The effects of anisotropy and non-adiabaticity on the evolution of the curvature perturbation

Sergio Andrés Vallejo Peña[1,†], Antonio Enea Romano[1,2],
Atsushi Naruko[3] and Misao Sasaki[4,5,6]

[1] *Instituto de Fisica, Universidad de Antioquia, A.A.1226, Medellin, Colombia*
[2] *Theoretical Physics Department, CERN, CH-1211 Geneva 23, Switzerland*
[3] *Frontier Research Institute for Interdisciplinary Sciences & Department of Physics, Tohoku University, Sendai 980-8578, Japan*
[4] *Kavli Institute for the Physics and Mathematics of the Universe, University of Tokyo, Kashiwa 277-8583, Japan*
[5] *Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan*
[6] *Leung Center for Cosmology and Particle Astrophysics, National Taiwan University, Taipei 10617, Taiwan*
[†] *E-mail: sergio.vallejo@udea.edu.co*

We derive a general equation for the evolution of the curvature perturbation on comoving slices $\mathcal{R}_c$ in the presence of anisotropy and non-adiabaticity in the energy-momentum tensor of matter fields. The equation is obtained by manipulating the perturbed Einstein equations in the comoving slicing. It could be used to study the evolution of perturbations for a system with an anisotropic energy-momentum tensor, such as in the presence of a vector field or in the presence of non-adiabaticity, such as in a multi-field system.

Keywords: Curvature perturbation, anisotropic stress, non-adiabatic pressure.

1. Introduction

The theory of cosmological perturbations is very useful to study the early stages of the Universe, especially during inflation, that is, an exponential expansion phase which the standard cosmological model hypothesizes to explain observations such as anisotropies in the cosmic microwave background radiation (CMB). One quantity which is particularly important in this context is the curvature perturbation on comoving slices, $\mathcal{R}_c$. In slow-roll single field inflationary models this quantity is conserved on super-horizon scales,[1,2] which has important implications on the relation between primordial perturbations and late-time observables such as CMB anisotropies. For a globally adiabatic system in a single field model this quantity may not be conserved.[3] Other possible causes of super-horizon evolution could be the presence of anisotropy or non-adiabaticity in the energy-momentum tensor. We derive the equations for the curvature perturbation on comoving slices, $\mathcal{R}_c$, including these two terms showing that they act, as expected, as source terms which are relevant also on super-horizon scales. Our approach is quite generic and can be applied to any system which can be described by an energy-momentum tensor of the form we use, not only to a multi-scalar fields system.

The derivation is based on manipulating the Einstein equations in order to obtain an equation involving only $\mathcal{R}_c$, the anisotropy and non-adiabatic pressure and background quantities. The equation can be used to study phenomenologically the effects of anisotropy and non-adiabaticity without assuming any specific model.

One useful application could be to study models which violate the non-Gaussianity consistency relation[4] that was derived in fact based on the assumption of the conservation of the comoving curvatue perturbation on superhorizon scales.

2. Evolution of comoving curvature perturbations

The Einstein equations in a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) background are

$$3\mathcal{H}^2 = a^2\,\rho\,, \tag{1}$$

$$2(\mathcal{H}' - \mathcal{H}^2) = -a^2\,(\rho + P)\,. \tag{2}$$

Here a prime denotes a derivative with respect to the conformal time η and $\mathcal{H}$ stands for the conformal Hubble parameter defined by $\mathcal{H} = a'/a$. ρ and P represent the background energy density and pressure of the matter field respectively. We use the units in which $8\pi G = c = 1$.

Scalar perturbations on a spatially flat FLRW metric can be written as

$$ds^2 = a^2\left[-(1+2A)\mathrm{d}\eta^2 + 2\partial_i B \mathrm{d}x^i \mathrm{d}\eta + \left\{(1+2C)\delta_{ij} + 2\partial_i\partial_j E\right\}\mathrm{d}x^i\mathrm{d}x^j\right], \tag{3}$$

where the Latin indices run from 1 to 3. The corresponding energy-momentum tensor takes the form:

$$T^0{}_0 = -(\rho + \delta\rho)\,, \quad T^0{}_i = \frac{\rho + P}{a}u_i\,, \quad T^i{}_j = (P + \delta P)\delta^i{}_j + \Pi^i{}_j\,, \tag{4}$$

where

$$u_i = a\,\partial_i(v + B)\,, \quad \Pi^i{}_j = \delta^{ik}\partial_k\partial_j\Pi - \frac{1}{3}\overset{(3)}{\Delta}\,\Pi\delta^i{}_j\,, \quad \Pi^i{}_i = 0\,. \tag{5}$$

In the above equations $\Pi^i{}_j$ is the anisotropic stress component of the energy-momentum tensor, v is the velocity potential, Π is the anisotropy potential and we have defined $\overset{(3)}{\Delta} \equiv \delta^{ij}\partial_i\partial_j$.

The curvature perturbation on comoving slices $\mathcal{R}_c$ is a gauge-invariant quantity defined as the curvature perturbation C evaluated on the hypersurfaces in which $v + B$ vanish. The spatial Fourier expansion of the linearly perturbed Einstein equations on comoving slices[5] takes the form :

$$2k^2(\mathcal{R}_c - \mathcal{H}\sigma_c) = a^2\delta\rho_c\,, \tag{6}$$

$$\mathcal{R}_c' - \mathcal{H}A_c = 0\,, \tag{7}$$

$$2(\mathcal{H}' - \mathcal{H}^2)A_c = a^2\left[\delta P_c - (2k^2/3)\Pi_c\right], \tag{8}$$

$$\sigma_c' + 2\mathcal{H}\sigma_c - A_c - \mathcal{R}_c = a^2\Pi_c\,, \tag{9}$$

where $\sigma = E' - B$ is the scalar shear.

In general we can decompose the pressure perturbation as

$$\delta P_c = c_s^2(\eta)\delta\rho_c + \Gamma_c\,, \tag{10}$$

where we can interpret c_s and Γ_c as the adiabatic sound speed and the non-adiabatic part of the pressure respectively. For a minimally coupled scalar field model $c_s = 1$ and Γ_c is zero, but in general one would expect that Γ_c could be non-vanishing. Our goal is to derive an equation for $\mathcal{R}_c$ in the presence of both anisotropic stress $\Pi^i{}_j$ and non-adiabatic pressure Γ_c.

First we use Eq. (7) to express A_c in terms of $\mathcal{R}_c$,

$$A_c = \frac{\mathcal{R}_c'}{\mathcal{H}} \,. \tag{11}$$

We substitute this A_c and δP_c given in Eq. (10) into Eq. (8), and solve it for $\delta\rho_c$:

$$\delta\rho_c = \frac{\mathcal{H}\left(2k^2\Pi_c - 3\Gamma_c\right) - 3(\rho+P)\mathcal{R}_c'}{3\mathcal{H}c_s^2} \,. \tag{12}$$

We then insert this into Eq. (6) to get an expression for σ_c :

$$\sigma_c = \frac{\mathcal{R}_c}{\mathcal{H}} - \frac{a^2\left[\mathcal{H}(2k^2\Pi_c - 3\Gamma_c) - 3(\rho+P)\mathcal{R}_c'\right]}{6k^2\mathcal{H}^2c_s^2} \,. \tag{13}$$

Finally we substitute A_c and σ_c given by Eqs. (11) and (13), respectively, into Eq. (9) to obtain

$$\mathcal{R}_c'' + \frac{(z^2)'}{z^2}\mathcal{R}_c' - c_s^2 \overset{(3)}{\Delta} \mathcal{R}_c + \frac{\mathcal{H}}{\rho+P}Y_c = 0 \,, \tag{14}$$

where we have defined

$$z^2 \equiv \frac{a^4(\rho+P)}{c_s^2\mathcal{H}^2} \,, \tag{15}$$

$$Y_c \equiv \left[\log\left(\frac{a^4}{\mathcal{H}c_s^2}\right)\right]'\left(\frac{2}{3}\overset{(3)}{\Delta}\Pi_c + \Gamma_c\right) + 2\mathcal{H}c_s^2\overset{(3)}{\Delta}\Pi_c + \frac{2}{3}\overset{(3)}{\Delta}\Pi_c' + \Gamma_c' \,. \tag{16}$$

As expected, for adiabatic ($\Gamma_c = 0$) and isotropic perturbations ($\Pi_c = 0$) the above equation takes the well-known form :

$$\mathcal{R}_c'' + \frac{(z^2)'}{z^2}\mathcal{R}_c' - c_s^2 \overset{(3)}{\Delta} \mathcal{R}_c = 0 \,. \tag{17}$$

3. Curvature perturbation for scalar fields

Given the generality of the form of the energy momentum tensor used in the derivation of Eq. (14) it can be applied to a wide class of physical scenarios, including multi-field systems. Let us consider the case of two minimally coupled scalar fields with Lagrangian

$$L = -\sum_{n=1}^{2} X_n - 2V(\Phi_1, \Phi_2) \,, \tag{18}$$

where $X_n = g^{\mu\nu}\partial_\mu\Phi_n\partial_\nu\Phi_n$ and $\Phi_n(x^\mu) = \phi_n(\eta) + \delta\phi_n(x^\mu)$. The perturbed energy-momentum tensor, without gauge fixing, is given by

$$\delta\rho = \frac{\phi_1'\delta\phi_1' + \phi_2'\delta\phi_2' - A({\phi_1'}^2 + {\phi_2'}^2)}{a^2} + V_1\delta\phi_1 + V_2\delta\phi_2\,,$$
$$\delta P = \frac{\phi_1'\delta\phi_1' + \phi_2'\delta\phi_2' - A({\phi_1'}^2 + {\phi_2'}^2)}{a^2} - V_1\delta\phi_1 - V_2\delta\phi_2\,,$$
$$\Pi = 0 \quad , \quad \delta T^0{}_i = \partial_i\left(-\frac{\phi_1'\delta\phi_1 + \phi_2'\delta\phi_2}{a^2}\right)\,, \tag{19}$$

where we denote the partial derivatives as $V_n = (\partial V/\partial\Phi_n)(\phi_1, \phi_2)$.

The field perturbations transform under an infinitesimal time translation $\eta \to \eta + \delta\eta$

$$\widetilde{\delta\phi_1} = \delta\phi_1 - \phi_1'\delta\eta \quad , \quad \widetilde{\delta\phi_2} = \delta\phi_2 - \phi_2'\delta\eta\,. \tag{20}$$

The time translation $\delta\eta_c$ necessary to define the comoving slices can be found by imposing the condition $(\delta T^0{}_i)_c \propto \phi_1'\widetilde{\delta\phi_1} + \phi_2'\widetilde{\delta\phi_2} = 0$, giving

$$\delta\eta_c = \frac{\phi_1'\delta\phi_1 + \phi_2'\delta\phi_2}{{\phi_1'}^2 + {\phi_2'}^2}\,. \tag{21}$$

The comoving curvature perturbation for the two scalar fields system is given by

$$\mathcal{R}_c = C - \mathcal{H}\delta\eta_c = C - \mathcal{H}\frac{\phi_1'\delta\phi_1 + \phi_2'\delta\phi_2}{{\phi_1'}^2 + {\phi_2'}^2}\,. \tag{22}$$

The gauge invariant field perturbations in the comoving slices now can be defined

$$U_1 = \delta\phi_1 - \phi_1'\frac{\phi_1'\delta\phi_1 + \phi_2'\delta\phi_2}{{\phi_1'}^2 + {\phi_2'}^2}\,, \tag{23}$$
$$U_2 = \delta\phi_2 - \phi_2'\frac{\phi_1'\delta\phi_1 + \phi_2'\delta\phi_2}{{\phi_1'}^2 + {\phi_2'}^2}\,, \tag{24}$$

and similarly for the pressure and energy perturbations in the comoving slices we get

$$\delta\rho_c = \frac{\phi_1'U_1' + \phi_2'U_2' - A_c({\phi_1'}^2 + {\phi_2'}^2)}{a^2} + V_1U_1 + V_2U_2\,, \tag{25}$$
$$\delta P_c = \frac{\phi_1'U_1' + \phi_2'U_2' - A_c({\phi_1'}^2 + {\phi_2'}^2)}{a^2} - V_1U_1 - V_2U_2\,. \tag{26}$$

Note that all the above quantities are gauge invariant by construction.

Combining Eqs. (23), (24) and the background field equations of motion we find

$$\frac{\phi_1'U_1' + \phi_2'U_2'}{a^2} = V_1U_1 + V_2U_2 = -\frac{{\phi_1'}^2 + {\phi_2'}^2}{4a^2}\Theta\,, \tag{27}$$

where we have defined the function Θ according to

$$\Theta = \left[\frac{\partial}{\partial\eta}\left(\frac{{\phi_1'}^2 - {\phi_2'}^2}{{\phi_1'}^2 + {\phi_2'}^2}\right)\right]\left(\frac{\delta\phi_1}{\phi_1'} - \frac{\delta\phi_2}{\phi_2'}\right)\,. \tag{28}$$

Assuming a classical field trajectory parameterized as $\phi_2 = \phi_2(\phi_1)$ we can write Θ in this form

$$\Theta = -4\frac{\mathrm{d}^2\phi_2}{\mathrm{d}\phi_1^2}\left[\left(\frac{\mathrm{d}\phi_2}{\mathrm{d}\phi_1}\right)^2 + 1\right]^{-2}\left(\frac{\mathrm{d}\phi_2}{\mathrm{d}\phi_1}\delta\phi_1 - \delta\phi_2\right). \tag{29}$$

From the above expression we can see that in order for Θ to be different from zero the trajectory has to have non vanishing second derivative, i.e. there must be some turn in the field space.

After replacing Eqs. (7) and (27) into Eqs. (25) and (26) we get

$$\delta\rho_c = -\frac{\phi_1'^{\,2} + \phi_2'^{\,2}}{a^2\mathcal{H}}\mathcal{R}_c' - \frac{\phi_1'^{\,2} + \phi_2'^{\,2}}{2a^2}\Theta\,, \tag{30}$$

$$\delta P_c = -\frac{\phi_1'^{\,2} + \phi_2'^{\,2}}{a^2\mathcal{H}}\mathcal{R}_c'\,. \tag{31}$$

It follows from Eqs. (30) and (31) that

$$\delta P_c = \delta\rho_c + \frac{\phi_1'^{\,2} + \phi_2'^{\,2}}{2a^2}\Theta\,, \tag{32}$$

and comparing this with Eq. (10) we obtain the sound speed and the entropy perturbations

$$c_s^2(\eta) = 1\,, \qquad \Gamma_c = \frac{\phi_1'^{\,2} + \phi_2'^{\,2}}{2a^2}\Theta\,. \tag{33}$$

From these relations we get

$$\mathcal{R}_c'' + 2\frac{z'}{z}\mathcal{R}_c' - \overset{(3)}{\Delta}\mathcal{R}_c = -\frac{a^2\mathcal{H}}{(\phi_1'^{\,2} + \phi_2'^{\,2})}Y_c\,, \tag{34}$$

where

$$z^2 = \frac{a^2(\phi_1'^{\,2} + \phi_2'^{\,2})}{\mathcal{H}^2}\,, \tag{35}$$

$$Y_c = \left[\log\left(\frac{a^4}{\mathcal{H}}\right)\right]'\Gamma_c + \Gamma_c'\,. \tag{36}$$

Eq. (34) is in agreement with Ref. [6], confirming that Eq. (14) is general and can also be applied to multi-field systems once the entropy has been appropriately defined. In general in order to use Eq. (14) it is first necessary to compute the energy momentum tensor in the comoving slices with a procedure similar to the one shown above for two fields.

4. Conclusions

We have derived a general equation for the evolution of the comoving curvature perturbation by taking into account the effects of anisotropy and non-adiabaticity in the energy-momentum tensor. The equation can be applied to multi-field systems. This approach does not require the decomposition of field perturbations in components parallel and perpendicular to the classical field trajectory, but is based just on the fundamental definition of non-adiabatic pressure. In the future it will be interesting to apply the equation to more complex systems where both entropy and anisotropy are present.

References

1. A. E. Romano, S. Mooij, and M. Sasaki, *Phys. Lett. B* **755**, 464 (2016), arXiv:gr-qc/1512.05757.
2. D. Wands, K. A. Malik, D. H. Lyth, A. R. Liddle, *Phys. Rev. D* **62**, 043527 (2000), arXiv:astro-ph/0003278.
3. A. E. Romano, S. Mooij, and M. Sasaki, *Phys. Lett. B* **761**, 119 (2016), arXiv:gr-qc/1606.04906.
4. J. M. Maldacena, *JHEP* **05**, 013 (2003), arXiv:astro-ph/0210603.
5. K. A. Malik and D. Wands, *Physics Reports* **475**, 1 (2009), arXiv:0809.4944.
6. J. García-Bellido and D. Wands, *Phys. Rev. D* **52**, 6739 (1995), arXiv:gr-qc/9506050.