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Crystalline chiral condensates in compact stars

Stefano Carignano

INFN, Laboratori Nazionali del Gran Sasso, Via G. Acitelli, 22, I-67100 Assergi (AQ), Italy

E-mail: `carignano@lngs.infn.it`

Abstract. I discuss the phenomenon of inhomogeneous chiral symmetry breaking in dense quark matter, with a particular emphasis on its relevance for the physics of compact stars. After briefly reviewing the formalism employed for the study of crystalline chiral condensates within effective models, I focus on their effects on the equation of state and the possible consequences for mass-radius relations. Finally, I discuss how model extensions which provide a more realistic description of matter inside compact stellar objects affect the formation of these inhomogeneous condensates.

1. Introduction

The study of the properties of quantum chromodynamics (QCD) at finite density is one of the most challenging tasks in contemporary nuclear physics. In spite of our little knowledge, the phase diagram of strong interaction is nevertheless expected to be extremely rich. A particularly interesting phenomenon which might be realized in dense quark matter is the formation of spatially inhomogeneous condensates formed by pairs of fermions carrying a finite momentum. This translates into an explicit spatial dependence of these condensates and the formation of crystalline structures. Prominent examples of inhomogeneous phases in dense quark matter are related to the phenomena of color-superconductivity and chiral symmetry breaking. Inhomogeneous color-superconductors might appear in isospin imbalanced quark matter, where the separation between the quark Fermi surfaces makes the formation of diquark Cooper pairs with finite momentum the energetically favored channel (see [1] for a recent review). Inhomogeneous chiral condensates on the other hand arise in the region between the low-density phase where chiral symmetry is spontaneously broken and the high-density restored one. In the usual picture limited to homogeneous chiral condensates made of quark-antiquark pairs, the presence of a chemical potential induces a stress on the formation of these pairs, eventually leading to chiral restoration. On the other hand, at finite densities the formation of pairs made of quarks and holes at the Fermi surface becomes a competitive condensation channel. Since pairs with large relative momenta are disfavored, the preferred pairing mechanism leads to a condensate with a non-vanishing net total momentum, i.e. a spatially inhomogeneous one (for a more detailed discussion, see e.g. [2]).

Explicit model calculations indeed support this hypothesis by suggesting that an inhomogeneous "island" where the favored structure for the chiral condensate is a spatially modulated one appears at low temperatures and intermediate densities (for a recent review, see [3]). A typical phase diagram resulting from this type of calculations is shown in Fig. 1. The actual extension of this window is still uncertain: some NJL and Dyson-Schwinger studies suggest that, at least at low temperatures, inhomogeneous chiral symmetry breaking might occur
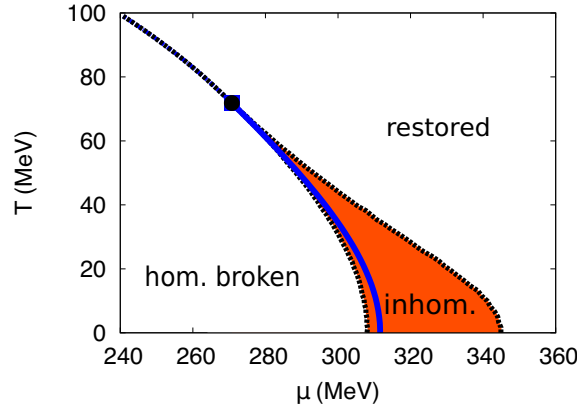
Figure 1. Typical phase diagram in the μ-T plane when allowing for inhomogeneous phases in model calculations. The shaded region indicates the inhomogenous phase, covering the first-order phase boundary (blue solid line) between the homogeneous chirally broken and restored phases.

in a very wide range of densities [4, 5], while other models predict a smaller size. In this sense, Fig. 1 could be seen as a rather conservative estimate.

At any rate, the realization of inhomogeneous chiral symmetry breaking appears to be a rather robust prediction for the region of the phase diagram which is expected to be relevant for the description of compact stars, which provide the only known realization of ultra-dense matter in nature. The formation of crystalline chiral condensates might therefore have significant consequences on the physical properties of these objects, possibly leading to new signatures.

2. Model description of inhomogeneous chiral symmetry breaking

The Nambu–Jona-Lasino (NJL) model provides one of the most commonly employed frameworks for the study of chiral symmetry breaking in dense quark matter. Starting point is the Lagrangian

$$\mathcal{L}_{\mathrm{NJL}} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi + G\left(\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi}i\gamma^5\tau^a\psi\right)^2\right), \tag{1}$$

where ψ denotes a quark field with bare mass m (which in this work will be taken for simplicity to zero, i.e. the chiral limit), interacting by local four-point vertices proportional to the dimensionful coupling constant G.

In order to analyze the thermodynamic properties at temperature T and quark chemical potential μ one typically calculates the grand potential Ω of the system within a mean-field approximation in the presence of static scalar and pseudoscalar condensates (here for simplicity the possibility of charged pion condensation is neglected):

$$S(\mathbf{x}) = \langle\bar{\psi}\psi\rangle \quad \text{and} \quad P(\mathbf{x}) = \langle\bar{\psi}i\gamma^5\tau^3\psi\rangle, \tag{2}$$

If the condensates are allowed to retain an explicit spatial dependence, the derivation of the mean-field thermodynamic potential becomes significantly more difficult than for space-independent condensates. As discussed in detail in [3], one finds

$$\Omega_{\mathrm{MF}}(T,\mu;S,P) = \Omega_{kin} + \Omega_{cond}, \tag{3}$$

with the condensate part

$$\Omega_{cond} = \frac{1}{V}\int_V d^3x\, G\left(S^2(\mathbf{x}) + P^2(\mathbf{x})\right) \tag{4}$$

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and the kinetic part

$$\Omega_{kin} = -\frac{1}{V} \sum_{\lambda} \left[\frac{E_{\lambda} - \mu}{2} + T \log \left(1 + e^{-\frac{E_{\lambda} - \mu}{T}} \right) \right],$$

(5)

where E_{λ} are the eigenvalues of the effective Dirac Hamiltonian

$$H(\mathbf{x}) = \gamma^0 \left[-i\gamma^i \partial_i + m - 2G(S(\mathbf{x}) + i\gamma^5 \tau^3 P(\mathbf{x})) \right],$$

(6)

In contrast to the homogeneous case, the diagonalization of H for arbitrary space-dependent condensate functions is rather difficult and must in general be performed numerically. After that, Ω_{MF} must be minimized with respect to the functions $S(\mathbf{x})$ and $P(\mathbf{x})$ in order to find the energetically favored ground state. Since this problem has not yet been solved in 3+1 dimensions, one considers simple ansätze for the condensate functions. This reduces the problem to finding the minimum of Ω_{MF} with respect to a limited number of variational parameters[1].

In this contribution I will focus on two popular one-dimensional ansätze which even allow for an analytical diagonalization of H. Choosing the modulation to be along the z-direction and introducing the complex mass function

$$M(z) = -2G\big(S(z) + iP(z)\big),$$

(7)

the most simple (and therefore most popular) one is the so-called (dual) chiral density wave (CDW) [6, 7]

$$M(z) = \Delta e^{iqz},$$

(8)

i.e. a single plane wave with amplitude Δ and wave number q, corresponding to a spatial period of $L = 2\pi/q$.

A more sophisticated ansatz is the "real kink crystal" (RKC), corresponding to an array of domain-wall solitons [8, 9]. The mass function reads

$$M(z) = \Delta\sqrt{\nu}\,\text{sn}(\Delta z | \nu)$$

(9)

and depends on two parameters, Δ and ν. Here $\text{sn}(\alpha|\nu)$ denotes a Jacobi elliptic function with elliptic modulus $\nu \in [0, 1]$, interpolating between $\tanh(\alpha)$ for $\nu = 1$ and $\sin(\alpha)$ for $\nu = 0$. The amplitude of the modulation is then given by $\Delta\sqrt{\nu}$ and its spatial period by $L = 4\mathbf{K}(\nu)/\Delta$, where $\mathbf{K}(\nu)$ is a complete elliptic integral of the first kind. In contrast to the CDW, the RKC is real (hence the name), i.e. the pseudoscalar condensate vanishes identically.

The resulting phase diagrams for the two modulations look qualitatively similar to Fig. 1, although one finds that the transition from the homogeneous broken to the inhomogeneous phase is first-order for the CDW while it is second-order for the RKC [3, 10]. Furthermore, the CDW is found to be disfavored against the RKC [9].

It is possible to calculate spatially averaged quark-number densities $\bar{n} = -\partial\Omega/\partial\mu$ for these two ansätze (see [11] for a more detailed discussion). For the RKC ansatz, the inhomogeneous phase starts essentially at $\bar{n} = 0$, corresponding to infinitely separated domain-wall solitons [12, 13]. For the CDW solution, on the other hand, the minimum density is around $1.4\,n_0$, where $n_0 \approx 0.5$ fm^{-3} is the quark number density in symmetric nuclear matter at saturation. In both cases the upper density of the inhomogeneous island reaches about $2.2\,n_0$. Keeping in mind that these numbers are model and parameter dependent and could also be higher, this lies in the ballpark of densities expected in a possible quark phase in compact stars.

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[1] Most parametrizations used in literature correspond to one-dimensional spatial modulations. Two- and higher-dimensional modulations have been studied as well, but were found to be disfavored against one-dimensional ones, at least within the mean-field approximation [14, 15]
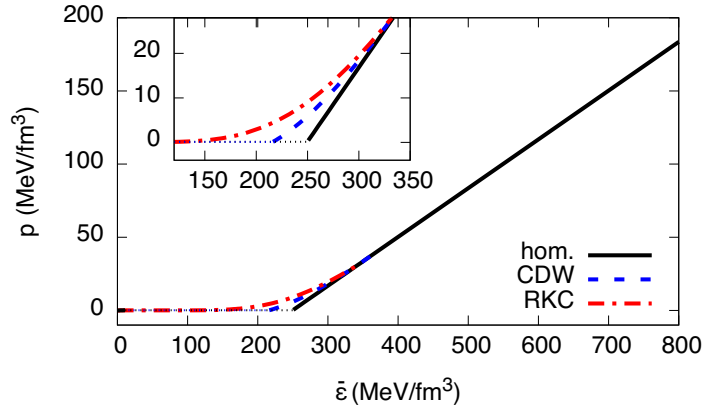
Figure 2. Equation of state (pressure p vs. spatially averaged energy density $\bar{\varepsilon}$) at $T = 0$ for homogeneous matter (solid) and in the inhomogeneous phase with CDW (dashed) or RKC (dash-dotted) modulations. All equations of state begin at $p = \bar{\varepsilon} = 0$ in the homogeneously broken phase. The dotted horizontal lines indicate discontinuities related to first-order transitions to the restored phase or to the CDW solution. The inset shows an enlarged detail of the transition region.

Knowing the density and the pressure for these solutions, it is possible to calculate the corresponding equations of state at $T = 0$, i.e. the pressure as a function of the spatially averaged energy density $\bar{\varepsilon} = -p + \mu\bar{n}$. They are shown in Fig. 2. In general, the inhomogeneous equations of state and in particular the RKC one are stiffer than the homogeneous one, although the differences are rather small. One can therefore anticipate that the effect of inhomogeneous phases on the mass-radius relation of compact stars will be negligible. This has been checked by solving the Tolman-Oppenheimer-Volkoff (TOV) equation using these equations of state of the system as an input [11]. The results of this calculation are shown in Fig. 3. In this context it is important to recall that the aim here is not to build a realistic $M(R)$ sequence which can reach the $2M_\odot$ ($M_\odot$ being the solar mass) observed values[2] [16, 17], but rather to examine the influence of inhomogeneous chiral condensation on the mass-radius relations. For this, only results for pure non self-bound quark stars with different crystalline structures are presented, neglecting effects like charge neutrality or the presence of magnetic fields. As expected, the presence of inhomogeneous condensates in the core of the stars makes very little difference on the resulting mass-radius sequences (except for the low-mass region which should become irrelevant once a more realistic description of the hadronic phase is implemented). It looks therefore that it would be impossible to infer the existence of an inhomogeneous phase on the basis of a mass-radius measurement, even if it was very precise.

3. Effects of model extensions on inhomogeneous chiral symmetry breaking
In order to provide a realistic description of matter inside compact stars, several extensions can be implemented on top of the simple NJL model considered in the previous section. Their influence on inhomogeneous chiral symmetry breaking are briefly described in the following. For simplicity, only the two light quark flavors will be considered.

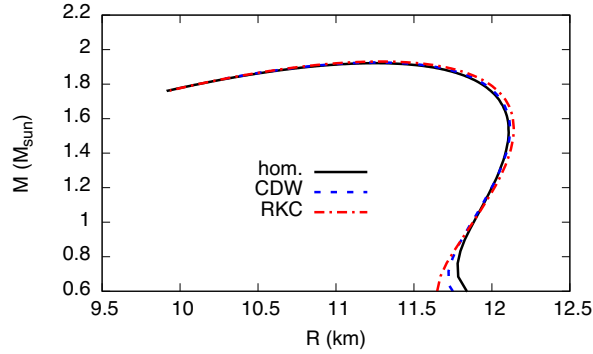[2] Such an analysis has been performed e.g. in [18].

Figure 3.   Mass-radius sequences for a pure quark star, neglecting electric neutrality and magnetic fields, for a RKC and CDW modulations as well as for homogeneous matter. The differences between the three curves turn out to be minimal.

3.1. Electric neutrality and isospin asymmetry

First of all, when describing compact stars it is necessary to take into account weak decays and ensure global electric neutrality. In order to describe this situation, electrons are included in the model[3] and the system is then characterized by two conserved global charges: the net quark number and the total electric charge. The thermodynamic potential Ω then depends on two independent chemical potentials, the quark number chemical potential μ and the electric charge chemical potential μ_Q. The chemical potentials of the individual particle species are given as linear combinations of μ and μ_Q, according to their quantum numbers, i.e.

$$\mu_u = \mu + \frac{2}{3}\mu_Q, \quad \mu_d = \mu_s = \mu - \frac{1}{3}\mu_Q, \quad \mu_\ell = -\mu_Q. \tag{10}$$

The condition for (spatially averaged) electric neutrality is then given by $\bar{n}_Q = -\partial\Omega/\partial\mu_Q = 0$. In the quark sector, μ_Q is equivalent to an isospin chemical potential $\mu_I \equiv \mu_u - \mu_d = \mu_Q$. The treatment of isospin-asymmetric matter within the NJL model is not entirely trivial, as it turns out that the exact isospin structure of the interaction becomes relevant in this case. The $\mathcal{L}_{\text{NJL}}$ of Eq. (1) can be viewed as a special case of the more general Lagrangian [19, 20]

$$\begin{aligned}
\mathcal{L} = \ & \bar{\psi}\big(i\gamma^\mu\partial_\mu - \hat{m}\big)\psi \\
& + G_1\Big((\bar{\psi}\psi)^2 + (\bar{\psi}\tau^a\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 + (\bar{\psi}i\gamma_5\tau^a\psi)^2\Big) \\
& + G_2\Big((\bar{\psi}\psi)^2 - (\bar{\psi}\tau^a\psi)^2 - (\bar{\psi}i\gamma_5\psi)^2 + (\bar{\psi}i\gamma_5\tau^a\psi)^2\Big),
\end{aligned} \tag{11}$$

with the current quark matrix $\hat{m} = \text{diag}_f(m_u, m_d)$ and two $SU(2)_L \times SU(2)_R \times U_V(1)$ symmetric interaction terms, proportional to the coupling constants G_1 and G_2, respectively. The first term respects an additional $U_A(1)$ symmetry, which is explicitly broken by the second. A detailed study of inhomogeneous phases in this framework is in progress and will be presented soon [21].

Finally, instead of considering a homogeneous leptonic background and imposing the neutrality condition for the spatially averaged charge density, one could in principle do better and allow for an inhomogeneous lepton density as well. While the density gradient would increase the kinetic energy of the electrons, it could lower the local Coulomb energy, and the optimal shape would result from balancing these two contributions. This possibility will also be analyzed in the near future [22].

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[3] Neutrinos can freely leave the star, so that it is sufficient to take into account charged leptons.

3.2. Magnetic fields

Another very important element characterizing compact stars is the presence of strong magnetic fields. For magnetars, surface fields of up to $10^{15}G$ have been measured, and even higher values are expected to be reached in their cores. Aside from the well known effects on the usual homogeneous chiral symmetry breaking mechanism, such as the phenomenon of magnetic catalysis (see [23, 24] for recent reviews), the presence of strong magnetic fields has a significant influence on the formation and the properties of inhomogeneous phases. For a CDW-type ansatz (Eq. (8)) in the presence of a static background magnetic field H pointing in the z direction, the quark energies for each flavor have been calculated in [25] and are given by

$$E_n^f = \pm \begin{cases} \sqrt{\Delta^2 + p_z^2} + q, & n = 0 \\ \sqrt{\left(\pm\sqrt{\Delta^2 + p_z^2} + q\right)^2 + 2|e_f H| n} & n > 0. \end{cases} \tag{12}$$

By inspecting this expression one can see that the quark momenta transverse with respect to the direction of H become quantized into $p_\perp^2 \sim 2n|e_f H|$, with the integer n labelling the so-called Landau levels. In turn, this implies that at the lowest Landau level (LLL), for which $n = 0$, the problem becomes effectively 1+1-dimensional, a configuration in which inhomogeneous chiral condensation is always favored [26, 27]. In particular, as a consequence of the spectral asymmetry at the LLL, at low temperatures the CDW becomes the thermodynamically favored state over homogeneous matter for any $\mu > 0$. Furthermore, it has been argued that due to the presence of magnetic fields new types of inhomogeneous condensates characterized by complex order parameters can become favored as well [28].

In summary, the presence of a magnetic field qualitatively alters the model phase structure at finite density, significantly enhancing the size of the inhomogeneous phase. Of course, these effects will be relatively small until the strength of the magnetic field approaches the scales given by the quark chemical potential (see e.g. [25, 11] for a discussion), which are typically around $H \gtrsim 10^{17}$G.

3.3. Vector interactions

Finally, in compact stars, where matter is expected to reach several times nuclear-matter density, vector-interaction effects could be particularly important. The effect of vector interactions on inhomogeneous matter has been investigated in [12]. To this end the NJL Lagrangian (Eq. (1)) was extended by the term

$$\mathcal{L}_V = -G_V (\bar\psi \gamma^\mu \psi)^2, \tag{13}$$

where G_V is the vector coupling constant.

Within the mean-field approximation, only the temporal part $V^0 = \langle \bar\psi \gamma^0 \psi \rangle$ of the vector condensate, which is identical to the quark number density n, was retained. The effect of this field can then be absorbed in a shift of the chemical potential[4]: $\mu \to \tilde\mu = \mu - 2G_V n$.

For $G_V = 0$, as seen in Fig. 1, the inhomogeneous phase covers the first-order phase boundary between the homogeneous broken and restored phases, and ends at the critical point [9, 29]. Since vector interactions weaken or even remove this homogeneous first-order phase transition, it was expected that the inhomogeneous phase becomes smaller or disappears as well when G_V is increased. It turned out, however, that the inhomogeneous phase becomes larger instead, keeping its extension in temperature and enhancing its size in the chemical potential direction [12].

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[4] When dealing with inhomogeneous condensates, the quark density is in general spatially modulated, implying that the shifted chemical potential also depends on the position. For a CDW this is not the case, as the density is constant, whereas it is for the RKC modulation. Nevertheless, in [12] the shift in the chemical potential was implemented only through the spatial average of the vector condensate, an approximation which is expected to be rather inaccurate close to the onset of the inhomogeneous island, but otherwise reasonably valid, particularly when approaching the restored phase.

4. Conclusions

I discussed some properties of the phenomenon of inhomogeneous chiral symmetry breaking, focusing on some of its implications for the physics of compact stars. Several models suggest that spatially modulated structures of the chiral order parameters become energetically favored in a window of the phase diagram which is relevant for cold and dense stellar matter. Furthermore, model extensions providing a more realistic description of matter inside compact stars, such as the inclusion of background magnetic fields or vector interactions have been shown to enlarge the inhomogeneous window significantly, suggesting that if quark matter is present at all inside these objects, it is likely in a crystalline phase.

The interplay of these crystalline chiral condensates with color superconductivity, on the other hand, is not yet completely clear. Preliminary NJL model studies have shown that inhomogeneous chiral condensates may coexist with a homogeneous diquark condensate if the diquark coupling is small, but become suppressed for larger couplings [30, 31]. This could change, however, if electric neutrality is imposed, since it strongly disfavors the BCS pairing of up and down quarks. As a possible result, the diquark condensates could become spatially modulated as well or be suppressed completely. In particular, the realization of a coexistence phase where both color-superconducting and chiral condensates are inhomogeneous is a fascinating possibility, which might be worth investigating.

The determination of clear signatures relating macroscopic observables with the microscopic properties of matter within stellar objects is obviously a very challenging task. While the influence of inhomogeneous condensates on mass-radius relations turned out to be marginal, more promising signatures can be expected from observables related to transport properties, since they are directly influenced by the crystalline structure of the matter and the corresponding low-energy excitations (phonons). Some of these signatures, like gravitational waves and glitches, have been investigated for crystalline color superconductors (see e.g. [1]), but not yet for the specific case of inhomogeneous chiral condensates[5].

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