THE LOBACHEVSKY SPACE IN RELATIVISTIC NUCLEAR PHYSICS

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Abstract

Relativistic nuclear collisions are considered in terms of relative 4-velocity and rapidity space (the Lobachevsky space). The connection between geometric relations in the Lobachevsky space and measurable (experimentally determined) kinematical characteristics are discussed. General properties of relativistic invariants distributions characterizing geometric position of particles in the Lobachevsky space are discussed. Possible applications of the obtained results for planning of experimental research and analysis of data on multiple particle production are discussed. The analysis is illustrated by processed experimental data.

1. Introduction

A desire to discover simple laws of Nature describing a wide range of phenomena, plays a progressive role of one of the basic principles of fundamental science. An important step in constructing theories is the selection of a set of variables for description of observed phenomena. That is why special attention is paid in this paper to the discussion of the variables used in analysis of relativistic particle collisions.

The theory of nuclear interactions is at present far from completeness. Essentially, it represents a set of phenomenological models and approaches describing the available experimental data. The most complicated from the point of view of theoretical description of nuclear matter is, in our opinion, the transition region between proton-neutron model of a nucleus and the region where excitation of internal quark-gluon degrees of freedom is essential.

One of the most important problems nowadays, as it was formulated by a distinguished scientist S.Nagamia in 1994, is the determination of the conditions in which hadrons lose their identity, and sub-nucleonic degrees of freedom begin to play a dominant role. A.M.Baldin proposed a classification of applicability of the notion "elementary particle" on the basis of a variable b_{ik} (square relative four-velocity between the considered objects) [1] introduced by him, in answer to the above problem.

Relativistic nuclear physics which originated at the interface between elementary particle physics and nuclear physics needs mathematically adequate space of variables for investigation of the processes of particle interaction and production. The Lobachevsky space is considered as such space in the present paper.

The investigation of the properties of the 4-velocity space allows to formulate general rules of particle distributions, to develop relativistically invariant methods of analysis of multiparticle production, and imposes a number of intrinsic limitations on the relativistic collision models. Long-term investigations (see, for example, [10-13]) are dedicated to the application of the Lobachevsky geometry in physics.

2. *The relative 4-velocity space. General characteristics of particle distributions*

When studying nuclear reactions the experimentally determined quantities are momentum, angle, type of registered particle, collision energy, reaction cross section, and their derivatives.

The relativistically invariant measurable scalar quantity $\frac{P_i P_j}{m_i m_j}$, where P_i, P_j are 4-momenta of particles i and j, and m_i, m_j are masses of these particles, underlies the determination of invariant mass, rapidity ρ, square relative 4-velocity b_{ik} and invariant cross section.

Rapidity ρ forms a metric space – the Lobachevsky space. Investigation of the properties of this space is necessary for understanding of the relation between the 4-dimensional energy-momentum space and the 3-dimensional Euclidean space of physical experiment.

The invariant variable described through measurable quantities is the particle 4-velocity:

$$U = \{U^0; U\}, \tag{1}$$

where $U^0 = \frac{E}{m}$, $U = \frac{\bar{p}}{m}$. Here E is the total energy, $\bar{p}$ is the 3-dimensional momentum, and m is the mass of particle.

The 3-dimensional Lobachevsky space is connected with the 4-dimensional velocity space by expressing the fourth component of the velocity through the first three:

$$U^0 = \pm\sqrt{1 + U_x^{\,2} + U_y^{\,2} + U_z^{\,2}} \tag{2}$$

The Lobachevsky geometry of the 3-dimensional rapidity space is defined on the upper sheet of the two-sheet hyperboloid (3). The relations between the components of the 4-velocity and rapidity are the following:

$$U^0 = \operatorname{ch}\rho \;;\; |\boldsymbol{U}| = \operatorname{sh}\rho . \tag{3}$$

So, the relation between the particle energy, momentum and mass $E^2 - \bar{p}^2 = m^2$ takes the following form in the rapidity space: $(\operatorname{ch}\rho)^2 - (\operatorname{sh}\rho)^2 = 1$.

The particle rapidity in the laboratory system can be expressed through measurable parameters as follows:

$$\rho = \frac{1}{2}\ln\frac{E + |\bar{p}|}{E - |\bar{p}|} \tag{4}$$

The invariant variable b_{ik} is defined as [1]:

$$b_{ik} = -(U_i - U_k)^2 = 2[(U_i U_k) - 1] = 2\left[\frac{E_i E_k - \bar{p}_i \bar{p}_k}{m_i m_k} - 1\right] \tag{5}$$

The relation between the variable b_{ik} and rapidity is evident:

$$b_{ik} = 2[(U_i U_k) - 1] = 2[\operatorname{ch}\rho_{ik} - 1] \tag{6}$$

Consider typical particle distributions over the variable b_{ik} for the data obtained using the propane bubble chamber illuminated by 4.2 GeV/c p, d, He, C beams at interaction of relativistic nuclei with matter [2]. The experimental data used hereafter were obtained by the collaboration [3,4] for investigations using the 2m propane chamber [5]. Fig.1 shows the normalized distributions of relative 4-velocities of pairs of particles (protons and π-mesons) registered in the reactions C+Ta, He+Ta, d+Ta, p+Ta. It is seen that the character of the distributions for all four reactions is similar. It is also seen that the number of particles with relative 4-velocities close to zero grows steeper than an exponent – in a pole-like way. The pole approximation in the form

$$\frac{d\sigma}{dN} \approx \frac{C}{(b_{ik} + \alpha)^2}, \text{ where } \alpha \approx 0.002 \tag{7}$$

was proposed for the first time for the cross sections of fragmentation processes in [1].
The experimentally observed change of the character of b_{ik} distributions from the pole-like to the exponent and power-like illustrates the classification of elementary particle interactions proposed by A.M.Baldin [6]:

- the region $0 \le b_{ik} \le 10^{-2}$ relates to non-relativistic nuclear physics, where nucleons can be considered as point objects;
- the region $b_{ik} \sim 1$ relates to excitation of internal degrees of freedom of hadrons;
- the region $b_{ik} >> 1$ should, in principle, be described by quantum chromodynamics.

A large number of publications (see, for example, [6,7]) are dedicated to investigation of particle distributions over b_{ik} and analysis of general properties of these distributions, in particular, the correlation depletion principle.

The analysis of b_{ik} distributions carried out by the authors showed that the shape of these distributions is independent of particle multiplicity in an event. Fig.2 shows the distributions of relative 4-velocities of all combinations of pairs of protons and π-mesons in the reaction C+Ta for the selected events arranged into five groups: for multiplicity in the intervals 16-20 particles, 26-30 particles, 36-40 particles, 46-50 particles, and 56-60 particles.

Independence of inclusive cross sections of meson production of multiplicity was noted by the authors in [8]. Independence of such distributions of experimentally observed particle characteristics of multiplicity indicates that the mechanism of independent nucleon-nucleon collisions prevails in multiple particle production. This general property should be taken into account in theoretical and computer models of nucleon-nucleon collisions and in planning of experiments aimed at investigation of exotic states of nuclear matter (quark-gluon plasma and other collective effects).

Consider particle – target relative 4-velocity distributions for protons, registered using the propane bubble chamber in the reactions C+Ta, p+C (Fig.3). The plots demonstrate the existence of transition to internal degrees of freedom of nucleons for b_{ik} close to unity. Note that this effect is the same for different interacting nuclei and different collision energies.

The transition to internal nucleon degrees of freedom can be demonstrated on the basis of the available data on total cross sections of hadron interactions (Fig.4) [9]. Thus, it is in the region b_{ik}~1, both for a pair target-registered proton (Fig.3), and a pair target-projectile (Fig.4), that sub-nucleonic degrees of freedom of nuclear matter are significant, and nucleons are no more point-like.

It should be noted that the variable b_{ik} does not form a metric space, i.e. the relation $b_{12} + b_{13} \geq b_{23}$ is, generally speaking, wrong. It can be illustrated using the experimental data of the collaboration for investigations using the 2m propane chamber. Fig.5 shows the distribution of the value $b_{13} + b_{23} - b_{12}$, where 1, 2 indicate the projectile and target, respectively, and 3 – the registered proton, for the reaction C+Ta. It is seen from Fig.5 that large part of protons tends to be displaced "close" to the projectile and target simultaneously. Rapidity ρ_{ik} has an advantage that, being, along with b_{ik}, the relativistic invariant, it forms, unlike b_{ik}, a metric space – the Lobachevsky space.

Total interaction cross sections of π-mesons, K-mesons, protons as functions of particle-target relative rapidity are shown in Fig.6. The rapidity range between 1 and 4, corresponding to the projectile momentum between 1 and 25 AGeV/c, defines the transition energy region between classical nuclear physics and quantum chromodynamics.

Thus, taking into account non-Euclidean character of the 4-velocity space is important already at relatively low hadron energies (starting from hundreds of MeV), and non-relativistic mechanistic images based on the notions of isotropy, thermalization, etc., have principle limitations related with the selection of a reference system.

3. *Geometric characteristics of particle distributions in the rapidity space*

Analysis of particle properties in terms of rapidity is more complete than consideration of its longitudinal and transversal components. In literature, however, experimental data are often presented as functions of longitudinal rapidity (projection on the reaction axis) and transversal momentum (or transversal mass). Longitudinal rapidity is defined as follows:

$$y = \frac{1}{2}\ln\frac{E + p_{\parallel}}{E - p_{\parallel}}, \tag{8}$$

and transversal mass as:

$$m_T = \sqrt{m^2 + p_T^{\ 2}}\,, \tag{9}$$

where p_T is transversal momentum.

Define transversal rapidity τ:

$$\mathrm{ch}\,\tau = \frac{m_T}{m}, \tag{10}$$

Total rapidity ρ is related with longitudinal and transversal rapidities by the Pythagorean theorem in the Lobachevsky space:

$$\operatorname{ch}\rho = \operatorname{ch} y \cdot \operatorname{ch}\tau . \quad (11)$$

The properties of the space pose certain limitations on the rapidity range (the consequence of metric characteristics of triangles with the sides - relative rapidities):

$$(\rho_{23})^{\max}_{\min} = |\rho_{12} \pm \rho_{13}|;\ (\rho_{13})^{\max}_{\min} = |\rho_{12} \pm \rho_{23}|;\ (\rho_{12})^{\max}_{\min} = |\rho_{23} \pm \rho_{13}| \quad (12)$$

The simplest geometric element is a triangle. Basic relations for a triangle with the vertices – rapidities in the Lobachevsky space (see Fig.7) are given below.

Two theorems can be used to define the relations between sides and angles of the triangle: the law of cosines:

$$ch(\rho_{12}) = ch(\rho_{13}) \cdot ch(\rho_{23}) - sh(\rho_{13}) \cdot sh(\rho_{23}) \cdot \cos(\alpha_3) \quad (13)$$

and the law of sines:

$$\frac{sh(\rho_{12})}{\sin(\alpha_3)} = \frac{sh(\rho_{13})}{\sin(\alpha_2)} = \frac{sh(\rho_{23})}{\sin(\alpha_1)} \quad (14)$$

Note, that the height of the triangle h (see Fig.7) is defined as:

$$sh(h) = sh(\rho_{23}) \cdot \sin(\alpha_2) = sh(\rho_{13}) \cdot \sin(\alpha_1)$$

Thus, h coincides with the transversal rapidity of particle 3, i.e. is a dimensionless relativistically invariant characteristic of transversal motion.

Usually, when analyzing experimental data, the registered particles are classified on the basis of the criterion of interaction «hardness». For example, the «evaporating» protons with momenta less than 300 MeV with respect to the target and «stripping» protons with momenta close to the projectile momentum and laboratory angles less than 4^0, are attributed to the results of "soft" interactions [2]. The analysis in the rapidity space allows to apply a unified relativistically invariant criterion for such classification using particle-target and particle-projectile relative rapidities. For "soft" interactions the upper limit of relative rapidity is ~0.3.

Note that such relativistically invariant analysis is valid for all and with respect to any registered particles, as well as, generally speaking, to all points of the rapidity space, rather than only two points corresponding to the colliding objects. Such approach is especially helpful in analysis of multiple particle production for their separation into groups (pair correlations, clusters, jets, etc.).

In any projective geometry, including the Lobachevsky geometry, the principle of duality is valid, according to which statements formulated in terms of distances between points are equivalent to statements formulated in terms of angles between beams.

Thus, the degree of «hardness» of interactions can be analyzed using the values of angles of the triangles in the rapidity space. Fig.8 shows the ρ_{23} distributions of protons for the selected angle α_3 intervals (see Fig. 7). The regions of ρ_{23} in the vicinity of 0 and 3 corresponding to the target and projectile fragmentation, respectively, can be extracted applying a selection criterion to the angle α_3 between the rapidities ρ_{12} and ρ_{13}.

A triangle is characterized by its defect, which is proportional to the area of the triangle (the constant of proportionality equals to square curvature of the space):

$$defect = \pi - \alpha_1 - \alpha_2 - \alpha_3 \qquad (15)$$

Angular defect is the scalar characteristic of relative position of trios of particles in the rapidity space. Fig.9 shows the distribution of defects of triangles formed by all combinations of protons and all combinations of π-mesons registered at interaction of 10 GeV/c protons with carbon. The defect distribution for proton trios, as seen from the figure, has an exponential shape, i.e. the probability to observe three protons «far» from each other (in terms of rapidity) drops exponentially. It should be noted that the data on protons from the RQMD simulation [14] agrees very well with the experiment. The defect distribution for π-mesons has another shape – these trios form triangles of larger area in the rapidity space, as compared to protons. Note, that the model adequately reproduces inclusive spectra both of protons and π-mesons. The distribution of trios of π-mesons, however, differs noticeably from the experimental data.

Let us illustrate another general property of particle distributions in the rapidity space. Consider combinations of three particles: point 1 – projectile, point 2 – target and point 3 – any registered particle. Fig.10 shows the defects of such triangles as functions of their perimeters calculated for the experimental data on π-meson production in the reaction p(10 GeV/c)+C. For a certain perimeter particles with maximum allowed defects are produced with higher probability. It is consistent with the known feature that cross sections grow towards the phase space boundary, and agrees with the simulation [14].

Let us consider defects as functions of the position of registered particle with respect to colliding nuclei (difference of rapidities, see Fig. 7). Fig.11 shows the plots of defect vs. $(\rho_{12}-\rho_{13})$ for the experimental and simulated protons produced in the reaction p(10 GeV/c)+C. It is seen that the model does not reproduce the peculiarities of the transition region, $\rho \sim 1$ (Fig.11 a,b). The target fragmentation region is shown in more detail in Fig.11 c,d. It is seen that the specific fine structure of proton distribution corresponding to symmetric configurations in the rapidity space is not reproduced by the model. In this region the peculiarities in the cross sections of the registered protons correspond to isosceles triangles, when relative target-projectile and projectile-registered particle rapidities are close. Higher probability of particle production is observed also when relative target-particle and projectile-particle rapidities become close (the symmetric position of the registered particle with respect to the colliding nuclei). Fig.12 illustrates the above idea for π-mesons.

Multiple particle production takes place when their relative velocities approach the light velocity. This suggests that the whole problem of particle production (birth) can be considered from the point of view of the fundamental limitation on experimental observation due to three-dimensional character of Euclidean space. The relationship between four-dimensional Minkowsky space in which energy and momentum conservation laws are formulated and three-dimensional Euclidean space of experiment is realized through the Lobachevsky space.

It is important to stress that, unlike the Euclidean space, the area-to-perimeter ratio for triangles in the Lobachevsky space is limited (see Fig.13). This fundamental difference can hardly be imagined on the basis of mechanistic three-dimensional images which, as a rule, underlie models pretending to describe particle interaction dynamics.

One of the most remarkable variables introduced by N.I.Lobachevsky in his geometry is the angle of parallelism

$$\Pi_L(h) = 2arctg\left(e^{-h}\right) \tag{16}$$

What is the meaning of this parameter and how can it be applied to analysis of experimental data? What rapidity corresponds to the angle of parallelism? Let us consider the height of the triangle *123* (see Fig. 7), where *1* and *2* denote colliding particles and *3* the registered particle. Fig.14 shows the probability to register protons and π-mesons as function of the variable $2\Pi_L - \alpha_3$ (the difference between doubled angle of parallelism and angle at the registered particle in the rapidity space). It is seen from the figure that there exists a strongly pronounced maximum of proton and π-meson production. It is important to note that this maximum corresponds to the angle of parallelism calculated for the value of relative rapidity of colliding objects.

Conclusion

The unified relativistically invariant criteria for particle classification, for example, selection of "stripping" and "evaporating" protons, can be formulated on the basis of spatial rapidities (angles). It is possible to select particles produced by different mechanisms using such characteristics in the Lobachevsky space as defect and perimeter.

The analysis of the data obtained using the propane bubble chamber showed that the general character of particle distributions in the 4-velocity space is similar for different reactions and does not depend on multiplicity.

The comparison of experimental data and model simulations showed that the model [14], while adequately reproducing integral characteristics of particle distributions – inclusive spectra, filling of phase space, is incapable of correct reproduction of two and three- particle correlations.

Taking into account the properties of the Lobachevsky space, in particular, that there is no geometric similarity (unlike the Euclidean geometry), is very important for analysis of experimental data and construction of models of multiple particle production. It is the author's opinion that the Lobachevsky space is the most adequate for description of the processes of particle interaction and production.

Calculation of areas and volumes of arbitrary figures in the Lobachevsky space is a complicated mathematical problem. The property of limited surface area – to - volume ratio is a fundamental property of the Lobachevsly space. It is the author's belief that this property is a key to deeper comprehension of the problem of confinement in strong interactions.

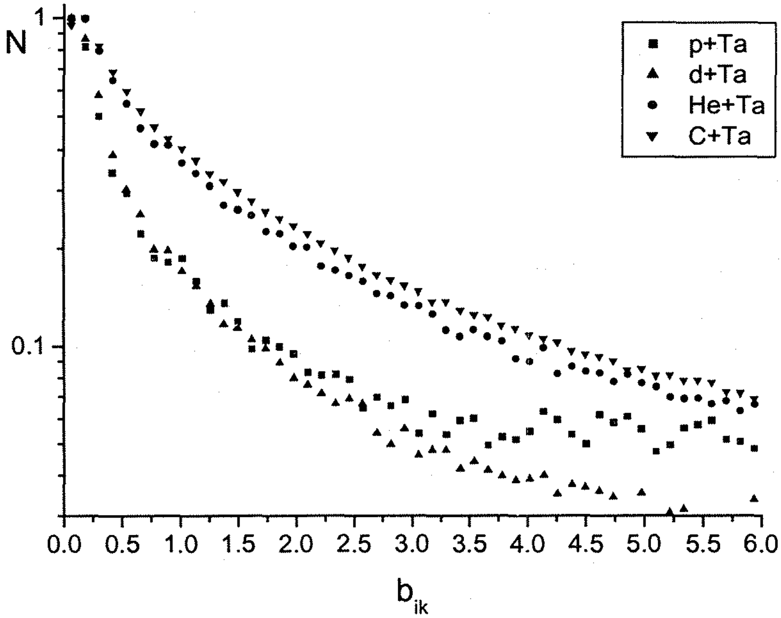

Fig.1. The normalized distributions of relative 4-velocities of the pairs of registered particles (p-p, p-π and π-π) in the reactions C+Ta, He+Ta, d+Ta, p+Ta

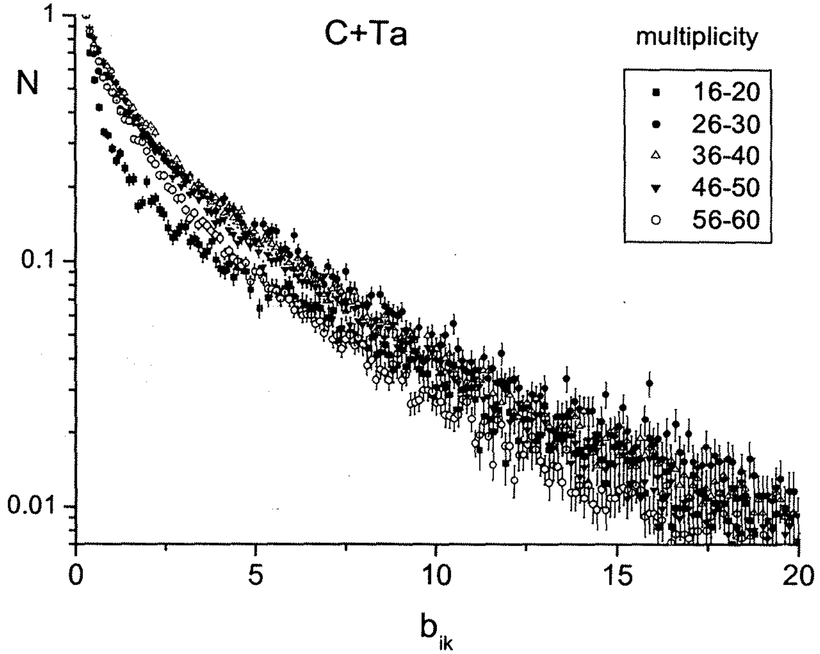

Fig.2. The normalized distributions of relative 4-velocities of the pairs of registered particles (p-p, p-π and π-π) in the reaction C+Ta for five groups of the selected events: with multiplicity in the intervals 16-20 particles, 26-30 particles, 36-40 particles, 46-50 particles, and 56-60 particles

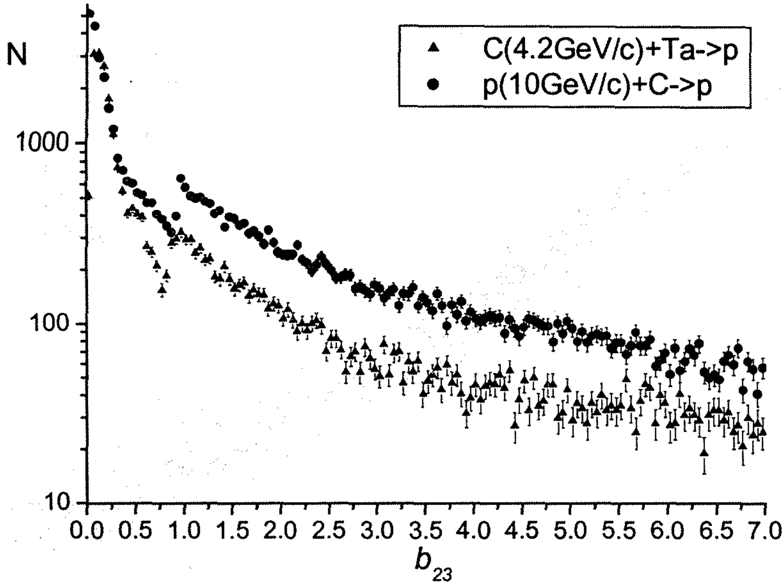

Fig.3. The particle – target relative 4-velocity distributions for the registered protons in the reactions C+Ta, p+C

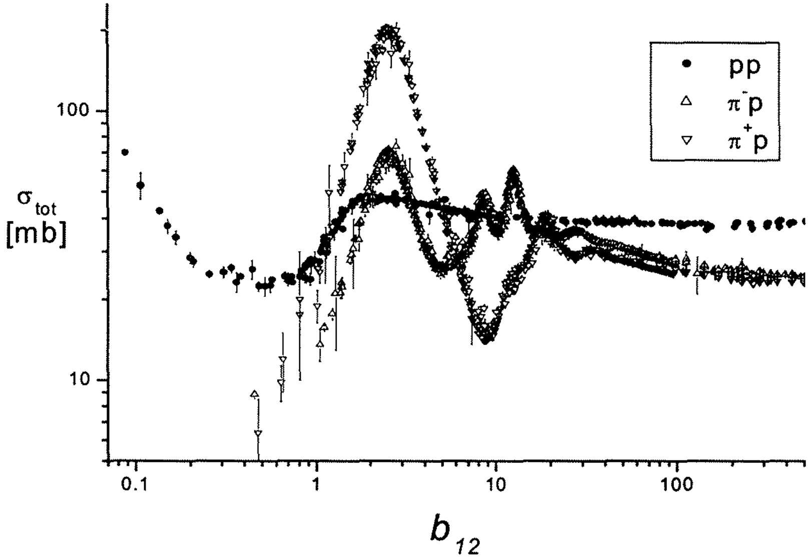

Fig.4. Total cross sections of hadron interactions as functions of relative four-velocity. The data are taken from [9]

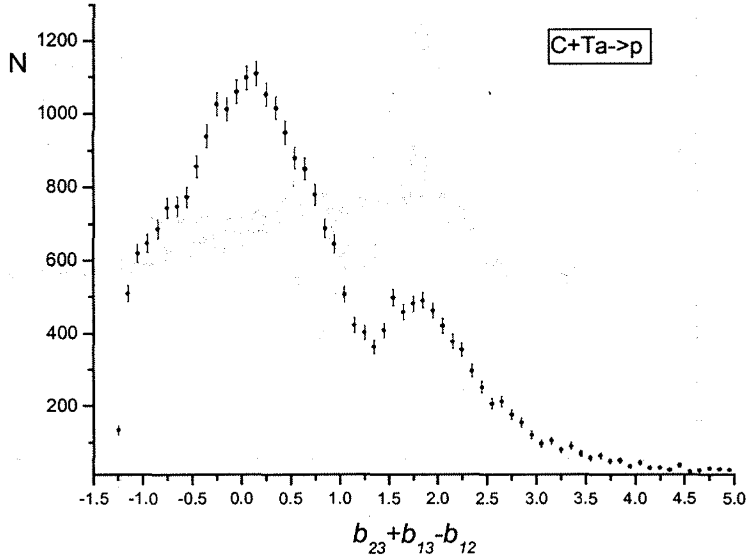

Fig.5. The distribution of relative four-velocities $b_{13}+b_{23}-b_{12}$, where 1, 2 are the projectile and target, respectively, and 3 is the registered proton, for the reaction C+Ta

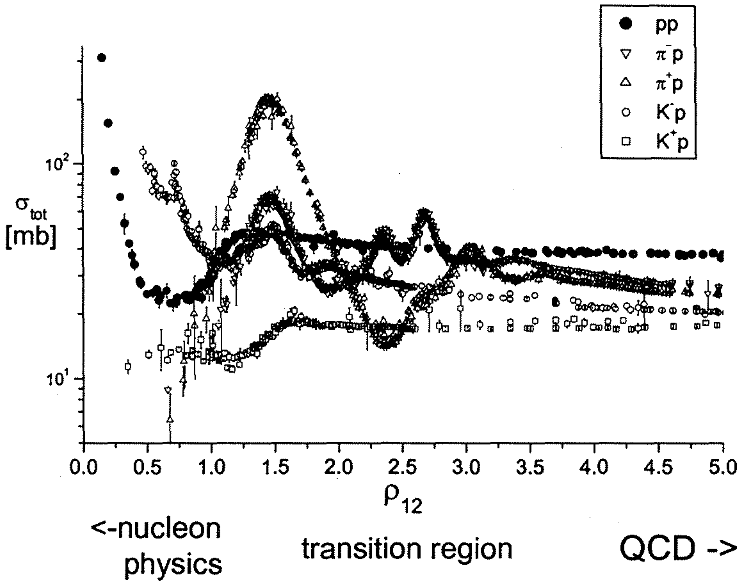

Fig.6. The total interaction cross sections of π-mesons, K-mesons, protons as functions of particle-target relative rapidity. The data are taken from [9]

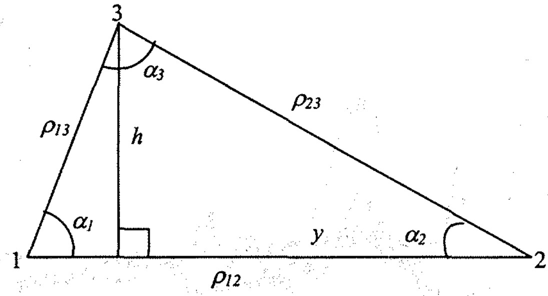

Fig.7. A simplex in the Lobachevsky space. Particles with rapidities ρ_1, ρ_2, ρ_3 correspond to the vertices of the triangle 123. The triangle sides ρ_{12}, ρ_{13}, ρ_{23} are relative rapidities of particles 1, 2, 3.

If 2 is a target at rest in the laboratory system, then the angle α_2 is equal to the laboratory angle of the registered particle

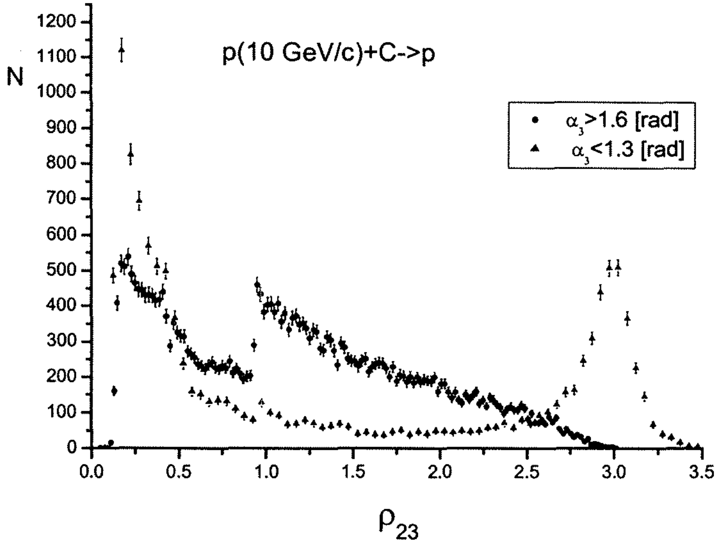

Fig.8. The ρ_{23} distributions of protons for two selected angle α_3 intervals: $\alpha_3>1.6$ rad and $\alpha_3<1.3$ rad in the reaction p(10GeV/c)+C.

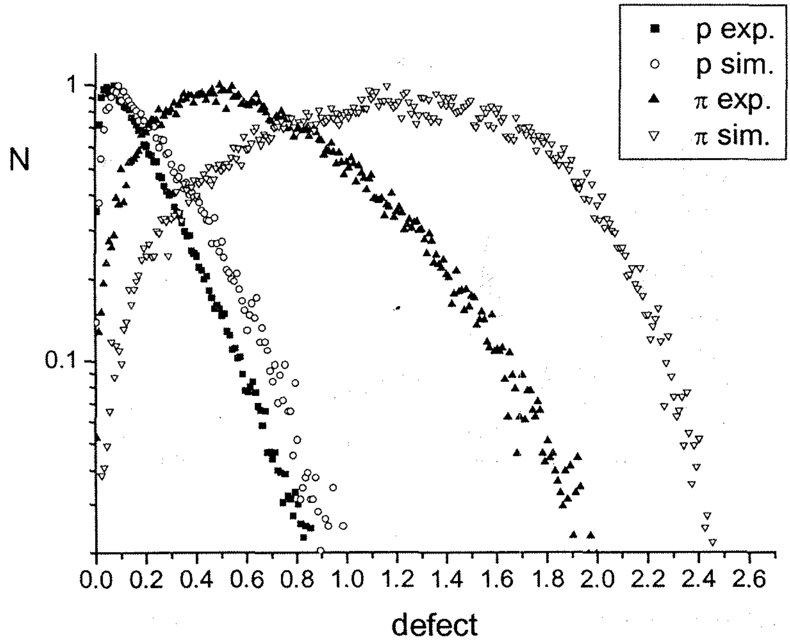

Fig.9. The normalized distributions of defects of triangles formed by all combinations of protons and all combinations of π-mesons registered in the reaction p(10GeV/c)+C

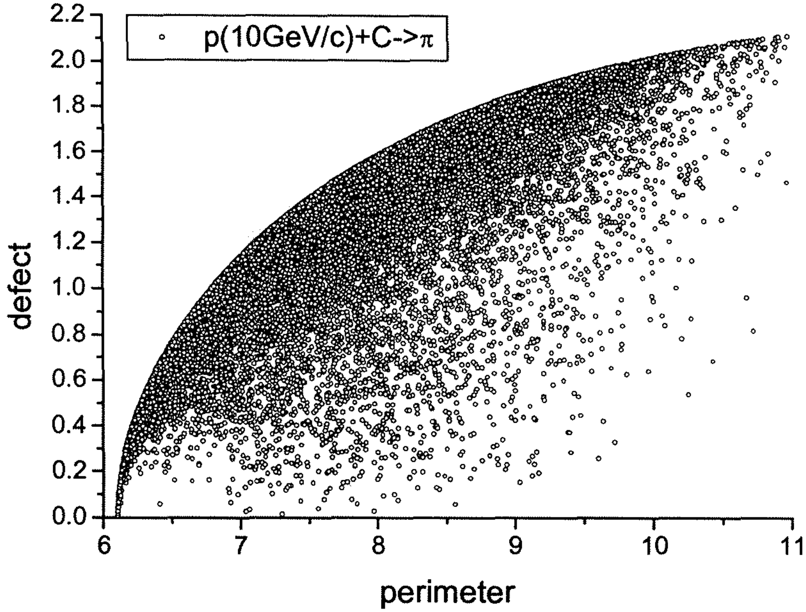

Fig. 10. Defect vs. perimeter for the triangles formed by combinations of three particles: point 1 – projectile, point 2 – target, point 3 – any π-meson registered in the reaction p(10 GeV/c)+C

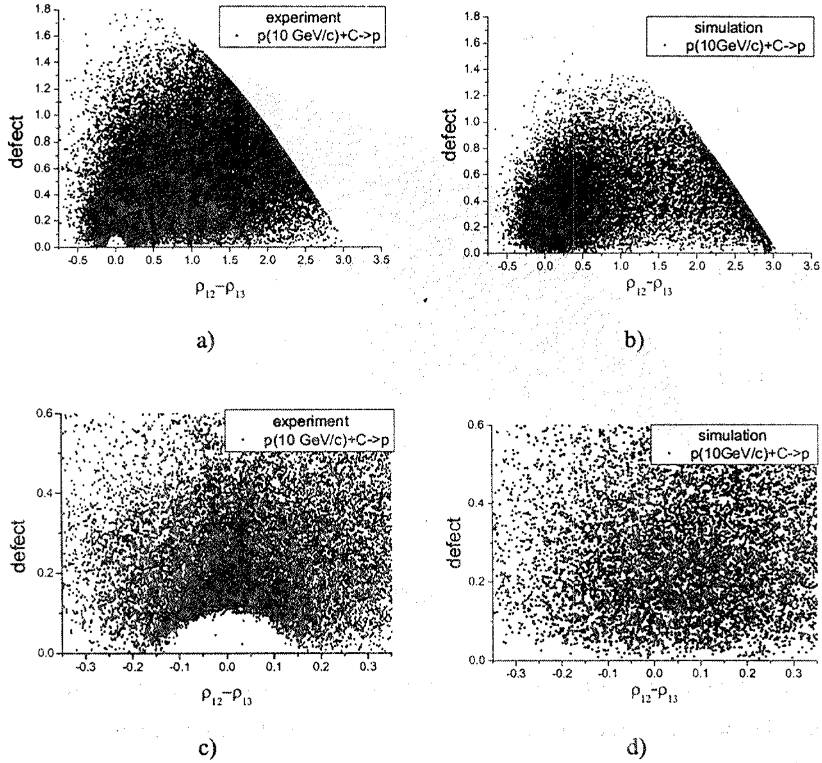

Fig.11. Defect vs. $(\rho_{12}-\rho_{13})$, where ρ_{12} is the projectile-target relative rapidity and ρ_{13} - the projectile-particle relative rapidity for protons produced in the reaction p(10 GeV/c)+C: the experimental (a) and simulated (b) data; and in more detail the region near zero: the experimental (c) and simulated (d) data.

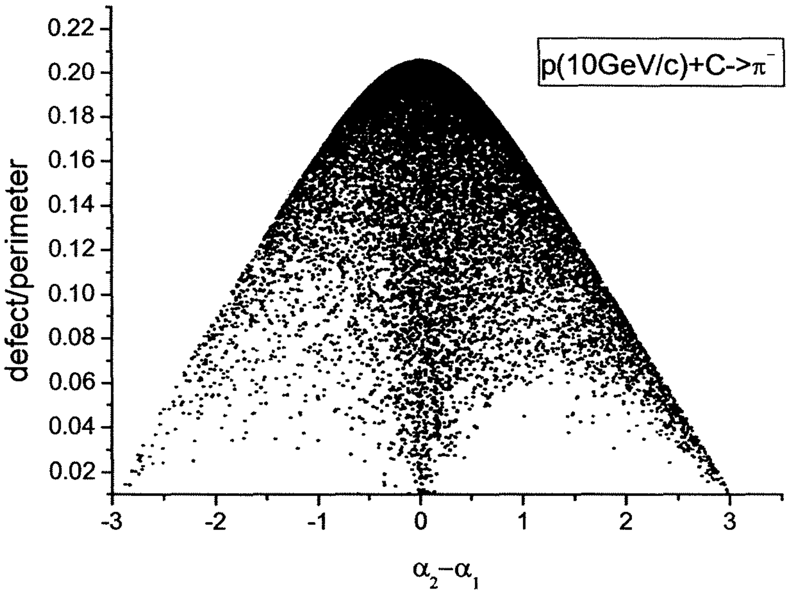

Fig.12. The ratio defect/perimeter vs. the difference of the angles at the target and at the projectile for π-mesons measured in the reaction p(10GeV/c)+C. Prevalence of certain symmetric structures is observed

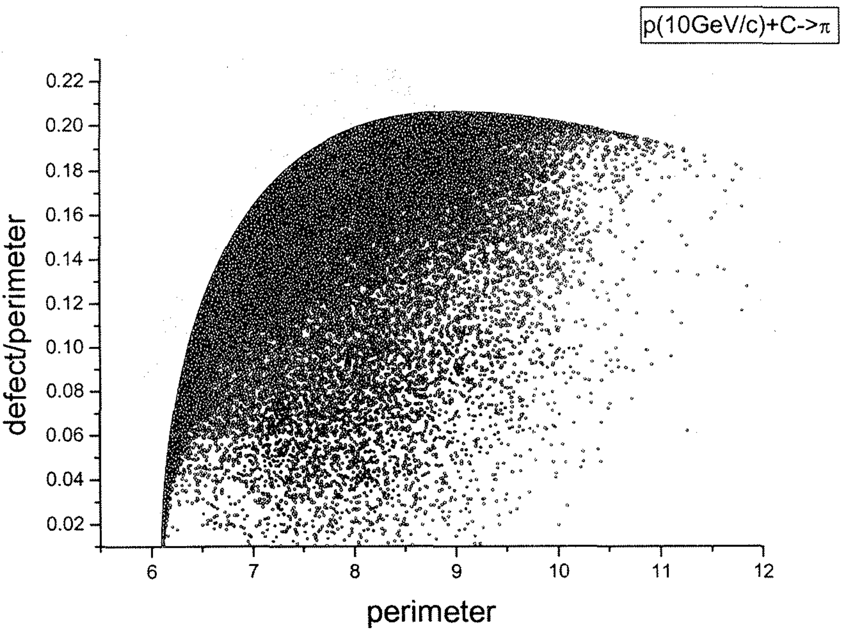

Fig. 13. The defect – to perimeter ratio for triangles *123* (see Fig.7), where *1,2* are interacting particles, *3* is a π-meson registered in the reaction p(10GeV/c)+C

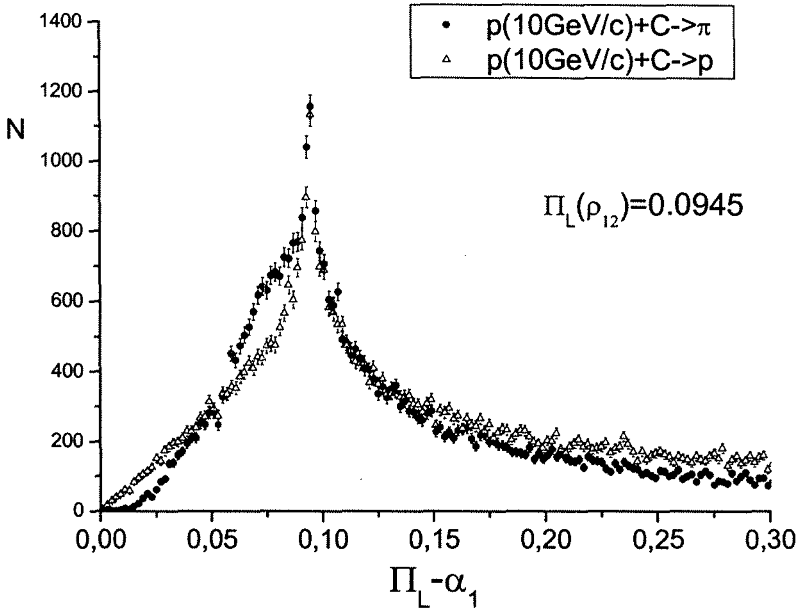

Fig. 14. Probabilities of proton and π-meson production in the reaction p(10GeV/c)+C as functions of the variable $\Pi_L - \alpha_3$, where Π_L is the angle of parallelism calculated for the height h of the triangle *123*, and α_3 is the angle at the registered particle (see Fig. 7)

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