the boundary $r = r_0 \to \infty$ we will not need to keep terms that vanish in this limit.

3.2.2 Conditions from variational principle

We first impose

$$\delta g^{(0)}_{uu} = \delta g^{(1)}_{uu} = \delta g^{(0)}_{ua} = \delta g^{(1)}_{ua} = 0 \tag{3.65}$$

for all configurations as these are part of the boundary conditions derived in Section (3.1). Furthermore, since we consider the class of geometries that are locally flat we restrict to the class (3.11) of geometries: $g^{(2+n)}_{\mu\nu} = \delta g^{(2+n)}_{\mu\nu} = 0$ for $n = 1, 2, \cdots$. We also impose the gauge conditions and $\mathrm{Tr}\, g^{(1)} = 0$ as well. Then we have

$$\delta\mathcal{L} = -2\, r\, \delta\left(\sqrt{g^{(0)}}\, g^{(2)}_{uu}\right) + r\,\sqrt{g^{(0)}}\,\left[\frac{1}{2}\, g^{ab}_{(0)}\partial_u g^{(1)}_{bc}\, g^{cd}_{(0)}\, \delta g^{(0)}_{da}\right] - \sqrt{g^{(0)}}\ \delta\mathcal{L}_0 + \mathcal{O}(1/r) \tag{3.66}$$

where,

$$\begin{aligned} \delta\mathcal{L}_0/\sqrt{g^{(0)}} = \frac{1}{2} g^{(2)}_{uu}\left(g^{ab}_{(1)}\delta g^{(0)}_{ab}\right) - \left(D_{(a} g^{(2)}_{b)u} - D^c g^{(2)}_{uc}\, g^{(0)}_{ab}\right)\, \delta g^{ab}_{(0)} \\ + \frac{1}{2}\left(\partial_u g^{(2)}_{ab}\delta g^{ab}_{(0)} - \mathrm{Tr}(\partial_u g^{(2)})\, \left(g^{(0)}_{ab}\ \delta g^{ab}_{(0)}\right)\right) + g^{(2)}_{uu}\, g^{ab}_{(0)}\delta g^{(1)}_{ab} - \frac{1}{2}\left(\mathrm{Tr}\left(g^{-1}_{(0)}\partial_u g^{(1)} g^{-1}_{(0)}\delta g^{(1)}\right)\right). \end{aligned} \tag{3.67}$$

We just have to ensure that this quantity $\delta\mathcal{L}$, when integrated over the boundary directions $(u, z, \bar{z})$, vanishes in the $r = r_0 \to \infty$. One more condition that we will impose for all the cases below is:

$$\delta \int d^2 z \left[\sqrt{g^{(0)}}\, g^{(2)}_{uu}\right] = 0 \tag{3.68}$$

which when using the relation $g^{(2)}_{uu} = -\frac{1}{2} R_0$ is equivalent to holding the Euler character of the metric on Σ_2 fixed under variations (see also [129]). Now we consider special cases of boundary conditions that ensure $\delta S = 0$.

Solutions with Dirichlet Boundary conditions

In this case we take $\delta g^{(0)}_{ab} = 0$. Since $g^{(2)}_{uu}$ is related to the curvature of $g^{(0)}_{ab}$ as given in (3.12) it follows that $\delta g^{(2)}_{uu} = 0$ and therefore $\delta\left(\sqrt{g^{(0)}}\, g^{(2)}_{uu}\right) = 0$. Using $\delta g^{(0)}_{ab} = 0$ in the rest of the terms leaves:

$$\delta\mathcal{L}_0 = -\sqrt{g^{(0)}}\,\left[g^{(2)}_{uu}\, g^{ab}_{(0)}\delta g^{(1)}_{ab} - \frac{1}{2}\left(\mathrm{Tr}\left(g^{-1}_{(0)}\partial_u g^{(1)} g^{-1}_{(0)}\delta g^{(1)}\right)\right)\right]. \tag{3.69}$$

The first term in the bracket is proportional to the variation of $g^{ab}_{(0)} g^{(1)}_{ab}$ which is held to zero. So the only non-trivial term in the variation of the Lagrangian is proportional to $g^{ab}_{(1,1)} \delta g^{(1)}_{ab}$, and the simplest (covariant and non-chiral) way to ensure the vanishing of this term is to set $g^{(1,1)}_{ab} = 0$. Thus, we conclude that the locally flat solutions that satisfy the variational principle with fixed metric on the boundary Σ_2 are given by the class of solutions in conformal gauge (3.106) with $\kappa(z) = \bar{\kappa}(\bar{z}) = 0$ for Ω corresponds to Σ_2 being $\mathbb{R}^2$ ($\Omega = 1$) or S^2 ($\Omega = 4\,(1 + z\bar{z})^{-2}$) or $\mathbb{H}^2$ ($\Omega = 4\,(1 - z\bar{z})^{-2}$).

There is a rather interesting consequence of this conclusion: The algebra of symmetries that enable us to move along the space of locally flat solutions, in this case, is exactly $\mathfrak{bms}_4$ [3, 4, 26], but do not allow for the $\mathfrak{bms}_4$ algebra to become the extended $\mathfrak{bms}_4$ [111, 132].

3.2.3 Polyakov gauge solutions

Next, we turn to the non-Dirichlet boundary metrics on Σ_2 and in particular restrict to the chiral Polyakov gauge for the metric on Σ_2: $\bar{f}(z, \bar{z}) = 0$ with Ω to be given a function of $(z, \bar{z})$. We will continue to impose the boundary conditions

$$\delta g^{(0)}_{uu} = \delta g^{(1)}_{uu} = \delta g^{(0)}_{ua} = \delta g^{(1)}_{ua} = \delta \int d^2 z \sqrt{g^{(0)}} R_0 = 0. \tag{3.70}$$

In this gauge even though $\delta g^{(0)}_{zz} \neq 0$ we still have $\delta \sqrt{\det g^{(0)}_{ab}} = 0$. We will need to ensure the remaining terms in (3.66, 3.67) also vanish.

1. From the $\mathcal{O}(r)$ term in $\delta\mathcal{L}$ that is proportional to $g^{(1,1)}_{\bar{z}\bar{z}} \delta F$ we need to set $g^{(1,1)}_{\bar{z}\bar{z}} = 0$.

2. From the first term in $\mathcal{O}(r^0)$ terms we further need to impose $g^{(1,0)}_{\bar{z}\bar{z}} = 0$.

3. Then imposing the conditions $g^{ab}_{(0)} g^{(1)}_{ab} = 0$ immediately implies $g^{(1)}_{z\bar{z}} = 0$.

4. Substituting $g^{(1)}_{z\bar{z}} = g^{(1)}_{\bar{z}\bar{z}} = 0$ into $g^{(2)}_{ab} = \frac{1}{4} g^{(1)}_{ac} g^{cd}_{(0)} g^{(1)}_{db}$ we find that $g^{(2)}_{ab} = 0$.

5. Further in our gauge $g^{ab}_{(0)}\, \delta g^{(0)}_{ab} = 0$, $g^{ab}_{(0)}\, \delta g^{(1)}_{ab} = 0$, and $\mathrm{Tr}\left(g^{-1}_{(0)} \partial_u g^{(1)} g^{-1}_{(0)} \delta g^{(1)} \right) = 0$.

6. Finally we need to check: $D_{\bar{z}} g^{(2)}_{\bar{z}u} = D_{\bar{z}} D^a g^{(1)}_{a\bar{z}}$. Using $g^{(1)}_{z\bar{z}} = g^{(1)}_{\bar{z}\bar{z}} = 0$ and (3.12) it is easy to see that this quantity also vanishes.

To summarise the variational principle in the Polyakov gauge can be solved by imposing:

$$g^{(1,0)}_{\bar{z}\bar{z}} = g^{(1,1)}_{\bar{z}\bar{z}} = 0 \tag{3.71}$$

on the class of locally flat geometries we found in the previous section.

In anticipation of this result, we have already presented the solutions satisfying these additional conditions in section (3.1.3). Here we present the locally flat solutions in closed form for two cases: $\Omega = 1$ and $\Omega = 4\,(1+z\bar{z})^{-2}$ that also are consistent with our variational principle.

- Locally flat solutions $\Omega = 1$:

$$\begin{aligned} ds^2_{M_2=\mathbb{R}^2} = &-2\,du\,dr + \,r^2\,dz\,d\bar{z} + r^2\,\left(-\eta_{ab}\,J^a\,\bar{z}^{1+b}\right)\,dz^2 \\ &+ r\,u\,\left(\frac{1}{u}\,\epsilon_{sr}\,C^r\,\bar{z}^{\frac{2s+1}{2}} - \kappa(z) + \frac{1}{2}\,\eta_{ab}\,J^a(z)\,J^b(z) + (1+a)\,\eta_{ab}\,\bar{z}^a\,\partial_z J^b\right)\,dz^2 \\ &+ 2\,\left(\epsilon_{rs}\,\frac{2s+1}{2}\,C^r\,\bar{z}^{\frac{2s-1}{2}} + \,u\,\eta_{ab}\,b\,(1+b)\,\partial_z J^a\,\bar{z}^{b-1}\right)\,du\,dz \\ &+ 2\,\eta_{ab}\,b\,(1+b)\,J^a\,\bar{z}^{b-1}\,du^2. \end{aligned} \tag{3.72}$$

- Locally flat solutions with $\Omega = 4\,(1+z\bar{z})^{-2}$:

$$\begin{aligned} ds^2_{\mathbb{M}^2=\mathbb{S}^2} = &-2\,du\,dr + \frac{4\,r^2}{(1+z\,\bar{z})^2}\,dz\,d\bar{z} - \frac{4\,r^2}{(1+z\,\bar{z})^2}\,\eta_{ab}\,J^a\,\bar{z}^{1+b}\,dz^2 \\ &+ r\,u\,\Bigg[-\kappa(z) + \frac{1}{2}\eta_{ab}J^a J^b + (1+a)\,\eta_{ab}\,\bar{z}^a\,\partial_z J^b \\ &+ \frac{2}{1+z\,\bar{z}}\,\left(\frac{1}{u}\epsilon_{sr}\,C^r\,\bar{z}^{\frac{2s+1}{2}} + \,\eta_{ab}\left[(b-1)\,J^a - z\,\partial_z J^a\right]\,\bar{z}^{b+1}\right)\Bigg]\,dz^2 \\ &+\left(-1+\eta_{ab}\,(-z)^{1-a}\,J^b\right)\,du^2 - \left[\epsilon_{rs}\,C^r\,(-z)^{\frac{1-2s}{2}} + u\,\eta_{ab}\,\partial_z\left((-z)^{1-a}\,J^b\right)\right]\,du\,dz. \end{aligned} \tag{3.73}$$

The solutions (3.72) and (3.73) are parameterised in terms of six holomorphic functions

$$\{J^a(z), C^s(z), \kappa(z)\} \quad \text{where } a \in 0, \pm 1\,, \quad s \in \{-\frac{1}{2}, \frac{1}{2}\} \tag{3.74}$$

where η_{ab} is defined in (3.33) and $\epsilon_{ss'}$ in (3.37). $\eta_{ab}, \epsilon_{ss'}$ can be used to lower and raise the indices on $J^a(z)$ and $C^s(z)$ respectively. Now that we have obtained the locally flat solutions in the Polyakov gauge with the boundary conditions consistent with the variational problem we turn to analysing the symmetry algebras of these solution spaces in the next section.

3.3 Asymptotic symmetries

As reviewed in section (2.1.1), after obtaining our solution space, we will now look for residual diffeomorphisms that keep us within the classes of solutions (3.72), (3.73). This will involve deriving vector fields that do not spoil the Newman-Unti gauge and other boundary conditions that we have imposed on our solution space. We will then compute their commutator algebra.

3.3.1 The vector fields

From the general analysis of asymptotic symmetries done in (A.1-A.3), the first few components of the vector fields that keep us within the class of (3.72, 3.73) are given by,

$$\xi^z_{(0)} = Y(z), \quad \xi^{\bar{z}}_{(0)} = \overline{Y}(z,\bar{z}), \quad \xi^u(u,z,\bar{z}) = \xi^u_{(0)}(z,\bar{z}) + \frac{u}{2}\, D_a\, V^a, \tag{3.75}$$

$$\xi^r_{(0)} = -\frac{1}{2}\, D_a\, V^a, \quad \xi^a_{(1)} = -g^{ab}_{(0)}\, D_b\, \xi^u, \tag{3.76}$$

$$\xi^a_{(2)} = \frac{1}{2} g^{ab}_{(0)}\, g^{(1)}_{bc}\, g^{cd}_{(0)}\, D_d\, \xi^u, \tag{3.77}$$

$$\xi^r_{(1)} = \frac{1}{2}\, g^{ab}_{(0)} D_a D_b\, \xi^u - \frac{1}{4}\, \xi^a_{(0)}\, D_a \left(g^{bc}_{(0)} g^{(1)}_{bc}\right) - \frac{1}{4}\, \partial_u \left(\xi^u\, g^{bc}_{(0)} g^{(1)}_{bc}\right). \tag{3.78}$$

In the above solution $V^a = (Y(z), \overline{Y}(z,\bar{z}))$. So far, three functions parameterise our vector fields,

$$\{Y(z), \overline{Y}(z,\bar{z}), \xi^u_{(0)}(z,\bar{z})\}. \tag{3.79}$$

The rest of the sub-leading terms in r in the vector field are given in terms of (3.79). Under the action of these vector fields the data $g^{(0)}_{ab}$ and $g^{(1)}_{ab}$ transform as follows,

$$\delta g^{(0)}_{ab} = \mathcal{L}_{V^c} g^{(0)}_{ab} - 2\, \partial_u\, \xi^u\, g^{(0)}_{ab}, \tag{3.80}$$

$$\delta g^{(1)}_{ab} = \mathcal{L}_{V^c} g^{(1)}_{ab} - \partial_u\, \xi^u\, g^{(1)}_{ab} + \xi^u\, \partial_u g^{(1)}_{ab} + g^{(0)}_{ab}\, g^{cd}_{(0)} D_c\, D_d\, \xi^u - 2\, D_a\, D_b\, \xi^u. \tag{3.81}$$

We now impose conditions coming from the variational principle: $g^{(1)}_{\bar{z}\bar{z}} = g^{(1,0)}_{\bar{z}\bar{z}} + u\, g^{(1,1)}_{\bar{z}\bar{z}} = 0$ as found in section (3.2.3) which will put constraints on (3.79). Considering $\delta g^{(1)}_{\bar{z}\bar{z}}$ component in eq (3.81) and setting it to zero gives

$$\delta g^{(1,0)}_{\bar{z}\bar{z}} + u\, \delta g^{(1,1)}_{\bar{z}\bar{z}} = -2\, D_{\bar{z}} D_{\bar{z}} \big(\xi^u_{(0)} + \frac{u}{2}\, D_a\, V^a \big) = 0 \tag{3.82}$$

This requires setting $D_{\bar{z}} D_{\bar{z}}\, \xi^u_{(0)} = 0$ and $D_{\bar{z}} D_{\bar{z}}\, D_a\, V^a = 0$ separately.

- The equation $D_{\bar{z}} D_{\bar{z}}\, \xi^u_{(0)} = 0$ can be solved for $\xi^u_{(0)}$ and we obtain

$$\begin{aligned} \Sigma_2 = \mathbb{R}^2 : \quad & \xi^u_{(0)} = P_{-\frac{1}{2}}(z) + \bar{z}\, P_{\frac{1}{2}}(z) \\ \Sigma_2 = \mathbb{S}^2 : \quad & \xi^u_{(0)} = \frac{2}{1 + z\bar{z}} \left(P_{-\frac{1}{2}}(z) + \bar{z}\, P_{\frac{1}{2}}(z) \right) \end{aligned} \tag{3.83}$$

- The equation $D_{\bar{z}} D_{\bar{z}}\, D_a\, V^a = 0$ allows us to solve for $\overline{Y}(z, \bar{z})$ as

$$\overline{Y}(z, \bar{z}) = Y_{-1}(z) + \bar{z}\, Y_0(z) + \bar{z}^2\, Y_1(z) \tag{3.84}$$

  in either case of Σ_2.

Therefore, the vector fields that keep us within our solution spaces are characterised by six holomorphic functions: $\{Y(z), Y_a(z), P_s(z)\}$ where $a \in \{0, \pm 1\}$, $s \in \{-\frac{1}{2}, \frac{1}{2}\}$. The sub-leading components of vector fields are then found in terms of these leading order components following the general analysis done in (A). If these holomorphic functions are allowed to have poles, then the supertranslation vector field $\xi^u_{(0)}$ for a specific choice of P_s,

$$P_{-\frac{1}{2}}(z) = -\frac{(1 + w\bar{w})\, \bar{w}}{2\,(z - w)}, \quad P_{\frac{1}{2}}(z) = \frac{(1 + w\bar{w})}{2\,(z - w)}$$

becomes,

$$\xi^u_{(0)} = \frac{1 + w\,\bar{w}}{z - w} \frac{(\bar{z} - \bar{w})}{1 + z\bar{z}}. \tag{3.85}$$

The vector field in equation (3.85) was used in [133, 59] to show the equivalence between supertranslation Ward identity and Weinberg leading soft theorem for positive helicity graviton localised at the reference direction $(w, \bar{w})$ on the sphere. The vector field in (3.85) induces transformation of $g^{(1,0)}_{\bar{z}\bar{z}}$ which is zero everywhere on the sphere except for $z = w$,

$$\delta g^{(1,0)}_{\bar{z}\bar{z}} = -\frac{4\pi\ (1 + w\,\bar{w})}{(1 + z\bar{z})}\delta^{(2)}(z - w). \tag{3.86}$$

In a similar spirit for the vector field in equation (3.84) the choice,

$$Y_{-1}(z) = \frac{\bar{w}^2}{(z - w)}\,,\quad Y_0 = -\frac{2\,\bar{w}}{(z - w)}\,,\quad Y_1 = \frac{1}{(z - w)} \tag{3.87}$$

coincides with the vector field used in [12] to show the equivalence between Diff($\mathbb{S}^2$) Ward identity and Cazchao-Strominger (CS) sub-leading soft theorem for positive helicity graviton (the conjugate vector field give rise to Ward identity that is equivalent to CS soft theorem for negative helicity graviton).[6] For this particular choice of a vector field,

$$\delta g^{(1;1)}_{\bar{z}\bar{z}} = 4\pi\delta^{(2)}(z - w) \tag{3.88}$$

The singularity in (3.86) and (3.88) is similar to the one generated by superrotation vector fields of the type

$$Y^z = \frac{1}{z - w}, \tag{3.89}$$

which change the sphere metric according to (3.80) at null infinity by adding singularities at isolated points [134].

3.3.2 Algebra of vector fields

The full set of vector fields that enable us to move in the space of solutions (3.72, 3.73) are given by,

$$\xi = \left(\xi^u_{(0)} + \frac{u}{2}\,D_a\,V^a\right)\partial_u + \left(V^a - \frac{1}{2\,r}\,g_0^{ab}\,D_b\,\xi^u\right)\partial_a - \frac{r}{2}\,D_a\,V^a\,\partial_r + \cdots\,, \tag{3.90}$$

where V^a and $\xi^u_{(0)}$ are given by (3.83, 3.84). The $\cdots$ denote the subleading terms of $\mathcal{O}(1/r)$ in the vector fields that are given in terms of $\xi^u_{(0)}, V^a$. We check that the Lie brackets between any two of these vector fields close under the modified commutator

[6] As mentioned in [133] this vector field acts as a kernel from which any smooth vector field can be constructed and therefore Ward identity associated with any Diff($\mathbb{S}^2$) vector field on the sphere can be obtained from the Ward identity associated with this vector field.

defined in (2.14). We find:

$$[\,\xi_1\,,\,\xi_2\,]_M = \left(\hat{\xi}^u_{(0)} + \frac{u}{2}\,D_a\,\hat{V}^a\right)\partial_u + \left(\hat{V}^a - \frac{1}{2\,r}\,g_0^{ab}\,D_b\,\hat{\xi}^u\right)\partial_a - \frac{r}{2}\,D_a\,\hat{V}^a\,\partial_r + \cdots \tag{3.91}$$

where

$$\hat{\xi}^u_{(0)} = V_1^a\,\partial_a\,\xi^u_{2\,(0)} + \frac{1}{2}\,\xi^u_{1\,(0)}\,D_a\,V_2^a - V_2^a\,\partial_a\,\xi^u_{1\,(0)} - \frac{1}{2}\,\xi^u_{2\,(0)}\,D_a\,V_1^a, \tag{3.92}$$

$$\hat{V}^a = V_1^b\,\partial_b\,V_2^a - V_2^b\,\partial_b\,V_1^a. \tag{3.93}$$

and $\cdots$ denote the subleading terms of $\mathcal{O}(1/r)$ in the resultant vector field.

- We denote the vector fields with $Y(z) = \xi^u_{(0)}(z,\bar{z}) = 0$ and following (3.84), $\overline{Y}(z,\bar{z}) = -z^n\,\bar{z}^{a+1}$ by $\mathcal{J}_{a,n}$ for $a \in \{0,\pm 1\}$ and $n \in \mathbb{Z}$. The commutator of these vector fields using (3.93) is then given by,

$$[\mathcal{J}_{a,m}\,,\,\mathcal{J}_{b,n}\,] = \big(a-b\big)\,\mathcal{J}_{a+b,m+n} \tag{3.94}$$

  This can be recognised as the $\mathfrak{sl}(2,\mathbb{R})$ current algebra.

- Denoting the vector field with $\overline{Y} = \xi^u_{(0)} = 0$ and $Y(z) = -z^{n+1}$ by $\mathcal{L}_n$ for $n \in \mathbb{Z}$ we find using (3.93):

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\,\mathcal{L}_{m+n}, \quad [\,\mathcal{L}_m\,,\mathcal{J}_{a,n}\,] = -n\,\mathcal{J}_{a,m+n} \tag{3.95}$$

  The first of these is simply the Witt algebra.

- Finally denoting the vector field with $Y = \overline{Y} = 0$ and

$$\xi^u_{(0)} = -z^{r+\frac{1}{2}}\,\bar{z}^{s+\frac{1}{2}} \;\text{ for }\; \Sigma_2 = \mathbb{R}^2, \quad \xi^u_{(0)} = -\frac{z^{r+\frac{1}{2}}\,\bar{z}^{s+\frac{1}{2}}}{1+z\bar{z}} \;\text{ for }\; \Sigma_2 = S^2 \tag{3.96}$$

  by $\mathcal{P}_{s,r}$ where $r \in \mathbb{Z} + \frac{1}{2}$ and $s \in \{-\frac{1}{2}, \frac{1}{2}\}$, the remaining commutators work out to be:

$$[\mathcal{L}_n, \mathcal{P}_{s,r}] = \frac{1}{2}(n-2\,r)\,\mathcal{P}_{s,n+r}, \quad [\mathcal{J}_{a,n}, \mathcal{P}_{s,r}] = \frac{1}{2}(a-2\,s)\,\mathcal{P}_{a+s,n+r}$$
$$[\mathcal{P}_{s,r}, \mathcal{P}_{s',r'}] = 0 \tag{3.97}$$

Thus the final result for the algebra of vector fields that preserve our spaces of locally flat solutions (3.72, 3.73) are (3.94, 3.95, 3.97). We refer to this algebra as chiral $\mathfrak{bms}_4$ algebra and it is identical to the symmetry algebra uncovered from the analysis of conformal soft theorem in the celestial CFT by [13, 14].

3.3.3 Variations of fields in locally flat solutions

Our final set of locally flat solutions (both for $\Sigma_2 = \mathbb{R}^2$ or $\mathbb{S}^2$) are characterised by the fields in (3.74). We compute the transformations of these fields under the symmetry algebra found in the previous subsection. It turns out that the transformations of the fields are identical for either of the solutions (3.72) and (3.73). So we will demonstrate this for the case of $\Sigma_2 = \mathbb{R}^2$. First we rewrite the field C^r as (like in [127, 125] for instance),

$$C^r(z) = -2\,\partial_z^2\,\mathcal{C}^r(z) + \left(\frac{1}{2}\,\eta_{ab}\,J^a(z)\,J^b(z) - \kappa(z)\right)\mathcal{C}^r(z) \\ + g^r_{a\,s}\big(-\partial\,J^a(z)\,\mathcal{C}^s(z) - 2\,J^a(z)\,\partial\mathcal{C}^s(z)\big). \tag{3.98}$$

Here the non-zero structure constant g^r_{is} are given by: $g^{\frac{1}{2}}_{0\,\frac{1}{2}} = g^{-\frac{1}{2}}_{-1\,\frac{1}{2}} = 1\,,\; g^{\frac{1}{2}}_{1\,-\frac{1}{2}} = g^{-\frac{1}{2}}_{0\,-\frac{1}{2}} = -1$. We write explicitly the vector field (3.90) that preserves the form of our solutions (up to $1/r$-order terms) as follows:

$$\xi^u(u,z,\bar z) = \epsilon_{rs}\,\bar z^{\frac{2r+1}{2}}\,P^s(z) + \frac{u}{2}\Big(\eta_{ab}\,(1+b)\,Y^a(z)\,\bar z^b + \partial_z Y(z)\Big), \tag{3.99}$$

$$\begin{aligned}\xi^r(r,u,z,\bar z) &= -\frac{r}{2}\Big(\eta_{ab}\,(1+b)\,Y^a(z)\,\bar z^b + \partial_z Y(z)\Big)\\ &+ \eta_{ab}\,(1+b)\,J^a(z)\,\bar z^b\,\Big(-\epsilon_{rs}\ (2s+1)\,P^r(z)\,\bar z^{\frac{2s-1}{2}} + u\,\eta_{ab}\,(1+b)\,b\,Y^a(z)\,\bar z^{b-1}\Big)\\ &- \epsilon_{rs}\,(2s+1)\,\partial_z\,P^r(z)\,\bar z^{\frac{2s-1}{2}} + u\,\eta_{ab}\,(1+b)\,b\,\partial_z Y^a(z)\,\bar z^{b-1}\\ &- \frac{1}{r}\,\Big[\epsilon_{rs}\,\tfrac{2s+1}{2}\,\bar z^{\frac{2s-1}{2}}\,(-2\,P^r(z) + C^r(z)) + u\,\eta_{ab}\,(1+b)\,b\,\bar z^{b-1}\,(Y^a(z) + \partial_z J^a(z))\Big] + \cdots\,,\end{aligned} \tag{3.100}$$

$$\xi^z(r,u,z,\bar z) = Y(z) + \frac{2}{r}\,\left[\epsilon_{rs}\,\frac{2s+1}{2}\,P^r(z)\,\bar z^{\frac{2s-1}{2}} - u\,\eta_{ab}\,(1+b)\,b\,Y^a(z)\,\bar z^{b-1}\right] + \cdots\,, \tag{3.101}$$

$$\begin{aligned}\xi^{\bar z}(r,u,z,\bar z) &= \eta_{ab}\,Y^a(z)\,\bar z^{1+b} + \frac{1}{r}\,\Big[2\,\eta_{ab}\,J^a(z)\,\bar z^{1+b}\,\epsilon_{rs}\,(2s+1)\,P^r(z)\,\bar z^{\frac{2s-1}{2}}\\ &+ 2\,\epsilon_{rs}\,\partial_z\,P^r(z)\,\bar z^{\frac{2s+1}{2}} - u\,(2\,\eta_{ab}\,J^a(z)\,\bar z^{1+b}\,\eta_{kl}\,(1+l)\,l\,Y^k(z)\,\bar z^{l-1}\\ &+ \eta_{ab}\,(1+b)\,\partial_z Y^a(z)\,\bar z^b + \partial_z^2\,Y(z)\,)\Big] + \cdots\,.\end{aligned} \tag{3.102}$$

Under these residual diffeomorphisms, the variation of the fields (3.74) are

$$\begin{aligned}\delta J^a(z) &= J^a(z)\,\partial\,Y(z) + Y(z)\,\partial\,J^a(z) + \ f^a{}_{bc}\,J^b(z)\,Y^c(z) - \partial\,Y^a(z),\\ \delta\kappa(z) &= Y(z)\,\partial\,\kappa(z) + 2\,\kappa(z)\,\partial Y(z) + \partial^3\,Y(z),\\ \delta\mathcal{C}^r(z) &= Y(z)\,\partial\,\mathcal{C}^r(z) - \frac{1}{2}\,\partial Y(z)\,\mathcal{C}^r(z) + g_a{}^r{}_s\,\frac{1}{2}\,Y^a(z)\,\mathcal{C}^s(z) + P^r(z).\end{aligned} \tag{3.103}$$

The values of the non-zero components of the structure constants $f^a{}_{bc} = -f^a{}_{cb}$ are $f^{-1}_{-1\,0} = -1\,,\ f^0{}_{1,-1} = \frac{1}{2}\,,\ f^1{}_{1,0} = 1$.

One expects that these $\delta_\xi\{\kappa(z), J^a(z), \mathcal{C}^s(z)\}$ are given by the Poisson brackets of putative charges $Q[\xi]$ corresponding to the vector field ξ with the fields. Assuming this interpretation to be true, one can comment on the nature of these fields by looking at their transformations (3.103). The triplet $J^a(z)$ transforms as a primary of weight $h=1$ under the left Virasaro transformation generator $Q[Y(z)]$. The field $\kappa(z)$ transforms like a quasi primary of weight $h=2$ with an inhomogeneous term similar to the chiral stress tensor of a $2d$ CFT. The $\mathcal{C}^s(z)$ are related to supertranslation current and transform like the primary of weight $h = -\frac{1}{2}$.[7] The fields $\kappa(z), J^a(z), \mathcal{C}^s(z)$ also transform as Kac-Moody primaries with $j = 0, 1, \frac{1}{2}$ respectively under the $\mathfrak{sl}(2,\mathbb{R})$ current algebra with $\mathcal{C}^s(z)$ forming a doublet under the current algebra which is also evident from looking at the commutation relation

$$[\mathcal{J}_{a,n}, \mathcal{P}_{s,r}] = \frac{1}{2}(a - 2\,s)\,\mathcal{P}_{a+s,n+r}$$

in (3.97).

3.3.4 Comments on the charges

The locally flat solutions that we have obtained parameterise the space of gravitational vacua. The fields $\{\kappa(z), J^i(z), \mathcal{C}^r(z)\}$ can be thought of as 'Goldstone' modes associated with spontaneous symmetry breaking of chiral $\mathfrak{bms}_4$ symmetry. To construct complete phase space we are required to construct soft charges that generate symmetry transformation (3.103) on these fields. From what we have seen the fields in the solutions behave like (quasi-) primaries under the chiral Virasoro generators as well as the current algebra ones and thus one expects the charges, if they exist, to be given more appropriately as line integrals as it is the norm for a chiral conformal field theory. It is not clear how to obtain such line integral charges from the $4d$ bulk perspective where the usual route of defining charges through covariant phase space formalism [120, 135, 136] or the cohomological formalism [118, 137, 138] lead to co-dimension two surface charges that are defined as integrals over the celestial sphere $\mathbb{S}^2$.

In [129] charges for gravitational vacua exhibiting supertranslation and Diff($\mathbb{S}^2$) symmetry were constructed. It was shown that these co-dimension two charges corresponding to supertranslations vanish but charges for superrotations which are proportional to terms quadratic in $g^{(1,0)}_{ab}$ and their derivatives are in general non-vanishing and conserved. These charges are evaluated to zero for our chiral solutions in (3.72, 3.73). Therefore one needs to find an alternative prescription to calculate charges that are line integrals and provide a representation of chiral $\mathfrak{bms}_4$ algebra and generate correct symmetry transformation on solution space.

One such potential partial resolution to get the charges as line integrals is to switch to the first-order formalism of gravity where these locally flat solutions can be written

[7] The field $C^r(z)$ in (3.98) will transform like $h = \frac{3}{2}$ conformal primary but continue to be a doublet of current algebra. Refer to (B.18) for its transformation.

as connections A_μ that take value in $\mathfrak{iso}(1,3)$ algebra. Please refer to (B) for details of this calculation. Here we provide the inferences drawn from that calculation. The condition of vanishing curvature for this connection ($F_{\mu\nu} = 0$) is equivalent to the local flatness condition that we imposed on our configuration space. Using gauge transformation, one can gauge away the r dependence and get an effective $3d$ flat connection. The sector within this $3d$ gauge connection parameterised by fields $\{\kappa(z), J^i(z)\}$ is independent of field $\mathcal{C}^r(z)$ and finds a natural interpretation in terms of $3d$ flat $\mathfrak{sl}(2,\mathbb{C})$ Chern-Simons connection. The residual gauge transformations that preserve the form of this connection are a subset of the asymptotic symmetries found in the previous section (the $\mathfrak{sl}(2,\mathbb{R})$ current algebra and the Witt algebra).[8]

The charges generating residual gauge transformations in a Chern-Simons theory are line integral given by,

$$\not{\delta} Q = \frac{k}{2\pi i} \int dx^\mu \, \mathrm{Tr}(\Lambda \, \delta A_\mu), \tag{3.104}$$

where k is the level and Λ is the gauge parameter with $\delta A = d\Lambda + [\Lambda, A]$. Using the formula (3.104) the charges generating $\mathfrak{sl}(2,\mathbb{R})$ current algebra and Virasaro symmetry are given by

$$Q = \frac{k}{2\pi i} \oint dz \, [\eta_{ab} \, Y^a(z) \, J^b(z) + Y(z)\kappa(z)]. \tag{3.105}$$

One can construct a consistent phase space with these currents $(J^a(z), \kappa(z))$ themselves and after deriving the OPEs between them one can show that the charges (3.105) generate the correct variation of $\delta J^a(z)$ and $\kappa(z)$ in (3.103) for some fixed value of k.

The part of gauge connection that has the information about the mode $\mathcal{C}^r(z)$ associated with supertranslation is gauged by the momentum generators ($P_{r,s}$) of Poincare algebra and if one attempts to calculate charges associated with supertranslation using (3.104) one has to consider full $\mathfrak{iso}(1,3)$ gauge connection as δA_μ gets non-trivial contribution from the commutators of $\mathfrak{sl}(2,\mathbb{C})$ generators with $P_{r,s}$.[9] Since Poincare algebra does not have a non-degenerate bi-linear invariant, therefore computation of charges for currents associated with supertranslations is not possible using the formula for charge in (3.104) and therefore one has to revisit some notions of covariant

[8]A similar computation was done in [139] where the Goldstone mode associated with Virasaro transformation and Diff($\mathbb{S}^2$) superrotation is described by two-dimensional Alekseev-Shatashvili theory obtained after Hamiltonian reduction of $3d$ Chern-Simons theory that parameterised the gravitational vacua associated with these symmetry transformations.

[9]The Poincare algebra in this basis is given in (B.2).

phase space analysis for our solutions that can provide the full set of line integral charge such that it reproduces the variations (3.103). This we leave to future works.

3.4 Locally flat solutions in Conformal gauge

In addition to the Polyakov gauge, one can parameterise the boundary metric (3.17) in a conformal gauge. This gauge corresponds to setting $f = \bar{f} = 0$, such that $g^{(0)}_{z\bar{z}} = \frac{\Omega(z,\bar{z})}{2}$. This was the choice of the boundary metric that authors considered in [111]. The variation of this conformal factor allows for Weyl rescaling of the boundary metric $g^{(0)}_{ab}$. The free data $g^{(0)}_{ab}$ and $g^{(1)}_{ab}$ in this gauge is given by,

$$\begin{aligned}
g^{(0)}_{zz} &= g^{(0)}_{\bar{z}\bar{z}} = 0, \quad g^{(0)}_{z\bar{z}} = \frac{1}{2}\,\Omega(z,\bar{z}), \quad g^{(1)}_{z\bar{z}} = 0, \\
g^{(1,1)}_{zz} &= \kappa(z) + \frac{\partial_z^2\Omega(z,\bar{z})}{\Omega(z,\bar{z})} - \frac{3\,(\partial_z\Omega(z,\bar{z}))^2}{2\,(\Omega(z,\bar{z}))^2}, \\
g^{(1,1)}_{\bar{z}\bar{z}} &= \bar{\kappa}(\bar{z}) + \frac{\partial_{\bar{z}}^2\Omega(z,\bar{z})}{\Omega(z,\bar{z})} - \frac{3\,(\partial_{\bar{z}}\Omega(z,\bar{z}))^2}{2\,(\Omega(z,\bar{z}))^2} \\
g^{(1,0)}_{zz} &= -2\left(\partial_z^2 C(z,\bar{z}) - \frac{\partial_z C(z,\bar{z})\,\partial_z\zeta(z,\bar{z})}{\zeta(z,\bar{z})}\right) - g^{(1,1)}_{zz}\,C(z,\bar{z}), \\
g^{(1,0)}_{\bar{z}\bar{z}} &= -2\left(\partial_{\bar{z}}^2 C(z,\bar{z}) - \frac{\partial_{\bar{z}} C(z,\bar{z})\,\partial_{\bar{z}}\zeta(z,\bar{z})}{\zeta(z,\bar{z})}\right) - g^{(1,1)}_{\bar{z}\bar{z}}\,C(z,\bar{z})
\end{aligned} \tag{3.106}$$

where $C(z,\bar{z})$ and $\Omega(z,\bar{z})$ are arbitrary functions. To obtain the above solution from the general solution of $g^{(1)}_{ab}$, we redefined the field $\upsilon(z,\bar{z})$ in (3.86) as follows,

$$\upsilon(z,\bar{z}) = \frac{C(z,\bar{z})}{\rho\,\sqrt{\Omega(z,\bar{z})}} \tag{3.107}$$

where $\rho = \left(\partial\zeta\bar{\partial}\bar{\zeta}\right)^{-1/2}$. Because in conformal gauge $f = \bar{f} = 0$, one has $\zeta(z,\bar{z}) = \zeta(z)$ and $\bar{\zeta}(z,\bar{z}) = \bar{\zeta}(\bar{z})$ and therefore we define

$$\kappa(z) = \{\zeta(z), z\}\,, \quad \bar{\kappa}(\bar{z}) = \{\bar{\zeta}(\bar{z}), \bar{z}\}. \tag{3.108}$$

$\{\zeta(z), z\}$ and $\{\bar{\zeta}(\bar{z}), \bar{z}\}$ are the Schwarzian derivatives defined in (3.19). Thus an element of this space of solutions is specified by the functions

$$(\kappa(z),\ \bar{\kappa}(\bar{z}),\ C(z,\bar{z}),\ \Omega(z,\bar{z})). \tag{3.109}$$

The $\Omega(z,\bar{z})$ dependent terms in $g^{(1,1)}_{zz}$ and $g^{(1,1)}_{\bar{z}\bar{z}}$ are simply the Schwarzian derivatives of $\Omega(z,\bar{z})$ and they vanish by themselves if and only if $\Omega(z,\bar{z}) = \frac{\alpha^2}{(\beta+z\bar{z})^2}$. The cases of the boundary metric being $\mathbb{R}^2$ corresponds to $\alpha = \beta \to \infty$ (that is, $\Omega(z,\bar{z}) = 1$) whereas the case of round unit S^2 corresponds to $\alpha = 2$ and $\beta = 1$, and $\alpha = 2$ and $\beta = -1$ corresponds to the case of $\mathbb{H}^2$. The space of solutions (3.106) gets constrained further by the imposition of a variational principle, which we turn to next.

3.4.1 Imposing variational principle

In this case we have $\delta g^{(0)}_{ab} = g^{(0)}_{ab}\,\Omega^{-1}\,\delta\Omega$. To satisfy the variational problem, we first look at (3.66), where in the first term we require that $\int_{M_2} d^2z\,\sqrt{g^{(0)}}\,R_0$ is fixed. This condition was already used to impose variational principle in previous sections. Using the tracelessness condition of $g^{(1)}_{ab}$ w.r.t $g^{(0)}_{ab}$ the left-over terms at $\mathcal{O}(r)$ in (3.66) read:[10]

$$\delta g^{(1)}_{z\bar{z}} = 0, \quad g^{(1,1)}_{zz} = g^{(1,1)}_{\bar{z}\bar{z}} = 0 \tag{3.110}$$

Demanding $g^{(1,1)}_{zz} = g^{(1,1)}_{\bar{z}\bar{z}} = 0$ requires

$$-\kappa(z) = \frac{\partial_z^2\Omega(z,\bar{z})}{\Omega(z,\bar{z})} - \frac{3\,(\partial_z\Omega(z,\bar{z}))^2}{2\,(\Omega(z,\bar{z}))^2}, \qquad -\bar{\kappa}(\bar{z}) = \frac{\partial_{\bar{z}}^2\Omega(z,\bar{z})}{\Omega(z,\bar{z})} - \frac{3\,(\partial_{\bar{z}}\Omega(z,\bar{z}))^2}{2\,(\Omega(z,\bar{z}))^2}. \tag{3.111}$$

However, for general $\Omega(z,\bar{z})$ these equations cannot be imposed as the right-hand side of $\kappa(z)$ ($\bar{\kappa}(\bar{z})$) is not necessarily holomorphic (anti-holomorphic), and satisfying these conditions imposes further conditions on $\Omega(z,\bar{z})$. To obtain what these conditions on $\Omega(z,\bar{z})$ are, we start by noticing that the scalar curvature of the boundary metric $ds^2_{\Sigma_2} = \Omega(z,\bar{z})\,dz\,d\bar{z}$ is

$$R_0 = \frac{4}{\Omega(z,\bar{z})^3}\left[\partial_z\Omega(z,\bar{z})\,\partial_{\bar{z}}\Omega(z,\bar{z}) - \Omega(z,\bar{z})\,\partial_z\partial_{\bar{z}}\Omega(z,\bar{z})\right]. \tag{3.112}$$

[10]This choice is a "non-chiral" way of solving the variational problem. There may be other "chiral" ways to do so that we do not consider.

Let us note the following identities:

$$\partial_{\bar{z}}\left[\frac{\partial_z^2\Omega(z,\bar{z})}{\Omega(z,\bar{z})}-\frac{3\,(\partial_z\Omega(z,\bar{z}))^2}{2\,(\Omega(z,\bar{z}))^2}\right]+\frac{1}{4}\Omega(z,\bar{z})\,\partial_z R_0=0,$$

$$\partial_z\left[\frac{\partial_{\bar{z}}^2\Omega(z,\bar{z})}{\Omega(z,\bar{z})}-\frac{3\,(\partial_{\bar{z}}\Omega(z,\bar{z}))^2}{2\,(\Omega(z,\bar{z}))^2}\right]+\frac{1}{4}\Omega(z,\bar{z})\,\partial_{\bar{z}} R_0=0\,. \qquad (3.113)$$

Thus to satisfy (3.111) we require that R_0 in (3.112) is a constant. We need to consider the cases of $R_0 = 0$ and $R_0 \neq 0$ separately.

- The case of $R_0 = 0$ means

  $$\partial_z\Omega(z,\bar{z})\,\partial_{\bar{z}}\Omega(z,\bar{z})=\Omega(z,\bar{z})\,\partial_z\partial_{\bar{z}}\Omega(z,\bar{z}). \qquad (3.114)$$

  Writing $\Omega(z,\bar{z}) = e^{\phi(z,\bar{z})}$ this equation is equivalent to $\partial_z\partial_{\bar{z}}\phi(z,\bar{z}) = 0$ which is immediately solved by taking $\phi(z,\bar{z}) = \varphi(z) + \varphi(\bar{z})$. Thus we write

  $$\Omega(z,\bar{z}) = f'(z)\bar{f}'(\bar{z}). \qquad (3.115)$$

- The case of $R_0 = r_0$ for non-zero constant r_0: Again writing $\Omega(z,\bar{z}) = e^{\phi(z,\bar{z})}$ this reads:

  $$\partial_z\partial_{\bar{z}}\phi(z,\bar{z}) + \frac{r_0}{4}e^{\phi(z,\bar{z})} = 0. \qquad (3.116)$$

This is the well-known Liouville equation on the complex plane. And a general solution is

$$\Omega(z,\bar{z}) = \frac{4\alpha^2 f'(z)\,\bar{f}'(\bar{z})}{(\beta + f(z)\,\bar{f}(\bar{z}))^2} \qquad (3.117)$$

for which the curvature is $r_0 = 2\beta/\alpha^2$. Whenever $r_0 > 0$ we set $\alpha = \beta = 1$ (which means we take $r_0 = 2$) and for $r_0 < 0$ we set $\alpha = -\beta = 1$ (which corresponds to $r_0 = -2$).

Let us also note that as a consequence of the consistent variational problem, we have $\delta R_0 = 0$. This still leaves two terms proportional to $g^{ab}_{(0)}\partial_u g^{(2)}_{ab}$ and $D^a g^{(2)}_{au}$ in (3.67). Since we impose $g^{(1,1)}_{ab} = 0$ we immediately have $\partial_u g^{(2)}_{ab} = 0$ and we just have to impose

$$D^a g^{(2)}_{au} = D^a D^b g^{(1,0)}_{ab} = 0. \qquad (3.118)$$

This condition restricts the choice of $C(z,\bar z)$ in (3.11).

- For $\Omega = f'(z)\bar f'(\bar z)$, this equation can be solved in generality by writing:

$$\kappa(z) = \frac{3f''(z)}{2\,f'(z)^2} - \frac{f'''(z)}{f'(z)}, \quad \bar\kappa(\bar z) = \frac{3\bar f''(\bar z)}{2\,\bar f'(\bar z)^2} - \frac{\bar f'''(\bar z)}{\bar f'(\bar z)},$$

$$C(z,\bar z) = f_0(z) + \bar f_0(\bar z) + f(z)\,\bar f_1(\bar z) + \bar f(\bar z)\,f_1(z). \tag{3.119}$$

- For $\Omega = (4f'(z)\bar f'(\bar z))/(1+f(z)\bar f(\bar z))^2$ we find:

$$\kappa(z) = \frac{3f''(z)}{2\,f'(z)^2} - \frac{f'''(z)}{f'(z)}, \quad \bar\kappa(\bar z) = \frac{3\bar f''(\bar z)}{2\,\bar f'(\bar z)^2} - \frac{\bar f'''(\bar z)}{\bar f'(\bar z)},$$

$$C(z,\bar z) = f_0(z) + \bar f_0(\bar z) + \frac{f(z)\,\bar f_1(\bar z) + \bar f(\bar z)\,f_1(z)}{1+f(z)\bar f(\bar z)}, \tag{3.120}$$

for arbitrary functions $(f_0(z), f_1(z), \bar f_0(\bar z), \bar f_1(\bar z))$. Thus in the conformal gauge, the locally flat solutions that satisfy the variational problem are specified by three holomorphic functions $(f(z), f_0(z), f_1(z))$ and three anti-holomorphic functions $\bar f(\bar z), \bar f_0(\bar z), \bar f_1(\bar z))$. We now turn to obtain the set of residual gauge transformations for this space of solution.

3.4.2 Residual gauge transformations

From (A.3), we write down,

$$\begin{aligned} \xi^u(u,z,\bar z) &= \xi^u_{(0)}(z,\bar z) + u\,\xi^u_{(1)}(z,\bar z), \quad \xi^a_{(0)} = \xi^a_{(0)}(z,\bar z), \\ \xi^r_{(0)} &= -\xi^u_{(1)}, \quad \xi^a_{(1)} = -g^{ab}_{(0)} D_b \xi^u, \quad \xi^a_{(2)} = \frac{1}{2} g^{ab}_{(0)} g^{(1)}_{bc}\, g^{cd}_{(0)} D_d \xi^u, \\ \xi^r_{(1)} &= \frac{1}{2} g^{ab} D_a\, D_b \xi^u. \end{aligned} \tag{3.121}$$

Now imposing $\delta g^{(0)}_{zz} = \delta g^{(0)}_{\bar z\bar z} = 0$ one obtains,

$$\xi^a_{(0)} = \{V^z(z), V^{\bar z}(\bar z)\}. \tag{3.122}$$

If we allow for Weyl rescaling of the boundary metric we have,

$$\delta g^{(0)}_{ab} = \omega\, g^{(0)}_{ab}\,, \quad \Longrightarrow \delta\Omega(z,\bar{z}) = \omega(z,\bar{z})\,\Omega(z,\bar{z}). \tag{3.123}$$

The $\mathcal{O}(r^2)$ terms in $\delta g_{z\bar{z}}$ gives,

$$\xi^u_{(1)} = \frac{1}{2} D_a V^a - \frac{1}{2}\,\omega(z,\bar{z}) \tag{3.124}$$

where the covariant derivation D_a is associated with the boundary metric in conformal gauge $ds^2_{\Sigma_2} = \Omega(z,\bar{z})\,dz\,d\bar{z}$. The resultant vector fields are parameterised by the following functions

$$\{V^z(z), V^{\bar{z}}(\bar{z}), \omega(z,\bar{z}), \xi^u_{(0)}(z,\bar{z})\}. \tag{3.125}$$

These vector fields constitute the vector fields of extended $\mathfrak{bms}_4$ algebra along with abelian Weyl scaling ω in [111].[11] So far we haven't imposed any conditions coming from the variational problem on our vector field. As we will see imposing these conditions will kill some of the extended $\mathfrak{bms}_4$ symmetries. We make a choice here to work with $\Omega = f'(z)\bar{f}'(\bar{z})$ and for the other choice of Ω in (3.120), the calculation is straightforward and for both choices of Ω the transformation of the fields are identical just like in Polyakov gauge.

Now the first condition that we impose is that the vector field should obey $\delta g^{(1,1)}_{ab} = 0$. From (3.81), this give us the following constraint equation on $\omega(z,\bar{z})$,

$$L_{ab} := g^{(0)}_{ab}\, g^{cd}_{(0)} D_c D_d\, \xi^u_{(1)} - 2\, D_a D_b\, \xi^u_{(1)} = 0 \tag{3.126}$$

where $\xi^u_{(1)}$ is given in terms of ω as in (3.124). The zz and $\bar{z}\bar{z}$ component of L_{ab} can be solved simultaneously for the following solution of ω,

$$\begin{aligned}\omega(z,\bar{z}) = \frac{V^z(z)\,f''(z)}{f'(z)} + \partial_z V^z(z) + \frac{V^{\bar{z}}(\bar{z})\,\bar{f}''(\bar{z})}{\bar{f}'(\bar{z})} + \partial_{\bar{z}} V^{\bar{z}}(\bar{z})\\ + a_1 + a_2\, f(z) + a_3\, \bar{f}(\bar{z}) + a_4\, f(z)\,\bar{f}(\bar{z})\end{aligned} \tag{3.127}$$

When one fixes the conformal factor $\Omega = 1$, this amounts to setting $f(z) = z\,, \bar{f}(\bar{z}) = \bar{z}$ and therefore if we do not set the constant parameters $\{a_1, a_2, a_3, a_4\}$ to zero, we will get, from (3.123), non-zero variation for a fixed conformal factor which is contradictory. Hence we set these constants to zero and therefore for $\Omega = f'(z)\bar{f}'(\bar{z})$,

[11] See also the derivation of this vector field from the Weyl BMS group in [48].

we finally get,

$$\omega(z,\bar{z}) = \frac{V^z(z)\, f''(z)}{f'(z)} + \partial_z V^z(z) + \frac{V^{\bar{z}}(\bar{z})\, \bar{f}''(\bar{z})}{\bar{f}'(\bar{z})} + \partial_{\bar{z}} V^{\bar{z}}(\bar{z}). \tag{3.128}$$

From (3.128) and variation in (3.123) one can deduce the transformation of $f(z)$ and $\bar{f}(\bar{z})$, which are

$$\delta f(z) = V^z\, \partial_z f(z)\,, \quad \delta \bar{f}(\bar{z}) = V^{\bar{z}} \partial_{\bar{z}}\, \bar{f}(\bar{z}). \tag{3.129}$$

From the u-independent part of (3.81), we obtain the transformation of $C(z,\bar{z})$ as

$$\delta C(z,\bar{z}) = V^z\, \partial_z C(z,\bar{z}) + V^{\bar{z}} \partial_{\bar{z}} C(z,\bar{z}) + \xi^u_{(0)}(z,\bar{z}). \tag{3.130}$$

For the form of $C(z,\bar{z})$ in (3.119), using (3.130) one can deduce $\xi^u_{(0)}$ that will lead to the correct variation of $C(z,\bar{z})$,

$$\xi^u_{(0)} = \mathfrak{h}_0(z) + \bar{\mathfrak{h}}_0(\bar{z}) + f(z)\, \bar{\mathfrak{h}}_1(\bar{z}) + \bar{f}(\bar{z})\, \mathfrak{h}_1(z). \tag{3.131}$$

The resultant transformation of the fields in (3.119) are,

$$\delta \bar{f}_0(\bar{z}) = V^{\bar{z}}(\bar{z})\, \partial_{\bar{z}}\, \bar{f}_0(\bar{z}) + \bar{\mathfrak{h}}_0(\bar{z})\,, \quad \delta \bar{f}_1(\bar{z}) = V^{\bar{z}}(\bar{z})\, \partial_{\bar{z}}\, \bar{f}_1(\bar{z}) + \bar{\mathfrak{h}}_1(\bar{z}), \tag{3.132}$$

$$\delta f_0(z) = V^z(z)\, \partial_z\, f_0(z) + \mathfrak{h}_0(z)\,, \quad \delta f_1(z) = V^z(z)\, \partial_z\, f_1(z) + \mathfrak{h}_1(\bar{z}). \tag{3.133}$$

Therefore the complete set of variations of the background fields induced by the residual diffeomorphisms are given in (3.129), (3.132) and (3.133). The final vector field is written as follows,

$$\begin{aligned} \xi = \big(\mathfrak{h}_0(z) + \bar{\mathfrak{h}}_0(\bar{z}) + f(z)\, \bar{\mathfrak{h}}_1(\bar{z}) + \bar{f}(\bar{z})\, \mathfrak{h}_1(z)\big)\, \partial_u + V^z(z)\, \partial_z + V^{\bar{z}}(\bar{z}) \partial_{\bar{z}} \\ + \frac{1}{r} \left(\frac{2\, \partial_{\bar{z}} \bar{\mathfrak{h}}_1(\bar{z})}{\bar{f}'(\bar{z})} + \frac{2\, \partial_z \mathfrak{h}_1(z)}{f'(z)} \right) \partial_r + \cdots . \end{aligned} \tag{3.134}$$

The $\cdots$ in (3.134) denote subleading terms in $\mathcal{O}(1/r)$. It is interesting to note that because of the imposition of variational problem, the $\mathcal{O}(r)$ component in the vector field along the radial direction r and the linear in u component of ξ^u has vanished. Just like the solution space in conformal gauge, the vector fields are also parameterised by 3 holomorphic and 3 anti-holomorphic functions. As the boundary vector fields in (3.134) have background dependence $f(z)$, $\bar{f}(\bar{z})$, one uses the Courant bracket defined in (2.14) to obtain the commutation relations between them. These vector fields close under this bracket and form a Lie algebra

$$[\xi_1,\xi_2] = \Big(\hat{\mathfrak{h}}_0(z) + \hat{\bar{\mathfrak{h}}}_0(\bar{z}) + f(z)\,\hat{\bar{\mathfrak{h}}}_1(\bar{z}) + \bar{f}(\bar{z})\,\hat{\mathfrak{h}}_1(z)\Big)\,\partial_u + \hat{V}^z(z)\,\partial_z + \hat{V}^{\bar{z}}(\bar{z})\partial_{\bar{z}} + \cdots\,, \tag{3.135}$$

where $\hat{\mathfrak{h}}_0(z), \hat{\bar{\mathfrak{h}}}_0(\bar{z}), \hat{\bar{\mathfrak{h}}}_1(\bar{z}), \hat{\mathfrak{h}}_1(z), \hat{V}^z(z), \hat{V}^{\bar{z}}(\bar{z})$ are,

$$\hat{V}^z(z) = V_1^z\,\partial_z V_2^z - V_2^z\,\partial_z V_1^z\,, \quad \hat{V}^{\bar{z}}(\bar{z}) = V_1^{\bar{z}}\,\partial_{\bar{z}} V_2^{\bar{z}} - V_2^{\bar{z}}\,\partial_{\bar{z}} V_1^{\bar{z}}, \tag{3.136}$$

$$\hat{\mathfrak{h}}_0(z) = V_1^z\,\partial_z\,\mathfrak{h}_0^2 - V_2^z\,\partial_z\,\mathfrak{h}_0^1\,, \quad \hat{\mathfrak{h}}_1(z) = V_1^z\,\partial_z\,\mathfrak{h}_1^2 - V_2^z\,\partial_z\,\mathfrak{h}_1^1, \tag{3.137}$$

$$\hat{\bar{\mathfrak{h}}}_0(\bar{z}) = V_1^{\bar{z}}\,\partial_{\bar{z}}\,\bar{\mathfrak{h}}_0^2 - V_2^{\bar{z}}\,\partial_z\,\bar{\mathfrak{h}}_0^1\,, \quad \hat{\bar{\mathfrak{h}}}_1(z) = V_1^{\bar{z}}\,\partial_z\,\bar{\mathfrak{h}}_1^2 - V_2^{\bar{z}}\,\partial_z\,\bar{\mathfrak{h}}_1^1. \tag{3.138}$$

Let us define the following doublets,

$$\begin{aligned}\mathfrak{h}_\gamma &= \{\mathfrak{h}_0(z), \mathfrak{h}_1(z)\}\,, \gamma \in \{0,1\}\\ \bar{\mathfrak{h}}_\gamma &= \{\bar{\mathfrak{h}}_0(\bar{z}), \bar{\mathfrak{h}}_1(\bar{z})\}\,, \gamma \in \{0,1\}\end{aligned} \tag{3.139}$$

- We denote the vector field with $V^{\bar{z}} = \mathfrak{h}_\gamma = \bar{\mathfrak{h}}_\gamma = 0$ and $V^z = -z^{n+1}$ as L_n and $V^z = \mathfrak{h}_\gamma = \bar{\mathfrak{h}}_\gamma = 0$ and $V^{\bar{z}} = -\bar{z}^{n+1}$ as $\bar{L}_n$ for $n \in \mathbb{Z}$ then the commutator of the vector fields are,

$$[L_n\,, L_m] = (n-m)\,L_{m+n}\,, \quad \big[\bar{L}_n\,, \bar{L}_m\big] = (n-m)\,\bar{L}_{m+n} \tag{3.140}$$

  These are two copies of Witt algebra.

- If we denote the vector field $V^{\bar{z}} = V^z = \bar{\mathfrak{h}}_\gamma = 0$ and $\mathfrak{h}_\gamma = -z^n$ as $\mathfrak{h}_{\gamma,n}$ and $V^{\bar{z}} = V^z = \mathfrak{h}_\gamma = 0$ and $\bar{\mathfrak{h}}_\gamma(z) = -\bar{z}^n$ as $\bar{\mathfrak{h}}_{\gamma,n}$ for $n \in \mathbb{Z}$ then the commutator of vector fields are,

$$[\mathfrak{h}_{\gamma,n}, \mathfrak{h}_{\beta,m}] = 0\,, \quad \big[\bar{\mathfrak{h}}_{\gamma,n}, \bar{\mathfrak{h}}_{\beta,m}\big] = 0\,, \big[\bar{\mathfrak{h}}_{\gamma,n}, \mathfrak{h}_{\beta,m}\big] = 0\,, \tag{3.141}$$

$$[L_n\,, \mathfrak{h}_{\gamma,m}] = -m\,\mathfrak{h}_{\gamma,m+n}\,, \quad \big[\bar{L}_n\,, \bar{\mathfrak{h}}_{\gamma,m}\big] = -m\,\bar{\mathfrak{h}}_{\gamma,m+n} \tag{3.142}$$

$$\big[L_n\,, \bar{\mathfrak{h}}_{\gamma,m}\big] = 0\,, \quad \big[\bar{L}_n\,, \mathfrak{h}_{\gamma,m}\big] = 0 \tag{3.143}$$

The asymptotic symmetry algebra for the locally flat solutions in conformal gauge consistent with the variational principle is given by (3.140), (3.141), (3.142), (3.143). This symmetry algebra factorises into a holomorphic and anti-holomorphic sector with each sector having a copy of Witt algebra and two abelian current algebras.

3.5 Further Comments and Discussions

Our variational problem is defined for $4d$ Einstein gravity with a specific set of boundary terms we proposed in the text. We constructed boundary action which is invariant under boundary-preserving diffeomorphisms in the NU gauge and is adapted for the null infinity in the limit $r \to \infty$. The boundary preserving diffeos are not three-dimensional unlike the AdS_4 case in the Fefferman-Graham gauge but a smaller set of symmetries. It will be interesting to compare our boundary action with those of [140] where the boundary action is constructed directly on the null boundary of the spacetime. To ensure $\delta S = 0$ for the allowed classical solutions we have chosen to impose additional boundary conditions on the configurations that already solve the bulk equations of motion.[12] We would like to emphasise that even in the case of Dirichlet boundary conditions (in which the metric on Σ_2 is held fixed) there are additional conditions imposed by our variational principle and solving them in a non-chiral fashion requires us to set $g^{(1,1)}_{zz} = g^{(1,1)}_{\bar{z}\bar{z}} = 0$. Even though we did not provide the details it can be seen easily that the residual large diffeomorphisms of such Dirichlet class of solutions do not permit extension of the Lorentz algebra part of the $\mathfrak{bms}_4$ into two copies of Witt algebra. Thus our boundary conditions would not allow the extended $\mathfrak{bms}_4$ as the asymptotic symmetry algebra in the Dirichlet case. It will be important to explore the consequences of this fact further.

Starting from our solution space we can generate three more classes of geometries by $T : u \leftrightarrow -v$ and/or $P : z \leftrightarrow -1/\bar{z}$ – which are time-reversal and spatial parity transformations respectively in the asymptotic flat spacetime (with $\Sigma_2 = S^2$). We suggest that one should think of these four classes of solutions as valid in four different coordinate patches of the fully extended (locally flat) geometries. Then these four patches are related to each other by the (P, T) transformations. One can, in principle, patch together these solutions that are (C) PT invariant. Just as there are four classes of solutions there are four corresponding chiral algebras. This is similar to the two copies of $\mathfrak{bms}_4$ symmetry algebras in the context of Dirichlet boundary conditions – one at the past null infinity and the other at the future null infinity. Just as an appropriate combination was found by Strominger (see [5] for a review) picked by the CPT invariance of the scattering amplitudes we expect that an appropriate combination of the four copies of our current algebras to emerge as the correct symmetry of the relevant scattering amplitudes.

[12] It is known that a consistent variational problem at null-infinities would not allow solutions representing gravitational radiation at infinity. Our boundary conditions, however, are weak enough to at least allow for a Schwarzchild black hole in the solution space.

The locally flat (LF) solutions (3.72) and (3.73) that we worked with after imposing chiral boundary conditions are complex in the $\mathbb{R}^{1,3}$ gravity (they are real however in the $\mathbb{R}^{2,2}$ gravity as $(z, \bar{z})$ will be light-cone coordinates (x^+, x^-)).[13] As a consequence the vector fields generating the asymptotic symmetries of this class of geometries are also complex. Since for LF solutions, the Riemann tensor vanishes, in principle these solutions can all be obtained (in open patches) by the finite complex coordinate transformation of metric $\eta_{\mu\nu}$ of the Minkowski spacetime $\mathbb{R}^{1,3}$. Expanding the LF solutions around $\eta_{\mu\nu}$ to the first order in the six holomorphic functions the perturbation $h_{\mu\nu}$ can be written as pure diffeomorphisms with the same complex vector fields that generate our asymptotic symmetries. In [59] it was shown that the conformal basis for graviton wave-functions $h^{\Delta,\pm}_{\mu\nu;a}$ of definite helicity (where $\pm$ superscript denotes whether the graviton is incoming/out-going and the subscript a denotes its helicity) in the soft limits $\Delta \to 1$, $\Delta \to 0$ become pure diffeomorphisms: $h^{\Delta,\pm}_{\mu\nu;a} = \nabla_\mu \zeta^{\Delta,\pm}_{\nu;a} + \nabla_\nu \zeta^{\Delta,\pm}_{\mu;a}$.[14] Near null infinity these vector fields $\zeta^{\Delta}_{\mu;\pm}$ in the soft limit become generators of supertranslation symmetry and Diff($\mathbb{S}^2$) symmetry and for positive helicity soft graviton take the form of (3.85), (3.87) respectively. The conformal primary wave functions given in terms of these vector fields are then interpreted as Goldstone modes of spontaneously broken asymptotic symmetries of gravity theory.[15] Similarly one can interpret our complex solutions as the 'condensates' of soft gravitons of positive (negative) helicity if the solutions are characterised by six holomorphic (anti-holomorphic) functions.

In this chapter, we used the definition of variational principle in the conventional sense that implies stationarity of the action $S_{EH+bdry.}$ on the solutions. As argued in [97], one modifies the definition of a well-defined variational principle in the presence of radiation to allow for some presymplectic flux through the boundary. However, we believe that even if we consider radiative solutions, then in the asymptotic regions unaffected by the radiation (for instance, if a pulse of radiation reaches the boundary during some finite interval in u, then sufficiently far away from this interval), the solutions should approach one of the vacuum solutions. And these vacuum solutions, we posit, have to be one of the classes of locally flat solutions for which the total action is stationary. It will be interesting to see whether our symmetry analysis still holds in the case of presymplectic flux, through the boundary, instead of the

[13]Gravity scattering amplitudes in such signature were explored recently by [141].

[14]Even though the analysis in [59] was done in harmonic gauge, the boundary vector fields that generate generalised BMS transformations at leading order are the same as that of Bondi gauge and hence NU gauge.

[15]Our solution spaces contain one additional holomorphic function $\kappa(z)$ associated with the vector field in (3.89) which is possibly the Goldstone mode of addition positive soft helicity mode (related to the shadow transform of negative helicity graviton) with $\tilde{\Delta} = 2$ in section 5.3.1 of [59]

stationarity condition of the action.

A bigger symmetry algebra ($w_{1+\infty}$) than the one we uncovered in this chapter is observed from the conformal soft limits of graviton operators in the celestial CFT [63, 66] recently. It will be interesting to explore whether our boundary conditions can be generalised to incorporate these extended symmetries or not.[16] Following the analysis of [142, 143], one expects that the vector fields generating $w_{1+\infty}$ are superleading in r as one approaches the boundary of spacetime. However, it can be checked from eq. (A.2) that such behaviour for vector fields in the NU gauge does not exist. Therefore, to extend the analysis of algebra in our present work to $w_{1+\infty}$ algebra, one may have to work in a different gauge. In [59], the authors derived the vector fields that generate arbitrary diffeomorphisms of the celestial sphere in harmonic gauge. One can do a similar analysis as theirs by demanding less restrictive boundary conditions that allow for metric perturbation around Minkowski spacetime to fall off as $\mathcal{O}(r^n)$ for $n \geq 2$ and obtain vector fields that are superleading in r near null infinity. It is plausible that imposing a variational problem for such configuration might further restrict the solution space, the residual gauge transformations of which could map to $w_{1+\infty}$. However, the interpretation of solutions with such superleading behaviour in r near the null infinity from the spacetime perspective is unclear as they do not obey the standard asymptotically flat boundary conditions.

We considered the metric on Σ_2 in the Polyakov gauge. The other natural choice is the conformal gauge in which we have found a class of locally flat solutions. The asymptotic symmetry algebra completely factorises into a holomorphic and anti-holomorphic sector. It is different from extended $\mathfrak{bms}_4$, as the supertranslation current reduces to two holomorphic and two anti-holomorphic currents with conformal weight $h = 0$ and $\bar{h} = 0$ under the left and right Virasaro transformations respectively. The charges corresponding to this symmetry algebra and the significance of such an algebra in the context of soft theorems are still under investigation.

The chiral gravity boundary conditions of [10] that gave rise to Polyakov's chiral $\mathfrak{sl}(2, \mathbb{R})$ current algebra from the AdS_3 gravity was generalised to the supersymmetric context in [144]. It should be possible to generalise the boundary conditions used in this chapter to $4d$ gravitational theories with supersymmetry to find supersymmetric extensions of the chiral $\mathfrak{bms}_4$ algebra found here. Very recently such supersymmetric extensions have been observed in the celestial amplitudes [145].

We have obtained our chiral algebra from the future null infinity. The generalised

[16] See [67] for the derivation of charges using vacuum Einstein equations that form the canonical representation of these $w_{1+\infty}$ algebra.

$\mathfrak{bms}_4$ has been shown to emerge from the asymptotic symmetry analysis around the time-like infinity as well [146]. It will be interesting to analyse our problem from the point of view of the time-like infinity.

Chapter 4

Constructing Carrollian CFTs

In this chapter, we turn our attention to another proposed dual theory to asymptotically flat spacetimes in $\mathbb{R}^{1,3}$ gravity, which is a $3d$ conformal field theory defined on null infinity, a quintessential example of Carrollian manifold. In particular, we will consider the construction of such theories for scalar fields on a general Carroll manifold. The Poincare algebra admits two interesting limiting algebras obtained by Inonu-Wigner contraction where one takes the speed of light $c \in [0, \infty)$ to either zero or infinity. When $c \to \infty$ one obtains the well-known Galilean algebra, and when $c \to 0$ one ends up with the so-called Carrollian algebra [74, 73]. The former case has played a very important role as it is relevant for a host of physical systems in which typical velocities involved are very small compared to the speed of light, such as Newtonian mechanics, many condensed matter systems etc. The Carroll limit is relevant if the velocities involved are all very close to c and it remained unexplored for a long time until the last decade.

An important realisation of the Poincare algebra arises from the isometries of the flat Minkowski spacetime with the standard line element:

$$ds^2 = -c^2\, dt^2 + d\mathbf{x} \cdot d\mathbf{x}$$

which is an example of a pseudo-Riemannian manifold, with non-degenerate Lorentzian metric $g_{\mu\nu} = \eta_{\mu\nu}$. If one takes either of the limits $c \to \infty$ (better done on $g^{\mu\nu}$) or $c \to 0$ (done on $g_{\mu\nu}$) one obtains a manifold with a degenerate metric tensor. The geometries that realise Galilean or Carrollian algebras are therefore not Riemannian manifolds but belong to a more general class of geometrical objects called Newton-Cartan manifolds (for $c \to \infty$ case) and Carroll manifolds (for $c \to 0$ case) respectively (see, for instance, [70, 147, 148]).

Just as the conformal algebra can be represented by conformal Killing vectors (CKV) of (conformally) flat spacetimes, such as $\mathbb{R} \times S^d$ with line element

$$ds^2 = -c^2\, dt^2 + d\Omega_d^2$$

or $\mathbb{R}^{1,d}$, the Galilean and Carrollian conformal algebras can be thought of as appropriately defined conformal algebras of (conformally) flat Galilei and Carroll spacetimes. The conformal algebra of (any spacetime conformal to) Minkowski spacetime M_{d+1} is finite-dimensional for $d \geq 2$ and therefore obtaining the corresponding Galilean or Carrollian conformal algebras via Inonu-Wigner contraction will necessarily result in finite dimensional algebras for $d \geq 2$. However, if one defines these algebras directly as conformal algebras of the corresponding Galilei and Carroll spacetimes one may get bigger algebras than those obtained by the contraction procedure [72]. There is a more general notion of conformal symmetry even in the flat Carroll spacetime C_{d+1}, parameterised by z, the analogue of the dynamical exponent of Galilean conformal transformations, which in turn is given by a non-negative integer k by $z = 2/k$ in [68, 69]. One obtains only the $z = 1$ ($k = 2$) case via the contraction procedure.

Also as was shown in [69] the Carrollian conformal algebra $\mathfrak{cca}_{d+1}^{(z)}$ for $d = 2$ and $z = 1$ is the $\mathfrak{bms}_4$ algebra [3, 4], which is infinite dimensional. So if one wants to describe some $(d+2)$-dimensional gravitational theory with asymptotically flat boundary conditions holographically then the holograms should be field theories on $\mathcal{I}^{\pm}$ with $\mathfrak{cca}_{d+1}^{(z=1)}$ as their global symmetries [11, 29]. This expectation has led to a lot of work seeking field theories on null-manifolds which are Carrollian CFTs (see for example [18, 17, 72]). The procedure followed there is typically to start with a CFT on $\mathcal{M}_{d+1}$ and take an ultra-relativistic limit. This has been a fruitful exercise and has resulted in quite a lot of interesting examples with $\mathfrak{cca}_{d+1}^{(z=1)}$ symmetries.

However, it is plausible that constructing field theories directly on a Carroll manifold $\mathcal{C}_{d+1}$ gives rise to a more general class of theories than those obtained by taking an ultra-relativistic limit of known CFTs on the parent pseudo-Riemannian manifold $\mathcal{M}_{d+1}$ [19]. Such a construction was known in the context of Carollian Fluid dynamics [75, 76]. The main aim of this chapter is to demonstrate that this expectation is indeed realised. We will construct classical scalar field theories on generic 3-dimensional Carroll manifolds that are invariant under both diffeomorphisms and Weyl transformations of the Carroll manifold. These theories can be put on any given Carroll manifold – of which the interesting examples include null boundaries of asymptotically flat spacetimes, horizons of black holes, boundaries of causal developments, etc [71, 149]. Then the residual symmetries should, by construction, make the resultant theory have $\mathfrak{cca}_3^{(z)}$ as the symmetry algebra for generic values of z. We will mainly concentrate on $d = 2$ for the most part (relegating the higher dimensional case to Appendix C). We obtain a larger class of theories for scalar fields going beyond what one obtains by the method of taking ultra-relativistic limit.

The results of this chapter include,

- In Section (4.1), we briefly review how one constructs a diffeomorphic and Weyl invariant theory of a scalar field in the background of a generic (pseudo) Riemann manifold.
- In Section (4.2) we repeat the steps of Section (4.1) to the case of scalar fields $\Phi(t, \mathbf{x})$ in the background of a Carroll manifold. Specifically, we construct two classes of diffeomorphic and Weyl covariant equations of motion, one with two-time derivatives of the field (time-like), and the other with (up to) two space derivatives (space-like). These exist for general values of z and conformal dimension δ of $\Phi(t, \mathbf{x})$. We show that the invariant actions exist only when $\delta = \frac{z}{2}$ for space-like action and $\delta = 1 - \frac{z}{2}$ for time-like action with both types of actions combining for $z = 1$.
- In Section (4.3) we show how a subset of results of Section (4.2) can be recovered starting from when one expands the equation of motion and action of a conformally coupled scalar field in pseudo-Riemannian geometry as a polynomial in c, the speed of light, using the Randers-Pappetrou parametrization of $3d$ metric [19]. We show that the coefficient of order $\mathcal{O}\left(\frac{1}{c^2}\right)$ corresponds to the time-like equation/action and the coefficient $\mathcal{O}(1)$ corresponds to the space-like equation/action mentioned in the second point for $z = 1$ and $\delta = \frac{1}{2}$ case with appropriate identification of parameters.
- In Section (4.4) we gauge fix our theories to recover and generalise the classical scalar field theories with $\mathfrak{cca}_3^{(z)}$ algebra worth of symmetries. We also show that alternative ways of gauge fixing lead to field theories with two distinct symmetry algebras isomorphic to $\mathbb{R} \oplus \mathcal{A}$ and generalised $\mathfrak{bms}_4$ algebra for $z = 1$, where $\mathcal{A}$ is the algebra of volume-preserving diffeomorphisms on the sphere.

We conclude with some comments and discussions in Section (4.5). The Appendix C contains some details of the case of arbitrary dimensions. This chapter is based on the [150].

4.1 Conformally coupled scalar field - revisited

In this section, we (re) construct the well-known conformally coupled scalar field equation of motion and its action in detail. This will provide us with the method we follow in the later sections. The result we want to re-derive is the diffeomorphic and Weyl invariant classical (free) scalar field theory in $d = 2+1$ dimensions in a general background with metric $g_{\mu\nu}$. This has the action:

$$\mathcal{S} = \int d^{2+1}x \left[-\frac{1}{2}\sqrt{-g} \left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \frac{1}{8}R\,\phi^2 \right) \right] \tag{4.1}$$

and the equation of motion

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{1}{8}R\,\phi = 0\,. \tag{4.2}$$

This action (4.1) is invariant and the equation of motion (4.2) is covariant under the Weyl transformations:

$$g'_{\mu\nu}(x) = \frac{1}{B^2}g_{\mu\nu}(x), \quad \phi'(x) = B^{\frac{1}{2}}\phi(x) \tag{4.3}$$

where B is an arbitrary function of the coordinates, and diffeomorphisms:

$$g'_{\mu\nu}(x') = \frac{dx^\alpha}{dx'^\mu}\frac{dx^\beta}{dx'^\nu}g_{\alpha\beta}(x), \quad \phi'(x') = \phi(x) \tag{4.4}$$

where $x \to x'^\mu(x)$ is a coordinate transformation. Let us rederive this result (see Wald [151] for instance). For this one starts with scalar (under diffeomorphisms in (4.4)) combinations that are linear in Φ and have (at most) two derivatives, namely, $R\,\Phi$ and $\nabla_\mu\nabla^\mu\Phi$. Their transformation properties under the Weyl transformations $g_{\mu\nu} \to B^{-2}\,g_{\mu\nu}$ and $\Phi \to B^\delta\,\Phi$, are

$$\begin{aligned} R\,\Phi &\longrightarrow B^\delta \left[B^2\,R + 4\,B\,g^{\mu\nu}\nabla_\mu\nabla_\nu B - 6\,g^{\mu\nu}\nabla_\mu B\,\nabla_\nu B \right] \Phi\,, \\ g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi &\longrightarrow B^\delta \Big[B^2\,g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi + \delta\,\Phi\,B\,g^{\mu\nu}\nabla_\mu\nabla_\nu B + \delta\,(\delta-2)\,\Phi\,g^{\mu\nu}\nabla_\mu B\,\nabla_\nu B \\ &\quad + (2\delta-1)\,B\,g^{\mu\nu}\nabla_\mu B\,\nabla_\nu\Phi \Big] \end{aligned} \tag{4.5}$$

where δ is the Weyl weight of the scalar Φ. For a linear combination of $R\,\Phi$ and $\Box\Phi$ to be covariant all the inhomogeneous terms in the Weyl transformation of that combination should cancel out. But in no linear combination the term containing $B\,g^{\mu\nu}\nabla_\mu B\,\nabla_\nu\Phi$ on the right hand side of $\Box\Phi$ in (4.5) gets canceled. So its coefficient

$(2\delta - 1)$ has to vanish identically, giving us $\delta = \frac{1}{2}$. Then the only linear combination that transforms homogeneously is

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi - \frac{1}{8}R\,\Phi \to B^{\frac{5}{2}}\left[g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi - \frac{1}{8}R\,\Phi\right] \tag{4.6}$$

showing that the equation (4.2) is the only covariant one. For the construction of an action one notes:

$$g^{\mu\nu}\nabla_\mu\Phi\,\nabla_\nu\Phi \longrightarrow \quad B\left[B^2\,g^{\mu\nu}\nabla_\mu\Phi\,\nabla_\nu\Phi + \frac{1}{4}\Phi^2\,g^{\mu\nu}\nabla_\mu B\,\nabla_\nu B + \Phi\,B\,g^{\mu\nu}\nabla_\mu\Phi\,\nabla_\nu B\right] \tag{4.7}$$

which can be used along with the first line of (4.5) to show that

$$\begin{aligned}\sqrt{g}\left[g^{\mu\nu}\nabla_\mu\Phi\,\nabla_\nu\Phi + \frac{1}{8}R\,\Phi^2\right] \longrightarrow & \sqrt{g}\left[g^{\mu\nu}\nabla_\mu\Phi\,\nabla_\nu\Phi + \frac{1}{8}R\,\Phi^2\right] \\ & + \partial_\mu\left[\frac{\Phi^2}{2B}\sqrt{g}\,g^{\mu\nu}\partial_\nu B\right].\end{aligned} \tag{4.8}$$

So the action (4.1) is also invariant under (4.3, 4.4) and results in the equation of motion (4.2). In the next section, we use the same procedure to construct analogous equations of motion and actions for scalar fields on 3-dimensional Carroll manifolds.

4.2 Scalar field theories on Carroll Spacetimes

Let us first review some essential aspects of Carrollian geometries. We will follow the notations and conventions of [27], [76] here. A Carroll spacetime is a fibre bundle $\mathcal{C}_{d+1}$ with a d-dimensional base $\mathcal{S}$ and one-dimensional fibre. We work with local coordinates $\mathbf{x}$ on the base and t on the fibre. Then the Carroll spacetime is specified by a non-degenerate d-dimensional metric $a_{ij}(t,\mathbf{x})$ on the base $\mathcal{S}$, the Ehresmann connection 1-form $b_i(t,\mathbf{x})$ and a scalar $\omega(t,\mathbf{x})$.[1] Then one defines the Carroll diffeomorphisms as those that keep this structure invariant. In our coordinates, they take the form:

$$t \to t'(t,\mathbf{x}), \quad \mathbf{x} \to \mathbf{x}'(\mathbf{x}). \tag{4.9}$$

[1]There is another formulation of Carroll geometry in terms of $3d$ degenerate metric $h_{\mu\nu}$ and a vector field ζ^μ defined everywhere [68, 70, 69]. Refer to the works [84, 82, 152, 85] which use this definition to construct Carrollian conformal field theories. Also, see the work [78] for a detailed comparison between the two formulations.

The Jacobian of these Carroll diffeomorphisms $(t, x^i) \to (t'(t, \mathbf{x}), x'^i(\mathbf{x}))$ is the matrix

$$\begin{pmatrix} J(t,\mathbf{x}) & J_i(t,\mathbf{x}) \\ 0 & J^j{}_i \end{pmatrix} \tag{4.10}$$

where $J = \frac{\partial t'}{\partial t}$, $J_i = \frac{\partial t'}{\partial x^i}$, and $J^j{}_i = \frac{\partial x'^j}{\partial x^i}$, with its inverse

$$\begin{pmatrix} J^{-1} & -J^{-1} J_k (J^{-1})^k{}_i \\ 0 & (J^{-1})^j{}_i \end{pmatrix} \tag{4.11}$$

where $(J^{-1})^i{}_j$ is the inverse of the matrix $J^i{}_j$. Under these transformations the geometrical data (a_{ij}, b_i, ω) of the Carroll spacetime transforms as:

$$a'_{ij}(t', \mathbf{x}') = a_{kl}(t, \mathbf{x})\, (J^{-1})^k{}_i (J^{-1})^l{}_j, \quad \omega'(t', \mathbf{x}') = J^{-1}\, \omega(t, \mathbf{x})$$

$$b'_k(t', \mathbf{x}') = \Big(b_i(t, \mathbf{x}) + J^{-1}\, J_i\, \omega(t, \mathbf{x})\Big)\, (J^{-1})^i{}_k \tag{4.12}$$

along with $\partial'_t = J^{-1}\partial_t$, $\partial'_j = (J^{-1})^i{}_j(\partial_i - J^{-1}\, J_i\, \partial_t)$. Now one can list the objects that are covariant under the Carroll diffeomorphisms. At the first derivative order, one has

$$\phi_i = \frac{1}{\omega}(\partial_i \omega + \partial_t b_i), \quad \hat{\gamma}^i_j = \frac{1}{2\omega} a^{ik} \partial_t a_{jk}, \quad \theta = \frac{1}{\omega}\partial_t \ln \sqrt{a} = \hat{\gamma}^i_i, \quad f_{ij} = 2\left(\partial_{[i} b_{j]} + b_{[i}\, \phi_{j]}\right). \tag{4.13}$$

Because these are covariant one can raise and lower the indices using a_{ij} and its inverse a^{ij}. One also has the following differential operators

$$\hat{\partial}_t = \frac{1}{\omega}\partial_t, \quad \hat{\partial}_i = \partial_i + \frac{b_i}{\omega}\, \partial_t \tag{4.14}$$

that are covariant. Then the Carroll-Christoffel connection

$$\hat{\gamma}^i_{jk} = \frac{1}{2} a^{il} (\hat{\partial}_j a_{lk} + \hat{\partial}_k a_{jl} - \hat{\partial}_l a_{jk}) \tag{4.15}$$

allows one to write down further sets of covariant objects. This connection transforms under Carroll diffeomorphisms in the same manner as the usual Christoffel connection in Riemannian geometry. We will define the Carroll tensors to transform as:

$$\Phi'(t',\mathbf{x}') = \Phi(t,x), \quad V'^i(t',\mathbf{x}') = J^i{}_j\, V^j(t,x), \quad V'_i(t',\mathbf{x}') = V_j(t,x)\,(J^{-1})^j{}_i, \quad etc. \tag{4.16}$$

Under the Carroll connection (4.15) a_{ij} and a^{ij} are covariantly constant. Another fact is that if one defines $\hat{\nabla}_t a_{ij} := \hat{\partial}_t a_{ij} - \hat{\gamma}_i^k a_{kj} - \hat{\gamma}_j^k a_{ik}$ and $\hat{\nabla}_t a^{ij} := \hat{\partial}_t a^{ij} + \hat{\gamma}_k^i a^{kj} + \hat{\gamma}_k^j a^{ik}$ then the metric a_{ij} and its inverse are covariantly constants under $\hat{\nabla}_t$ as well.

At the second derivative order, one can list the following invariant objects:

$$\theta^2, \quad \hat{\partial}_t\theta = \frac{1}{\omega}\partial_t\theta, \quad \hat{\gamma}_j^i\hat{\gamma}_i^j, \quad \hat{r}, \quad a^{ij}\hat{\nabla}_i\phi_j, \quad a^{ij}\phi_i\phi_j \tag{4.17}$$

where $\hat{r}$ is the Carroll Ricci scalar defined [76] as follows:

$$\hat{r}^i_{jkl} = \hat{\partial}_k\hat{\gamma}^i_{lj} - \hat{\partial}_l\hat{\gamma}^i_{kj} + \hat{\gamma}^i_{km}\hat{\gamma}^m_{lj} - \hat{\gamma}^i_{lm}\hat{\gamma}^m_{kj}, \quad \hat{r}_{ij} = \hat{r}^k_{ikj}, \quad \hat{r} = a^{ij}\hat{r}_{ij}\,. \tag{4.18}$$

Next, we consider Carroll Weyl transformations. Following [27] we define this as

$$\tilde{a}_{ij}(t,\mathbf{x}) = (B(t,\mathbf{x}))^{-2}a_{ij}(t,\mathbf{x}), \quad \tilde{\omega}(t,\mathbf{x}) = (B(t,\mathbf{x}))^{-z}\omega(t,\mathbf{x}),$$
$$\tilde{b}_i(t,\mathbf{x}) = (B(t,\mathbf{x}))^{-z}b_i(t,\mathbf{x}) \tag{4.19}$$

where $B(t,\mathbf{x})$ is an arbitrary function and z is a non-zero real number. We are now ready to emulate the steps of Section (4.1) and construct equations of motion that are covariant under the Carroll diffeomorphisms (4.9) and Weyl transformations (4.19).

4.2.1 Constructing equations of motion

For this, we first start by listing the transformation properties of our Carroll diffeomorphism invariants (4.17) under (4.19). One finds:

$$\begin{aligned}
\theta &\longrightarrow B^{z-1}\Big[B\,\theta - 2\,\hat{\partial}_t B\Big]\,,\\
\hat{\partial}_t\theta &\longrightarrow B^{2\,z-2}\Big[B^2\,\hat{\partial}_t\theta + z\,B\,\theta\,\hat{\partial}_t B - 2\,(z-1)\,(\hat{\partial}_t B)^2 - 2B\,\hat{\partial}_t\hat{\partial}_t B\Big]\,,\\
\hat{\gamma}^i_j\hat{\gamma}^j_i &\longrightarrow B^{2\,z-2}\Big[B^2\,\hat{\gamma}^i_j\hat{\gamma}^j_i - 2\,B\,\theta\,\hat{\partial}_t B + 2\,(\hat{\partial}_t B)^2\Big]\,,\\
\hat{r} &\longrightarrow B^2\,\hat{r} + 2B\,a^{ij}\hat{\nabla}_i\hat{\partial}_j B - 2a^{ij}\hat{\partial}_i B\hat{\partial}_j B\,,\\
a^{ij}\hat{\nabla}_i\phi_j &\longrightarrow B^2\,a^{ij}\hat{\nabla}_i\phi_j - z\,B\,a^{ij}\hat{\nabla}_i\hat{\partial}_j B + z\,a^{ij}\hat{\partial}_i B\hat{\partial}_j B\,,\\
a^{ij}\phi_i\phi_j &\longrightarrow B^2\,a^{ij}\phi_i\phi_j - 2z\,B\,\phi^i\hat{\partial}_i B + z^2\,a^{ij}\hat{\partial}_i B\hat{\partial}_j B\,.
\end{aligned}\tag{4.20}$$

Three combinations that transform homogeneously are:

$$\begin{aligned}
\hat{r} + \frac{2}{z}a^{ij}\hat{\nabla}_i\phi_j &\longrightarrow B^2\left(\hat{r} + \frac{2}{z}a^{ij}\hat{\nabla}_i\phi_j\right)\,,\\
\hat{\gamma}^i_j\hat{\gamma}^j_i - \frac{1}{2}\theta^2 &\longrightarrow B^{2\,z}(\hat{\gamma}^i_j\hat{\gamma}^j_i - \frac{1}{2}\theta^2)\,,\\
f_{ij}f^{ij} &\longrightarrow B^{4-2\,z}\,f_{ij}f^{ij}\,.
\end{aligned}\tag{4.21}$$

Now we are ready to include matter fields. Let us consider a real scalar field $\Phi(t,\mathbf{x})$ for simplicity. We seek to construct equations of motion which are Carroll Weyl invariant. For this we start by listing Carroll diffeomorphism invariants up to two derivatives (on both sets of objects (a_{ij}, b_i, ω) and Φ), and linear in Φ:

$$\begin{aligned}
&\Phi,\ \ \theta\,\Phi,\ \ \theta^2\,\Phi,\ \ \hat{\gamma}^i_j\hat{\gamma}^j_i\,\Phi,\ \ \hat{\partial}_t\theta\,\Phi,\ \ \hat{r}\,\Phi,\ \ \hat{\nabla}^i\phi_i\,\Phi,\ \ \phi^i\phi_i\,\Phi,\ \ f_{ij}f^{ij}\Phi\\
&\qquad\qquad\hat{\partial}_t\Phi,\ \ \theta\,\hat{\partial}_t\Phi,\ \ \phi^i\,\hat{\partial}_i\Phi,\ \ \hat{\partial}_t\hat{\partial}_t\Phi,\ \ \hat{\nabla}^i\hat{\partial}_i\Phi
\end{aligned}\tag{4.22}$$

Defining that the scalar Φ transforms as $\Phi \to B^\delta\Phi$ under Weyl transformations we can find how the objects in (4.22) transform. We find:

$$\begin{aligned}
\hat{\partial}_t\Phi &\longrightarrow B^{z+\delta-1}\Big[B\,\hat{\partial}_t\Phi + \delta\,\Phi\,\hat{\partial}_t B\Big]\,,\\
\hat{\partial}_t\hat{\partial}_t\Phi &\longrightarrow B^{\delta+2\,z-2}\Big[B^2\,\hat{\partial}_t\hat{\partial}_t\Phi + (z+2\,\delta)\,B\,\partial_t B\,\partial_t\Phi + \delta\,B\,\Phi\,\hat{\partial}_t\hat{\partial}_t B + \delta\,(\delta+z-1)\Phi\,(\hat{\partial}_t B)^2\Big]\,,\\
\phi^i\,\hat{\partial}_i\Phi &\longrightarrow B^\delta\Big[B^2\,\phi^i\,\hat{\partial}_i\Phi + \delta\,B\,\Phi\,\phi^i\,\hat{\partial}_i B - \delta\,z\,\Phi\,\hat{\partial}_i B\,\hat{\partial}^i B - z\,B\,\hat{\partial}^i\Phi\,\hat{\partial}_i B\Big]\,,\\
\hat{\nabla}^i\hat{\partial}_i\Phi &\longrightarrow B^\delta\Big[B^2\,\hat{\nabla}^i\hat{\partial}_i\Phi + \delta(\delta-1)\Phi\hat{\partial}^i B\,\hat{\partial}_i B + 2\,\delta\,B\hat{\partial}^i B\,\hat{\partial}_i\Phi + \delta\,B\,\hat{\nabla}^i\hat{\partial}_i B\Big]\,.
\end{aligned}\tag{4.23}$$

The simplest and only Weyl covariant object linear and homogeneous in Φ at the first derivative order is :

$$\hat{\partial}_t\Phi + \frac{\delta}{2}\theta\,\Phi \longrightarrow B^{z+\delta}(\hat{\partial}_t\Phi + \frac{\delta}{2}\theta\,\Phi)\,. \tag{4.24}$$

This combination was already known in [76] in the context of $z = 1$. At the second derivative order, we find two such covariant objects:

$$\begin{aligned} &\hat{\partial}_t^2\Phi + \frac{1}{2}(z+2\,\delta)\,\theta\,\hat{\partial}_t\Phi + \frac{\delta}{4}\left[(z+\delta)\,\theta^2 + 2\,\hat{\partial}_t\theta\right]\,\Phi \\ \longrightarrow B^{2\,z+\delta}&\left(\hat{\partial}_t^2\Phi + \frac{1}{2}(z+2\,\delta)\,\theta\,\hat{\partial}_t\Phi + \frac{\delta}{4}\left[(z+\delta)\,\theta^2 + 2\,\hat{\partial}_t\theta\right]\,\Phi\right) \end{aligned} \tag{4.25}$$

$$\begin{aligned} &\hat{\nabla}^i\hat{\partial}_i\Phi + \frac{2\,\delta}{z}\,\phi^i\,\partial_i\Phi - \frac{\delta}{2}\left[\hat{r} - \frac{2\,\delta}{z^2}\,\phi^i\,\phi_i\right]\,\Phi \\ \longrightarrow B^{2+\delta}&\left(\hat{\nabla}^i\hat{\partial}_i\Phi + \frac{2\,\delta}{z}\,\phi^i\,\hat{\partial}_i\Phi - \frac{\delta}{2}\left[\hat{r} - \frac{2\,\delta}{z^2}\,\phi^i\,\phi_i\right]\,\Phi\right)\,. \end{aligned} \tag{4.26}$$

We found two distinct possibilities for the covariant equations of motion: one with weight $2z+\delta$, and the other with weight $2+\delta$. We will refer to these as *time-like* case and *space-like* case respectively.

These are not yet the full set of covariant combinations. At the linear order in Φ, one is free to consider linear combination of (4.25) with $\left(\hat{\gamma}^i_j\hat{\gamma}^j_i - \frac{1}{2}\theta^2\right)\Phi$ and (4.26) with $(\hat{r} + \frac{2}{z}a^{ij}\hat{\nabla}_i\phi_j)\,\Phi$. At the level of interactions we may consider linear combinations of appropriate powers of Φ multiplying $\{1, \hat{r} + \frac{2}{z}a^{ij}\hat{\nabla}_i\phi_j,\ f_{ij}f^{ij}\}$ added to (4.25) and $\{1, \hat{\gamma}^i_j\hat{\gamma}^j_i - \frac{1}{2}\theta^2,\ f_{ij}f^{ij}\}$ added to (4.26).

In the interaction terms the powers of Φ have to be chosen so that the combinations have the right Weyl weights and the potentials are preferably polynomials in Φ. This latter requirement constrains the data (z,δ). For instance, whenever $\frac{2\,z+\delta}{\delta}$ $(:= N_{\lambda_0} - 1)$ is a positive integer then one can consider $\Phi^{\frac{2\,z+\delta}{\delta}}$ along with (4.25). Similarly in the *space-like* case one can add $\Phi^{\frac{2+\delta}{\delta}}$ whenever $\frac{2+\delta}{\delta}$ $(:= N_{\lambda_1} - 1)$ is a positive integer. Such terms are expected to contribute to the equations of motion when there are monomial potential of the type $\Phi^{N_{\lambda_0}}$ $(\Phi^{N_{\lambda_1}})$ in the action for Φ.

There are two special values of z, namely, $z = 1$ and $z = 2$, that have to be treated more carefully because in these cases we can consider more general covariant combinations.

- $z = 1$: We can consider constant linear combinations of the two quantities (4.25) and (4.26) – for in this case the Weyl weights of both of these quantities become equal to $2+\delta$. Furthermore precisely in this case the Weyl weights of $\left(\hat{\gamma}^i_j\hat{\gamma}^j_i - \frac{1}{2}\theta^2\right)\Phi$, $\left(\hat{r} + \frac{2}{z}a^{ij}\hat{\nabla}_i\phi_j\right)\Phi$ and $f_{ij}f^{ij}\,\Phi$, (which are all linear in Φ) also become $2+\delta$ and so can be added.

- $z=2$: We can consider linear combinations of the first order time-derivative object (4.24) with the *space-like* derivative object in (4.26) as both their Weyl weights become equal to $2+\delta$.

We can summarise our results so far for various types of Carroll diffeomorphic and Weyl invariant equations of motion for a scalar field Φ as follows:

- For $z=1$ we can take the covariant equation to be:

$$\begin{aligned}&\kappa_0\left[\hat{\partial}_t^2\Phi + \frac{1}{2}(1+2\,\delta)\,\theta\,\hat{\partial}_t\Phi + \frac{\delta}{4}\left[(1+\delta)\,\theta^2 + 2\,\hat{\partial}_t\theta\right]\,\Phi\right]\\ &+\ \kappa_1\left[\hat{\nabla}^i\hat{\partial}_i\Phi + 2\,\delta\,\phi^i\,\hat{\partial}_i\Phi - \frac{\delta}{2}\left[\hat{r} - 2\,\delta\,\phi^i\phi_i\,\right]\,\Phi\right]\\ &+\left[\sigma_0\,\left(\hat{\gamma}^i_j\hat{\gamma}^j_i - \frac{1}{2}\theta^2\right) + \sigma_1\,\left(\hat{r} + 2\,a^{ij}\hat{\nabla}_i\phi_j\right) + \sigma_2\,f_{ij}f^{ij}\right]\,\Phi + \lambda\,\Phi^{\frac{2+\delta}{\delta}} = 0.\end{aligned} \tag{4.27}$$

  This equation has five independent real parameters – since $(\kappa_0, \kappa_1, \sigma_0, \sigma_1, \sigma_2, \lambda)$ are equivalent to $\alpha\,(\kappa_0, \kappa_1, \sigma_0, \sigma_1, \sigma_2, \lambda)$ for any non-zero real α. We may assume that $(\kappa_0, \kappa_1) \neq (0,0)$ so that we have a differential equation. Note also that there is no reason to fix δ at this stage beyond the assumption that $\frac{2+\delta}{\delta}$ is (preferably an odd) positive integer, whenever $\lambda \neq 0$.

- For $z=2$ we can take the covariant equation to be:

$$\begin{aligned}&\kappa_3\,\left[\hat{\partial}_t\Phi + \frac{\delta}{2}\theta\,\Phi\right] + \tau_3\,\left[\hat{\nabla}^i\hat{\partial}_i\Phi + \delta\,\phi^i\,\hat{\partial}_i\Phi - \frac{\delta}{2}\left(\hat{r} - \frac{\delta}{2}\,\phi^i\phi_i\right)\,\Phi\right]\\ &+\sigma_3\,\left(\hat{r} + a^{ij}\hat{\nabla}_i\phi_j\right)\,\Phi + \omega_3\,\left(\hat{\gamma}^i_j\hat{\gamma}^j_i - \frac{1}{2}\theta^2\right)\,\Phi^{1-\frac{2}{\delta}} + \left(\lambda_3 + \mu_3\,f_{ij}f^{ij}\right)\,\Phi^{\frac{2+\delta}{\delta}} = 0,\end{aligned} \tag{4.28}$$

  with Weyl weight $2+\delta$ and (at most) five independent parameters, and/or

$$\begin{aligned}&\hat{\partial}_t^2\Phi + (1+\delta)\,\theta\,\hat{\partial}_t\Phi + \frac{\delta}{4}\left[(2+\delta)\,\theta^2 + 2\,\hat{\partial}_t\theta\right]\,\Phi + \sigma_0\,\left(\hat{\gamma}^i_j\hat{\gamma}^j_i - \frac{1}{2}\theta^2\right)\Phi\\ &\qquad + \kappa_0\,\left(\hat{r} + a^{ij}\hat{\nabla}_i\phi_j\right)\,\Phi^{1+\frac{2}{\delta}} + (\lambda_0\, + \mu_0\,f_{ij}f^{ij})\,\Phi^{\frac{4+\delta}{\delta}} = 0,\end{aligned} \tag{4.29}$$

with Weyl weight $\delta + 4$ and (at most) four parameters.

- For general values of z we have the following two types of Carroll diffeomorphism and Weyl invariant equations of motion

  1.
$$\hat{\partial}_t^2\Phi + \frac{1}{2}(z + 2\,\delta)\,\theta\,\hat{\partial}_t\Phi + \frac{\delta}{4}\left[(z+\delta)\,\theta^2 + 2\,\hat{\partial}_t\theta\right]\,\Phi + \sigma_0\,\left(\hat{\gamma}^i_j\hat{\gamma}^j_i - \frac{1}{2}\theta^2\right)\,\Phi$$
$$+ \mu_0\,\Phi^{1+\frac{4}{\delta}(z-1)} f_{ij}f^{ij} + \kappa_0\,\left(\hat{r} + \frac{2}{z}a^{ij}\hat{\nabla}_i\phi_j\right)\,\Phi^{1+\frac{2}{\delta}(z-1)} + \lambda_0\,\Phi^{\frac{2z+\delta}{\delta}} = 0, \tag{4.30}$$

  2.
$$\hat{\nabla}^i\hat{\partial}_i\Phi + \frac{2\,\delta}{z}\,\phi^i\,\hat{\partial}_i\Phi - \frac{\delta}{2}\left[\hat{r} - \frac{2\,\delta}{z^2}\,\phi^i\phi_i\right]\,\Phi + \sigma_1\,\left(\hat{r} + \frac{2}{z}a^{ij}\hat{\nabla}_i\phi_j\right)\,\Phi$$
$$+ \mu_1\,f_{ij}f^{ij}\,\Phi^{1+\frac{2}{\delta}(z-1)} + \kappa_1\,\left(\hat{\gamma}^i_j\hat{\gamma}^j_i - \frac{1}{2}\theta^2\right)\,\Phi^{1+\frac{2}{\delta}(1-z)} + \lambda_1\,\Phi^{\frac{2+\delta}{\delta}} = 0, \tag{4.31}$$

  where we assume that the coefficient (μ_0, κ_0) and (μ_1, κ_1) are non-vanishing only if the powers of Φ multiplying them are positive integers.

Let us make some comments on the nature of interaction terms in the equations (4.30, 4.31). In the next section, we will attempt to construct actions that produce these equations. We will assume that the interaction terms can be obtained from an action with a potential that is a polynomial in Φ and bounded below. This requires the potential to be a linear combination of positive integer powers of Φ with appropriate coefficients and the highest power (the degree of the polynomial) being an even number. Further, we will assume that the Weyl scaling δ of Φ is positive.

- In the *time-like* case let us denote the potential terms in (4.30) by

$$\mu_0\,f_{ij}f^{ij}\,\Phi^{N_{\mu_0}-1} + \kappa_0\,\left(\hat{r} + \frac{2}{z}a^{ij}\hat{\nabla}_i\phi_j\right)\,\Phi^{N_{\kappa_0}-1} + \lambda_0\,\Phi^{N_{\lambda_0}-1} \tag{4.32}$$

- Then we have $N_{\lambda_0} - N_{\kappa_0} = \frac{2}{\delta}$, $N_{\mu_0} - N_{\kappa_0} = \frac{2}{\delta}(z-1)$ and $N_{\mu_0} - N_{\lambda_0} = \frac{2}{\delta}(z-2)$. Therefore

  1. for $z < 1$ we have $N_{\mu_0} < N_{\kappa_0} < N_{\lambda_0}$,

2. for $1 < z < 2$ we have $N_{\kappa_0} < N_{\mu_0} < N_{\lambda_0}$, and
3. for $z > 2$ we have $N_{\kappa_0} < N_{\lambda_0} < N_{\mu_0}$.

Similarly

- for the *space-like* case let us denote the interaction terms in (4.31) by

$$\mu_1 \, f_{ij} f^{ij} \, \Phi^{N_{\mu_1}-1} + \kappa_1 \left(\hat{\gamma}^i_j \hat{\gamma}^j_i - \frac{1}{2}\theta^2 \right) \Phi^{N_{\kappa_1}-1} + \lambda_1 \, \Phi^{N_{\lambda_1}-1} \, . \tag{4.33}$$

- Then we have $N_{\lambda_1} - N_{\kappa_1} = \frac{2z}{\delta}$, $N_{\lambda_1} - N_{\mu_1} = \frac{2}{\delta}(2-z)$ and $N_{\kappa_1} - N_{\mu_1} = \frac{4}{\delta}(1-z)$. Hence,

1. for $z < 0$ we have $N_{\mu_1} < N_{\lambda_1} < N_{\kappa_1}$,
2. for $0 < z < 1$ we have $N_{\mu_1} < N_{\kappa_1} < N_{\lambda_1}$,
3. for $1 < z < 2$ we have $N_{\kappa_1} < N_{\mu_1} < N_{\lambda_1}$, and
4. for $z > 2$ we have $N_{\kappa_1} < N_{\lambda_1} < N_{\mu_1}$.

Therefore depending on the range of z the degree of the polynomial is different. Demanding that the powers of Φ are positive integers implies non-trivial constraints on the set of values (z, δ) can take. As we will see in the next section the existence of actions implies further constraints on these data. So we postpone further discussion of this aspect for later.

4.2.2 Constructing actions

Now that we have derived the most general two-derivative diffeomorphic and Weyl covariant equations of motion for a single real[2] scalar field Φ in the background of generic Carroll geometry, we would like to turn to construct the corresponding actions next. We should seek actions that produce each of the equations (4.27, 4.28, 4.29, 4.30, 4.31) as their Euler-Lagrange equations for the scalar field Φ.

Note that our equations of motion in the absence of interactions are linear in Φ and have up to two derivatives on Φ. If we are to be able to derive them from some actions then they should be quadratic in Φ and should contain up to two derivatives.

[2]For complex Φ one has to replace the potentials to be appropriate real combinations, such as $|\Phi|^{2n}$ etc.

Therefore let us again start by listing all such invariants – now counting derivatives both on the background geometric quantities (a_{ij}, b_i, ω) and on Φ.

$$
\begin{aligned}
&\theta\,\Phi^2, \quad \Phi\,\hat{\partial}_t\Phi,\\
\hat{\gamma}^i_j\hat{\gamma}^j_i\,\Phi^2, \quad \theta^2\,\Phi^2, \quad \theta\,\Phi\,\hat{\partial}_t\Phi, \quad \hat{\partial}_t\theta\,\Phi^2, \quad (\hat{\partial}_t\Phi)^2, \quad \Phi\,\hat{\partial}_t^2\Phi,\\
\hat{r}\,\Phi^2, \quad \hat{\nabla}_i\phi^i\,\Phi^2, \quad \phi^i\phi_i\,\Phi^2, \quad f_{ij}f^{ij}\,\Phi^2, \quad \Phi\,\phi^i\hat{\partial}_i\Phi, \quad \hat{\partial}_i\Phi\hat{\partial}_i\Phi, \quad \Phi\,\hat{\nabla}^i\hat{\partial}_i\,\Phi\,. \qquad (4.34)
\end{aligned}
$$

We have already listed in (4.23) all the transformations required under the Carroll Weyl transformations and we simply have to find the combinations of quantities in (4.34) that transform homogeneously.

After a straightforward analysis using the results of the previous subsection, we find the following combinations up to quadratic order in Φ :

1. At first order in time-derivatives we have the unique combination with weight $z + 2\delta$

$$\Phi\,(\hat{\partial}_t\Phi + \frac{\delta}{2}\theta\,\Phi) \qquad (4.35)$$

2. At second order in time-derivatives we find three covariant combinations with weight $2(z + \delta)$.

$$\Big(\hat{\partial}_t\Phi + \frac{\delta}{2}\theta\,\Phi\Big)^2, \qquad \Big(\hat{\gamma}^i_j\hat{\gamma}^j_i - \frac{1}{2}\theta^2\Big)\,\Phi^2, \qquad (4.36)$$

$$(\hat{\partial}_t\theta + \frac{z}{2}\hat{\gamma}^i_j\hat{\gamma}^j_i)\,\Phi^2 + \frac{2}{\delta}\Phi\,\hat{\partial}_t^2\Phi - \frac{2\delta + z}{\delta^2}(\hat{\partial}_t\Phi)^2. \qquad (4.37)$$

We will refer to these as *time-like* combinations.

3. At second order in space-derivatives we find three combinations with Weyl weight $2(1 + \delta)$:

$$(\hat{\partial}_i\Phi + \frac{\delta}{z}\,\phi_i\Phi)(\hat{\partial}^i\Phi + \frac{\delta}{z}\,\phi^i\Phi), \qquad (\hat{r} + \frac{2}{z}\hat{\nabla}_i\phi^i)\,\Phi^2, \qquad (4.38)$$

$$\Phi\hat{\nabla}_i\hat{\partial}^i\Phi + 2\frac{\delta}{z}\Phi\,\phi^i\hat{\partial}_i\Phi + \frac{\delta}{z}(\hat{\nabla}_i\phi^i + \frac{\delta}{z}\,\phi_i\phi^i)\,\Phi^2. \qquad (4.39)$$

We refer to these as *space-like* combinations.

At higher orders in Φ we can consider potentials with coefficients from any of

$$1, \quad f_{ij}f^{ij}, \quad \hat{r} + \frac{2}{z}\hat{\nabla}_i\phi^i, \quad \hat{\gamma}^i_j\hat{\gamma}^j_i - \frac{1}{2}\theta^2\,. \tag{4.40}$$

Now any candidate action has to be an integral over the coordinates $(t, \mathbf{x})$ of our Carroll manifold:

$$S = \int dt\, d^2\mathbf{x}\, \mathcal{L}\,. \tag{4.41}$$

For the action to be invariant under diffeomorphisms the Lagrangian density should transform as $\mathcal{L} \to \mathcal{L}'$, such that

$$\int dt\, d^2\mathbf{x}\ \mathcal{L} = \int dt' d^2\mathbf{x}'\ \mathcal{L}'\,. \tag{4.42}$$

From (4.9) we have $dt'd^2\mathbf{x}' = J \det J^i_j\ dt\, d^2\mathbf{x}$. So the Lagrangian density $\mathcal{L}$ has to transform as $\mathcal{L} \to \mathcal{L}' = J^{-1}\det((J^{-1})^i_j)\ \mathcal{L}$ – *i.e,* as a scalar density of weight 3, equal to the dimension of the manifold – under the relevant Carroll diffeomorphisms. The combinations we listed above in (4.35 - 4.39) are all scalars under Carroll diffeomorphisms, even though they have non-trivial weights under Carroll Weyl transformations. So to make them densities of suitable weights we need to multiply them by $\omega\sqrt{a}$ as this is the only combination of the Carroll geometry without derivatives and transforms as desired:

$$\omega\sqrt{a} \to J^{-1}\det((J^{-1})^i_j)\ \omega\sqrt{a}\,, \tag{4.43}$$

where a is the determinant of the metric a_{ij} on the base space. For instance

$$S = \int dt\, d^2\mathbf{x}\ \omega\sqrt{a}\ \left[\hat{\partial}_i\Phi\hat{\partial}^i\Phi + 2\frac{\delta}{z}\Phi\,\phi^i\hat{\partial}_i\Phi + \frac{\delta^2}{z^2}\phi_i\phi^i\,\Phi^2\right] \tag{4.44}$$

is a good action for a Carroll diffeomorphism invariant theory. However, for it to be also a Weyl invariant theory the Lagrangian density $\mathcal{L}$ has to be invariant by itself (up to total-divergence terms) under (4.19). Let us check this for (4.44): the measure transforms as $\omega\sqrt{a} \to B^{-2-z}\omega\sqrt{a}$ and the quantity in square-brackets transforms as $[\cdots] \to B^{2+2\delta}[\cdots]$ under the Weyl transformations (4.19). So demanding that the Lagrangian density $\mathcal{L}$ in (4.44) is Weyl invariant requires

$$2 + 2\delta = 2 + z \implies \delta = \frac{z}{2}. \tag{4.45}$$

This conclusion is valid for all actions constructed using the *space-like* combinations with Weyl weight $2+2\delta$. Also $\delta > 0$ requires $z > 0$.

Similarly, if we use any of the Carroll diffeomorphism invariants and Carroll Weyl covariant *time-like* combinations in (4.36) we need to fix the weight δ of the scalar such that:

$$2+z = 2z+2\delta \implies \delta = 1-\frac{z}{2}\,. \tag{4.46}$$

In this case, for $\delta > 0$ we need to have $z < 2$.

Note that this is unlike the conformally coupled scalar in the background of a (pseudo) Riemann manifold where the Weyl weight of the scalar is fixed at the level of the equation of motion itself (as reviewed in Section (4.1)), whereas, for the scalar in Carroll geometry, it is fixed (even for a free scalar) at the level of the existence of an action.

When we include interactions, we have to continue to impose the constraints (4.46) in *time-like* case and (4.45) in the *space-like* case. So we have the potential :

- in the *time-like* case is ($\omega\sqrt{a}$ times)

$$\frac{1}{N_{\mu_0}}\mu_0\, f_{ij}f^{ij}\,\Phi^{N_{\mu_0}} + \frac{1}{N_{\kappa_0}}\kappa_0\left(\hat{r}+\frac{2}{z}a^{ij}\hat{\nabla}_i\phi_j\right)\Phi^{N_{\kappa_0}} + \frac{1}{N_{\lambda_0}}\lambda_0\,\Phi^{N_{\lambda_0}} \tag{4.47}$$

  with $N_{\mu_0} = \frac{4}{\delta}-6$, $N_{\kappa_0} = \frac{2}{\delta}-2$ and $N_{\lambda_0} = \frac{4}{\delta}-2$. We further assume that $N_{\mu_0}, N_{\kappa_0}, N_{\lambda_0} \geq 2$. For $N_{\mu_0} \geq 2$ and $N_{\kappa_0} \geq 2$ we need $\delta \leq \frac{1}{2}$ and $N_{\lambda_0} \geq 2$ requires $\delta \leq 1$. Thus we can have all three terms in the potential only when $0 \leq \delta \leq \frac{1}{2}$ with N_{λ_0} being the largest power. When $\frac{1}{2} < \delta \leq 1$ we can only have λ_0 non-zero and if $\delta > 1$ no potential is possible.

- In the *space-like* case it is ($\omega\sqrt{a}$ times)

$$\frac{1}{N_{\mu_1}}\mu_1\, f_{ij}f^{ij}\,\Phi^{N_{\mu_1}} + \frac{1}{N_{\kappa_1}}\kappa_1\left(\hat{\gamma}^i_j\hat{\gamma}^j_i-\frac{1}{2}\theta^2\right)\Phi^{N_{\kappa_1}} + \frac{1}{N_{\lambda_1}}\lambda_1\,\Phi^{N_{\lambda_1}} \tag{4.48}$$

  with $N_{\mu_1} = 6-\frac{2}{\delta}$, $N_{\kappa_1} = \frac{2}{\delta}-2$ and $N_{\lambda_1} = \frac{2}{\delta}+2$. We again assume that $N_{\mu_1}, N_{\kappa_1}, N_{\lambda_1} \geq 2$. For $N_{\mu_1} \geq 2$ requires $\delta \geq \frac{1}{2}$, for $N_{\kappa_1} \geq 2$ requires $\delta \leq \frac{1}{2}$, and for $N_{\lambda_1} \geq 2$ requires $\delta \geq 0$. So when we take $0 \leq \delta < \frac{1}{2}$ (which means $0 \leq z < 1$) we need to set $\mu_1 = 0$, and we have N_{λ_1} as the largest power. When $\frac{1}{2} < \delta \leq 1$ we cannot have κ_1 non-zero and the highest power of the potential is again given by N_{λ_1}. When $\delta > 1$ we still have $\kappa_1 = 0$ and the largest power of Φ in the potential will be N_{μ_1}. Only when $\delta = \frac{1}{2}$ all three terms are possible with $N_{\lambda_1} = 6$.

To summarise in both *time-like* and *space-like* cases the different types of interactions are possible depending on the values of δ (determined in terms of z via (4.46, 4.45)). When $0 \leq \delta \leq 1$ the dominant power in the potential comes from the monomial $\lambda_0 \Phi^{N_{\lambda_0}}$ ($\lambda_1 \Phi^{N_{\lambda_1}}$) in the *time-like* (*space-like*) case. When $\delta > 1$ no potential is possible in the *time-like* case and in the *space-like* case the dominant power is given by N_{μ_1}. Finally, we have to impose that the dominant power of the potential is even.

- In the *time-like* case, whenever the potential is allowed the degree of the potential is N_{λ_0}. For the potential to be bounded below we require that N_{λ_0} is even, i.e., $N_{\lambda_0} = 2n_t$ for some $n_t \geq 1$. Thus the data (z, δ) are restricted to:

$$\delta = 1 - \frac{z}{2} = \frac{z}{n_t - 1} \implies z = 2\frac{n_t - 1}{n_t + 1} \;\; \& \;\delta = \frac{2}{n_t + 1}, \quad n_t \geq 1\,. \qquad (4.49)$$

  In this case the range of z is between 0 (when $n_t = 1$) and 2 (when $n_t \to \infty$). The case of $n_t = 3$ gives $z = 1$ with $\delta = 1/2$, and this is what one obtains by taking the ultra-relativistic limit of conformally coupled scalar in 3 dimensions – as we show in Section (4.3).

- In the *space-like* case when $0 < \delta \leq 1$ we take $N_{\lambda_1} = 2n_s$ for some $n_s \geq 2$ (since when $0 < \delta \leq 1$ we have $N_{\lambda_1} \geq 4$). Then we have

$$\delta = \frac{z}{2} = \frac{1}{n_s - 1} \implies z = \frac{2}{n_s - 1} \;\; \& \;\; \delta = \frac{1}{n_s - 1}, \quad n_s \geq 2\,. \qquad (4.50)$$

  This is the set of values assumed for z by Duval *et al* [69], where they denoted $k = n_s - 1$ with $k \geq 1$. In this case the range of z is between zero (when $n_s \to \infty$) and 2 (when $n_s = 2$) with the corresponding values of δ ranging between 0 and 1. Again the special value $z = 1$ gives $\delta = 1/2$.

  When $\delta > 1$ we need to take $N_{\mu_1} = 2n_\mu$ for some $n_\mu \geq 3$ since $N_{\mu_1} = 6 - \frac{2}{\delta}$. At the same time N_{μ_1} is bounded above by 6 and so the only non-trivial possibility is $n_\mu = 3$. But when $n_\mu = 3$ both z and δ are infinite which we do not consider.

So we conclude that the existence of (i) bounded and polynomial potentials and (ii) invariant actions implies discrete and specific rational values for both z and δ.

Actions for Equations:

We now propose actions which produce (4.27 - 4.31) as their equations of motion.

- For the *time-like* case with $z = 2\,(1-\delta)$ our action is

$$S_t = \int dt\, d^2\mathbf{x}\ \ \omega\sqrt{a}\left[\alpha_1\left(\hat{\partial}_t\Phi + \frac{\delta}{2}\theta\,\Phi\right)^2 + \beta_1\left(\hat{\gamma}^i_j\hat{\gamma}^j_i - \frac{1}{2}\theta^2\right)\Phi^2 + \lambda_1\,\Phi^{2+\frac{2\,z}{\delta}}\right], \tag{4.51}$$

which gives the equation of motion for Φ to be:

$$\begin{aligned}
&-2\,\alpha_1\left[\hat{\partial}^2_t\Phi + \theta\,\hat{\partial}_t\Phi - \frac{1}{4}\delta\,(\delta-2)\,\theta^2\,\Phi + \frac{\delta}{2}\hat{\partial}_t\theta\,\Phi\right] \\
&+2\beta_1\left(\hat{\gamma}^i_j\hat{\gamma}^j_i - \frac{1}{2}\theta^2\right)\Phi + 2\,\lambda_1\left(\frac{2}{\delta}-1\right)\Phi^{\frac{4}{\delta}-3} = 0
\end{aligned} \tag{4.52}$$

- For the *space-like* case with $z = 2\delta$ our action is

$$\begin{aligned}
S_s = \int dt\, d^2\mathbf{x}\ \omega\,\sqrt{a}\,\Bigg[&\alpha_2\left(\hat{\partial}_i\Phi + \frac{\delta}{z}\,\phi_i\Phi\right)\left(\hat{\partial}^i\Phi + \frac{\delta}{z}\,\phi^i\Phi\right) \\
&+\beta_2\left(\hat{r} + \frac{2}{z}\hat{\nabla}_i\phi^i\right)\Phi^2 + \lambda_2\,\Phi^{2+\frac{2}{\delta}}\Bigg]
\end{aligned} \tag{4.53}$$

which gives rise to

$$\begin{aligned}
&-2\,\alpha_2\left[\hat{\Box}\Phi + \phi^i\hat{\nabla}_i\Phi + \frac{1}{4}\phi^i\phi_i\Phi + \frac{1}{2}\hat{\nabla}_i\phi^i\,\Phi\right] \\
&+2\beta_2\left(\hat{r} + \frac{2}{z}\hat{\nabla}_i\phi^i\right)\Phi + 2\lambda_2\left(1+\frac{1}{\delta}\right)\Phi^{1+\frac{2}{\delta}} = 0\,.
\end{aligned} \tag{4.54}$$

It is easy to see that these equations are the same as (4.30 - 4.31) with appropriate identifications of the coefficients.[3]

For the special case $z = 1$ we can consider linear combinations of (4.51), (4.53) with (4.39), which can be seen to generate equations of the form (4.27).

The other special value of z, namely $z = 2$ needs some more attention. In this case, we either have to consider a complex Φ or two real fields Φ_1 and Φ_2. To see this we first note that in this case, we expect an action to reproduce an equation of motion of the type (4.28) that includes (whenever $\kappa_3 \neq 0$) first-order time-derivatives. Using

[3] One can easily incorporate other interaction terms into the actions to produce potential terms in (4.30), (4.31) with non-constant coefficients as well whenever they are allowed.

a single real scalar a candidate action that could have produced a first-order time-derivative term is $\omega\sqrt{a}\,\Phi\,(\hat{\partial}_t\Phi + \frac{\delta}{2}\theta\,\Phi)$. But such an action does not lead to a non-trivial equation of motion in $z = 2$ case (which in turn requires $\delta = 1$) since $\omega\sqrt{a}\,\Phi\,(\hat{\partial}_t\Phi + \frac{1}{2}\theta\,\Phi) = \partial_t(\frac{1}{2}\sqrt{a}\,\Phi^2)$ is a total derivative. A simple way to overcome this is to consider $\omega\sqrt{a}\,\Phi^\star\,(\hat{\partial}_t\Phi + \frac{\delta}{2}\theta\,\Phi)$ for a complex Φ with $\Phi^\star$ being its complex conjugate. We will not pursue this further here.

4.3 A Carroll CFT from a conformally coupled scalar

Now we show that some special cases of the equations of motion and their corresponding actions we derived in the last section arise in the ultra-relativistic limit of the conformally coupled scalar reviewed in Section (4.1).

Our starting point is the diffeomorphic and Weyl invariant scalar field theory in 2+1 dimensional spacetime. The background metric is taken in the so-called Randers-Papapetrou form (see for instance [76]), with the line element:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -c^2\omega^2(dt - \omega^{-1}b_i\,dx^i)^2 + a_{ij}\,dx^i\,dx^j, \tag{4.55}$$

where $x^\mu = (t, \mathbf{x})$. This geometry in the $c \to 0$ limit is expected to produce a Carroll geometry. Moreover, the subset of all diffeomorphisms that leave this metric form-invariant is precisely the Carroll diffeomorphisms (4.9) and the quantities $\{a_{ij}, b_i, \omega\}$ transform under (4.9) as in (4.12).

It can be seen that the Ricci scalar of this geometry is

$$R = c^{-2}\left(\theta^2 + \hat{\gamma}^i_j\hat{\gamma}^j_i + 2\,\hat{\partial}_t\theta\right) + \left(\hat{r} - 2\hat{\nabla}_i\phi^i - 2\,\phi_i\phi^i\right) + \frac{c^2}{4}f_{ij}f^{ij}\,. \tag{4.56}$$

Consider the Weyl invariant Klein-Gordon scalar field equation on a general background in three dimensions:

$$g^{ab}\nabla_a\nabla_b\,\Phi - \frac{1}{8}R\,\Phi = 0. \tag{4.57}$$

Using Randers-Papapetrou metric ansatz (4.55) one can reduce this equation to a polynomial in c. This equation admits an expansion in terms of the combinations in (4.27 - 4.31) with $(z, \delta) = (1, \frac{1}{2})$. In particular:

$$
\begin{aligned}
&\hat{\Box}\Phi - \frac{1}{8}\, R\, \Phi \\
&\qquad = -\frac{1}{c^2}\left[\hat{\partial}_t^2\Phi + \theta\,\hat{\partial}_t\Phi + \frac{1}{16}\left(3\,\theta^2 + 4\,\hat{\partial}_t\theta\right)\,\Phi\right] - \frac{1}{8\,c^2}\left(\hat{\gamma}^i_j\hat{\gamma}^j_i - \frac{1}{2}\theta^2\right) \\
&\qquad\quad + \left[\hat{\Box}\Phi + \phi^i\hat{\nabla}_i\Phi - \frac{1}{4}\left(\hat{r} - \phi^i\phi_i\right)\Phi\right] + \frac{1}{8}\left(\hat{r} + 2\,\hat{\nabla}_i\phi^i\right)\,\Phi \\
&\qquad\quad - \frac{c^2}{32} f_{ij} f^{ij}\,\Phi\,.
\end{aligned}
\tag{4.58}
$$

Notice that if we had set any of these terms at any given order in powers of c to zero, it would have given us a Carroll diffeomorphic and Weyl covariant equation. We do not have to just restrict to the lowest order term (the leading term in the ultra-relativistic limit – as was done in the previous works [70, 69, 68, 27, 75, 76, 18, 17, 72, 19]). However, if we took the linear combination exactly as in (4.58) we can re-package the equation into a fully diffeomorphic and Weyl covariant equation (4.57) in the pseudo-Riemannian space with metric (4.55).

Let us now turn to the action in this special case. For this, we start with the Lagrangian density of the conformally coupled scalar to the background Randers-Papapetrou metric (4.55) and again expand it in powers of c. This gives us:

$$
\begin{aligned}
S =& \frac{1}{c}\int dt\, d^2\mathbf{x}\;\omega\sqrt{a}\left[(\hat{\partial}_t\Phi)^2 - \frac{1}{8}\left(\theta^2 + \hat{\gamma}^i_j\hat{\gamma}^j_i + 2\,\hat{\partial}_t\theta\right)\Phi^2\right] \\
& - c\int dt\, d^2\mathbf{x}\;\omega\sqrt{a}\left[\hat{\nabla}_i\Phi\,\hat{\nabla}^i\Phi + \frac{1}{8}(\hat{r} - 2\,\hat{\nabla}_i\phi^i - 2\,\phi_i\phi^i)\,\Phi^2\right] \\
& - \frac{c^3}{32}\int dt\, d^2\mathbf{x}\;\omega\sqrt{a}\, f_{ij} f^{ij}\,\Phi^2\,.
\end{aligned}
\tag{4.59}
$$

If we put $\alpha_1 = \frac{1}{c}$ and $\beta_1 = -\frac{1}{8c}$ (with $\lambda_1 = 0$) and use

$$
\partial_t(\sqrt{a}\,\theta\,\Phi^2) = \omega\,\sqrt{a}\,\left[2\,\theta\,\Phi\,\hat{\partial}_t\Phi + (\theta^2 + \hat{\partial}_t\theta)\,\Phi^2\right]
\tag{4.60}
$$

we see that the $\mathcal{O}(1/c)$ term of this action is a special case of (4.51) with $(z, \delta) = (1, \frac{1}{2})$ up to a total derivative. Similarly noticing that

$$
\partial_i\left[\omega\sqrt{a}\,\phi^i\,\Phi^2\right] + \partial_t\left[\sqrt{a}\,b_i\phi^i\,\Phi^2\right] = \omega\,\sqrt{a}\,\left[2\,\Phi\phi^i\hat{\partial}_i\Phi + (\phi^i\phi_i + \hat{\nabla}_i\phi^i)\,\Phi^2\right]
\tag{4.61}
$$

and setting $\alpha_2 = 1$, $\beta_2 = \frac{1}{8}$ we can see that the $\mathcal{O}(c)$ term in this action is again a special case of (4.53). Finally, the order c^3 term can be added for free again in this case of $z = 1$ and $\delta = 1/2$ as before.

4.4 Gauge fixing and residual symmetries

So far we have been using both Carroll diffeomorphisms and Carroll Weyl symmetries to constrain our theories. However, our main interest is in getting down to Carrollian CFTs. For this, we need to fix the background spacetime geometry as much as possible using the local symmetries. In the standard construction of CFTs in the background of a (pseudo) Riemannian manifold one takes the background metric and then uses the local symmetries in $(d+1)$-dimensional case ($(d+1)$ diffeomorphisms and 1 Weyl transformation) to fix $(d+2)$ components of the metric completely. Furthermore, if one wants to get a conformal field theory with symmetry algebra $so(2,d+1)$, then the background should be conformally flat. This condition requires that the metric $g_{\mu\nu}$ should have vanishing Weyl tensor: $W_{\mu\nu\sigma\lambda}=0$ (or the Cotton tensor $C_{\mu\nu}=0$ in $d=2$ case).

Turning now to the Carroll case we again have in a generic geometry, specified by (a_{ij},b_i,ω), as many as $\frac{1}{2}(d+1)(d+2)$ components which are all arbitrary functions of $(t,\mathbf{x})$. But the local symmetries have two functions $(t'(t,\mathbf{x}),\ B(t,\mathbf{x}))$ of both space and time and d functions $(x'^i(\mathbf{x}))$ of space alone.

We are looking for 3-dimensional Carrollian CFTs with symmetry algebra $\mathfrak{cca}_3^{(z)}$ that contains the conformal algebra of the two-dimensional base space $\mathfrak{so}(1,3)$ – so we would like to be able to completely gauge fix a_{ij} to some fixed time-independent metric. One possibility is that we restrict to Carroll geometries where a_{ij} is of the form:

$$a_{ij}(t,\mathbf{x})=e^{\chi(t,\mathbf{x})}a^{(0)}_{ij}(\mathbf{x}),\qquad \det a^{(0)}_{ij}(\mathbf{x})=\text{fixed}. \tag{4.62}$$

Note that the choice of two-dimensional metric in (4.62) is the same as the choice of boundary metric on codimension-two surface Σ_2 in Newman-Unti gauge considered in Chapter 3 for asymptotically locally flat solutions. In other words the metric a_{ij} is conformally time-independent. Then we will have two components in $a^{(0)}_{ij}(\mathbf{x})$ which can be gauge fixed completely using the spatial diffeomorphisms alone. We can use the temporal diffeomorphism and the Weyl symmetry to gauge fix $\chi(t,\mathbf{x})=0$ and $\omega(t,\mathbf{x})=1$. It turns out (as shown later on in this section) that this will be sufficient to ensure that the residual symmetry algebra is $\mathfrak{cca}_3^{(z)}$.[4]

Some examples of Carroll spacetimes include: Flat Carroll spacetime [70, 69, 68, 27]

[4]For the choice of metric in (4.62), the complete set of Carrollian analogue of $3d$ Cotton tensor do not go to zero as shown in [81] and one has to further impose some notion of conformal flatness on the one-dimensional fibre of Carrollian manifold $\mathcal{C}_{2+1}$.

given by

$$a_{ij} = \delta_{ij}, \quad b_i = b_i^{(0)}, \quad \omega = 1 \tag{4.63}$$

where $b_i^{(0)}$ are constants. For this we have $\theta = \phi_i = \hat{\gamma}^i_j = f_{ij} = \hat{\gamma}^i_{jk} = 0$. The *time-like* action (4.51) is what people obtained by the ultra-relativistic limit of the Klein-Gordon scalar in 3-dimensional Minkowski spacetime M_{2+1}. The *space-like* action (4.53) becomes, for $z = 2\delta$

$$\int dt\, d^2\mathbf{x}\, \left[\hat{\partial}_i\Phi\hat{\partial}_i\Phi + \lambda\,\Phi^{2+\frac{2}{\delta}}\right] . \tag{4.64}$$

More general Carroll manifolds that include some of the interesting Carroll spacetimes, such as null infinities, black hole horizons etc, and their conformal symmetries are discussed in detail in [27]. To study the symmetries of the gauge fixed actions one takes the vector field that generates Carroll diffeomorphism to be of the form:

$$\begin{aligned} \xi &= f(t,\mathbf{x})\,\hat{\partial}_t + \xi^i(\mathbf{x})\,\hat{\partial}_i \\ &= \omega^{-1}\,(f + \xi^i\, b_i)\,\partial_t + \xi^i\partial_i\,. \end{aligned} \tag{4.65}$$

Under the infinitesimal coordinate transformation

$$t' = t + \omega^{-1}\,(f + \xi^i\, b_i) + \cdots\,, \quad x'^i = x^i + \xi^i + \cdots \tag{4.66}$$

the background data (a_{ij}, b_i, ω) and matter field Φ transform as:

$$\begin{aligned} \delta_\xi a_{ij} &= -2f\,\hat{\gamma}_{ij} - (\hat{\nabla}_i\xi_j + \hat{\nabla}_j\xi_i)\,, \\ \omega^{-1}\,\delta_\xi\omega &= -\hat{\partial}_t f - \phi_i\,\xi^i\,, \\ \delta_\xi b_i &= -b_i\,(\phi_j\xi^j + \hat{\partial}_t f) + f_{ij}\,\xi^j + (\hat{\partial}_i - \phi_i)\,f\,, \\ \delta_\xi\Phi &= -(f\,\hat{\partial}_t\Phi + \xi^i\hat{\partial}_i\Phi)\,. \end{aligned} \tag{4.67}$$

Under the infinitesimal Weyl transformations with $B = e^\sigma$, we have

$$\begin{aligned} \delta_\sigma a_{ij} &= -2\,\sigma\,a_{ij}, \quad \delta_\sigma\omega = -z\,\sigma\,\omega, \\ \delta_\sigma b_i &= -z\,\sigma\,b_i, \quad \delta_\sigma\Phi = \delta\,\sigma\,\Phi\,. \end{aligned} \tag{4.68}$$

One first demands that the metric on the base a_{ij} is invariant under the combined action (4.67, 4.68):

$$(\delta_\xi + \delta_\sigma)\, a_{ij} = 0 \tag{4.69}$$

and this leads to

$$2\, f\left[\hat{\gamma}_{ij} - \frac{1}{2}\theta\, a_{ij}\right] + \hat{\nabla}_i \xi_j + \hat{\nabla}_j \xi_i - \hat{\nabla}_k \xi^k\, a_{ij} = 0, \quad \sigma = -\frac{1}{2}(f\,\theta + \hat{\nabla}_i \xi^i)\,. \tag{4.70}$$

Now following [27] we will also choose to impose that the traceless symmetric tensor $\zeta_{ij} = \hat{\gamma}_{ij} - \frac{1}{2}\theta\, a_{ij}$ (referred to as the Carroll shear) vanishes. This condition can be solved for and it implies (4.62). Next we choose to impose $(\delta_\xi + \delta_\sigma)\,\omega = 0$ and this leads to

$$(\hat{\partial}_t - \frac{z}{2}\theta)\, f - \frac{z}{2}(\hat{\nabla}_i - \frac{2}{z}\phi_i)\, \xi^i = 0\,,$$

$$\text{along with} \qquad \hat{\nabla}_i \xi_j + \hat{\nabla}_j \xi_i - \hat{\nabla}_k \xi^k\, a_{ij} = 0\,. \tag{4.71}$$

Finally noticing that these equations are Carroll Weyl invariant one chooses a_{ij} to be a completely time-independent fixed metric. This implies that $\theta = 0$ and the Carroll Levi-Civita connection $\hat{\gamma}^i_{jk}$ reduces to the Christoffel connection for (now time independent) a_{ij}. One also chooses a fixed background value for ω (say, $\omega = 1$) without loss of generality. Then the residual symmetries have to satisfy

$$\begin{aligned} &\nabla_i \xi_j + \nabla_j \xi_i - \nabla_k \xi^k\, a_{ij} = 0\,, \\ &\partial_t\, f - \frac{z}{2}(\nabla_i - \frac{2}{z}\phi_i)\, \xi^i = 0\,. \end{aligned} \tag{4.72}$$

One can integrate these equations completely and the result is given in terms of $\{T(\mathbf{x}), Y^i(\mathbf{x}\}$ where $\xi^i = Y^i(\mathbf{x})$ are the conformal Killing vectors on a_{ij} and $T(\mathbf{x})$ is arbitrary:

$$\begin{aligned} f(t,\mathbf{x}) &= T(\mathbf{x}) + \frac{z}{2}\int^t dt' \left[\nabla_i Y^i(\mathbf{x}) - \frac{2}{z}\phi_i\, Y^i(\mathbf{x})\right] \\ &= T(\mathbf{x}) + \frac{z}{2} t\, \nabla_i Y^i(\mathbf{x}) - b_i(t,\mathbf{x}) Y^i(\mathbf{x}) \end{aligned} \tag{4.73}$$

where we have set $\omega(t,\mathbf{x}) = 1$ and $\phi_i = \partial_t b_i$. So the Carrollian conformal Killing vector is

$$\xi = (f + \xi^i\, b_i)\,\partial_t + \xi^i \partial_i$$
$$= \left[T(\mathbf{x}) + \frac{z}{2} t\, \nabla_i Y^i(\mathbf{x})\right]\, \partial_t + Y^i(\mathbf{x})\,\partial_i\,. \tag{4.74}$$

It is argued in [27] that the algebra of the Carrollian CKV does not depend on the choice of ω either. The fact that the b_i dependence is cancelled out in the final answer (4.74) is also general enough that we do not need to fix b_i to define the residual symmetries. Therefore the residual fields $b_i(t, \mathbf{x})$ along with Φ transform under the Carrollian conformal transformations as:

$$\delta b_i = -b_i\,(\phi_j \xi^j + \hat{\partial}_t f) + f_{ij}\,\xi^j + (\hat{\partial}_i - \phi_i)\, f - z\,\sigma\, b_i\,,$$
$$\delta\Phi = -(f\,\hat{\partial}_t \Phi + \xi^i \hat{\partial}_i \Phi) + \delta\,\sigma\,\Phi\,, \tag{4.75}$$

where we have to use a_{ij} to be fixed and time-independent (for instance that of a round S^2), $\omega = 1$ (and hence $\phi_i = \partial_t b_i$), $\xi^i = Y^i(\mathbf{x})$ and $f(t, \mathbf{x})$ as in (4.73). These form the symmetries of our theories by construction when we gauge fix as above. This means that our $\mathfrak{cca}_3^{(z)}$ symmetric scalar field theories, after gauge fixing the Carroll diffeomorphisms and Weyl symmetries would have the matter field $\Phi(t, \mathbf{x})$ along with the geometric fields $b_i(t, \mathbf{x})$ dynamical. Then the transformations (4.75) are their symmetries and can be used to study the conserved currents and charges *etc.* of our theories both at the classical level and beyond.

There are other interesting possibilities to gauge fix the local symmetries with apparently different residual symmetry algebras - we discuss one such gauge now. Let us consider restricting the background Carroll geometric data (a_{ij}, b_i, ω) by imposing $\phi_i = 0$. This condition can be solved as follows:

$$\frac{1}{\omega}(\partial_i \omega + \partial_t b_i) = 0 \implies b_i(t, \mathbf{x}) = b_i^{(0)}(\mathbf{x}) - \partial_i \int^t dt'\, \omega(t', \mathbf{x}) \tag{4.76}$$

If we further gauge fix $\omega = 1$ as in the previous gauge choice we conclude that $\phi_i = 0$ implies that b_i is independent of time. So we can use the spatial Carroll diffeomorphisms to gauge fix $b_i = 0$.[5] Finally, we can use the Weyl symmetry to fix the determinant of a_{ij}, say $a = 1$, which in turn will imply $\theta = 0$. So in this gauge we have $\phi_i = \theta = b_i = f_{ij} = 0$. Using $(\delta_\xi + \delta_\sigma)\, b_i = 0$ leads to $\partial_i f = 0$. From $(\delta_\xi + \delta_\sigma)\,\omega = 0$ we find $\sigma = -\frac{1}{z}\partial_t f(t)$. This gauge leaves $a_{ij}(t, \mathbf{x})$ to fluctuate subject to the condition $a = 1$ along with $\Phi(t, \mathbf{x})$. The condition $a = 1$ also implies

[5] A weaker condition is to fix b_i to be such that $f_{ij} = 0$.

$\sigma = -\frac{1}{2}\nabla_i\xi^i$. Since $\sigma \sim \partial_t f(t)$ the only consistent choices are $f(t) = \alpha + \beta\, t$ and $\nabla_i\xi^i = \frac{2}{z}\beta$, for constant α and β. One may further choose to set $\beta = 0$ (though this is not necessary in all the cases) as in [15]. Then one has $\xi = \alpha\,\partial_t + \xi^i\partial_i$ with constant α and $\nabla_i\xi^i = 0$, giving rise to an algebra isomorphic to $\mathbb{R}\oplus\mathcal{A}$ where $\mathcal{A}$ is the algebra of volume-preserving (smooth) diffeomorphisms of a_{ij} (say $\mathbb{R}^2$ or round S^2). In the $z = 1$ case, such an algebra has appeared recently [15] in a different context.

If we allow for partial gauge fixing of the local symmetries of our theory and only gauge fix the determinant of a_{ij} to 1 and further gauge fix $\omega = 1$ like in the previous paragraph with no further conditions on b_i's then in this case from $(\delta_\xi + \delta_\sigma)\omega = 0$, one obtains,

$$-\partial_t f - \phi_i\xi^i - z\,\sigma = 0 \tag{4.77}$$

and the determinant condition $a^{ij}\,(\delta_\xi + \delta_\sigma)\,a_{ij} = 0$ leads to,

$$\sigma = -\frac{1}{2}(f\,\theta + \hat{\nabla}_i\xi^i) \tag{4.78}$$

As we are considering geometry with vanishing Carroll shear, the only time dependence appearing in the metric is the conformal factor. Using Weyl symmetry to fix the determinant of the metric implies that the metric is now time independent and therefore one can set $\partial_t a_{ij} = 0$, which means that the Carroll Levi-Civita connection reduces to the Christoffel connection. Solving for f in (4.4) one obtains,

$$\begin{aligned} f &= T(x) + \frac{z}{2}\int^t dt'\left[\nabla_i\xi^i - \frac{2}{z}\,\phi_i\xi^i\right] \\ &= T(x) + \frac{z}{2}\,t\,\nabla_i\xi^i - b_i\xi^i \end{aligned} \tag{4.79}$$

This is similar to (4.73) except that the CKV condition on ξ^i is dropped. So the resultant vector field that generates Carroll diffeomorphisms for this particular partial gauge fixing is of the form,

$$\xi = \left(T(x) + \frac{z}{2}t\nabla_i\xi^i\right)\partial_t + \xi^i\,\partial_i \tag{4.80}$$

with no further condition on ξ^i. For $z = 1$, this vector field is identical to the boundary vector field that forms the representation of generalised $\mathfrak{bms}_4$ symmetry algebra [47, 153] where $T(x)$ is the supertranslations and ξ^i are vector fields associated with arbitrary diffeomorphisms on S^2, namely, Diff($\mathbb{S}^2$). Therefore we see that to obtain generalised $\mathfrak{bms}_4$ symmetry algebra from the residual gauge transformations of Carrollian diffeomorphisms one has to resort to partial gauge fixing of background geometric quantities because any attempt to gauge fix b_i's will eventually reduce this symmetry algebra to $\mathbb{R} \oplus \mathcal{A}$. We leave the determination of the exact nature of field theories having generalised $\mathfrak{bms}_4$ global symmetries to future endeavours.

4.5 Further Comments and Discussions

We presented equations of motion for a scalar field coupled to a generic 3-dimensional Carroll geometry that are Carroll diffeomorphic and Weyl covariant. Even though we have demonstrated our methods most explicitly for 3-dimensional Carroll space-times the generalisation to arbitrary (higher) dimensions is straightforward – see the Appendix C for some technical details of this exercise. Also one can extend our construction following similar methods to include Carroll like Maxwell, Yang-Mills fields and other matter fields.[6]

We have concentrated on classical theories in this chapter. It will be interesting to explore the quantum aspects of our theories – including renormalisation, anomalies etc. We need to compute the Noether charges for the symmetries and show that they form $\mathfrak{cca}_3^{(z)}$ [19]. In particular, we will need to construct the soft charges – which may lead to Ward identities of celestial amplitudes in connection with the soft graviton theorems (see [134, 53, 6]). See [7, 88, 154, 87] that explore the connection between Carrollian CFTs and Celestial CFT.

There are some features expected of holographic dual of flat space gravity [55, 56, 88, 154]. It will be interesting to see if these are borne out in theories of the type we constructed here. In this connection, we anticipate that the existence of additional fields, either $b_i(t, \mathbf{x})$ (or $a_{ij}(t, \mathbf{x})$ with $a = 1$) may play a useful role. For example, the emergence of chiral $\mathfrak{bms}_4$ from this set-up would be extremely interesting to study and constitute one of the future directions. One also expects such requirements to impose further cuts on the spaces of classical theories we find, along with any other possible quantum consistency conditions.

In this work, we concentrated on Carrollian theories. However, the same techniques can be used for constructing the Galilean theories as well. There is a curious duality between Carrollian and Galilean field theories [70]. It would be interesting to check

[6] See [81] for the construction of the Carrollian Chern Simons gravity actions and equations of motion from the 3d Chern Simons gravity using the technique we provide in (4.3).

if such duality exists between our theories and the corresponding Galilean ones.

Eventually one would like to construct fully consistent (perhaps supersymmetric) Carrollian CFTs which can potentially be useful to describe flat space gravity/string theories holographically. We hope that our methods will lead to more avenues to explore than the method of taking ultra-relativistic limits of known theories.

Chapter 5

A chiral $\mathcal{W}$-algebra extension from AdS$_4$ gravity

One of the main results of the first part of the thesis was the realisation of chiral $\mathfrak{bms}_4$ algebra, uncovered in celestial CFT [13, 14] from the bulk asymptotic symmetries of $\mathbb{R}^{1,3}$ gravity. In this part of the thesis motivated by the existence of chiral $\mathfrak{bms}_4$ algebra defined in (3.1), we look for the chiral extensions of $\mathfrak{so}(2,3)$ that play a role in AdS$_4$ gravity. Until recently it was considered that the symmetry algebra of asymptotically AdS$_{d+1}$ spacetimes for $d \geq 3$ is finite-dimensional $\mathfrak{so}(2,d)$ for a class of boundary conditions [155, 95].[1] As mentioned in the introduction, in [15, 16], the authors proposed new boundary conditions for generic asymptotically AdS$_4$ spacetimes in Bondi gauge, where they uncover the symmetry algebra of residual gauge transformations to be an infinite dimensional Lie algebroid,[2] the Λ-$\mathfrak{bms}_4$. Specifically, they fixed only part of the boundary metric using the residual symmetries. The Λ-$\mathfrak{bms}_4$ and its corresponding phase space in the flat space limit ($\Lambda \to 0$) coincide with the generalised $\mathfrak{bms}_4$ algebra and the phase space associated with it. To impose a well-defined variational principle, they further set part of the holographic stress tensor to zero giving rise to reduced symmetry algebra which is a direct sum of $\mathbb{R} \oplus \mathcal{A}$, where $\mathbb{R}$ denotes the abelian time translations and $\mathcal{A}$ is the algebra of 2-dimensional area-preserving diffeomorphisms.[3]

In this chapter, we answer the following two very natural questions that arise due

[1] Special cases of infinite dimensional enhancement of symmetry algebra in the case of AdS$_4$ were considered in [156, 157].

[2] In Lie algebroid, the structure constants are background field dependent [15].

[3] Note that such a symmetry algebra was also uncovered from the Carrollian CFT in Chapter 4.

to the existence of chiral $\mathfrak{bms}_4$ in celestial CFT and its realisation from flat space $\mathbb{R}^{1,3}$ gravity,

- Is there a chiral extension of $\mathfrak{so}(2,3)$ algebra that in some appropriate flat space limit reduces to the chiral $\mathfrak{bms}_4$?
- Can one obtain such a chiral version of Λ-$\mathfrak{bms}_4$ algebra from the AdS$_4$ gravity?

To obtain such a chiral extension of $\mathfrak{so}(2,3)$ from AdS$_4$ gravity, inspired by the chiral boundary conditions that we used in Chapter 3, we propose chiral boundary conditions for solutions in Newman-Unti gauge in AdS$_4$ gravity.

The boundary conditions that we employ in this chapter form the subset of Neumann boundary conditions [99]. We consider locally AdS$_4$ solutions for which the components associated with the holographic stress tensor go to zero and the boundary metric is conformally flat [105].[4] Because the boundary metric is conformally flat, these solutions are referred to as asymptotically AdS$_4$ solutions in the literature [95, 158]. We obtain an infinite-dimensional algebra of asymptotic symmetries associated with these solutions after imposing chiral boundary conditions consistent with a well-defined variational principle. We denote this algebra by chiral Λ-$\mathfrak{bms}_4$. Furthermore, the resultant chiral locally AdS$_4$ solutions that we obtained are in a specific form such that in the flat space limit ($l \to \infty$, where l is the AdS length), one obtains chiral locally flat solutions derived in Chapter 3. The chiral Λ-$\mathfrak{bms}_4$ is a Lie algebroid and in the flat space limit it reduces to the chiral $\mathfrak{bms}_4$ algebra (3.1). The following are the results of this chapter:

- We obtain locally AdS$_4$ solutions parameterized by six holomorphic functions $\{J_a(z), T(z), G_s(z)\}\quad a \in \{0, \pm 1\}, s \in \{\pm\frac{1}{2}\}$ with the algebra of the residual diffeomorphisms being a Lie algebroid. These diffeomorphisms induce nontrivial transformations on the holomorphic functions similar to currents in a $2d$ CFT.
- We postulate a line integral charge as that of $2d$ chiral CFT and derive the operator product expansions (OPEs) between the currents (of the previous paragraph) such that they produce the same variations of these currents. These OPEs (and the mode commutation relations obtained from them) give us the semi-classical limit of a $\mathcal{W}$-algebra.

[4]The necessary and sufficient condition for $d = 3$ boundary metric to be conformally flat is the vanishing Cotton tensor.

- We derive the quantum version of the aforementioned $\mathcal{W}$-algebra in a $2d$ CFT by imposing Jacobi identities on the commutators of the modes of the chiral (quasi-) primaries, a method developed by Nahm et al. ([25] and references therein) as an equivalent implementation of constraints from OPE associativity [159].[5] We denote this algebra by $\mathcal{W}(2;(3/2)^2,1^3)$ and show that in the semi-classical limit with proper identification of parameters, this algebra matches with the chiral $\mathcal{W}$-algebra symmetry of AdS$_4$ gravity alluded to above.

- The full commutator algebra of $\mathcal{W}(2;(3/2)^2,1^3)$ that we derive in this chapter is given as follows,

$$
\begin{aligned}
&[L_m, L_n] = (m-n)\,L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\,,\\
&[L_m, J_{a,n}] = -n\,J_{a,m+n}, \quad [L_m, G_{s,r}] = \frac{1}{2}(n-2r)\,G_{s,n+r}\,,\\
&[J_{a,m}, J_{b,n}] = -\frac{1}{2}\kappa\, m\,\eta_{ab}\,\delta_{m+n,0} + f_{ab}{}^{c}\,J_{c,m+n}, \quad [J_{a,n}, G_{s,r}] = G_{s',n+r}(\lambda_a)^{s'}{}_{s}\,,\\
&[G_{s,r},\, G_{s',r'}] = \epsilon_{ss'}\Big[\alpha\Big(r^2-\frac{1}{4}\Big)\delta_{r+r',0} + \beta\, L_{r+r'} + \gamma\,(J^2)_{r+r'}\Big] + \delta(r-r')J_{a,r+r'}(\lambda^a)_{ss'}
\end{aligned}
$$

$$
\text{with}\;\; c = \frac{6\kappa\,(1+2\kappa)}{5+2\kappa},\;\; \alpha = -\frac{1}{4}\gamma\kappa(3+2\kappa),\;\; \beta = \frac{1}{4}\gamma(5+2\kappa),\;\; \delta = -\frac{1}{2}\gamma(3+2\kappa) \tag{5.1}
$$

where $\eta_{ab} = (3a^2-1)\delta_{a+b,0}$, $f_{ab}{}^{c} = (a-b)\delta^{c}_{a+b}$, $(\lambda^a)_{ss'} = \frac{1}{2}\delta^{a}_{s+s'}$, $(\lambda_a)_{ss'} = \eta_{ab}(\lambda^b)_{ss'}$, $\kappa \neq -5/2$, and γ can be fixed to be any non-zero function of κ by rescaling the $G_{s,r}$ appropriately. Finally $(J^2)_n$ are the modes of the normal ordered quasi-primary $\eta^{ab}(J_aJ_b)(z)$. This algebra can be identified with the $N=1$ case of an infinite series of algebra defined for $2N$ bosonic spin-$\frac{3}{2}$ currents and spin-1 Kac-Moody current for $\mathfrak{sp}(2N)$ derived by Romans in a different context [24]..

- We also write down all the 3 and 4-point functions of the currents in operator product algebra (5.1) that can be shown to admit all the expected crossing symmetries.

The rest of this chapter is organised as follows: In Section (5.1) we present the calculations in locally AdS$_4$ solution and show the emergence of chiral Λ- $\mathfrak{bms}_4$ as the asymptotic symmetry algebra after imposing chiral boundary conditions. We then show that the algebra of charges that generate this symmetry algebra is a type

[5]See [160], [161] and also the comprehensive review [162] of everything to do with OPEs.

of $\mathcal{W}$-algebra. In Section (5.2) we derive the quantum version of this $\mathcal{W}$-algebra from $2d$ CFT techniques. We implement the OPE associativity of operator product algebra through the Jacobi Identities of modes of symmetry-generating currents of this $\mathcal{W}$-algebra. Section (5.3) contains the details of the 3 and 4-point functions of the symmetry currents of our $\mathcal{W}$-algebra. We conclude in Section (5.4). In Appendix D, we provide the coordinate transformations that allow us to write the chiral locally AdS_4 solutions in Fefferman-Graham gauge and Appendix F provide some details on the imposition of the associativity constraints on 3 and 4-point correlation functions of symmetry generating currents. This chapter is partly based on [163]

5.1 Locally AdS_4 Solutions

To obtain a candidate chiral Λ-$\mathfrak{bms}_4$ algebra from AdS_4 gravity, it is sufficient to analyse solutions to Einstien equations that are locally AdS_4 ($LAdS_4$). Therefore we consider $LAdS_4$ geometries, which in addition to being the solution to the Einstien equation for negative cosmological constant in four dimensions,

$$E_{\mu\nu} = R_{\mu\nu} + \frac{3}{l^2}\, g_{\mu\nu} = 0 \tag{5.2}$$

also satisfies locally AdS_4 condition given by

$$A_{\mu\nu\sigma\lambda} = R_{\mu\nu\sigma\lambda} + \frac{1}{l^2}\left(g_{\mu\sigma}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\sigma}\right) = 0. \tag{5.3}$$

Equation (5.3) implies equation of motion (5.2). Taking the flat space limit $l \to \infty$ of these two equations we obtain the conditions for locally flat (LF) geometries that were used in section (3.1.1). To obtain solutions satisfying the conditions (5.2) and (5.3), we work in Newman-Unti (NU) gauge with coordinates $(u, r, z, \bar{z})$. We prefer this gauge over Fefferman-Graham (FG) gauge because it is straightforward to compare $LAdS_4$ solution with the locally flat solutions of (3.72) and (3.73) in the limit ($l \to \infty$). Using coordinate transformations one can map solutions in the NU gauge to the solutions in the FG gauge as shown in the Appendix D. Recall that this gauge is defined by imposing the following conditions on the metric $g_{\mu\nu}$,

$$g_{rr} = g_{rz} = g_{r\bar{z}} = 0, \quad g_{ur} = -1. \tag{5.4}$$

The line element is then given by,

$$ds^2_{lAdS} = -2\,du\,dr + g_{ij}(r, x^k)\,dx^i\,dx^j, \tag{5.5}$$

where r is the radial coordinate and i labels the boundary directions $(u, z, \bar{z})$. The metric components g_{ij} for $i, j \in \{u, z, \bar{z}\}$ are expanded in the following form:

$$g_{ij}(r, u, z, \bar{z}) = \sum_{n=0}^{\infty} r^{2-n}\, g^{(n)}_{ij}(u, z, \bar{z}). \tag{5.6}$$

Imposing the condition (5.3) one finds that the solution terminates at order $\mathcal{O}(1)$ in power series expansion in r such that for LAdS_4 solutions in $4d$ we have,

$$g_{ij}(r, u, z, \bar{z}) = \sum_{n=0}^{2} r^{2-n}\, g^{(n)}_{ij}(u, z, \bar{z}). \tag{5.7}$$

We begin by solving Einstein equations $E_{\mu\nu} = 0$ for the solution ansatz (5.5) in NU gauge at each order in r near the boundary $(r \to \infty)$. From $\mathcal{O}(r^2)$ term of the component E_{uu} one obtains the condition,

$$Det\,(g^{(0)}_{ij}) = -\frac{1}{l^2}\, Det\,(g^{(0)}_{ab}), \tag{5.8}$$

where $a \in \{z, \bar{z}\}$ labels the index on codimension-two hypersurface. This condition is solved by taking $g^{(0)}_{uu}$ to be of the form,

$$g^{(0)}_{uu} = -\frac{1}{l^2} + v_a\, v^a, \tag{5.9}$$

where we define $v_a \coloneqq g^{(0)}_{ua}$ in this chapter and the indices $\{a, b\}$ are lowered and raised by boundary metric $g^{(0)}_{ab}$ and $g^{ab}_{(0)}$ respectively. The algebraic condition at $\mathcal{O}\left(\frac{1}{r}\right)$ in E_{ra} gives the expression for $g^{(1)}_{ua}$ as follows,

$$g^{(1)}_{ua} = g^{(1)}_{ab}\, v^a. \tag{5.10}$$

Similarly the order $\mathcal{O}\left(\frac{1}{r}\right)$ term of E_{ur} and E_{ra} can be solved respectively to obtain $g^{(1)}_{uu}$ and $g^{(2)}_{ua}$,

$$g^{(1)}_{uu} = -\frac{\theta}{2} + D_a v^a - 2\, g^{(1)}_{ab}\, v^a\, v^b + 3\, v^a g^{(1)}_{ua} - \frac{1}{2l^2}\, g^{ab}_{(0)}\, g^{(1)}_{ab}, \tag{5.11}$$

$$g^{(2)}_{ua} = \frac{1}{2} g^{(2)}_{ab} v^b + \frac{1}{2}\, g^{cb}_{(0)}\, D_c g^{(1)}_{ab} - \frac{1}{2}\, g^{cb}_{(0)}\, D_a\, g^{(1)}_{cb}, \tag{5.12}$$

where $\theta = g^{ab}_{(0)}\,\partial_u g^{(0)}_{ab}$ and $D_a v^a$ is the covariant derivative defined with respect to the $2d$ boundary metric $g^{(0)}_{ab}$. At $\mathcal{O}\,(r)$ in E_{ab} we obtain the following condition that will be used to solve for $g^{(1)}_{ab}$,

$$\frac{1}{l^2} g^{(1)}_{ab} - \left[D_a\, v_b + D_b\, v_a - D_c v^c\, g^{(0)}_{ab} + \frac{1}{2}\,\theta g^{(0)}_{ab} - \partial_u g^{(0)}_{ab} \right] - \frac{1}{2\,l^2}\, g^{cd}_{(0)}\, g^{(1)}_{cd}\, g^{(0)}_{ab} = 0. \tag{5.13}$$

The $\mathcal{O}(1)$ term in E_{ab} gives us the following condition,

$$\begin{aligned} g^{(0)}_{ab} \Bigg[g^{(2)}_{uu} &+ \frac{R_0}{2} - \frac{1}{4\,l^2}\, g^{(1)}_{cd}\, g^{cd}_{(1)} + \frac{1}{2} g^{cd}_{(0)}\, \partial_u g^{(1)}_{cd} - \frac{1}{4}\, g^{ab}_{(0)}\, g^{(1)}_{ab}\, \theta - \frac{1}{2}\, g^{cd}_{(0)}\, v^e\, D_e\, g^{(1)}_{cd} \\ &+ v^c\, v^d\, g^{(2)}_{cd} + \frac{1}{l^2}\, g^{ab}_{(0)}\, g^{(2)}_{ab} - 2\, g^{(2)}_{uc}\, v^c + \frac{1}{2}\, g^{ab}_{(0)}\, g^{(1)}_{ab}\, D_c v^c - g^{cd}_{(1)} D_c v_d \Bigg] \\ &+ \frac{1}{4}\, Tr\, g^{ab}_{(0)}\, g^{(1)}_{ab}\, \partial_u g^{(0)}_{ab} + \frac{1}{l^2} g^{(2)}_{ab} - \frac{1}{2}\, g^{ab}_{(0)}\, g^{(1)}_{ab}\, D_{(a}\, v_{b)} = 0 \end{aligned} \tag{5.14}$$

where R_0 is the Ricci scalar of metric $g^{(0)}_{ab}$. Taking the trace of this equation allows us to solve for $g^{(2)}_{uu}$,

$$\begin{aligned} g^{(2)}_{uu} = &-\frac{R^{(0)}}{2} + \frac{1}{4\,l^2}\, g^{(1)}_{ab} g^{ab}_{(1)} - \frac{1}{2}\, g^{ab}_{(0)}\, \partial_u g^{(1)}_{ab} + \frac{1}{8}\, g^{ab}_{(0)}\, g^{(1)}_{ab}\, \theta + \frac{1}{2}\, g^{ab}_{(0)}\, v^e\, D_e\, g^{(1)}_{ab} - v^a\, v^b\, g^{(2)}_{ab} \\ &- \frac{3}{2\,l^2}\, g^{ab}_{(0)}\, g^{(2)}_{ab} + 2\, g^{(2)}_{ua}\, v^a - \frac{1}{4}\, g^{cd}_{(0)}\, g^{(1)}_{cd}\, D_a\, v^a + g^{ab}_{(1)}\, D_a v_b. \end{aligned} \tag{5.15}$$

In order to obtain $g^{(2)}_{ab}$, we impose locally AdS condition of (5.3) which is easier to implement in the form

$$\det(g^{(4d)})\, A_{\mu\nu\rho\sigma} = 0, \tag{5.16}$$

as the series expansion in r is a positive power with highest order 8. $\det(g^{(4d)})$ is the determinant of full $4d$ metric in NU gauge. The $\mathcal{O}(r^2)$ term of component $\det(g^{(4d)})\, A_{urur}$ gives $g^{(2)}_{ab}$ as follows,

$$g^{(2)}_{ab} = \frac{1}{4} g^{(1)}_{ac}\, g^{cd}_{(0)}\, g^{(1)}_{db}\,. \tag{5.17}$$

The pullback of the metric in the Newman-Unti gauge at the boundary $\mathcal{I}$ at $r \to \infty$ is given by,

$$ds^2_{bdry} = \left(-\frac{1}{l^2} + v_a v^a \right) du^2 + 2\, v_a\, du\, dx^a + g^{(0)}_{ab}\, dx^a\, dx^b. \tag{5.18}$$

Some comments about our solution are in order,

1. In the flat space limit the condition (5.8) implies that g_{ij} is degenerate and another way to solve it is to set $g^{(0)}_{uu} = v_a = 0$, as we did in (3.4).

2. The condition in equation (5.13) implies that $g^{(1)}_{ab}$ is not free data on the boundary and is given in terms of v_a and $g^{(0)}_{ab}$. This is in contrast to the locally flat solution where $g^{(1)}_{ab}$ is free data at the boundary null infinity. This can be seen from taking the flat limit of (5.13) where the term proportional to $g^{(1)}_{ab}$ drops out and after putting $v_a = 0$ one is left with the condition constraining the time dependence of the boundary metric $g^{(0)}_{ab}$. For a locally flat solution, this equation implied that the boundary metric is conformally time-dependent, i.e, ($g^{(0)}_{ab} = \Omega(u, z, \bar{z})\, q_{ab}(z, \bar{z})$) where we choose Ω to be u-independent such that $\theta = 0$. For LAdS$_4$ solutions we do not have any such restriction on $g^{(0)}_{ab}$.

3. We further set $g^{ab}_{(0)} g^{(1)}_{ab} = 0$ as part of our gauge condition. This is standard practice for solutions in NU and Bondi gauge and is equivalent to the condition of fixing the origin of an affine parameter of the null geodesic [164]. The final metric components are then given by,

$$
\begin{aligned}
g^{(1)}_{uu} &= -\frac{\theta}{2} + D_a v^a + g^{(1)}_{ab}\, v^a\, v^b, \\
g^{(2)}_{ua} &= \frac{1}{2} g^{(2)}_{ab} v^b + \frac{1}{2}\, g^{cb}_{(0)}\, D_c g^{(1)}_{ab}, \\
g^{(1)}_{ab} &= l^2 \left[D_a\, v_b + D_b\, v_a - D_c v^c\, g^{(0)}_{ab} + \frac{1}{2}\, \theta g^{(0)}_{ab} - \partial_u g^{(0)}_{ab} \right], \\
g^{(2)}_{uu} &= -\frac{R_0}{2} - \frac{1}{8\, l^2}\, g^{(1)}_{ab} g^{ab}_{(1)} - \frac{1}{2}\, g^{ab}_{(0)}\, \partial_u g^{(1)}_{ab} + g^{bc}_{(0)}\, v^a\, D_c\, g^{(1)}_{ab} + g^{ab}_{(1)}\, D_a v_b
\end{aligned}
\tag{5.19}
$$

   along with equations (5.9) and (5.10).

4. One can check that for $v_a = 0$ and $\theta = 0$ in the limit $l \to \infty$ the metric components coincide with those of locally flat solution in (3.12) except that $g^{(1)}_{ab}$ is free data.

For LAdS$_4$ solution the boundary metric $g^{(0)}_{ab}$ and v_a are the free data on the boundary,[6] in terms of which all other metric components are given. For our solution to completely solve equations (5.2) and (5.3), the Cotton tensor of the boundary metric (5.18) necessarily vanishes following the analysis of [105]. This imposes constraints on boundary data $g^{(0)}_{ab}$ and v_a. We do not provide these constraints nor their general solutions in this chapter as it is not needed for our analysis. We will solve the con-

[6] Note that $\{g^{(0)}_{ab}, v^a\}$ constitute only five independent metric components of $3d$ boundary metric. The sixth component is fixed due to the condition (5.8).

straints only after imposing further boundary conditions on our free data $\{g^{(0)}_{ab}\,, v_a\}$, which we obtain after imposing a well-defined variational principle.

5.1.1 Variational principle

In this section, we will find boundary conditions for the fields in configuration space motivated by the variational principle which will then allow us to completely solve the constraint conditions on the free data. The action for the Einstien equation in $4d$ for non-zero cosmological constant is given by,

$$S = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,(R+\frac{6}{l^2}). \tag{5.20}$$

The variation of this action gives an equation of motion and a boundary term which depends on the normal derivative of the boundary metric. To solve the variational principle one usually adds Gibbons Hawking term and various counter terms such that in the end total variation of the action is only proportional to the boundary component and not its derivative. Since we are working in Newman-Unti gauge we use the boundary terms found in (3.55).[7] The boundary action that we add to the Einstein-Hilbert action is,

$$\begin{aligned} S_{bdry} &= (16\pi G)^{-1}\sqrt{\sigma}\left[2\,V^a\partial_r V_a - \partial_r\omega - \left(\omega\,\sigma^{ab} + V^aV^b - V^cV_c\,\sigma^{ab}\right)\partial_r\sigma_{ab} - \sigma^{ab}\,\partial_u\sigma_{ab}\right] \\ &= \sqrt{\sigma}\,\partial_r(V^aV_a-\omega) + 2\left[(V^aV_a-\omega)\,\partial_r - \partial_u\right]\sqrt{\sigma} \end{aligned} \tag{5.21}$$

and the total divergence term in $\delta S_{EH+bdry}$ is $(16\pi G)^{-1}$ times

$$\partial_a\left[\sqrt{\sigma}\,\left(\delta V^a - V^a\,\sigma_{bc}\,\delta\sigma^{bc}\right)\right] \tag{5.22}$$

that we ignore (this amounts to assuming that the geometry of Σ_2 with coordinates $(z,\bar{z})$ is either compact or, when it is not, the integrand falls off fast enough near its asymptotes). Here $V_a = g_{ua}\,, \omega = g_{uu}\,, \sigma_{ab} = g_{ab}$. Substituting the series expansion of these components near the boundary $\mathcal{I}$, one obtains the following expansion of

[7]Normally the boundary action for AdS_{d+1} gravity in Fefferman-Graham gauge [101, 102] or the gauge used to describe 'boosted black brane' [165] is invariant under d-dimensional boundary preserving diffeomorphisms. We expect the variational problem that gives rise to chiral symmetry algebra in NU gauge to hold for these gauges as well.

the variation of total Lagrangian density after adding boundary action,

$$\delta\mathcal{L}_{EH+bdry} = -2\,r^3\sqrt{g^{(0)}}\,\delta g^{(0)}_{uu} - r^2\sqrt{g^{(0)}}\left[\frac{1}{2}\left(g^{cd}_{(0)}g^{(1)}_{cd}\right)\,\delta g^{(0)}_{uu} + 2\,\delta g^{(1)}_{uu}\right]$$
$$- r\sqrt{g^{(0)}}\Big[2\,\delta g^{(2)}_{uu} + g^{(2)}_{uu}\,g^{ab}_{(0)}\,\delta g^{(0)}_{ab} + \frac{1}{2}\left(g^{cd}_{(0)}g^{(1)}_{cd}\right)\,\delta g^{(1)}_{uu} - 2\,g^{(2)}_{ua}\,g^{ab}_{(0)}\delta g^{(0)}_{ub}$$
$$- \frac{1}{2}\left(g^{ab}_{(0)}\partial_u g^{(1)}_{bc}\,g^{cd}_{(0)}\,\delta g^{(0)}_{da} - \left(g^{cd}_{(0)}\partial_u g^{(1)}_{cd}\right)\left(g^{ab}_{(0)}\,\delta g^{(0)}_{ab}\right)\right)\Big] - \sqrt{g^{(0)}}\,\delta\mathcal{L}_0 + \mathcal{O}(1/r) \tag{5.23}$$

where:

$$\delta\mathcal{L}_0 = \frac{1}{2}\left(g^{cd}_{(0)}g^{(1)}_{cd}\right)\,\delta g^{(2)}_{uu} + \frac{1}{2}g^{(2)}_{uu}\left(g^{ab}_{(1)}\delta g^{(0)}_{ab}\right)$$
$$- \frac{1}{2}\Big[\frac{1}{2}\mathrm{Tr}\,(\mathrm{g}^{-1}_{(0)}\,\mathrm{g}^{(1)})\,\mathrm{Tr}\,(\mathrm{g}^{-1}_{(0)}\mathrm{g}^{(2)}) - \mathrm{Tr}\left(\mathrm{g}^{(1)}\mathrm{g}^{-1}_{(0)}\mathrm{g}^{(2)}\mathrm{g}^{-1}_{(0)}\right)$$
$$- \frac{1}{8}\mathrm{Tr}\,(\mathrm{g}^{-1}_{(0)}\,\mathrm{g}^{(1)})\mathrm{Tr}\left(\mathrm{g}^{(1)}\mathrm{g}^{-1}_{(0)}\mathrm{g}^{(1)}\mathrm{g}^{-1}_{(0)}\right) + \frac{1}{4}\mathrm{Tr}\left(\mathrm{g}^{(1)}\mathrm{g}^{-1}_{(0)}\right)^3\Big]\,\delta\mathrm{g}^{(0)}_{\mathrm{uu}} - \left(\mathrm{D}_{(\mathrm{a}}\mathrm{g}^{(2)}_{\mathrm{b)u}} - \mathrm{D}^{\mathrm{c}}\mathrm{g}^{(2)}_{\mathrm{uc}}\,\mathrm{g}^{(0)}_{\mathrm{ab}}\right)\,\delta\mathrm{g}^{\mathrm{ab}}_{(0)}$$
$$- \frac{1}{4}\mathrm{Tr}(\mathrm{g}^{-1}_{(0)}\,\mathrm{g}^{(1)})\,\left(\partial_{\mathrm{u}}\mathrm{g}^{(1)}_{\mathrm{ab}} - \partial_{\mathrm{u}}(\mathrm{Tr}\mathrm{g}^{(1)})\mathrm{g}^{(0)}_{\mathrm{ab}}\right)\,\delta\mathrm{g}^{\mathrm{ab}}_{(0)} + \frac{1}{2}\left(\partial_{\mathrm{u}}\mathrm{g}^{(2)}_{\mathrm{ab}}\delta\mathrm{g}^{\mathrm{ab}}_{(0)} - \mathrm{Tr}(\partial_{\mathrm{u}}\mathrm{g}^{(2)})\,\left(\mathrm{g}^{(0)}_{\mathrm{ab}}\,\delta\mathrm{g}^{\mathrm{ab}}_{(0)}\right)\right)$$
$$+ g^{(2)}_{ua}\,g^{ab}_{(1)}\delta g^{(0)}_{bu} - 2g^{(2)}_{au}\,g^{ab}_{(0)}\,\delta g^{(1)}_{bu} + g^{(2)}_{uu}\,g^{ab}_{(0)}\delta g^{(1)}_{ab}$$
$$- \frac{1}{2}\left(\mathrm{Tr}\left(\mathrm{g}^{-1}_{(0)}\partial_{\mathrm{u}}\mathrm{g}^{(1)}\mathrm{g}^{-1}_{(0)}\delta\mathrm{g}^{(1)}\right) - \mathrm{Tr}\left(\mathrm{g}^{-1}_{(0)}\partial_{\mathrm{u}}\mathrm{g}^{(1)}\right)\mathrm{Tr}\left(\mathrm{g}^{-1}_{(0)}\delta\mathrm{g}^{(1)}\right)\right) \tag{5.24}$$

with $\mathrm{Tr}\left(g^{-1}_{(0)}\,g^{(2)}\right) = g^{ab}_{(0)}g^{(2)}_{ab}$ and so on. The variation is calculated at $r = r_0$ hypersurface and in the end, we take the limit $r_0 \to \infty$. The terms at $\mathcal{O}\left(\frac{1}{r^n_{(0)}}\right)$ for $n \geq 1$ vanish in this limit and therefore one does not need to consider these terms for our analysis.

Now we impose boundary conditions on the boundary data such that the total variation in (5.23) is zero on the solution space.

1. For the terms at $\mathcal{O}(r^3)$ and $\mathcal{O}(r^2)$ to vanish one has to impose $\delta g^{(0)}_{uu} = \delta g^{(1)}_{uu} = 0$. This condition implies that $v_a = \delta v_a = 0$ for our solution. It also states that $\delta\theta = 0$ which is solved by keeping determinant of the $2d$ spatial boundary metric fixed such that $\delta\sqrt{det\,g^{(0)}_{ab}} = 0$. To impose such a condition we parameterize our boundary metric $g^{(0)}_{ab}$ in chiral Polyakov gauge where we set $g^{(0)}_{\bar{z}\bar{z}} = 0$ and $g^{(0)}_{z\bar{z}} = \Omega(z,\bar{z})$ and $\delta g^{(0)}_{\bar{z}\bar{z}} = \delta g^{(0)}_{z\bar{z}} = 0$. $\Omega(z,\bar{z})$ is the conformal factor of $2d$ metric which is $\frac{1}{2}$ for $\mathbb{R}^2$ and $\frac{4}{(1+z\bar{z})^2}$ for $\mathbb{S}^2$. In rest of the analysis we work with $\Omega(z,\bar{z}) = \frac{1}{2}$. It is straightforward to extend the solution for the case of

$\mathbb{S}^2$. The line element of the resultant $2d$ boundary metric is given as follows,

$$ds^2_{2d} = g^{(0)}_{zz}(u, z, \bar{z})\, dz^2 + dz\, d\bar{z} \tag{5.25}$$

These chiral boundary conditions on the boundary metric are similar to what was imposed in (3.25) except now the boundary metric has u dependence as well.

2. At order $\mathcal{O}(r)$ we impose one more condition which is,

$$\delta \int d^2z \left[\sqrt{g^{(0)}}\, R_0\right] = 0. \tag{5.26}$$

This condition was also used for locally flat case in Chapter 3. It corresponds to holding the Euler character of metric $g^{(0)}_{ab}$ fixed [129].

The above conditions along with $g^{ab}_{(0)}\, g^{(1)}_{ab} = 0$ ensure that $\delta\mathcal{L}_{EH+bdy}$ vanishes on the solution space that shares the same boundary conditions. In literature the boundary conditions $v_a = \delta v_a = \delta\sqrt{det\, g^{(0)}_{ab}} = 0$ were also imposed in [15] where the symmetry algebra of asymptotically locally AdS_4 was found to be Λ-$\mathfrak{bms}_4$. These conditions were realised using the gauge freedom of the boundary diffeomorphisms and Weyl scaling of boundary metric data. Instead of imposing chiral boundary conditions, their treatment kept the boundary metric $g^{(0)}_{ab}$ general with determinant fixed.

Once we impose these boundary conditions on our configuration space we see that $g^{(0)}_{zz}$ is the only free data on the boundary. One can now easily solve (5.2) and (5.3) completely by solving the following constraint equations that are obtained from the $\mathcal{O}(r^3)$ terms of $\det(g^{(4d)})\, A_{uru\bar{z}}$, $\det(g^{(4d)})\, A_{uruz}$ and $\mathcal{O}(r^5)$ terms of $\det(g^{(4d)})\, A_{uzuz}$ respectively,

$$\partial_u\, \partial^2_{\bar{z}}\, g^{(0)}_{zz} = 0, \quad \partial_z\, \partial^2_{\bar{z}}\, g^{(0)}_{zz} = l^2\, \partial^2_u\, \partial_{\bar{z}} g^{(0)}_{zz}, \tag{5.27}$$

$$\partial_u \left[\frac{1}{2}\left(\partial_{\bar{z}} g^{(0)}_{zz}\right)^2 - g^{(0)}_{zz}\, \partial^2_{\bar{z}}\, g^{(0)}_{zz} + \partial_z\, \partial_{\bar{z}}\, g^{(0)}_{zz} - \frac{l^2}{2}\partial^2_u\, g^{(0)}_{zz}\right] = 0. \tag{5.28}$$

It is important to keep track of the correct powers of l in the solution because $g^{(0)}_{ab}$ should have no length dimension and hence any power of u should be accompanied by $\frac{1}{l}$ to make it dimensionless. Being careful about the correct scaling factor plays an important role while taking the flat space limit. Since $g^{(1)}_{ab}$ has length dimension one and should also survive the flat space limit, we write down the solution in a way that it scales as $\sqrt{G}$(G is the Newton constant) in the limit $l \to \infty$.[8] Keeping these in mind, the solution to the equations (5.27) and (5.28) is given by,

$$g^{(0)}_{zz} = \Gamma(u,z) + J_{-1}(z) + \bar{z}^2\, J_1(z) + \bar{z}\left(J_0(z) - \frac{\sqrt{G}\,u}{l^2} G_{1/2}(z) + \frac{u^2}{l^2}\,\partial J_1(z)\right) \quad (5.29)$$

$$\begin{aligned}\Gamma(u,z) = &\frac{u\sqrt{G}}{l^2}\left(\frac{3\,G_{1/2}(z)\,\partial J_1(z)}{4\,J_1(z)^2} - \frac{\partial G_{1/2}(z)}{2\,J_1(z)} - \frac{G_{1/2}(z)\,J_0(z)}{2\,J_1(z)}\right)\\ &+\frac{u^2}{l^2}\left(\frac{G\,G_{1/2}(z)^2}{4\,l^2\,J_1(z)} + \frac{J_0(z)\,\partial J_1(z)}{2\,J_1(z)} - \frac{3\,(\partial J_1(z))^2}{4\,J_1(z)^2} + \frac{\partial^2 J_1(z)}{2\,J_1(z)}\right)\\ -&\frac{u^3\sqrt{G}\,G_{1/2}\,\partial J_1(z)}{2\,l^4\,J_1(z)} + \frac{u^4\,(\partial J_1(z))^2}{4\,l^4\,J_1(z)} + \frac{\sin\left(\frac{2\,u\sqrt{J_1(z)}}{l}\right)}{\sqrt{J_1(z)}}K_1(z) + \frac{\sin^2\left(\frac{u\sqrt{J_1(z)}}{l}\right)}{J_1(z)}K_2(z)\end{aligned} \quad (5.30)$$

where,

$$\begin{aligned}K_1(z) = \frac{\sqrt{G}}{8\,l\,J_1(z)^2}&\big[2\,J_1(z)\,\left(\partial\,G_{1/2}(z) - 2\,G_{-1/2}(z)\,J_1(z)\right)\\ &+ G_{1/2}(z)\,\left(2\,J_0(z)\,J_1(z) - 3\,\partial J_1(z)\right)\big],\end{aligned} \quad (5.31)$$

$$\begin{aligned}K_2(z) = &-\frac{1}{2}\left(\kappa(z) - \frac{1}{2}\,J_0(z)^2 - \partial J_0(z) + 2\,J_1(z)\,J_{-1}(z)\right)\\ &+\frac{1}{4\,J_1(z)^2}\left[3\,(\partial J_1)^2 - J_0(z)\,\partial(J_1(z)^2) - 2\,J_1(z)\,\partial^2\,J_1(z) - \frac{G}{l^2}\,J_1(z)\,G_{1/2}(z)^2\right].\end{aligned} \quad (5.32)$$

Let us discuss some of the properties of this solution,

1. As mentioned before the only free data is $g^{(0)}_{zz}$. Rest all other metric components are obtained from $g^{(0)}_{zz}$.
2. The solution is parameterised by six holomorphic functions,

$$\{J_a(z)\,, \kappa(z)\,, G_s(z)\} \;\; \text{for} \;\; a \in \{0, \pm 1\} \;\; \text{and} \;\; s \in \{\pm 1/2\}. \quad (5.33)$$

3. As can be seen from the solution, function $J_1(z)$ appears in the denominator at several places. However, the solution has a well-defined limit as $J_1(z) \to 0$

[8]One can also use Planck length l_{pl}.

along with the other five functions in (5.33) such that the solution does not diverge. One gets back to the global AdS_4 solution when all the six holomorphic functions approach zero value.

4. The solution has a well defined flat space limit ($l \to \infty$) such that,

$$ds^2_{LAdS_4} = ds^2_{LF} + \mathcal{O}\left(\frac{1}{l^2}\right) + \cdots \tag{5.34}$$

where $\cdots$ indicate higher order terms in $\frac{1}{l^2}$ with,

$$\begin{aligned} ds^2_{LF} =& -\ 2\,du\,dr + \ r^2\,dz\,d\bar{z} + r^2\left(J_p(z)\,\bar{z}^{1+p}\right)dz^2 \\ &+ r\,u\left(\frac{1}{u}\sqrt{G}\,G_s(z)\,\bar{z}^{\frac{2s+1}{2}} + \kappa(z) + \frac{1}{2}J_p(z)J^p(z) - (1+p)\,\bar{z}^p\,\partial_z\,J_p(z)\right)dz^2 \\ &+ 2\left(\frac{2s+1}{2}\sqrt{G}\,G_s\,\bar{z}^{\frac{2s-1}{2}} -\ u\,p\,(1+p)\,\partial_z J_p\,\bar{z}^{p-1}\right)du\,dz \\ &- 2\,p\,(1+p)\,J_p\,\bar{z}^{p-1}\,du^2, \quad p \in \{-1,0,1\},\ r \in \{-\frac{1}{2},\frac{1}{2}\} \end{aligned} \tag{5.35}$$

The expression of ds^2_{LF} coincides with the locally flat solution found in (3.72), where the topology of the boundary metric at the boundary of flat space is $\mathbb{R}^2$ with the identification $\{J_p \to -J^p\,, \kappa(z) \to -\kappa(z), G_s \to \frac{C_r}{\sqrt{G}}\}$.

Now that we have obtained $LAdS_4$ solution with chiral boundary conditions we turn next to calculating its asymptotic symmetry algebra, i.e., a set of residual diffeomorphisms that keep this solution within the same equivalence class. In the process, we will derive the variations of the six holomorphic functions (5.33) due to these diffeomorphisms.

5.1.2 Asymptotic symmetry algebra

We look for residual diffeomorphisms that take us from one $LAdS_4$ solution to another without spoiling the gauge and boundary conditions of our initial solution. To preserve the gauge conditions of the NU gauge the vector fields of the form $\xi^\mu\partial_\mu$ should satisfy,

$$\partial_r\xi^u = 0, \quad \partial_u\xi^u + \partial_r\xi^r = g_{ua}\,\partial_r\xi^a, \quad \partial_a\xi^u = g_{ab}\partial_r\xi^b \tag{5.36}$$

From first of the equation in (5.36) we see that $\xi^u = \xi^u_{(0)}(u, z, \bar{z})$. The other two equations can further be solved in an asymptotic expansion around large-r. We expand ξ^r and ξ^a near the boundary as follows,

$$\xi^r = \sum_{n=0}^{\infty} r^{1-n} \xi^r_{(n)}(u,z,\bar{z}), \quad \xi^a = \sum_{n=0}^{\infty} r^{-n} \xi^a_{(n)}(u,z,\bar{z}), \quad a \in \{z, \bar{z}\} \tag{5.37}$$

At each order in r the equations (5.36) are solved recursively to obtain the terms at subleading order in r of ξ^r and ξ^i in terms of leading coefficients $\xi^r_{(0)}$ and $\xi^i_{(0)}$ and $\xi^u_{(0)}$. The first few terms are given by,

$$\begin{aligned}
\xi^r_{(0)} &= -\partial_u \xi^u_{(0)} + v^a\,\partial_a \xi^u_{(0)}\,, \quad \xi^r_{(2)} = g^{(2)}_{ua}\,\xi^a_{(1)} + 2\,\xi^a_{(2)}\,g^{(1)}_{ua}\,, \\
\xi^r_{(n)} &= \xi^a_{(n-1)}\,g^{(2)}_{ua}\,, \quad \text{for } n = 3, 4, \cdots\,, \\
\xi^a_{(1)} &= -g^{ab}_{(0)}\,\partial_b \xi^u_{(0)}\,, \quad \xi^a_{(2)} = \frac{1}{2}\,g^{ab}_{(0)}\,g^{(1)}_{bc}\,g^{cd}_{(0)}\,D_d\,\xi^u_{(0)}.
\end{aligned} \tag{5.38}$$

Also demanding that the vector fields preserve the traceless condition of $g^{(1)}_{ab}$ allows us to solve for $\xi^r_{(1)}$ which is given by,

$$\xi^r_{(1)} = \frac{1}{2}\Box\,\xi^u - \frac{1}{4}\xi^a_{(0)} D_a \left(g^{bc}_{(0)} g^{(1)}_{bc}\right) - \frac{1}{4}\partial_u \left(\xi^u\,g^{bc}_{(0)} g^{(1)}_{bc}\right) + \frac{1}{4}\,v^c\,\partial_c\,\xi^u_{(0)}\,g^{ab}_{(0)}\,g^{(1)}_{ab}. \tag{5.39}$$

Here $\xi^a_{(0)}$ and $\xi^u_{(0)}$ are unconstrained functions of $(u, z, \bar{z})$ and constitute boundary vector fields that are obtained by the pullback of ξ on the boundary

$$\xi_{bdry} = \xi^u_{(0)}\,\partial_u + \xi^a_{(0)}\,\partial_a. \tag{5.40}$$

The variation of free data $g^{(0)}_{ab}$ and v_a in terms of these components of boundary vector fields are given by,

$$\delta g^{(0)}_{ab} = \xi^u_{(0)} \partial_u\,g^{(0)}_{ab} + \mathcal{L}_{\xi^c_{(0)}}\,g^{(0)}_{ab} + 2\,v_{(a}\partial_{b)}\,\xi^u_{(0)} - 2\,g^{(0)}_{ab}\,\left(\partial_u\,\xi^u_{(0)} - v^c\,\partial_c\,\xi^u_{(0)}\right), \tag{5.41}$$

$$\begin{aligned}
\delta v_a = {}& \xi^u_{(0)}\,\partial_u v_a + \mathcal{L}_{\xi^c_{(0)}}\,v_a + v_a\,\left(2\,v^c\,\partial_c\,\xi^u_{(0)} - \partial_u\,\xi^u_{(0)}\right) + g^{(0)}_{ab}\,\partial_u \xi^b_{(0)} \\
& - \frac{1}{l^2}\partial_a \xi^u_{(0)}\,g^{(0)}_{uu} + \partial_a\,\xi^u_{(0)}\,v^b\,v_b.
\end{aligned} \tag{5.42}$$

Here $\mathcal{L}_{\xi^c_{(0)}}\,g^{(0)}_{ab}$ is the Lie derivative of boundary metric $g^{(0)}_{ab}$ under boundary vector field $\xi^c_{(0)}$. Demanding that the transformation of free data at the boundary $\{g^{(0)}_{ab}\,, v_a\}$ under (5.40) should obey the conditions obtained by solving variational principle will give us constraints on these boundary vector fields. The conditions that we impose are,

$$\delta g^{(0)}_{\bar{z}\bar{z}} = 0\,,\ \delta g^{(0)}_{z\bar{z}} = 0\,,\ \delta v_a = 0 \tag{5.43}$$

along with $g^{(0)}_{\bar z\bar z} = v_a = 0,\ g^{(0)}_{z\bar z} = \frac{1}{2}$. From $\delta g^{(0)}_{\bar z\bar z} = 0$ one gets,

$$\xi^z_{(0)}(u, z, \bar z) = Y^z(u, z). \tag{5.44}$$

From $\delta v_a = 0$ one gets the equation

$$\partial_a \xi^u_{(0)}\, g^{(0)}_{uu} = -g^{(0)}_{ab}\, \partial_u \xi^b_{(0)}. \tag{5.45}$$

The $\bar z$ component of equation (5.45) can be solved to give

$$\xi^u_{(0)} = f(u, z) + \frac{l^2}{2}\, \bar z\, \partial_u Y^z(u, z). \tag{5.46}$$

Imposing $\delta g^{(0)}_{z\bar z} = 0$ allows one to write $\xi^{\bar z}$ as follows,

$$\xi^{\bar z}_{(0)} = Y^{\bar z}(u, z) + \bar z\, (2\, \partial_u f - \partial_z Y^z) + \frac{l^2}{2} \bar z^2\, \partial_u^2\, Y^z. \tag{5.47}$$

Also, we have $\xi^r_{(0)} = -\partial_u \xi^u_{(0)}$. Note the explicit $\bar z$ dependence of the boundary vector fields. Using these expressions one finds that the z component of (5.45) is a polynomial in $\bar z$ with the highest power of $\bar z$ as 2. Setting the coefficient of each power of $\bar z$ in δv_z to zero separately one gets the following conditions,

$$\frac{l^2}{4}\, \partial_u^3\, Y^z + J_1(z)\, \partial_u Y^z(u, z) = 0, \tag{5.48}$$

$$\begin{aligned} &\partial_u^2\, f(u, z) + J_0(z)\, \partial_u Y^z(u, z) - \partial_u\, \partial_z\, Y^z(u, z) \\ &+ \frac{1}{l^2} \left(u^2\, \partial\, J_1(z)\, \partial_u\, Y^z(u, z) - u\, \sqrt{G}\, G_{1/2}(z)\, \partial_u Y^z\right) = 0, \end{aligned} \tag{5.49}$$

$$\frac{1}{2}\, \partial_u\, Y^{\bar z}(u, z) - \frac{1}{l^2}\, \partial_z\, f(u, z) + \Gamma(u, z)\, \partial_u Y^z(u, z) = 0 \tag{5.50}$$

where $\Gamma(u, z)$ was defined in equation (5.30). After completely determining $\bar z$ dependence of these vector fields one can solve these differential conditions to obtain the explicit u dependence. The solution to equation (5.48) is given by

$$Y^z(u, z) = Y(z) + \frac{\sqrt{G}\, P_{1/2}(z)}{l}\, \frac{\sin\left(\frac{2\, u\, \sqrt{J_1(z)}}{l}\right)}{\sqrt{J_1(z)}} + Y_1(z)\, \frac{\sin^2\left(\frac{u\, \sqrt{J_1(z)}}{l}\right)}{J_1(z)}. \tag{5.51}$$

As the order of the differential equation is three, the solution has three undetermined integration functions $\{Y(z), P_{1/2}(z), Y_1(z)\}$. Similarly to obtain $f(u, z)$ one solves equation (5.49) to find

$$f(u,z) = \sqrt{G}\,P_{-1/2}(z) + \frac{\sin\left(\frac{2\,u\,\sqrt{J_1(z)}}{l}\right)}{\sqrt{J_1(z)}}\,L_1(u,z) + \frac{\cos\left(\frac{2\,u\,\sqrt{J_1(z)}}{l}\right)}{J_1(z)}\,L_2(u,z)$$
$$+ \frac{\sqrt{G}}{4\,J_1(z)^2}\,\left(2\,J_1(z)\,(\partial P_{1/2}(z) - J_0(z)\,P_{1/2}(z)) + G_{1/2}\,Y_1(z) + P_{1/2}\,\partial J_1(z)\right)$$
$$+ \frac{u}{4\,J_1(z)^2}\,\left(2\,J_1(z)\,\left(\partial Y_1(z) - \frac{G}{l^2}\,G_{1/2}(z)P_{1/2}(z) - J_0(z)\,Y_1(z)\right) - Y_1(z)\,\partial J_1(z)\right)$$
$$+ \frac{u}{2}\,(\partial Y(z) + Y_0(z)) \tag{5.52}$$

where $L_1(u,z)\,,L_2(u,z)$ are ,

$$L_1(u,z) = \frac{G}{4\,l\,J_1(z)}\,\left[2\,G_{1/2}(z)\,P_{1/2}(z) + \frac{l^2}{G}\,(J_0(z)\,Y_1(z) - \partial Y_1(z))\right]$$
$$- \frac{\sqrt{G}\,u}{4\,l\,J_1(z)}\,\left[G_{1/2}(z)Y_1(z) + 2\,P_{1/2}(z)\,\partial J_1(z)\right] + \frac{u^2}{4\,l\,J_1(z)}\,Y_1(z)\,\partial J_1(z), \tag{5.53}$$
$$L_2(u,z) = \frac{\sqrt{G}}{4\,J_1(z)}\,\left[2\,J_1(z)\,J_0(z)\,P_{1/2}(z) - G_{1/2}(z)\,Y_1(z) - P_{1/2}(z)\,\partial J_1(z) - \frac{1}{2}\,\partial P_{1/2}(z)\right]$$
$$+ \frac{u}{4\,l^2\,J_1(z)}\,\left[l^2\,Y_1(z)\,\partial J_1(z) - 2\,G\,J_1(z)\,G_{1/2}(z)\,P_{1/2}(z)\right] + \frac{\sqrt{G}\,u^2}{2\,l^2}\,P_{1/2}(z)\,\partial J_1(z). \tag{5.54}$$

The solution has two undetermined integration functions $P_{-1/2}(z)$ and $Y_0(z)$. Now that we know $Y^z(u,z)$ and $f(u,z)$ in closed form, one can use them in equation (5.50) to obtain $Y^{\bar{z}}$ as follows,

$$Y^{\bar{z}}(u,z) = Y_{-1}(z) + \frac{2}{l^2}\int du\,\partial_z\,f(u,z) - 2\int du\,\left(\Gamma(u,z) + J_{-1}(z)\right)\,\partial_u Y^z(u,z) + L_3(z) \tag{5.55}$$

where $L_3(z)$ is given by,

$$L_3(z) = \frac{\sqrt{G}\, P_{1/2}(z)}{2\,l\, J_1(z)}\, K_1(z) + \frac{3\, Y_1(z)}{8\, J_1(z)^2}\, K_2(z) + \frac{\partial Y_1(z)}{4\, J_1(z)^3}\, \left(J_1(z)\, J_0(z) + \partial J_1(z)\right)$$
$$+ \frac{G\, P_{1/2}(z)}{2\, l^2\, J_1(z)^3} \left[J_1(z)^2\, G_{-1/2}(z) + \frac{3}{4}\, G_{1/2}(z)\, \partial J_1(z) - \frac{3}{2}\, G_{1/2}(z)\, J_1(z)\, J_0(z)\right]$$
$$+ \frac{3\, Y_1(z)}{8\, J_1(z)^2} \left[\frac{2}{3}\, \partial J_0(z) + \frac{J_0(z)\, \partial J_1(z)}{3\, J_1(z)} - \frac{3\, \partial J_1(z)^2}{2\, J_1(z)^2} + \frac{\partial^2\, J_1(z)}{J_1(z)} - \frac{8}{3}\, J_{-1}(z)\, J_1(z)\right.$$
$$\left. + \frac{5\, G\, G_{1/2}(z)}{6\, l^2\, J_1(z)}\right] + \frac{1}{4\, J_1(z)^2} \left(\frac{3G}{l^2}\, G_{1/2}(z)\, \partial P_{1/2}(z) - \partial^2\, Y_1(z)\right) \tag{5.56}$$

where $Y_{-1}(z)$ is another integration function and the integral over u can be performed easily to obtain the closed form. $L_3(z)$ is written in a way such that the flat space limit of this vector field agrees with that of locally flat solutions in (3.99).

In conclusion, the boundary vector fields that generate residual diffeomorphisms on the chiral LAdS$_4$ solution are parameterised by six holomorphic functions

$$\{Y_a(z)\, , Y(z)\, , P_s(z)\, \} \quad \text{for} \quad a \in \{0, \pm 1\} \quad \text{and} \quad s \in \{\pm 1/2\}.. \tag{5.57}$$

The subleading terms in $\frac{1}{l^2}$ as well as in r expansion are given in terms of these six holomorphic free functions. The solution to equations (5.48, 5.49, 5.50) is constructed in such a way that it has a well-defined flat space limit and the behaviour is smooth if $J_1(z)$ is taken to zero. Also one needs to take care of the length dimension of vector fields. ξ^z and $\xi^{\bar{z}}$ is dimensionless but ξ^u has dimension one. In the flat space limit ξ^u gets scaled by $\sqrt{G}$ like we did for $g^{(1)}_{ab}$. After taking care of these subtleties one finds that in the large l limit the components of boundary vector fields behave as follows,

$$\xi^u_{(0)} = \sqrt{G}\, \big(P_{-1/2}(z) + \bar{z}\, P_{1/2}(z)\big) + \frac{u}{2}\, \big(Y_{(0)}(z) + \partial Y(z) + 2\, \bar{z}\, Y_1(z)\big) + \cdots \tag{5.58}$$
$$\xi^z_{(0)} = Y(z) + \cdots \tag{5.59}$$
$$\xi^{\bar{z}}_{(0)} = Y_{-1}(z) + \bar{z}\, Y_0(z) + \bar{z}^2\, Y_1(z) + \cdots\, . \tag{5.60}$$

where $\cdots$ denotes higher order terms in $\frac{1}{l^2}$. One observes that in this limit the leading order terms in $\frac{1}{l^2}$ expansion of the boundary vector field is equal to the boundary vector field that was found in the context of locally flat solutions in (3.99) with appropriate identification of $P_s(z)$. If we now inspect $\delta g^{(0)}_{zz}$ then we can deduce the transformations of background fields (5.33) in terms of (5.57). We do the following redefinition on the fields:

$$J_1(z) \to \frac{1}{2}\,J_1(z)\,, \quad J_{-1}(z) \to \frac{1}{2}\,J_{-1}(z)\,, \quad Y_1(z) \to \frac{1}{2}\,Y_1(z)\,, \quad Y_{-1}(z) \to \frac{1}{2}\,Y_{-1}(z).$$

These variations are then given by

$$\delta J_a(z) = \frac{2\,G}{l^2}\,(\tau_a)^{ss'}\,G_s(z)\,P_{s'}(z) + \partial Y_a(z) + f_a{}^{bc}\,J_b(z)\,Y_c(z) + \partial(J_a(z)Y(z)), \tag{5.61}$$

$$\begin{aligned}\delta\kappa(z) &= Y(z)\,\partial\,\kappa(z) + 2\kappa(z)\,\partial Y(z) - \partial^3\,Y(z)\\ &+ \frac{2\,G}{l^2}\,(\tau^a)^{ss'}\,J_a(z)\,G_s(z)\,P_{s'}(z) + \frac{3G}{l^2}\,\epsilon^{ss'}\,G_s\,\partial P_{s'} + \frac{G}{l^2}\,\epsilon^{ss'}\,\partial G_s\,P_{s'},\end{aligned} \tag{5.62}$$

$$\begin{aligned}\delta G_s(z) &= \frac{3}{2}\,G_s(z)\,\partial Y(z) + Y(z)\,\partial G_s(z) - 2\,\partial^2\,P_s(z) + P_s(z)\left(\kappa(z) + \frac{1}{2}\,\eta^{ab}\,J_a(z)\,J_b(z)\right)\\ &+\, G_{s'}(z)\,(\tau^a)^{s'}{}_s\,Y_a(z) + 4\,\partial P_{s'}(z)\,(\tau^a)^{s'}{}_s\,J_a(z) + 2\,P_{s'}(z)\,(\tau^a)^{s'}{}_s\,\partial J_a(z).\end{aligned} \tag{5.63}$$

Here $a, b, c.. \in \{-1, 0, 1\}$ and $s, s' \in \{-\frac{1}{2}, \frac{1}{2}\}$ and $f_a{}^{bc}$ and $(\tau^a)^r{}_s$ are structure constants defined as,

$$f_{ab}{}^c = (a-b)\,\delta^c_{a+b}\,, \quad (\tau_a)^r{}_s = \frac{1}{2}(a-2s)\,\delta^r_{a+s}. \tag{5.64}$$

The indices s, s' are raised and lowered by $\epsilon^{ss'}$ and $\epsilon_{ss'}$ whereas indices $a, b, c, ..$ are raised and lowered by η^{ab} and η_{ab}. These are further defined as,

$$\eta_{ab} = -(1-3a^2)\,\delta_{a+b,0}, \quad \eta^{ab} = -\frac{1}{(1-3a^2)}\,\delta^{a+b,0}, \tag{5.65}$$

$$\epsilon_{ss'} = 2s'\,\delta_{s+s',0}, \quad \epsilon^{ss'} = -2s'\,\delta^{s+s',0}. \tag{5.66}$$

5.1.3 Algebra of vector fields

In the previous section, we derived full set of vector fields that enable us to move in the class of $LAdS_4$ solutions. Now we check whether the Courant brackets defined in (2.14) between any two of these vector fields close under the modified commutator

$$[\,\xi_1\,,\,\xi_2\,]_M := \xi_1^\sigma\,\partial_\sigma\,\xi_2 - \xi_2^\sigma\,\partial_\sigma\,\xi_1 - \delta_{\xi_1}\,\xi_2 + \delta_{\xi_2}\,\xi_1. \tag{5.67}$$

Calculating the bracket as in eq (5.67) we find:

$$[\,\xi_1(P_r^1, Y^1, Y_a^1)\,,\,\xi_2(P_r^2, Y^2, Y_a^2)\,]_M = \hat{\xi}^u_{(0)}\,\partial_u + \hat{\xi}^z_{(0)}\,\partial_z + \hat{\xi}^{\bar{z}}_{(0)}\,\partial_{\bar{z}} + \hat{\xi}^r_{(0)}\partial_r + \cdots \quad (5.68)$$

where $\{\hat{\xi}^u_{(0)}\,,\hat{\xi}^z_{(0)}\,,\hat{\xi}^{\bar{z}}_{(0)}\,,\hat{\xi}^r_{(0)}\}$ are functions of $\hat{P}_r\,,\hat{Y}\,,\hat{Y}_a$ which are defined as follows:

$$\begin{aligned}
\hat{Y} &= Y^1(z)\,\partial Y^2(z) - Y^2(z)\,\partial Y^1(z) - \frac{2\,G}{l^2}\,\epsilon^{rs}\,P_r^1(z)\,P_s^2(z), \\
\hat{Y}_a &= Y^1(z)\,\partial Y_a^2 - Y^2(z)\,\partial Y_a^1 - f_a{}^{bc}\,Y_b^1(z) Y_c^2(z) - \frac{4\,G}{l^2}\,(\tau_a)^{rs}\,P_r^1\,\partial P_s^2 \\
&\quad + \frac{4\,G}{l^2}\,(\tau_a)^{rs}\,P_r^2\,\partial P_s^1 + \frac{4\,G}{l^2}\,J_a(z)\,\epsilon^{rs} P_r^1\,P_s^2, \\
\hat{P}_r &= -(\tau^a)^s{}_r\,\big(P_s^1\,Y_a^2 - P_s^2\,Y_a^1\big) + Y^1\,\partial P_r^2 - Y^2\,\partial P_r^1 + \frac{1}{2}\,P_r^1\,\partial Y^2 + \frac{1}{2}\,P_r^2\,\partial Y^1.
\end{aligned} \quad (5.69)$$

The vector fields on the L.H.S of (5.67) are parameterised by two sets of six holomorphic functions written down in (5.57) and under this modified commutator the form of a resultant vector field on the R.H.S does not change and have the same functional dependence on a combination of these two sets as written down in (5.69). Note the dependence of $\hat{Y}^a$ on the background function J_a. Using (5.69) one can write down the commutators between various modes of (5.57) as follows,

- Denoting the parameter $Y_a(z) = -z^n$ by $\mathcal{J}_n^a$ for $a \in \{-1, 0, 1\}$ and $n \in \mathbb{Z}$. The commutator of these is given by,

  $$[\,\mathcal{J}_m^a\,,\,\mathcal{J}_n^b\,] = \big(a - b\big)\,\mathcal{J}_{m+n}^{a+b} \quad (5.70)$$

  This can be recognised as the $\mathfrak{sl}_2$ current algebra.

- Denoting the parameter $Y(z)$ as $Y(z) = -z^{n+1}$ by $\mathcal{L}_n$ for $n \in \mathbb{Z}$ we find:

  $$[\mathcal{L}_m, \mathcal{L}_n] = (m - n)\,\mathcal{L}_{m+n}, \quad [\,\mathcal{L}_m\,,\mathcal{J}_n^a\,] = -n\,\mathcal{J}_{m+n}^a \quad (5.71)$$

  which are Witt algebra, with $\mathcal{J}^a(z)$ being $h = 1$ currents.

- Finally denoting the parameter $P_s(z)$ as $\mathcal{P}_{s,r}(z) = -z^{r+\frac{1}{2}}$ where $r \in \mathbb{Z} + \frac{1}{2}$ and $s \in \{-\frac{1}{2}, \frac{1}{2}\}$, the remaining commutators work out to be:

  $$[\mathcal{L}_n, \mathcal{P}_{s,r}] = \frac{1}{2}(n - 2\,r)\,\mathcal{P}_{s,n+r}, \quad [\mathcal{J}_n^a, \mathcal{P}_{s,r}] = \frac{1}{2}(a - 2\,s)\,\mathcal{P}_{a+s,n+r} = 0 \quad (5.72)$$

$$\left[\mathcal{P}_{s,r},\mathcal{P}_{w,t}\right]=\frac{2G}{l^2}(r-t)\,\mathcal{J}_{s+w,t+r}-\frac{2G}{l^2}\epsilon_{sw}\,\mathcal{L}_{r+t}+\frac{4G}{l^2}\epsilon_{sw}\,\left(\mathcal{J}^a_{t+r+1}J_a(z)\right) \tag{5.73}$$

Thus the final result of the algebra of the vector fields that preserve our space of locally AdS_4 solutions are (5.70, 5.71, 5.72, 5.73). This algebra is one of the main results of this chapter. The symmetry algebra can be considered as the chiral version of Λ-$\mathfrak{bms}_4$ [15, 16] and is a Lie algebroid as indicated by the presence of background field ($J_a(z)$) dependent structure constant in the commutator of two $\mathcal{P}_{r,s}$. We denote this algebra as chiral Λ-$\mathfrak{bms}_4$. In the limit $l\to\infty$ one can see that $\mathcal{P}_{s,r}$ commute and resultant algebra becomes the chiral $\mathfrak{bms}_4$ (3.1).

5.1.4 Charges

If we perform the following redefinition of the field $\kappa(z)$ and introduce a new function $T(z)$ as follows,

$$T=\kappa(z)-\frac{1}{2}\,\eta^{ab}\,J_a(z)\,J_b(z)$$

then the resultant transformations of the field $T(z)$ and $G_s(z)$ are,

$$\begin{aligned}\delta T(z)&=Y(z)\,\partial\,T(z)+2\,T(z)\,\partial Y(z)-\partial^3\,Y(z)\\&\quad-\eta^{ab}\,J_a(z)\,\partial Y_b(z)+\frac{3G}{l^2}\,\epsilon^{ss'}\,G_s(z)\,\partial P_{s'}(z)+\frac{G}{l^2}\,\epsilon^{ss'}\,\partial G_s(z)\,P_{s'}(z),\end{aligned} \tag{5.74}$$

$$\begin{aligned}\delta G_s(z)&=\frac{3}{2}\,G_s(z)\,\partial Y(z)+Y(z)\,\partial G_s(z)-2\,\partial^2\,P_s(z)+P_s(z)\left(T(z)+\eta^{ab}\,J_a(z)\,J_b(z)\right)\\&\quad+\,G_{s'}(z)\,(\tau^a)^{s'}{}_{s}\,Y_a(z)+4\,\partial P_{s'}(z)\,(\tau^a)^{s'}{}_{s}\,J_a(z)+2\,P_{s'}(z)\,(\tau^a)^{s'}{}_{s}\,\partial J_a(z).\end{aligned} \tag{5.75}$$

The variation of holomorphic functions $\{T(z),J_a(z),G_s(z)\}$ in (5.61, 5.74, 5.75) is very similar to the transformation of chiral currents in $2d$ CFT. For instance, under $Y(z)$, $T(z)$ transforms like a conformal stress tensor of weight 2 of a $2d$ CFT. Therefore, we treat it as a stress tensor and $Y(z)$ as the generator of Virasaro transformations. Then under these Virasaro transformations $J_a(z)$ and $G_s(z)$ transform as conformal primaries of weights 1 and $\frac{3}{2}$ respectively. The transformation of $J_a(z)$ under Y_a and the form of structure constant $({f_a}^{bc})$ suggest that these are $\mathfrak{sl}(2,\mathbb{R})$ Kac-Moody currents at some (undetermined) level and therefore the parameters $Y_a(z)$ generate this current algebra symmetry. The transformation of $G_s(z)$ under this current algebra indicates that these $G_s(z)$ are a doublet of current algebra primaries. To summarize, the background fields can be classified as a chiral stress tensor $T(z)$, a triplet of $\mathfrak{sl}(2,\mathbb{R})$ Kac Moody currents $J_a(z)$ and a doublet ($h = \frac{3}{2}$) of conformal primary as well as current algebra primaries $G_s(z)$.

We now promote the fields in (5.33) to conformal operators and propose a line integral charge just like in a $2d$ CFT that will induce the transformations (5.61, 5.74, 5.75). Once we propose such a charge, we derive the OPEs between various operators in (5.33) that will give rise to these transformations. The conjectured line integral charge is,

$$Q = \oint \left[T(z)\,Y(z) - \eta^{ab}\,J_a(z)Y_b(z) + \frac{2G}{l^2}\epsilon^{ss'}\,G_s(z)\,P_{s'}(z)\right]\frac{dz}{2\pi i} \tag{5.76}$$

Here we assume that the underlying symmetry algebra is at a fixed level of central charge and current level.[9] The operator product expansions between various operators in (5.33) are given by

$$\begin{aligned}
T(z)\,J_a(w) =& \frac{J_a(w)}{(z-w)^2} + \frac{\partial J_a(w)}{(z-w)},\\
J_a(z)\,J_b(w) =& -\frac{\eta_{ab}}{(z-w)^2} - \frac{f^c_{ab}\,J_c(w)}{(z-w)},\\
T(z)\,T(w) =& -\frac{6}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)},\\
J_a(z)\,G_s(w) =& -\frac{G_{s'}\,(\tau_a)_s^{s'}}{(z-w)},\\
T(z)\,G_s(w) =& \frac{3\,G_s(w)}{2\,(z-w)^2} + \frac{\partial G_s(w)}{(z-w)},\\
\frac{2G}{l^2}\,G_s(z)\,G_{s'}(w) =& \frac{4\,\epsilon_{ss'}}{(z-w)^3} - \frac{\epsilon_{ss'}\,T(w)}{(z-w)} - \frac{\epsilon_{ss'}\,J^2(w)}{(z-w)}\\
& -2\,(\tau)^a{}_{ss'}\left[\frac{\partial J_a(w)}{(z-w)} + \frac{2\,J_a(w)}{(z-w)^2}\right].
\end{aligned} \tag{5.77}$$

These OPEs are obtained using standard $2d$ conformal field theory techniques. Next, we write the operators $T(z)$, $J_a(z)$ and $G_s(z)$ in terms of their modes as

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1}T(z), \quad J_{a,n} = \oint \frac{dz}{2\pi i} z^n\,J_a(z), \quad G_{s,r} = \oint \frac{dz}{2\pi i} z^{r+\frac{1}{2}}G_s(z). \tag{5.78}$$

[9]The derivation of the charge (5.76) and the value of the current algebra level and central charge c in terms of dimensionless parameters of the bulk is an open problem.

The OPEs (5.77) are then translated to the following commutators,

$$
\begin{aligned}
&[L_m, L_n] = (m-n)\, L_{m+n} - \frac{1}{2} m(m^2-1)\delta_{m+n,0} \\
&[L_m, J_{a,n}] = -n\, J_{a,m+n}, \quad [L_m, G_{s,r}] = \frac{1}{2}(n-2r)\, G_{s,n+r} \\
&[J_{a,m}, J_{b,n}] = -m\, \eta_{ab}\, \delta_{m+n,0} - f_{ab}{}^{c}\, J_{c,m+n}\,, \quad [J_{a,n}\,, G_{s,r}] = -G_{s',n+r}(\lambda_a)^{s'}{}_{s} \\
\frac{2G}{l^2}&[G_{s,r}, G_{s',r'}] = \epsilon_{ss'}\left[2\left(r^2 - \frac{1}{4}\right)\delta_{r+r',0} - L_{r+r'} - (J^2)_{r+r'}\right] - (r-r')J_{a,r+r'}(\lambda^a)_{ss'}
\end{aligned}
\tag{5.79}
$$

where we shifted the zero mode of L_n as,

$$
L_n \to L_n + \frac{1}{2}\delta_{n,0}. \tag{5.80}
$$

The commutator algebra in (5.79) is another main result of this chapter in addition to (5.70, 5.71, 5.72, 5.73). Under a suitable scaling of operator $G_s \to \frac{l^2}{2G} G_s$ and taking $l \to \infty$, the algebra (5.79) reduces to chiral $\mathfrak{bms}_4$ with non-zero central charge and current level.

We have derived a charge algebra from the bulk AdS_4 gravity after postulating the existence of line integral charges that generate the residual diffeomorphisms which themselves obey the algebra (5.70, 5.71, 5.72, 5.73). This charge algebra is non-linear as can be seen by the presence of $(J^2)_{r+r'}$ term on the RHS of commutator $[G_{s,r}, G_{s',r'}]$ and is a semi-classical limit of a $\mathcal{W}$-algebra at a specific value of central charge and current level. The operator contents of this $\mathcal{W}$- algebra include,

1. Chiral stress tensor $T(z)$ with central charge c.

2. Spin-1 $\mathfrak{sl}(2,\mathbb{R})$ current algebra primary J_a for $a \in \{-1, 0, 1\}$ with level κ.

3. A doublet of spin $\frac{3}{2}$ chiral operators $G_s(z)$ for $s \in \{-\frac{1}{2}, \frac{1}{2}\}$.

To validate our findings, in the next section, we will derive a complete quantum version of the algebra (5.79) for finite central charge and current level using tools of $2d$ CFT such as associativity constraints.

5.2 Derivation of $\mathcal{W}$-algebra

The problem of finding extensions of conformal symmetry of a $2d$ chiral CFT to those with an extended set of symmetry currents has a long history. Such algebras can be described by the operator product expansions of the set of all the conformal primary chiral fields involved. The singular parts of the OPE result in Lie brackets of the modes with the regular parts forming the normal ordered products. These OPEs are demanded to be consistent with conformal symmetry as well as to satisfy the OPE associativity constraints [159]. One can implement the associativity constraints in terms of modes ϕ_n in which each chiral quasi-primary $\phi(z)$ with dimension h_ϕ is written as

$$\phi(z) = \sum_{n\in\mathbb{Z}} z^{-n-h_\phi}\phi_n \quad \text{for} \quad h_\phi \in \mathbb{Z}_{>0}$$

or

$$\phi(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} z^{-r-h_\phi}\phi_r \quad \text{for} \quad h_\phi \in \mathbb{Z}_{\geq 0} + \frac{1}{2}.$$

This procedure was laid out fully by Nahm et al. [25] (see also [161]) developing further on earlier works such as [166]. Even though these techniques of construction of new algebra are standard, we review them and show their working explicitly to derive the $\mathcal{W}$-algebra of our interest for the convenience of the readers. We briefly list the relevant steps involved.

1. One starts by listing out all simple Virasoro primary fields.[10] In our case these are, the identity operator $\mathbb{I}$ ($h = 0$), the chiral stress tensor $T(z)$ ($h = 2$), the $\mathfrak{sl}(2,\mathbb{R})$ currents $J_a(z)$ ($h = 1$) and the current algebra primary doublet $G_s(z)$ ($h = 3/2$).

2. Then one constructs all the quasi-primaries that are formed by considering normal ordered products of simple primaries, that can appear on the right-hand side of the OPE of simple primaries with dimensions up to $2h - 1$ where h is the largest conformal dimension in the set of simple primaries. In our case, the highest dimension is that of $T(z)$ which has $h = 2$ and therefore we only have to consider quadratic normal ordered quasi-primaries of dimension up to 2.

3. Then one uses global conformal symmetry (and other symmetries postulated) to solve for the structure constants that appear between simple primaries and

[10] A primary is said to be simple if the corresponding state is orthogonal (with overlap defined using the BPZ duals) to those of all the normal ordered quasi-primaries [25].

global descendants of quasi-primaries completely. However, those between three simple primaries remain free parameters at this stage.

4. Finally one needs to impose the Jacobi identities on the corresponding commutators of modes of the simple primaries. This, in principle, determines the remaining structure constants.

The theorems of Nahm et al. [25] then guarantee that the resultant chiral algebra satisfies all the necessary constraints of $2d$ chiral CFTs.

Normal ordered quasi-primaries of interest

We have already declared our set of simple primaries. Now we need the list of relevant quasi-primaries that are normal ordered products between any two of these primaries. In terms of the modes of a primary

$$\phi_{i,n} = \oint \frac{dz}{2\pi i} z^{n+h_i-1}\phi_i(z) \tag{5.81}$$

the normal ordered product is defined by

$$(\phi_i\phi_j)_n = \sum_{k>-h_i} \phi_{j,n-k}\phi_{i,k} + \sum_{k\leq -h_i} \phi_{i,k}\phi_{j,n-k} \ . \tag{5.82}$$

We need to consider normal ordered products such as

$$(TT)(w),\ (TJ_a)(w),\ (TG_s)(w),\ (J_aJ_b)(w),\ (J_aG_s)(w),\ (G_sG_{s'})(w), (J_aJ_bJ_c)(w),\ \cdots$$

and their derivatives. Among these, there is exactly one chiral operator of dimension $h \leq 2$ and it is $(J_aJ_b)(w)$ which has $h = 2$. It is easy to see that the symmetric tensor $(J_{(a}J_{b)})(w)$ is a quasi-primary (whereas the anti-symmetric combination is not). This quasi-primary belongs to a reducible representation of the $\mathfrak{sl}(2,\mathbb{R})$ algebra generated by the zero modes of $J_a(z)$; which can be decomposed into irreps by separating the trace (constructed using the Killing form η_{ab} of $\mathfrak{sl}(2,\mathbb{R})$) and the traceless parts. So the only normal ordered quasi-primaries we need to consider are:

$$(J^2)(w) := \eta^{ab}(J_aJ_b)(w) \ \text{ and } \ (J_{(a}J_{b)})(w) - \frac{1}{3}\eta_{ab}(J^2)(w).$$

Note that these are a singlet ($j = 0$) and a quintuplet ($j = 2$) respectively of the $\mathfrak{sl}(2,\mathbb{R})$ algebra.

5.2.1 The OPE Ansatz

To construct the algebra that we seek we can start with all the OPEs that are unambiguously fixed already by the nature of the chiral operators we have postulated, namely, one $T(z)$ $(h=2)$, one $J_a(z)$ $(h=1)$ and one $G_s(z)$ $(h=3/2)$. These are:

$$T(z)T(w) \sim \frac{c}{2}(z-w)^{-4} + 2\,(z-w)^{-2}\,T(w) + (z-w)^{-1}\,\partial T(w)\,,$$

$$T(z)J_a(w) \sim (z-w)^{-2}J_a(w) + (z-w)^{-1}\partial J_a(w)\,,$$

$$T(z)G_s(w) \sim (3/2)(z-w)^{-2}G_s(w) + (z-w)^{-1}\partial G_s(w) \tag{5.83}$$

and, following the conventions of Polyakov in [112] for the $\mathfrak{sl}(2,\mathbb{R})$ current algebra

$$J_a(z)J_b(w) \sim -\frac{\kappa}{2}\eta_{ab}(z-w)^{-2} + (z-w)^{-1}f_{ab}{}^{c}J_c(w),$$

$$J_a(z)G_s(w) \sim (z-w)^{-1}G_{s'}(w)(\lambda_a)^{s'}{}_{s} \tag{5.84}$$

where $a,b,\cdots = 0,\pm 1$ and $s,s',\cdots = \pm 1/2$. The matrices λ_a form a 2×2 matrix representation of $\mathfrak{sl}(2,\mathbb{R})$ algebra which we take to be:

$$[\lambda_a,\lambda_b] = (a-b)\,\lambda_{a+b}\,. \tag{5.85}$$

We use the following explicit form of these matrices

$$(\lambda_a)^{s}{}_{s'} = \frac{1}{2}(a-2s')\delta^{s}_{a+s'}\,. \tag{5.86}$$

We define the $\mathfrak{sl}(2,\mathbb{R})$ Killing form η_{ab} via $\mathrm{Tr}\,(\lambda_a\lambda_b) = \frac{1}{2}\eta_{ab}$. Using the λ_a in (5.86) we have

$$\mathrm{Tr}(\lambda_a\lambda_b) = (\lambda_a)^{s'}{}_{s}(\lambda_b)^{s}{}_{s'} = \frac{1}{2}(3a^2-1)\delta_{a+b,0}, \tag{5.87}$$

which means $\eta_{ab} = (3a^2-1)\delta_{a+b,0}$ along with its inverse $\eta^{ab} = (3a^2-1)^{-1}\delta_{a+b,0}$.

There is a unique invariant tensor in the tensor product of two spinor $(j=1/2)$ representations of $\mathfrak{sl}(2,\mathbb{R})$, namely the antisymmetric $\epsilon_{ss'}$ and we define it such that $\epsilon_{-\frac{1}{2},+\frac{1}{2}} = 1$. We can lower and raise the fundamental/spinor indices on $(\lambda_a)^{s}{}_{\tilde{s}}$ using $\epsilon_{s\tilde{s}}$ and its inverse $\epsilon^{s\tilde{s}}$, and the adjoint indices using η_{ab} and its inverse η^{ab}. We will also need the matrices $(\lambda^a)_{s\tilde{s}} = \eta^{ab}\epsilon_{ss'}(\lambda_b)^{s'}{}_{\tilde{s}}$ which have

$$(\lambda^a)_{ss'} = \eta^{ab}(\lambda_b)_{ss'} = \frac{1}{2}\delta^a_{s+s'} \tag{5.88}$$

and so are symmetric. Some more useful identities satisfied by the matrices λ_a are

$$(\lambda_a)^r{}_s(\lambda_b)^{r'}{}_{s'}\eta^{ab} = -\frac{1}{2}\epsilon^{rr'}\epsilon_{ss'} - \frac{1}{4}\delta^r_s\delta^{r'}_{s'}\,,$$

$$(\lambda_a)^s{}_{\tilde{s}}(\lambda_b)^{\tilde{s}}{}_{s'} = -\frac{1}{4}\eta_{ab}\,\delta^s{}_{s'} + \frac{1}{2}f_{ab}{}^c(\lambda_c)^s{}_{s'}\,. \tag{5.89}$$

From these one also has $\eta^{ab}(\lambda_a)^s{}_{\tilde{s}}(\lambda_b)^{\tilde{s}}{}_{s'} = -(3/4)\,\delta^s_{s'}$. One more identity that λ_a matrices satisfy is:

$$(\lambda_a)^s{}_{\hat{s}}\epsilon_{s'\tilde{s}} + (\lambda_a)^s{}_{\tilde{s}}\epsilon_{\hat{s}s'} + (\lambda_a)^s{}_{s'}\epsilon_{\tilde{s}\hat{s}} = 0. \tag{5.90}$$

Ansatz for $G_s(z)G_{s'}(w)$ OPE :

The only remaining OPE is that of $G_s(z)G_{s'}(w)$ and using global conformal invariance we can write the right-hand side in terms of the quasi-primaries of dimension up to (and including) $h = 2$ and their global descendants. The list of such quasi-primaries is

$$\Big\{\mathbb{I},\ J_a(w),\ G_s(w),\ T(w),\ (J^2)(w),\ (J_{(a}J_{b)})(w) - (1/3)\eta_{ab}(J^2)(w)\Big\} \tag{5.91}$$

Apart from the (global) conformal symmetry, this OPE has to be consistent with the $\mathfrak{sl}(2,\mathbb{R})$ symmetry (the global part of the $\mathfrak{sl}(2,\mathbb{R})$ current algebra) as well. From the OPE of $J_a(z)$ with $G_s(w)$ we know that the index s is a doublet ($j = 1/2$) index of this $\mathfrak{sl}(2,\mathbb{R})$. Therefore the quasi-primaries that can appear in the OPE $G_s(z)G_{s'}(w)$ can only carry $\mathfrak{sl}(2,\mathbb{R})$ indices of irreps in the tensor product of two doublets, namely the singlet ($j = 0$) and the triplet ($j = 1$). Therefore, in our list (5.91) of quasi-primaries neither $G_s(w)$ nor $(J_{(a}J_{b)})(w) - (1/3)\eta_{ab}(J^2)(w)$ can appear as they correspond to $j = 1/2$ and $j = 2$ irreps of $\mathfrak{sl}(2,\mathbb{R})$ respectively.

- The coefficients in front of the $j = 0$ quasi-primary operators $\{\mathbb{I}, T(w), (J^2)(w)\}$ have to be proportional to the Clebsch–Gordan (CG) coefficients of $\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} \to \mathbf{0}$, namely $\epsilon_{ss'}$.
- The coefficient in front of the $j = 1$ quasi-primary operator $J_a(w)$ has to be

proportional to the CG coefficients of $\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} \to \mathbf{1}$, namely $(\lambda^a)_{ss'}$.

Thus after using all the global symmetries we arrive at the ansatz:

$$G_s(z)G_{s'}(w) = \quad 2\alpha\, \epsilon_{ss'}(z-w)^{-3} + \delta\,(z-w)^{-2}\,(\lambda^a)_{ss'}\,[2\,J_a(w) + (z-w)\,J_a{}'(w)] \\ +\beta\,\epsilon_{ss'}(z-w)^{-1}\,T(w) + \gamma\,(z-w)^{-1}\,\epsilon_{ss'}(J^2)(w)\ . \tag{5.92}$$

The parameters (α, β, γ, δ) in (5.92) as well as the central charges (c, κ) in (5.83, 5.84) are to be constrained using OPE associativity, which we implement as Jacobi identities of commutators of the modes of our simple primaries. Writing the operators $T(z)$, $J_a(z)$ and $G_r(z)$ in terms of their modes as

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z), \quad J_{a,n} = \oint \frac{dz}{2\pi i} z^n J_a(z), \quad G_{s,r} = \oint \frac{dz}{2\pi i} z^{r+\frac{1}{2}} G_s(z) \tag{5.93}$$

with $n \in \mathbb{Z}$ and $r \in \mathbb{Z} + \frac{1}{2}$, the OPEs (5.83, 5.84, 5.92) translate to:

1. **Virasoro algebra:**

$$[L_m, L_n] = (m-n)\,L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \tag{5.94}$$

2. **Virasoro primaries:**

$$[L_m, J_{a,n}] = -n\,J_{a,m+n}, \quad [L_m, G_{s,r}] = \frac{1}{2}(n-2r)\,G_{s,n+r} \tag{5.95}$$

3. $\mathfrak{sl}(2,\mathbb{R})$ **current algebra:**

$$[J_{a,m}, J_{b,n}] = -\frac{1}{2}\kappa\, m\, \eta_{ab}\,\delta_{m+n,0} + f_{ab}{}^c\, J_{c,m+n} \tag{5.96}$$

4. **Current algebra primary:**

$$[J_{a,n}, G_{s,r}] = G_{s',n+r}(\lambda_a)^{s'}{}_s \tag{5.97}$$

5. **Commutator** $[G_{s,r}, G_{s',r'}]$:

$$[G_{s,r},G_{s',r'}]=\epsilon_{ss'}\left[\alpha\left(r^2-\frac{1}{4}\right)\delta_{r+r',0}+\beta\,L_{r+r'}+\gamma\,(J^2)_{r+r'}\right]+\delta(r-r')J_{a,r+r'}(\lambda^a)_{ss'} \tag{5.98}$$

Finally, following [25] we have to impose the Jacobi identities on these commutation relations. We turn to this next.

5.2.2 Imposing Jacobi Identities

Following our analysis of the imposition of Jacobi Identities, the details of which are presented in Appendix E, the parameters $(\alpha, \beta, \gamma, \delta, c, \kappa)$ must satisfy

$$\alpha-\frac{c}{6}\beta+\gamma\,\frac{\kappa}{2}=0,\quad \alpha-\delta\,\frac{\kappa}{2}=0,\quad \beta-\gamma\,(\kappa+2)-\frac{1}{2}\delta=0,\quad \beta-\frac{\gamma-\delta}{2}=0\ . \tag{5.99}$$

These are linear and homogeneous equations in the variables $(\alpha, \beta, \gamma, \delta)$. So for some non-trivial solutions for $(\alpha, \beta, \gamma, \delta)$ to exist the determinant of the coefficient matrix

$$\begin{pmatrix} 1 & -\frac{c}{6} & \frac{\kappa}{2} & 0 \\ 1 & 0 & 0 & -\frac{\kappa}{2} \\ 0 & 1 & -(\kappa+2) & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

has to vanish. This gives $c(2\kappa+5)+6\kappa\,(2\kappa+1)=0$ which can be solved for c (for $\kappa \neq -5/2$) as

$$c=-\frac{6\kappa\,(1+2\kappa)}{5+2\kappa}\ . \tag{5.100}$$

When this holds, for generic values of κ the coefficient matrix is of rank 3 and so there exists one non-trivial solution, which can be written (taking γ to be the independent variable) as

$$\alpha=-\frac{1}{4}\gamma\kappa(3+2\kappa),\quad \beta=\frac{1}{4}\gamma(5+2\kappa),\quad \delta=-\frac{1}{2}\gamma(3+2\kappa)\,. \tag{5.101}$$

This is our final result valid for generic values of κ ($\neq -5/2$). The full commutator algebra is (5.1) which we denoted by $\mathcal{W}(2;(3/2)^2\,,1^3)$. Some observations are in order:

- As mentioned in the introduction, a $\mathcal{W}$-algebra identical to that of $\mathcal{W}(2;(3/2)^2,1^3)$ already existed in the literature in a different context [24]. The current algebra in [24] was $\mathfrak{sp}(2)$ which is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

- When $c \neq -\frac{6\kappa\,(1+2\kappa)}{5+2\kappa}$ we have to set $\alpha=\beta=\gamma=\delta=0$, and the corresponding algebra is a chiral extension of $\mathfrak{iso}(1,3)$ with arbitrary c and κ.

  This case corresponds to the algebra found in [13, 14] as the central charges there remained undetermined.

- We may also consider the limit $\gamma\to 0$ in the non-trivial solution (5.101). This corresponds to a contraction of the algebra (5.98) to the chiral extension of $\mathfrak{iso}(1,3)$, but with the relation (5.100) still in place.

  The chiral $\mathfrak{bms}_4$ algebra obtained from the analysis of chiral boundary conditions on $\mathbb{R}^{1,3}$ or $\mathbb{R}^{2,2}$ gravity appears to belong to this class.

- When $\gamma\neq 0$ its value can be fixed to be any non-zero function of κ by an appropriate normalisation of the generators $G_{s,r}$. Then the algebra is determined completely in terms of one parameter (say) κ.

Relation to $\mathfrak{so}(2,3)$

The non-trivial solution (5.100, 5.101) makes the algebra in (5.94 - 5.98) like the $\mathcal{W}$-algebras of yesteryears, particularly close to the Bershadsky-Polyakov algebra $\mathcal{W}_3^{(2)}$ [167, 168]. Now we will argue that just as the $\mathcal{W}_3^{(2)}$ algebra is based on $\mathfrak{sl}(3,\mathbb{R})$ algebra, this new algebra is based on (and is an extension of) $\mathfrak{so}(2,3)$. For this, first note that $\mathfrak{so}(2,3)$ algebra can be written more suggestively as follows:

$$\begin{aligned}
&[L_m,L_n]=(m-n)L_{m+n},\quad [\bar{L}_m,\bar{L}_n]=(m-n)\,\bar{L}_{m+n},\quad [L_m,\bar{L}_n]=0\\
&[L_n,G_{s,r}]=\frac{1}{2}(n-2r)\,G_{s,n+r},\quad [\bar{L}_n,G_{s,r}]=\frac{1}{2}(n-2s)\,G_{n+s,r},\\
&\qquad [G_{s,r},G_{s',r'}]=2\,\epsilon_{rr'}\,\bar{L}_{s+s'}+2\,\epsilon_{ss'}\,L_{r+r'}
\end{aligned}\tag{5.102}$$

for $m,n\dots=0,\pm1$, $r,s,\dots=\pm1/2$. Here one may think of L_n and $\bar{L}_n$ to generate either the $\mathfrak{so}(2,2)=\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ subalgebra or $\mathfrak{so}(1,3)=\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})^c$ subalgebra of $\mathfrak{so}(2,3)$. With the identification: $\bar{L}_a=J_{a,0}$ and noticing that ${f_{ab}}^c=(a-b)\delta^c_{a+b}$, the first two lines have the same form as those of our algebra (now restricted to $m,n\dots=0,\pm1$, $r,s,\dots=\pm1/2$). To compare the $[G_{s,r},G_{s',r'}]$ let us first take the large-κ (the "classical") limit in (5.98). In this limit, from (5.100, 5.101) we have[11]

$$\alpha \to -\frac{1}{2}\gamma\kappa^2, \quad \beta \to \frac{1}{2}\gamma\kappa, \quad \delta \to -\gamma\kappa\,, \quad c \to -6\,\kappa\,. \tag{5.103}$$

Further in this limit by choosing $\gamma \to -\frac{2}{\kappa}$ we arrive at

$$\alpha \to \kappa, \quad \beta \to -1, \quad \gamma \to -\frac{2}{\kappa}\,, \quad \delta \to 2\,, \quad c \to -6\,\kappa\,. \tag{5.104}$$

Finally noticing that in (5.98) the first term proportional to α drops out when restricted to $r, r' = \pm 1/2$. The term proportional to δ, for $r, r' = \pm 1/2$ can be rewritten, using $(r - r') = -\epsilon_{rr'}$ and $(\lambda^a)_{ss'} = \frac{1}{2}\delta^a_{s+s'}$, as

$$\delta\,(r - r')\,J_{a,r+r'}\,(\lambda^a)_{ss'} \to -\epsilon_{rr'}\bar{L}_{s+s'}\ . \tag{5.105}$$

Therefore we conclude that the commutator (5.98) in the large-κ limit and restricted to the global modes $(r, r', s, s' \cdots = \pm 1/2)$ reads

$$[G_{s,r}, G_{s',r'}] = 2\,\epsilon_{rr'}\,\bar{L}_{s+s'} + 2\,\epsilon_{ss'}\,L_{r+r'} + \mathcal{O}(1/\kappa).$$

Thus the full algebra (5.1) is therefore an extension (and a deformation around large-κ) of $\mathfrak{so}(2,3)$ algebra. This completes our analysis.

Comparison with the $\mathcal{W}$-algebra of AdS$_4$ gravity

In the large κ-limit, the coefficients $\{\alpha, \beta, \delta, c\}$ take the value as in (5.104). In this limit the OPEs become,

$$\begin{aligned} G_s(z)\,G_{s'}(w) &= \frac{2\kappa\,\epsilon_{ss'}}{(z-w)^3} - \frac{\epsilon_{ss'}\,T(w)}{(z-w)} - \frac{\epsilon_{ss'}\,\frac{2}{\kappa}\,J^2(w)}{(z-w)} \\ &\quad + 2\,(\lambda)^a_{ss'}\left[\frac{\partial J_a(w)}{(z-w)} + \frac{2\,J_a(w)}{(z-w)^2}\right] \end{aligned} \tag{5.106}$$

$$T(z)\,T(w) = -\frac{3\,\kappa}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{5.107}$$

The rest of the other OPEs in this limit do not change. For the fixed value of $\kappa = 2$ and $c = -12$, these OPEs coincide with the OPEs in (5.77) (with the following redefinition $G_s \to \frac{\sqrt{G}}{l}G_s$) that were derived using the postulated line integral charge

[11] It is curious that in this classical limit, the Virasoro central charge approaches the same value as in [10, 169, 170], namely $c = -6\kappa$.

(5.76). Therefore the currents that form the representation of chiral Λ-$\mathfrak{bms}_4$ in AdS$_4$ gravity obey a $\mathcal{W}$-algebra which is the semi-classical limit of $\mathcal{W}(2;(3/2)^2,1^3)$ that we derived in this section.

5.3 The 3 and 4-point functions of currents

We have seen in previous sections how one can use Jacobi identities on the commutators of the modes of the chiral (quasi-) primaries to impose the associativity of OPEs of the corresponding chiral operators. Following [171], one can equivalently impose the associativity of Operator Product Algebra by demanding the conformal invariance and crossing symmetry conditions of 3 and 4- point functions of symmetry generating currents $\{T(z), J_a(z), G_s(z)\}$. In this section, we present all the possible three-point and four-point correlation functions with the help of OPE relations (5.83), (5.84) and (5.92), calculated using the method spelt out in [171]. Here we also assume that one point function is zero for all the local operators. One demands that these correlation functions follow two principles,

1. The n-point correlation function of primaries transform as follows under global $SL(2,\mathbb{R})$ conformal transformations $z \to \frac{a\,z+b}{c\,z+d}$,

$$\langle \phi'_1(z'_1)\,\phi'_2(z'_2)\,\cdots\phi'_n(z'_n)\rangle = \left(\frac{dz'_1}{dz_1}\right)^{-h_1}\dots\left(\frac{dz'_n}{dz_n}\right)^{-h_n}\langle \phi_1(z_1)\,\phi_2(z_2)\,\cdots\phi_n(z_n)\rangle \tag{5.108}$$

   with the condition $ad-bc=1$ imposed on the parameters $\{a,b,c,d\}$.

2. Correlation functions should be invariant under the rearrangement of quasi-primaries in the correlation function, *i.e.*, permutation symmetry.

It can be shown that imposing the above two conditions on these correlation functions also leads to the constraint equations (5.99). The details of this calculation are outlined in Appendix F. Here we only list these correlation functions resulting after imposing equations (5.99). The non-zero 2-point correlation functions are,

$$\langle T(z_1)\,T(z_2)\rangle = \frac{\frac{c}{2}}{z_{12}^4}\,,\;\langle J_{a_1}(z_1)\,J_{a_2}(z_2)\rangle = -\frac{\frac{\kappa}{2}\,\eta_{a_1a_2}}{z_{12}^2}\,,\;\langle G_{s_1}(z_1)\,G_{s_2}(z_2)\rangle = \frac{2\,\alpha\,\epsilon_{s_1s_2}}{z_{12}^3} \tag{5.109}$$

where $z_{ij}=z_i-z_j$.

5.3.1 3-point functions

One expects the correlation functions to respect the global part of $\mathfrak{sl}(2,\mathbb{R})$ current algebra as well under $J_a(z)$ is a triplet and $G_s(z)$ is a doublet. Therefore any 3-point correlator involving J_a's and G_s's should be given in terms of invariant tensors carrying their representation indices of $\mathfrak{sl}(2,\mathbb{R})$ and should vanish if no such invariant tensors exist. This immediately leads to the expectation that any 3-point function with one $J_a(z)$ or odd number of $G_s(z)$'s vanishes. It is easy to verify that this is indeed the case.

Now we list the remaining 3-point functions. The 3-point correlation functions of the type $\langle TTT\rangle\,,\langle JJJ\rangle$ are:

$$\langle T(z_1)\,T(z_2)\,T(z_3)\rangle = \frac{c}{z_{12}^2\,z_{13}^2\,z_{23}^2}\;,\quad \langle J_{a_1}(z_1)\,J_{a_2}(z_2)\,J_{a_3}(z_3)\rangle = -\frac{\frac{\kappa}{2}\,f_{a_1a_2a_3}}{z_{12}\,z_{13}\,z_{23}}\;, \tag{5.110}$$

These are invariant under the exchange of the insertion points (z_i) and corresponding indices (a_i) and transform as (5.108) under global $SL(2,\mathbb{R})$ conformal transformation with $h=2$ and $h=1$ respectively. The correlation functions of the type $\langle TJJ\rangle$ are:

$$\begin{aligned}\langle T(z_1)\,J_{a_2}(z_2)\,J_{a_3}(z_3)\rangle &= \langle J_{a_2}(z_2)\,T(z_1)\,J_{a_3}(z_3)\rangle = \langle J_{a_2}(z_2)\,J_{a_3}(z_3)\,T(z_1)\rangle\\ &= -\frac{\kappa\,\eta_{a_2a_3}}{2\,z_{12}^2\,z_{13}^2}\end{aligned} \tag{5.111}$$

The correlation function (5.111) is invariant under the permutation of the z_i and corresponding a_i index as expected. The 3-point correlation functions that contain two $G's$ are given by,

$$\begin{aligned}\langle T(z_3)\,G_{s_1}(z_1)\,G_{s_2}(z_2)\rangle &= \langle G_{s_1}(z_1)\,T(z_3)\,G_{s_2}(z_2)\rangle = \langle G_{s_1}(z_1)\,G_{s_2}(z_2)\,T(z_3)\rangle\\ &= \frac{3\,\alpha\,\epsilon_{s_1s_2}}{z_{12}\,{z_{13}}^2\,z_{23}^2}\end{aligned} \tag{5.112}$$

and

$$\begin{aligned}\langle J_a(z_3)\,G_{s_1}(z_1)\,G_{s_2}(z_2)\rangle &= \langle G_{s_1}(z_1)\,J_a(z_3)\,G_{s_2}(z_2)\rangle = \langle G_{s_1}(z_1)\,G_{s_2}(z_2)\,J_a(z_3)\rangle\\ &= -\frac{2\alpha\,\left(\lambda_a\right)_{s_1s_2}}{z_{13}\,z_{23}\,{z_{12}}^2}.\end{aligned} \tag{5.113}$$

These are invariant under $(z_1,s_1)\leftrightarrow(z_2,s_2)$ interchange as expected.

5.3.2 4-point functions

Eight types of 4-point functions contain one J_a or odd number of G_s and they all vanish as expected. This leaves us with seven types of 4-point functions that are non-trivial and we provide their answers below.

The correlation function of four $T(z)$'s is given by,

$$\begin{aligned}\langle T(z_1)\, T(z_2)\, T(z_3)\, T(z_4)\rangle = & \frac{2c}{(z_1-z_2)^4\,(z_3-z_4)^4}\left[\frac{x^4}{(1-x)^2}+\frac{x^2}{(1-x)}\right] \\ & + \frac{c^2}{4\,(z_1-z_2)^4(z_3-z_4)^4}\left[1+x^4+\frac{x^4}{(1-x)^4}\right], \end{aligned} \tag{5.114}$$

where x is the cross ratio defined as

$$x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}.$$

This correlation function is expected to be invariant under each of the permutations of z_i. To see this it is sufficient to check that the answer is invariant under each of the exchanges $z_1 \leftrightarrow z_4$ ($x \to \frac{1}{x}$) and $z_1 \leftrightarrow z_3$ ($x \to 1-x$). The conformal factor outside the bracket in (5.114) under these transformations compensates for the changes inside the bracket such that the resultant answer is invariant.

The 4-point function with all $J_a(z)$ is given by

$$\langle J_{a_1}(z_1)\, J_{a_2}(z_2)\, J_{a_3}(z_3)\, J_{a_4}(z_4)\rangle = \frac{1}{z_{12}^2\, z_{34}^2}\,\left(\kappa^2\, I^{(1)}_{a_1a_2a_3a_4}(x) + \kappa\, I^{(2)}_{a_1a_2a_3a_4}(x)\right) \tag{5.115}$$

where

$$I^{(1)}_{a_1a_2a_3a_4}(x) = \frac{1}{4}\left[\eta_{a_1a_2}\,\eta_{a_3a_4} + x^2\,\eta_{a_1a_3}\,\eta_{a_2a_4} + \frac{x^2}{(1-x)^2}\,\eta_{a_1a_4}\,\eta_{a_2a_3}\right], \tag{5.116}$$

$$I^{(2)}_{a_1a_2a_3a_4}(x) = -\frac{1}{2}\left[\frac{x\,(2-x)}{1-x} f_{a_1a_2}{}^{a_5}\, f_{a_5a_3a_4} - \frac{x}{1-x}\, f_{a_1a_3}{}^{a_5}\, f_{a_5a_2a_4} + x\, f_{a_1a_4}{}^{a_5}\, f_{a_5a_2a_3}\right]. \tag{5.117}$$

This correlator is expected to be invariant under the permutation of (z_i, a_i). For this, it is sufficient to check that the answer (5.115) is independently invariant under $(x \leftrightarrow \frac{1}{x}\ \ \&\ \ a_1 \leftrightarrow a_4)$ and $(x \leftrightarrow 1-x\ \ \&\ \ a_1 \leftrightarrow a_3)$. One can use Jacobi identity for the structure constants $f_{ab}{}^c$

$$f_{a_1a_2}{}^{a_5}\, f_{a_5a_3a_4} - f_{a_1a_3}{}^{a_5}\, f_{a_5a_2a_4} + f_{a_1a_4}{}^{a_5}\, f_{a_5a_2a_3} = 0 \tag{5.118}$$

to show that (5.115) is indeed invariant under these transformations.

The next non-trivial 4-point functions are

$$\langle T(z_1)\, J_{a_2}(z_2)\, J_{a_3}(z_3)\, J_{a_4}(z_4)\rangle = \frac{z_{24}}{z_{14}\, z_{12}^3\, z_{34}^2}\, \frac{\kappa}{2} f_{a_2 a_3 a_4} \left[\frac{x^3}{x-1} + \frac{x}{x-1} - \frac{x^2}{x-1}\right] \tag{5.119}$$

This correlation function (5.119) is invariant under all the permutations of (z_i, a_i) for the three $J_{a_i}(z_i)$ insertions. One can further show that the correlation functions $\langle JTJJ\rangle$, $\langle JJTJ\rangle$ and $\langle JJJT\rangle$ are all equal to (5.119), thus establishing its invariance under all the possible rearrangements of the four operators involved.

The correlation functions with two $T(z)$'s and two $J_a(z)$'s such as $\langle TTJJ\rangle$, $\langle JTTJ\rangle$, $\langle JJTT\rangle$, $\langle TJJT\rangle$,$\langle JTJT\rangle$ and $\langle TJJT\rangle$ are all equal to each other and given by,

$$\langle T(z_1)\, T(z_2)\, J_{a_3}(z_3)\, J_{a_4}(z_4)\rangle = -\frac{\kappa\, \eta_{a_3 a_4}}{z_{12}^4\, z_{34}^2} \left[\frac{1}{2}\left(x^2 + \frac{x^2}{(x-1)^2}\right) + \frac{c}{4}\right] \tag{5.120}$$

This expression (5.120) is invariant under $(z_1 \leftrightarrow z_2)$ and/or $(z_3, a_3) \leftrightarrow (z_4, a_4)$.

The next non-trivial 4-point function is with two G and two T given by,

$$\langle T(z_1)\, T(z_2)\, G_{s_3}(z_3)\, G_{s_4}(z_4)\rangle = \frac{3\,\alpha\, \epsilon_{s_3 s_4}}{z_{12}^4\, z_{34}^3} \left(\frac{c}{3} + \frac{2\,x^2}{(1-x)^2} - \frac{x^3}{2(1-x)^2} - \frac{3\,x^3}{2(1-x)}\right) \tag{5.121}$$

The correlations obtained through permutations, such as $\langle GTTG\rangle$, $\langle GGTT\rangle$, $\langle GTGT\rangle$, $\langle TGTG\rangle$, $\langle TGGT\rangle$ are all equal to $\langle TTGG\rangle$ in eq. (5.121) as required. Next, we give the expression for $\langle JJGG\rangle$ type correlator:

$$\langle J_{a_1}(z_1)\, J_{a_2}(z_2)\, G_{s_3}(z_3)\, G_{s_4}(z_4)\rangle = -\frac{\alpha}{z_{12}^2\, z_{34}^3}\, \mathcal{J}_{a_1 a_2 s_3 s_4} \tag{5.122}$$

$$\mathcal{J}_{a_1 a_2 s_3 s_4} = \epsilon_{s_3\, s_4}\, \eta_{a_1 a_2} \left(\kappa + \frac{x^2}{2(1-x)}\right) + f_{a_1 a_2}{}^{a_3}\, (\lambda_{a_3})_{s_3\, s_4} \left(\frac{2x - x^2}{1-x}\right) \tag{5.123}$$

which can also be shown to be invariant under all permutations of the four operators involved.

Finally, we present the correlation function involving four $G_s(z)$ operators. After imposing the constraints (5.99), one can write this correlation function as follows,

$$\langle G_{s_1}(z_1)\, G_{s_2}(z_2)\, G_{s_3}(z_3)\, G_{s_4}(z_4)\rangle = \frac{2\,\alpha^2}{z_{12}^3\, z_{34}^3}\, \left(2\, F^{(1)}_{s_1s_2s_3s_4}(x) - \frac{1}{\kappa}\, F^{(2)}_{s_1s_2s_3s_4}(x)\right) \tag{5.124}$$

where

$$F^{(1)}_{s_1s_2s_3s_4}(x) = \epsilon_{s_1s_2}\, \epsilon_{s_3s_4}\, \left(1+x^3\right) + \epsilon_{s_1s_4}\, \epsilon_{s_2s_3}\, \left(x^3 + \frac{x^3}{(1+x)^3}\right)\,, \tag{5.125}$$

$$\begin{aligned} F^{(2)}_{s_1s_2s_3s_4}(x) &= \epsilon_{s_1s_2}\, \epsilon_{s_2s_4}\, \left(x + x^2 + \frac{x^2}{(1-x)^2} + \frac{x^3}{(1-x)^2}\right) \\ &\quad \epsilon_{s_1s_4}\, \epsilon_{s_2s_3}\, \left(x + \frac{x}{1-x} + \frac{x^3}{(1-x)} + \frac{x^3}{(1-x)^2}\right)\,. \end{aligned} \tag{5.126}$$

This correlator is expected to be invariant under all the permutations of the pairs (z_i, a_i) which can be shown to be the case using the the following identity

$$\epsilon_{s_1s_3}\, \epsilon_{s_2s_4} - \epsilon_{s_1s_4}\, \epsilon_{s_2s_3} - \epsilon_{s_1s_2}\, \epsilon_{s_3s_4} = 0. \tag{5.127}$$

This completes our verification of the expectation that the new algebra of the quasi-primaries $\{T(z), J_a(z), G_s(z)\}$ does lead to 3 and 4-point correlation functions of these currents that are invariant under appropriate permutations.

5.4 Further Comments and Discussions

In this chapter, we found an analogue of chiral $\mathfrak{bms}_4$ algebra for $\mathfrak{so}(2,3)$ Lie algebra. This chiral extension of $\mathfrak{so}(2,3)$ which we denote as chiral Λ-$\mathfrak{bms}_4$ was found to be the asymptotic symmetry algebra of locally AdS$_4$ solutions parameterised by six holomorphic functions $(T(z), J_a(z), G_s(z))$ for $a = 0, \pm 1$ and $s = \pm 1/2$ after imposing chiral boundary conditions on codimension-two hypersurface consistent with the well defined variational problem. We postulated line integral charge from the bulk AdS$_4$ gravity that induces the variations of these holomorphic currents. The charge algebra obeyed a symmetry algebra which turned out to be the semi-classical limit of a $\mathcal{W}$-algebra. To validate our findings we derive such a $\mathcal{W}$-algebra from $2d$ chiral CFT techniques. We show that there exists a one-parameter family of chiral $\mathcal{W}$-algebras generated by six chiral operators $(T(z), J_a(z), G_s(z))$ for $a = 0, \pm 1$ and $s = \pm 1/2$ with dimensions $(2, 1, 3/2)$ which in semi-classical limit matches with the $\mathcal{W}$-algebra derived from AdS$_4$ gravity. This $\mathcal{W}$-algebra is identical to the one derived by Romans [24]. In the $\kappa \to \infty$ limit it admits $\mathfrak{so}(2,3)$ as a subalgebra. Just as $\mathfrak{so}(2,3)$ admits a contraction to $\mathfrak{iso}(1,3)$ our $\mathcal{W}$-algebra (5.1) also admits a contraction ($\gamma \to 0$ for finite κ) to chiral $\mathfrak{bms}_4$ algebra that appeared in the studies of graviton soft-theorems [13, 14].

Using the nomenclature of [172] we refer to this new algebra (5.1) as $\mathcal{W}(2;,(3/2)^2,1^3)$. It turns out this algebra is similar to the algebra found by Romans [24] in a different context. The spin-1 Kac-Moody current in [24] was $\mathfrak{sp}(2)$ in contrast to our $\mathfrak{sl}(2,\mathbb{R})$ current algebra. Even though $\mathcal{W}(2;,(3/2)^2,1^3)$ has a chiral primary $G_s(z)$ of half-integer dimension, it is not a superconformal algebra such as the one of Knizhnik [173]. The reason for this is that even though the current G_s has $h = 3/2$ it still has an integer spin defined by $L_0 - J_0$. Our algebra (5.1) is more akin to the Bershadsky-Polyakov algebra $\mathcal{W}_3^{(2)}$ – where by considering a twisted stress tensor one can make $G_s(z)$ to appear bosonic (integral dimension).

The existence of line integral charges (5.76) from bulk AdS_4 gravity needs to be understood better. Usual ways of calculating charges through covariant phase space formalism [120, 135, 136] or the cohomological formalism [118, 137, 138] lead to co-dimension two surface charges. The charges for asymptotically locally AdS_4 solution as computed in [95, 99, 16] are proportional to holographic stress tensor which vanishes for our locally AdS_4 solution. However, as shown by authors in [99], for the Neumann boundary condition, the effective boundary theory is an induced gravity theory for which it is expected that such codimension-two charges will vanish. In that case, the calculation of this induced gravity action at the boundary can lead to such line integral charges. For locally AdS_4 solution as shown by Skenderis and Solodukhin in [105], the Weyl and diffeomorphism invariant boundary effective action is zero therefore one needs to revisit the arguments of [105] and provide a prescription to calculate such an effective action for locally AdS_4 configurations. The possibility of adding other boundary terms to the bulk action may also provide a solution for such a problem as will be touched upon in the next chapter where one can add a Pontryagin term to the bulk Einstein-Hilbert action that gives rise to Chern-Simons gravity action at the boundary.

Originally the (finite-k) $\mathcal{W}_3^{(2)}$ algebra was realised in the constrained WZW model with $\mathfrak{sl}(3,\mathbb{R})$ algebra [167]. One expects that a similar realisation should exist for the algebra presented here as well via Drinfeld-Sokolov reduction of $\mathfrak{so}(2,3)$ current algebra (See for e.g [174, 175]). It will be interesting to exhibit this. Another chiral $\mathcal{W}$-algebra extension of $\mathfrak{so}(2,3)$ in the classical limit was obtained in [20] from the asymptotic symmetry analysis of $3d$ conformal gravity. This algebra was called conformal $\mathfrak{bms}_3$ or $W_{(2,2,2,1)}$.[12]

One should also study possible representations of the chiral algebra constructed here. Using the usual notions of inner products and unitarity of $2d$ CFTs, one does not expect the algebra $\mathcal{W}(2;,(3/2)^2,1^3)$ to admit unitary representations without imposing further constraints. This is because $\mathfrak{sl}(2,\mathbb{R})$ current algebra at non-zero level does not admit any unitary representations, the algebra will contain negative norm states as can be seen from the two-point function of $\langle J_a(z)J_b(w)\rangle$ in equation (5.109). The authors in [169] alluded to this issue, where $\mathfrak{sl}(2,\mathbb{R})$ current algebra arises in the context of $2d$ Polyakov-induced gravity. There exist two possible scenarios to preserve unitarity in this case. One can impose constraints to remove the negative normed states, which require setting some of the currents $J_a(z)$ to zero (see for example [176]). Alternatively, one can set the level κ of the current algebra to zero. One example of level zero $\mathfrak{sl}_2$ current algebra is [13], which appears in the context of celestial holography.[13]

Apart from the $J_a(z)$ of the $\mathfrak{sl}(2,\mathbb{R})$ current algebra at non-zero level, one can check that the action of the $G_s(w)$ operators on the vacuum state will also generate states with the negative norm. These additional negative normed states are similar to the ones encountered in the case of Feigin-Semikhatov algebra $W_n^{(2)}$, which for $n=3$ is the Berdshasky-Polyakov algebra. In this case, one tries to give a less limiting notion of unitarity to construct unitary representation for even n [178]. Alternatively, we note that setting the level $\kappa=0$ for current algebra will also make the two-point function $\langle G_s(z)G_{s'}(w)\rangle$ vanishing (the parameter α in (5.109) goes to zero as can be seen from the first equation of (5.101)) thus circumventing the problem of negative normed states for this algebra. It will be interesting to explore these topics further.

[12]We will provide the derivation of the full quantum version of this algebra for finite c and κ and other possible chiral $\mathcal{W}$-algebra extensions of $\mathfrak{so}(2,3)$ in the next chapter.

[13]See also [177] where the authors have shown that level zero $\mathfrak{su}(2)$ Kac-Moody conformal field theory is topological.

Chapter 6

All chiral $\mathcal{W}$-algebra extensions of $\mathfrak{so}(2,3)$ algebra and their holographic realisations

In the previous chapter, we showed that the charge algebra associated with chiral Λ-$\mathfrak{bms}_4$ symmetry of AdS$_4$ gravity is a $\mathcal{W}$-algebra. There, we also derived the complete quantum version of this $\mathcal{W}$-algebra from standard $2d$ chiral CFT techniques, which we denoted by $\mathcal{W}(2;(3/2)^2,1^3)$. Now it becomes important to ask if this is the only such chiral $\mathcal{W}$-algebra extension of $\mathfrak{so}(2,3)$. In this chapter, we address this question and show that $\mathcal{W}(2;(3/2)^2,1^3)$ found in Chapter 5 is one of the four such chiral $\mathcal{W}$-algebras. We identify the remaining three explicitly completing the list. The construction of these $\mathcal{W}$-algebras rely on rewriting $\mathfrak{so}(2,3)$ in four different ways by following these simple steps:

1. To start with, we show that there are exactly four inequivalent ways to embed a copy of $\mathfrak{sl}(2,\mathbb{R})$ inside $\mathfrak{so}(2,3)$.

2. Then we identify in each case the maximal sub-algebra $\mathfrak{h} \subset \mathfrak{so}(2,3)$ that commutes with the $\mathfrak{sl}(2,\mathbb{R})$ of the first step.

3. The rest of the generators arrange themselves into finite dimensional irreducible representations of the sub-algebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{h}$. These steps facilitate writing $\mathfrak{so}(2,3)$ Lie algebra in four avatars.

The four inequivalent copies of $\mathfrak{sl}(2,\mathbb{R})$ embedded into $\mathfrak{so}(2,3)$ have $\mathfrak{h}$ as $\emptyset$, $\mathfrak{so}(1,1)$, $\mathfrak{so}(2)$ and $\mathfrak{sl}(2,\mathbb{R})$. Each of these leads to, a chiral $\mathcal{W}$-algebra extension. The resultant algebra corresponding to the case with $\mathfrak{h} = \emptyset$ can be identified to be the $\mathcal{W}(2,4)$ algebra found long ago [179]. Another, corresponding to $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{R})$ is $\mathcal{W}(2;(3/2)^2,1^3)$ that we derived in Chapter 5. We will complete this list by constructing the remaining two, with $\mathfrak{h} = \mathfrak{so}(1,1)$ and $\mathfrak{h} = \mathfrak{so}(2)$. The one with $\mathfrak{h} = \mathfrak{so}(1,1)$ may be called the conformal $\mathfrak{bms}_3$ whose semi-classical limit (that is, valid only for large values of central charges c and κ) has been obtained from a $3d$ conformal gravity computation by [20].

Furthermore, we will derive the semi-classical limit of $\mathcal{W}(2,4)$ and $\mathcal{W}(2;(3/2)^2,1^3)$ from the asymptotic symmetries of $3d$ $\mathfrak{so}(2,3)$ Chern-Simons gauge theory which is equivalent to $3d$ conformal gravity in the first order formulation as shown by Witten and Horne in [103]. Using the standard methodology reviewed in (2.1.4), we provide boundary conditions and gauge fixing of $\mathfrak{so}(2,3)$ gauge connection that is compatible with variational principle and show that the symmetry algebra of charges that generate residual gauge transformations of these gauge connections form representations of $\mathcal{W}(2,4), \mathcal{W}(2;(3/2)^2,1^3)$ algebras in the semi-classical limit. The analogous calculation for $\mathcal{W}(2;2^2,1)$ already existed in the literature [20], which can be extended trivially for the case of $\widetilde{\mathcal{W}}(2;2^2,1)$. See [21, 22, 23] for previous works on asymptotic symmetries of $3d$ conformal gravity which do not include global conformal symmetry algebra $\mathfrak{so}(2,3)$.

We also consider $3d$ gravitational Chern-Simons action (6.153), an equivalent second-order formulation of $3d$ conformal gravity. This action is usually added to the Einstein-Hilbert action where the resultant theory describes a topologically massive gravity [113, 114, 180, 181, 182]. However, this action alone has not been explored much in comparison to the equivalent first-order formulation of conformal gravity in the language of $\mathfrak{so}(2,3)$ gauge theory. In [21], the authors studied holography for theory described by gravitational Chern Simons action. They proposed the Dirichlet boundary conditions and showed the existence of two copies of Virasaro algebra with abelian Weyl scaling as the asymptotic symmetry algebra when the Weyl factor of the bulk metric is not kept fixed. In this chapter after providing arguments for the potential role played by this gravitational Chern Simons action in AdS_4 gravity, we analyse the asymptotic symmetries of such theory by considering mixed boundary conditions on the configuration space such that the solutions satisfy the variational principle. The treatment of [21] included the calculation of charges using first-order formulation language. We further derive charges associated with Weyl transformation and diffeomorphisms using the modified covariant phase space formalism proposed by Tachikawa [28] as reviewed in (2.1.5) and show that the symmetry algebra that charges obey is semi-classical limit of $\mathcal{W}(2;(3/2)^2,1^3)$.

The rest of this chapter is organised as follows. In Section (6.1) we provide a simple classification of all the inequivalent embedding of $\mathfrak{sl}(2,\mathbb{R})$ inside $\mathfrak{so}(2,3)$. This results in four different ways to write the algebra $\mathfrak{so}(2,3)$ which makes the embedded $\mathfrak{sl}(2,\mathbb{R})$ manifest. In Section (6.2) we provide a derivation of two novel $\mathcal{W}$-algebra extension of $\mathfrak{so}(2,3)$ completing the list of such algebras along with the two already known from [179] and (5.1) (See also [24]). In Section (6.3) and (6.4) we provide the derivation of $\mathcal{W}(2;(3/2)^2,1^3)$ and $\mathcal{W}(2,4)$ respectively from the asymptotic symmetries of $\mathfrak{so}(2,3)$ gauge theory. In Section (6.5), we discuss gravitational Chern-Simons ($3d$ Weyl gravity) action and its application in AdS_4/CFT_3 holography and show that the set of residual diffeomorphisms and Weyl symmetry form an infinite dimensional algebra which is $\mathcal{W}(2;(3/2)^2,1^3)$. We conclude with a discussion and future directions in Section (6.6). In Appendix G we provide a similar derivation of chiral algebra extensions of $\mathfrak{so}(2,2)$ and show that they are isomorphic to the Brown-Henneaux algebra [2] and the conformal induced gravity algebra of Polyakov [112] as realised in [10, 169]. This chapter will be partly based on [183].

6.1 Embedding $\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{so}(2,3)$

We are interested in promoting the algebra $\mathfrak{so}(2,3)$ to an infinite dimensional one that contains at least one Virasoro algebra. Each such Virasoro can in turn be thought of as an infinite dimensional extension of an $\mathfrak{sl}(2,\mathbb{R})$ subalgebra of $\mathfrak{so}(2,3)$. Therefore we would end up with a potentially distinct chiral $\mathcal{W}$-algebra extension of $\mathfrak{so}(2,3)$ depending on which $\mathfrak{sl}(2,\mathbb{R})\subset\mathfrak{so}(2,3)$ is promoted to a Virasoro. Therefore the first question to answer is: what are all the inequivalent embeddings of a copy of $\mathfrak{sl}(2,\mathbb{R})$ in the $\mathfrak{so}(2,3)$ algebra?

This can be answered in any faithful matrix representation of the algebra $\mathfrak{so}(2,3)$. We choose to work with the matrix representation of $\mathfrak{so}(2,3)$ when the representation space is $\mathbb{R}^{2,3}$ (with the pseudo-Cartesian coordinates (x_0,x_1,x_2,x_3,x_4) where x^0,x^4 are time-like and the rest space-like) on which the $\mathfrak{so}(2,3)$ algebra elements generate linear homogeneous transformations $x^\mu\to\Lambda^\mu{}_\nu x^\nu$ that preserve the line element $ds^2=\eta_{\mu\nu}dx^\mu dx^\nu$, i.e., $\{\Lambda\in M_5(\mathbb{R}):\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\eta_{\mu\nu}=\eta_{\alpha\beta}\ \&\ \det\Lambda=1\}$ where $\eta_{\mu\nu}=\mathrm{diag}(-1,1,1,1,-1)$.[1] Here the algebra of $\mathfrak{so}(2,3)$ is realised in terms of 5×5 real

[1]Another simple choice would have been to work with the 4×4 real matrix generators of the $\mathfrak{sp}(4,\mathbb{R})$ which is isomorphic to $\mathfrak{so}(2,3)$.

matrices M satisfying $M^{\mathrm{T}}\eta + \eta M = 0$. We will use the basis $L_{\mu\nu}$

$$(L_{\mu\nu})^{\alpha}{}_{\beta} = \delta^{\alpha}_{\mu}\eta_{\nu\beta} - \delta^{\alpha}_{\nu}\eta_{\mu\beta}, \qquad \mu,\nu,\alpha,\beta \cdots \in \{0,1,2,3,4\} \tag{6.1}$$

satisfying

$$[L_{\mu\nu}, L_{\mu'\nu'}] = \eta_{\mu\nu'}L_{\nu\mu'} + \eta_{\nu\mu'}L_{\mu\nu'} - \eta_{\mu\mu'}L_{\nu\nu'} - \eta_{\nu\nu'}L_{\mu\mu'}\,. \tag{6.2}$$

A general element in the Lie algebra is $M = \frac{1}{2}\omega^{\mu\nu}L_{\mu\nu}$ for $\omega^{\mu\nu} = -\omega^{\nu\mu} \in \mathbb{R}$, and in the space of M we would like to identify choices of triplets $(L_0, L_{\pm 1})$ that form $\mathfrak{sl}(2,\mathbb{R})$ algebra

$$[L_m, L_n] = (m-n)\, L_{m+n}\,. \tag{6.3}$$

We want all such possible choices that are not equivalent under the action of the group $O(2,3)$ which acts on M as $M \to \Lambda^{-1}M\Lambda$ where the 5×5 real matrices Λ satisfy $\Lambda^{\mathrm{T}}\eta\Lambda = \eta$.[2] These realisations of L_n will therefore be a five dimensional representation of $\mathfrak{sl}(2,\mathbb{R})$.

Before proceeding further let us recall some known facts about finite-dimensional representations of $\mathfrak{sl}(2,\mathbb{R})$. The irreducible realisations of the $\mathfrak{sl}(2,\mathbb{R})$ are given in terms of $(2j+1)\times(2j+1)$ dimensional real matrices, one for each $j \in \frac{1}{2}\mathbb{N}$ (along with $j=0$, the singlet). Their matrix elements may be written as:

$$\begin{aligned}&(L_0)_{qp} = p\,\delta_{qp}, \quad (L_{-1})_{qp} = h(j,p)(p-j)\delta_{q,p+1},\\ &(L_1)_{qp} = h(j,q)^{-1}(p+j)\delta_{q+1,p}\,, \quad \text{for} \quad p,q \in \{-j,-j+1,\cdots,j-1,j\}\end{aligned} \tag{6.4}$$

for any arbitrary non-zero choice of $h(j,p)$ – which can be fixed by appropriate actions of the group $GL(2,\mathbb{R})$. The simplest choices of $h(j,p)$ include: (i) $h(j,p)=1$ (or any other non-zero real number), (ii) $h(j,p) = \frac{1}{p-j}$, or $h(j,p) = p+j+1$ etc.[3] In all these cases the L_0 is diagonal with rank $2j$ $(2j+1)$ for $2j$ is even (odd), and with real eigenvalues. Its eigenvalues range over $-j,\cdots,j$. The only input we will take from here is that L_0 is diagonalisable with real eigenvalues, whereas $L_{\pm 1}$ are expected to be nilpotent (such that $L_{\pm}^{k} = 0$ for some k).

[2]We do not require the $\det\Lambda = 1$ condition to keep $\mathfrak{so}(2,3)$ invariant under $M \to \Lambda^{-1}M\Lambda$.

[3]The choice $h(j,p) = \sqrt{\frac{p+j+1}{p-j}}$ is the one related to the discrete series unitary representations of $\mathfrak{sl}(2,\mathbb{R})$ when j is analytically continued using the replacement $-j \to h$ for positive h [184]. However, this choice of course does not keep $L_{\pm 1}$ real.

Next, we seek all possible candidates in the algebra $\mathfrak{so}(2,3)$ in its vector representation (6.1) that can play the role of L_0. This requires us to consider the class of all M that admits only real eigenvalues. We can take any such matrix M, using the freedom of acting with the group $O(2,3)$, to be a linear combination of any two commuting boost generators (L_{0a}, L_{b4}) for $a \neq b$. There are six such choices which are all equivalent to each other under $O(2,3)$. So we choose:

$$L_0 = \lambda\, L_{01} + \lambda'\, L_{34} \tag{6.5}$$

for $\lambda, \lambda' \in \mathbb{R}$ with real eigenvalues $(0, \pm\lambda, \pm\lambda')$. One can independently make the exchanges $\lambda \leftrightarrow -\lambda$, $\lambda' \leftrightarrow -\lambda'$ and $\lambda \leftrightarrow \lambda'$ with appropriate $O(2,3)$ transformations.[4] So we further choose $\lambda \geq \lambda' \geq 0$. Now taking this as the general form of L_0 we can seek candidate $L_{\pm 1}$ such that they form the algebra (6.3) of $\mathfrak{sl}(2,\mathbb{R})$. There are, depending on the values of (λ, λ'), further $O(2,3)$ transformations that leave L_0 invariant and we can use these residual symmetries to put the candidate $L_{\pm 1}$ into their simplest forms. This exercise can be carried out systematically and straightforwardly. We find that, at the first instance, the existence of non-trivial $L_{\pm 1}$ with $[L_0, L_{\pm 1}] = \mp L_{\pm 1}$ requires that (λ, λ') have to satisfy one of the following conditions: (i) $\lambda = 1$, (ii) $\lambda' = 1$, (iii) $\lambda' = 1 - \lambda$ and (iv) $\lambda' = \lambda - 1$. We can then impose the last condition $[L_1, L_{-1}] = 2L_0$ in each of these four cases. We simply list the results of this straightforward analysis:

1. $\lambda = 1$ leads to three sub-cases (i) $\lambda' = 1$, (ii) $0 < \lambda' < 1$ and (iii) $\lambda' = 0$.
   (a) In the cases $\lambda' = 1$ and $0 < \lambda' < 1$ there is no solution for $L_{\pm 1}$.
   (b) When $\lambda = 1$ and $\lambda' = 0$, with $L_0 = L_{01}$ the most general $L_{\pm 1}$ are:

$$\begin{aligned}(c_{12}^2 + c_{13}^2 - c_{04}^2)\, L_1 &= c_{12}(L_{12} - L_{02}) + c_{13}(L_{13} - L_{03}) + c_{04}(L_{14} - L_{04}), \\ L_{-1} &= c_{12}(L_{02} + L_{12}) + c_{13}(L_{03} + L_{13}) + c_{04}(L_{04} + L_{14}),\end{aligned} \tag{6.6}$$

   for $c_{12}^2 + c_{13}^2 - c_{04}^2 \neq 0$. It can be shown that the residual symmetries ($O(2,3)$ transformations that leave L_0 invariant) leave $c_{12}^2 + c_{13}^2 - c_{04}^2$ invariant up to a positive scale. Thus depending on the sign of $c_{12}^2 + c_{13}^2 - c_{04}^2$ we can take (c_{12}, c_{13}, c_{04}) to be either $(1,0,0)$ or $(0,0,1)$ and these two will be the only inequivalent choices in this case.

2. $\lambda' = 1$ and $\lambda > 1$ leads to two sub cases: (i) $\lambda = 2$ and (ii) $\lambda \neq 2$.

[4] These matrices are $\Lambda = \mathrm{diag}(-1,1,-1,1,1)$, $\mathrm{diag}(1,1,-1,1,-1)$, $\Lambda_{mn} = \delta_{m+n}$ respectively.

(a) For $\lambda = 2$ and $\lambda' = 1$ with $L_0 = 2L_{01} + L_{34}$, we find:

$$\begin{aligned} b_{04}b_{23}L_1 &= b_{23}(L_{04} - L_{03} + L_{13} - L_{14}) + 3b_{04}(L_{23} + L_{24}), \\ L_{-1} &= b_{04}(L_{03} + L_{04}) + (L_{13} + L_{14}) + b_{23}(L_{23} - L_{24}). \end{aligned} \tag{6.7}$$

The residual symmetries can be used again to set $b_{23} = b_{04} = 1$.

(b) For $\lambda > 1$ and $\lambda \neq 2$ leads to no solution.

3. $\lambda' = 1 - \lambda$ with $\frac{1}{2} \leq \lambda < 1$ (since $\lambda' = 0$ is already considered) also leads to two sub-cases: (i) $\lambda = 1/2$ and (ii) $1/2 < \lambda < 1$.

(a) $\lambda = 1/2$ with $L_0 = \frac{1}{2}(L_{01} + L_{34})$ leads to a solution

$$\begin{aligned} 4\, a_{04}\, L_{-1} &= -L_{03} + L_{04} - L_{13} + L_{14}, \\ a_{04}^{-1}\, L_1 &= L_{03} + L_{04} - L_{13} - L_{14}, \end{aligned} \tag{6.8}$$

again we can use the residual symmetries of L_0 to set $a_{04} = 1/2$.

(b) $1/2 < \lambda < 1$ – leads to no solution.

4. The final case is $\lambda' = \lambda - 1$. This means $\lambda \geq 1$. We have already covered the cases of $\lambda = 1$ and $\lambda = 2$. So we restrict to $\lambda > 1$ and not equal to 2. And one can show that there is no solution in this case.

Thus we have arrived at the result that there are precisely four inequivalent embeddings of $\mathfrak{sl}(2,\mathbb{R})$ into $\mathfrak{so}(2,3)$: (i) $(\lambda,\lambda') = (1,0)$ with $(c_{12}, c_{13}, c_{04}) = (1,0,0)$, (ii) $(\lambda,\lambda') = (1,0)$ with $(c_{12}, c_{13}, c_{04}) = (0,0,1)$, (iii) $(\lambda,\lambda') = (2,1)$, and (iv) $(\lambda,\lambda') = (1/2,1/2)$. As the next step, we will write down the algebra of $\mathfrak{so}(2,3)$ from the perspective of each of these embeddings. We choose to characterise these four avatars of $\mathfrak{so}(2,3)$ by their maximal subalgebras $\mathfrak{h}$ that commute with the embedded $\mathfrak{sl}(2,\mathbb{R})$.

6.1.1 Four avatars of $\mathfrak{so}(2,3)$

$\mathfrak{h} = \mathfrak{so}(1,1)$

This case corresponds to $(\lambda, \lambda') = (1,0)$ & $(c_{12}, c_{13}, c_{04}) = (1,0,0)$ with the generators

$$
\begin{aligned}
&L_{-1} = L_{02} + L_{12}, \quad L_0 = L_{01}, \quad L_1 = -L_{02} + L_{12}, \quad H = L_{34}, \\
&P^0_{-1} = -L_{04} - L_{14}, \quad P^0_0 = L_{24}, \quad P^0_1 = -L_{04} + L_{14}\,, \\
&P^1_{-1} = -L_{03} - L_{13}, \quad P^0_0 = L_{23}, \quad P^0_1 = -L_{03} + L_{13}\,,
\end{aligned} \tag{6.9}
$$

satisfying

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n}\,, \\
[L_m, P^a_n] &= (m-n)P^a_{m+n}\,, \\
[P^a_m, P^b_n] &= -\eta^{ab}(m-n)L_{m+n} - \epsilon^{ab}\eta_{mn}\,H\,, \\
[H, P^a_m] &= P^b_m\,\epsilon_b{}^a\,,
\end{aligned} \tag{6.10}
$$

for $m, n, \cdots \in \{0, \pm 1\}$ and $a, b, \cdots \in \{0, 1\}$. Further $\epsilon^{01} = -\epsilon^{10} = 1$, $\eta_{ab} = \text{diag}(-1,1) = \eta^{ab}$ and $\epsilon_b{}^a = \eta_{bc}\epsilon^{ca}$, and $\eta_{mn} = (3m^2-1)\delta_{m+n} = m^2 + n^2 - mn - 1$. This avatar is related to the way $\mathfrak{so}(2,3)$ is realised as the conformal algebra of $\mathbb{R}^{1,2}$. Clearly, in this case the subalgebra $\mathfrak{h}$ that commutes with the subalgebra $\mathfrak{sl}(2,\mathbb{R})$ is generated by the boost generator $H = L_{34}$ and hence is $\mathfrak{so}(1,1)$.

$\mathfrak{h} = \mathfrak{so}(2)$

This case corresponds to $(\lambda, \lambda') = (1,0)$ & $(c_{12}, c_{13}, c_{04}) = (0,0,1)$ with the generators

$$
\begin{aligned}
&L_0 = L_{01}, \quad L_1 = L_{04} - L_{14}, \quad L_{-1} = L_{04} + L_{14}, \quad R = L_{23}, \\
&P^1_{-1} = -L_{03} - L_{13}, \quad P^1_0 = L_{34}, \quad P^1_1 = L_{03} - L_{13}, \\
&P^2_{-1} = -L_{02} - L_{12}, \quad P^2_0 = L_{24}, \quad P^2_1 = L_{02} - L_{12}
\end{aligned} \tag{6.11}
$$

satisfying

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n}, \\
[L_m, P^a_n] &= (m-n)P^a_{m+n}, \\
[P^a_m, P^b_n] &= \delta^{ab}(m-n)L_{m+n} + \epsilon^{ab}\eta_{mn}\, R\,, \\
[R, P^a_m] &= \epsilon^{ab}P^b_m\,,
\end{aligned}
\tag{6.12}
$$

for $m, n, \cdots \in \{0, \pm 1\}$ and $a, b, \cdots \in \{1, 2\}$. Further $\epsilon^{12} = 1 = -\epsilon^{21}$. Contrary to the previous case ($\mathfrak{h} = \mathfrak{so}(1,1)$), the subalgebra $\mathfrak{h}$ is generated by the rotation generator L_{23} and hence is $\mathfrak{so}(2)$. Thus, even though this case is very similar, it is not equivalent to the one with $\mathfrak{h} = \mathfrak{so}(1,1)$.

$\mathfrak{h} = \emptyset$

This case corresponds to $(\lambda, \lambda') = (2, 1)$. The generators are

$$
\begin{aligned}
L_0 &= 2L_{01} + L_{34}, \\
L_1 &= -L_{03} + L_{04} + L_{13} - L_{14} + 3L_{23} + 3L_{24}, \\
L_{-1} &= L_{03} + L_{04} + L_{13} + L_{14} + L_{23} - L_{24},
\end{aligned}
\tag{6.13}
$$

$$
\begin{aligned}
W_{-3} &= \beta(L_{03} - L_{04} + L_{13} - L_{14}), \\
W_{-2} &= \beta(L_{02} + L_{12}), \\
W_{-1} &= \beta\frac{3}{5}(-L_{03} - L_{04} - L_{13} - L_{14} + \frac{2}{3}(L_{23} - L_{24})), \\
W_0 &= \beta\frac{3}{5}(-L_{01} + 2L_{34}), \\
W_1 &= \beta\frac{3}{5}(L_{03} - L_{04} - L_{13} + L_{14} + 2(L_{23} + L_{24})), \\
W_2 &= 3\beta(L_{02} - L_{12}), \\
W_3 &= 9\beta(-L_{03} - L_{04} + L_{13} + L_{14}),
\end{aligned}
\tag{6.14}
$$

satisfying

$$
\begin{aligned}
[L_m, L_n] &= (m-n)\, L_{m+n}, \\
[L_m, W_n] &= (3m-n)\, W_{m+n}, \\
[W_m, W_n] &= \frac{\beta^2}{100}\, A(m,n)L_{m+n} + \frac{\beta}{10}\, B(m,n)W_{m+n},
\end{aligned}
\tag{6.15}
$$

where

$$
\begin{aligned}
A(m,n) &= (m-n)\Big(3(m^4+n^4) - 2mn(m-n)^2 - 39(m^2+n^2) + 20mn + 108\Big),\\
B(m,n) &= (m-n)(n^2+m^2-mn-7)\,. \qquad (6.16)
\end{aligned}
$$

We can fix β to be any non-zero real number by a residual $O(2,3)$ transformation – and we choose $\beta = 10/\sqrt{1680}$ for later convenience. This avatar of $\mathfrak{so}(2,3)$ is special as the corresponding embedding of $\mathfrak{sl}(2,\mathbb{R})$ in it is what can be called the principal embedding – as the rest of the generators form a single 5-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$.

$\mathfrak{h} = \mathfrak{sl}(2,\mathbb{R})$

This case corresponds to $(\lambda,\lambda') = (1/2,1/2)$. The generators are

$$
\begin{aligned}
L_0 &= \frac{1}{2}(L_{01}+L_{34}), \quad L_1 = \frac{1}{2}(L_{03}+L_{04}-L_{13}-L_{14}),\\
L_{-1} &= \frac{1}{2}(-L_{03}+L_{04}-L_{13}+L_{14}), \quad J_0 = \frac{1}{2}(L_{01}-L_{34}),\\
J_1 &= \frac{1}{2}(-L_{03}-L_{14}+L_{04}+L_{13}), \quad J_{-1} = \frac{1}{2}(L_{03}+L_{14}+L_{04}+L_{13}),\\
G_{\frac{1}{2},\frac{1}{2}} &= \alpha(-L_{02}+L_{12}), \quad G_{-\frac{1}{2},\frac{1}{2}} = \alpha(-L_{23}-L_{24}),\\
G_{\frac{1}{2},-\frac{1}{2}} &= \alpha(L_{23}-L_{24}), \quad G_{-\frac{1}{2},-\frac{1}{2}} = \alpha(L_{02}+L_{12}), \qquad (6.17)
\end{aligned}
$$

satisfying

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n}, \quad [J_a, J_b] = (a-b)J_{a+b},\\
[J_a, G_{sr}] &= \frac{1}{2}(a-2s)\,G_{a+s,r}, \quad [L_m, G_{sr}] = \frac{1}{2}(m-2r)\,G_{s,m+r},\\
[G_{sr}, G_{s'r'}] &= -2\alpha^2(\epsilon_{rr'}J_{s+s'} + \epsilon_{ss'}L_{r+r'}) \qquad (6.18)
\end{aligned}
$$

where $m,n,\dots \in \{0,\pm1\}$, $a,b,\dots \in \{0,\pm1\}$ and $r,s,\dots \in \{\pm1/2\}$, with $\epsilon_{-\frac{1}{2},\frac{1}{2}} = 1$. Again we can choose α to be any non-zero real number and we fix it to be unity. This completes the exercise of finding the inequivalent embeddings of $\mathfrak{sl}(2,\mathbb{R})$ inside $\mathfrak{so}(2,3)$ and writing the $\mathfrak{so}(2,3)$ algebra that makes the corresponding $\mathfrak{sl}(2,\mathbb{R})$ manifest. Each such embedding is based on the identification of a sub-algebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{h} \subset \mathfrak{so}(2,3)$.[5]

[5]These sub-algebras are also in one-to-one correspondence with the maximal sub-algebras of $\mathfrak{so}(2,3)$ listed in [185] that contain at least one copy of $\mathfrak{sl}(2,\mathbb{R})$ in them.

Next, we would like to promote each of these avatars (6.10, 6.12, 6.15, 6.18) of $\mathfrak{so}(2,3)$ into an infinite dimensional algebra that consists one copy of Virasoro algebra. We will show in the next section that the first two avatars above admit minimal chiral $\mathcal{W}$-algebra extensions that are novel. The third avatar leads to the known $\mathcal{W}(2,4)$ algebra, and the extension of the fourth one is what was done in Chapter 5.

6.2 The four $\mathcal{W}$-algebra extensions

We now would like to find the minimal infinite dimensional extensions of the four avatars of $\mathfrak{so}(2,3)$ found in Section (6.1) such that they contain modes of a chiral stress tensor. We implement the following set of rules for this:

- For every copy of $\mathfrak{sl}(2,\mathbb{R})$ with generators L_n (for $n = 0, \pm 1$) we postulate a chiral quasi-primary stress tensor $T(z)$ with dimension $h = 2$ and central charge c, and modes L_n (for $n \in \mathbb{Z}$).
- For the generators of the subalgebra $\mathfrak{h}$ we postulate a level-κ current with $h = 1$.
- For every set of generators that form a non-trivial finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{R})$ of dimension k we postulate a primary of dimension $h = \frac{1}{2}(k+1)$.
- We write down the most general OPEs among the chiral operators as obtained above imposing the global symmetries at hand.
- Finally we fix the undermined coefficients in these OPEs using the OPE associativity - implemented in terms of commutators of the modes and their Jacobi identities explicitly.

Following these rules we consider one copy of a chiral stress tensor: $T(z)$ – a quasi-primary with $h = 2$ and central charge c for each $\mathfrak{sl}(2,\mathbb{R})$ subalgebra we identified in the previous section with

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\,. \tag{6.19}$$

Then each of the four avatars of $\mathfrak{so}(2,3)$ lead to the following sets of chiral operators.

1. In the $\mathfrak{h} = \mathfrak{so}(1,1)$ case we have, along with $T(z)$,

(a) An $\mathfrak{so}(1,1)$ current $H(z)$ which is a conformal primary with $h = 1$ and level-κ.

$$\begin{aligned} T(z)H(w) &\sim \frac{H(w)}{(z-w)^2} + \frac{\partial H(w)}{z-w}, \\ H(z)H(w) &\sim \frac{\kappa}{(z-w)^2} \,. \end{aligned} \tag{6.20}$$

(b) A pair of chiral operators $P^a(z)$ that are conformal primaries with $h = 2$ as well as a doublet of $\mathfrak{so}(1,1)$ current $H(z)$.[6]

$$\begin{aligned} T(z)P^a(w) &\sim \frac{2P^a(w)}{(z-w)^2} + \frac{\partial P^a(w)}{z-w}\,, \\ H(z)P^a(w) &\sim \frac{P^b(w)\,\epsilon_b{}^a}{z-w} \,. \end{aligned} \tag{6.21}$$

This leaves the OPE $P^a(z)P^b(w)$ to be determined. We will refer to the resultant algebra to be $\mathcal{W}(2;2^2,1)$.

2. In the $\mathfrak{h} = \mathfrak{so}(2)$ avatar we have, along with $T(z)$,

(a) An $\mathfrak{so}(2)$ current $R(z)$ which is a conformal primary with $h = 1$ and level-κ.

$$\begin{aligned} T(z)R(w) &\sim \frac{R(w)}{(z-w)^2} + \frac{\partial R(w)}{z-w}, \\ R(z)R(w) &\sim \frac{\kappa}{(z-w)^2} \,. \end{aligned} \tag{6.22}$$

(b) A pair of chiral operators $P^a(z)$ that are conformal primaries with $h = 2$ as well as a doublet of $\mathfrak{so}(2)$ current $R(z)$.

$$\begin{aligned} T(z)P^a(w) &\sim \frac{2P^a(w)}{(z-w)^2} + \frac{\partial P^a(w)}{z-w}\,, \\ R(z)P^a(w) &\sim \frac{\epsilon^{ab}P^b(w)}{z-w} \,. \end{aligned} \tag{6.23}$$

The OPE $P^a(z)P^b(w)$ will be determined later. We will refer to the resultant algebra to be $\widetilde{\mathcal{W}}(2;2^2,1)$.

3. In the case with $\mathfrak{h} = \emptyset$ we have a stress tensor $T(z)$ and

[6] One could have postulated that $P^a(z)$ are only quasi-primaries at this stage. It turns out, however, that consistency of the final algebra with associativity does not allow central terms in $T(z)P^a(w)$ OPE and hence we preclude this.

(a) A chiral operator $W(z)$ that is a conformal primary of dimension $h = 4$.

$$T(z)W(w) \sim \frac{4W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} \tag{6.24}$$

The $W(z)W(w)$ will be determined later. We will identify the resultant algebra with the known $\mathcal{W}(2,4)$ algebra of [179].

4. In the case with $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{R})$ we have a $T(z)$ and

(a) A triplet $J_a(z)$ with $a = 0, \pm 1$, which are conformal primaries with $h = 1$

$$T(z)J_a(w) \sim (z-w)^{-2} J_a(w) + (z-w)^{-1}\partial J_a(w)\,, \tag{6.25}$$

and among themselves form a level-κ $\mathfrak{sl}(2,\mathbb{R})$ algebra,

$$J_a(z)J_b(w) \sim -\frac{\kappa}{2}\eta_{ab}(z-w)^{-2} + (z-w)^{-1} f_{ab}{}^{c} J_c(w)\,. \tag{6.26}$$

(b) A pair $G_s(z)$ for $s = \pm 1/2$ that are conformal primaries with $h = 3/2$

$$T(z)G_s(w) \sim (3/2)(z-w)^{-2} G_s(w) + (z-w)^{-1}\partial G_s(w) \tag{6.27}$$

as well as a doublet of $\mathfrak{sl}(2,\mathbb{R})$ current algebra

$$J_a(z)G_s(w) \sim (z-w)^{-1} G_{s'}(w) (\lambda_a)^{s'}{}_{s} \tag{6.28}$$

where $(\lambda_a)^{s}{}_{s'} = \frac{1}{2}(a - 2s')\delta^{s}_{a+s'}$. This case was already dealt with by us in Chapter 5 and the OPE $G_s(z)G_{s'}(w)$ was determined. The resultant algebra is our $\mathcal{W}(2;(3/2)^2,1^3)$.

We will now turn to obtaining the underdetermined OPEs in each of these cases. We will provide only the essential details of the derivation of the first of these algebras and the results for the remaining three.

6.2.1 $\mathcal{W}(2;2^2,1)$

In this case, the only undetermined OPE is that of $P^a(z)P^b(w)$ and following the conformal invariance one expects that all quasi-primaries of dimension up to three and their global descendants can appear in the singular terms on the RHS. These quasi-primaries can also be composites (normal ordered products) of simple (quasi-) primaries $(T(z), P^a(z), H(z))$.

One can construct the necessary quasi-primaries using the OPEs

$$\begin{aligned}
T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \\
T(z)H(w) &\sim \frac{H(w)}{(z-w)^2} + \frac{\partial H(w)}{z-w}, \qquad H(z)T(w) \sim \frac{H(w)}{(z-w)^2}, \\
H(z)H(w) &\sim \frac{\kappa}{(z-w)^2}, \\
T(z)P^a(w) &\sim \frac{2P^a(w)}{(z-w)^2} + \frac{\partial P^a(w)}{z-w} \sim P^a(z)T(w), \\
H(z)P^a(w) &\sim \frac{P^b(w)\,\epsilon_b{}^a}{z-w} \sim -P^a(z)H(w)\,.
\end{aligned} \tag{6.29}$$

The relevant composite quasi-primaries are as follows.

- At $h = 2$ we have only one normal-ordered composite quasi-primary $(H^2)(z)$ with

$$T(z)(H^2)(w) \sim \frac{\kappa}{(z-w)^4} + \frac{2}{(z-w)^2}(H^2)(w) + \frac{1}{(z-w)}\partial(H^2)(w)\,. \tag{6.30}$$

  The corresponding Sugawara stress tensor $\frac{1}{2\kappa}(H^2)(z)$ has $c = 1$.

- At $h = 3$ we have two composite scalar quasi-primaries

$$\begin{aligned}
\Lambda(z) &:= (TH)(z) - \frac{1}{2}\partial^2 H(z), \\
\Sigma(z) &:= \frac{1}{2\kappa}((H^2)H)(z) - \frac{1}{2}\partial^2 H(z).
\end{aligned} \tag{6.31}$$

  Their OPEs with $T(z)$ are given by

$$\begin{aligned}
T(z)\Lambda(w) &\sim \frac{2+c}{2(z-w)^4}H(w) + \frac{3}{(z-w)^2}\Lambda(w) + \frac{1}{(z-w)}\partial\Lambda(w)\,, \\
T(z)\Sigma(w) &\sim \frac{3}{2(z-w)^4}H(w) + \frac{3}{(z-w)^2}\Sigma(w) + \frac{1}{(z-w)}\partial\Sigma(w)\,.
\end{aligned} \tag{6.32}$$

  Note that the linear combination $3\,\Lambda(z) - (2+c)\,\Sigma(z)$ is a primary.

- At $h = 3$ we have a doublet of primaries (not just quasi-primaries)

$$[P^a H](z) := (P^a H)(z) + \frac{3}{4}\epsilon_b{}^a \partial P^b = (HP^a)(z) - \frac{1}{4}\epsilon_b{}^a \partial P^b \tag{6.33}$$

This is the full list of composite quasi-primaries that are covariant under $\mathfrak{so}(1,1)$ and dimension $h \leq 3$. In particular, let us note that there is no rank-2 symmetric traceless tensor of $\mathfrak{so}(1,1)$ with $h \leq 3$. There are other quasi-primaries of higher dimensions that we will introduce as and when required.

We now make use of the global symmetries to constrain the OPE $P^a(z)P^b(w)$.

- Since the two operators involved carry vector indices $a, b, \cdots$ of $\mathfrak{so}(1,1)$ their OPE should respect this symmetry. This means that the rhs can only involve representations of $\mathfrak{so}(1,1)$ that can appear in the tensor product of two vectors.
- There are in general four independent components in a rank-2 tensor of $\mathfrak{so}(1,1)$. Arranging them into covariant irreducible objects, there are
  1. two singlets - with the CG coefficients being the symmetric tensor η^{ab} and the anti-symmetric tensor ϵ^{ab}. These can be multiplied by $\mathfrak{so}(1,1)$ invariants.
  2. a symmetric traceless rank-2 tensor with two independent components.
- One consequence of these tensor structures is that we cannot have a covariant rank-1 object (vector) on the rhs and that rules out (6.33). Also the fact that there are no composite operators with $h \leq 3$ that are rank-2 traceless symmetric tensors leaves us with, in the increasing order of conformal dimensions $\{I, H(z), T(z), (H^2)(z), \Lambda, \Sigma\}$ that can appear on the rhs.
- Another fact is that the OPE is expected to be invariant under the simultaneous exchange $(a, z) \leftrightarrow (b, w)$. It is easy to see that this requires that the terms containing η^{ab} should have even dimensional operators, namely, $(I, T(z))$ (and their descendants) and terms containing ϵ^{ab} should have odd dimensional operators, namely, $(H(z), \Lambda(z), \Sigma(z))$ and their descendants.
- The relative coefficients of a quasi-primary and its global conformal descendants are fixed by the global conformal invariance. This leaves only six real numbers to be undetermined.

Thus we arrive at the following ansatz:

$$
\begin{aligned}
P^a(z)P^b(w) \sim \eta^{ab}\Big[&\alpha\,(z-w)^{-4}\,I + \beta\,(z-w)^{-2}\,\Big(2T(w) + (z-w)\,\partial T(w)\Big) \\
&+ \gamma(z-w)^{-2}\,\Big(2(H^2)(w) + (z-w)\partial(H^2)(w)\Big)\Big] \\
&+ \epsilon^{ab}\Big[\delta\,(z-w)^{-3}\Big(6H(w) + 3\,(z-w)\partial H(w) + \,(z-w)^2\,\partial^2 H(w)\Big) \\
&+ \sigma\,(z-w)^{-1}\Lambda(w) + \omega\,(z-w)^{-1}\Sigma(w)\Big]
\end{aligned}
\tag{6.34}
$$

The undetermined real numbers $(\alpha, \beta, \gamma, \delta, \sigma, \omega)$ (that can depend on the central charges (c, κ)) are to be determined by imposing the OPE associativity. This we do using Jacobi identities in terms of modes (as in Chapter 5) following [25]. To proceed further we need the mode expansions of these quasi-primaries. Using the definition from [160, 161]

$$
(AB)_m = \sum_{n\leq -h_A} A_n B_{m-n} + \sum_{n>-h_A} B_{m-n} A_n \tag{6.35}
$$

for the modes of normal ordered products we can write

$$
(H^2)_n = \sum_{k>-1} H_{n-k} H_k + \sum_{k\leq -1} H_k H_{n-k}\,. \tag{6.36}
$$

Similarly we have $\Lambda(z) = \sum_{n\in\mathbb{Z}} \frac{\Lambda_n}{z^{n+3}}$ where

$$
\Lambda_n = \sum_{k>-2} H_{n-k} L_k + \sum_{k\leq -2} L_k H_{n-k} - \frac{1}{2}(n+1)(n+2) H_n \tag{6.37}
$$

and $\Sigma(z) = \sum_{n\in\mathbb{Z}} \frac{\Sigma_n}{z^{n+3}}$ where

$$
\Sigma_n = \sum_{k>-2} H_{n-k} \tilde{L}_k + \sum_{k\leq -2} \tilde{L}_k H_{n-k} - \frac{1}{2}(n+1)(n+2) H_n \tag{6.38}
$$

with $\tilde{L}_n = \frac{1}{2\kappa}(H^2)_n$.

Jacobi identities

Let us convert all the OPEs into commutators of modes. The modes are

$$L_n = \oint_0 \frac{dz}{2\pi i} z^{n+1} T(z), \quad P^a_n = \oint_0 \frac{dz}{2\pi i} z^{n+1} P^a(z), \quad H_n = \oint_0 \frac{dz}{2\pi i} z^n \, H(z) \tag{6.39}$$

resulting in

1. **Virasoro Algebra**:

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{c}{12} m(m^2-1)\, \delta_{m+n,0} \tag{6.40}$$

2. **Virasoro Primaries**

$$[L_m, P^a_n] = (m-n)\, P^a_{m+n}, \quad [L_m, H_n] = -n\, H_{m+n} \tag{6.41}$$

3. $\mathfrak{so}(1,1)_\kappa$ **Current Algebra**

$$[H_m, H_n] = \kappa\, m\, \delta_{m+n,0} \tag{6.42}$$

4. **Current algebra primaries**

$$[H_m, P^a_n] = P^b_{m+n}\, \epsilon_b{}^a \tag{6.43}$$

5. **Commutator** $[P^a_m, P^b_n]$

$$\begin{aligned}[P^a_m, P^b_n] = \eta^{ab} \left[\frac{\alpha}{6} m(m^2-1)\delta_{m+n,0} + (m-n)\Big(\beta\, L_{m+n} + \gamma\, (H^2)_{m+n}\Big)\right] \\ + \epsilon^{ab}\Big[\delta\, (m^2+n^2-mn-1)\, H_{m+n} + \sigma\, \Lambda_{m+n} + \omega\, \Sigma_{m+n}\Big]\end{aligned} \tag{6.44}$$

It remains to simply determine the unknowns $(\alpha, \beta, \gamma, \delta, \sigma, \omega)$ in terms of c and κ using the Jacobi identities. As in Chapter 5 we schematically denote the Jacobi identity involving the modes of the operators $A(z), B(z), C(z)$ as identity (ABC). There are a total of 10 identities to be considered. The four identities (TTT), (TTH), (THH), (HHH) are automatically satisfied. Next there are three with one P^a each: (TTP^a), (THP^a), (HHP^a) – which can also be seen to be satisfied identically.

The remaining three Jacobi identities contain at least two P^a_n's: (TPP), (HPP) and (PPP). These involve commutators between two P^a_m's. To impose these we need the commutators of the modes of the simple primaries (L_m, H_m, P^a_m) with $((H^2)_n, \Lambda_n, \Sigma_n)$.

Commutators with $(H^2)_n$

It is easy to see

$$[L_m, (H^2)_n] = (m-n)\,(H^2)_{m+n} + \frac{\kappa}{6}\,m(m^2-1)\,\delta_{m+n,0}\,,$$
$$[H_m, (H^2)_n] = 2\kappa\, m\, H_{m+n}\,. \tag{6.45}$$

For $[P^a_m, (H^2)_n]$ we first note that the OPE between $P^a(z)$ and $(H^2)(w)$ can be written in terms of quasi-primaries and their descendants as follows:

$$P^a(z)(H^2)(w) \sim \frac{P^a(w)}{(z-w)^2} + \frac{\partial P^a(w)}{2(z-w)} - 2\epsilon_b{}^a \frac{[P^b H](w)}{z-w} \tag{6.46}$$

where

$$[P^a H](z) := (P^a H)(z) + \frac{3}{4}\epsilon_b{}^a \partial P^a \tag{6.47}$$

is a primary. In obtaining this we used the identity: $(HP^a)(z) - (P^a H)(z) - \epsilon_b{}^a \partial P^b(z) = 0$. Using this OPE we can immediately write the commutator of the corresponding modes as

$$[P^a_m, (H^2)_n] := -2\epsilon_b{}^a [P^b H]_{m+n} + \frac{1}{2}(m-n)\,P^a_{m+n}\,. \tag{6.48}$$

Commutators with Λ_n

It is straightforward to find

$$[L_m, \Lambda_n] = (2m-n)\Lambda_{m+n} + \frac{1}{12}(c+2)m(m^2-1)\,H_{m+n},$$

$$[H_m, \Lambda_n] = \kappa\, m\, L_{m+n} + m\,(H^2)_{m+n}\,. \tag{6.49}$$

The next non-trivial commutator is $[P^a_m, \Lambda_n]$. For this again we write the corresponding OPE $P^a(z)\Lambda(w)$ in terms of quasi-primaries and their global descendants. We find

$$P^a(z)\Lambda(w) \sim \frac{-2\epsilon_b{}^a}{(z-w)^3}\left[P^b(w) + \frac{1}{4}(z-w)\partial P^b(w) + \frac{1}{20}(z-w)^2\partial^2 P^b(w)\right]$$

$$+\frac{2[P^aH](w)}{(z-w)^2} + \frac{2\partial[P^aH](w)}{3(z-w)} - \epsilon_b{}^a\frac{[TP^b](w)}{z-w} + \frac{[\partial P^aH](w)}{3(z-w)} \tag{6.50}$$

where the relevant composite quasi-primaries are

$$\begin{aligned}
h=3 \quad &: \quad [P^aH](z) := (P^aH)(z) + \frac{3}{4}\epsilon_b{}^a\partial P^b(z),\\
&\quad [HP^a](z) := (HP^a)(z) - \frac{1}{4}\epsilon_b{}^a\partial P^b(z),\\
h=4 \quad &: \quad [H\partial P^a](z) := (H\partial P^a)(z) - 2\,(P^a\partial H)(z) - \frac{11}{10}\partial^2 P^b(z)\epsilon_b{}^a\\
&\quad = (\partial P^aH)(z) - 2(P^a\partial H)(z) - \frac{3}{5}\partial^2 P^b\epsilon_b{}^a := [\partial P^aH](z),\\
h=4 \quad &: \quad [TP^a](z) := (TP^a)(z) - \frac{3}{10}\partial^2 P^a(z) = (P^aT)(z) - \frac{3}{10}\partial^2 P^a(z) := [P^aT](z).
\end{aligned} \tag{6.51}$$

Converting the OPE (6.50) into a commutator we find

$$\begin{aligned}
[P^a_m, \Lambda_n] =& -\epsilon_b{}^a[TP^b]_{m+n} + \frac{1}{3}[H\partial P^a]_{m+n}\\
&- \frac{1}{10}(6m^2+n^2-3mn-4)\epsilon_b{}^aP^b_{m+n} + \frac{2}{3}(2m-n)[P^aH]_{m+n}\,.
\end{aligned} \tag{6.52}$$

Commutators with Σ_n

It is easy to see that

$$\begin{aligned}
[L_m, \Sigma_n] &= (2m-n)\Sigma_{m+n} + \frac{1}{4}m(m^2-1)\,H_{m+n},\\
[H_m, \Sigma_n] &= \frac{3}{2}m\,(H^2)_{m+n}\,.
\end{aligned} \tag{6.53}$$

To find $[P^a_m, \Sigma_n]$ we again write the OPE:

$$\begin{aligned}
P^a(z)\Sigma(w) \sim\ & \frac{-\epsilon_b{}^a}{2\kappa(z-w)^3}\left[P^b(w) + \frac{1}{4}(z-w)\partial P^b(w) + \frac{1}{20}(z-w)^2\partial^2 P^b(w)\right] \\
& + \frac{3}{2\kappa(z-w)^2}\left[[P^aH](w) + \frac{1}{3}(z-w)\partial[P^aH](w)\right] - \frac{1}{4\kappa(z-w)}[\partial P^aH](w) \\
& - \frac{1}{2\kappa(z-w)}\epsilon_b{}^a\Big[[HHP^b](w) + [P^aHH](w) + [HP^aH](w)\Big]
\end{aligned} \tag{6.54}$$

where the additional quasi-primaries with $h = 4$ are:

$$\begin{aligned}
&[HHP^a](z) := ((H^2)P^a)(z) - \epsilon_b{}^a\frac{3}{4}(H\partial P^b)(z) - \epsilon_b{}^a\frac{3}{2}(P^b\partial H)(z) - \frac{27}{40}\partial^2 P^a(z)\,, \\
&[HP^aH](z) := ((HP^a)H)(z) + \frac{5}{4}\epsilon_b{}^a(H\partial P^b) - \frac{1}{2}\epsilon_b{}^a(P^b\partial H)(z) - \frac{1}{40}(11+20\kappa)\partial^2 P^a, \\
&[P^aHH](z) := ((P^aH)H)(z) + 4\epsilon_b{}^a(P^b\partial H)(z) - \frac{1}{10}(5\kappa - 17)\partial^2 P^a,
\end{aligned} \tag{6.55}$$

and we used the identities

$$\begin{aligned}
(HP^a)(z) - (P^aH)(z) - \epsilon_b{}^a\partial P^b(z) = 0\,, \\
(\partial P^aH)(z) - (H\partial P^a)(z) + \frac{1}{2}\epsilon_b{}^a\partial^2 P^b(z) = 0\,.
\end{aligned} \tag{6.56}$$

Converting the OPE (6.54) into commutator modes we find:

$$\begin{aligned}
[P^a_m, \Sigma_n] =\ & -\frac{1}{40\kappa}\epsilon_b{}^a(6m^2 + n^2 - 3mn - 4)P^b_{m+n} + \frac{1}{2\kappa}(2m-n)[P^aH]_{m+n} \\
& - \frac{1}{4\kappa}[\partial P^aH]_{m+n} - \frac{1}{2\kappa}\epsilon_b{}^a\Big([HHP^b]_{m+n} + [HP^bH]_{m+n} + [P^bHH]_{m+n}\Big).
\end{aligned} \tag{6.57}$$

Result of Jacobi identities

We are ready to impose the (TPP) and (HPP) identities. The computations are straightforward and we find that (TPP) identity requires

$$2\alpha - c\beta - 2\kappa\,\gamma = 0, \quad 24\delta - (2+c)\sigma - 3\omega = 0\,. \tag{6.58}$$

The (HPP) identity requires

$$\begin{aligned}
\beta + 3\delta + 2\kappa\,\gamma = 0, \quad \alpha + 6\kappa\,\delta = 0\,, \\
2\beta + \kappa\,\sigma = 0, \quad 2\gamma + \sigma + \frac{3}{2}\omega = 0\,.
\end{aligned} \tag{6.59}$$

Finally, we need to impose (PPP) Jacobi identity. For this, first we write $[P^c_k,[P^a_m,P^b_n]]$ as

$$\left(\beta+\frac{\gamma}{2}\right)\eta^{ab}(m-n)(k-m-n)P^c_{m+n+k}$$
$$-\left[\delta\left(m^2+n^2-mn-1\right)+\frac{1}{10}\left(\sigma+\frac{\omega}{4\kappa}\right)\left(6k^2+(m+n)^2-3k(m+n)-4\right)\right]\epsilon^{ab}\epsilon_d{}^cP^d_{m+n+k}$$
$$+\left(\frac{\omega}{2\kappa}+\frac{2\sigma}{3}\right)(2k-m-n)\epsilon^{ab}[P^cH]_{m+n+k}-2\gamma(m-n)\eta^{ab}\epsilon_d{}^c[P^dH]_{m+n+k}$$
$$-\sigma\epsilon^{ab}\,\epsilon_d{}^c[TP^d]_{m+n+k}+\left(\frac{\sigma}{3}-\frac{\omega}{4\kappa}\right)\epsilon^{ab}[H\partial P^c]_{m+n+k}$$
$$-\frac{\omega}{2\kappa}\epsilon^{ab}\epsilon_d{}^c\Big([HHP^b]_{m+n+k}+[HP^bH]_{m+n+k}+[P^bHH]_{m+n+k}\Big)\,. \tag{6.60}$$

The Jacobi identity to be imposed is

$$[P^c_k,[P^a_m,P^b_n]]+\Big((c,k)\to(a,m)\to(b,n)\to(c,k)\Big)$$
$$+\Big((c,k)\to(b,n)\to(a,m)\to(c,k)\Big)=0. \tag{6.61}$$

The terms that contain $\epsilon^{ab}\epsilon_d{}^c\mathcal{O}^d_{m+n+k}$ and $\epsilon^{ab}\mathcal{O}^c_{m+n+k}$ can be dropped as they vanish after summing over the cyclic permutations above. Then there are two types of terms left which contain (i) P^c_{k+m+n}, (ii) $[P^cH]_{m+n+k}$.

1. Setting terms containing $[P^aH]_{m+n+k}$ to zero we find one condition

$$4\kappa(\gamma-\sigma)-3\omega=0. \tag{6.62}$$

2. Setting terms containing P^a_{m+n+k} to zero gives the last condition

$$4\kappa(2\beta+\gamma-2\delta+\sigma)+\omega=0. \tag{6.63}$$

Thus the full set of conditions obtained from the Jacobi identities is (6.58, 6.59, 6.62, 6.63). Out of these eight only six equations are linearly independent. These equations admit a non-trivial two-parameter solution.[7] Taking these parameters to

[7] There is of course the trivial solution: $\alpha=\beta=\gamma=\delta=\sigma=\omega=0$, for arbitrary c and κ.

be (σ, κ) we find:

$$\begin{aligned}
\alpha &= \frac{3\kappa^2(\kappa-1)}{\kappa+1}\sigma, \quad \beta = -\frac{\kappa}{2}\sigma, \\
\gamma &= \frac{2\kappa-1}{2(\kappa+1)}\sigma, \quad \delta = -\frac{\kappa(\kappa-1)}{2(\kappa+1)}\sigma, \\
\omega &= -\frac{2\kappa}{\kappa+1}\sigma, \quad c = -\frac{2\,(6\kappa^2-8\kappa+1)}{\kappa+1}\,.
\end{aligned} \tag{6.64}$$

This completes our derivation of the algebra $\mathcal{W}(2;2^2,1)$. In the large-κ limit these become

$$\begin{aligned}
&\alpha \to 3\kappa^2\sigma, \quad \beta \to -\frac{\kappa}{2}\sigma, \quad \gamma \to \sigma, \\
&\delta \to -\frac{\kappa}{2}\sigma, \quad \omega \to -2\,\sigma, \quad c \to -12\,\kappa\,.
\end{aligned} \tag{6.65}$$

By further taking $\sigma \to \frac{2}{\kappa}$ it can be seen that this $\mathcal{W}$ algebra when restricted to the modes of $\mathfrak{so}(2,3)$ is a deformation of (6.10) around the $\kappa \to \infty$ limit.

Relation between $\mathcal{W}(2;2^2,1)$ and conformal $\mathfrak{bms}_3$ of [20]

Before we turn to the next case let us briefly demonstrate that the conformal $\mathfrak{bms}_3$ algebra of [20] arises in the semiclassical limit of our $\mathcal{W}(2;2^2,1)$. For this we define

$$\mathcal{P}_m = P^0_m + P^1_m\,, \;\; \mathcal{K}_m = P^0_m - P^1_m \tag{6.66}$$

and re-write the commutator (6.44) in terms of $\mathcal{P}_m$ and $\mathcal{K}_n$. This results in, for instance, the commutator $[\mathcal{P}_m\,, \mathcal{K}_n]$

$$\begin{aligned}
[\mathcal{P}_m\,, \mathcal{K}_n] &= -\frac{\alpha}{3}\,m(m^2-1)\delta_{m+n,0} - 2(m-n)\left(\beta\,L_{m+n} + \gamma\,(H^2)_{m+n}\right) \\
&- 2\,\delta\,(m^2+n^2-mn-1)\,H_{m+n} - 2\,\sigma\,\Lambda_{m+n} - 2\,\omega\,\Sigma_{m+n}\Big]
\end{aligned} \tag{6.67}$$

Next, we take the large-κ limit which amounts to using (6.65). If one further replaces $c \to 12\,\tilde{c}$, $\kappa \to -\tilde{c}$ and $\sigma \to 2/\tilde{c}$ one obtains the conformal $\mathfrak{bms}_3$ algebra as written down in [20] after the following identifications

$$H_p \to i\mathcal{D}_p\,, \quad L_n \to \mathcal{J}_n. \tag{6.68}$$

The normal ordered operators in this large-κ limit after regularisation become

$$(H^2)_m \to -\sum_p \mathcal{D}_{m-p}\,\mathcal{D}_p\,, \quad \Lambda_m \to i\sum_p \mathcal{J}_{m-p}\,\mathcal{D}_p\,, \quad \Sigma_m \to -\frac{i}{2\tilde{c}}\sum_{n,l}\mathcal{D}_{m-n-l}\mathcal{D}_n\mathcal{D}_l \tag{6.69}$$

completing the identification with the non-linear terms of (6.67) with the expressions in (2.2–2.4) of [20]. This makes it clear that the conformal $\mathfrak{bms}_3$ algebra of [20] is only valid for large values of their central charge.

6.2.2 $\widetilde{\mathcal{W}}(2;2^2,1)$

Now we turn to the second way the $\mathfrak{sl}(2,\mathbb{R})$ is embedded in $\mathfrak{so}(2,3)$ where P^a instead of being a vector in $\mathfrak{so}(1,1)$ algebra is vector under $\mathfrak{so}(2)$ algebra. This gives rise to a different $\mathcal{W}$-algebra extension which closely resembles $\mathcal{W}(2;2^2,1)$ algebra found in the previous section. The structure of all the quasi-primaries and primaries are expected to remain the same with some minor modifications (such as $H(z) \to R(z)$) and therefore we provide only the result here.

1. **Virasoro Algebra**:

$$[L_m, L_n] = (m-n)\,L_{m+n} + \frac{c}{12}m(m^2-1)\,\delta_{m+n,0} \tag{6.70}$$

2. **Virasoro Primaries**

$$[L_m, P^a_n] = (m-n)\,P^a_{m+n}, \quad [L_m, R_n] = -n\,R_{m+n} \tag{6.71}$$

3. $\mathfrak{so}(2)_\kappa$ **Current Algebra**

$$[R_m, R_n] = \kappa\, m\,\delta_{m+n,0} \tag{6.72}$$

4. **Current algebra primaries**

$$[R_m, P^a_n] = \epsilon^{ab} P^b_{m+n} \tag{6.73}$$

5. **Commutator** $[P^a_m, P^b_n]$

$$[P^a_m, P^b_n] = \delta^{ab}\left[\frac{\alpha}{6}m(m^2-1)\delta_{m+n,0} + (m-n)\Big(\beta\, L_{m+n} + \gamma\,(R^2)_{m+n}\Big)\right]$$

$$+\,\epsilon^{ab}\Big[\delta\,(m^2+n^2-mn-1)\,R_{m+n} + \sigma\,\Lambda_{m+n} + \omega\,\Sigma_{m+n}\Big] \tag{6.74}$$

where Λ_m and Σ_m are the modes of those in (6.31) with H replaced by R. As before we impose Jacobi identities to solve for unknown parameters in the $\left[P^a_m, P^b_n\right]$ commutator. From Jacobi Identity (TPP) we get the following equations

$$2\alpha - c\,\beta - 2\gamma\,\kappa = 0, \quad 24\,\delta - (2+c)\,\sigma - 3\,\omega = 0 \tag{6.75}$$

The (RPP) Jacobi Identity demands

$$\beta + 3\,\delta + 2\,\gamma\,\kappa = 0, \quad \alpha - 6\,\kappa\,\delta = 0,$$
$$2\,\beta - \kappa\,\sigma = 0, \quad 2\gamma - \sigma - \frac{3}{2}\,\omega = 0. \tag{6.76}$$

In imposing the Jacobi Identity (PPP) we find similar structures that we saw before resulting in the following equations,

$$4\,\kappa\,(\gamma+\sigma) - 3\,\omega = 0, \quad 4\,\kappa\,(2\beta - \gamma - 2\,\delta + \sigma) - \omega = 0. \tag{6.77}$$

These equations (6.75, 6.76, 6.77) with replacements $\gamma \to -\gamma$ and $\kappa \to -\kappa$ match with the equations (6.58, 6.59, 6.62, 6.63) obtained in $\mathfrak{so}(1,1)$ case. Hence we obtain the new algebra $\widetilde{\mathcal{W}}(2; 2^2, 1)$ where the parameters are given by (6.64) after the replacements $\gamma \to -\gamma$ and $\kappa \to -\kappa$.

6.2.3 $\mathcal{W}(2,\, 4)$

Here we start with the third avatar of $\mathfrak{so}(2,3)$ given by,

$$[L_m, L_n] = (m-n)\,L_{m+n}, \quad [L_m, W_n] = (3m-n)\,W_{m+n},$$
$$[W_m, W_n] = \frac{1}{1680}A(m,n)\,L_{m+n} + \frac{1}{\sqrt{1680}}B(m,n)\,W_{m+n} \tag{6.78}$$

where $A(m,n)$ and $B(m,n)$ are as given in (6.16). Using the method that was spelt out earlier, to obtain the infinite-dimensional extension of this algebra we start with the following ansatz of $[W_m, W_n]$,

$$\begin{aligned}[W_m, W_n] = \; & \alpha\,\tilde{\alpha}(m,n) + \beta\, A(m,n)\, L_{m+n} + B(m,n)\,\left(\gamma\, W_{m+n} + \delta\,\Lambda_{m+n}\right) \\ & + (m-n)\left[\omega\,\Delta_{m+n} + \tau\,\Gamma_{m+n} + \sigma\,\Omega_{m+n}\right] \end{aligned} \tag{6.79}$$

where,

$$\tilde{\alpha}(m,n) = \frac{m\,(m^6 - 14m^4 + 49m^2 - 36)}{5040}\,\delta_{m+n,0} \tag{6.80}$$

The modes Λ_m are associated with the $h = 4$ quasi-primary

$$\Lambda(z) = (TT)\,(z) - \frac{3}{10}\partial^2 T(z). \tag{6.81}$$

The modes $\Delta_m, \Gamma_m\,, \Omega_m$ are those of the following three $h = 6$ quasi-primaries:

$$\begin{aligned} \Delta(z) &= (T\,\Lambda)\,(z) - \frac{1}{6}\,\partial^2\Lambda(z), \\ \Gamma(z) &= \left(\partial^2 TT\right)(z) - \partial(\partial TT)(z) + \frac{2}{9}\,\partial^2(TT)(z) - \frac{1}{42}\partial^4 T(z), \\ \Omega(z) &= (TW)(z) - \frac{1}{6}\partial^2 W(z)\,. \end{aligned} \tag{6.82}$$

To obtain equations constraining the parameters $\{\alpha, \beta, \gamma, \delta, \omega, \sigma, \tau\}$ in (6.79) one imposes Jacobi identities as for the earlier cases. From the Jacobi identity of (TWW) one gets the following equations,

$$\begin{aligned} & \alpha - 420\,c\,\beta = 0, \quad 1960\,\beta - (22+5c)\,\delta = 0, \quad 14\,\gamma - \frac{1}{12}(24+c)\,\sigma = 0, \\ & 14\delta - \frac{1}{180}(160\,\tau + 3\,\omega(15c+164)) = 0, \\ & 737100\,\beta - 315(22+5c)\,\delta - 5(29+70c)\,\kappa - 42(22+5c)\,\omega = 0\,. \end{aligned} \tag{6.83}$$

From imposing (WWW) Jacobi Identity one gets,

$$18900\,\beta - 8820\,\delta - 315\,\gamma\,(60\,\gamma - 7\,\sigma) + 95\,\tau + 1029\,\omega = 0. \tag{6.84}$$

The solution to these equations are as follows,

$$\alpha = 420\, c\, \beta\,, \quad \delta = \frac{1960\, \beta}{22+5c}\,, \quad \omega = \frac{20160(13+72\, c)\, \beta}{70c^3+953c^2+2498\, c-1496}\,,$$

$$\tau = \frac{252\,(-524+19c)\,\beta}{14c^2+129c-68}\,, \quad \sigma = \frac{168\,\gamma}{24+c}\,, \quad \gamma^2 = \frac{70\,\beta(c^3-148c^2-3932c+4704)}{70c^3+953c^2+2498c-1496} \tag{6.85}$$

where β is a free parameter, the value of which can be fixed. Choosing $\beta = \frac{1}{1680}$ one can check that the above algebra is $\mathcal{W}(2,4)$ algebra as was written down in [179].

6.2.4 $\mathcal{W}(2;(3/2)^2,1^3)$

This case was dealt with in detail already by us recently in Chapter 5 motivated from the algebra derived in the context of symmetries responsible for soft theorems in the context of MHV graviton amplitudes in $4d$ flat spacetimes. For the convenience of the readers we simply quote the result for the only non-trivial OPE

$$G_s(z)G_{s'}(w) = 2\alpha\,\epsilon_{ss'}(z-w)^{-3} + \delta\,(z-w)^{-2}\,(\lambda^a)_{ss'}\,[2\,J_a(w)+(z-w)\,J_a{}'(w)]$$
$$+\,\beta\,\epsilon_{ss'}(z-w)^{-1}\,T(w) + \gamma\,(z-w)^{-1}\,\epsilon_{ss'}(J^2)(w)\,, \tag{6.86}$$

which in terms of modes reads

$$[G_{s,r},G_{s',r'}] = \epsilon_{ss'}\Big[\alpha\Big(r^2-\frac{1}{4}\Big)\delta_{r+r',0}+\beta\,L_{r+r'}+\gamma\,(J^2)_{r+r'}\Big]+\delta(r-r')J_{a,r+r'}(\lambda^a)_{ss'} \tag{6.87}$$

where $(J^2)_n$ are the modes of the normal ordered quasi-primary $\eta^{ab}(J_aJ_b)(z)$:

$$(J^2)_n = \eta^{ab}\left[\sum_{k>-1} J_{b,n-k}J_{a,k} + \sum_{k\leq -1} J_{a,k}J_{b,n-k}\right]$$

The result of the imposition of the Jacobi identities is the following set of constraints on the parameters $(\alpha, \beta, \gamma, \delta, c, \kappa)$

$$\alpha - \frac{c}{6}\beta + \gamma\,\frac{\kappa}{2} = 0, \quad \alpha - \delta\,\frac{\kappa}{2} = 0, \quad \beta - \gamma\,(\kappa+2) - \frac{1}{2}\delta = 0, \quad \beta - \frac{\gamma - \delta}{2} = 0\ . \tag{6.88}$$

whose solution exists provided

$$c = -\frac{6\kappa\,(1+2\kappa)}{5+2\kappa}\ . \tag{6.89}$$

When this holds, for generic values of κ ($\neq -5/2$) the solution can be written (taking γ to be the independent variable) as

$$\alpha = -\frac{1}{4}\gamma\kappa(3+2\kappa), \quad \beta = \frac{1}{4}\gamma(5+2\kappa), \quad \delta = -\frac{1}{2}\gamma(3+2\kappa)\,. \tag{6.90}$$

This is the final result valid for generic ($\kappa \neq -5/2$) values of κ [163].

This completes our exercise of finding all minimal chiral $\mathcal{W}$-algebra extensions of $\mathfrak{so}(2,3)$.

6.3 $\mathcal{W}(2;(3/2)^2,1^3)$ from $\mathfrak{so}(2,3)$ gauge theory

In this section, we show that the semi-classical limit of the $\mathcal{W}(2;(3/2)^2,1^3)$ is the asymptotic symmetry algebra of the $3d$ conformal gravity – formulated as the Chern-Simons gauge theory for the gauge algebra $\mathfrak{so}(2,3)$ – with a consistent set of boundary conditions. The standard steps and methodology regarding the calculations in $\mathfrak{so}(2,3)$ gauge theory was reviewed in (2.1.4).

We start with the Chern-Simons theory in $3d$ with gauge algebra $\mathfrak{so}(2,3)$ written as (6.18). The $3d$ gauge connection $\mathcal{A} = \mathfrak{b}\,d\mathfrak{b}^{-1} + \mathfrak{b}\mathfrak{a}\mathfrak{b}^{-1}$ where $\mathfrak{b} = e^{(J_0 - L_0)\,\ln(r/\ell)}$ and $\mathfrak{a}$ is a $2d$ connection whose coordinates are $x^{\pm}$. We write the gauge connection $\mathfrak{a}$ as

$$\mathfrak{a} = A^{(m)}L_m + \bar{A}^{(a)}J_a + \Phi^{(s,r)}G_{s,r}. \tag{6.91}$$

Then the $3d$ flatness conditions $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0 \implies d\mathfrak{a} + \mathfrak{a} \wedge \mathfrak{a} = 0$

$$\begin{aligned}
\partial_i A_j^{(p)} - \partial_j A_i^{(p)} + (m-n)\delta^p_{m+n} A_i^{(m)}\, A_j^{(n)} + 2\epsilon_{ss'}\Phi_i^{(s,r)}\Phi_j^{(s',r')}\delta^p_{r+r'} &= 0, \\
\partial_i \bar{A}_j^{(c)} - \partial_j \bar{A}_i^{(c)} + (a-b)\delta^c_{a+b} \bar{A}_i^{(a)}\, \bar{A}_j^{(b)} + 2\epsilon_{rr'}\Phi_i^{(s,r)}\Phi_j^{(s',r')}\delta^c_{s+s'} &= 0, \\
\partial_i \Phi_j^{(s,r)} + \frac{1}{2}(m-2r')\delta^r_{m+r'} A_i^{(m)}\Phi_j^{(s,r')} + \frac{1}{2}(a-2s')\delta^s_{a+s'}\bar{A}_i^{(a)}\Phi_j^{(s',r)} - (i \leftrightarrow j) &= 0.
\end{aligned} \tag{6.92}$$

6.3.1 The constraints and a variational principle

The action is

$$S = \frac{k}{4\pi}\int_M \mathrm{Tr}\Big(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\Big) \tag{6.93}$$

whose variation is

$$\delta S = \frac{k}{4\pi}\int \mathrm{Tr}\Big(2\mathcal{F} \wedge \delta\mathcal{A}\Big) - \frac{k}{4\pi}\int_{\partial M} \mathrm{Tr}\Big(\mathcal{A} \wedge \delta\mathcal{A}\Big) \tag{6.94}$$

Around an on-shell (*i.e.*, with $\mathcal{F} = 0$) configuration under general variation $\delta\mathcal{A}$

$$\delta S\Big|_{on-shell} = -\frac{k}{4\pi}\int_{\partial M} \mathrm{Tr}\Big(\mathcal{A} \wedge \delta\mathcal{A}\Big) \tag{6.95}$$

Using the traces $\mathrm{Tr}(L_m L_n) = -\eta_{mn}$, $\mathrm{Tr}(J_a J_b) = -\eta_{ab}$ and $\mathrm{Tr}(G_{(s,r)}G_{(s',r')}) = 4\epsilon_{ss'}\epsilon_{rr'}$ this becomes

$$\begin{aligned}
\delta S\Big|_{on-shell} = \frac{k}{8\pi}\int_{r=r_0} d^2x \Big[&\eta_{mn}\left(A_-^{(m)}\delta A_+^{(n)} - A_+^{(m)}\delta A_-^{(n)}\right) \\
&+ \eta_{ab}\left(\bar{A}_-^{(a)}\delta \bar{A}_+^{(b)} - \bar{A}_+^{(a)}\delta \bar{A}_-^{(b)}\right) \\
&- 4\,\epsilon_{ss'}\epsilon_{rr'}\left(\Phi_-^{(s,r)}\delta\Phi_+^{(s',r')} - \Phi_+^{(s,r)}\delta\Phi_-^{(s',r')}\right)\Big] + \mathcal{O}(1/r_0)
\end{aligned} \tag{6.96}$$

We first impose the following constraints:[8]

$$\begin{aligned}
&A_-^{(-)} = 1, \quad A_-^{(0)} = 0, \quad \bar{A}_+^{(+)} = 1, \quad \bar{A}_-^{(0)} = 0, \\
&\Phi_+^{(+,-)} = \Phi_-^{(+,-)} = \Phi_-^{(-,-)} = \Phi_-^{(-,+)} = 0\,, \\
&\bar{A}_+^{(0)} = A_+^{(0)}\,.
\end{aligned} \tag{6.97}$$

The constraints imposed on $A^{(m)}$ and $\bar{A}^{(m)}$ in (6.97) are the same as those in [170] for AdS$_3$ gravity. Then some of the equations of motion (flatness of $\mathcal{A}$) lead to the following additional constraints:

$$\begin{aligned}
&A_+^{(0)} = \partial_- A_+^{(-)}, \quad \Phi_+^{(+,+)} = \Phi_-^{(+,+)} A_+^{(-)}, \quad \Phi_+^{(-,+)} = \partial_- \Phi_+^{(-,-)}, \\
&\bar{A}_-^{(-)} = \frac{1}{2}\partial_- \bar{A}_+^{(0)} - \Phi_+^{(-,-)}\Phi_-^{(+,+)}, \\
&A_+^{(+)} = \frac{1}{2}\partial_-^2 A_+^{(-)} + A_+^{(-)} A_-^{(+)} - \Phi_+^{(-,-)}\Phi_-^{(+,+)}
\end{aligned} \tag{6.98}$$

and the rest of the equations of motion lead to the following equations among the remaining six fields $(A_+^{(-)}, A_-^{(+)}, \bar{A}_+^{(-)}, \bar{A}_-^{(+)}, \Phi_+^{(-,-)}, \Phi_-^{(+,+)})$:

$$\frac{1}{2}\partial_-^3 A_+^{(-)} + \left(A_+^{(-)}\partial_- + 2\partial_- A_+^{(-)} - \partial_+\right) A_-^{(+)} = 3\Phi_-^{(+,+)}\partial_-\Phi_+^{(-,-)} + \Phi_+^{(-,-)}\partial_-\Phi_-^{(+,+)}, \tag{6.99}$$

$$\begin{aligned}
&\partial_- \left[\bar{A}_+^{(-)} - \frac{1}{4}\left(\partial_- A_+^{(-)}\right)^2 - \frac{1}{2}\partial_-\partial_+ A_+^{(-)}\right] \\
&+ \partial_+ \left[\Phi_+^{(-,-)}\Phi_-^{(+,+)} + \bar{A}_-^{(+)}\bar{A}_+^{(-)}\right] + \left[\Phi_+^{(-,-)}\Phi_-^{(+,+)} + \bar{A}_-^{(+)}\bar{A}_+^{(-)}\right] \partial_- A_+^{(-)} = 0,
\end{aligned} \tag{6.100}$$

$$\partial_+\Phi_-^{(+,+)} - A_+^{(-)}\partial_-\Phi_-^{(+,+)} - 2\Phi_-^{(+,+)}\partial_- A_+^{(-)} - 2\bar{A}_-^{(+)}\partial_-\Phi_+^{(-,-)} - \Phi_+^{(-,-)}\partial_-\bar{A}_-^{(+)} = 0, \tag{6.101}$$

[8] The $\{+,-\}$ subscript of one-forms $\{A^{(m)}, \bar{A}^{(a)}, \Phi^{(r,s)}\}$ denote the $\{x^+, x^-\}$ component respectively. In the context of field $\Phi^{(r,s)}$ even though $r, s \in \{-1/2, 1/2\}$, we will denote it by $\{-,+\}$ for notational simplicity.

$$\partial_-^2 \Phi_+^{(-,-)} - \left[\Phi_-^{(+,+)} A_+^{(-)} + \Phi_+^{(-,-)} \bar{A}_-^{(+)}\right] \left[-\Phi_+^{(-,-)} \Phi_-^{(+,+)} + \bar{A}_-^{(+)} \bar{A}_+^{(-)} + \frac{1}{2}\partial_-^2 A_+^{(-)}\right]$$
$$+ \left[\Phi_-^{(+,+)} \bar{A}_+^{(-)} + A_+^{(+)} \Phi_+^{(-,-)}\right] = 0. \tag{6.102}$$

Now we consider an arbitrary configuration that satisfies the constraints (6.97, 6.98) and impose these constraints at the level of variations about such a configuration. It is seen that the variation of the action around such configuration and variations of fields gives:

$$\delta S\Big|_{on-shell} = -\frac{k}{4\pi}\int_{r=r_0} 4\, d\bar{z} \wedge dz \Big[\bar{A}_+^{(-)} \delta \bar{A}_-^{(+)} - A_-^{(+)} \delta A_+^{(-)} - \Phi_+^{(-,-)} \delta \Phi_-^{(+,+)} + \Phi_-^{(+,+)} \delta \Phi_+^{(-,-)}\Big]\,. \tag{6.103}$$

We will solve this in a chiral way by choosing[9]

$$A_-^{(+)} = 1/4, \quad \bar{A}_-^{(+)} = 0, \quad \Phi_-^{(+,+)} = 0 \tag{6.104}$$

which leads to the solutions with

$$\begin{aligned} A_+^{(-)}(x^\pm) &= J^{(-1)}(x^+)\, e^{-ix^-} + J^{(0)}(x^+) + J^{(1)}(x^+) e^{ix^-} \\ \Phi_+^{(-,-)} &= G^{(-1/2)}(x^+)\, e^{-\frac{i}{2}x^-} + G^{(1/2)}(x^+)\, e^{\frac{i}{2}x^-} \\ \bar{A}_+^{(-1)} &= T(x^+) + \frac{1}{2}\partial_+\partial_- A_+^{(-1)} + \frac{1}{4}\left(\partial_- A_+^{(-1)}\right)^2 . \end{aligned} \tag{6.105}$$

The residual gauge transformations that leave the constraints (6.97, 6.104) invariant can be worked out easily. We write the gauge parameter as follows,

$$\Lambda = \Lambda^{(m)} L_m + \bar{\Lambda}^{(a)} J_a + \Lambda^{(s,r)} G_{s,r}, \tag{6.106}$$

where $\{\Lambda^{(m)}, \bar{\Lambda}^{(a)}, \Lambda^{(s,r)}\}$ are general function of $x^\pm$. The gauge transformation of the connection is given by,

$$\delta \mathcal{A}_i = \partial_i \Lambda + [\mathcal{A}_i, \Lambda]\,. \tag{6.107}$$

The gauge parameter corresponding to the residual gauge transformations is expected to preserve the constraints (6.97, 6.104) and leave the connection $\mathcal{A}$ form-invariant. We find that such a Λ can be written in terms of three free functions of

[9] The first of these requires adding a further boundary term which can be done easily as in [170].

$x^{\pm}$, $\{\Lambda^{(-)}, \bar{\Lambda}^{(+)}, \Lambda^{(+,-)}\}$ as follows,

$$\begin{aligned}
&\Lambda^{(0)} = -4\,\partial_-\Lambda^{(-)}, \quad \Lambda^{(1)} = \frac{1}{2}\lambda^{(0)}(x^+) - \frac{1}{4}\Lambda^{(-)}(x^+, x^-)\,, \; \Lambda^{(+,+)} = \partial_-\Lambda^{(+,-)},\\
&\Lambda^{(-,+)} = \frac{1}{2}\Lambda^{(+,-)}\left[J^{(-1)}(x^+)\, e^{-ix^-} + J^{(1)}(x^+)\, e^{ix^-}\right] + \partial_-\Lambda^{(-,-)},\\
&\Lambda^{(-,-)} = \bar{\lambda}^{(1)}\,\Phi_+^{(-,-)} - \partial_+\Lambda^{(+,-)} + \partial_-\Lambda^{(+,-)}A_+^{(-)},\\
&\partial_-\bar{\Lambda}^{(0)} = -\bar{\Lambda}^{(+)}\left[J^{(-1)}(x^+)e^{-ix^-} + J^{(+1)}(x^+)e^{ix^-}\right],\\
&\partial_-\bar{\Lambda}^{(-)} = -\frac{1}{2}\bar{\Lambda}^{(0)}\left[J^{(-1)}(x^+)e^{-ix^-} + J^{(+1)}(x^+)e^{ix^-}\right],
\end{aligned} \tag{6.108}$$

which are further constrained to satisfy

$$\begin{aligned}
&\partial_-^2\Lambda^{(-)}(x^+, x^-) + \Lambda^{(-)}(x^+, x^-) - \lambda^{(0)}(x^+) = 0,\\
&\partial_-\bar{\Lambda}^{(+)} = 0\,, \quad \partial_-^2\,\Lambda^{(+,-)} + \frac{1}{4}\Lambda^{(+,-)} = 0.
\end{aligned} \tag{6.109}$$

These constraints in (6.109) can be solved to obtain

$$\begin{aligned}
&\Lambda^{(-)} = \lambda^{(-1)}(x^+)\, e^{-ix^-} + \lambda^{(0)}(x^+) + \lambda^{(1)}(x^+)e^{ix^-},\\
&\bar{\Lambda}^{(+)} = \lambda(x^+),\\
&\Lambda^{(+,-)} = \chi^{(-1/2)}(x^+)e^{-\frac{i}{2}x^-} + \chi^{(1/2)}(x^+)e^{\frac{i}{2}x^-}.
\end{aligned} \tag{6.110}$$

Thus the residual gauge parameter is given in terms of the six chiral functions $\{\lambda^a(x^+), \lambda(x^+), \chi^s(x^+)\}$ and they can be shown to produce the following variations:

$$\begin{aligned}
\delta T &= 2T\,\lambda' + \lambda\,T' + \frac{1}{2}\lambda''' + i\epsilon^{rs}(3G_r\chi_s' + G_r'\chi_s) + 2(\lambda^a)^{rs}J_aG_r\chi_s,\\
\delta J_a &= \lambda_a' + i\,f^{bc}{}_a J_b\lambda_c + 4(\lambda_a)^{rs}G_r\chi_s,\\
\delta G_s &= \lambda G_s' + \frac{3}{2}G_s\lambda' - \chi_s'' - i(\lambda^a)^r{}_s(J_a\,\lambda - \lambda_a)\,G_r - \left(\frac{1}{4}\eta^{ab}J_aJ_b + T\right)\chi_s\\
&\quad + i(\lambda^a)^r{}_s\,(2J_a\,\chi_r' + J_a'\,\chi_r)\,.
\end{aligned} \tag{6.111}$$

We also have

$$\begin{aligned}
\mathrm{Tr}\,(\Lambda\delta A_-) &= \lambda^{(1)}\,(e^{-ix^-}\,\delta J^{(-1)} + e^{ix^-}\,\delta J^{(1)}),\\
\mathrm{Tr}\,(\Lambda\delta A_+) &= \eta^{ab}\delta J_a\lambda_b - 4i\,\epsilon^{rs}\chi_s\delta G_r - 2\lambda\,\delta T\\
&\quad - \partial_+\partial_-\left(\lambda^{(1)}\,(e^{-ix^-}\,\delta J^{(-1)} + e^{ix^-}\,\delta J^{(1)})\right).
\end{aligned} \tag{6.112}$$

Using the coordinates $x^{\pm} = \phi \pm t$ we have $\mathcal{A}_\phi = \mathcal{A}_+ + \mathcal{A}_-$ and $\partial_\phi = \partial_+ + \partial_-$. The charge is [186]

$$\begin{aligned}\not{\delta} Q_\Lambda &= -\frac{k}{2\pi}\int_0^{2\pi} d\phi\, \mathrm{Tr}\,[\Lambda\, \delta \mathcal{A}_\phi] \\ &= -\frac{k}{2\pi}\int_0^{2\pi} d\phi\, \Big[\eta^{ab}\delta J_a \lambda_b + 4i\,\epsilon^{sr}\chi_s \delta G_r - 2\lambda\,\delta T \\ &+ \partial_\phi\Big(i\,\lambda^{(1)}\,(e^{-ix^-}\delta J^{(-1)} - e^{ix^-}\delta J^{(1)})\Big)\Big].\end{aligned} \tag{6.113}$$

Assuming that $\delta J^{(\pm 1)}(x^+)$ and $\lambda^{(1)}(x^+)$ are periodic in $\phi \to \phi + 2\pi$ we will drop the total derivative terms here in the second line. Before carrying on we will make the following replacements

$$\begin{aligned}\lambda_a &\to \lambda_a + J_a\,\lambda \\ \delta T &\to \delta T + \frac{1}{2}\eta^{ab} J_a\,\delta J_b\end{aligned} \tag{6.114}$$

which changes the transformations (6.111) to

$$\begin{aligned}\delta T &= 2T\,\lambda' + \lambda\,T' + \frac{1}{2}\lambda''' + i\epsilon^{rs}(3G_r\chi_s' + G_r'\chi_s) - \frac{1}{2}\eta^{ab}J_a\lambda_b', \\ \delta J_a &= (\lambda\,J_a)' + \lambda_a' + i\,f^{bc}{}_a J_b\,\lambda_c + 4(\lambda_a)^{rs}G_r\chi_s, \\ \delta G_s &= \lambda G_s' + \frac{3}{2}G_s\lambda' - \chi_s'' - \left(\frac{1}{2}\eta^{ab}J_aJ_b + T\right)\chi_s + i(\lambda^a)^r{}_s\,(2J_a\,\chi_r' + J_a'\,\chi_r + \lambda_a\,G_r)\,.\end{aligned} \tag{6.115}$$

and $\not{\delta} Q_\Lambda$, after replacements $\delta T \to \delta T + \frac{1}{2}\eta^{ab}J_a\delta J_b$ and $\lambda_a \to \lambda_a + J_a\,\lambda$ remains form-invariant. Thus we see it is integrable and we write

$$Q_\Lambda = \frac{k}{2\pi}\int_0^{2\pi} d\phi\,\Big(2\lambda\,T - \eta^{ab}J_a\lambda_b + 4i\,\epsilon^{rs}\,G_r\,\chi_s\Big)\,. \tag{6.116}$$

Using the fact that this charge Q_Λ is expected to induce the change $\delta_\Lambda f = \{Q_\Lambda, f\}$, one can read out brackets among $(T(\phi), J_a(\phi), G_r(\phi))$. This results in

$$\begin{aligned}\frac{k}{\pi}\{T(\phi), T(\psi)\} &= 2\,T(\psi)\,\delta'(\psi-\phi) + \delta(\psi-\phi)\,T'(\psi) + \frac{1}{2}\delta'''(\psi-\phi), \\ \frac{k}{\pi}\{T(\phi), J_a(\psi)\} &= J_a(\psi)\,\delta'(\psi-\phi) + J_a'(\psi)\,\delta(\psi-\phi),\end{aligned} \tag{6.117}$$

$$\frac{k}{\pi}\{T(\phi), G_r(\psi)\} = \frac{3}{2}G_r(\psi)\,\delta'(\psi-\phi) + G_r'(\psi)\,\delta(\psi-\phi),$$

$$-\frac{k}{2\pi}\{J_a(\phi), J_b(\psi)\} = if_{ab}{}^c J_c(\psi)\,\delta(\psi-\phi) + \eta_{ab}\,\delta'(\psi-\phi),$$

$$-\frac{k}{2\pi}\{J_a(\phi), G_r(\psi)\} = i(\lambda_a)^s{}_r G_s(\psi)\,\delta(\psi-\phi),$$

$$-\frac{k}{\pi}\{G_r(\phi), G_s(\psi)\} = \frac{i}{2}\epsilon_{rs}\left(T(\psi) + \frac{1}{2}J^2(\psi)\right)\delta(\psi-\phi) + \frac{i}{2}\epsilon_{rs}\,\delta''(\psi-\phi)$$
$$+ (\lambda^a)_{rs}\left[J_a(\psi)\,\delta'(\psi-\phi) + \frac{1}{2}J_a'(\psi)\,\delta(\psi-\phi)\right], \qquad (6.118)$$

where all derivatives denoted by primes are with respect to the arguments. We use the mode expansions

$$T(\phi) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} T_n\, e^{in\phi} \iff L_n = \int_0^{2\pi} d\phi\, T(\phi)e^{-in\phi},$$
$$J_a(\phi) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} J_{a,n}\, e^{in\phi} \iff J_{a,n} = \int_0^{2\pi} d\phi\, J_a(\phi)e^{-in\phi},$$
$$G_s(\phi) = \frac{1}{2\pi}\sum_{r\in\mathbb{Z}+\frac{1}{2}} G_{s,r}\, e^{ir\phi} \iff G_{s,r} = \int_0^{2\pi} d\phi\, G_s(\phi)e^{-ir\phi}\,. \qquad (6.119)$$

Note that we have postulated anti-periodic boundary conditions for $G_r(\phi)$ which immediately follows from demanding that the gauge field component $\Phi_+^{(-,-)}$ is periodic. Using the mode expansion of $J_a(\phi)$ we can write the mode expansion of $\eta^{ab}J_a(\phi)J_b(\phi)$ as follows:

$$\eta^{ab}J_a(\phi)J_b(\phi) = \frac{1}{4\pi^2}\sum_{m,n\in\mathbb{Z}} \eta^{ab}J_{a,m}J_{b,n}\, e^{i(m+n)\phi}$$
$$= \frac{1}{4\pi^2}\sum_{m+n=p\in\mathbb{Z}}\left(\sum_{m\in\mathbb{Z}}\eta^{ab}J_{a,m}J_{b,p-m}\right)e^{ip\phi} := \frac{1}{4\pi^2}\sum_{p\in\mathbb{Z}}(J^2)_p\, e^{ip\phi}$$
$$\implies (J^2)_p = 2\pi\int_0^{2\pi} d\phi\, \eta^{ab}J_a(\phi)J_b(\phi)\, e^{-ip\phi}. \qquad (6.120)$$

Replacing the Poisson brackets $\{A, B\}$ in terms of the Dirac brackets $[A, B]$ using $\{A, B\} = -i[A, B]$ and $L_n \to \frac{\pi}{k} L_n$, $J_{a,n} \to \frac{2\pi}{k} J_{a,n}$ and $G_{s,r} \to \frac{\pi}{\sqrt{-\alpha k}} G_{s,r}$ we find

$$[L_m, L_n] = (m-n)L_{m+n} + k\, n^3 \delta_{m+n,0},$$

$$[L_m, J_{a,n}] = -n\, J_{a,m+n}, \quad [L_m, G_{s,r}] = \frac{1}{2}(m-2r)G_{s,m+r},$$

$$[J_{a,m}, J_{b,n}] = f_{ab}{}^{c} J_{c,m+n} + k\, n\, \eta_{ab}\delta_{m+n,0}, \quad [J_{a,m}, G_{s,r}] = (\lambda_a)^{s'}{}_{s} G_{s',m+r},$$

$$\alpha^{-1}[G_{s,r}, G_{s',r'}] = \epsilon_{ss'}\left[r^2\delta_{r+r',0} - \frac{1}{2k}L_{r+r'} - \frac{1}{2k^2}(J^2)_{r+r'}\right] + \frac{1}{k}(\lambda^a)_{ss'}(r-r')J_{a,r+r'}. \tag{6.121}$$

Finally to compare with earlier results (5.1) we need to replace $L_n \to L_n + \frac{k}{2}\delta_{n,0}$ and set $k = \frac{\kappa}{2}$. Then this algebra is seen to be in perfect agreement with the large-κ limit of the $\mathcal{W}(2, (3/2)^2, 1^3)$ algebra.

6.4 $\mathcal{W}(2,4)$ from $\mathfrak{so}(2,3)$ gauge theory

In (6.15), the Lie algebra $\mathfrak{so}(2,3)$ split into a $\mathfrak{sl}(2,\mathbb{R})$ given by a set $\{L_1, L_0, L_1\}$ and a 7 dimensional vector multiplet under $\mathfrak{sl}(2,\mathbb{R})$ denoted by W_i where $i \in \{\pm 3, \pm 2, \pm 1\}$. We write this algebra again with α as an arbitrary parameter,

$$\left[L_m\,, L_n\right] = (m-n)L_{m+n}\,, \quad \left[L_m\,, W_n\right] = (3m-n)W_{m+n} \tag{6.122}$$

$$\left[W_m, W_n\right] = \alpha^2\, A(m,n)\, L_{m+n} - \alpha\, B(m,n) W_{m+n} \tag{6.123}$$

which can be chosen to coincide (6.123) with the commutation relations in (6.15) but for now we will keep it unfixed. From boundary terms in the variation of Chern-Simons action (6.95), we choose to impose $\mathcal{A}_- = 0$ at the boundary to solve the variational problem. In what follows we will use the boundary conditions and gauge fixing method used in [116, 117], where the asymptotic symmetries of $3d$ higher spin gauge theories described by $SL(N) \times SL(N)$ Chern-Simons theories were evaluated to be $\mathcal{W}_N$ algebras. One gauge fix $\mathcal{A}_r$ as,

$$\mathcal{A}_r = f^{-1}(r)\,\partial_r\, f(r) \tag{6.124}$$

Then following the arguments of [116, 117], one can gauge fix $\mathcal{A}_- = 0$ everywhere on-shell and the resultant gauge connection $\mathcal{A}$ can be written in terms of $2d$ connection $\mathfrak{a}$ as follows,

$$\mathcal{A}_- = 0\,, \quad \mathcal{A}_+ = f^{-1}(r)\,\mathfrak{a}(x^+)\,f(r). \tag{6.125}$$

where $\mathfrak{a}$ is given as

$$\mathfrak{a} = a^{(m)}(x^+)\,L_m + w^{(i)}(x^+)\,W_i \tag{6.126}$$

We further gauge fix $\mathfrak{a}$ according to the highest weight gauge used in [116, 117]. The resultant $\mathfrak{a}$ is then given by,

$$\mathfrak{a} = L_1 + T(x^+)\,L_{-1} + \mathcal{W}(x^+)\,W_{-3} \tag{6.127}$$

For different choices of $T(x^+)$ and $\mathcal{W}(x^+)$, $\mathfrak{a}$ represents physically distinct solutions to the equation of motion. To find the residual gauge transformations that do not spoil the gauge and boundary conditions, we use the following form of gauge parameter that takes value in $\mathfrak{so}(2,3)$ Lie algebra,

$$\Lambda = \lambda^{(m)}(x^+)\,L_m + \chi^{(i)}(x^+)\,W_i. \tag{6.128}$$

Using the gauge transformation $\delta\mathfrak{a} = d\mathfrak{a} + [\mathfrak{a}, \Lambda]$ one can solve for various components of the gauge parameters (6.128) to find

$$\lambda^{(0)} = -\partial\lambda^{(1)}\,,\ \lambda^{(-1)} = \frac{1}{2}\left(2\,T\,\lambda^{(1)} + 2160\,\alpha^2\,\mathcal{W}\,\chi^{(3)} + \partial^2\lambda^{(1)}\right),$$

$$\chi^{(2)} = -\partial\chi^{(3)}\,,\ \chi^{(-3)} = \frac{1}{2}\left(6\,T\chi^{(3)} + \partial^2\chi^{(3)}\right),$$

$$\chi^{(-2)} = \frac{1}{6}\left(-6\,\chi^{(3)}\,\partial T - 16\,T\,\partial\chi^{(3)} - \partial^3\chi^{(3)}\right),$$

$$\chi^{(1)} = \frac{1}{24}\Big(72\,T^2\,\chi^{(3)} - 720\,\alpha\,\mathcal{W}\,\chi^{(3)} + 22\,\partial T\,\partial\chi^{(3)} + 6\,\chi^{(3)}\,\partial^2 T(x^+) + 28\,T(x^+)\,\partial^2\chi^{(3)} + \partial^4\chi^{(3)}\Big),$$

$$\chi^{(0)} = \frac{1}{120}\Big(-264\,T^2\,\partial\chi^{(3)} + 2160\,\alpha\,\mathcal{W}\,\partial\chi^{(3)} - 28\,\partial\chi^{(3)}\,\partial^2 T - 50\,\partial T\,\partial^2\chi^{(3)} + \chi^{(3)}\left(720\,\alpha\,\partial\mathcal{W} - 6\,\partial^3\,T\right) - 8T\,\left(27\,\chi^{(3)}\partial T + 5\,\chi^{(3)}\right) - \partial^5\chi^{(3)}\Big), \tag{6.129}$$

$$\chi^{(-1)} = \frac{1}{720}\Big(720\,T^3\,\chi^{(3)} + 216\,\chi^{(3)}\,(\partial T)^2 - 2880\,\alpha\,\partial\mathcal{W}\,\partial\chi^{(3)} - 720\,\alpha\,\chi^{(3)}\,\partial^2\mathcal{W} + 544\,T^2\,\partial^2\chi^{(3)} + 78\,\partial^2 T\,\partial^2\,\chi^{(3)} + 720\,\mathcal{W}\,\left(\lambda^{(1)} - 22\,\alpha\,T\,\chi 6(3) - 5\,\alpha\,\partial^2\chi^{(3)}\right) + 34\,\partial\chi^{(3)}\partial^3 T + 90\,\partial T\,\partial^3\chi^{(3)} + 6\,\chi^{(3)}\,\partial^4 T + T\left(964\,\partial T\,\partial\chi^{(3)} + 276\,\chi^{(3)}\,\partial^2 T + 50\,\partial^4\chi^{(3)}\right) + \partial^{(6)}\chi^{(3)}\Big). \tag{6.130}$$

In the expression (6.130), the functional dependence of fields on x^+ is suppressed. All the components of gauge parameter (6.128) are given in terms of two fields $\{\lambda^{(1)}(x^+), \chi^{(3)}(x^+)\}$. These residual gauge transformations will also induce variation on background fields $\{T(x^+), \mathcal{W}(x^+)\}$ given by,

$$\delta T = \lambda\,\partial T + 2\,\partial\lambda T + \frac{1}{2}\,\partial^3\,\lambda + 1080\,\alpha^2\,\chi\,\partial\mathcal{W} + 1440\,\alpha^2\,\mathcal{W}\,\partial\chi, \tag{6.131}$$

$$\begin{aligned}\delta\mathcal{W} &= \lambda\,\partial\mathcal{W} + 4\,\mathcal{W}\,\partial\lambda + \chi\left[\frac{24}{5}T^2\,\partial T - 28\,\alpha\,\partial\,(\mathcal{W}T) + \frac{59}{60}\partial T\partial^2 T + \frac{13}{30}T\,\partial^3 T\right.\\ &\left. - \alpha\,\partial^3\mathcal{W} + \frac{1}{120}\,\partial^5 T\right] + \partial\chi\left[\frac{16}{5}T^3 - 56\,\alpha\,T\,\mathcal{W} + \frac{59}{36}(\partial T)^2 + \frac{88}{45}T\,\partial^2 T - 5\,\alpha\,\partial^2\mathcal{W}\right.\\ &\left. + \frac{1}{18}\,\partial^4 T\right] + \partial^2\chi\left[\frac{49\,T\,\partial T}{15} - 9\,\alpha\partial\mathcal{W} + \frac{7\,\partial^3 T}{45}\right] + \partial^3\chi\left[\frac{49}{45}T^2 - 6\,\alpha\,\mathcal{W} + \frac{7\,\partial^2 T}{30}\right]\\ &+ \frac{7}{36}\,\partial T\,\partial^4\chi + \frac{7}{90}\,T\,\partial^5\chi + \frac{1}{720}\,\partial^7\chi\end{aligned} \tag{6.132}$$

where we define $\lambda^{(1)} := \lambda\,, \chi^{(3)} = \chi$. Using the coordinates $x^{\pm} = \phi \pm t$ we have $\mathcal{A}_\phi = \mathcal{A}_+ + \mathcal{A}_-$ and $\partial_\phi = \partial_+ + \partial_-$. The charges that generate the variations (6.131), (6.132) are given by

$$\begin{aligned}\delta\!\!\!/ Q_\Lambda &= -\frac{k}{2\pi}\int_0^{2\pi} d\phi\,\mathrm{Tr}\,[\Lambda\,\delta\mathcal{A}_\phi]\\ \Longrightarrow Q &= \frac{k}{2\,\pi}\int_0^{2\pi} d\phi\,\left[20\,T(\phi)\,\lambda(\phi) + 7200\,\alpha^2\,\mathcal{W}(\phi)\,\chi(\phi)\right].\end{aligned} \tag{6.133}$$

Using the charges (6.133) and variations (6.131), (6.132) induced by them, one can find the Poisson bracket between fields $\{T(\phi), \mathcal{W}(\phi)\}$,

$$\frac{10\,k}{\pi}\{T(\phi), T(\psi)\} = 2\,T(\psi)\,\delta'(\psi-\phi) + \delta(\psi-\phi)\,T'(\psi) + \frac{1}{2}\delta'''(\psi-\phi), \tag{6.134}$$

$$\begin{aligned}\frac{10\,k}{\pi}\{T(\phi), \mathcal{W}(\psi)\} &= 4\,\mathcal{W}(\psi)\,\delta'(\psi-\phi) + \mathcal{W}'(\psi)\,\delta(\psi-\phi),\\ \frac{10\,k}{\pi}\{\mathcal{W}(\phi), T(\psi)\} &= 4\,\mathcal{W}(\psi)\,\delta'(\psi-\phi) + 3\,\mathcal{W}'(\psi)\,\delta(\psi-\phi),\end{aligned} \tag{6.135}$$

$$\begin{aligned}\frac{3600\,\alpha^2\,k}{\pi}\{\mathcal{W}(\phi), \mathcal{W}(\psi)\} &= \delta(\psi-\phi)\Big[\frac{24}{5}T^2\,T' - 28\,\alpha\,(\mathcal{W}T)' + \frac{59}{60}T'T'' + \frac{13}{30}T\,T'''\\ &- \alpha\,\mathcal{W}''' + \frac{1}{120}\,T'''''\Big] + \frac{7\,T}{90}\,\delta^{(5)}(\psi-\phi) + \frac{1}{720}\delta^{(7)}(\psi-\phi) + \frac{7\,T'}{36}\,\delta^{(4)}(\psi-\phi)\\ &+ \delta'(\psi-\phi)\left[\frac{16}{5}T^3 - 56\,\alpha\,T\,\mathcal{W} + \frac{59}{36}(T')^2 + \frac{88}{45}T\,T'' - 5\,\alpha\,\mathcal{W}'' + \frac{1}{18}\,T''''\right]\\ &+ \delta''(\psi-\phi)\left[\frac{49\,T\,T'}{15} - 9\,\alpha\mathcal{W}' + \frac{7\,T'''}{45}\right] + \delta'''(\psi-\phi)\,\left[\frac{49}{45}T^2 - 6\,\alpha\,\mathcal{W} + \frac{7\,T''}{30}\right].\end{aligned} \tag{6.136}$$

We use the following mode expansions,

$$\begin{aligned}T(\phi) &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} T_n\,e^{in\phi} \iff L_n = \int_0^{2\pi} d\phi\,T(\phi)e^{-in\phi},\\ \mathcal{W}(\phi) &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} W_n\,e^{in\phi} \iff W_n = \int_0^{2\pi} d\phi\,\mathcal{W}(\phi)e^{-in\phi}.\end{aligned} \tag{6.137}$$

Now the mode expansion $\{W_m, W_n\}$ is calculated as follows,

$$\{\mathcal{W}_m, \mathcal{W}_n\} = \frac{\pi}{3600\,\alpha^2\,k}\int_0^{2\pi} d\phi\int_0^{2\pi} d\psi e^{-im\,\phi}e^{-in\,\psi}\frac{3600\,\alpha^2\,k}{\pi}\{\mathcal{W}(\phi), \mathcal{W}(\psi)\} \tag{6.138}$$

In RHS of (6.138), terms that are first order in fields are as follows,

$$\frac{\pi}{3600\,\alpha^2\,k}\int_0^{2\pi}d\phi\int_0^{2\pi}d\psi e^{-i\,m\,\phi}e^{-i\,n\,\psi}\Big[\delta(\psi-\phi)\,\left(-\alpha\,\partial^3\mathcal{W}+\frac{1}{120}\,\partial^5 T\right)$$
$$\delta^{(1)}(\psi-\phi)\left(-5\,\alpha\,\partial^2\,\mathcal{W}+\frac{1}{18}\,\partial^4 T\right)+\delta^{(2)}(\psi-\phi)\left(-9\,\alpha\,\partial\,\mathcal{W}+\frac{7}{45}\,\partial^3 T\right)$$
$$+\,\delta^{(3)}(\psi-\phi)\left(-6\,\alpha\,\mathcal{W}+\frac{7}{30}\,\partial^2 T\right)+\frac{7}{36}\partial T\,\delta^{(4)}(\psi-\phi)+\frac{7}{90}T\,\delta^{(5)}(\psi-\phi)\Big]. \tag{6.139}$$

Terms that have quadratic field dependence are given by,

$$\frac{\pi}{3600\,\alpha^2\,k}\int_0^{2\pi}d\phi\int_0^{2\pi}d\psi e^{-i(m\phi+n\psi)}\Big[\delta(\psi-\phi)\big(-28\,\alpha\partial(\mathcal{W}T)+\frac{11}{80}\partial^3 T^2$$
$$+\frac{19}{120}\partial(T\partial^2 T)\big)\delta^{(1)}(\psi-\phi)\left(-56\,\alpha\mathcal{W}T+\frac{59}{72}\partial^2 T^2+\frac{19}{60}\,T\partial^2 T\right)$$
$$+\frac{49}{30}\delta^{(2)}(\psi-\phi)\,\partial(T^2)+\frac{49}{45}\delta^{(3)}(\psi-\phi)\,T^2\Big] \tag{6.140}$$

Terms with cubic order in fields are as follows,

$$\frac{\pi}{3600\,\alpha^2\,k}\int_0^{2\pi}d\phi\int_0^{2\pi}d\psi e^{-i\,m\,\phi}e^{-i\,n\,\psi}\left[\frac{24}{15}\,\delta(\psi-\phi)\partial(T^3)+\frac{16}{5}\delta^{(1)}(\psi-\phi)T^3\right]. \tag{6.141}$$

The full Poisson bracket after collecting all the terms is given by,

$$\{W_m\,,W_n\}=-\frac{i\,\pi}{k\,\alpha^2}\,\tau^{L}_{WW}\,L_{m+n}-\frac{i\,\pi}{\alpha\,k}\,\tau^{W}_{WW}\,W_{m+n}+\frac{i}{\alpha^2\,k}\,\tau^{\Lambda}_{WW}\,(LL)_{m+n}$$
$$-\,\frac{i}{\pi\,k\,\alpha^2}\,\tau^{\Delta}_{WW}\,\sum_p\sum_q\,L_{m+n-p-q}L_pL_q+\frac{i}{\alpha^2\,k}\,\tau^{\Gamma}_{WW}\,\sum_p\,p^2\,L_{m+n-p}L_p$$
$$+\,\frac{i}{\alpha\,k}\,\tau^{\Omega}_{WW}\,(WL)_{m+n}-\frac{i\,\pi^2}{\alpha^2\,k}d_{WW}\,\delta_{m+n,0}. \tag{6.142}$$

where

$$(WL)_p = \sum_m W_{p-m} L_m = 2\pi \int_0^{2\pi} d\phi \, \mathcal{W}(\phi) T(\phi) e^{-ip\phi},$$
$$(LL)_p = \sum_m L_{p-m} L_m = 2\pi \int_0^{2\pi} d\phi \, (\phi) T(\phi)^2 e^{-ip\phi},$$
$$(T\partial^2 T)_p = \sum_m m^2 \, L_{p-m} \, L_m = -2\pi \int d\phi \, e^{-ip\phi} \, T(\phi) \partial^2 \, T(\phi),$$
$$(T^3)_p = \sum_m \sum_n L_{p-n-m} L_n \, L_m = 4\pi^2 \int_0^{2\pi} T^3(\phi) \, e^{-ip\phi} \, d\phi, \tag{6.143}$$

and

$$\tau^L_{WW} = \frac{1}{1296000} \left(3m^5 - 5m^4 \, n + 6 \, m^3 \, n^2 - 6 \, m^2 \, n^3 + 5 \, m \, n^4 - 3 \, n^5\right),$$
$$\tau^W_{WW} = \frac{1}{3600} \left(m^3 - 2m^2 \, n + 2m \, n^2 - n^3\right),$$
$$\tau^\Lambda_{WW} = \frac{1}{5184000} \left(99m^3 - 293m^2 \, n + 293 \, m \, n^2 - 99 \, n^3\right),$$
$$\tau^\Delta_{WW} = \frac{1}{9000} \, (m-n) \, ,$$
$$\tau^\Gamma_{WW} = \frac{19}{864000} \, (m-n) \, ,$$
$$\tau^\Omega_{WW} = \frac{7}{1800} (m-n), \quad d_{WW} = \frac{1}{1296000} \, n^7. \tag{6.144}$$

To go from Poisson bracket to commutator one replaces $i \{W_m, W_n\} \to [W_m, W_n]$ with the following identifications,

$$L_m \to \frac{\pi}{10\,k} \, L_m, \; W_m \to \frac{\pi}{k} \, W_m \, , \quad k \to -\frac{c}{120},$$

$$[L_m \, L_n] = (m-n) L_{m+n} + \frac{c}{12} \, m^3 \, \delta_{m+n,0}, \tag{6.145}$$
$$[L_m \, W_n] = (3m-n) \, W_{m+n}, \tag{6.146}$$

$$[W_m\,,W_n] = \frac{1}{10\,\alpha^2}\,\tau^{L}_{WW}\,L_{m+n} + \frac{1}{\alpha}\,\tau^{W}_{WW}\,W_{m+n} - \frac{1}{100\,k\,\alpha^2}\,\tau^{\Lambda}_{WW}\,(LL)_{m+n}$$

$$+ \frac{1}{1000\,k^2\,\alpha^2}\,\tau^{\Delta}_{WW}\,\sum_p\sum_q L_{m+n-p-q}L_pL_q - \frac{1}{100\,\alpha^2\,k}\,\tau^{\Gamma}_{WW}\,\sum_p p^2\,L_{m+n-p}L_p$$

$$- \frac{1}{10\,\alpha\,k}\,\tau^{\Omega}_{WW}\,(WL)_{m+n} + \frac{k}{\alpha^2}d_{WW}\,\delta_{m+n,0}. \tag{6.147}$$

To compare (6.144) with the classical limit (large central charge c) of coefficients in the $\mathcal{W}(2,4)$ algebra in (6.2.3), we redefine L_n as follows,

$$L_n \to L_n - \frac{c}{24}\,\delta_{n,0} \tag{6.148}$$

which lead to the following changes in the mode expansions of the composite operators:

$$(LL)_{m+n} \to (LL)_{m+n} - \frac{c}{12}\,L_{m+n} + \frac{c^2}{24^2}\,\delta_{m+n,0},$$

$$\sum_p\sum_q L_{m+n-p-q}L_pL_q \to \sum_p\sum_q L_{m+n-p-q}L_pL_q - \frac{3c}{24}\,(LL)_{m+n} + \frac{3c^2}{24^2}L_{m+n} - \frac{c^3}{24^3}\,\delta_{m+n,0},$$

$$\sum_p p^2\,L_{m+n-p}L_p \to \sum_p p^2\,L_{m+n-p}L_p - \frac{c}{24}(m+n)^2\,L_{m+n},$$

$$(WL)_{m+n} \to (WL)_{m+n} - \frac{c}{24}\,W_{m+n}. \tag{6.149}$$

Substituting these changes, the resultant commutator becomes,

$$[W_m\,,W_n] = \tilde{\tau}^{L}_{WW}\,L_{m+n} + \tilde{\tau}^{W}_{WW}\,W_{m+n} + \tilde{\tau}^{\Lambda}_{WW}\,\Lambda_{m+n} + \tilde{\tau}^{\Omega}_{WW}\,\Omega_{m+n}$$

$$+ \tilde{\tau}^{\Delta}_{WW}\,\Delta_{m+n} + \tilde{\tau}^{\Gamma}_{WW}\,\Gamma_{m+n} + \tilde{d}_{WW}\,\delta_{m+n,0} \tag{6.150}$$

where

$$\Omega_{m+n} = (WL)_{m+n}\;, \quad \Delta_{m+n} = \sum_p\sum_q L_{m+n-p-q}\,L_p\,L_q,$$

$$\Gamma_{m+n} = \left(\sum_p p^2\,L_{m+n-p}\,L_p - \frac{5}{18}(m+n)^2\,(LL)_{m+n} + \frac{4}{9}(LL)_{m+n}\right). \tag{6.151}$$

These expressions are the mode expansions of quasi-primaries defined in (6.82) in the semi-classical limit of large central charge c. The final coefficients are,

$$\tilde{\tau}^{L}_{WW} = \frac{1}{1680}(m-n)\Big(3(m^4+n^4) - 2mn(m-n)^2 - 39(m^2+n^2) + 20mn + 108\Big),$$

$$\tilde{\tau}^{W}_{WW} = \frac{\sqrt{105}}{420}(m-n)(n^2+m^2-mn-7)\,, \quad \tilde{\tau}^{\Delta}_{WW} = \frac{432\,(m-n)}{35\,c^2},$$

$$\tilde{\tau}^{\Lambda}_{WW} = \frac{7(m-n)}{30\,c}\left(m^2+n^2-mn-7\right)\,, \quad \tilde{\tau}^{\Gamma}_{WW} = \frac{57\,(m-n)}{280\,c},$$

$$\tilde{\tau}^{\Omega}_{WW} = \sqrt{\frac{42}{5}} \times \frac{(m-n)}{c}\,, \quad \tilde{d}_{WW} = \frac{c\,m\,(m^6 - 14m^4 + 49m^2 - 36)}{20160}\,\delta_{m+n,0} \tag{6.152}$$

where we fixed the value of $\alpha = \frac{1}{60}\sqrt{\frac{7}{15}}$. These coefficients in large central charge c limit match with the coefficients in (6.85).

6.5 Gravitational Chern Simon term

In the previous sections, we only focused on the $3d$ conformal gravity formulated as $\mathfrak{so}(2,3)$ Chern-Simons gauge theory. However, there is also an equivalent second-order formulation of three-dimensional conformal gravity given by the following action,

$$S_{CSG} = \frac{k}{2}\int d^3x\,\sqrt{-G}\,\epsilon^{\lambda\mu\nu}\left(\Gamma^{\rho}_{\lambda\sigma}\,\partial_\mu\Gamma^{\sigma}_{\rho\nu} + \frac{2}{3}\,\Gamma^{\rho}_{\lambda\sigma}\,\Gamma^{\sigma}_{\mu\tau}\,\Gamma^{\tau}_{\nu\rho}\right) \tag{6.153}$$

where k is a dimensionless parameter, which is the level of the Chern-Simons action and G is the determinant of the metric $G_{\mu\nu}$. The Christoffel symbol Γ is the Levi-Civita connection associated with the $G_{\mu\nu}$. In our notation, the $3d$ Levi-Civita tensor $\epsilon^{\mu_1\mu_2\mu_3}$ on a curved manifold is defined as follows,

$$\epsilon_{\mu_1\mu_2\mu_3} = \sqrt{-G}\,\hat{\epsilon}_{\mu_1\mu_2\mu_3}\,, \quad \epsilon^{\mu_1\mu_2\mu_3} = \frac{1}{\sqrt{-G}}\hat{\epsilon}^{\mu_1\mu_2\mu_3} \tag{6.154}$$

where $\hat{\epsilon}^{\mu_1\mu_2\mu_3}$ is a pseudo-tensor defined as follows,

$$dx^{\mu_1}\wedge dx^{\mu_2}\wedge x^{\mu_3} = \hat{\epsilon}^{\mu_1\mu_2\mu_3}\,dx^1\wedge dx^2\wedge dx^3 = \hat{\epsilon}^{\mu_1\mu_2\mu_3}\,d^3x \tag{6.155}$$

that takes the value $\hat{\epsilon}^{012} = 1$ and is antisymmetric in all indices. Therefore the volume element on the $3d$ manifold can be written as,

$$dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} = \sqrt{-G}\, \epsilon^{\mu_1\mu_2\mu_3} d^3x = \hat{\epsilon}^{\mu_1\mu_2\mu_3}\, d^3x. \tag{6.156}$$

The equation of motion derived from the action (6.153) is

$$C_{\mu\nu} = \epsilon_\mu{}^{\sigma\rho}\nabla_\sigma \left(R_{\rho\nu} - \frac{1}{4} R\, g_{\rho\nu}\right) = 0. \tag{6.157}$$

The tensor $C_{\mu\nu}$ is the Cotton tensor and the equation (6.157) is a sufficient and necessary condition for $G_{\mu\nu}$ to be conformally flat in 3 dimensions. The tensor $S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}\, Rg_{\mu\nu}$ is the Schouten tensor. Under Weyl transformations

$$G_{\mu\nu} \to e^{-2\sigma} G_{\mu\nu}$$

the Cotton tensor $C_{\mu\nu}$ is invariant (but this is not true in general for $d \geq 4$). The Lagrangian density in (6.153) is invariant under diffeomorphisms and Weyl transformations up to the total derivative, which plays an important role while calculating charges associated with these theories using modified covariant phase space formalism [28, 119].

This action becomes relevant when one quantizes the AdS_4 gravity using the Neumann boundary condition where the holographic stress tensor $T_{ij} = 0$, and one expects the boundary action to be an induced gravity action [99]. One can add a bulk Pontryagin term to the action (6.153)

$$S_{pontryagin} = k \int d^4x\, \sqrt{-g} \epsilon^{\mu\nu\sigma\lambda}\, R^\alpha{}_{\beta\mu\nu}\, R^\beta_{\alpha\sigma\nu} \tag{6.158}$$

with an arbitrary coupling constant. The Pontryagin term is a total derivative, equivalent to Chern-Simons Gravity action (6.153) at the boundary. Therefore the variation of the action after adding all the boundary terms, necessary counter terms and Pontryagin term (6.158) will be given by,

$$\delta S_{total} \sim \int d^3x\, (T^{ij} + k\, C^{ij})\, \delta g^{(0)}_{ij} \tag{6.159}$$

where $g^{(0)}_{ij}$ is the induced metric at the boundary. For non-Dirichlet boundary conditions where $\delta g^{(0)}_{ij} \neq 0$, one sets the modified holographic stress tensor to zero. However, in the large k limit, the term proportional to T^{ij} can be neglected, and we can solve the variational principle by setting $C^{ij} = 0$, which means we look for only those solutions whose boundary metric is conformally flat. Furthermore, if one calculates boundary effective action then the action (6.153) dominates the induced gravity action in the large-k limit at the boundary and thus becomes relevant for AdS_4 holography. As alluded to in the conclusion of Chapter 5, for locally AdS_4 the induced effective action goes to zero [105] and in that case addition of Pontryagin term will provide an effective gravitational action at the boundary (6.153), that can be used to calculate the expectation value of the symmetry generating currents in the bulk AdS_4 gravity. This can potentially also explain the emergence of line integral charges from the bulk AdS_4 gravity because the effective action is a gravitational theory which will have its codimension-two charges making the charges codimension-three from the bulk perspective. As it turns out we will see that the charges we derive for the action (6.153) are similar to the conjectured charges from the AdS_4 gravity (5.76).

Having dispensed with the motivations, we now turn to the analysis of $3d$ gravitational Chern-Simons action leaving further implications on bulk AdS_4 gravity for future works.

6.5.1 Imposing variational principle

The variation of the action (6.153) yields,

$$\delta S = \frac{k}{2}\int d^3x\,\sqrt{-G}\,C^{\mu\nu}\delta G_{\mu\nu} + \int d^2x\,\sqrt{-\gamma}\,n_\mu\Theta^\mu \tag{6.160}$$

where $n_\mu = -\frac{l}{r}\,\delta^r_\mu$ is normal to time-like boundary $r = r_0$ and γ_{ab} is the induced metric on this hypersurface. The boundary coordinates are labelled by $x^a \in \{x^+, x^-\}$. The presymplectic potential is given by,

$$\Theta^\mu = \frac{k}{2}\epsilon^{\lambda\mu\nu}\,\Gamma^\alpha_{\lambda\rho}\,\delta\Gamma^\rho_{\alpha\nu} + \frac{k}{2}\,\epsilon^{\lambda\sigma\nu}\,R^{\rho\mu}{}_{\nu\sigma}\,\delta G_{\lambda\rho} \tag{6.161}$$

The metric in the Fefferman-Graham gauge is,

$$ds^2 = \frac{l^2}{r^2}\,dr^2 + \frac{l^2}{r^2}\,g_{ab}\,dx^a dx^b. \tag{6.162}$$

Using the relation

$$\sqrt{-\gamma} = \frac{r}{l}\sqrt{-G}$$

the boundary term in (6.160) can be written as,

$$\begin{aligned}\int d^2x\,\left(-\delta^r_\mu\,\Theta^\mu\right) &= -\int d^2x\,\frac{k}{2}\left[\hat\epsilon^{\lambda r\nu}\,\Gamma^\alpha_{\lambda\rho}\,\delta\Gamma^\rho_{\alpha\nu} + \hat\epsilon^{\lambda\sigma\nu}\,R^{\rho r}{}_{\nu\sigma}\,\delta g_{\lambda\rho}\right]\\ &= \int d^2x\,k\left[\frac{\hat\epsilon^{ab}}{2}\,\Gamma^\alpha_{a\rho}\delta\Gamma^\rho_{\alpha b} - \hat\epsilon^{ab}\,R^{cr}{}_{rb}\,\delta g_{ac}\right]\end{aligned} \tag{6.163}$$

where we set $\hat\epsilon^{rab} = \hat\epsilon^{ab}$ The first term can be expanded as follows,

$$\hat\epsilon^{ab}\,\Gamma^\alpha_{a\rho}\delta\Gamma^\rho_{\alpha b} = \hat\epsilon^{ab}\left[\Gamma^r_{ac}\,\delta\Gamma^c_{rb} + \Gamma^c_{ar}\,\delta\Gamma^r_{cb} + \Gamma^c_{ad}\delta\Gamma^d_{cb}\right]. \tag{6.164}$$

In the above expression, we have used $\Gamma^r_{ar} = 0$. Putting in the metric (6.162) the first two terms in (6.164) have only r derivatives on the variation of the boundary metric. The third term $\hat{\epsilon}^{ab}\,\Gamma^c_{ad}\,\delta\Gamma^d_{cb}$ can be written as follows,

$$\hat{\epsilon}^{ab}\,\Gamma^c_{ad}\,\delta\Gamma^d_{cb} = \Big[\partial_e\left(\epsilon^{ab}\Gamma^e_{ad}\,g^{dc}\delta g_{cb}\right) + \partial_b\left(\epsilon^{ab}\Gamma^e_{ac}\,g^{cd}\delta g_{de}\right) - \partial_d\left(\epsilon^{ab}\Gamma^e_{ac}\,g^{cd}\delta g_{eb}\right)\Big]$$
$$+\,\epsilon^{ab}\,g^{de}\left[-\nabla_c\Gamma^c_{ad}\,\delta g_{eb} - \nabla_b\Gamma^c_{ad}\,\delta g_{ce} + \nabla_e\Gamma^c_{ad}\,\delta g_{cb}\right]. \tag{6.165}$$

We can drop the total derivative terms and will only be left with the second line of (6.165). Near the boundary, one expands the metric g_{ij} in (6.162) as follows,

$$g_{ij} = g^{(0)}_{ij} + \frac{l}{r}g^{(1)}_{ij} + \frac{l^2}{r^2}\,g^{(2)}_{ij} + \cdots \tag{6.166}$$

Substituting this in (6.163) and expanding in r, one can check that at $\mathcal{O}(r)$ the first two terms in (6.164) cancel the terms coming from the second term in (6.163), and therefore, there is no divergent boundary term in the variation of the action when (6.163) is expanded near the boundary. At $\mathcal{O}(1)$ we find the following terms,

$$\hat{\epsilon}^{cb}\left(\frac{1}{l^2}\,g^{(0)}_{dc}\,g^{ad}_{(2)} - \frac{1}{4l^2}\,g^{ef}_{(0)}g^{(1)}_{ef}\,g^{(0)}_{dc}\,g^{ad}_{(1)}\right)\delta g^{(0)}_{ab} - \hat{\epsilon}^{cb}\left(\frac{1}{4l^2}\,g^{(0)}_{dc}\,g^{ad}_{(1)}\delta g^{(1)}_{ab}\right)$$
$$-\,\frac{\hat{\epsilon}^{ab}}{4}\,g^{ed}_{(0)}\left(D_c\tilde{\Gamma}^c_{ad}\,\delta g^{(0)}_{eb} + D_b\tilde{\Gamma}^c_{ae}\,\delta g^{(0)}_{cd} - D_e\tilde{\Gamma}^c_{ad}\delta g^{(0)}_{cb}\right) \tag{6.167}$$

where D_a is the covariant derivative associated with $g^{(0)}_{ab}$ and $\tilde{\Gamma}$ is the Christoeffel connection associated with it. We parameterise our boundary metric $g^{(0)}_{ab}$ in Polyakov chiral gauge such that the boundary line element is given as,

$$ds^2_{bdry} = -\,dx^+dx^- + g^{(0)}_{++}(x^+,x^-)(dx^+)^2 \tag{6.168}$$

With this choice of boundary metric the first term in (6.167) becomes,

$$\frac{1}{2l^2}\,g^{++}_{(2)}\,\delta g^{(0)}_{++} + \frac{1}{8l^2}\,g^{ef}_{(0)}g^{(1)}_{ef}\,g^{++}_{(1)}\delta g^{(0)}_{++} \tag{6.169}$$

Following [8, 10], one sets $g^{++}_{(2)} = g^{(2)}_{--} = -\frac{N^2}{4}$ such that the first term in (6.169) becomes

$$-\frac{N^2}{8l^2}\,\delta^a_+\delta^b_+\,\delta g^{(0)}_{ab}$$

which can be written as a total variation $\delta\left(-\frac{N^2}{8l^2}\,\delta^a_+\delta^b_+\,g^{(0)}_{ab}\right)$. To cancel this term we follow the prescription of [10], where one adds the following boundary term to the action (6.153)

$$S' = k\int d^2x\,\mathcal{T}^{ab}\,g^{(0)}_{ab}\;,\text{with }\;\mathcal{T}^{ab} = \frac{N^2}{8l^2}\,\delta^a_+\,\delta^b_+. \tag{6.170}$$

such that from the variation of the total action $\delta S_{total} = \delta S_{CSG} + \delta S'$ the term $-\frac{N^2}{8l^2}\,\delta^a_+\delta^b_+\,\delta g^{(0)}_{ab}$ drops out. The second term in (6.169) will go to zero from the gauge choice (6.174). Now we come to the second term in (6.167) which becomes,

$$\frac{1}{2l^2}\left(g^{(1)}_{++}\,\delta g^{(1)}_{--} - g^{(1)}_{--}\,\delta g^{(1)}_{++}\right) + \frac{1}{l^2}\,g^{(0)}_{++}\left(g^{(1)}_{+-}\,\delta g^{(1)}_{--} - g^{(1)}_{--}\,\delta g^{(1)}_{+-}\right). \tag{6.171}$$

There is a clear choice of boundary condition for which all of the above terms will go to zero and that is $g^{(1)}_{--} = \delta g^{(1)}_{--} = 0$. Also due to the chiral conditions imposed on the boundary metric as in (6.168) all the terms in the second line of (6.167) will go to zero. To summarise we solve the variational problem as follows,

- Parameterise the boundary metric in the Polyakov gauge as

$$g^{(0)}_{+-} = -\frac{1}{2}\,, g^{(0)}_{--} = 0\,, \quad g^{(0)}_{++} = g^{(0)}_{++}(x^+, x^-). \tag{6.172}$$

- Fix the value of $g^{(2)}_{--} = -\frac{N^2}{4}$ following [10] and add a boundary term $S' = k\int d^2x\,\mathcal{T}^{ab}\,g^{(0)}_{ab}$ to the action (6.153).
- Impose the condition $g^{(1)}_{--} = 0$.

6.5.2 Solutions to conformal gravity

We solve the equation $C_{\mu\nu} = 0$ near the boundary at each order in r after writing the metric in Fefferman-Graham gauge as follows,

$$ds^2 = \frac{l^2}{r^2}\,dr^2 + \frac{r^2}{l^2}\left(g^{(0)}_{ab} + \frac{l}{r}\,g^{(1)}_{ab} + \frac{l^2}{r^2}\,g^{(2)}_{ab} + \cdots\right)dx^a\,dx^b \tag{6.173}$$

Further, we also impose the gauge condition to fix Weyl symmetry as follows,

$$\partial_r\left(\frac{det(G)}{r^2}\right) = 0 \tag{6.174}$$

where $det(G)$ is the determinant of the full metric in (6.173). This condition fixes the off-diagonal component of $g^{(0)}_{ab}, g^{(1)}_{ab}, g^{(2)}_{ab}$ and so on. The expansion is similar to that of [8, 10, 9] except for the presence of $g^{(1)}_{ab}$ term. We start by imposing the chiral boundary conditions that satisfy the variational principle,

$$g^{(0)}_{--} = 0\,,\ g^{(0)}_{+-} = -\frac{1}{2}\,,\ g^{(1)}_{--} = 0\,,\ g^{(2)}_{--} = -\frac{N^2}{4}.$$

With these boundary conditions, one can solve for various metric components

$$\begin{aligned}
&g^{(1)}_{--} = 0\,,\; g^{(2)}_{-+} = \frac{N^2}{4}\,,\; g^{(3)}_{+-} = \frac{N^2}{4}\,g^{(1)}_{+-},\\
&g^{(2)}_{++} = T(x^+) - \frac{1}{2}l^2\left(\partial_- g^{(0)}_{++}\right)^2 + l^2\, g^{(0)}_{++}\partial_-^2\, g^{(0)}_{++} + \frac{1}{2}l^2\,\partial_-\partial_+ g^{(0)}_{++}.
\end{aligned} \tag{6.175}$$

The free data that characterise the solutions are $\{g^{(0)}_{++}, g^{(1)}_{++}, T(x^+)\}$, where $g^{(0)}_{++}, g^{(1)}_{++}$ satisfies the following constraint equations.

$$\begin{aligned}
&N^2\,\partial_- g^{(0)}_{++}(x^+,x^-) + l^2\,\partial_-^3 g^{(0)}_{++}(x^+,x^-) = 0,\\
&N^2\, g^{(1)}_{++}(x^+,x^-) + 4\,l^2\,\partial_-^2 g^{(1)}_{++}(x^+,x^-) = 0.
\end{aligned} \tag{6.176}$$

The solution to the constraint equations (6.176) are

$$g^{(0)}_{++} = J^-(x^+)\,e^{-\frac{i\,N\,x^-}{l}} + J^0(x^+) + J^+(x^+)\,e^{\frac{i\,N\,x^-}{l}} = J^a(x^+)\;e^{a\,\frac{i\,N\,x^-}{l}}, \tag{6.177}$$

$$g^{(1)}_{++} = G^{-1/2}(x^+)\,e^{-\frac{i\,N\,x^-}{2\,l}} + G^{1/2}(x^+)\,e^{\frac{i\,N\,x^-}{2\,l}} = G^s(x^+)\;e^{s\,\frac{i\,N\,x^-}{l}}. \tag{6.178}$$

Thus the complete solution space for $C_{\mu\nu} = 0$ with boundary conditions (6.175) is parameterised by six chiral functions $\{J^a(x^+), G^s(x^+), \kappa(x^+)\}\,,\quad a \in \{0, \pm 1\}$ and $s \in \{\pm\frac{1}{2}\}$.

Asymptotic symmetries

Since the $3d$ conformal gravity theory defined by the action (6.153) is diffeomorphic and Weyl invariant, the asymptotic asymmetries associated with such a theory constitute residual diffeomorphisms and residual Weyl transformations that do not spoil the gauge and boundary conditions. The variation of metric due to these residual gauge transformations is

$$\delta G_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - 2\sigma G_{\mu\nu} = 0. \tag{6.179}$$

Imposing the gauge condition $\delta G_{rr} = 0$ leads to the determination of the Weyl parameter σ on the radial component of the vector field ξ as,

$$\sigma = -\frac{\xi^r}{r} + \partial_r \xi^r. \tag{6.180}$$

We further expand the vector field component near the boundary as,

$$\begin{aligned}
\xi^r &= r^2\,\xi^r_{(-2)}(x^a) + r\,\xi^r_{(-1)}(x^a) + \xi^r_{(0)}(x^a) + \cdots\\
\xi^a &= \sum_{n=0} r^{-n}\xi^a_{(0)}(x^b)
\end{aligned} \tag{6.181}$$

From the gauge condition $\delta G_{ra} = 0$ at $\mathcal{O}(1)$, one gets

$$\xi^a_{(1)} = l^4\, g^{ab}_{(0)}\, \partial_b \xi^r_{(-2)}. \tag{6.182}$$

From $\delta G_{--} = 0$ at $\mathcal{O}(r^2)$ and $\mathcal{O}(r)$, one gets

$$\xi^+_{(0)} = \lambda(x^+)\,, \quad \frac{N^2}{2}\,\xi^r_{(-2)} + 2\,l^2\,\partial_-^2 \xi^r_{(-2)} = 0. \tag{6.183}$$

The equation (6.183) can be solved for $\xi^r_{(-2)}$ to find,

$$\xi^r_{(-2)} = \chi^{1/2}(x^+)\; e^{\frac{i\,N\,x^-}{2l}} + \chi^{-1/2}(x^+)\; e^{-\frac{i\,N\,x^-}{2l}}. \tag{6.184}$$

From δG_{+-} at $\mathcal{O}(r^2)$ one obtains

$$\xi^r_{(-1)} = -\frac{1}{2} D_a \xi^a_{(0)}. \tag{6.185}$$

The gauge condition $\delta G_{r-} = 0$ at the $\mathcal{O}\left(\frac{1}{r}\right)$ is given by

$$\xi^+_{(2)} = -l^4\, \partial_- \xi^r_{(-1)}. \tag{6.186}$$

Using this at $\delta G_{--} = 0$ at $\mathcal{O}(1)$ one obtains the following constraint equation,

$$N^2\, \partial_- \xi^-_{(0)} + l^2\, \partial_-^3 \xi^-_{(0)} = 0 \tag{6.187}$$

which can be solved as

$$\xi^-_{(0)} = \lambda^{(-1)}(x^+)\, e^{\frac{i\,N\,x^-}{l}} + \lambda^{(0)}(x^+) + \lambda^{(1)}(x^+)\, e^{-\frac{i\,N\,x^-}{l}}. \tag{6.188}$$

Therefore the final residual diffeomorphisms and Weyl parameters are,

$$\begin{aligned}
&\xi^+ = \lambda(x^+) + \cdots\,, \quad \xi^- = \lambda^a(x^+)\; e^{a\,\frac{i\,N\,x^-}{l}} + \cdots \quad a \in \{\pm 1, 0\}, \\
&\xi^r = r^2\, \chi_s(x^+)\; e^{s\,\frac{i\,N\,x^-}{l}} + \cdots \quad s \in \{\pm\frac{1}{2}\}, \\
&\sigma = r\, \chi_s(x^+)\; e^{s\,\frac{i\,N\,x^-}{l}} + \cdots\,.
\end{aligned}$$

These induce the following variations on the background fields,

$$\delta J_a = \lambda\,\partial J_a + \partial\lambda J_a + \partial\lambda_a + i\,f_a{}^{bc}\,J_b\lambda_c - 4\,l^3\,N\,(\lambda_a)^{ss'}\;C_s\,\chi_{s'},$$

$$\delta C_s = \lambda\partial C_s + \frac{3}{2}C_s\,\partial\lambda - \partial^2\chi_s - (T + \frac{1}{2}\eta^{ab}\,J_aJ_b)\chi_s$$
$$+\,i\,(\lambda^a)^{s'}{}_s\,(2\,J_a\,\partial\chi_{s'} + \partial J_a\,\chi_{s'} + \lambda_a\,C_{s'})\,,$$

$$\delta T = 2\,T\,\partial\lambda + \lambda\,\partial T + \frac{1}{2}\,\partial^3 Y - \frac{1}{2}\eta^{ab}J_a\,\partial\lambda_b - i\,l^3\,N\,\epsilon^{ss'}\;(3\,C_s\,\partial\chi_{s'} + \partial C_s\,\chi_{s'})\,. \tag{6.189}$$

These transformations are identical to (6.115) from the first order formulation of conformal gravity for $N = -\frac{1}{l^3}$.

6.5.3 Calculation of Charges

We now turn to calculate the charges that induce variations of the background field (6.189) due to residual diffeomorphisms and Weyl symmetries. As mentioned before, the Lagrangian density is invariant up to the total derivative under the diffeomorphism parameter ξ and the Weyl transformation $g_{\mu\nu} \to e^{-2\sigma}\,g_{\mu\nu}$. To calculate the charges we use a modified covariant phase space formalism proposed by Tachikawa [28] which we reviewed in (2.1.5). The symmetry parameters for our theory are $\chi \in \{\xi, \sigma\}$. Let us calculate the charges associated with the residual Weyl symmetry σ. For this, we calculate the variation of Lagrangian density under Weyl transformation

$$\delta_\sigma L = \partial_\mu \Xi^\mu_{(\sigma)} \tag{6.190}$$

where

$$\Xi^\mu_{(\sigma)} = \frac{k}{2}\,\epsilon^{\lambda\mu\nu} g^{\alpha\beta}\,\partial_\alpha\sigma\,\partial_\lambda g_{\beta\nu}. \tag{6.191}$$

The Noether current J^μ for Weyl transformation is given by

$$J^\mu_{(\sigma)} = \theta^\mu_{(\sigma)} - \Xi^\mu_{(\sigma)} \tag{6.192}$$

It can be checked that $J^{\mu}_{(\sigma)}$ is conserved on-shell. One can use Noether current to obtain $K^{\mu\nu}$ such that,

$$J^{\mu}_{(\sigma)} \simeq \partial_\mu K^{\mu\nu}_{(\sigma)} \tag{6.193}$$
$$K^{\mu\nu}_{(\sigma)} = -\frac{k}{4}\epsilon^{\lambda\mu\nu} g^{\rho\alpha}\partial_\lambda g_{\rho\alpha}\,\sigma \tag{6.194}$$

Under Weyl transformation, the symplectic potential transforms as follows,

$$\delta_\sigma \theta^\mu = \Pi^{\mu}_{(\sigma)}, \tag{6.195}$$
$$\begin{aligned}\Pi^{\mu}_{(\sigma)} = \frac{k}{2}\epsilon^{\lambda\mu\nu}\Big(&- \partial_\lambda\sigma\delta\Gamma^{\alpha}_{\ \alpha\nu} + \partial_\beta\sigma\, g^{\alpha\beta}\partial_\nu\delta g_{\alpha\lambda} + \partial_\beta\sigma\,\partial_\lambda g_{\nu\rho}\,\delta g^{\beta\rho} \\ &+ 2\,\partial_\beta\sigma\, g^{\rho\beta}\,\Gamma^{\alpha}_{\ \lambda\rho}\,\delta g_{\alpha\nu} - \partial_\nu\delta\sigma\,\Gamma^{\rho}_{\ \rho\lambda} + \partial_\tau\delta\sigma\,\partial_\lambda g_{\nu\rho}\, g^{\rho\tau}\Big).\end{aligned} \tag{6.196}$$

Now the formula (2.58) states that one should be able to write $\delta\Xi^{\mu}_{(\sigma)} - \Xi^{\mu}_{(\delta\sigma)} - \Pi^{\mu}_{(\sigma)}$ as a total derivative on-shell. We find that up to order $\mathcal{O}(1)$ terms in r one can show that on shell,

$$\delta\Xi^{\mu}_{(\sigma)} - \Xi^{\mu}_{(\delta\sigma)} - \Pi^{\mu}_{(\sigma)} \simeq \partial_\nu \Sigma^{\mu\nu}_{(\sigma)} \tag{6.197}$$
$$= -k\,\epsilon^{\mu\nu\lambda}\partial_\beta\sigma\, g^{\alpha\beta}\delta g_{\lambda\beta}. \tag{6.198}$$

Then the Weyl charge using the formula (2.60) is given by,

$$Q^{\mu\nu}_{(\sigma)} = \frac{1}{16\pi}\left(-k\,\epsilon^{\mu\nu\lambda}\partial_\beta\sigma\, g^{\alpha\beta}\delta g_{\lambda\beta}\right) \tag{6.199}$$

Note that the current $K^{\mu\nu}_{(\sigma)}$ for Weyl transformation in (6.194) vanishes for the chiral parameterisation of the boundary metric (6.168). Therefore the contribution to the Weyl charge is entirely from (6.198). The calculation of the charges for diffeomorphism symmetries has been carried out in [28, 119] in the context of topologically massive gravity. For diffeomorphisms, the full charge is written as follows [28, 119],

$$\begin{aligned}K^{\mu\nu}_{\xi} &= -2k\,\hat\epsilon^{\lambda\mu\nu}\left(S_{\lambda\alpha}\xi^\alpha - \frac{1}{4}\Gamma^{\alpha}_{\lambda\beta}\nabla_\alpha\xi^\beta\right), \\ \Sigma^{\mu\nu}_{\xi} &= \frac{k}{2}\,\hat\epsilon^{\lambda\mu\nu}\,\delta\Gamma^{\beta}_{\lambda\alpha}\partial_\beta\xi^\alpha,\end{aligned} \tag{6.200}$$

$$16\pi Q^{\mu\nu}_{\xi} = -2\,k\,\hat\epsilon^{\mu\nu\rho}\left(\delta S_{\rho\alpha}\xi^\alpha - \frac{1}{2}\delta\Gamma^{\alpha}_{\rho\beta}\,\nabla_\alpha\xi^\beta + \xi^\beta\, R^{\alpha}_{[\beta}\delta g_{\rho]\alpha}\right). \tag{6.201}$$

Therefore the total charge becomes,

$$16\pi Q_{\xi}^{\mu\nu} + 16\pi Q_{(\sigma)}^{\mu\nu} = -2\,k\,\hat{\epsilon}^{\mu\nu\rho}\left(\delta S_{\rho\alpha}\xi^{\alpha} - \frac{1}{2}\delta\Gamma^{\alpha}_{\rho\beta}\,\nabla_{\alpha}\xi^{\beta} + \xi^{\beta}\,R^{\alpha}_{[\beta}\delta g_{\rho]\alpha} + \frac{1}{2}\partial_{\beta}\sigma\,g^{\alpha\beta}\delta g_{\rho\beta}\right). \tag{6.202}$$

We integrate this charge on a constant t surface at the boundary and use the coordinates $x^{\pm} = \phi \pm t$ to obtain,

$$16\pi \not\delta Q = \int_0^{2\pi} d\phi\; Q^{rt} = \int_0^{2\pi} d\phi\,\frac{1}{2}\left(Q^{r+} - Q^{r-}\right). \tag{6.203}$$

The Weyl charges $Q^{\mu\nu}_{(\sigma)}$ cancel the divergent piece at $\mathcal{O}(r)$ from the diffeomorphism charges $\mathcal{Q}^{\mu\nu}_{\xi}$ and the resultant charge is given by

$$16\pi\not\delta Q = \frac{k}{2}\int d\phi\left(2\,\delta T(x^{+})\,\lambda - \,\eta^{ab}\delta J_a\,\lambda_b - 4\,i\,l^3\,N\,\epsilon^{ss'}\,\delta C_s\,\chi_{s'}\right) \tag{6.204}$$

This charge is trivially integrable and therefore the final charge is

$$Q = \frac{k}{32\pi}\int d\phi\left(2\,T(x^{+})\,\lambda(x^{+}) - \eta^{ab}\,J_a\,\lambda_b - 4\,i\,l^3\,N\,\epsilon^{ss'}\,C_s\,\chi_{s'}\right) \tag{6.205}$$

For $N = -\frac{1}{l^3}$, this charge matches with that derived from the first-order formalism in (6.116). The Poisson bracket between currents will be identical to (6.117) and (6.118) after substituting $N = -\frac{1}{l^3}$ and therefore to avoid redundancy, we do not write it here. One can redefine $C_s \to \frac{C_s}{\sqrt{l^3 N}}$ so that the resultant commutation relations are given by,

$$[L_m, L_n] = (n-m)L_{m+n} + \frac{k}{8}n^3\,\delta_{m+n,0} \tag{6.206}$$

$$[L_m, J_{a,n}] = -n\,J_{a,n+m}, \quad [L_m, C_{s,r}] = \frac{1}{2}(m-2r)C_{s,m+r} \tag{6.207}$$

$$\begin{aligned}
&[J_{a,m}, J_{b,n}] = f_{ab}^{\;c}J_c + \frac{k}{8}\,n\,\eta_{ab}\delta_{m+n,0}\,, \quad [J_{a,m}\,, C_{s,r}] = \frac{1}{2}(a-2s)\,C_{s+a,r+m}\\
&\left[C_{s,r'}, C_{s',r'}\right] = \epsilon_{ss'}\left[\left(r^2 - \frac{1}{4}\right)\,\delta_{r+r'} - \frac{4}{k}L_{r+r'} - \frac{32}{k^2}\left(J^2\right)_{r+r'}\right]\\
&+ \frac{8}{k}\left(\lambda^a\right)_{ss'}(r-r')J_{a,r+r'}
\end{aligned} \tag{6.208}$$

For $k = 4\kappa$ and $C_{s,r} \to G_{s,r}$ this symmetry algebra matches with the semi-classical limit of $\mathcal{W}(2,(3/2)^2,1^3)$ in (5.1). This concludes our treatment of gravitational Chern-Simons action and showing the chiral $\mathcal{W}(2,(3/2)^2,1^3)$ symmetry of it.

6.6 Further Comments and Discussions

In this chapter one of the questions we have answered is:

What are all the chiral $\mathcal{W}$-algebra extensions of $\mathfrak{so}(2,3)$?

As the first step, we showed that there are exactly four ways to embed a copy of $\mathfrak{sl}(2,\mathbb{R})$ into $\mathfrak{so}(2,3)$ algebra. Each such embedding results in a distinct chiral $\mathcal{W}$-algebra extension and we have constructed two new ones, and recovered the other two that are known from [179] and (5.1) – thus completing the list of such chiral extension of $\mathfrak{so}(2,3)$. One of our new algebras (namely $\mathcal{W}(2;2^2,1)$ with $\mathfrak{h}=\mathfrak{so}(1,1)$) is the conformal $\mathfrak{bms}_3$ whose semi-classical version was first found in [20].

Just as the $\mathcal{W}_3$ algebra of Zamalodchikov [171] and the $W_3^{(2)}$ algebra of Bershadsky-Polyakov [168, 167] could be realised [116, 187, 117] in the Chern-Simons theory with $\mathfrak{sl}(3,\mathbb{R})$ gauge algebra, we realised in this chapter $\mathcal{W}(2,4)$ and $\mathcal{W}(2;(3/2)^2,1^3)$ as the asymptotic symmetry algebra of $3d$ conformal gravity written as $\mathfrak{so}(2,3)$ Chern Simons gauge theory after proposing appropriate boundary conditions. The case of $\mathcal{W}(2;2^2,1)$ or conformal $\mathfrak{bms}_3$ from $3d$ conformal gravity already existed in the literature [20].

Each extension we considered depends on the list of simple primaries [25] that are declared. As we have seen there are primaries of higher dimensions in each chiral algebra and we could probably have worked with other sets of primaries as the simple ones and these might lead to other extensions $\mathfrak{so}(2,3)$. The choices we made of sets of simple primaries are with minimal conformal dimensions, and therefore the extensions we considered are in a sense minimal. It will be interesting to construct the non-minimal algebras as well (if they exist) and study them.

As we have seen, in each of the four chiral $\mathcal{W}$-algebras we constructed, the constraints from Jacobi identities admit trivial solutions as well where all the parameters vanish and central charges are arbitrary. These trivial solutions do not correspond to extensions of $\mathfrak{so}(2,3)$ but instead of different contractions of it. For instance in the case of $\mathcal{W}(2;(3/2)^2,1^3)$ it corresponds to a chiral conformal algebra extension of $\mathfrak{iso}(1,3)$ ($\mathfrak{iso}(2,2)$), namely the Poincare algebra – which is the isometry algebra of $\mathbb{R}^{1,3}$ ($\mathbb{R}^{2,2}$). In the other three cases, the analogous chiral conformal extensions are of other 10-dimensional algebras that are still contractions of $\mathfrak{so}(2,3)$. It will be interesting to identify spacetimes which admit these contractions of $\mathfrak{so}(2,3)$ as their symmetry algebras and ask if the corresponding infinite-dimensional chiral conformal algebra extensions have any important role in them.

Even though we aimed to construct chiral $\mathcal{W}$-algebras it does not mean that we miss out on algebras that are not chiral. As an example, we show in the Appendix G that if we repeat the same exercise for $\mathfrak{so}(2,2)$ we end up with two chiral infinite dimensional algebras. One of them is indeed a chiral algebra which is an extension of CIG algebra of Polyakov [112] as derived by [10]. The other can be identified with the non-chiral Brown-Henneaux algebra though seen from the perspective of the diagonal Virasoro. It will be interesting to explore this question for our extensions of $\mathfrak{so}(2,3)$ as well.

Our chiral extensions of $\mathfrak{so}(2,3)$ should be thought of as the analogues of Polyakov's symmetries of chiral induced gravity [112] to three dimensions. Therefore, it is expected that they may be realised as such. This leads us to predict that these symmetries should be realised as asymptotic symmetries of AdS_4 gravity just as the Polyakov symmetry algebra was realised as the asymptotic symmetries of AdS_3 gravity in [10]. We demonstrated this for $\mathcal{W}(2;(3/2)^2,1^3)$ where in Chapter 5, we show the existence of this $\mathcal{W}$-algebra from AdS_4 gravity. It is highly desirable to see other chiral extensions of $\mathfrak{so}(2,3)$ emerging from the symmetries of AdS_4 gravity.

The next step in analysing the algebras of this chapter and their utility in the context of holography would be to construct their representations in terms of generalised primary operators and study the Ward identities of their correlation functions. Once that is achieved it will be important to provide a prescription to compute them holographically, either in the $3d$ conformal gravity theories or AdS_4-gravity theories along the lines of [188]. We hope to address these in the near future.

Chapter 7

Future Directions

The findings in the context of celestial conformal field theory (CFT) suggest the existence of a significantly larger infinite-dimensional $w_{1+\infty}$ symmetry algebra associated with asymptotically flat spacetimes [189]. Notably, the chiral $\mathfrak{bms}_4$ that we derived from bulk $\mathbb{R}^{1,3}$ gravity in Section 3 is identified as a constituent of this algebra. One of the future directions is to investigate the potential extension of the chiral boundary conditions within the framework of Einstein gravity in $\mathbb{R}^{1,3}$ or $\mathbb{R}^{2,2}$ to encompass this $w_{1+\infty}$ symmetry algebra. The prior analysis, conducted in Section 3, focused on vacuum solutions; hence, the intent is to include excited state solutions (radiative solutions) that manifest these chiral symmetry algebras. One way this is possible is due to the results of the reduced phase space analysis of generalised $\mathfrak{bms}_4$ algebra for linearised gravity [126]. The constraints (3.14), (3.16) that we obtained from locally flat conditions are a subset of the constraints considered in [126] and one would like to extend the construction of phase space analysis to the chiral radiative solutions following their analysis. Additionally, it will be interesting to adapt the techniques of covariant phase space analysis for the derivation of the charges associated with the chiral $\mathfrak{bms}_4$ algebra and $w_{1+\infty}$ as we expect these charges to be line integral just like the charges in $2d$ CFT.

In Section 4, we provided a general prescription to construct actions and equations of motion of Carrollian conformal field theories for scalars that one can extend for the construction of conformal coupled field theories with higher spin Carroll fields and Carroll fermions [79, 80, 190, 191]. It will be interesting to investigate the possible duality between the bulk $4d$ gravitational theories with non-Dirichlet boundary conditions and induced conformal Carrollian gravity theories on null infinity [81]. One expects that the gravitational theory on the boundary will give rise to a line integral charge that will obey symmetry algebra identical to chiral $\mathfrak{bms}_4$ algebra. Another interesting problem that we would like to address in future is the emergence of chiral $\mathfrak{bms}_4$ and $w_{1+\infty}$ algebra from the global symmetries of field theories intrinsically defined on the Carrollian manifold. See [192], where the scalar field coupled to $2d$ gravity in Polyakov gauge exhibit a hidden $\mathfrak{sl}(2,\mathbb{R})$ current algebra symmetry.

The analysis in Section 5 of chiral locally AdS$_4$ solutions and the emergence of $\mathcal{W}(2;(3/2)^2,1^3)$ algebra from the asymptotic symmetry algebra points to the fact that by imposing appropriate bulk boundary conditions, one obtains a topological sector in AdS$_4$ which is governed holographically by the symmetry algebra of $2d$ chiral CFT. These preliminary computations suggest the possibility of a type of codimension-two holography for AdS$_4$ gravity similar to the celestial holography of $\mathbb{R}^{1,3}$ gravity. Another interesting open problem is the derivation of the postulated line integral charges that obey the symmetry algebra $\mathcal{W}(2;(3/2)^2,1^3)$ directly from bulk AdS$_4$ gravity. Because $\mathcal{W}(2;(3/2)^2,1^3)$ contracts to chiral $\mathfrak{bms}_4$ algebra in flat space limit whose currents give rise to conformal soft theorems, the connection of $\mathcal{W}(2;(3/2)^2,1^3)$ algebra in probing the infrared effects of AdS$_4$ needs to be understood better. For example, it will be interesting to link the works [193, 194] addressing the correction to the soft factor in the presence of small cosmological constant to $\mathcal{W}(2;(3/2)^2,1^3)$ algebra.

In Section 6, we found exactly four chiral $\mathcal{W}$-algebra extensions of $\mathfrak{so}(2,3)$ Lie algebra and show that some of these algebras in the semi-classical limit emerge as the asymptotic symmetry algebras of $3d$ $\mathfrak{so}(2,3)$ Chern-Simons gauge theory [20]. We also show the emergence of $\mathcal{W}(2;(3/2)^2,1^3)$ algebra from the second order formalism of $3d$ conformal gravity, which we argue to describe holographically AdS$_4$ gravity for some specific bulk configurations and boundary conditions. We would like to derive these $\mathcal{W}$-algebra from the asymptotic symmetries of AdS$_4$ gravity and understand the relevance of these $\mathcal{W}$-algebras better from the perspective of AdS$_4$ holography.

Appendix A

Residual symmetries in NU gauge

Here we consider vector fields $\xi^\mu \partial_\mu$ which leave our boundary conditions and gauge choice invariant. First imposing the gauge conditions $\delta g_{ru} = \delta g_{rr} = \delta g_{ra} = 0$ require that the vector field components satisfy:

$$\partial_r \xi^u = 0, \quad \partial_u \xi^u + \partial_r \xi^r = g_{ua}\, \partial_r \xi^a, \quad \partial_a \xi^u = g_{ab} \partial_r \xi^b \tag{A.1}$$

These equations can again be solved in an asymptotic expansion around large-r. Expanding ξ^r and ξ^a as

$$\xi^r = \sum_{n=0}^{\infty} r^{1-n} \xi^r_{(n)}(u, z, \bar{z}), \quad \xi^a = \sum_{n=0}^{\infty} r^{-n} \xi^a_{(n)}(u, z, \bar{z}) \tag{A.2}$$

and imposing the conditions obtained from form-invariance of our class of geometries ($g^{(0)}_{ua} = g^{(0)}_{uu} = g^{(1)}_{ua} = g^{(1)}_{uu} = 0$) leads to the following conditions on the first few coefficients:

$$\begin{aligned} \xi^u(u, z, \bar{z}) &= \xi^u_{(0)}(z, \bar{z}) + u\, \xi^u_{(1)}(z, \bar{z}), \quad \xi^a_{(0)} = \xi^a_{(0)}(z, \bar{z}) \\ \xi^r_{(0)} &= -\xi^u_{(1)}, \quad \xi^a_{(1)} = -g^{ab}_{(0)} D_b \xi^u, \quad \xi^a_{(2)} = \frac{1}{2} g^{ab}_{(0)} g^{(1)}_{bc} g^{cd}_{(0)} D_d \xi^u \end{aligned} \tag{A.3}$$

and so on. At the leading order, there are four undetermined functions in these vector fields: $\{\xi^u_{(0)}(z, \bar{z}), \xi^u_{(1)}(z, \bar{z}), \xi^a_{(0)}(z, \bar{z})\}$. If we further assume that the variation of $g^{ab}_{(0)} g^{(1)}_{ab}$ vanishes leads to determination of $\xi^r_{(1)}$ as well to be:

$$\xi^r_{(1)} = \frac{1}{2} \Box\, \xi^u - \frac{1}{4} \xi^a_{(0)} D_a \left(g^{bc}_{(0)} g^{(1)}_{bc} \right) - \frac{1}{4} \partial_u \left(\xi^u\, g^{bc}_{(0)} g^{(1)}_{bc} \right) \tag{A.4}$$

As in [92] we assume the gauge choice $g^{ab}_{(0)} g^{(1)}_{ab} = 0$ for the background geometries. This choice of $\xi^r_{(1)}$ also ensures that the transformed $g^{(1)}$ is also traceless and remains at most linear in u. Under such a diffeomorphism the data $g^{(0)}_{ab}$, $g^{(1)}_{ab}$ etc transform as follows:

$$\delta g^{(0)}_{ab} = \mathcal{L}_{\xi^a_{(0)}} g^{(0)}_{ab} - 2\,\partial_u \xi^u\, g^{(0)}_{ab} \tag{A.5}$$

which means that part of the bulk diffeomorphism acts as a diffeo and a Weyl transformation of $2d$ geometry $g^{(0)}_{ab}$. Then the data at $\mathcal{O}(r)$ transforms as:

$$\begin{aligned}\delta g^{(1)}_{ab} &= -\partial_u \xi^u g^{(1)}_{ab} + \xi^u\, \partial_u g^{(1)}_{ab} + \xi^c D_c g^{(1)}_{ab} + g^{(1)}_{ac} D_b \xi^c + g^{(1)}_{bc} D_a \xi^c \\ &\quad + g^{(0)}_{ab}\, \Box \xi^u - 2\, D_a D_b \xi^u.\end{aligned} \tag{A.6}$$

Appendix B

Locally Flat solution as $\mathfrak{iso}(1,3)$ gauge connection

The locally flat solutions of (3.72) and (3.73) can be written in the language of $\mathfrak{iso}(1,3)$ gauge connection in the first-order formulation of gravity. For this we start with a $2d$ gauge connection which takes value in Lie algebra $\mathfrak{iso}(1,3)$,

$$\mathcal{A}_{2d} = A^{(m)}\, L_m + \bar{A}^{(m)}\, \bar{L}_m + \Phi^{(r,s)}\, P_{r,s} \tag{B.1}$$

where L_m and $\bar{L}_m$ with $m \in \{-1, 0, 1\}$ are the $\mathfrak{sl}(2,\mathbb{C})$ generators and $P_{r,s}$ with $\{r, s\} \in \{-\frac{1}{2}, \frac{1}{2}\}$ are momentum generators written in different basis. The commutators between various operators are given by,

$$\begin{aligned} &[L_m\,, L_n] = (m-n)\, L_{m+n} \quad \left[\bar{L}_a, \bar{L}_b\right] = (a-b)\, \bar{L}_{a+b} \quad [L_m, P_{s,r}] = \frac{1}{2}(m-2s)P_{s+m,r} \\ &\qquad\qquad \left[\bar{L}_m, P_{s,r}\right] = \frac{1}{2}(m-2r)P_{s,r+m}\,, \quad \left[P_{s,r}\,, P_{s',r'}\right] = 0. \end{aligned} \tag{B.2}$$

One can solve the equation of motion $d\mathcal{A}_{2d} + \mathcal{A}_{2d} \wedge \mathcal{A}_{2d} = 0$ after imposing various gauge and boundary conditions. Here we will only provide the solution to this equation that is equivalent to the locally flat solution of (3.72) but the same method will extend to locally flat solutions in (3.73). If the $2d$ coordinates on Σ_2 are $(z, \bar{z})$ then various connection components of $\mathcal{A}_{2d}$ are given as,

$$A_{\bar{z}}^{(-1)} = A_{\bar{z}}^{(0)} = \bar{A}_{\bar{z}}^{(0)} = \bar{A}_{\bar{z}}^{(1)} = 0, \tag{B.3}$$

$$\Phi_{\bar{z}}^{(r,s)} = 0\,, \quad \text{for all} \quad r, s \in \{-\frac{1}{2}, \frac{1}{2}\}\,, \quad \Phi_{z}^{(-1/2,-1/2)} = \Phi_{z}^{(-1/2,1/2)} = 0. \tag{B.4}$$

Rest all other non-zero components are

$$\Phi_z^{(1/2,-1/2)} = \frac{1}{\sqrt{2}}\, g(z,\bar z)\,, \quad \Phi_z^{(1/2,1/2)} = -\sqrt{2}\,\partial_{\bar z}\, g(z,\bar z), \tag{B.5}$$

$$A_{\bar z}^{(1)} = \bar A_z^{(1)} = \frac{1}{\sqrt{2}}\,\partial_{\bar z}^2\, f(z,\bar z)\,, \quad A_z^{(-1)} = \bar A_{\bar z}^{(-1)} = -\frac{1}{\sqrt{2}}, \tag{B.6}$$

$$A_z^{(0)} = \bar A_{\bar z}^{(0)} = \partial_{\bar z} f(z,\bar z)\,, \quad \bar A_z^{(-1)} = \frac{1}{\sqrt{2}} f(z,\bar z), \tag{B.7}$$

$$A_z^{(1)} = \frac{1}{\sqrt{2}}\,\left(-f(z,\bar z)\,\partial_{\bar z}^2\, f(z,\bar z) - \kappa(z) + \partial_z\,\partial_{\bar z}^2\, f(z,\bar z)\right), \tag{B.8}$$

where $g(z,\bar z) = C_{-1/2}(z) + \bar z\, C_{1/2}(z)$ and $f(z,\bar z) = J_{-1}(z) + \bar z\, J_0 + \bar z^2\, J_1(z)$. This locally flat $2d$ gauge connection is parameterized by six holomorphic fields,

$$\{\kappa(z)\,, J^a\,, C^s\} \quad \text{for} \quad a \in \{-1,0,1\}\,, \quad s \in \{-\frac{1}{2},\frac{1}{2}\}$$

just like the locally flat solutions. The function C^s can be written in terms of $\mathcal{C}^s$ using (3.98). The $2d$ gauge connection can be dressed with u and r dependence using successive gauge transformation such that one gets a resultant $4d$ gauge connection for bulk gravity theory in coordinates $\{u, r, z, \bar z\}$. We do the following operation on $\mathcal{A}_{2d}$,

$$\mathcal{A}_{3d} = \mathfrak{h}^{-1}\,\mathcal{A}_{2d}\,\mathfrak{h} + \mathfrak{h}^{-1} d\,\mathfrak{h} \tag{B.9}$$

where,

$$\mathfrak{h} = e^{u\left(P_{-\frac{1}{2},-\frac{1}{2}} - 2\,J_1(z)\,P_{\frac{1}{2},\frac{1}{2}}\right)} \tag{B.10}$$

and

$$\mathcal{A}_{4d} = \mathfrak{h}'^{-1}\,\mathcal{A}_{3d}\,\mathfrak{h}' + \mathfrak{h}'^{-1} d\,\mathfrak{h}' \tag{B.11}$$

where,

$$\mathfrak{h}' = e^{r\,P_{\frac{1}{2},\frac{1}{2}}}\,. \tag{B.12}$$

The resultant $4d$ gauge connection can then be written in the form,

$$\mathcal{A}_{4d} = E^{(r,s)}\,P_{r,s} + \omega^m\,L_m + \bar\omega^m\,\bar L_m \tag{B.13}$$

where $E^{(r,s)}$ are the vielbeins and $\omega^m, \bar\omega^m$ are the spin connection coefficient. Using vielbeins one can recover the locally flat solution (3.72) as follows,

$$ds^2_{LF} = \epsilon_{rs}\,\epsilon_{r's'}\,E^{(s',r)}\,E^{(r',s)}.$$

Therefore we have shown that one can recover exactly the complete locally flat solution from the $2d$ gauge connection defined on $\mathbb{R}^2$ by uplifting to $4d$ using the gauge transformations (B.10) and (B.12). One thing to note is that all the analysis has been done so far at the level of the equation of motion. For $3d$ gauge connection $\mathcal{A}_{3d}$ one can write down the Chern Simons action that gives the equation of motion [115], however, we had to indirectly get the $4d$ gauge connection since unlike $3d$ gravity there is no equivalent Chern Simons formulation of Einstien gravity in $4d$ where the Einstien gravity action in $4d$ can be shown equivalent to a BF-theory action with equation of motion $d\mathcal{A}_{4d} + \mathcal{A}_{4d} \wedge \mathcal{A}_{4d} = 0$.

We find the residual gauge transformations that preserve the gauge and boundary conditions of our $2d$ gauge connection $\mathcal{A}_{2d}$. The transformation of $\mathcal{A}_{2d}$ under these transformations is given by

$$\delta\mathcal{A}_{2d} = d\Lambda + [\mathcal{A}_{2d}, \Lambda]\,, \tag{B.14}$$

where Λ is the gauge parameter which takes value in $\mathfrak{so}(2,3)$ Lie algebra as

$$\Lambda = \Lambda^m\, L_m + \bar{\Lambda}^a\, \bar{L}_a + \Lambda^{(r,s)}\, P_{r,s}. \tag{B.15}$$

Using (B.15) and the solution $\mathcal{A}_{2d}$ in (B.14) one obtains various constraints on the gauge parameter and also the transformation induced by them on the background fields are

$$\delta J^a(z) = \partial Y^a(z) + f^a_{bc}\, J^b(z)\, Y^c(z) + \partial(J^a(z) Y(z)), \tag{B.16}$$

$$\delta\kappa(z) = Y(z)\,\partial\,\kappa(z) + 2\kappa(z)\,\partial Y(z) - \partial^3\, Y(z), \tag{B.17}$$

$$\begin{aligned}\delta C^r(z) =& \frac{3}{2}\, C^r(z)\,\partial Y(z) + Y(z)\,\partial C^r(z) - 2\,\partial^2\, P^r(z) + P^r(z)\left(\kappa(z) + \frac{1}{2}\,\eta_{ab}\, J^a\, J^b\right) \\ &+ C^s(z)\; g_a{}^r{}_s\, Y^a(z) + 4\,\partial P^s(z)\; g_a{}^r{}_s\, J^a(z) + 2\, P^s(z)\; g_a{}^r{}_s\,\partial J^a(z).\end{aligned} \tag{B.18}$$

Here we have appropriately identified our gauge parameters in (B.15) in terms of $\{Y_a, Y(z), P_r\}$. Now we ask what type of charges generate such transformations. It is difficult to find an answer in a conventional sense for $\mathfrak{iso}(1,3)$ gauge connection because of the way charges are computed in CS gauge theory. For instance, the expression for charges in CS theory is given by,

$$Q = -\frac{k}{2\pi}\int_{\partial\Sigma} \mathrm{Tr}\,(\Lambda\,\delta\mathcal{A}_{\mathrm{3d}}). \tag{B.19}$$

Because $\mathfrak{iso}(1,3)$ has a degenerate bi-linear invariant i.e. ($\mathrm{Tr}(\mathrm{P_{r,s}\,P_{r',s'}}) = 0$), one cannot find line integral charges completely that generate transformation in δC_r. This issue was also discussed before in Appendix of [139]. One way to calculate these charges is to get a hint from the equation of motion $d\mathcal{A}_{3d} + \mathcal{A}_{3d} \wedge \mathcal{A}_{3d} = 0$ written down in component form as follows,

$$\begin{aligned}
\partial_i A_j^{(p)} - \partial_j A_i^{(p)} + (m-n)\delta^p_{m+n} A_i^{(m)}\, A_j^{(n)} &= 0,\\
\partial_i \bar{A}_j^{(c)} - \partial_j \bar{A}_i^{(c)} + (a-b)\delta^c_{a+b} \bar{A}_i^{(a)}\, \bar{A}_j^{(b)} &= 0,\\
\partial_i \Phi_j^{(s,r)} + \frac{1}{2}(m-2r')\delta^s_{m+r'} A_i^{(m)} \Phi_j^{(r',r)} + \frac{1}{2}(m-2r')\delta^r_{m+r'} \bar{A}_i^{(m)} \Phi_j^{(s,r')} - (i \leftrightarrow j) &= 0.
\end{aligned} \tag{B.20}$$

In the case of $\mathfrak{iso}(1,3)$ flat connection the equation of motion split into two sectors where equations governing $A^{(m)}$ and $\bar{A}^{(m)}$ are independent of $\Phi^{(r,s)}$ and for equation determining $\Phi^{(r,s)}$, A and $\bar{A}$ are needed. One can think of $\Phi^{(r,s)}$ as some matter field in the background geometry defined by A and $\bar{A}$. We consider the following matrix representation of $L_n, \bar{L}_n$

$$\left(L_n\right)^r{}_s = -\frac{1}{2}(n-2r)\,\delta_{n+r,s}\,,\quad \left(\bar{L}_n\right)^r{}_s = -\frac{1}{2}(n-2r)\,\delta_{n+r,s}, \tag{B.21}$$

and one define A and $\bar{A}$ in the matrix representation of $\mathfrak{sl}(2,\mathbb{C})$ as follows,

$$\left(A_j\right)^r{}_s = A_i^{(m)}\left(L_m\right)^r{}_s\,,\ (\bar{A}_j)^r{}_s = \bar{A}_i^{(m)}\,(\bar{L}_m)^r{}_s. \tag{B.22}$$

Therefore instead of defining a single $\mathfrak{iso}(1,3)$ Chern-Simons gauge action for these equations, one can think of these equations coming from two separate actions, one $\mathfrak{sl}(2,\mathbb{C})$ Chern Simons action which gives first two equations of (B.20) and another first-order action for matter $\Phi^{(r,s)}$ in bi-adjoint representation of L_n and $\bar{L}_n$. The latter action is given by

$$\mathcal{S}_{matter} = \frac{\mu}{4\pi}\int \epsilon^{ijk}\,\Phi_{i(r,r')}\left(\partial_j\,\Phi_k^{(r,r')} + \left(A_j\right)^r{}_s\,\Phi_k^{(s,r')} + \Phi_k^{(r,s)}\,(\bar{A}_j)^{r'}{}_s\right). \tag{B.23}$$

The variation of Lagrangian under $\mathfrak{iso}(1,3)$ gauge transformation defined in (B.14) is,

$$\delta\mathcal{L} = \frac{1}{2}\,\epsilon^{ijk}\left[\partial_k\left(D_i\,\Phi_j^{(r,r')}\,\Lambda_{(r,r')}\right) - 2\,\Lambda_{(r,r')}\left(\left(F_{ij}\right)^r{}_s\,\Phi_k^{(s,r')} + \Phi_k^{(r,s)}\,(\bar{F}_{ij})^{r'}{}_s\right)\right]. \tag{B.24}$$

The Lagrangian is only invariant up to the total derivative when the field strength F_{ij} and $\bar{F}_{ij}$ corresponding to $(A_j)^r{}_s$ and $(\bar{A}_j)^r{}_s$ vanishes. Now to calculate charges we use the cohomological formalism as reviewed in [186] and compute the term $\frac{\delta\mathcal{L}}{\delta\Phi_i^{(r,r')}}\,\delta_\Lambda\Phi_i^{(r,r')}$

$$\frac{\delta\mathcal{L}}{\delta\Phi_i^{(r,r')}}\,\delta_\Lambda\Phi_i^{(r,r')} = \frac{\mu}{2\pi}\epsilon^{ijk}\,\delta_\Lambda\Phi_{i_{(r,r')}}\,D_j\,\Phi_k^{(r,r')} \tag{B.25}$$

$$\begin{aligned}
&= \frac{\mu}{2\pi}\left[\partial_k\,S^k - \epsilon^{ijk}\,\Lambda_{(r,r')}\left(\left(F_{ij}\right)^r{}_s\,\Phi_k^{(s,r')} + \Phi_k^{(r,s)}\,(\bar{F}_{ij})^{r'}{}_s\right)\right]\\
&- \frac{\mu}{2\pi}\,\epsilon^{ijk}\,\Phi_{i_{(r,r')}}\left[\left(\delta A_j\right)^r{}_s\,\Phi_k^{(s,r')} - \Phi_k^{(r,s)}\,(\delta\bar{A}_j)^{r'}{}_s\right]
\end{aligned} \tag{B.26}$$

where S^k is given by,

$$S^k = \epsilon^{ijk}\left(\frac{1}{2}\,\Lambda_{(r,r')}\,D_i\,\Phi_j^{r\,r'} + \Phi_{i_{(r,r')}}\delta\Phi_j^{(r,r')}\right). \tag{B.27}$$

The term in (B.26) is total derivative only when the variation of the background field A and $\bar{A}$ are held fixed and their equations of motion are satisfied. The variation of the matter field using gauge transformation is given by,

$$\delta\Phi_i^{(r,r')} = \partial_i\Lambda^{(r,r')} + (A_i)^r{}_s\,\Lambda^{(s,r')} + \Lambda^{(r,s)}\,(\bar{A}_i)^{r'}{}_s - \tilde{\Lambda}^r{}_s\,\Phi_i^{(s,r')} - \Phi_i^{(r,s)}\,\tilde{\bar{\Lambda}}^{r'}{}_s, \tag{B.28}$$

where $\tilde{\Lambda}$ and Λ are gauge parameter associated with $\mathfrak{sl}(2,\mathbb{C})$ gauge connection in (6.106) written as

$$(\tilde{\Lambda})^r{}_s = \Lambda^{(m)}\,(L_m)^r{}_s\,,\ (\tilde{\bar{\Lambda}})^r{}_s = \bar{\Lambda}^{(m)}\,(\bar{L}_m)^r{}_s. \tag{B.29}$$

From S^k, we can define the surface charge two-form using the method in [186]

$$k = I^2_{\delta\Phi_i^{rr'}}S = \frac{\mu}{4\,\pi}\,\delta\Phi_i^{(r,r')}\,\frac{\partial}{\partial\partial_j\,\Phi_i^{(r,r')}}\,\frac{\partial}{\partial\,dx^j}\Big(S^k(d^2x)_k\Big), \tag{B.30}$$

where $S^k(d^2x)_k = \frac{1}{2}\,S^k\,\epsilon_{kmn}\,dx^m\wedge dx^n$. Therefore,

$$k = \frac{\mu}{2\pi}\,\delta\Phi_i^{(r,r')}\,\Lambda^{(r,r')}dx^i \tag{B.31}$$

For locally flat solution (B.8), the charge from (B.31) is given by,

$$Q = \frac{1}{2}\int dz\,\epsilon^{rs}\,C_r(z)P_s(z) \tag{B.32}$$

where one scales $\Lambda^{(-1/2,1/2)} \to -\sqrt{2}\,\Lambda^{(-1/2,1/2)}$ and then identifies, $\Lambda^{(-1/2,1/2)} = P_{-1/2}(z)$ and $\Lambda^{(-1/2,-1/2)} = P_{1/2}(z)$. So we have managed to derive the line integral charges associated with the supertranslation current but this assumes that the background geometry to which these matter fields were coupled is fixed and does not vary. Additionally, it remains undetermined the OPEs between the functions $C_s(z)$ such that it will give the correct variation $\delta C^s(z)$ in (B.18) for the charges (B.32). We hope to give insights into these issues in our future works.

Appendix C

Carroll-Weyl and Carroll Diffeomorphisms covariant blocks in general dimensions

In this appendix, we provide some technical details of the extension of the $d = 2$ computations of this paper to general d. Let us start by providing the transformation properties of various Carroll diffeomorphic quantities under Carroll Weyl transformations - generalising those of (4.20).

$$
\begin{aligned}
\theta &\longrightarrow B^{z-1}\Big[B\,\theta - d\,\hat{\partial}_t B\Big], \quad \theta^2 \longrightarrow B^{2z-2}\Big[B^2\theta - 2B\theta\, d\,\hat{\partial}_t B + d^2(\hat{\partial}_t B)^2\Big] \\
\hat{\gamma}^i_j\hat{\gamma}^j_i &\longrightarrow B^{2z-2}\Big[B^2\hat{\gamma}^i_j\hat{\gamma}^j_i - 2B\,\theta\hat{\partial}_t B + d\,(\hat{\partial}_t B)^2\Big] \\
\hat{r} &\longrightarrow B^2\,\hat{r} - d(d-1)a^{ij}\hat{\nabla}_i B\,\hat{\nabla}_j B + 2(d-1)a^{ij}B\,\hat{\nabla}_i\hat{\nabla}_j B \\
a^{ij}\hat{\nabla}_i\phi_j &\longrightarrow B^2\,a^{ij}\hat{\nabla}_i\phi_j + z(d-1)a^{ij}\hat{\nabla}_i B\,\hat{\nabla}_j B - z\,B\,a^{ij}\hat{\nabla}_i\hat{\nabla}_j B - B(d-2)a^{ij}\phi_i\hat{\nabla}_j B \\
a^{ij}\phi_i\phi_j &\longrightarrow B^2 a^{ij}\phi_i\phi_j - 2zBa^{ij}\phi_i\hat{\nabla}_j B + z^2 a^{ij}\hat{\nabla}_i B\,\hat{\nabla}_j B
\end{aligned}
$$

The combinations that transform homogeneously are

$$
\begin{aligned}
&\hat{r} + \frac{2}{z}(d-1)a^{ij}\hat{\nabla}_i\phi_j - \frac{(d-2)(d-1)}{z^2}a^{ij}\phi_i\phi_j \\
&\qquad \longrightarrow B^2\left(\hat{r} + \frac{2}{z}(d-1)a^{ij}\hat{\nabla}_i\phi_j - \frac{(d-2)(d-1)}{z^2}a^{ij}\phi_i\phi_j\right) \\
&\hat{\gamma}^i_j\hat{\gamma}^j_i - \frac{\theta^2}{d} \longrightarrow B^{2z}\left[\hat{\gamma}^i_j\hat{\gamma}^j_i - \frac{\theta^2}{d}\right]
\end{aligned} \tag{C.1}
$$

Taking that the scalar Φ transforms as $\phi \to B^{\delta}\Phi$ under Weyl transformations various expression transforms as follows

$$
\begin{aligned}
a^{ij}\phi_i\hat{\partial}_j\Phi &\longrightarrow B^{\delta}\big[B^2\, a^{ij}\phi_j\hat{\partial}_i\Phi + \delta B\, a^{ij}\phi_i\hat{\partial}_j B\,\Phi - \delta z a^{ij}\hat{\nabla}_i B\hat{\nabla}_j B\,\Phi - zBa^{ij}\hat{\partial}_i B\hat{\partial}_j\Phi\big]\\
\hat{\nabla}_i\hat{\partial}^i\Phi &\longrightarrow B^{\delta}\big[B^2\hat{\nabla}_i\hat{\partial}^i\Phi + (\delta^2 - d\delta + \delta)a^{ij}\hat{\nabla}_i B\hat{\nabla}^i B\,\Phi + \delta B\hat{\nabla}_i\hat{\nabla}^i B\,\Phi\\
&\qquad + (2 + 2\delta - d)B\,\hat{\partial}_i B\,\hat{\partial}^i\Phi\big]
\end{aligned}
$$

The simplest Weyl covariant object at first order derivative is

$$
\hat{\partial}_t\Phi + \frac{\delta}{d}\theta\,\Phi \longrightarrow B^{z+\delta}(\hat{\partial}_t\Phi + \frac{\delta}{d}\theta\,\Phi) \tag{C.2}
$$

At second order we find:

$$
\begin{aligned}
&\hat{\partial}_t^2\Phi + \frac{1}{d}(2\delta + z)\,\theta\,\hat{\partial}_t\Phi + \frac{\delta}{d}\left[\frac{1}{d}(z+\delta)\,\theta^2 + \hat{\partial}_t\theta\right]\Phi\\
&\longrightarrow B^{2z+\delta}\left(\hat{\partial}_t^2 + \frac{1}{d}(2\delta + z)\,\theta\,\hat{\partial}_t\Phi + \frac{\delta}{d}\left[\frac{1}{d}(z+\delta)\,\theta^2 + \hat{\partial}_t\theta\,\Phi\right]\right) &\text{(C.3)}\\
&\hat{\nabla}^i\hat{\partial}_i\Phi + \frac{(2-d+2\delta)}{z}\phi^i\hat{\partial}_i\Phi - \frac{\delta}{2}\left[\frac{1}{d-1}\hat{r} - \frac{1}{z^2}(2-d+2\delta)\phi^i\phi_i\right]\Phi\\
&\longrightarrow B^{2+\delta}\left(\hat{\nabla}^i\hat{\partial}_i\Phi + \frac{(2-d+2\delta)}{z}\phi^i\hat{\partial}_i\Phi - \frac{\delta}{2}\left[\frac{1}{d-1}\hat{r} - \frac{1}{z^2}(2-d+2\delta)\phi^i\phi_i\right]\Phi\right) &\text{(C.4)}
\end{aligned}
$$

For $z = 1$ the covariant equation can be written as,

$$
\begin{aligned}
&\kappa_0\left[\hat{\partial}_t^2\Phi + \frac{1}{d}(2\delta + z)\,\theta\,\hat{\partial}_t\Phi + \frac{\delta}{d}\left[\frac{1}{d}(z+\delta)\,\theta^2 + \hat{\partial}_t\theta\right]\Phi\right]\\
&+ \kappa_1\left[\hat{\nabla}^i\hat{\partial}_i\Phi + \frac{(2-d+2\delta)}{z}\phi^i\hat{\partial}_i\Phi - \frac{\delta}{2}\left[\frac{1}{d-1}\hat{r} - \frac{1}{z^2}(2-d+2\delta)\phi^i\phi_i\right]\Phi\right]\\
&+ \left[\sigma_0\left(\hat{\gamma}^i_j\hat{\gamma}^j_i - \frac{\theta^2}{d}\right) + \sigma_1\left(\hat{r} + \frac{2}{z}(d-1)a^{ij}\hat{\nabla}_i\phi_j - \frac{(d-2)(d-1)}{z^2}a^{ij}\phi_i\phi_j\right) + \sigma_2 f_{ij}f^{ij}\right]\Phi\\
&+ \lambda\Phi^{\frac{2+\delta}{\delta}} = 0
\end{aligned} \tag{C.5}
$$

The analogous general d expressions of invariants in (4.35-4.39) that can be used to construct actions are

$$
\Phi(\hat{\partial}_t\Phi + \frac{\delta}{d}\,\theta\,\Phi) \tag{C.6}
$$

$$
(\hat{\partial}_t\theta + \frac{z}{2}\hat{\gamma}^i_j\hat{\gamma}^j_i)\,\Phi^2 + \frac{d}{\delta}\Phi\hat{\partial}_t^2\Phi - \frac{d(z+2\delta)}{2\delta^2}(\hat{\partial}_t\Phi)^2 \tag{C.7}
$$

$$
(\hat{\partial}_t\Phi)^2 + \delta\,\theta\Phi\hat{\partial}_t\Phi + \frac{\delta^2}{d}\gamma^i_j\hat{\gamma}^j_i\Phi^2 \tag{C.8}
$$

At second order in space derivatives there are three more combinations with Weyl weight $2+2\delta$:

$$\left[\hat{r}+\frac{2}{z}(d-1)a^{ij}\hat{\nabla}_i\phi_j-\frac{(d-2)(d-1)}{z^2}a^{ij}\phi_i\phi_j\right]\Phi^2 \tag{C.9}$$

$$\hat{\partial}_i\Phi\hat{\partial}^i\Phi+2\frac{\delta}{z}\Phi\phi^i\hat{\partial}_i\Phi+\frac{\delta^2}{z^2}\phi^i\phi_i\Phi^2 \tag{C.10}$$

$$\Phi\hat{\nabla}^i\hat{\partial}_i\Phi-\frac{d-2(1+\delta)}{z}\Phi\phi^i\hat{\partial}_i\Phi+\frac{\delta}{z}\hat{\nabla}^i\phi_j\Phi^2+\frac{\delta}{z^2}(2-d+\delta)\phi^i\phi_i\Phi^2 \tag{C.11}$$

These can be used to construct Carroll diffeomorphic and Weyl invariant actions for general d straightforwardly.

Appendix D

LAdS_4 solution in Fefferman-Graham gauge

In this appendix, we will find coordinate transformations that take our chiral LAdS_4 solution from the Newman-Unti gauge to the Fefferman Graham gauge. We use the road map outlined in the appendix of [15] and would encourage readers to refer to it for a more general analysis. We start by quickly reviewing some properties of asymptotically locally AdS solution in Fefferman Graham gauge in $4d$ [95, 105]. The line element is written as follows,

$$ds^2 = \frac{1}{\rho^2}\,\left(l^2\,d\rho^2 + \gamma_{ij}(t,\rho,z,\bar{z})\,dx^i\,dx^j\right). \tag{D.1}$$

ρ is the radial coordinate near the boundary and x^i for $\{i,j\} \in \{t,z,\bar{z}\}$ are coordinates on hypersurface that are orthogonal to ρ. These become boundary coordinates in the limit $\rho \to 0$. γ_{ij} can further be expanded in power series of ρ,

$$\gamma_{ij}(t,\rho,z,\bar{z}) = \gamma^{(0)}_{ij} + \rho\,\gamma^{(1)}_{ij} + \rho^2\gamma^{(2)}_{ij} + \rho^3\,\gamma^{(3)}_{ij} + \cdots \tag{D.2}$$

$\cdots$ indicate terms at higher order in ρ and $\gamma^{(n)}_{ij}$ for $n \in \{0,1,2,3,\cdots\}$ are functions of $(t,z,\bar{z})$. Using this ansatz to solve for Einstien equation (5.2) one finds that the free data are γ^{0}_{ij} and $\gamma^{(3)}_{ij}$. The equation of motion impose $\gamma^{(1)}_{ij} = 0$ whereas $\gamma^{(2)}_{ij}$ is given in terms of $\gamma^{(0)}_{ij}$ as follows,

$$\gamma^{(2)}_{ij} = -l^2\,\left(R^{(0)}_{ij} - \frac{1}{4}\,R^{(0)}\,\gamma^{(0)}_{ij}\right) \tag{D.3}$$

$\gamma_{ij}^{(n)}$ at higher order are given in terms of $\gamma_{ij}^{(0)}$ and $\gamma_{ij}^{(3)}$. There are some constraints on $\gamma_{ij}^{(3)}$ that are given by,

$$\gamma_{(0)}^{ij}\,\gamma_{ij}^{(3)} = 0, \quad D^i\,\gamma_{ij}^{(0)} = 0 \tag{D.4}$$

where D^i is covariant derivative with respect to three dimensional boundary metric $\gamma_{ij}^{(0)}$. If one further impose the locally AdS condition as given in equation (5.3) one finds that the expansion of γ_{ij} terminates at ρ^4 [105] and $\gamma_{ij}^{(3)} = 0$ with $\gamma_{ij}^{(4)}$ given in terms of $\gamma^{(2)}$ and $\gamma^{(0)}$ as follows,

$$\gamma_{ij}^{(4)} = \frac{1}{4}\,\gamma_{ik}^{(2)}\,\gamma_{(0)}^{kl}\,\gamma_{lj}^{(2)} \tag{D.5}$$

Now If we start with our chiral locally AdS$_4$ solution in Neumann Unti gauge with coordinates $\{u, r, Z, \bar{Z}\}$ then we can do the following coordinate transformation to go to FG gauge,

$$u = T(\rho, t, z, \bar{z}) - R(\rho, t, z, \bar{z}) \tag{D.6}$$

$$r = -\frac{l^2}{R(\rho, t, z, \bar{z})} \tag{D.7}$$

$$Z = z_s(\rho, t, z, \bar{z}) \tag{D.8}$$

$$\bar{Z} = \bar{z}_s(\rho, t, z, \bar{z}) \tag{D.9}$$

These functions $X^\mu = \{T, R, z_s, \bar{z}_s\}$ are further expanded as power series expansion in ρ near the boundary $\rho \to 0$ as follows,

$$X^\mu = x^\mu + \sum_{n=1} X_{(n)}^\mu\,\rho^n \tag{D.10}$$

with $x^\mu = \{t, 0, z, \bar{z}\}$. If we demand that the FG gauge conditions (D.1) are preserved after doing the coordinate transformation (D.10) on the solution in NU gauge then one obtains the following form of the functions X^μ,

$$R(\rho,t,z,\bar{z}) = \frac{l^2\,\rho}{1-l^2\,\rho^2\,J_1(z)} \tag{D.11}$$

$$T(\rho,t,z,\bar{z}) = t + \left[R(\rho,t,z,\bar{z}) - \frac{l\,\arctan\left(\frac{2\,R(\rho,t,z,\bar{z})\,\sqrt{J_1(z)}}{l}\right)}{2\,\sqrt{J_1(z)}}\right] \tag{D.12}$$

$$z_s(\rho,t,z,\bar{z}) = z \tag{D.13}$$

$$\bar{z}_s(\rho,t,z,\bar{z}) = \bar{z} + \frac{2}{l^2}\,\left(\sqrt{G}\,C_{1/2}(z) - 2\,t\,\partial J_1(z)\right)\left[\frac{R(\rho,t,z,\bar{z})}{4\,J_1(z)} - \frac{l\,\arctan\left(\frac{2\,R(\rho,t,z,\bar{z})\,\sqrt{J_1(z)}}{l}\right)}{2\,\sqrt{J_1(z)}}\right]$$
$$+\bar{Z}(\rho,z) \tag{D.14}$$

where $\bar{Z}(\rho,z)$ obeys the following differential condition,

$$\partial_\rho\bar{Z}(\rho,z) + \frac{2\,\partial_z R(\rho,t,z,\bar{z})}{(1+l^2\,\rho^2\,J_1(z))}\left[1 - \frac{1}{\rho\,l}\,\frac{\arctan\left(\frac{2\,R(\rho,t,z,\bar{z})\,\sqrt{J_1(z)}}{l}\right)}{\sqrt{J_1(z)}}\right] = 0 \tag{D.15}$$

This differential equation can be solved easily to give $\bar{Z}$ in the closed form. The solution that we obtain after this exercise has only $\gamma^{(0)}_{ij}$ as free data whose metric components are given by

$$\gamma^{(0)}_{tt} = -\frac{1}{l^2}\,, \quad \gamma^{(0)}_{tz} = \gamma^{(0)}_{t\bar{z}} = 0 \tag{D.16}$$

$$\gamma^{(0)}_{zz} = \Gamma(t,z) + J_{-1}(z) + \bar{z}^2\,J_1(z) + \bar{z}\left(J_0(z) - \frac{\sqrt{G}\,t}{l^2}C_{1/2}(z) + \frac{t^2}{l^2}\,\partial J_1(z)\right) \tag{D.17}$$

$$\gamma^{(0)}_{\bar{z}\bar{z}} = 0\,, \ \gamma^{(0)}_{z\,\bar{z}} = \frac{1}{2} \tag{D.18}$$

The rest of the solution can be constructed from this using equations (D.3, D.5). As expected $\gamma^{(3)}_{ij}$ which is interpreted as a holographic stress tensor at the boundary which sources the metric $g^{(0)}_{ij}$ is zero for locally AdS_4 solutions [105].

Appendix E

Computation of Jacobi identities

There are three species of modes $\{L_m, J_{a,n}, G_{s,r}\}$ and so a total of ten classes of Jacobi identities to impose. We choose to denote them schematically by (LLL), (LLJ), (LJJ), (JJJ), etc. in a self-evident notation. Out of these, there are seven that involve at most one G and they can all be checked easily to be satisfied without providing any constraints.

Next, two Jacobi identities have two G's each: (LGG) and (JGG). To impose these we need the commutators $[L_m, (J^2)_n]$ and $[(J^2)_m, J_{a,n}]$. These can be computed easily using the definition

$$\eta^{ab}(J_a J_b)_n = \eta^{ab}\left[\sum_{k>-1} J_{b,n-k}J_{a,k} + \sum_{k\leq -1} J_{a,k}J_{b,n-k}\right] \tag{E.1}$$

of $(J^2)_n$ and repeatedly using (5.94 – 5.98). We find

$$[L_m, (J^2)_n] = (m-n)\,(J^2)_{m+n} - \frac{\kappa}{4}m(m^2-1)\delta_{m+n,0}\ , \tag{E.2}$$

$$[(J^2)_m, J_{a,n}] = n\,(\kappa+2)\,J_{a,m+n}\ . \tag{E.3}$$

We first impose the (LGG) Jacobi identity:

$$[L_m, [G_{s,r}, G_{s',r'}]] + [G_{s',r'}, [L_m, G_{s,r}]] - [G_{s,r}, [L_m, G_{s',r'}]] = 0\ . \tag{E.4}$$

For this we find:

- $[L_m, [G_{s,r}, G_{s',r'}]] =$

$$\beta\, \epsilon_{ss'} \left[(m-r-r')\, L_{m+r+r'} + \frac{c}{12} m(m^2-1)\, \delta_{m+r+r'}\right]$$
$$+\gamma\, \epsilon_{ss'} \left[(m-r-r')\, (J^2)_{m+r+r'} - \frac{\kappa}{4} m(m^2-1)\, \delta_{m+r+r'}\right]$$
$$-\delta\, (r-r')\, (r+r')\, J_{a,m+r+r'} (\lambda^a)_{ss'}$$

- $[G_{s',r'}, [L_m, G_{s,r}]] =$

$$\frac{1}{2}(m-2r)\Big[-\epsilon_{ss'} \left(\alpha \left((m+r)^2 - \frac{1}{4}\right) \delta_{m+r+r'} + \beta\, L_{m+r+r'} + \gamma\, (J^2)_{m+r+r'}\right)$$
$$+\, \delta\, (r'-r-m)\, J_{a,m+r+r'}\, (\lambda^a)_{s's}\Big]$$

- $-[G_{s,r}, [L_m, G_{s',r'}]] =$

$$-\frac{1}{2}(m-2r')\Big[\epsilon_{ss'} \left(\alpha \left((m+r')^2 - \frac{1}{4}\right) \delta_{m+r+r'} + \beta\, L_{m+r+r'} + \gamma\, (J^2)_{m+r+r'}\right)$$
$$+\delta\, (r-r'-m)\, J_{a,m+r+r'}\, (\lambda^a)_{ss'}\Big]$$

substituting these into (E.4) we find that the coefficient of $\epsilon_{ss'}\, \delta_{m+r+r'}$ is

$$\frac{1}{2} m(m^2-1) \left[\beta\, \frac{c}{6} - \gamma\, \frac{\kappa}{2}\, - \alpha\right]$$

which vanishes (for generic values of m) only if

$$\alpha = \frac{c}{6}\beta - \gamma\, \frac{\kappa}{2}\ . \tag{E.5}$$

The coefficients of $\beta\, L_{m+r+r'}$, $\delta\, J_{a,m+r+r'}(\lambda^a)_{ss'}$, and $\gamma\, \epsilon_{ss'}\, (J^2)_{m+r+r'}$ vanish without further conditions. Next we consider the (JGG) Jacobi identity:

$$[J_{a,n}, [G_{s,r}, G_{s',r'}]] + [G_{s',r'}, [J_{a,n}, G_{s,r}]] - [G_{s,r}, [J_{a,n}, G_{s',r'}]] = 0\ . \tag{E.6}$$

For this we find

- $[J_{a,n}, [G_{s,r}, G_{s',r'}]]$ evaluates to

$$\epsilon_{ss'}\, [\beta - \gamma(\kappa+2)]\ n\, J_{a,n+r+r'} + (r-r')\, \delta\, \left[f_{ab}{}^{c} J_{c,n+r+r'} (\lambda^b)_{ss'} - \frac{\kappa}{2}\, n\, (\lambda_a)_{ss'} \delta_{n+r+r',0}\right]$$

- $[G_{s',r'},[J_{a,n},G_{s,r}]]$ evaluates to (using $(\lambda^b)_{s'\tilde{s}}(\lambda_a)^{\tilde{s}}{}_s = -\frac{1}{4}\delta^b_a\epsilon_{s's} + \frac{1}{2}f_{ad}{}^b(\lambda^d)_{s's}$):

$$\left[\alpha\left(r'^2-\frac{1}{4}\right)\delta_{n+r+r',0}+\beta\,L_{n+r+r'}+\gamma\,(J^2)_{n+r+r'}\right](\lambda_a)_{s's}$$
$$+\delta(r'-n-r)\left[\frac{1}{4}\epsilon_{ss'}J_{a,n+r+r'}+\frac{1}{2}f_{ad}{}^b(\lambda^d)_{ss'}J_{b,n+r+r'}\right]$$

- $-[G_{s,r},[J_{a,n},G_{s',r'}]]$ evaluates to

$$-\left[\alpha\left(r^2-\frac{1}{4}\right)\delta_{n+r+r',0}+\beta\,L_{n+r+r'}+\gamma\,(J^2)_{n+r+r'}\right](\lambda_a)_{ss'}$$
$$-\delta(r-n-r')\left[-\frac{1}{4}J_{b,n+r+r'}\epsilon_{ss'}+\frac{1}{2}f_{ad}{}^b(\lambda^d)_{ss'}J_{b,n+r+r'}\right]$$

Substituting these back into (E.6) and collecting terms, the coefficients of $L_{n+r+r'}$, $(J^2)_{n+r+r'}$ and $f_{ab}{}^c J_{c,n+r+r'}(\lambda^b)_{ss'}$ vanish.

- Coefficient of $\delta_{n+r+r',0}(\lambda_a)_{ss'}$

$$-\delta\,\frac{\kappa}{2}\,n(r-r')+\alpha\left[\left(r'^2-\frac{1}{4}\right)-\left(r^2-\frac{1}{4}\right)\right]=-n(r'-r)\left[\alpha-\delta\,\frac{\kappa}{2}\right]$$

  which imposes

$$\alpha-\delta\,\frac{\kappa}{2}=0\;. \tag{E.7}$$

- Coefficient of $\epsilon_{ss'}\,J_{a,n+r+r'}$ is

$$n\,[\beta-\gamma(\kappa+2)]+\frac{1}{4}\delta(r'-n-r)+\frac{1}{4}\delta(r-n-r')=n\left[\beta-\gamma(\kappa+2)-\frac{1}{2}\delta\right]$$

  which imposes

$$\beta-\gamma\,(\kappa+2)-\frac{1}{2}\delta=0\;. \tag{E.8}$$

The last Jacobi identity left is:

$$[G_{s,r},[G_{s',r'},G_{\tilde{s},\tilde{r}}]]+[G_{\tilde{s},\tilde{r}},[G_{s,r},G_{s',r'}]]-[G_{s',r'},[G_{s,r},G_{\tilde{s},\tilde{r}}]]=0\;. \tag{E.9}$$

To impose this we further need the commutator $[(J^2)_n,G_{s,r}]$. Using the standard methods we find two different expressions for this commutator, namely

$$[(J^2)_n,G_{s,r}]=\eta^{ab}\Big[(J_aG_{s'})_{n+r}(\lambda_b)^{s'}{}_s+(G_{s'}J_b)_{n+r}(\lambda_a)^{s'}{}_s\Big]+\frac{3}{4}\left(r+\frac{1}{2}\right)G_{s,n+r} \tag{E.10}$$

and

$$[(J^2)_n, G_{s,r}] = \qquad 2\eta^{ab}(J_a G_{s'})_{n+r}(\lambda_b)^{s'}{}_s - \frac{3}{4}(n+1)\, G_{s,n+r} \ . \tag{E.11}$$

Taking the linear combination (E.11) +2 × (E.10) we arrive at:[1]

$$[(J^2)_n, G_{s,r}] = \eta^{ab}\left[\frac{4}{3}\,(J_a G_{s'})_{n+r}(\lambda_b)^{s'}{}_s + \frac{2}{3}\,(G_{s'} J_a)_{n+r}(\lambda_b)^{s'}{}_s\right] - \frac{1}{4}(n-2r) G_{s,n+r} \,. \tag{E.12}$$

Using (E.12) along with the other commutators we find

$$\begin{aligned}[G_{s,r}, [G_{s',r'}, G_{\tilde{s},\tilde{r}}]] = \epsilon_{s'\tilde{s}}\, G_{s,R}\left[-\left(\frac{\beta}{2}+\frac{\gamma}{4}\right)(R-3r) + \frac{\delta}{4}(r'-\tilde{r})\right] \\ -\frac{\delta}{2}(r'-\tilde{r})\, G_{s',R}\epsilon_{s\tilde{s}} - \gamma\left[\frac{4}{3}\,(J_a G_{\hat{s}})_R + \frac{2}{3}\,(G_{\hat{s}} J_a)_R\right](\lambda^a)^{\hat{s}}{}_s \epsilon_{s'\tilde{s}}\end{aligned}$$

where $R = r + r' + \tilde{r}$. The left-hand side of the Jacobi identity (E.9) is obtained by adding to the above expression two more terms that are obtained by cyclic permutations of pairs of indices $(s, r) \to (s', r') \to (\tilde{s}, \tilde{r}) \to (s, r)$. Luckily the terms containing $(J_a G_{\hat{s}})_R$, and $(G_{\hat{s}} J_a)_R$ do not survive after this sum because of the identity (5.90). The rest are proportional to $G_{s,R}$ and we find

$$\begin{aligned}\epsilon_{s'\tilde{s}}\, G_{s,R}\left[\left(\frac{\beta}{2}-\frac{\gamma-\delta}{4}\right)(R-3r)\right] \\ + \epsilon_{\tilde{s}s}\, G_{s',R}\left[\left(\frac{\beta}{2}-\frac{\gamma-\delta}{4}\right)(R-3r')\right] \\ + \epsilon_{ss'}\, G_{\tilde{s},R}\left[\left(\frac{\beta}{2}-\frac{\gamma-\delta}{4}\right)(R-3\tilde{r})\right] \ .\end{aligned}$$

This can vanish, for generic values of the floating indices, only if

$$\beta = \frac{\gamma-\delta}{2}. \tag{E.13}$$

[1]This particular linear combination dictated by the requirement that the right-hand side is written in terms of modes of the quasi-primaries $G_s(z)$ and $(J_a G_s)(z) + (1/2)(G_s J_a)(z)$.

Appendix F

Implementing the associativity of 3 and 4-point correlation functions

In this appendix, we will give the details of the computation of only those 3- and 4-point correlation functions which after imposing associativity conditions defined in the main body of the thesis give rise to constraint equations (5.99). If one calculates the three-point function symbolically denoted by $\langle GTG \rangle$ or $\langle GGT \rangle$ using (5.83), (5.84) and (5.92) one finds,

$$\begin{aligned} G_{r_1}(z_1)\, T(z_3)\, G_{r_2}(z_2) &= \frac{\epsilon_{r_1 r_2}\, (-6\,\alpha + \beta\, c - 3\,\gamma\,\kappa)}{2\, z_{12}\, {z_{23}}^4} + \frac{3\,\alpha\, \epsilon_{r_1 r_2}}{z_{12} {z_{13}}^2 {z_{23}}^2} \\ &= G_{r_1}(z_1)\, G_{r_2}(z_2)\, T(z_3) \end{aligned} \tag{F.1}$$

For these correlation functions to be equal to the correlation function in (5.112) one needs to impose the condition,

$$6\alpha + 3\gamma\kappa - c\beta = 0. \tag{F.2}$$

The correlation function involving one J and two G such as $\langle JGG \rangle$ is given by,

$$\langle J_a(z_3)\, G_{r_1}(z_1)\, G_{r_2}(z_2) \rangle = -\frac{2\alpha\ (\lambda_a)_{r_1 r_2}}{z_{13}\, z_{23}\, {z_{12}}^2} \tag{F.3}$$

whereas the correlation functions $\langle GJG \rangle$ and $\langle GGJ \rangle$ are given by

$$\begin{aligned} \langle G_{r_1}(z_1)\, J_a(z_3)\, G_{r_2}(z_2) \rangle &= -\frac{2\alpha\ (\lambda_a)_{r_1 r_2}}{z_{13}\, z_{23}\, {z_{12}}^2} + \frac{(\lambda_a)_{r_1 r_2}}{z_{13}\, z_{23}} \left[\frac{1}{z_{23}^2} - \frac{1}{z_{12}^2} \right] (\delta\kappa - 2\alpha) \\ &= \langle G_{r_1}(z_1)\, G_{r_2}(z_2)\, J_a(z_3) \rangle \end{aligned} \tag{F.4}$$

If these correlation functions have to be equal then one has to impose the condition,

$$\delta = \frac{2\alpha}{\kappa} \tag{F.5}$$

The imposition of associativity on three-point correlation functions does not give any more constraint equations; therefore, we move on to calculating the 4-point correlation functions. Calculating the 4-point correlation functions with two J's and two G's we get,

$$\begin{aligned}\langle J_{a_1}(z_1)\, J_{a_2}(z_2)\, G_{r_3}(z_3)\, G_{r_4}(z_4)\rangle = &-\frac{\alpha\, \epsilon_{r_3\, r_4}\, \eta_{a_1 a_2}}{z_{12}^2\, z_{34}^3}\left(\kappa + \frac{x^2}{2(1-x)}\right) \\ &- \frac{\alpha\, f_{a_1 a_2}^{\;a_3}\, (\lambda_{a_3})_{r_3\, r_4}}{z_{12}^2\, z_{34}^3}\left(\frac{2x - x^2}{1-x}\right)\end{aligned} \tag{F.6}$$

$$\begin{aligned}\langle G_{r_3}(z_3)\, J_{a_1}(z_1)\, J_{a_2}(z_2)\, G_{r_4}(z_4)\rangle &= -\frac{\alpha\, \epsilon_{r_3\, r_4}\, \eta_{a_1 a_2}}{z_{12}^2\, z_{34}^3}\left(\kappa + \frac{x^2}{2(1-x)}\right) \\ &- \frac{\alpha\, f_{a_1 a_2}^{\;a_3}\, (\lambda_{a_3})_{r_3\, r_4}}{z_{12}^2\, z_{34}^3}\left(\frac{2x - x^2}{1-x}\right) + \frac{\kappa\, \eta_{a_1 a_2}\, \epsilon_{r_3 r_4}}{2\, z_{34}\, z_{14}^2\, z_{24}^2}\left(-\beta + \gamma(\kappa+2) + \frac{\delta}{2}\right) \\ &- \frac{(\delta\, \kappa - 2\, \alpha)}{z_{13}\, z_{12}\, z_{24}^3}\left(\frac{1}{4}\, \eta_{a_1 a_2}\, \epsilon_{r_3 r_4} - \frac{1}{2}\, f_{a_1 a_2}^{\;a_3}\, (\lambda_a)_{r_3 r_4}\right) \\ &- \frac{\alpha\, \eta_{a_1 a_2}\, \epsilon_{r_3 r_4}}{2\, z_{23}\, z_{24}\, z_{34}\, z_{13}\, z_{14}}\left(\frac{(2\alpha - \delta\, \kappa)}{2\, \alpha}\left[\frac{z_{23}\, (z_{34} - z_{12})}{z_{12} z_{24}}\right]\right) \\ &+ \frac{f_{a_1 a_2}^{\;a_3}\, (\lambda_{a_3})_{r_3 r_4}\, (2\, \alpha - \delta\, \kappa)}{2\, z_{12}\, z_{13}\, z_{14}\, z_{24}^2\, z_{34}^2}\, (z_{13}\, z_{34} + z_{14}\, z_{24} + z_{13}\, z_{24})\end{aligned} \tag{F.7}$$

For (F.6) to be equal to (F.7), we need to impose the following two conditions,

$$-\beta + \gamma(\kappa+2) + \frac{\delta}{2} = 0\,, \quad 2\alpha - \delta\kappa = 0 \tag{F.8}$$

The first of the two constraints above is a new one. Now we compute the four-point correlation function that involves all the G fields. To compute it we require,

$$\begin{aligned}G_{s_1}(z) : J_a\, G_{s_2}(w) := &(\lambda_a)_{s_1\, s_2}\left[\frac{2\, \alpha}{(z-w)^4} + \frac{\beta\, T(w)}{(z-w)^2} + \frac{\gamma\; : J^2(w) :}{(z-w)^2}\right] \\ &+ \epsilon_{s_1 s_2}\left[\frac{2\, \alpha\, J_a(w)}{(z-w)^3} + \frac{\beta\; : J_a T(w) :}{z-w} + \frac{\gamma\; : J_a\, J^2(w) :}{z-w}\right] - \frac{\delta\, (\lambda_a)^{s_3}_{\;s_1}\; : G_{s_3}\, G_{s_2}(w) :}{(z-w)} \\ &- \frac{\delta\, (\lambda_a)^{s_3}_{\;s_1}\, (\lambda^b)_{s_3 s_2}}{(z-w)^2}\left[\partial J_b(w) + \frac{2\, J_b(w)}{(z-w)}\right] + \frac{\delta\, (\lambda^b)_{s_1 s_2}}{(z-w)}\left[\frac{2\; : J_a J_b(w) :}{(z-w)} + : J_a\, \partial J_b(w) :\right]\end{aligned} \tag{F.9}$$

$$G_{s_1}(z):G_{s_2}\,J_a(w):=\frac{\epsilon_{s_1s_2}\,J_a(w)}{(z-w)^3}\left(2\alpha+\beta-(\kappa+2)\,\gamma\right)+\frac{\partial J^a(w)\,\epsilon_{s_1s_2}}{(z-w)^2}\left(\beta-(\kappa+2)\,\gamma\right)$$
$$+\frac{\epsilon_{s_1s_2}}{(z-w)}\left(\beta\ :TJ_a(w):+\gamma:J^2\,J_a(w):\right)+\frac{\delta(\lambda^b)_{s_1s_2}}{(z-w)^2}\left[:J_bJ_a(w):+\frac{f_{ba}{}^{c}\,J_c(w)}{(z-w)}-\frac{2\,\kappa\,\eta_{ab}}{(z-w)^2}\right]$$
$$-\frac{(\lambda_a)^{s_3}{}_{s_1}\ :G_{s_2}\,G_{s_3}(w):}{(z-w)}\tag{F.10}$$

Using this one can write down the expression of $\langle GGGG\rangle$ compactly as follows,

$$\langle G_{s_1}(z_1)\,G_{s_2}(z_2)\,G_{s_3}(z_3)\,G_{s_4}(z_4)\rangle=\frac{1}{z_{12}^3\,z_{34}^3}\left[\epsilon_{s_1\,s_2}\epsilon_{s_3\,s_4}F_1(x)+\epsilon_{s_1\,s_3}\epsilon_{s_2\,s_4}F_2(x)\right.$$

$$\left.+\,\epsilon_{s_1\,s_4}\epsilon_{s_2\,s_3}F_3(x)\right]+\Gamma^{(1)}_{s_1\,s_2\,s_3\,s_4}+\Gamma^{(2)}_{s_1\,s_2\,s_3\,s_4}+\Gamma^{(3)}_{s_1\,s_2\,s_3\,s_4}\tag{F.11}$$

where,

$$F_1(x)=4\alpha^2\left(2+x^3-\frac{x^3}{(1-x)^3}\right)+\alpha\,\delta\left(\frac{x}{1-x}-x-2x^2-\frac{2x^2}{(1-x)^2}\right.$$
$$\left.-\frac{x^3}{(1-x)^2}+\frac{x^3}{1-x}\right)$$
$$F_2(x)=4\,\alpha^2\left(\frac{x^3}{(1-x)^3}-1\right)+\alpha\,\delta\left(x^2-\frac{x}{1-x}+\frac{x^2}{(1-x)^2}-\frac{x^3}{1-x}\right)$$
$$F_3(x)=4\,\alpha^2\left(1+x^3\right)-\alpha\,\delta\left(x+x^2+\frac{x^2}{(1-x)^2}+\frac{x^3}{(1-x)^2}\right)\tag{F.12}$$

$$\Gamma^{(1)}_{s_1\,s_2\,s_3\,s_4}=-\frac{3\,\beta\epsilon_{s_1s_3}\,\epsilon_{s_2s_4}}{z_{13}\,z_{24}\,z_{34}^4}\left(\alpha-\frac{\beta\,c}{6}+\frac{\gamma\,\kappa}{2}\right)-\frac{3\,\kappa\,\gamma\,\epsilon_{s_1s_3}\,\epsilon_{s_2s_4}}{z_{13}\,z_{24}\,z_{34}^4}\left(\beta-\gamma(2+\kappa)-\frac{\delta}{2}\right)$$
$$-\frac{(\lambda^a)_{s_1s_3}\,(\lambda_a)_{s_2s_4}}{z_{23}\,z_{13}\,z_{43}^3}\,\delta\,\left(\delta\,\kappa-2\,\alpha\right)\left(\frac{2}{z_{13}}+\frac{1}{z_{23}}+\frac{3}{z_{34}}\right)+(r_3,z_3)\leftrightarrow(r_4,z_4)$$
$$+\frac{\epsilon_{s_1\,s_2}\epsilon_{s_3\,s_4}}{2(z_{23}\,z_{24}\,z_{34}^3\,z_{12})}\gamma\,\left(2\,\alpha-\delta\,\kappa\right)\tag{F.13}$$

$$
\begin{aligned}
\Gamma^{(2)}_{s_1\,s_2\,s_3\,s_4} = \epsilon_{s_1\,s_2}\epsilon_{s_3\,s_4}\Bigg[&(\delta\,\kappa-2\,\alpha)\left(\frac{\delta}{z_{14}^2\,z_{24}\,z_{34}\,z_{23}^2}+\frac{\delta}{2\,z_{14}\,z_{24}\,z_{34}^2\,z_{23}^2}+\frac{\delta}{2\,z_{14}\,z_{24}^2\,z_{34}\,z_{23}^2}\right.\\
&\left.-\frac{\delta}{2\,z_{13}\,z_{23}\,z_{24}^2\,z_{34}^2}+\frac{\delta}{z_{13}^2\,z_{23}\,z_{24}^2\,z_{34}}+\frac{\delta}{2\,z_{13}z_{24}^2\,z_{34}\,z_{23}^2}-\frac{\gamma}{2\,z_{23}\,z_{24}\,z_{34}^3\,z_{12}}\right)\\
&+\frac{3\,\alpha\,\left(\beta-\frac{\gamma}{2}+\frac{\delta}{2}\right)}{z_{12}\,z_{34}\,z_{23}^2\,z_{24}^2}\Bigg]+\epsilon_{s_1\,s_3}\epsilon_{s_2\,s_4}\Bigg[(\delta\,\kappa-2\,\alpha)\Bigg(-\frac{\delta}{2\,z_{13}^2\,z_{23}\,z_{24}^2\,z_{34}}+\frac{\delta}{4\,z_{13}\,z_{23}\,z_{24}^2\,z_{34}^2}\\
&-\frac{\delta}{4\,z_{13}\,z_{23}^2\,z_{24}^2\,z_{34}}-\frac{\gamma}{4\,z_{13}\,z_{34}^4\,z_{24}}\Bigg)+\left(\frac{3\,\alpha}{z_{13}z_{24}z_{23}^2z_{43}^2}+\frac{3\,\kappa\,\gamma}{2\,z_{24}\,z_{13}\,z_{34}^4}\right)\left(\beta-\frac{\gamma}{2}+\frac{\delta}{2}\right)\Bigg]\\
&+\epsilon_{s_1\,s_4}\epsilon_{s_2\,s_3}\Bigg[(\delta\,\kappa-2\,\alpha)\Bigg(\frac{\delta}{2\,z_{23}^2\,z_{24}\,z_{14}^2\,z_{34}}+\frac{\delta}{4\,z_{14}\,z_{23}^2\,z_{24}\,z_{34}^2}+\frac{\delta}{4\,z_{14}\,z_{23}^2\,z_{24}^2\,z_{34}}\Bigg)\\
&+\left(\frac{3\,\alpha}{z_{14}z_{23}z_{24}^2z_{43}^2}+\frac{3\,\kappa\,\gamma}{2\,z_{23}\,z_{14}\,z_{34}^4}\right)\left(\beta-\frac{\gamma}{2}+\frac{\delta}{2}\right)\Bigg]
\end{aligned}
\tag{F.14}
$$

$$
\begin{aligned}
\Gamma^{(3)}_{s_1\,s_2\,s_3\,s_4} = \epsilon_{s_1\,s_3}\epsilon_{s_2\,s_4}\Bigg[&\gamma\,(\delta\,\kappa-2\,\alpha)\left(\frac{1}{z_{13}\,z_{23}\,z_{24}\,z_{34}^3}+\frac{1}{2\,z_{13}\,z_{24}\,z_{34}^4}\right)\\
&-\frac{3\,\kappa\,\gamma}{2\,z_{13}\,z_{24}\,z_{34}^4}\left(-\beta+\frac{\delta}{2}+\gamma(2+\kappa)+\beta-\frac{\gamma}{2}+\frac{\delta}{2}\right)\Bigg]\\
+\epsilon_{s_1\,s_3}\epsilon_{s_2\,s_4}\Bigg[&\gamma\,(\delta\,\kappa-2\,\alpha)\left(\frac{1}{z_{14}\,z_{24}\,z_{23}\,z_{43}^3}+\frac{1}{4\,z_{14}\,z_{23}\,z_{34}^4}\right)\\
&-\frac{3\,\kappa\,\gamma}{2\,z_{14}\,z_{23}\,z_{34}^4}\left(-\beta+\frac{\delta}{2}+\gamma(2+\kappa)+\beta-\frac{\gamma}{2}+\frac{\delta}{2}\right)\Bigg]
\end{aligned}
\tag{F.15}
$$

For $\langle GGGG\rangle$ to be crossing symmetry invariant all the $\Gamma's$ have to vanish. From ${\Gamma^{(2)}}_{s_1 s_2 s_3 s_4}$, one gets the fourth constraint equation,

$$
\beta-\frac{\gamma}{2}+\frac{\delta}{2}=0 \tag{F.16}
$$

Therefore we have managed to derive all the constraint equations in (5.99) from the associativity of OPEs. If all these constraints are satisfied one can see that all the Γ's vanish and the resultant four-point correlation function is invariant under crossing symmetry.

Appendix G

Chiral conformal extensions of $\mathfrak{so}(2,2)$

In this appendix, we construct the chiral conformal/$\mathcal{W}$-algebra extensions of $\mathfrak{so}(2,2)$. Repeating the steps of Section (6.1), we can take the L_0 to belong to the Cartan sub-algebra and we choose it such that the eigenvalues are real. Taking the generator to be $L_{\mu\nu}$ for $\mu, \nu \in \{0,1,2,3\}$ we take

$$L_0 = \lambda L_{01} + \lambda' L_{23} \tag{G.1}$$

We can using $O(2,2)$ transformations

1. $(\lambda, \lambda') \leftrightarrow (-\lambda, \lambda')$ using $\Lambda = \text{diag.}(1,-1,1,1)$
2. $(\lambda, \lambda') \leftrightarrow (-\lambda, \lambda')$ using $\Lambda = \text{diag.}(1,1,-1,1)$
3. $(\lambda, \lambda') \leftrightarrow (\lambda', \lambda)$ using $\Lambda_{mn} = (-1)^m \delta_{m+n,3}$

So we can again choose $\lambda \geq \lambda' \geq 0$. Then the existence of $L_{\pm 1}$ demands that (λ, λ') satisfy one of the following four conditions: (i) $\lambda + \lambda' = 1$, (ii) $\lambda - \lambda' = 1$, (iii) $\lambda + \lambda' = -1$, (iv) $\lambda - \lambda' = -1$. Since we have already assumed $\lambda \geq \lambda' \geq 0$ we need to consider only two cases: (i) $\lambda + \lambda' = 1$ and (ii) $\lambda - \lambda' = 1$. In each case we can solve the constraints to find the following results:

1. $\lambda = 1$ & $\lambda' = 0$.

$$\begin{aligned} L_0 = L_{01}, \quad (b_{03}^2 - b_{12}^2) L_1 &= b_{12}(L_{02} - L_{12}) + b_{03}(L_{03} - L_{13}) \\ L_{-1} &= b_{12}(L_{02} + L_{12}) + b_{03}(L_{03} + L_{13}) \end{aligned} \tag{G.2}$$

The residual $O(2,3)$ transformations (that keep these form-invariant) can be seen to leave $(b_{03}^2 - b_{12}^2)$ invariant up to a positive scaling: $(b_{03}^2 - b_{12}^2) \to e^{\beta}(b_{03}^2 - b_{12}^2)$. Using the discrete elements of $O(2,3)$ we can change the signs of b_{03} and b_{12} independently. Thus using these residual transformations we can choose (b_{03}, b_{12}) to be either $(1,0)$ or $(0,1)$.

2. $\lambda = 1/2 = \lambda'$

$$\begin{aligned} L_0 &= \frac{1}{2}(L_{01} + L_{23}), \quad L_1 = -a_{12}(L02 + L_{03} - L_{12} - L_{13}), \\ L_{-1} &= \frac{1}{4a_{12}}(L_{02} - L_{03} + L_{12} - L_{13}) \end{aligned} \tag{G.3}$$

The residual transformations can be used to fix $a_{12} = 1/2$.

This analysis gives us three inequivalent embeddings of $\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{so}(2,2)$ – resulting in three different avatars of $\mathfrak{so}(2,2)$.

1. $\mathfrak{h} = \emptyset$ with $(b_{03}, b_{12}) = (1,0)$

$$\begin{aligned} L_1 &= L_{03} - L_{13}, \quad L_0 = L_{01}, \quad L_{-1} = L_{03} + L_{13}, \\ P_1 &= L_{02} - L_{12}, \quad P_0 = L_{23}, \quad P_{-1} = -L_{02} - L_{12}. \end{aligned} \tag{G.4}$$

$$[L_m, L_n] = (m-n)\, L_{m+n} = [P_m, P_n], \quad [L_m, P_n] = (m-n) P_{m+n}, \tag{G.5}$$

2. $\mathfrak{h} = \emptyset$ with $(b_{03}, b_{12}) = (0,1)$

$$\begin{aligned} L_1 &= -L_{02} + L_{12}, \quad L_0 = L_{01}, \quad L_{-1} = L_{02} + L_{12}, \\ P_1 &= L_{03} - L_{13}, \quad P_0 = -L_{23}, \quad P_{-1} = L_{03} + L_{13}. \end{aligned} \tag{G.6}$$

$$[L_m, L_n] = (m-n)\, L_{m+n} = [P_m, P_n], \quad [L_m, P_n] = (m-n) P_{m+n}, \tag{G.7}$$

3. $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{R})$

$$\begin{aligned} L_1 &= -\frac{1}{2}(L_{02}+L_{03}-L_{12}-L_{13}), \\ L_{-1} &= \frac{1}{2}(L_{02}-L_{03}+L_{12}-L_{13}), \quad L_0 = \frac{1}{2}(L_{01}+L_{23}) \\ J_1 &= \frac{1}{2}(-L_{02}+L_{03}+L_{12}-L_{13}), \\ J_0 &= \frac{1}{2}(L_{01}-L_{03}), \quad L_{-1} = \frac{1}{2}(L_{02}+L_{03}+L_{12}+L_{13}). \end{aligned} \tag{G.8}$$

$$[L_m, L_n] = (m-n)L_{m+n}, \quad [J_a, J_b] = (a-b)J_{a+b}, \quad [L_m, J_a] = 0 \tag{G.9}$$

Unlike in the $\mathfrak{so}(2,3)$ case the first two avatars above with $\mathfrak{h} = \emptyset$, even though cannot be mapped to each other by any $O(2,3)$ transformations nevertheless give rise to identical algebras. So we have two distinct avatars of $\mathfrak{so}(2,2)$ which can be extended to the following chiral algebras:

1. $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{R})$

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \\ [J_{a,m}, J_{b,n}] &= (a-b)J_{a+b,m+n} - \frac{1}{2}\kappa(a^2+b^2-ab-1)m\,\delta_{m+n,0}, \\ [L_m, J_{a,n}] &= -n\,J_{a,m+n} \end{aligned} \tag{G.10}$$

   The Jacobi identity imposes no constraints on c and κ.

2. $\mathfrak{h} = \emptyset$

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \\ [L_m, P_n] &= (m-n)P_{m+n} + \frac{\tilde{c}}{12}m(m^2-1)\delta_{m+n,0}, \\ [P_m, P_n] &= (m-n)L_{m+n} + \gamma\,m(m^2-1)\delta_{m+n,0} \end{aligned} \tag{G.11}$$

   Imposing the Jacobi identity here leads to the condition that $\gamma = \frac{c}{12}$.

Thus we arrive at the result that there are exactly two chiral conformal algebra extensions of $\mathfrak{so}(2,2)$. The first one is the CIG algebra [10, 169] (however, with no relation between c and κ) and the second can be mapped to the non-chiral Brown-Henneaux one (with no relation between left and right Virasoro central charges) after appropriate redefinitions. Specifically, if we take $L_n \to \ell_n + \bar{\ell}_n$ and $P_n \to \ell_n - \bar{\ell}_n$ we will recover two commuting copies of Virasaro generated by ℓ_n with central charge $(c+\tilde{c})/2$ and $\bar{\ell}_n$ with central charge $(c-\tilde{c})/2$ respectively. It is interesting that even though we started with chiral conformal/W-algebra extensions we seemed to get both chiral and non-chiral extensions of $\mathfrak{so}(2,2)$ already known.

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