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Amontova, Sofia

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UNIVERSITÉ DE GENÈVE
Séction de mathématiques

FACULTÉ DES SCIENCES
Dr. Michelle BUCHER

**On Invariants for Representations
of Complex Hyperbolic Lattices**

THÈSE

Présentée à la Faculté des Sciences de l'Université de Genève
pour obtenir le grade de Docteur ès Sciences, mention Mathématiques

par

Sofia AMONTOVA

de

Tsukuba (Japon)

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«On Invariants for Representations of Complex Hyperbolic Lattices»

La Faculté des sciences, sur le préavis de

Madame M. BUCHER, docteure et directrice de thèse
Section de mathématiques

Monsieur A. SZENES, professeur ordinaire
Section de mathématiques

Monsieur T. HARTNICK, professeur
Institut for Algebra und Geometrie, Fakultät für Mathematik, Karlsruher Institut für Technologie
Karlsruhe, Allemagne

Monsieur M. DERAUX, docteur
Institut Fourier, Université Grenoble Alpes, Grenoble, France

autorise l'impression de la présente thèse, sans exprimer d'opinion sur les propositions qui y sont énoncées.

Genève, le 23 janvier 2026

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La Doyenne

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Résumé

Cette thèse contribue à la compréhension des composantes connexes des variétés de caractères pour les représentations, par l'étude des invariants numériques associés, qui présentent un intérêt en eux-mêmes, en utilisant la cohomologie bornée.

Nous nous concentrons sur deux invariants classiques : le *nombre d'Euler*, associé aux représentations des réseaux hyperboliques complexes sans torsion dans $\mathrm{PU}(n, 1)$, et l'*invariant de Toledo*, défini plus généralement pour les représentations dans des groupes de Lie hermitiens simples non compacts à centre fini. Ces deux invariants admettent des interprétations de géométrie différentielle, algèbro-géométriques et cohomologiques. Ils constituent des outils puissants pour établir des résultats de rigidité et jouent un rôle important dans la théorie de Teichmüller supérieure.

L'objectif central de ce travail est d'analyser la nature des valeurs prises par le nombre d'Euler et par l'invariant de Toledo. Pour les réseaux de dimension supérieure, nous montrons que ces deux invariants prennent des valeurs entières (à un facteur de normalisation près). De plus, en démontrant leur continuité, nous concluons qu'ils sont constants sur les composantes connexes des variétés de caractères correspondantes.

La cohomologie bornée joue un rôle essentiel dans notre approche, car elle fournit un cadre permettant d'étendre les définitions classiques des invariants des réseaux uniformes au cas *non uniforme*. Parmi les principales contributions de cette thèse figure l'identification d'une formulation appropriée de l'invariant de Toledo, compris comme le *degré* re-normalisé, et placé dans un cadre unifié aux côtés de la définition cohomologique bornée du nombre d'Euler, introduite et étudiée auparavant comme le *volume* re-normalisé dans le cas hyperbolique réel par Bucher, Burger et Iozzi [BBI13; BBI21]. Cette formulation unifiée permet ensuite d'établir nos résultats d'intégralité, qui résultent de l'interaction entre différentes théories cohomologiques et des constructions faisant intervenir plusieurs variantes des classes de Chern.

Enfin, nous introduisons de nouveaux invariants associés à des représentations mixtes, inspirés par l'étude des volumes de réseaux dans le contexte de la géométrie d'Anti-de Sitter et de ses généralisations par Tholozan [Tho16; Tho18]. Pour ces invariants également, nous montrons que le phénomène d'intégralité persiste.

Summary

This thesis contributes to the understanding of connected components of character varieties for representations through the study of associated numerical invariants, which are of interest in their own right, using bounded cohomology.

We focus on two classical invariants: the *Euler number*, associated to representations of torsion-free complex hyperbolic lattices into $\mathrm{PU}(n, 1)$, and the *Toledo invariant*, defined more generally for representations into simple non-compact Hermitian Lie groups with finite center. Both invariants admit differential-geometric, algebro-geometric, and cohomological interpretations. They serve as powerful tools in establishing rigidity results and play an important role in higher Teichmüller theory.

The central objective of this work is to analyze the nature of the values taken by the Euler number and the Toledo invariant. For higher-dimensional lattices, we show that both invariants take integral values (up to rescaling). Furthermore, by proving their continuity, we conclude that they are constant on connected components of the corresponding character varieties.

Bounded cohomology plays a crucial role in our approach, as it provides a framework that extends the classical definitions of the invariants from uniform to *non-uniform* lattices. Among the key contributions of this thesis is the identification of a suitable formulation for the Toledo invariant, understood as the rescaled *degree*, and placed within a unified setting alongside the bounded-cohomological definition of the Euler number, which was previously introduced and studied as the rescaled *volume* in the real hyperbolic case by Bucher, Burger and Iozzi [BBI13; BBI21]. This unified formulation, in turn, allows to establish our integrality results that emerge from an interplay between different cohomology theories and constructions involving several variants of Chern classes.

Finally, we introduce new invariants associated with mixed representations, inspired by the study of volumes for lattices in the context of Anti-de-Sitter geometry and its generalizations by Tholozan [Tho16; Tho18]. For these invariants as well, we show that the integrality phenomenon persists.

Introduction

Let Γ be a finitely generated group and G a Lie group. A *representation* of Γ into G is a group homomorphism $\rho : \Gamma \rightarrow G$. A natural question to ask is the following:

How can such objects be classified depending on the data (Γ, G) ?

An influential approach to this question can be considered to be Goldman's classification [Gol80] who studied representations of the fundamental group $\pi_1(S)$ of a closed oriented real hyperbolic surface S of genus $g \geq 2$, into its ambient Lie group:

$$\rho : \pi_1(S) \longrightarrow \mathrm{PSL}_2(\mathbb{R}).$$

Goldman's approach relies on the existence of a characteristic number associated to each such representation, namely the *Euler number*, which provides a numerical invariant. It takes integral values, that are bounded in absolute value by the Euler characteristic $|\chi(S)|$ due to the Milnor-Wood inequality [Mil58; Woo71]. Moreover, the Euler number is continuous on the space of representations $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{R}))$ equipped with natural topology, is invariant under conjugation, therefore descending to the orbit space with the quotient topology

$$\mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{R})) / \mathrm{PSL}_2(\mathbb{R}).$$

In particular, its continuity then also holds on the Hausdorffification¹ of this quotient, referred to as the (*Hausdorff*) *character variety* and denoted by

$$\mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{R})) // \mathrm{PSL}_2(\mathbb{R}).$$

Consequently, the Euler number is constant on the connected components of this character variety, thereby labeling unions of connected component with an integer in $\mathbb{Z} \cap [\chi(S), -\chi(S)]$ (due to [Gol88] the Euler number labels in fact *each* connected component). The extremal connected components consist precisely of all discrete and faithful representations, and in fact, they can be identified with two copies Teichmüller space of S , the parameter space of marked hyperbolic structures on S up to isotopy: one corresponding to S with the chosen orientation on S , and the other with its reversed orientation. Notably, the *classical Teichmüller theory* is a rich and well-established field in its own right, due to its links to complex analysis, geometric topology and dynamical systems (cf. [Hub06]).

¹A priori the quotient $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{R})) / \mathrm{PSL}_2(\mathbb{R})$ is not Hausdorff, so its connected components need not be closed, and their closures may intersect.

As such, Goldman's work, among others, influenced the modern field of *higher Teichmüller theory*, which studies connected components of character varieties consisting of discrete and faithful representations, where the target groups are higher-dimensional or higher-rank Lie groups, and S may be generalized to a broader class of manifolds (cf. [Wie18]). In these settings, it is of interest to consider suitable generalizations of the Euler number as labeling invariants for connected components of such character varieties. Subsequently, individual connected components can then possibly be studied for algebraic and geometric properties. In particular, one seeks to understand the nature of the values taken by such invariants, that are, under certain circumstances, governed by the generalization of S one considers. For instance, in case S is extended to higher-dimensional *closed* manifolds, often the appropriate component-wise invariants are characteristic numbers of representations, which take integral values [Gol82; Gro82; Sul76]. In case S is a complete *non-compact* finite-volume real hyperbolic surface, the corresponding characteristic numbers of representations vanish and therefore such classical constructions do not yield meaningful invariants. This was circumvented by Burger, Iozzi and Wienhard [BIW10] who introduced a bounded-cohomological generalization of Goldman's Euler number, known as the *Toledo invariant* - a construction that relies on the vanishing of bounded cohomology for the amenable groups modelling the surface's cusps. Here in contrast, integrality fails to hold, since the Toledo invariant associated with representations $\pi_1(S) \rightarrow G$ into a simple non-compact Hermitian Lie group with finite center, realises its full range as a closed interval continuously. Replacing now S by higher dimensional non-compact analogues, the Toledo invariant can be generalized in different ways depending on the perspective one takes:

In Goldman's classification above, regarding

$$\mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}_{\mathbb{R}}^2)$$

as the group of orientation-preserving isometries of the real hyperbolic plane, a natural generalization is to replace $\pi_1(S)$ with a non-uniform torsion-free lattice Γ in $\mathrm{SO}^+(n, 1) \cong \mathrm{Isom}^+(\mathbb{H}_{\mathbb{R}}^n)$ for $n > 2$. By extending the bounded-cohomological definition [BIW10], Bucher, Burger and Iozzi [BBI13; BBI21] associated the *volume* $\mathrm{Vol}_{\mathrm{Dup}}(\rho)$ with any representation $\rho : \Gamma \rightarrow \mathrm{SO}^+(n, 1)$ that relies on the existence of a bounded class represented by the Dupont volume cocycle. The authors proved in [BBI21] integrality of $\mathrm{Vol}_{\mathrm{Dup}}(\rho)$ for even integers $n \geq 4$, by viewing it as a rescaled version of the *Euler number* $E(\rho)$ - an invariant originating from the top Euler class in the same framework - through the relation

$$E(\rho) = (-1)^{n/2} \frac{2}{\mathrm{Vol}(S^n)} \cdot \mathrm{Vol}_{\mathrm{Dup}}(\rho),$$

where $\mathrm{Vol}(S^n)$ denotes the riemannian volume of the n -sphere S^n of constant curvature 1. For $n = 3$, integrality fails to hold due to Dehn fillings of cusps. Using different methods, [KK16] showed that $\mathrm{Vol}_{\mathrm{Dup}}(\cdot)$ is constant on connected components of the corresponding character variety for all integers $n \geq 4$. We remark that there are several equivalent definitions of the volume in this real hyperbolic setting [BBI13; KK14; Dun99; FK06], which have been used to prove volume rigidity recovering Mostow-Prasad rigidity. All these definitions have been shown to be equivalent in more general settings in [Kim16].

On the other hand, one may regard

$$\mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{Isom}^\circ(\mathbb{H}_{\mathbb{C}}^1)$$

as the group of holomorphic isometries of the complex hyperbolic 1-space, and consider instead of $\pi_1(S)$ a non-uniform torsion-free lattice Γ in $\mathrm{PU}(n, 1) \cong \mathrm{Isom}^\circ(\mathbb{H}_{\mathbb{C}}^n)$ for $n > 1$. Like in the real-hyperbolic case, the Euler number $E(\rho)$ can be associated to any representation $\rho : \Gamma \rightarrow \mathrm{PU}(n, 1)$. The values of $E(\cdot)$ in this complex hyperbolic setting have not yet been studied. This naturally leads to the following open question:

Open question (A):

What is the nature of the values of the Euler number $E(\rho)$ for $\rho : \Gamma \rightarrow \mathrm{PU}(n, 1)$ with $n \geq 2$?

Apart from that, there is another well-known extension, still called the *Toledo invariant*, that can be considered for representations of complex hyperbolic lattices into a simple non-compact Hermitian Lie group with finite center G , with $\mathrm{PU}(n, 1)$ as a prototype. The invariant is classically considered in terms of bounded transfer-maps [Mon01, Proposition 8.6.2, pp. 106–107], [Poz15, p. 1299] and equivalently, in terms of L_2 -cohomology [BI07b]. For uniform lattices, this definition reduces to the differential-geometric Toledo invariant, which has been important in the context of rigidity statements and study of maximal representations, i.e., representations that realise extremal values of the Toledo invariant [Tol89; Cor88; BGG03; BGG06; KM08b; KM17]. For non-uniform lattices, these statements have been extended in [BI07b; BIW10; KM08a]. Further, using Chern-Weil theory, Burger and Iozzi [BI07a, p. 4] have established the Toledo invariant's constancy on connected components of the corresponding character variety. However, the indexation of the components remains open:

Open question (B):

What is the nature of the values for the Toledo invariant for $\rho : \Gamma \rightarrow G$ with $n \geq 2$?

The central objective of this thesis is to provide answers towards open questions (A) and (B). We first note that Milnor-Wood type inequalities hold in both cases, providing lower and upper bounds for the volume [KK14] and Toledo invariant [BI07b].

To understand the nature of values for $E(\cdot)$, we will see that certain methods from [BIW10] and [BBI21] provide a guiding framework for our setting of higher dimensional complex hyperbolic lattices. With this in mind, for the transfer-theoretic Toledo invariant $t(\cdot)$ from [Poz15, p. 1299], [BI07a], we introduce an alternative definition, denoted by $\mathrm{Tol}(\cdot)$, and place it in a unified setting alongside $E(\cdot)$. In particular, relying on some of the machinery developed in [BBI13, §3], we prove the following:

Main Lemma. *Let $n \geq 1$, let $\Gamma < \mathrm{PU}(n, 1)$ be a torsion-free lattice and $\rho : \Gamma \rightarrow G$ an arbitrary representation into a simple non-compact Hermitian Lie group with finite center. Then it holds that*

$$\mathrm{Tol}(\rho) = n! \cdot \mathrm{Vol}(\Gamma \backslash \mathbb{H}_{\mathbb{C}}^n) \cdot t(\rho).$$

We emphasize that the invariants under consideration might appear under different terminologies and normalizations in the literature. They can be considered as arising either from

differential forms or Chern classes, differing by a constant that depends on the target of the representation. With this in mind, the Euler number $E(\cdot)$ can be rescaled to a volume-type invariant $\text{Vol}(\cdot)$, while we distinguish the Toledo invariant $\text{Tol}(\cdot)$ from its normalization - the *degree* $\text{Deg}(\cdot)$. Importantly, we emphasize that our notion of $\text{Vol}(\cdot)$ relies on a choice of bounded volume class distinct from the canonical one represented by the Dupont volume cocycle, which gives rise to the classical volume $\text{Vol}_{\text{Dup}}(\cdot)$. However, conjecturally both invariants should coincide. Our notion of degree $\text{Deg}(\cdot)$ recovers, up to sign, the classical algebro-geometric degree for representations of closed manifolds.

We now state our main theorem, keeping the chosen normalizations in view:

Main Theorem. *Let $n \geq 2$ and let $\Gamma < \text{PU}(n, 1)$ be a non-uniform torsion-free lattice.*

(A) *For $n \geq 3$ and any representation $\rho : \Gamma \rightarrow \text{PU}(n, 1)$, there exists a constant $\text{Cst}(\Gamma) \in \mathbb{N}^*$ such that*

$$E(\rho) = h_n \cdot \text{Vol}(\rho) \in \frac{1}{\text{Cst}(\Gamma)} \cdot \mathbb{Z}.$$

where $h_n = (-1)^n \frac{\chi(\mathbb{C}\mathbb{P}^n)}{\text{Vol}(\mathbb{C}\mathbb{P}^n)}$ is the Hirzebruch proportionality constant.

In case of $n = 2$, the same statement holds with the right-hand side replaced by $\frac{1}{2 \cdot \text{Cst}(\Gamma)} \cdot \mathbb{Z}$.

(B) *For any representation $\rho : \Gamma \rightarrow G$ into a simple non-compact Hermitian Lie group with finite center, there exists a constant $\text{Cst}(n) \in \mathbb{N}^*$, such that*

$$\text{Tol}(\rho) = \bar{\lambda} \cdot \lambda^{n-1} \cdot \text{Deg}(\rho) \in \frac{\bar{\lambda} \cdot \lambda^{n-1}}{\text{Cst}(n)} \cdot \mathbb{Z}$$

where $\lambda = -\frac{4\pi}{n+1}$ and $\bar{\lambda} = \bar{\lambda}(G) \in \mathbb{R} \setminus \{0\}$ is a universal constant depending on the group G .

Moreover, if all cusps of the manifold modelled by Γ are nilmanifolds, then $\text{Cst}(n) = 1$.

We note that part (A) of our main theorem is a similar statement to [BBI21, Theorem 1.1] for $\text{Vol}_{\text{Dup}}(\cdot)$ in the real hyperbolic setting, where the constant $\text{Cst}(\Gamma)$ is determined by the tori covers of each cusp associated to Γ , which are guaranteed to be finite by the Bieberbach theorem [Bie11; Bie12].

In our complex hyperbolic setting, the generalized Bieberbach theorem [Aus60] ensures that the cusps associated with Γ , which are infranilmanifolds, admit finite coverings by nilmanifolds. In part (A), additional finite covers must be taken to obtain the constant $\text{Cst}(\Gamma)$, whereas for part (B), one uses the fact that all nilmanifold covers are of bounded degree [BK81, p. 10], to obtain the constant $\text{Cst}(n)$. In fact, if Γ is the fundamental group of a one-cusped complex hyperbolic surface, then for any representation $\rho : \Gamma \rightarrow G$, one has $\text{Deg}(\rho) \in \mathbb{Z}$, since the cusp is necessarily a nilmanifold due to [Kam07]. Remarkably, a geometrically explicit example of such surfaces has only been constructed recently by Deraux and Stover in [DS24], with arbitrary large Euler characteristics that takes the values $-6d$ for odd $d \geq 1$.

For part (A) of our theorem, we expect that in the case of $n = 2$ the additional $1/2$ factor on the right-hand side can be omitted.

Furthermore, we prove that all of our invariants are continuous in the spirit of the machinery introduced in [BBI13, §3] and [BBI21, Appendix A]. This yields the following corollary:

Main Corollary. *Let $n \geq 2$, let $\Gamma < \mathrm{PU}(n, 1)$ be a non-uniform torsion-free lattice and G a simple non-compact Hermitian Lie group with finite center. Then it holds that*

- *the invariant $E(\rho)$ and $\mathrm{Vol}(\rho)$ are constant on connected components of the character variety*

$$\mathrm{Hom}(\Gamma, \mathrm{PU}(n, 1)) // \mathrm{PU}(n, 1).$$

- *the invariants $\mathrm{Tol}(\rho)$ and $\mathrm{Deg}(\rho)$ are constant on connected components of the character variety*

$$\mathrm{Hom}(\Gamma, G) // G.$$

In particular, the second part of the corollary recovers the constancy of connected components proved in [BI07a, p. 4].

Last but not least, we demonstrate the potential of our methods developed in **Main Theorem** to address related questions. In particular, we draw inspiration from constructions of manifolds motivated by Anti-de-Sitter geometry [KR85; Kli96; Kas08; Kob93; Gué+17] - essentially quotients by suitable products of representations of a torsion-free lattice of interest - and from the integrality questions on their volumes [Tho16; Tho18]. To this end, we consider a real-valued invariant associated with mixed representations of complex hyperbolic lattices, placed in a unified setting alongside the invariants $E(\cdot)$, $\mathrm{Vol}(\cdot)$ and $\mathrm{Tol}(\cdot)$, $\mathrm{Deg}(\cdot)$, and prove the following statement:

Complementary Main Theorem. *Let $n \geq 2$, let $\Gamma < \mathrm{PU}(n, 1)$ be a non-uniform torsion-free lattice. Let $\rho_i : \Gamma \rightarrow G$ be representations with target a simple non-compact Hermitian Lie group with finite center for $1 \leq i \leq n$. Then there exists a certain real-valued invariant $I_{\rho_1 \times \dots \times \rho_n}$ associated to the representation*

$$\rho_1 \times \dots \times \rho_n : \Gamma \longrightarrow G^n$$

that satisfies the following properties:

- *If all representations restrict to $\rho_i : \Gamma \rightarrow \mathrm{PU}(n, 1)$ for $1 \leq i \leq n$, then there exists a constant $\mathrm{Cst}'(\Gamma) \in \mathbb{N}_{>0}$ such that*

$$I_{\rho_1 \times \dots \times \rho_n} \in \frac{1}{\mathrm{Cst}'(\Gamma)} \cdot \mathbb{Z}.$$

- *If among $\rho_i : \Gamma \rightarrow G$ for $1 \leq i \leq n$, at least one representation is given by the canonical inclusion $\Gamma \hookrightarrow \mathrm{PU}(n, 1)$, then there exists a constant $\mathrm{Cst}(n) \in \mathbb{N}_{>0}$ such that*

$$I_{\rho_1 \times \dots \times \rho_n} \in \frac{1}{\mathrm{Cst}(n)} \cdot \mathbb{Z}.$$

Moreover, if all cusps of the manifold modelled by Γ are nilmanifolds, then $\mathrm{Cst}(n) = 1$.

In particular, the volume invariant considered in [Tho16, Theorem 4] is an example of the invariant $I_{\rho_1 \times \dots \times \rho_n}$ in the above theorem.

This thesis is structured as follows: We begin **Chapter I** with a brief introduction to the basic language needed for the source and target groups of representations of interest. We then formalize the classification problem for representations and (Hausdorff) character varieties in

this setting. Building on this, we motivate higher Teichmüller theory, as an instance for an active direction of research for studying of certain connected components. Next, we give a short preparation on smooth manifolds, introduce fundamental notions of Riemannian manifolds and present Kähler manifolds as structurally rich examples. We then turn to non-positively curved Riemannian manifolds, introducing the notion of boundary, the classification of isometries and elementary groups. This culminates in the thin-thick decomposition, which in particular allows to formalize the notion of cusps of complete non-compact finite-volume Riemannian manifolds of negative curvature. Subsequently, we discuss amenable groups from a geometrical viewpoint shortly. We then switch gears, and introduce symmetric spaces, outline their duality to Lie groups and include a brief discussion on maximal compact and minimal parabolic subgroups. This allows to give important properties of Hermitian symmetric spaces, that lie at the intersection of symmetric spaces and Kähler geometry. The chapter concludes with an important example: the complex hyperbolic space. It can be studied in terms of the projective model, ball model or the Siegel domain. The latter model is particularly useful for giving a group-theoretic characterization for cusps of complete non-compact finite-volume complex hyperbolic manifolds; these are modelled by certain uniform lattices in the group of Heisenberg similarities, namely the stabilizer group of points on the boundary of the complex hyperbolic space, up to conjugation.

Chapter II gives an overview of the bounded and unbounded cohomology theories with trivial coefficients in real, integral and circle group coefficient groups. The focus is on both groups and manifolds. Alongside essential features, such as comparison maps, cup products, long exact sequences, straightening maps and Poincaré dualities, we highlight relationships between different cohomology theories through classical isomorphism theorems. In the final section, we define the Kähler and volume classes, together with their bounded counterparts relying on work by [DT87],[CØ03],[BM99] and [HO12].

In **Chapter III**, we briefly discuss principal bundles, then introduce Chern classes both axiomatically but also from a universal perspective. This provides the foundation for defining group-cohomological Chern classes - integral and real - along with their bounded counterparts. The Hirzebruch proportionality principle allows to relate the first real Chern class to the Kähler class and the top real Chern class to the volume class. Next, we compute the cohomologies for $PU(n, 1)$ and its point-stabilizer subgroups. Our main focus is on the cohomological properties of the Heisenberg similarities and its subgroups, and in particular on the role of Chern classes in this context. This technical discussion provides pivotal steps for our main theorems presented in the following chapter.

Chapter IV is the core of the thesis. We introduce the invariants Euler $E(\cdot)$ and degree $\text{Deg}(\cdot)$, defined in terms of the group-cohomological Chern classes. We first present the case of representations of *uniform* torsion-free complex hyperbolic lattices, which motivates the necessity of bounded cohomology for the *non-uniform* case. The definition exploits the fact that the source group models a non-compact manifold with *amenable* cusps and that, in particular, the fact that bounded cohomology in positive degrees vanishes on such cusps. We place the invariants in a unified framework and adopt an elegant method, namely the *congruence relation*, from [BIW10, §8.1] and [BBI21, §4.2], which reduces the integrality question to establishing whether certain vanishing results hold. First, we prove such a vanishing result for the Euler number $E(\cdot)$

of representations whose target is $\mathrm{PU}(n, 1)$. Following the first step of the strategy in [BBI21, §5], we show that it suffices to assume that the target group is one of the point-stabilizer subgroups - either the maximal compact or the minimal parabolic subgroup - of $\mathrm{PU}(n, 1)$. In both cases, however, different arguments are needed to reach the desired conclusion, in contrast to the real hyperbolic setting. In [BBI21], the key idea is that cusps admit tori as finite covers, and the vanishing result is obtained by explicitly evaluating cup products of orientation and rotation cocycles on the torus' fundamental class. In the complex hyperbolic setting, especially in the case where the target of our representation is the maximal compact subgroup $\mathrm{U}(n)$ and $n \geq 3$, our argument essentially relies on the representation being conjugated into an abelian group (a maximal torus) and on the cusps admitting finite covers by certain nilmanifolds whose fundamental groups allow us to perform an abelianization trick to obtain the vanishing result. Now, for the degree $\mathrm{Deg}(\cdot)$ the corresponding vanishing result goes through in a more general setting, namely for representations with the target group being any simple non-compact Hermitian Lie group with finite center. Our proof has a fundamentally different strategy compared to the one employed for $\mathrm{E}(\cdot)$ and is particularly concise. Both integrality statements are stated in detailed forms in [Theorem A](#) and [Theorem B](#). One immediately obtains [Corollary C](#) - the corresponding integrality result for our volume-type invariant $\mathrm{Vol}(\cdot)$ and our Toledo invariant $\mathrm{Tol}(\cdot)$, which completes the proof of our [Main Theorem](#). Motivated by lattice constructions and the corresponding volumes arising from Anti-de-Sitter geometry and its generalizations, we introduce new invariants associated with certain mixed representations of complex hyperbolic lattices. In this setting, we immediately obtain [Corollary D](#) and subsequently establish [Theorem E](#), which together yield the proof [Complementary Main Theorem](#). The technical arguments of this chapter rely substantially on the computations developed in the final section of Chapter III.

Last but not least, [Chapter V](#) deals with transfer-theoretic versions of the invariants $\mathrm{E}(\cdot)$, $\mathrm{Vol}(\cdot)$ and $\mathrm{Tol}(\cdot)$, $\mathrm{Deg}(\cdot)$, that we have considered up to that point. After having introduced bounded transfer maps that we use to define the invariants $\mathrm{t}(\cdot)$, $\mathrm{d}(\cdot)$ and $\mathrm{e}(\cdot)$, we translate the machinery established in [BBI13, §3] for the real hyperbolic setting to our complex hyperbolic one (we note that in contrast to [BBI13] cohomologies with twisted coefficients are not needed). We rely on this machinery to relate all of our the invariants from Chapter IV to the transfer-theoretic ones: $\mathrm{E}(\cdot)$, $\mathrm{Vol}(\cdot)$ to $\mathrm{e}(\cdot)$; $\mathrm{Deg}(\cdot)$ to $\mathrm{d}(\cdot)$ and $\mathrm{Tol}(\cdot)$ to $\mathrm{t}(\cdot)$. In particular, we note that an additional simple commutativity property of bounded transfers and cup products is needed to prove our closed formula between $\mathrm{Tol}(\cdot)$ and $\mathrm{t}(\cdot)$ and complete the [Main Lemma](#). We then discuss values values for constants λ and $\bar{\lambda}$ appearing in [Theorem B](#). Finally, the transfer-theoretic perspective enables us to prove the continuity of all invariants under consideration. While we adopt a strategy from [BBI21, Appendix] for our initial steps, our argument ultimately departs from it, employing a different approach to reach the conclusion (in the real hyperbolic case, the argument exploits the absence of coboundaries among $\mathrm{Isom}(\mathbb{H}_{\mathbb{R}}^n)$ -invariant continuous bounded cochains in top degree; this property, however is not known to hold in the complex hyperbolic setting). We are then in position to establish the [Main Corollary](#).

We conclude the thesis by presenting several open questions that arise from the preceding discussion.

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Chapter I

Preliminaries

Broadly, we are interested in group homomorphisms from finitely generated groups to Lie groups, called *representations*. The goal of this chapter is to impose increasingly more structure on both source and target group of such representations, focusing on the algebraic and geometric aspects that are relevant to the present work.

I.1 Representations and the Classification Problem

We first set up the algebraic groundwork, fixing terminology for the source and target groups of our representations - the basic vocabulary relevant for the remainder of the chapter. Afterwards, this allows us in particular, to give a brief introduction to the classification of representations.

I.1.1 Target Group : Lie Groups

We begin by recalling some basic notions from Lie theory. The following definition sets the stage by introducing Lie groups and Lie algebras.

Definition I.1. A (real/complex) *Lie group* is a smooth manifold G , possibly disconnected, equipped with a group structure such that both the multiplication and inversion operations are smooth (real differentiable/holomorphic) maps. We shall denote by G° the connected component that contains the neutral element $e \in G$, called the *identity component*.

A (real/complex) *Lie algebra* is a (real/complex) vector space \mathfrak{g} equipped with a *Lie bracket*, i.e. a bilinear and antisymmetric operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ that satisfies the Jacobi identity $[[v, w], u] + [[w, u], v] + [[u, v], w] = 0$ for all $v, w, u \in \mathfrak{g}$.

We give a fundamental example relating Lie groups and Lie algebras: *The Lie algebra of a Lie group G* , denoted by $\text{Lie}(G)$ or \mathfrak{g} as before, is the tangent space $T_e G$, equipped with an operation induced by the commutator of left-invariant vector fields on G (for vector fields, see later section in Definition I.12). This operation is indeed a Lie bracket and therefore the terminology is justified. To pass back from $\mathfrak{g} = \text{Lie}(G)$ to its Lie group, we require moreover the following notion:

Definition I.2. The *exponential* map $\exp : \mathfrak{g} \rightarrow G$ is given by $\exp(v) = \gamma(1)$, where $\gamma : \mathbb{R} \rightarrow G$ is the unique one-parameter subgroup of G whose tangent vector at $e \in G$ is given by v .

If G is a connected Lie group with Lie algebra \mathfrak{g} , then G acts on itself by conjugation $I_h(g) = hgh^{-1}$ for all $g, h \in G$. Differentiating this action at the identity element $e \in G$ yields a linear automorphism of \mathfrak{g} , namely the *adjoint action*

$$\text{Ad}_h := d(I_h)_e : \mathfrak{g} \rightarrow \mathfrak{g},$$

which is a Lie algebra automorphism satisfying $\text{Ad}_h([v, w]) = [\text{Ad}_h(v), \text{Ad}_h(w)]$ for all $v, w \in \mathfrak{g}$. Then we may introduce the following:

Definition I.3. The analytic group homomorphism $\text{Ad}_G : G \rightarrow \text{GL}(\mathfrak{g})$ is called the *adjoint representation* of G .

Finally, we introduce standard classes of Lie groups that will appear repeatedly in this work, particularly relevant for our discussion on symmetric spaces, later in Section I.3 (for classical references we refer to [Hel78; Kna96]):

Definition I.4. Let G be a connected Lie group with Lie algebra \mathfrak{g} .

- The Lie algebra \mathfrak{g} is called *simple* if it is non-abelian and has no nontrivial ideals. A connected Lie group G is called *simple* if its Lie algebra is simple.
- The Lie algebra \mathfrak{g} is called *semisimple* if it is a direct sum of simple Lie algebras. A connected Lie group G is called *semisimple* if its Lie algebra is semisimple.
- The Lie algebra \mathfrak{g} is called *reductive* if

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}],$$

equivalently if its radical equals its center. A connected Lie group G is called *reductive* if its Lie algebra is reductive.

An important invariant of connected semisimple Lie groups is their *rank*:

Definition I.5. Let G be a connected semisimple (real) Lie group. A *real split torus* in G is a connected, abelian subgroup consisting of semisimple elements whose adjoint action on \mathfrak{g} is diagonalizable over \mathbb{R} . A real split torus $A \leq G$ is *maximal* if it is not properly contained in any larger real split torus. The (*real*) *rank* of G is defined as

$$\text{rank}_{\mathbb{R}}(G) := \dim A,$$

where A is a maximal real split torus of G .

I.1.2 Source Group : Lattices

Let Γ be finitely generated group. The definition of a *lattice* relies on topological and measure-theoretic properties of the quotient of the ambient group G by Γ , namely:

Definition I.6. Let G be a locally compact topological group.

- A lattice $\Gamma < G$ is a discrete subgroup such that the quotient space $\Gamma \backslash G$ admits a finite G -invariant measure, i.e. there exists a right-invariant Borel measure¹ μ on G such that $\mu(\Gamma \backslash G) < \infty$ and for any open subset $U \subset \Gamma \backslash G$ one has that $\mu(gU) = \mu(U)$ for all $g \in G$.
- A lattice $\Gamma < G$ is *uniform*, if the quotient $\Gamma \backslash G$ is compact, otherwise, if the quotient is non-compact of finite volume, it is *non-uniform*.

A lattice Γ acts properly discontinuously but not necessarily freely on G , so that the quotient $\Gamma \backslash G$ is generally an orbifold with singularities coming from finite non-trivial stabilizer subgroups. Assuming that Γ a *torsion-free* lattice ensures that the quotient $\Gamma \backslash G$ is a manifold.

We note that by Selberg’s lemma, any finitely generated linear group admits a torsion-free subgroup of finite index [Sel60]. An important class of Lie groups that admit uniform as well as non-uniform lattices are non-compact semi-simple Lie groups, a fundamental result due to Borel and Harish-Chandra [Bor63], [BH62].

The quotient $\Gamma \backslash G$ can be understood in terms of fundamental domains, which serve as measurable building blocks for G ; they intersect each orbit of the Γ -action exactly once. We give the precise definition:

Definition I.7. Let Γ be a lattice in a locally compact group G equipped with a G -invariant finite Borel measure. A (*measure-theoretic*) *Borel fundamental domain* is a Borel measurable subset $\mathcal{D} \subset G$ for the action of Γ on G with the following properties:

- $\mu(G \setminus \bigcup_{\gamma \in \Gamma} \gamma \mathcal{D}) = 0$,
- $\mu(\gamma_1 \mathcal{D} \cap \gamma_2 \mathcal{D}) = 0$ for all distinct $\gamma_1, \gamma_2 \in \Gamma$.

Moreover, in this work, we often view manifolds as quotients of simply-connected topological spaces by lattice actions, in the following sense: Let G be a locally compact topological group and X be a simply connected topological space on which G acts transitively. Suppose further that G admits a torsion-free lattice Γ . Then the action of Γ on X is properly discontinuous and free, so the quotient map $\pi : X \rightarrow \Gamma \backslash X$ is a covering map. Hence the quotient

$$M := \Gamma \backslash X$$

is a manifold. One refers to X as the *universal cover* of M , equipped with the natural universal projection map $\pi : X \rightarrow M$. In particular, $\Gamma \cong \pi_1(M)$ is the fundamental group of M and acts by deck transformations on the universal cover. If we further assume $X \cong G/K$ for some closed subgroup $K < G$, then M is compact if Γ is uniform, and non-compact if Γ is non-uniform. For non-compact manifolds, it is often useful to reduce them to compact ones in the following sense:

Definition I.8. A *compact core* of a non-compact manifold X is a compact submanifold $N \subset X$ with boundary ∂N such X deformation retracts onto N

Similarly to Definition I.7 the quotient $M = \Gamma \backslash X$ can be understood in terms of fundamental domains:

¹This is in fact the left-invariant Haar measure on G , which is unique up to scaling.

Definition I.9. Let Γ be a lattice that acts on a topological space X . A (*topological*) *Borel fundamental domain* is a Borel measurable subset $D \subset X$ for the action Γ on X with the following properties

- $\bigcup_{\gamma \in \Gamma} \gamma D = X$
- $\gamma_1 D^\circ \cap \gamma_2 D^\circ = \emptyset$ for all distinct $\gamma_1, \gamma_2 \in \Gamma$,

where D° denotes the interior of D .

I.1.3 Classification Problem and Character Varieties

Let Γ be a finitely generated group, not necessarily a lattice, and G a finite-dimensional Lie group. The classification problem for representation from Γ into G can be approached by first considering the set of all representations

$$\mathrm{Hom}(\Gamma, G),$$

and then promoting this set to a moduli space, i.e. a topological space that parametrizes representations up to a natural equivalence relation. We formalize this construction in the present section, and refer to the lecture notes [Mar25] for further details.

Topologically, we may view $\mathrm{Hom}(\Gamma, G)$ as a subspace of G^Γ , the space of all maps from Γ to G endowed with the product topology. We note that since Γ is discrete, this topology coincides with the compact-open topology, which corresponds to the topology of pointwise convergence: a sequence $\rho_i \rightarrow \rho$ converges if and only if $\rho_i(\gamma) \rightarrow \rho(\gamma)$ converges in G for every $\gamma \in \Gamma$.

Thus, as a topological space, $\mathrm{Hom}(\Gamma, G)$ decomposes into connected components. This decomposition provides a meaningful way to organize representations, as each connected component consists of those that can be continuously deformed into one another (since $\mathrm{Hom}(\Gamma, G)$ is locally path-connected, this implies that connected and path-connected components coincide). While this decomposition already offers a natural topological classification of representations, it may still be regarded as somewhat coarse, since two representations lying in the same component can nevertheless be viewed as equivalent from another, more algebraic standpoint: There is a natural action of the group of inner automorphisms $\mathrm{Inn}(G) < \mathrm{Aut}(G)$ by conjugation

$$\mathrm{Inn}(G) \times \mathrm{Hom}(\Gamma, G) \longrightarrow \mathrm{Hom}(\Gamma, G), \quad (g, \rho) \longmapsto (\gamma \mapsto g \rho(\gamma) g^{-1}).$$

We declare two representations in $\mathrm{Hom}(\Gamma, G)$ to be *equivalent* if they are conjugate to one another (for a geometrical motivation, see Example I.10). Consequently, we aim to refine our classification problem by passing to the quotient

$$\mathrm{Hom}(\Gamma, G) / \mathrm{Inn}(G).$$

However, this quotient is not Hausdorff in general, as the conjugation action may not be proper and orbits may therefore fail to be closed. A topological approach to remedy this, is to consider the *Hausdorffification* of the above quotient space, denoted as

$$\mathrm{Hom}(\Gamma, G) //_G := \mathrm{Haus}(\mathrm{Hom}(\Gamma, G) / \mathrm{Inn}(G)), \quad (\text{I.1})$$

where $\text{Haus}(X)$ denotes the Hausdorffification of a topological space X . This is the maximal Hausdorff quotient of X , characterized by the universal property that every continuous map from X to a Hausdorff space factors uniquely through $\text{Haus}(X)$; Hausdorffification exists for any topological space and is unique up to canonical homeomorphism.

As a result, the space $\text{Hom}(\Gamma, G) // G$ is often referred to as the (*Hausdorff*) *character variety*, provides a well-behaved topological setting for a moduli space of representations that are continuous deformable into one another up to conjugation.

Remark. Classically, the term *character variety*, is used whenever G is a complex reductive algebraic²group, such as $\text{SL}_2(\mathbb{C})$, for the algebraic GIT quotient

$$\text{Spec}(\mathbb{C}[\text{Hom}(\Gamma, G)]^G),$$

namely the spectrum of the algebra of G -invariant regular functions on the representation space $\text{Hom}(\Gamma, G)$. This is an affine algebraic variety that is homeomorphic to the Hausdorff quotient (I.1) (see e.g. [LM85]).

When G is a real reductive group, such as $\text{PSL}_2(\mathbb{R})$, $\text{SO}^+(n, 1)$ or $\text{PU}(n, 1)$, the GIT construction does not apply directly. In this case, the Hausdorff quotient (I.1) is the real locus inside the complex character variety of the complexification of G . Following standard usage in the literature (see e.g. [Mar25]), we nevertheless refer to it as a *character variety*.

It is of interest to understand connected components of character varieties. For instance, studying connected components of $\text{Hom}(\Gamma, G) // G$ formed by representations that are discrete and faithful, for a Lie group G of higher-dimension and of higher rank, is considered a rich modern field, often referred to as *higher Teichmüller theory* (cf. [Wie18]). A motivation for these special connected components classically comes from Goldman’s thesis [Gol80] for the case when $G = \text{PSL}_2(\mathbb{R})$ and Γ is a uniform torsion-free lattice therein :

Example I.10. We denote by $\mathbb{H}_{\mathbb{R}}^2$ the real hyperbolic two-plane (cf. Subsection I.4.1), whose orientation-preserving isometries form a group that can be identified with

$$\text{Isom}^+(\mathbb{H}_{\mathbb{R}}^2) \cong \text{PSL}_2(\mathbb{R}).$$

Given a hyperbolic structure S on a closed surface, one can associate the *developing map*, which is a local isometry

$$\text{dev} : \tilde{S} \longrightarrow \mathbb{H}_{\mathbb{R}}^2,$$

obtained by analytically continuing local coordinate charts into $\mathbb{H}_{\mathbb{R}}^2$ along paths in S . The *holonomy representation* is a homomorphism

$$\rho : \pi_1(S) \longrightarrow \text{PSL}_2(\mathbb{R}),$$

that records the monodromy of the developing map, namely dev is ρ -equivariant:

$$\text{dev}(\gamma \cdot x) = \rho(\gamma) \text{dev}(x) \quad \text{for all } \gamma \in \pi_1(S), x \in \tilde{S}.$$

²An *algebraic group* over a field \mathbb{K} is an algebraic variety equipped with a group structure such that both the multiplication and inversion operations are given by regular polynomial maps.

The pair (dev, ρ) is unique up to post-composition of dev by an isometry of $\mathbb{H}_{\mathbb{R}}^2$, which corresponds to conjugating ρ inside $\text{PSL}_2(\mathbb{R})$. Thus, a hyperbolic structure S determines a well-defined conjugacy class $[\rho]$ of representations. Two hyperbolic structures that are isotopic yield conjugate holonomy representations.

Goldman [Gol80] showed that the Euler number, a real-valued continuous invariant of representations $\pi_1(S) \rightarrow \text{PSL}_2(\mathbb{R})$ that is contained the interval $[\chi(S), -\chi(S)]$ by the Milnor-Wood inequality [Mil58; Woo71], takes integral values and thus labels unions of connected components of the corresponding character variety. He proved that the extremal components consist precisely of discrete and faithful representations that are the two connected components forming the set of conjugacy classes

$$\left\{ [\rho] \in \text{Hom}(\pi_1(S), \text{PSL}_2(\mathbb{R})) / \text{PSL}_2(\mathbb{R}) \mid \rho \text{ is the holonomy of some hyperbolic structure} \right\},$$

and identified with the two copies of the *classical Teichmüller space* of S , i.e. the parametrization space of hyperbolic structures of S up to isotopy, corresponding to the two possible orientations of the surface. In [Gol88] Goldman proved that the Euler number in fact identifies the character variety's connected components.

In Chapters IV and V we establish a general formalism for invariants that label unions of connected components, for which the Goldman's Euler number is the prototype.

I.2 Riemannian Manifolds

We dive deeper into the geometric background required for the thesis. We set the ground with classical notions from differential geometry to then discuss Riemannian manifolds and the special class of Kähler manifolds. Then our focus will be on manifolds of non-positive curvature, in particular Hadamard manifolds, where we can formalize the notion of boundary. This then permits to discuss elementary groups reaching a discussion on the thick-thin decomposition- We also briefly discuss amenable groups in this geometric context.

I.2.1 Smooth Manifolds

In order to state interesting properties of Riemannian manifolds, we first require some basic concepts and notation from differential geometry; a classical source for this material is [Lee12].

For this chapter we fix X to be a smooth manifold of real dimension m . We begin with the following important notion:

Definition I.11. For a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, a \mathbb{K} -*vector bundle* of \mathbb{K} -rank n over a smooth manifold X is a smooth surjective map $\pi : E \rightarrow X$ such that each fiber $\pi^{-1}(p)$ is an n -dimensional vector space \mathbb{K}^n , and E is locally isomorphic to $U \times \mathbb{K}^n$ for some open neighbourhood $U \subset X$ of each point $p \in X$. In particular, if E is globally isomorphic to $X \times \mathbb{K}^n$, then $\pi : E \rightarrow X$ is called a *trivial bundle*.

The main example for this chapter will be the *tangent bundle* TX of a smooth m -dimensional manifold X , a real vector bundle of rank m whose fiber over $p \in X$ is the tangent space T_pX .

From TX one naturally obtains other bundles, such as the *cotangent bundle* T^*X , whose fiber at p is the dual space T_p^*X . For $1 \leq k \leq m$, the k -th exterior powers $\Lambda^k TX$ and $\Lambda^k T^*X$ are again a real vector bundles; in particular $\det(TX) = \Lambda^m TX$ is called the *determinant line bundle* and $K_X = \det(T^*X) = \Lambda^m T^*X$ is the *canonical line bundle*.

As a short preparation, we introduce vector fields and differential forms, that provide a natural language for the differential geometry we will use throughout. Vector fields over smooth manifolds assign to each point of X a tangent vector:

Definition I.12. Let X be a smooth manifold. The space of *vector fields* on X is defined as the set of smooth section of the tangent bundle:

$$\mathfrak{X}(X) := \Gamma(X, TX).$$

In order to differentiate vector fields on a manifold in a way compatible with its smooth structure, we introduce the following notion:

Definition I.13. An (*affine*) *connection* ∇ on a smooth manifold X is a map

$$\nabla : \mathfrak{X}(X) \times \mathfrak{X}(X) \rightarrow \mathfrak{X}(X),$$

which assigns to any pair of vector fields $v, w \in \mathfrak{X}(X)$ the *covariant derivative of w along v* , denoted by $\nabla_v w$, and satisfies the following properties:

- *Linearity:* $\nabla_v(aw + bu) = a\nabla_v w + b\nabla_v u$ for all $a, b \in \mathbb{R}$ and $w, u \in \mathfrak{X}(X)$,
- *Leibniz rule:* $\nabla_{fv} w = f\nabla_v w$, $\nabla_v(fw) = (vf)w + f\nabla_v w$ for all $f \in C^\infty(X)$.

Further, a vector field $v \in \mathfrak{X}(X)$ is *parallel* to $w \in \mathfrak{X}(X)$, if $\nabla_v w = 0$.

We define geodesics in terms of parallel transport along curves:

Definition I.14. Let X be a smooth manifold equipped with an affine connection ∇ . A smooth curve $\gamma : (0, 1) \rightarrow X$ is called a *geodesic* if it satisfies

$$\nabla_{\gamma'(t)} \gamma'(t) = 0$$

for all $t \in (0, 1)$, where γ' is the velocity vector, i.e., $\gamma'(t) \in T_{\gamma(t)}X$ for each t .

Intuitively, this means that the velocity vector is parallel transported along the curve, so the curve moves without “turning” relative to the connection ∇ . Next, we introduce the exponential map for manifolds, which will be a reoccurring notion throughout this chapter:

Definition I.15. Let X be a smooth manifold with an affine connection ∇ , and let $p \in X$. For each $v \in T_p X$, let $\gamma_v : (-\epsilon, \epsilon) \rightarrow X$ be the unique³ geodesic satisfying

$$\gamma_v(0) = p, \quad \gamma'_v(0) = v.$$

³Uniqueness of γ_v is guaranteed by the standard existence and uniqueness theorem for ODEs applied to the geodesic equation $\nabla_{\gamma'_v} \gamma'_v = 0$ with initial conditions $\gamma_v(0) = p$ and $\gamma'_v(0) = v$.

The *exponential map* at p is defined by

$$\exp_p : U \rightarrow X, \quad \exp_p(v) := \gamma_v(1),$$

where $U \subseteq T_p X$ is an open neighborhood of the origin $0_p \in T_p X$ consisting of vectors v for which the geodesic $\gamma_v(t)$ is defined at $t = 1$.

We note that the map \exp_p is smooth and satisfies $\exp_p(0_p) = p$. Moreover, the differential pushforward at the origin, $(d\exp_p)_{0_p} : T_p X \rightarrow T_p X$ is the identity map. Consequently, \exp_p is a local diffeomorphism at the origin 0_p .

Dual notion to vector fields are so-called *1-forms*, and more generally:

Definition I.16. Let X be a smooth m -manifold. We define

- $\Omega^k(X)$ as the set of all smooth differential k -forms on X , i.e.

$$\Omega^k(X) = \Gamma(X, \Lambda^k T^* X),$$

i.e. the set of all smooth sections of the k -th exterior power of the cotangent bundle of X .

- The space of differential forms $\Omega^*(X) = \bigoplus_{k=0}^m \Omega^k(X)$ is equipped with the *wedge product*,

$$\wedge : \Omega^k(X) \times \Omega^q(X) \longrightarrow \Omega^{k+q}(X),$$

a bilinear, associative, and graded-commutative product satisfying

$$\alpha \wedge \beta = (-1)^{kq} \beta \wedge \alpha,$$

for all $\alpha \in \Omega^k(X)$ and $\beta \in \Omega^q(X)$.

- The *exterior derivative* is a linear operator

$$d^k : \Omega^k(X) \longrightarrow \Omega^{k+1}(X)$$

that is closed, i.e. $d^{k+1} \circ d^k = 0$, and satisfies the *Leibniz rule*:

$$d(\alpha \wedge \beta) = d^k \alpha \wedge \beta + (-1)^k \alpha \wedge d^q \beta \quad \text{for } \alpha \in \Omega^k(X), \beta \in \Omega^q(X).$$

In particular, $(\Omega^*(X), \wedge, d^*)$ is a *differential graded algebra*. Apart from serving to express several geometric concepts in the present chapter, the latter also forms a complex giving rise to *de Rham cohomology*; we return to this in Subsection II.2.1 in the chapter on cohomology theories. Finally, we formalize orientability (cf. [Zhe00, §3.6]):

Definition I.17. A smooth m -manifold X is *orientable* if there exists a nowhere-vanishing global m -form in $\Omega^m(X)$, or equivalently, the canonical line bundle K_X is isomorphic to the trivial line bundle $X \times \mathbb{R}$.

If α, α' are any two choices of non-vanishing m -forms, then there exists a smooth function $f : X \rightarrow \mathbb{R} \setminus \{0\}$ such that

$$\alpha = f \cdot \alpha',$$

and is hence either positive or negative everywhere. We can therefore define:

Definition I.18. Let X be an orientable smooth manifold. Then an *orientation* on X is a choice of connected component of the set of nowhere-vanishing m -forms $\Omega^m(X)$, or equivalently, it is an equivalence class of such forms under multiplication by positive smooth functions.

An alternative in homological terms will be given in Definition II.6 in the next chapter.

I.2.2 Fundamentals

For a discussion on the fundamentals on Riemannian geometry in this section, we refer to several sources [Hel78, Chapter 1, §10], [Zhe00], [Bal95]. First and foremost, any smooth manifold can be endowed with a Riemannian structure:

Definition I.19. A *Riemannian manifold* (X, g) is a smooth manifold X such that the tangent space $T_p X$ at any point $p \in X$ is equipped with a smoothly varying symmetric bilinear form

$$g_p : T_p X \times T_p X \rightarrow \mathbb{R}.$$

This metric gives a notion of length of curves on X , and hence of a distance function:

Definition I.20. Let (X, g) be a connected Riemannian manifold. Then the *distance function*

$$\text{dist} : X \times X \rightarrow \mathbb{R}_{\geq 0}$$

is defined by

$$\text{dist}(x, y) := \inf_{\gamma} \int_0^1 \|\gamma'(t)\|_g dt,$$

where the infimum is taken over all piecewise smooth curves $\gamma : [0, 1] \rightarrow X$ such that $\gamma(0) = x$ and $\gamma(1) = y$ and where for $\gamma'(t) \in T_{\gamma(t)} X$ the norm is given by

$$\|\gamma'(t)\|_g = \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}.$$

Geodesics, as defined previously in Definition I.14, locally minimize the distance function. A geodesic γ realizing the distance function between x and y in X is called a *minimizing geodesic*. If a connected Riemannian manifold (X, g) is *complete*, i.e. every geodesic extends indefinitely, then by the Hopf-Rinow theorem [HR32], any two points in X can be joined by a minimizing geodesic. We will mostly work with complete manifolds and so by geodesics, we will mean the minimizing ones. Moreover, from now and on we assume that all Riemannian manifolds are connected without further mention.

In case (X, g) is an oriented Riemannian manifold, the Riemannian metric g induces a *canonical* nowhere-vanishing differential n -form called the *volume form* (cf. [Zhe00, §3.6]):

Definition I.21. Let (X, g) be an oriented Riemannian m -manifold. The *Riemannian volume form* $\text{vol} \in \Omega^m(X)$ is the unique smooth m -form such that, for every point $p \in X$, and every positively oriented orthonormal basis (v_1, \dots, v_m) of $T_p X$, one has that

$$\text{vol}(v_1, \dots, v_m)_p = 1.$$

Since $\text{vol} \in \Omega^m(X)$ is a smooth top-degree form on the oriented m -manifold X , it can be integrated over X . This allows to define:

Definition I.22. Let (X, g) be a closed (i.e. compact with empty boundary), oriented Riemannian manifold with the volume form $\text{vol} \in \Omega^m(X)$. The *volume* of X is given by

$$\text{Vol}(X) := \int_X \text{vol}.$$

Since X is closed, the integral is well-defined and finite smooth functions are bounded on compact sets, and integration over a finite atlas yields a finite value.

Isometry Group

We will now take a group-theoretic perspective on Riemannian manifolds (cf. [Hel78, Ch. 1, §11]) and give the following classical definition:

Definition I.23. The *isometry group* of a Riemannian manifold (X, g) is given

$$\text{Isom}(X) = \left\{ \phi : X \rightarrow X \mid \begin{array}{l} \phi \text{ is a diffeomorphism and} \\ g_p(v, w) = g_{\phi(p)}(d\phi_p(v), d\phi_p(w)) \end{array} \right\},$$

where $d\phi : T_p X \rightarrow T_{\phi(p)} X$ is the differential pushforward of ϕ at the point p .

We note that $\text{Isom}(X)$ is a finite dimensional Lie group due to [MS39]. If there exists a Lie subgroup $G \leq \text{Isom}(X)$ that acts transitively on a Riemannian manifold X , then X is called a *homogeneous Riemannian manifold* and one has that

$$X \cong G / \text{Stab}_G(p),$$

where $\text{Stab}_G(p) = \{g \in G \mid g \cdot p = p\}$ is the stabilizer group of the basepoint p . We note that any two stabilizer groups for distinct basepoints are conjugate to each other. Moreover, if X is complete, then the stabilizer $\text{Stab}_G(p)$ is a *compact* subgroup. (cf. [Zhe00, §3.5] or [Hel78, Ch. 1, §3], [Hel78, Ch. III, §2, Theorem 2.5 (b)]).

Remark. If X is a simply connected Riemannian manifold and $\Gamma < G$ a discrete group that acts freely and properly discontinuously on X , then the quotient $\Gamma \backslash X$ inherits the Riemannian metric such that the projection map is a local isometry.

We will further need the following notion on the Riemannian manifold's decomposibility:

Definition I.24. Let (X_1, g_1) and (X_2, g_2) be Riemannian manifolds. Their *Riemannian product* is the manifold $X_1 \times X_2$ equipped with the product metric

$$g = \pi_1^* g_1 + \pi_2^* g_2,$$

where $\pi_1 : M_1 \times M_2 \rightarrow M_1$ and $\pi_2 : M_1 \times M_2 \rightarrow M_2$ are the natural projections. A Riemannian manifold (X, g) is called *irreducible* if it is not isometric to a Riemannian product of two Riemannian manifolds of positive dimension.

Levi-Civita Connection, Curvature and Rank

On a Riemannian manifold in particular, among all affine connections, we characterize a distinguished one:

Proposition I.25. *Let (X, g) be a Riemannian manifold. Then there exists a unique connection ∇ on TX , called the Levi-Civita connection, that satisfies for all $v, w, u \in \mathfrak{X}(X)$:*

- *Metric compatibility:* $\nabla g(w, u) = g(\nabla_v w, u) + g(w, \nabla_v u)$
- *Torsion-freeness:* $\nabla_v w - \nabla_w v = [v, w]$.

Proof. For a proof we refer to [Hel78, Theorem 9.1]. □

This allows us to formalize the sectional curvature (cf. [Hel78, Ch. 1, §12]):

Definition I.26. Let X be a Riemannian manifold. For any $p \in X$, let $\sigma \subset T_p X$ be a two-dimensional subspace and linearly independent vectors $v, w \in T_p X$ spanning σ .

- The *sectional curvature* $\text{sec}(\sigma)$ associated to g is defined as

$$\text{sec}(\sigma) = \frac{g_p(R(v, w)w, v)}{g_p(v, v) \cdot g_p(w, w) - g_p(v, w)^2},$$

where R is the Riemannian curvature tensor given in terms of the Levi-Civita connection ∇ by

$$R(v, w)u = \nabla_v \nabla_w u - \nabla_w \nabla_v u + \nabla_{[v, w]} u \quad \text{for all } v, w, u \in \mathfrak{X}(X)$$

- We say that if for all $p \in X$ and for all two-dimensional linear subspaces $\sigma \in T_p X$

$$\text{sec}(\sigma) = \begin{cases} < 0, & \text{then } X \text{ is of } \textit{negative curvature}. \\ > 0, & \text{then } X \text{ is of } \textit{positive curvature}, \\ = 0, & \text{then } X \text{ is } \textit{flat}. \end{cases}$$

We can also classify Riemannian manifolds using the notion of rank, defined in terms of totally geodesic flat submanifolds.

Definition I.27. Let (X, g) be a complete Riemannian manifold. The *rank* of X is the largest integer $k \geq 1$ such that X contains a totally geodesic⁴, isometrically embedded flat submanifold of dimension k , i.e., a submanifold isometric to the Euclidean space \mathbb{R}^k .

Clearly, in *rank one* manifolds' maximal totally geodesic flat submanifolds are geodesics.

⁴A submanifold $Y \subset X$ is *totally geodesic* if every geodesic of Y (with the induced metric) is also a geodesic of X .

I.2.3 Kähler Manifolds

The Riemannian structure can interact with additional geometric structures to yield a particularly rich and interesting classes of manifolds, for example Kähler manifolds. Classical references for Kähler manifolds are [Bal06; Huy05; Zhe00]. We take a step back from Riemannian manifolds and assume that X is a smooth manifold and gradually impose more structure on it for the necessary framework:

A *complex n -manifold* is a smooth manifold of real dimension $2n$ equipped with an atlas of charts to \mathbb{C}^n whose transition maps are holomorphic. To every complex manifold X , one can associate a natural *almost complex structure*, which is a smooth bundle map $J:TX \rightarrow TX$ with $J^2 = -\text{Id}$, that is *integrable*, i.e. it arises as multiplication by i in the complex coordinate chart. In fact, any complex manifold can be uniquely encoded by the data (X, J) .

Remark. For a complex manifold X , the integrable almost complex structure J extends complex-linearly to the complexified tangent bundle $T_{\mathbb{C}}X = TX \otimes_{\mathbb{R}} \mathbb{C}$. Since $J^2 = -\text{Id}$, the eigenvalues of J acting on $T_{\mathbb{C}}X$ are $\pm i$. This gives a decomposition into eigenspaces

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X,$$

called the *holomorphic and antiholomorphic tangent bundle* respectively. By slight abuse of notation, we shall refer to $TX = T^{1,0}X$ as the holomorphic tangent bundle in the context of complex manifolds.

We now state a notion that intertwines Riemannian and complex structure:

Definition I.28. A *Hermitian manifold* (X, g, J) is a complex manifold (X, J) equipped with a Riemannian metric g such that

$$g(Jv, Jw) = g(v, w) \quad \text{for all } v, w \in \mathfrak{X}(X).$$

We call g with this property a *Hermitian metric* on X .

Definition I.29. A *Kähler manifold* (X, g, J, ω) is a complex manifold (X, J) equipped with a Hermitian metric g such that the associated *Kähler form* $\omega \in \Omega^2(X)$ defined by

$$\omega(v, w) := g(Jv, w) \quad \text{for all } v, w \in \mathfrak{X}(X)$$

is closed, i.e. $d\omega = 0$.

We note that the above condition $d\omega = 0$ is equivalent to $\nabla J = 0$ for the Levi-Civita connection (see Definition I.25) satisfying $(\nabla_v J)(w) = \nabla_v(Jw) - J(\nabla_v w) = 0$ for any $v, w \in \mathfrak{X}(X)$ (see [Bal06, Theorem 4.17]).

Remark. For any Kähler form ω on X , one has that $\omega \in \Omega^{1,1}(X)$, where

$$\Omega^{1,1}(X) = \Gamma(X, (T^{1,0}X)^* \otimes (T^{0,1}X)^*),$$

i.e. the space of smooth sections of the tensor product of the dual bundles $(T^{1,0}X)^*$ and $(T^{0,1}X)^*$.

A distinctive feature of the Kähler form is its multiplicative relation with the volume form, namely the *Wirtinger formula*:

Proposition I.30. *Let (X, g, J, ω) be a Kähler n -manifold. Then the volume form $\text{vol} \in \Omega^{2n}(X)$ can be expressed as*

$$\text{vol} = \frac{\omega^n}{n!},$$

where $\omega^n = \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}}$ denotes the top wedge power of the Kähler form ω .

Proof. For a proof we refer to [Bal06, §4.1]. □

Moreover, Kähler manifolds admit yet another type of curvature:

Definition I.31. Let (X, g, J, ω) be a Kähler manifold. For any point $p \in X$, let $\sigma \subset T_p X$ be a complex line (a complex 1-dimensional subspace) and let $v \in \sigma$ be any non-zero vector spanning σ . The *holomorphic sectional curvature* of σ is defined by

$$\text{sec}_{\text{hol}}(\sigma) = \frac{g_p(R(v, Jv)Jv, v)}{g_p(v, v)^2},$$

where R is the Riemann curvature tensor of (X, g) from Definition I.26.

We note that $\text{sec}_{\text{hol}}(\sigma)$ is independent of the vector v spanning σ . We further observe that holomorphic sectional curvature coincides with the sectional curvature of the real two-plane spanned by v and Jv .

I.2.4 Hadamard Manifolds

We now return to ordinary Riemannian manifolds and discuss these under non-positive curvature assumptions; we refer to [BGS85, Lecture I, §2].

Definition I.32. A *Hadamard manifold* is a complete simply-connected Riemannian manifold X with non-positive sectional curvature.

Hadamard manifolds are central objects in differential geometry, exhibiting a wide range of phenomena with the most fundamental property being the following:

Proposition I.33. *Let X be a Hadamard manifold. Then the following holds:*

- (i) *Any two points in X can be joined by a unique geodesic.*
- (ii) *The distance function $\text{dist} : X \times X \rightarrow \mathbb{R}_{\geq 0}$ is convex along geodesics; that is, for any geodesic $\gamma : [0, 1] \rightarrow X$ and any fixed $y \in X$ the function $f(t) = \text{dist}(\gamma(t), y)$ satisfies*

$$f(t) \leq (1-t)f(0) + tf(1) \quad \text{for all } t \in [0, 1].$$

Sketch of the Proof. The Hadamard-Cartan theorem implies (i) and further that the Hadamard manifold is convex [BGS85, §2]. On convex sets of Riemannian manifolds of negative curvature the distance function is convex [BGS85, Theorem 1.3], which proves (ii). □

Boundary

The uniqueness property of geodesics permits to make sense of the notion of *boundary* for Hadamard manifolds, whose intricate structure we shall discuss in this paragraph; we refer to [BH99, Ch. II.8], [BGS85, Lecture I, §3].

Definition I.34. Let X be a Hadamard manifold. A *ray* is defined as a unit-speed geodesic

$$\gamma : [0, \infty) \rightarrow X.$$

Further, two rays γ_1, γ_2 are said to be *asymptotic* if the function

$$t \mapsto \text{dist}(\gamma_1(t), \gamma_2(t))$$

is bounded as $t \rightarrow \infty$. This defines an equivalence relation on the set of rays starting at a fixed basepoint $x \in X$, called an *ideal point*. The set of equivalence classes of asymptotic rays is called the *boundary (at infinity)* of X , denoted by ∂X .

The set $\bar{X} := X \cup \partial X$ is known as the *Eberlein-O'Neill compactification* of X , which can be endowed with natural topology, namely the *cone topology* [EO73].

We now take the group-theoretic perspective (cf. Definition I.23) and state a result that fully classifies elements of the isometry group $\text{Isom}(X)$ into three types.

Theorem I.35. *Let X be a Hadamard manifold. We consider the displacement function $\text{dist}_g(x) := \text{dist}(x, gx)$ for any $g \in \text{Isom}(X)$. Any non-trivial element $g \in \text{Isom}(X)$ falls into exactly one of the following types:*

- g has a fixed point in X , called *elliptic*
- g is such that $\inf_{x \in X} \text{dist}_g(x)$ is not attained in X , called *parabolic*.
- g is such that $\inf_{x \in X} \text{dist}_g(x)$ is positive and attained in X , called *hyperbolic*.

Proof. For a proof we refer to [BGS85, §6.1]. □

We note that if Hadamard manifolds are of strictly negative curvature, one can describe parabolic and hyperbolic elements also in terms of their fixed point behaviour on the boundary. In this case X is a *visibility manifold*, i.e. a Hadamard manifold for which any two points on ∂X can be joined by a geodesic. Using Brower's fixed point theorem, one sees that parabolic elements are those that have exactly one fixed point on the boundary ∂X , whereas hyperbolic elements are those that have exactly two fixed points on boundary ∂X , preserving the geodesic joining them (cf. [BGS85, Lemma 6.5]).

This classification of isometries plays for instance a role in the context of the following distinguished subsets of Hadamard manifolds X centered at ideal points in ∂X ; see [BGS85, Lecture I, §3.3].

Definition I.36. Let X be a Hadamard manifold, let $\xi \in \partial X$ and let $s \in \mathbb{R}_{>0}$. A *horosphere* $H_s(\xi)$ and *horoball* $B_s(\xi)$ of radius s centered in ξ , are respectively given by

$$H_s(\xi) := \{x \in X \mid b_\xi(x) = s\} \quad \text{and} \quad B_s(\xi) := \{x \in X \mid b_\xi(x) \leq s\},$$

where b_ξ is the *Busemann function* defined for a ray $\gamma : [0, \infty) \rightarrow X$ representing ξ , by

$$b_\xi(x) := \lim_{t \rightarrow \infty} (d(x, \gamma(t)) - t).$$

We note that $b_\xi(x)$ is independent of the choice of ray representing ξ . We observe that horospheres and horoballs have the following properties:

Proposition I.37. *Let X be a Hadamard manifold. Then:*

- (i) *Any horoball in X is a convex subset.*
- (ii) *Let $g \in \text{Isom}(X)$ be a parabolic element. Then there exists a point $\xi \in \partial X$ fixed by g . Moreover, g preserves each horosphere $H_s(\xi)$ for $s \in \mathbb{R}_{>0}$.*

Sketch of the Proof. By Proposition I.33 the Busemann function is convex which proves statement (i). For statement (ii) we refer to [BGS85, §3 and Lemma 6.6]. \square

Next, we characterize subgroups in $\text{Isom}(X)$ through their accumulation points on the boundary:

Definition I.38. Let X be a Hadamard manifold and let $H < \text{Isom}(X)$ be a subgroup. The *limit set* of H , is defined as the set

$$\Lambda_H = \overline{H \cdot p} \cap \partial X$$

for any $p \in X$ and where $\overline{H \cdot p}$ the closure of the orbit $H \cdot p \subset X$ in the compactification \overline{X} with the cone topology. The group H is called *elementary* if the limit set Λ_H consists of at most two points.

Remark. We remark that Λ_H is independent of the choice of basepoint p . Notably, if H is non-elementary, then $|\Lambda_H| = \infty$.

We are more specifically interested in discrete subgroups of the stabilizer group of some ideal point $\xi \in \partial X$, namely

$$\text{Stab}(\xi) = \{g \in \text{Isom}(X) \mid g \cdot \xi = \xi\}.$$

This is a closed subgroup of the Lie group $\text{Isom}(X)$, so it is a Lie group in its own right. Now, any discrete subgroup $\Gamma < \text{Stab}(\xi)$ is elementary when it satisfies one of the following three conditions:

- Γ is finite, so that $|\Lambda_\Gamma| = 0$,
- Γ is infinite, fixes a single ideal point in ∂X , and preserves setwise each horosphere (cf. Proposition I.37), so that $|\Lambda_\Gamma| = 1$. In this case, Γ is called a *discrete parabolic group*.
- Γ is infinite and contains a hyperbolic element so that $|\Lambda_\Gamma| = 2$, and is called in this case a *discrete hyperbolic group*.

Thin-Thick Decomposition

The thin-thick decomposition is fundamental for the study of manifolds of negative curvature. For this section we refer to [Bow95, §3.5] and [AX04, §5-6].

We assume now that X is a Hadamard manifold of *pinched negative curvature*, i.e. $\sec(\sigma)$ lies between two negative constants, for all tangent two-planes $\sigma \in T_p X$ and any $p \in X$. Moreover, let M be a finite-volume manifold obtained as the quotient of X by a discrete torsion-free subgroup $\Gamma < \text{Isom}(X)$. There is a standard way to decompose M into so-called *thick* and *thin* parts. For this, we first recall the exponential map introduced in Definition I.15; the extent to which this map remains a diffeomorphism in a neighbourhood of the origin of each tangent space is quantified by the injectivity radius:

Definition I.39. Let M be a complete Riemannian manifold (with no assumption on the curvature). The *injectivity radius* at a point $p \in M$ is defined as

$$\text{inj}(p) := \sup\{r > 0 \mid \exp_p : B_r(0_p) \rightarrow M \text{ is a diffeomorphism onto its image}\},$$

where $B_r(0_p) \subseteq T_p M$ is the open ball of radius r centered at the origin 0_p in $T_p M$. For any $\varepsilon > 0$, we define the ε -*thin part* and ε -*thick part* of M respectively as

$$M_{\text{thick}}(\varepsilon) := \{p \in M : \text{inj}(p) < \varepsilon\}, \quad \text{and} \quad M_{\text{thin}}(\varepsilon) := \{p \in M : \text{inj}(p) \geq \varepsilon\}.$$

Notably, the thin and thick parts in this disjoint decomposition of M have the following concrete characterizations:

Theorem I.40. *Let M be a complete finite-volume Riemannian m -manifold of pinched negative curvature. Then there exists a universal (Margulis) constant ϵ_X such that for any $\varepsilon < \epsilon_X$ the following holds:*

- *The ε -thick part $M_{\text{thick}}(\varepsilon)$ is a non-empty compact connected manifold, with possibly non-smooth boundary.*
- *Then ε -thin part $M_{\text{thin}}(\varepsilon)$ is a disjoint union of connected components that come in the following two types:*
 - ***Margulis tubes** are tubular neighbourhoods of closed geodesics of length at most 2ε . In particular, these are homeomorphic to $B^{m-1} \times S^1$ with B^{m-1} denoting a closed ball contained in \mathbb{R}^{m-1} .*
 - ***Margulis cusps** are quotients of horoballs in X by discrete parabolic subgroups of $\text{Isom}(X)$ of rank $m - 1$. In particular these are homeomorphic to $\Sigma \times [0, \infty)$ where Σ is a closed manifold of dimension $m - 1$, that is topologically identified with the corresponding component of $\partial M_{\text{thick}}(\varepsilon)$.*

Proof. For a proof we refer to [Bow95, §3.5] (see also [AX04, §5]), for which Margulis' lemma plays a central role. □

We note that if M is compact then $M = M_{\text{thick}}(\varepsilon)$ and $M_{\text{thin}}(\varepsilon) = \emptyset$. If M is non-compact, we observe that [BCD93, Lemma 2.4] provides a positive lower bound on the length of closed

geodesics determining the Margulis tubes. Therefore, by choosing ε sufficiently small, one can ensure that the ε -thin part $M_{\text{thin}}(\varepsilon)$ consists solely of finitely many Margulis cusps, with no Margulis tubes present. From now and on, we shall fix such an ε and further fix the notation

$$N := M_{\text{thick}}(\varepsilon) \quad \text{so that} \quad M \setminus N = M_{\text{thin}}(\varepsilon).$$

In particular, N is a compact core of M (cf. Definition I.8), whose boundary ∂N can be assumed to be smooth due to [BCD93]. On the group-theoretic side, the quotient $M \cong \Gamma \backslash X$ admits a discrete parabolic group $\Gamma_\xi = \text{Stab}(\xi) \cap \Gamma$ that fixes an ideal point $\xi \in \partial X$. There exists a sufficiently large $s \in \mathbb{R}_{>0}$ such that Γ_ξ acts freely and properly discontinuously on the horoball $B_s(\xi)$ and the horosphere $H_s(\xi)$, with a cocompact quotient. We introduce (cf. [AX04, §6]):

Definition I.41. Let M be a non-compact finite-volume Riemannian manifold of pinched negative curvature. For a discrete parabolic group $\Gamma_\xi < \pi_1(M)$ fixing an ideal point $\xi \in \partial X$ and a sufficiently large $s \in \mathbb{R}_{>0}$ we define the *cuspidal region* and the corresponding *cuspidal cross section* of M respectively by the quotients $\mathcal{C} := \Gamma_\xi \backslash B_s(\xi)$ and $\mathcal{S} := \Gamma_\xi \backslash H_s(\xi)$.

To summarize, for sufficiently small $\varepsilon > 0$, the ε -thin part of M is homeomorphic to a disjoint union of finitely many cuspidal regions \mathcal{C}_i for $1 \leq i \leq \ell$, namely

$$M \setminus N = M_{\text{thin}}(\varepsilon) \simeq \bigsqcup_{i=1}^{\ell} \mathcal{C}_i,$$

as illustrated in Figure I.1. This geometrical setup will play a role in Example II.10 in the context of bounded cohomology.

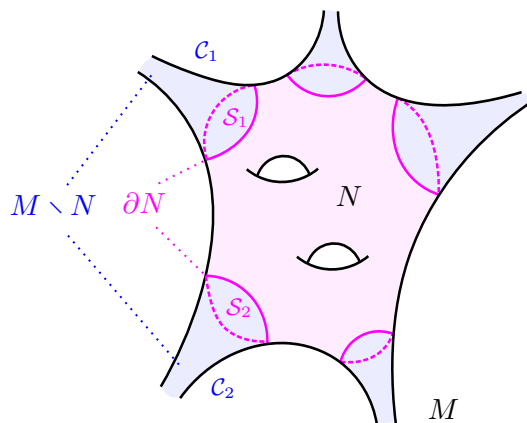


Figure I.1: Illustration of a manifold M in Definition I.41 that admits a thin-thick decomposition into $N = M_{\text{thick}}(\varepsilon)$ and $M \setminus N = M_{\text{thin}}(\varepsilon)$ for sufficiently small ε .

Amenable Groups

We briefly introduce the notion of amenable groups with a focus on their role in our geometric framework. We state the original definition due to von Neumann [Neu29]:

Definition I.42. A locally compact topological group H is *amenable* if for every continuous action of H on a compact topological space X there exists a positive H -invariant measure on X .

The prominent bounded-cohomological vanishing property of amenable groups will be stated in Theorem II.3, which plays a significant role in the present work. Standard examples of amenable groups include finite, abelian, nilpotent, solvable. Moreover, virtually amenable groups are again amenable groups. For these facts we refer to [BHV08, Appendix G],[Pat88]. On the other hand, free groups of rank at least two, and more generally any group that contains a non-abelian free subgroup, provide standard examples of non-amenable groups. An important class of amenable group we encountered so far are discrete parabolic groups. Indeed, these are virtually nilpotent due to [Bow95, Proposition 4.1] and hence amenable. Another example relevant to us, are *minimal parabolic groups* that we introduce later in Definition I.51 (in the rank-1 case).

It is noteworthy that amenable groups acting on the boundary of Hadamard manifolds enjoy the following property:

Theorem I.43 ([BS87, Theorem 2]). *Let X be a Hadamard manifold and let H be an amenable subgroup of $\text{Isom}(X)$. Then one, or possibly both, of the following statements hold:*

- (i) H fixes a point in ∂X .
- (ii) H preserves a totally geodesic, isometrically embedded Euclidean subspace $E \subset X$, that is H acts by Euclidean isometries on $E \cong \mathbb{R}^k$ for $k \in \mathbb{N}$.

This recovers the following nice characterization:

Corollary I.44 ([Ave70],[Zim83]). *Let X be a complete Riemannian manifold of non-positive curvature and finite volume with amenable fundamental group. Then X is Euclidean.*

I.3 Symmetric Spaces

We now go back to Riemannian manifolds of arbitrary curvature but with an additional strong symmetry condition to obtain so-called *symmetric spaces*. These spaces are interesting for their rich interplay between differential geometry and Lie theory, as we shall see in this section. Following our previous discussion, our focus will lie on symmetric spaces that exhibit negative curvature. For this section we refer to [Hel78, Ch. III, §3], [Wol11, Ch. 8] [Zhe00, §3.5].

Both, the exponential map from Definition I.15 and the injectivity radius from Definition I.39 plays a role to formalize a symmetry condition on the reversal of geodesics of a manifold:

Definition I.45. Let X be a Riemannian manifold. For any point $p \in X$ a diffeomorphism s_p from the ball $B_{\text{inj}(p)}(0_p)$ onto itself defined by

$$s_p(\exp_p(v)) = \exp_p(-v) \quad \forall v \in B_{\text{inj}(p)}(0_p) \subset T_p X$$

is called a *geodesic symmetry* at p .

We note that any geodesic symmetry s_p has p as an isolated fixed point and $d(s_p)_p = -\text{id}_{T_p X}$. We are now in position to give the following definition, that captures the idea that a manifold is locally symmetric under reversal of geodesics:

Definition I.46. Let (X, g) be a Riemannian manifold. We say that $X = (X, g, s)$ is a *locally symmetric space* if for any $p \in X$ the geodesic symmetry s_p is an isometry on $B_{\text{inj}(p)}(p)$. Moreover, we say that X is a (*globally*) *symmetric space* if X is complete and if for any $p \in X$ there exists an isometry in $\text{Isom}(X)$ that restricts to s_p on $B_{\text{inj}(p)}(p)$.

Clearly, every globally symmetric space is locally symmetric. Conversely, a complete and simply connected locally symmetric space is globally symmetric. Moreover, we note that symmetric spaces are homogeneous.

I.3.1 Symmetric Spaces versus Lie Groups

We now give a Lie-theoretic characterization for symmetric spaces. For this we require some preparation first.

Let G be a connected Lie group and $\sigma \in \text{Aut}(G)$ an involution. We consider the fixed-point set

$$G^\sigma = \{g \in G \mid \sigma(g) = g\}$$

and denote by $(G^\sigma)^\circ$ the identity component of G . We further recall Definition I.3 of the adjoint representation Ad_G in order to define the following:

Definition I.47. Let G be a connected Lie group and $K < G$ a closed subgroup such that

- $\text{Ad}_G(K)$ is a compact subgroup of $\text{GL}(\mathfrak{g})$,
- there exists an involution $\sigma \in \text{Aut}(G)$ such that $(G^\sigma)^\circ \leq K \leq G^\sigma$.

Then (G, K) is called a *symmetric pair* with involution σ .

We can now state the following duality:

Theorem I.48. *There is a natural assignment in both directions:*

$$\{(X, g, s) \mid \text{symmetric space}\} \begin{array}{c} \xrightarrow{\text{-----}} \\ \xleftarrow{\text{-----}} \end{array} \{(G, K, \sigma) \mid \begin{array}{c} (G, K) \text{ a symmetric pair,} \\ \text{with involution } \sigma \end{array}\},$$

allowing one to obtain (G, K, σ) from (X, g, s) and vice versa.

Proof. On the one hand, let (X, g, s) be a symmetric space and we set $G = \text{Isom}^\circ(X)$ to be the identity component of $\text{Isom}(X)$. Then G admits the involutive automorphism σ given by

$$\begin{aligned} \sigma = \sigma_p : G &\longrightarrow G, \\ g &\longmapsto s_p g s_p, \end{aligned}$$

where s_p is the geodesic symmetry for every $p \in X$. Further setting $K = \text{Stab}_G(p)$ for any basepoint $p \in X$, then (G, K) is a symmetric pair by [Hel78, Ch. IV, Theorem 3.3 (ii)].

On the other hand, given a symmetric pair (G, K) with involution σ , then the quotient $X = G/K$ is a symmetric space with respect to any G -invariant Riemannian metric g . Further, the geodesic symmetry s_p is obtained through the relation

$$s_p \circ \pi = \pi \circ \sigma,$$

where $p = \pi(K) \in G/K$ for the natural projection $\pi : G \rightarrow G/K$. For details we refer to [Hel78, Ch. II, Proposition 4.3 and Ch. IV, Proposition 3.4]. \square

Remark. If G is a connected semisimple Lie group acting effectively on the quotient $X = G/K$, then the above assignment gives a bijection between isometry classes of symmetric spaces and isomorphism classes of effective symmetric pairs (G, K) (cf. [Hel78, Ch. III, Theorem 4.1]).

Cartan Decomposition

On the Lie algebraic side, given a connected Lie group G together with an involutive automorphism $\sigma \in \text{Aut}(G)$, the differential at identity

$$\theta = (d\sigma)_e : \mathfrak{g} \longrightarrow \mathfrak{g}$$

is an involutive automorphism of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$, called the *Cartan involution*. Then the Lie algebra \mathfrak{g} decomposes into ± 1 -eigenspaces of θ , which allows to state the following definition:

Definition I.49. Let G be a connected Lie group with Cartan involution θ .

- The *Cartan decomposition* of $\mathfrak{g} = \text{Lie}(G)$ with respect to θ is given by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where \mathfrak{k} and \mathfrak{p} are the ± 1 eigenspace, with

$$\mathfrak{k} = \{v \in \mathfrak{g} \mid \theta(X) = X\}, \quad \mathfrak{p} = \{v \in \mathfrak{g} \mid \theta(X) = -X\}.$$

- The *compact real form* of the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is given by

$$\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p} \subset \mathfrak{g}^{\mathbb{C}}$$

and its corresponding connected Lie group is denoted by G_u .

For a symmetric space X with associated symmetric pair (G, K) the corresponding involutive automorphism σ is unique up to conjugation by an element of K . Consequently, the associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is unique up to the same conjugation. Furthermore, we recall Definition I.2 and note that in fact $\exp(\mathfrak{k}) = K$.

The following duality⁵ principle follows from Cartan's work [Car09]:

Theorem I.50. *If $X = G/K$ is an irreducible symmetric space of non-positive curvature. Then the compact dual of G given by*

$$X_u := G_u/K$$

is an irreducible symmetric space of non-negative curvature.

⁵The duality is classically expressed through spaces of compact and non-compact type, which requires additional notions: orthogonal symmetric Lie algebras and Killing forms [Hel78, Ch. V, §1]. Instead, we adopt the equivalent characterization through curvature, which suffices for our purposes.

Proof. For a proof we refer to [Hel78, Ch. V, Proposition 2.1] and the discussion in [Hel78, Ch. V, Theorem 3.1 and §4-5]. \square

Rank

For symmetric spaces, the algebraic notion of *rank* from Definition I.5 and the geometric one in Definition I.27 coincide (see [Hel78, Ch. V, Proposition 6.1 and Theorem 6.2]). In fact, unlike for general Riemannian manifolds, where such phenomena are rare, the rank is a significant invariant for symmetric spaces, with many distinct, non-trivial flats through each point.

Remarkably a result due to [Hei74] asserts that symmetric space of negative curvature with uniform upper bound strictly less than zero must have rank one. These spaces have been fully classified. We illustrate the irreducible simply connected symmetric spaces of rank 1, in view of Theorem I.50, in Table I.1. We note that irreducible symmetric spaces of non-positive curvature are necessarily of pinched negative curvature [Hei74].

Lie Group G	Symmetric Space G/K	Compact Dual G_u/K
$\mathrm{SO}^+(n, 1)$	$\mathbb{H}_{\mathbb{R}}^n$ $= \mathrm{SO}^+(n, 1)/\mathrm{SO}(n)$	\mathbb{S}^n $= \mathrm{SO}(n+1)/\mathrm{SO}(n)$
$\mathrm{SU}(n, 1)$	$\mathbb{H}_{\mathbb{C}}^n$ $= \mathrm{SU}(n, 1)/\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$	$\mathbb{C}\mathbb{P}^n$ $= \mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$
$\mathrm{Sp}(n, 1)$	$\mathbb{H}_{\mathbb{H}}^n$ $= \mathrm{Sp}(n, 1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1)$	$\mathbb{H}\mathbb{P}^n$ $= \mathrm{Sp}(n+1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1)$
F_4^{-20}	$\mathbb{H}_{\mathbb{O}}^2$ $= F_4^{-20}/\mathrm{Spin}(9)$	$\mathbb{O}\mathbb{P}^2$ $= F_4/\mathrm{Spin}(9)$

Table I.1: Full classification for irreducible rank 1 simply connected symmetric spaces of negative curvature G/K and the compact duals G_u/K (cf. [Hel78, Ch. X]).

Maximal Compact and Minimal Parabolic Subgroups

We assume that X is an irreducible simply-connected symmetric space of non-positive curvature and let $G = \mathrm{Isom}^\circ(X)$ be the identity component of its isometry group. Then for any point $p \in X$, the stabilizer $K = \mathrm{Stab}_G(p)$ is a *maximal* compact subgroup of G . Moreover, K is connected, and any other maximal compact subgroup G is conjugate to K [Hel78, Ch. VI, §2, Theorems 2.1 and 2.2].

Now, just like we can represent $X = G/K$ as a homogeneous space, in the same spirit this can be done for the boundary ∂X , even though more intricate considerations are required. However, for the context of this work, it is sufficient to assume that X is of rank 1. Restricting ourselves to this setting we define:

Definition I.51. Let $X = G/K$ be a rank 1 irreducible simply-connected symmetric space of non-positive curvature. A *minimal parabolic subgroup* $P \subset G$ is defined as the stabilizer of a basepoint $\xi \in \partial X$:

$$P := \mathrm{Stab}_G(\xi) = \{g \in G \mid g \cdot \xi = \xi\}.$$

All minimal parabolic subgroups in G are conjugate among each other, and for any minimal parabolic subgroup $P < G$ there is a G -equivariant homeomorphism $\partial X \cong G/P$.

To analyze the boundary structure and compactifications of X , it is essential to understand the structure of minimal parabolic subgroups. This is provided by the *Langlands decomposition* (coincides with the Iwasawa decomposition in rank 1):

Theorem I.52. *Let G be a connected semisimple Lie group of rank 1 and let $P \leq G$ be a minimal parabolic subgroup. Then P admits a decomposition*

$$P = MAN,$$

where M is a compact subgroup, A is a one-dimensional maximal \mathbb{R} -split torus, N is a connected, simply connected, nilpotent subgroup.

Proof. The proof can be found in [Kna96, §VII.7, p. 418]. □

For higher rank semi-simple Lie groups, the definition of minimal parabolic subgroups is more subtle and involves choosing special points called regular points at infinity, related to the geometry of Weyl chambers. It is in fact more common to take the characterization in Theorem I.52 as the definition [War72, Ch. 1.2]

Remark. In light of our previous discussion on amenable groups, we note that maximal compact subgroups are clearly amenable, whereas for minimal parabolic groups in rank 1 amenability follows from the Langland's decomposition: each of the factors is amenable and further amenability is preserved under extensions.

I.3.2 Hermitian Symmetric Space

We now discuss an important class of manifolds in the intersection of symmetric spaces and Kähler manifolds; we refer to [Hel78, Ch. VIII, §4, p. 372]. In particular, we extend our Definition I.46 to the following one:

Definition I.53. Let (X, g, J) be a Hermitian manifold. We say that (X, g, s, J) is a *locally Hermitian symmetric space* (X, g, s, J) if the underlying Riemannian manifold (X, g) is a locally symmetric space and moreover for $p \in X$, the geodesic symmetry $s_p : B_{\text{inj}(p)}(0_p) \rightarrow B_{\text{inj}(p)}(0_p)$ is holomorphic, i.e. $s_p^* J = J$.

We say that (X, g, s, J) is a (*globally*) *Hermitian symmetric space* if (X, g) is complete and if for any $p \in X$ there exists a holomorphic isometry in $\text{Isom}(X)$ that restricts to a geodesic symmetry on a neighbourhood of p .

We note that if X is a Hermitian symmetric space that is not flat is simply connected (cf. [Ch. VIII, §4, Theorem 4.6][Hel78]). If further X is irreducible and has non-positive curvature, there is a negative lower bound on the holomorphic sectional curvature depending on its rank. It is customary to normalize the minimal holomorphic sectional curvature to -1 .

Characterization as Kähler Manifolds

In view of Subsection I.2.3, since both the metric g and complex structure J are preserved by geodesic symmetries, then $\nabla J = 0$. But this is equivalent to a Hermitian manifold being Kähler which gives immediately the following corollary (cf. [Hel78, Ch. VIII, Proposition 4.1]).

Corollary I.54. *Any Hermitian symmetric space is a Kähler manifold.*

Any irreducible Hermitian symmetric space of non-positive curvature can be realized by a *bounded symmetric domain*⁶ in \mathbb{C}^n that is irreducible. Notably, this realization allows to give an explicit expression to the corresponding Kähler form: Any domain D in \mathbb{C}^n admits the so-called *Bergman kernel* $K : D \times D \rightarrow \mathbb{C}$, which is the reproducing kernel function for the Hilbert space of square-integrable holomorphic functions on D (cf. [Hel78, Ch. VIII, §3]). The Bergman kernel defines a natural Hermitian metric h on the holomorphic tangent bundle TX , which gives rise to the *Bergman-Kähler form* ω_X given by

$$\omega_X = i\partial\bar{\partial}\log K(z, \bar{z}).$$

Since ω_X is closed and positive-definite, it is indeed a Kähler form.

Therefore, any irreducible Hermitian symmetric space of non-positive curvature X , with $G = \text{Isom}^\circ(X)$, admits a canonical G -invariant Kähler form, namely the *Bergman-Kähler form*. For a detailed discussion we refer to [Hel78, Ch. VIII, Theorem 7.1].

Simple Non-Compact Hermitian Lie Groups

We now summarize Lie theoretic properties for irreducible Hermitian symmetric of non-positive curvature required for this work.

Theorem I.55. *Irreducible Hermitian symmetric spaces of non-positive curvature are precisely the homogeneous spaces G/K where G is a connected non-compact simple Lie group with finite center, and K is a maximal compact subgroup whose center is isomorphic to $U(1)$.*

Proof. For a proof we refer to [Hel78, Ch. VIII, Theorem 6.1 (ii) and Proposition 6.2]. \square

Remark. (i) In [Hel78, Ch. VIII, §6], Helgason proves the classification assuming that G has trivial center, but the result remains valid when $Z(G)$ is merely finite. In that case the quotient map $G \rightarrow G/Z(G)$ induces an isomorphism of symmetric spaces

$$G/K \cong G/Z(G)/K/Z(G),$$

so one may consider the adjoint group $G/Z(G)$, which satisfies Helgason's hypotheses. Since quotienting by a finite central subgroup does not affect the $U(1)$ -factor in the center of K , all arguments carry over.

⁶A bounded symmetric domain D is connected open set in \mathbb{C}^n that is bounded and for every $z \in D$ there exists a biholomorphic map on D which is involutive and has z as isolated fixed point.

- (ii) We notice that from the symmetric pair (G, K) with properties as in the theorem above, the complex structure of $X = G/K$ can be reconstructed as follows: For the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the center $\mathfrak{z}(\mathfrak{k})$ of \mathfrak{k} is one-dimensional generated by some $Z \in \mathfrak{z}(\mathfrak{k})$. Then setting $J := \text{ad}(Z)|_{\mathfrak{p}}$ defines G -invariant complex structure. For details, we refer to [Hel78, Ch. VIII, Proposition 4.2, Theorem 4.5].

We will assume for the entirety of this work that the holomorphic sectional curvature of X , as in Theorem I.55, is normalized to -1 , and we refer to the associated Lie group G for simplicity, as the *simple non-compact Hermitian Lie group with finite center*. In this thesis, it will be considered as a target group of the set of representations $\text{Hom}(\Gamma, G)$.

We note that, up to finite cover, the following list

$$\text{SU}(p, q) \text{ for } p, q > 0, \quad \text{Sp}(n, \mathbb{R}), \quad \text{SO}^*(2n), \quad \text{SO}(2, n) \text{ for } n > 0, \quad E_{6(-14)}, E_{7(-25)},$$

provides a full classification of simple non-compact Hermitian Lie groups with finite center (cf. [Hel78, Ch. X]).

Finally, the following property of maximal compact subgroups will be crucial later on for Subsection III.3.1:

Lemma I.56. *Let G be a simple non-compact Hermitian Lie group with finite center and K a maximal compact subgroup. Then it holds that*

$$K/[K, K] \cong \text{U}(1).$$

Proof. Since K is a compact Lie group, then by [Kna96, Corollary 4.25] K is in particular reductive. Hence, it holds due to [Mil17, Proposition 12.46, Example 19.25] that

$$K = Z(K) \cdot [K, K].$$

Moreover, we observe that

$$K/[K, K] \cong Z(K)/(Z(K) \cap [K, K]) \simeq Z(K),$$

where we first use the second isomorphism theorem for groups and for the second we use that $Z(K) \cap [K, K]$ is a finite group (see [Mil17, Proposition 19.21]) allowing for the quotient to deformation retract to $Z(K)$. By Theorem I.55, the center of K is isomorphic to $\text{U}(1)$, which yields the proof. \square

I.4 Complex Hyperbolic Geometry

The symmetric spaces of rank one in Table I.1 may be considered the simplest and most geometrically accessible class of symmetric spaces, making them a natural starting point for the study of cusp structures arising from non-uniform torsion-free lattices.

Among these, the complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^n$ is, on the one hand, the next natural generalization of the well-studied real hyperbolic space $\mathbb{H}_{\mathbb{R}}^n$, and on the other hand, it may be regarded as the simplest example of an irreducible Hermitian symmetric space of non-positive curvature.

I.4.1 Complex Hyperbolic Space

We first introduce the projective model of the complex hyperbolic space, and then consider - via appropriate affine transformations - the ball model and the Siegel domain. The latter is the viewpoint to introduce algebraic properties of the boundary of the complex hyperbolic space for the present work. As classical sources for this subject we refer to [Gol99] and [Par03].

Projective Model

Upon introducing the projective model of the complex hyperbolic space, we identify $\mathrm{PU}(n, 1)$ as the group of holomorphic isometries and subsequently deduce that it is a simple non-compact Hermitian Lie group with trivial center.

Let $n \geq 1$. We consider \mathbb{C}^{n+1} endowed with the Hermitian signature $(n, 1)$ -form $\langle \cdot, \cdot \rangle_1$ given by

$$\langle z, w \rangle_1 := \sum_{i=1}^n z_i \bar{w}_i - z_{n+1} \bar{w}_{n+1},$$

for all $z, w \in \mathbb{C}^{n+1}$. The space of *negative vectors* and *nil-vectors* are given by

$$V_- = \{z \in \mathbb{C}^{n+1} \mid \langle z, z \rangle_1 < 0\} \quad \text{and} \quad V_0 = \{z \in \mathbb{C}^{n+1} \mid \langle z, z \rangle_1 = 0\},$$

respectively, allowing to introduce the following:

Definition I.57. The *complex hyperbolic space* and its boundary are defined as projectivizations of V_- and V_0 respectively:

$$\mathbb{H}_{\mathbb{C}}^n := \mathbb{P}(V_-) \quad \text{and} \quad \partial \mathbb{H}_{\mathbb{C}}^n := \mathbb{P}(V_0).$$

Remark. The projective model is useful for uniformly describing irreducible rank-one symmetric spaces of non-positive curvature. In particular, in the same spirit one obtains the real hyperbolic space as

$$\mathbb{H}_{\mathbb{R}}^n := \mathbb{P}(\{z \in \mathbb{R}^{n+1} : \langle z, z \rangle < 0\})$$

where $\langle \cdot, \cdot \rangle$ is a non-degenerate symmetric bilinear form of signature $(n, 1)$. Then for any $[z] \in \mathbb{H}_{\mathbb{R}}^n$ with representative z , one may normalize z such that $\langle z, z \rangle = -1$, and choosing the representative with as positive last coordinate z_{n+1} selects the upper sheet, so that this gives the well known *upper sheet hyperboloid*:

$$\mathbb{H}_{\mathbb{R}}^n \cong \{z \in \mathbb{R}^{n+1} \mid \langle z, z \rangle = -1, \quad z_{n+1} > 0\}.$$

We now turn to group-theoretic aspects of the complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^n$ in order to characterize it as homogeneous space.

Definition I.58. The *projective unitary group* is defined by

$$\mathrm{PU}(n, 1) := \mathrm{U}(n, 1) / \mathrm{U}(1) \cdot \mathrm{Id}_{n+1}$$

the quotient of the unitary group preserving the Hermitian form $\langle \cdot, \cdot \rangle_1$ on $\mathbb{C}^{n,1}$:

$$U(n, 1) := \left\{ g \in \mathrm{GL}_{n+1}(\mathbb{C}) \mid \langle gz, gw \rangle_1 = \langle z, w \rangle_1 \quad \forall z, w \in \mathbb{C}^{n+1} \right\},$$

by its center consisting of the scalar matrices $\lambda \cdot \mathrm{Id}_{n+1}$ with $\lambda \in U(1)$.

In fact, the identity component of the isometry group $\mathrm{Isom}(\mathbb{H}_{\mathbb{C}}^n)$ is $\mathrm{PU}(n, 1)$, which is isomorphic to the group of holomorphic isometries $\mathrm{Isom}^\circ(\mathbb{H}_{\mathbb{C}}^n)$. More precisely,

$$\mathrm{Isom}(\mathbb{H}_{\mathbb{C}}^n) \cong \mathrm{PU}(n, 1) \rtimes \mathbb{Z}_2,$$

where the \mathbb{Z}_2 -factor acts by complex conjugation, interchanging the holomorphic and anti-holomorphic isometries (see [Gol99, Lemma 3.1.3]).

Furthermore, by [Gol99, Lemma 3.1.6] the group $\mathrm{PU}(n, 1)$ acts transitively on $\mathbb{H}_{\mathbb{C}}^n$ (and on its boundary $\partial\mathbb{H}_{\mathbb{C}}^n$). Under this action the stabilizer of any basepoint $p \in \mathbb{H}_{\mathbb{C}}^n$ is

$$\mathrm{Stab}_{\mathrm{PU}(n,1)}(p) \cong P(U(n) \times U(1)) \cong U(n),$$

which is a maximal compact subgroup of $\mathrm{PU}(n, 1)$. We therefore obtain the following identification of $\mathbb{H}_{\mathbb{C}}^n$ as a homogeneous space:

$$\mathbb{H}_{\mathbb{C}}^n = \mathrm{PU}(n, 1) / P(U(n) \times U(1)).$$

From Theorem I.55 we deduce that $\mathrm{PU}(n, 1)$ is a simple non-compact Hermitian Lie group with trivial center and that $\mathbb{H}_{\mathbb{C}}^n$ is an irreducible Hermitian symmetric space of non-positive curvature.

In particular, $\mathbb{H}_{\mathbb{C}}^n$ is a Kähler manifold, and the Hermitian form $\langle \cdot, \cdot \rangle_1$ induces a Riemannian metric compatible with the complex structure in the sense of Definition I.29. Explicitly, following [Gol99, pp. 73–74], the metric is expressed as follows: For a point $[z] \in \mathbb{H}_{\mathbb{C}}^n$, represented by a vector $z \in V_-$, the tangent space $T_{[z]}\mathbb{H}_{\mathbb{C}}^n$ is identified with the quotient $z^\perp/\mathbb{C}z$, where the orthogonal complement is taken with respect to $\langle \cdot, \cdot \rangle_1$. The Riemannian metric g at $[z]$ is then given by the positive-definite bilinear form

$$g_{[z]}(v, w) = \frac{\langle v, w \rangle_1}{\langle z, z \rangle_1} - \frac{\langle v, z \rangle_1 \langle z, w \rangle_1}{\langle z, z \rangle_1^2}, \quad v, w \in z^\perp. \quad (\text{I.2})$$

This metric yields a holomorphic sectional curvature normalized to -1 (see Definition I.31), while the corresponding real sectional curvature (see Definition I.26) is pinched in the interval

$$-1 \leq \sec(\sigma) \leq -\frac{1}{4},$$

for any real two-plane $\sigma \subset T_p\mathbb{H}_{\mathbb{C}}^n$ (cf. [Gol99, p. 93]). The upper bound $-1/4$ is attained on a totally real Lagrangian plane and the lower bound -1 is attained on a complex line (for definitions of the totally geodesic subspaces of $\mathbb{H}_{\mathbb{C}}^n$, see [Gol99, pp. 74, 80]).

Remark. We note that in case $n = 1$, the complex hyperbolic space can be identified with the real hyperbolic plane, and the associated isometry groups coincide: For $n = 1$, the projective

unitary group admits the identification

$$\mathrm{PU}(1,1) \cong \mathrm{SO}^+(2,1) \cong \mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}_{\mathbb{R}}^2),$$

where the latter denotes the group of orientation-preserving isometries of the real hyperbolic space. In particular, the complex hyperbolic line $\mathbb{H}_{\mathbb{C}}^1$ is isometric to the real hyperbolic plane $\mathbb{H}_{\mathbb{R}}^2$.

Lie Theoretic Characterization

We aim to characterize the compact dual of $\mathrm{PU}(n,1)$, for which we pass to the Lie algebraic side (cf. [Gol99, §4.1.3]). We have that $\mathrm{Lie}(\mathrm{PU}(n,1)) \cong \mathfrak{su}(n,1)$ with

$$\mathfrak{su}(n,1) = \{X \in \mathfrak{su}(n+1, \mathbb{C}) \mid X^* J + JX = 0\}. \quad \text{where } J = \mathrm{diag}(\mathrm{Id}_n, -1).$$

The Cartan involution $\theta(X) = -X^*$ induces the Cartan decomposition

$$\mathfrak{su}(n,1) = \mathfrak{k} \oplus \mathfrak{p},$$

with the $+1$ -eigenspace \mathfrak{k} and the -1 -eigenspace \mathfrak{p} given by

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & i\alpha \end{pmatrix} : A \in \mathfrak{u}(n), \alpha \in \mathbb{R}, \mathrm{tr} A + i\alpha = 0 \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix} : v \in \mathbb{C}^n \right\},$$

In particular, we see that the projectivizations of $K = \exp(\mathfrak{k})$ and $S(\mathrm{U}(n) \times \mathrm{U}(1))$ are isomorphic (recovering K as the stabiliser of a basepoint in $\mathbb{H}_{\mathbb{C}}^n$). For the compact real form of \mathfrak{g} given by $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$, one can check that $\mathfrak{g}_c \cong \mathfrak{su}(n+1)$. Therefore, the compact dual of $\mathrm{PU}(n,1)$ is given by

$$\mathrm{PSU}(n+1) / P(S(\mathrm{U}(n) \times \mathrm{U}(1))) \cong \mathrm{SU}(n+1) / S(\mathrm{U}(n) \times \mathrm{U}(1)).$$

The quotient is isomorphic to the complex projective space $\mathbb{C}\mathbb{P}^n$ and is of holomorphic sectional curvature 1.

Ball Model

One alternative model of the complex hyperbolic space is given by the *ball model* $\mathbb{B}_{\mathbb{C}}^n$ defined as

$$\mathbb{B}_{\mathbb{C}}^n := \{z \in \mathbb{C}^n : \langle z, z \rangle_{\mathbb{C}^n} < 1\},$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^n}$ denotes the standard Hermitian form, namely $\langle z, w \rangle_{\mathbb{C}^n} = \sum_{j=1}^n z_j \bar{w}_j$. In fact, the affine map $A_1: \mathbb{C}^n \rightarrow \mathbb{P}(\mathbb{C}^{n+1})$, defined by

$$A_1(z) = \left[\begin{array}{c} z \\ 1 \end{array} \right]$$

with respect to the Hermitian $(n, 1)$ -form $\langle \cdot, \cdot \rangle_1$, realizes the biholomorphism $\mathbb{B}_{\mathbb{C}}^n \cong \mathbb{H}_{\mathbb{C}}^n$ (with a similar computation as later in Proposition I.60). One can then endow $\mathbb{B}_{\mathbb{C}}^n$ with the metric A_1^*g by pulling back the metric in (I.2) via the map A_1 . An illustration of this model is provided in Figure I.2. In view of the remark in Subsection I.3.2, we observe that $\mathbb{B}_{\mathbb{C}}^n$ is in fact a bounded symmetric domain in \mathbb{C}^n and due to the discussion in [Gol99, §3.1.3]), it can be derived that the Bergman-Kähler form on $\mathbb{B}_{\mathbb{C}}^n$ is given by

$$\omega_{\mathbb{H}_{\mathbb{C}}^n} = i\partial\bar{\partial}\log(1 - \langle z, z \rangle_{\mathbb{C}^n}).$$

Siegel Domain

We consider yet another model, namely the Siegel domain, which is the interior of a paraboloid - a particularly useful model to study the boundary of the complex hyperbolic space. In particular, we wish to first provide a characterization of the horospheres that foliate the Siegel domain that provides a natural coordinate system for studying the boundary. For this section we refer to conventions⁷ in [Par03, §2.2, §4].

Definition I.59. The *Siegel domain* \mathfrak{S}^n and its boundary $\partial\mathfrak{S}^n$ are defined respectively, as follows:

$$\mathfrak{S}^n := \left\{ \begin{pmatrix} z_1 \\ z' \end{pmatrix} \in \mathbb{C} \times \mathbb{C}^{n-1} \mid 2\operatorname{Re}z_1 + \langle z', z' \rangle_{\mathbb{C}^{n-1}} < 0 \right\},$$

$$\text{and } \partial\mathfrak{S}^n := \left\{ \begin{pmatrix} z_1 \\ z' \end{pmatrix} \in \mathbb{C} \times \mathbb{C}^{n-1} \mid 2\operatorname{Re}z_1 + \langle z', z' \rangle_{\mathbb{C}^{n-1}} = 0 \right\}.$$

This indeed just another model of $\mathbb{H}_{\mathbb{C}}^n$ and one uses a similar affine map as for the ball model with respect to a different Hermitian form:

Lemma I.60. The affine map $A_2: \mathbb{C}^n \rightarrow \mathbb{P}(\mathbb{C}^{n+1})$, defined by

$$A_2(z) = \left[\begin{array}{c} z \\ 1 \end{array} \right]$$

with respect to the Hermitian $(n, 1)$ -form $\langle \cdot, \cdot \rangle_2: \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ given by

$$\langle z, w \rangle_2 = z_1 \bar{w}_{n+1} + \sum_{i=2}^n z_i \bar{w}_i + z_{n+1} \bar{w}_1,$$

⁷We would like to point out that Goldman uses different conventions for the Siegel model [Gol99, §4.1], where no change of coordinates is used.

realizes a bilomorphism $\mathfrak{S}^n \cong \mathbb{H}_{\mathbb{C}}^n$ and extends continuously to a homeomorphism between the compactified paraboloid $\partial\mathfrak{S}^n \cup \infty$ (Alexandroff's one-point compactification) and the boundary $\partial\mathbb{H}_{\mathbb{C}}^n$.

Proof. Let $[z] \in \mathbb{H}_{\mathbb{C}}^n \subset \mathbb{P}(\mathbb{C}^{n+1})$, and write

$$z = \begin{pmatrix} z_1 \\ z' \\ z_{n+1} \end{pmatrix} \in \mathbb{C}^{n+1}, \quad \text{where } z' = (z_2, \dots, z_n)^t.$$

Then we have that

$$\langle z, z \rangle_2 = z_1 \bar{z}_{n+1} + \langle z', z' \rangle_{\mathbb{C}^{n-1}} + z_{n+1} \bar{z}_1 = 2\operatorname{Re}(z_1 \bar{z}_{n+1}) + \langle z', z' \rangle_{\mathbb{C}^{n-1}}.$$

The condition $\langle z, z \rangle_2 < 0$ implies $z_{n+1} \neq 0$, so that

$$[z] = \left[\frac{1}{z_{n+1}} z \right] = \left[\left(\frac{z_1}{z_{n+1}}, \frac{z'}{z_{n+1}}, 1 \right)^t \right] \in \operatorname{Im}(A_2),$$

and in particular, A_2 realizes an bilomorphism between \mathfrak{S}^n and $\mathbb{H}_{\mathbb{C}}^n$.

Next, we note that A_2 extends continuously to the boundary $\partial\mathbb{H}_{\mathbb{C}}^n$. By considering the case $\langle z, z \rangle_2 = 0$ with $z_{n+1} \neq 0$, we again have

$$[z] = \left[\left(\frac{z_1}{z_{n+1}}, \frac{z'}{z_{n+1}}, 1 \right)^t \right] \in \operatorname{Im}(A_2).$$

On the other hand, if $z_{n+1} = 0$, then $\langle z', z' \rangle_{\mathbb{C}^{n-1}} = 0$, which implies $z' = 0$ and hence $z = (z_1, 0, 0)$. Consequently, the point $\infty \in \partial\mathfrak{S}^n$ corresponds to the vector $[1, 0, \dots, 0]^t \in \mathbb{H}_{\mathbb{C}}^n$ in the projective model. Therefore, we have the homeomorphism $\partial\mathfrak{S}^n = A_2^{-1}(\partial\mathbb{H}_{\mathbb{C}}^n) \cong \partial\mathbb{H}_{\mathbb{C}}^n$. \square

The Siegel domain \mathfrak{S}^n can be endowed with the Riemannian metric A_2^*g by pulling back g in (I.2) along the map A_1 (cf. [Gol99, §4.1.2, (4.4)]). This pullback metric induces the distance function on the Siegel domain, which we take as the definition of the complex hyperbolic distance in the Siegel domain model (we note that the formula is straightforwardly obtained directly from the distance formula for the projective model [Gol99, §3.1.6, (3.4)]):

Definition I.61. For any $(z_1, z')^t, (w_1, w')^t \in \mathfrak{S}^n$, the distance function $\operatorname{dist}(\cdot, \cdot)$ is given by

$$\cosh^2 \left(\frac{\operatorname{dist}((z_1, z')^t, (w_1, w')^t)}{2} \right) = \frac{|\bar{z}_1 + \langle z', w' \rangle_{\mathbb{C}^{n-1}} + w_1|^2}{(2\operatorname{Re}(z_1) + \langle z', z' \rangle_{\mathbb{C}^{n-1}}) \cdot (2\operatorname{Re}(w_1) + \langle w', w' \rangle_{\mathbb{C}^{n-1}})}.$$

Using the Busemann function, we obtain an explicit form for the horoballs and horospheres (cf. Definition I.36):

Proposition I.62. The Busemann function centered at $\infty \in \partial\mathbb{H}_{\mathbb{C}}^n$ takes the form

$$b_{\infty}((z_1, z')) = 2\operatorname{Re}(z_1) + \langle z', z' \rangle_{\mathbb{C}^{n-1}} \quad \forall (z_1, z') \in \mathbb{C} \times \mathbb{C}^{n-1}$$

and hence horoballs and horospheres in the Siegel domain centered at ∞ and of radius $s \in \mathbb{R}_{>0}$, are given by

$$H_s(\infty) = \left\{ \begin{pmatrix} z_1 \\ z' \end{pmatrix} \in \mathbb{C}^n \mid 2\operatorname{Re}(z_1) + \langle z', z' \rangle_{\mathbb{C}^{n-1}} = s \right\}$$

$$\text{and } B_s(\infty) = \left\{ \begin{pmatrix} z_1 \\ z' \end{pmatrix} \in \mathbb{C}^n \mid 2\operatorname{Re}(z_1) + \langle z', z' \rangle_{\mathbb{C}^{n-1}} < s \right\}.$$

Proof. For a proof we refer to [Gol99, §4.1.2]. □

This is to say that horospheres centered at ∞ are paraboloids that are obtained by vertically shifting the boundary $\partial\mathfrak{S}^n$ (see Figure I.2). We note that the Siegel domain can be viewed as the analogue of the upper-half space model of the real hyperbolic space, where horospheres are level sets of a height function.

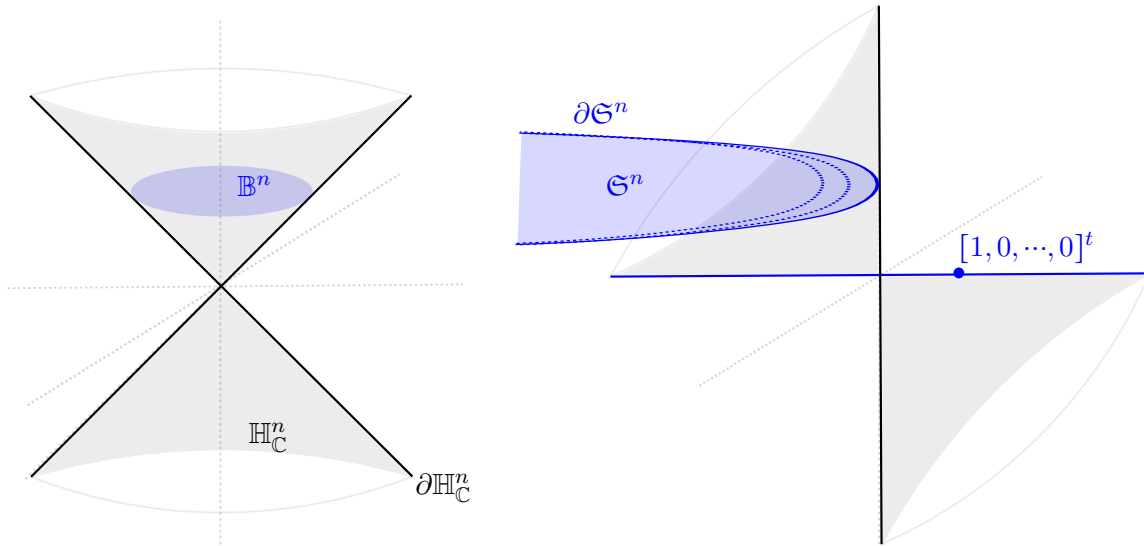


Figure I.2: On the left: projectivized cone lines and the ball model \mathbb{B}^n . On the right: projectivized cone lines under coordinate change and the Siegel domain \mathfrak{S}^n together with its boundary $\partial\mathfrak{S}^n$ and a distinguished point corresponding to ∞ . Horospheres centered at ∞ appear as dashed paraboloids.

Heisenberg Group

We observe that the boundary of the Siegel domain $\partial\mathfrak{S}^n$, can be alternatively expressed as

$$\partial\mathfrak{S}^n = \left\{ \begin{pmatrix} -\frac{1}{2}\langle z', z' \rangle_{\mathbb{C}^{n-1}} + it \\ z' \end{pmatrix} \in \mathbb{C} \times \mathbb{C}^{n-1} \mid z' \in \mathbb{C}^{n-1}, t \in \mathbb{R} \right\},$$

so that $\partial\mathfrak{S}^n$ admits the following parametrization:

$$\begin{aligned} \mathbb{C}^{n-1} \times \mathbb{R} &\leftrightarrow \partial\mathfrak{S}^n, \\ (z, t) &\mapsto \begin{pmatrix} -\langle z', z' \rangle_{\mathbb{C}^{n-1}} + it \\ \sqrt{2}z \end{pmatrix}. \end{aligned}$$

This set can be endowed with the following group structure:

Definition I.63. The *real Heisenberg group* $\mathcal{H}_{\mathbb{R}}^{2n-1}$ is given by the set $\mathbb{C}^{n-1} \times \mathbb{R}$ equipped with the following group law

$$(z, t) \bullet (w, s) := (z + w, t + s + 2\text{Im}\langle z, w \rangle_{\mathbb{C}^{n-1}}).$$

We observe that $[\mathcal{H}_{\mathbb{R}}^{2n-1}, \mathcal{H}_{\mathbb{R}}^{2n-1}] \cong \mathbb{R}$ and in particular $\mathcal{H}_{\mathbb{R}}^{2n-1}$ is a 2-step nilpotent Lie group of dimension $2n - 1$. Further, clearly the Heisenberg group $\mathcal{H}_{\mathbb{R}}^{2n-1}$ naturally acts on itself, and thus on the boundary $\partial\mathfrak{S}^n$ by left-translations. This is one kind of a diffeomorphisms that preserves the group structure (up to scaling); the others act by unitary rotations and dilations, more precisely (cf. [Gol99, pp. 121–122]):

Definition I.64. The *Heisenberg similarities* $\text{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$ is the set of transformations on $\mathcal{H}_{\mathbb{R}}^{2n-1} \cong \mathbb{C}^{n-1} \times \mathbb{R}$ of the form:

$$\begin{aligned} (z, t) &\mapsto h \bullet (\lambda U z, \lambda^2 t) \\ &= (\lambda U z + z_0, \lambda^2 t + t_0 + 2\text{Im}\langle \lambda U z, z_0 \rangle_{\mathbb{C}^{n-1}}), \end{aligned}$$

where $h = (z_0, t_0) \in \mathcal{H}_{\mathbb{R}}^{2n-1}$, $U \in \text{U}(n - 1)$, and $\lambda \in \mathbb{R}_{>0}$. The *Heisenberg isometries* $\text{Isom}(\mathcal{H}_{\mathbb{R}}^{2n-1})$ consist of all those transformation with trivial dilations, that is $\lambda = 1$.

From this definition, we may show that $\text{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$ is endowed with a structure of a semi-direct product:

Lemma I.65. *The Heisenberg similarities $\text{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$ is a group given by the semi-direct product*

$$\text{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1}) \cong \mathcal{H}_{\mathbb{R}}^{2n-1} \rtimes (\text{U}(n - 1) \times \mathbb{R}_{>0}).$$

In particular, the Heisenberg isometries $\text{Isom}(\mathcal{H}_{\mathbb{R}}^{2n-1})$ is a subgroup of $\text{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$ given by

$$\text{Isom}(\mathcal{H}_{\mathbb{R}}^{2n-1}) = \mathcal{H}_{\mathbb{R}}^{2n-1} \rtimes \text{U}(n - 1).$$

Proof. By definition, every similarity is a transformation of the form

$$(z, t) \mapsto h \bullet (\lambda U z, \lambda^2 t),$$

where $h \in \mathcal{H}_{\mathbb{R}}^{2n-1}$, $U \in \text{U}(n - 1)$, and $\lambda \in \mathbb{R}_{>0}$. Hence, each element of $\text{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$ can be uniquely expressed as a pair $(h, (U, \lambda))$. We define a homomorphism

$$\phi : \text{U}(n - 1) \times \mathbb{R}_{>0} \rightarrow \text{Aut}(\mathcal{H}_{\mathbb{R}}^{2n-1}), \quad \phi(U, \lambda)(z, t) := (\lambda U z, \lambda^2 t),$$

which describes how the dilation and unitary part acts on the Heisenberg group by automorphisms. The group operation then satisfies

$$(h_1, (U_1, \lambda_1)) \cdot (h_2, (U_2, \lambda_2)) = (h_1 \bullet \phi(U_1, \lambda_1)(h_2), (U_1 U_2, \lambda_1 \lambda_2)),$$

which is precisely the group law for the semidirect product $\mathcal{H}_{\mathbb{R}}^{2n-1} \rtimes_{\phi} (\mathrm{U}(n-1) \times \mathbb{R}_{>0})$.

Setting $\lambda = 1$, recovers the Heisenberg isometries. \square

In view of Theorem I.52, the group of Heisenberg similarities is in fact a minimal parabolic subgroup in P in $\mathrm{PU}(n, 1)$ (see Definition I.51), where the decomposition of P given above corresponds to the Langland's decomposition:

Proposition I.66. *The following isomorphism holds:*

$$\mathrm{Stab}_{\mathrm{PU}(n,1)}(\infty) \cong \mathrm{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$$

Proof. We refer to [Gol99, §4.2.2]. \square

We note that in analogy to the upper-half space in the real hyperbolic case, we have the analogous decomposition

$$\mathrm{Stab}_{\mathrm{SO}^+(n,1)}(\infty) \cong \mathbb{R}^{n-1} \rtimes (\mathrm{SO}(n) \times \mathbb{R}_{>0}),$$

Owing to the presence of the nilpotent $\mathcal{H}_{\mathbb{R}}^{2n-1}$ in place of the abelian group \mathbb{R}^{n-1} , the boundary structure of the complex hyperbolic space is more intricate than that of the real hyperbolic space. This will show in particular in the next section, in the discussion of the cusp-structure of complete non-compact complex hyperbolic manifolds of finite volume.

I.4.2 Complex Hyperbolic Lattices and Cusps

Since $\mathrm{PU}(n, 1)$ is a semi-simple Lie group, it admits torsion-free lattices (as mentioned previously in Subsection I.1.2). Any torsion-free lattice $\Gamma < \mathrm{PU}(n, 1)$ falls into one of the following two types (cf. Definition I.6):

$$M = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n \text{ is a } \begin{cases} \text{compact complex hyperbolic } n\text{-manifold,} & \text{if } \Gamma \text{ is uniform,} \\ \text{non-compact complex hyperbolic } n\text{-manifold of finite volume,} & \text{if } \Gamma \text{ is non-uniform.} \end{cases}$$

Following our discussion on the thick-thin decomposition, the quotient $M = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$, where $\Gamma < \mathrm{PU}(n, 1)$ is a non-uniform torsion-free lattice, admits a decomposition into a compact core N and its complement $M \setminus N$, which is homeomorphic to a disjoint union of cusps. We recall that for sufficiently large $s \in \mathbb{R}_{>0}$, the cusp is realized as a horoball based at a point $\xi \in \partial \mathbb{H}_{\mathbb{C}}^n$

$$\mathcal{C} = E \backslash B_s(\xi),$$

where $E := \Gamma \cap \mathrm{Stab}_{\mathrm{PU}(n,1)}(\xi)$ is a uniform torsion-free lattice. In fact, we may assume that $E < \mathrm{Stab}_{\mathrm{PU}(n,1)}(\infty)$, up to conjugation. In particular, E is a discrete parabolic subgroup and

thus preserves horospheres setwise. Now, any horosphere $H_s(\infty)$ in Proposition I.62 can be viewed as

$$H_s(\infty) = \left\{ \left(\begin{array}{c} -\langle z', z' \rangle_{\mathbb{C}^{n-1}} + it + \frac{s}{2} \\ \sqrt{2}z \end{array} \right) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid z' \in \mathbb{C}^{n-1}, t \in \mathbb{R} \right\},$$

parametrized by $\mathbb{C}^{n-1} \times \mathbb{R}$ and endowed with the Heisenberg group law \bullet , similar as for the boundary $\partial\mathfrak{S}^n$. By considering explicitly the action of $\text{Stab}_{\text{PU}(n,1)}(\infty)$ on $\mathbb{C}^{n-1} \times \mathbb{R}$ (see Definition I.64), clearly the groups $\mathcal{H}_{\mathbb{R}}^{2n-1}$ and $\text{U}(n-1)$ preserve $H_s(\infty)$. As for a non-trivial dilation $\lambda \in \mathbb{R}_{>0} \setminus \{1\}$, for the corresponding action given by $(z, t) \mapsto (\lambda z, \lambda^2 t)$, we see that

$$-\langle \lambda z', \lambda z' \rangle_{\mathbb{C}^{n-1}} + \lambda^2 t + \frac{s}{2} = \lambda^2 (-\langle z', z' \rangle_{\mathbb{C}^{n-1}} + t) + \frac{s}{2} \neq -\langle z', z' \rangle_{\mathbb{C}^{n-1}} + t + \frac{s}{2}.$$

It follows that E is necessarily contained in the subgroup of Heisenberg isometries:

$$\text{Isom}(\mathcal{H}_{\mathbb{R}}^{2n-1}) = \mathcal{H}_{\mathbb{R}}^{2n-1} \rtimes \text{U}(n-1).$$

We introduce common terminology for these cusp regions following [Dek96, §2.2]:

Definition I.67. Let $B_s(\infty) \subset \mathbb{H}_{\mathbb{C}}^n \cup \partial\mathbb{H}_{\mathbb{C}}^n$ be a horoball for sufficiently large $s \in \mathbb{R}_{>0}$. Then the horoball quotient

$$\mathcal{C} = E \backslash B_s(\infty)$$

- is an *infrnilmanifold*, if E is a uniform torsion-free lattice contained in $\mathcal{H}_{\mathbb{R}}^{2n-1} \rtimes \text{U}(n-1)$. In this case E is called an *almost-Bieberbach group*.
- is a *nilmanifold*, if E is a uniform torsion-free lattice contained in $\mathcal{H}_{\mathbb{R}}^{2n-1}$. In this case E is called a *nilpotent almost-Bieberbach group*.

There is a generalization of the classical Bieberbach theorem [Bie11; Bie12] to simply connected nilpotent groups due to Auslander, which we adopt directly for our setting in terms of the real Heisenberg group $\mathcal{H}_{\mathbb{R}}^{2n-1}$ (cf. [Dek96, Theorem 2.2.1]):

Theorem I.68 ([Aus60],[Gro78],[BK81, p. 10],[HP96, Proposition 5.8]).

Let $E < \mathcal{H}_{\mathbb{R}}^{2n-1} \rtimes \text{U}(n-1)$ be a uniform torsion-free lattice. Then the following statements hold:

- $\Delta := E \cap \mathcal{H}_{\mathbb{R}}^{2n-1}$ is a uniform lattice in $\mathcal{H}_{\mathbb{R}}^{2n-1}$,
- There exists a universal constant $I_n \in \mathbb{N}_{>0}$ such that

$$[E : \Delta] < I_n$$

where $I_n \leq 2(6\pi)^{(2n-1)(n-1)}$ and $I_2 = 6$.

That is to say, that any infrnilmanifold has a finite cover that is a nilmanifold. This is in analogy to the case of real hyperbolic manifolds with cusps, where each cusp cross-section is a compact flat manifold, which admits a finite cover that is a torus.

It is important to note for the context of this thesis, that Auslander provides the following explicit characterization for nilmanifolds:

Theorem I.69 ([Aus77, §1.2, Theorems I.2.6.-I.2.9.]). *Any uniform torsion-free lattice $\Delta < \mathcal{H}_{\mathbb{R}}^{2n-1}$ is isomorphic as an abstract group to*

$$\Delta := \Delta_{2n-1}(d, r_1, \dots, r_{n-1}) = \left\langle \begin{array}{l} x_1, \dots, x_{n-1}, \\ y_1, \dots, y_{n-1}, z \end{array} \left| \begin{array}{l} [x_i, y_i] = z^{k_i} \quad \text{for } i = 1, \dots, n-1, \\ [x_i, x_j] = [y_i, y_j] = [x_i, y_j] = e \quad \text{for } i \neq j, \\ [z, x_i] = [z, y_i] = e \quad \text{for all } i \end{array} \right. \right\rangle$$

such that $k_i = dr_i$ where $r_1 = 1$, and $r_i \mid r_{i+1}$ for all $i = 1, \dots, n-2$ and $d \in \mathbb{Z}_{>0}$ is the index of $[\Delta, \Delta]$ in the center $Z(\Delta)$.

When $k_i = 1$ for all $1 \leq i \leq n$, this reduces to the standard integer Heisenberg lattice (cf. [Kap19, Example 8.3]). In particular, from the above presentation, one directly deduces that the abelianisation of any uniform torsion-free lattice $\Delta < \mathcal{H}_{\mathbb{R}}^{2n-1}$ is given by:

$$\Delta / [\Delta, \Delta] \cong \mathbb{Z}_d \oplus \mathbb{Z}^{2n-2},$$

where \mathbb{Z}_d denotes the finite cyclic group of order d , as specified in the theorem. This fact will be important for the some of the central results of this thesis (see Proposition IV.9, Corollary, IV.10 and Theorem E).

Example I.70. For $n = 2$, all families of almost-Bieberbach groups in $\text{Isom}(\mathcal{H}_{\mathbb{R}}^3)$ are listed in [Dek96, pp. 159–167]; we display these with corresponding names used in [PS24] and the corresponding indices of the associated finite-index nilpotent subgroups in view of the generalized Bieberbach theorem I.68:

- (i) *Nil-tori* encompassing all instances of nilpotent almost-Bieberbach groups (cf. Theorem I.69):

$$\Delta := \Delta_3(1, 1, 1) = \langle a, b, c \mid [b, a] = c^k, [c, a] = [c, b] = 1 \rangle, \quad \forall k \in \mathbb{N}.$$

- (ii) *Vertical half-twist E* with index $[E : \Delta] = 2$:

$$E = \langle a, b, c, \alpha \mid [b, a] = c^k, [c, a] = [c, b] = [c, \alpha] = 1, \alpha a = a^{-1}\alpha, \alpha b = b^{-1}\alpha, \alpha^2 = c^i \rangle, \quad \forall k \in 2\mathbb{N}.$$

- (iii) *Horizontal half-twist E* with index $[E : \Delta] = 2$:

$$E = \langle a, b, c, \alpha \mid [b, a] = c^{2k}, [c, a] = [c, b] = [a, \alpha] = 1, \alpha b = b^{-1}\alpha c^{-k}, \alpha c = c^{-1}\alpha, \alpha^2 = a^i \rangle, \quad \forall k \in \mathbb{N}.$$

- (iv) *Double half-twist* with index $[E : \Delta] = 4$:

$$\begin{aligned} E = \langle a, b, c, \alpha, \beta \mid [b, a] = c^{2k}, [c, a] = [c, b] = [c, \alpha] = [a, \beta] = 1, \\ \alpha a = a^{-1}\alpha c^k, \alpha b = b^{-1}\alpha c^{-k}, \\ \beta b = b^{-1}\beta c^{-k}, \beta c = c^{-1}\beta, \\ \alpha \beta = a^{-1}b^{-1}\beta \alpha c^{-k-1}, \alpha^2 = c, \beta^2 = a^i \rangle, \quad \forall k \in 2\mathbb{N}. \end{aligned}$$

(v) $1/4$ -twist E with index $[E : \Delta] = 4$:

$$E = \langle a, b, c, \alpha \mid [b, a] = c^k, [c, a] = [c, b] = [c, \alpha] = 1, \alpha a = b\alpha, \alpha b = a^{-1}\alpha, \alpha^4 = c^{pi} \rangle$$

with $k \in 2\mathbb{N}$ and either $p = 1$ or $k \in 4\mathbb{N}$, $p = 3$.

(vi) $1/3$ -twist E with index $[E : \Delta] = 3$:

$$E = \langle a, b, c, \alpha \mid [b, a] = c^k, [c, a] = [c, b] = [c, \alpha] = 1, \alpha a = b\alpha c^{k_1}, \alpha b = a^{-1}b^{-1}\alpha, \alpha^3 = c^{k_2 i} \rangle$$

with $k > 0$ and:

- if $k \equiv 0 \pmod{3}$, then $k_1 = 0$, $k_2 = 1$ or 2 ;
- if $k \equiv 1, 2 \pmod{3}$, then $k_1 = 1$, $k_2 = 1$.

(vii) $1/6$ -twist E with index $[E : \Delta] = 6$:

$$E = \langle a, b, c, \alpha \mid [b, a] = c^k, [c, a] = [c, b] = [c, \alpha] = 1, \alpha a = ab\alpha, \alpha b = a^{-1}\alpha, \alpha^6 = c^{k_1 i} \rangle$$

with $k > 0$ and:

- if $k \equiv 0, 4 \pmod{6}$, then $k_1 = 1$;
- if $k \equiv 0, 2 \pmod{6}$, then $k_1 = 5$.

As noted in [PS24, p. 4], each of these families modelling infranilmanifolds corresponds to one of the six orientable flat 3-manifolds, which were fully classified by Bieberbach (see [Bie11; Bie12] and cf. [Wol11, Ch. III]). In particular, by setting $k = 0$ and choosing suitable values of p , k_1 and k_2 , one obtains exactly these six manifolds.

In the real hyperbolic setting, Long and Reid showed in [LR00] that only certain flat 3-manifolds occur as the cusp cross-section of one-cusped finite-volume, complete real hyperbolic 4-manifolds. The first explicit example of such a 4-manifold was later constructed in [KM13]. For higher dimensions $n \geq 5$, the existence of one-cusped real-hyperbolic $(n+1)$ -manifolds remains an open question. In contrast, in higher dimensions allowing for multiple cusps, any flat n -manifold can occur as a cusp cross-section of a real hyperbolic $(n+1)$ -manifold [LR02; McR09], at least whenever these are modelled by arithmetic lattices.

In the complex hyperbolic setting, there have been some parallel developments. Remarkably, a geometrically explicit of a single-cusped complex hyperbolic surface was only provided recently in an elegant construction by Deraux and Stover in [DS24], that produces a one cusped complex hyperbolic surfaces of volume $16\pi^2 \cdot d$ for odd $d \geq 1$. The existence of a variety of cusped complex hyperbolic surfaces with various properties, including the structure of cusps, has been shown using computer-based methods with Magma by Deraux in [Der24]. Moreover, allowing for multiple cusps, Paupert and Sell showed in [PS24] that any of the infranilmanifolds modelled by lattices in Example I.70 can occur as a cusp cross-section. In higher dimensions, allowing for multiple cusps, any infranilmanifold can occur as cusp cross-section of a complex hyperbolic manifold [McR04; McR09], again at least for those modelled by arithmetic lattices.

Remark. Over the past decades, several sophisticated algebraic and arithmetic approaches have been developed for constructing lattices in $\mathrm{PU}(2,1)$, including Deligne–Mostow lattices and fake projective planes [Par09]. While these methods are diverse and provide existence results, they are generally non-explicit, in the sense that they often rely on abstract arithmetic or geometric arguments rather than providing a concrete geometric description of the lattice or its fundamental domain.

Chapter II

Variants of Cohomology Theories

The results in this thesis rely on a confluence of a whole variety of cohomology theories we present in this chapter. For this we give an overview of cohomologies for both groups and manifolds restricting ourselves to trivial coefficient groups, more specifically real, integral and circle group coefficient groups. We highlight their features, methods and structures relevant to this thesis, and give fundamental relationships between these cohomologies that will be used in the remaining chapters. Finally, we dedicate the last section to the *Kähler and volume classes* as well as their bounded counterparts, which are instrumental to define the volume-type and the Toledo invariant in Chapter IV.

II.1 Cohomology for Groups

We first give a brief account of all group cohomologies that we will require and then state important features. Classical sources are [Bro82], [Fri17] and [Mon01].

II.1.1 Definitions and Notation

Let G be a locally compact second countable group and let A be an abelian group equipped with the trivial G -action, that is $g \cdot a = a$ for all $g \in G$ and $a \in A$. Such a G -module is referred to as having *trivial coefficients*; common examples relevant for us are \mathbb{R}, \mathbb{Z} and \mathbb{R}/\mathbb{Z} , all equipped with the trivial G -action.

For the sake of overview and uniformity, we give a template for different variants of cohomologies by imposing additional structure on G and A together with increasingly stronger regularity condition \mathcal{R} , possibly vacuous (i.e. imposing no restriction), in the following sense: For all $k \in \mathbb{N}$, we define

$$C_{\mathcal{R}}^k(G, A)^G := \left\{ f : G^{k+1} \rightarrow A \mid \begin{array}{l} f \text{ is } G\text{-invariant satisfying} \\ \text{regularity condition } \mathcal{R} \end{array} \right\} \quad (\text{II.1})$$

where the action of G on functions is the pullback diagonal action

$$(g \cdot f)(x_0, \dots, x_k) = g \cdot f(g^{-1}x_0, \dots, g^{-1}x_k) = f(g^{-1}x_0, \dots, g^{-1}x_k).$$

The homogeneous differential $d^k : C_{\mathcal{R}}^k(G, A)^G \rightarrow C_{\mathcal{R}}^{k+1}(G, A)^G$ is then given by

$$(d^k f)(x_0, \dots, x_{k+1}) = \sum_{i=0}^{k+1} (-1)^i f(x_0, \dots, \hat{x}_i, \dots, x_{k+1}), \quad (\text{II.2})$$

where \hat{x}_i indicates that x_i is omitted, and this operation preserves the regularity condition \mathcal{R} .

(Bounded) Group Cohomology

We assume first that the regularity condition \mathcal{R} is vacuous: A straightforward calculation shows that $d^k \circ d^{k-1} = 0$ for all $k \in \mathbb{N}$, so that $(C^*(G, A), d^*)$ defines a *homogeneous cochain complex* and we may define the corresponding *k-th group cohomology* as

$$H^k(G, A) = \text{Ker}(d^k) / \text{Im}(d^{k-1}),$$

where $\text{ker}(d^k)$ are referred to as *k-cocycles* and $\text{Im}(d^{k-1})$ are referred to as *k-coboundaries*.

From here, one can naturally define the bounded version of this group cohomology:

$$C_b^k(G, A)^G := \{f \in C^k(G, A)^G \mid \|f\|_{\infty} < \infty\},$$

where we mean the supremum norm

$$\|f\|_{\infty} := \sup_{(x_0, \dots, x_k) \in G^{k+1}} \|f(x_0, \dots, x_k)\|.$$

The differential d restricts to this set still satisfying $d^k \circ d^{k-1} = 0$, so that $(C_b^*(G, A), d^*)$ forms a subcomplexes of $(C^*(G, A), d^*)$ and we denote its group cohomology as $H_b^k(G, A)$ called *k-th bounded group cohomology*.

We now consider non-vacuous regularity conditions for \mathcal{R} ; for each case it is straightforward to see that the differential operator d restricts to (II.1) and still satisfies $d^k \circ d^{k-1} = 0$.

Borel (Bounded) Group Cohomology

We consider the locally compact second countable group G together with left Haar measure μ and A a Polish abelian group (i.e. separable completely metrizable abelian topological group). We denote

$$L^0(G^{k+1}, A) := \{f : G^{k+1} \rightarrow A \mid f \text{ is Borel measurable}\} / \sim,$$

where $f \sim g$, if $\mu^{\otimes(k+1)}(\{x \in G^{k+1} : f(x) \neq g(x)\}) = 0$. We now take for $\mathcal{R} = B$ the Borel measurable condition, that is:

$$C_B^k(G, A)^G := \left\{ f \in L^0(G^{k+1}, A) \mid \begin{array}{l} f \text{ is } G\text{-invariant} \\ \text{almost everywhere} \end{array} \right\}.$$

Together with the restricted differentials d , it forms a homogeneous Borel cochain complex whose *k-th cohomology* we denote as $H_B^k(G, A)$, called *k-th Borel group cohomology*.

The differentials further restricts to the Borel bounded k -cochains

$$C_{Bb}^k(G, A)^G := C_B^k(G, A)^G \cap C_b^k(G, A)^G$$

that give rise to $H_{Bb}^k(G, A)$, called k -th Borel bounded group cohomology.

Continuous (Bounded) Group Cohomology

Assume that G is a topological group and A a topological G -module. Then take $\mathcal{R} = c$ to be the continuity condition:

$$C_c^k(G, A)^G := \left\{ f : G^{k+1} \rightarrow A \mid \begin{array}{l} f \text{ is } G\text{-invariant} \\ \text{and continuous} \end{array} \right\}.$$

Together with the differential operator d , it forms a homogeneous continuous cochain complex whose cohomology we denote as $H_c^k(G, A)$, called k -th continuous group cohomology. The differential then also restricts to the bounded k -cochains $C_{cb}^k(G, A)^G := C_c^k(G, A)^G \cap C_b^k(G, A)^G$ which gives to $H_{cb}^k(G, A)$, called k -th continuous bounded group cohomology.

Clearly the standard group cohomology coincides with the Borel and continuous group cohomologies, whenever G is a discrete group; analogously for the respective bounded group cohomologies.

II.1.2 Properties

We discuss features of the cohomology theories we presented for varying regularity conditions $\mathcal{R} = Bb, B, cb, c, b$ and vacuous. We introduce comparison maps that naturally relate bounded and unbounded cohomologies and highlight some important differences between these types of cohomology theories in different frameworks. The cohomologies can moreover be endowed with a product structure, known as the cup product.

Subsequently we discuss a central method in the context of Borel cohomologies, which will find its primary application in Chapter III, where we compute Borel bounded cohomologies for point-stabilizer subgroups of $\text{PU}(n, 1)$. Further properties are given towards the end of this chapter.

Comparison map

Let $k \in \mathbb{N}^*$. For $\mathcal{R} = B, c$ and the vacuous case, with suitable assumptions on the group G and the coefficient group A as specified in the respective paragraphs of Subsection II.1.1, we observe that the inclusion of complexes

$$C_{\mathcal{R}b}^k(G, A)^G \subset C_{\mathcal{R}}^k(G, A)^G$$

induces a map on the corresponding cohomology groups

$$c_A : H_{\mathcal{R}b}^k(G, A) \longrightarrow H_{\mathcal{R}}^k(G, A), \tag{II.3}$$

called the *comparison map*.

We now specialize to the case $A = \mathbb{R}$ as trivial coefficients and set $\mathcal{R} = c$ to be continuous. We address a conjecture central to the field of bounded cohomology on the nature of the comparison map $c_{\mathbb{R}} : H_{cb}^*(G, \mathbb{R}) \rightarrow H_c^*(G, \mathbb{R})$, which has been a major topic of research in the last decades. In particular, Monod [Mon06] conjectured that $c_{\mathbb{R}}$ is in fact an isomorphism for G a connected semisimple Lie group with finite center.

It is straightforward to verify this conjecture in degree 0. In degree 1, every continuous 1-cocycle is a homomorphism $G \rightarrow \mathbb{R}$, but since G has no non-trivial abelian quotients, such a homomorphism is trivial, so that $H_{cb}^1(G; \mathbb{R}) \cong H_b^1(G, \mathbb{R}) = 0$. Now in degree 2, which is particularly relevant for the subsequent chapters, we have the following important result:

Theorem II.1 ([DT87],[CØ03],[BM99]). *For G a semisimple non-compact Hermitian Lie group with finite center, the comparison map $c_{\mathbb{R}} : H_{cb}^2(G, \mathbb{R}) \rightarrow H_c^2(G, \mathbb{R})$ is an isomorphism.*

Before Monod's conjecture, we note that Dupont [Dup79] had already posed the question whether the comparison map $c_{\mathbb{R}}$ is surjective for these groups. This has been settled affirmatively by Hartnick and Ott in the same framework as Theorem II.1, which in particular plays a pivotal role in this thesis:

Theorem II.2 ([HO12]). *For G a semisimple non-compact Hermitian Lie group with finite center, the comparison map $c_{\mathbb{R}} : H_{cb}^k(G, \mathbb{R}) \rightarrow H_c^k(G, \mathbb{R})$ is surjective for all $k \geq 0$.*

Next, we consider the case where G is amenable (see Definition I.42) for which the following classical vanishing theorem holds:

Theorem II.3 (Trauber (unpublished);[Joh72; Gro82], [BM02, p. 220, Example 3]). *For any amenable locally compact second-countable group G one has that $H_{cb}^{>0}(G, \mathbb{R}) = 0$.*

This vanishing result does not hold for unbounded cohomology in general. For example, for the non-compact Lie group \mathbb{R} , one has that $H_c^1(\mathbb{R}, \mathbb{R}) \neq 0$. On the other hand, if K is a compact Lie group, then we have that $H_c^{>0}(K, \mathbb{R}) = 0$ [Gui80, Ch. III, §2.1, Corollary 2.1].

Remark. While for a discrete group G , it is a straightforward to see that $H_b^0(G, \mathbb{R}) \cong \mathbb{R}$ and $H_b^1(G, \mathbb{R}) = 0$, bounded cohomology is generally considered to be hard to compute in higher degrees, even in the case of the non-abelian free group: In degrees 2, 3 and 4 bounded cohomology is infinite-dimensional, with the latter having been shown only recently [Kas25].

Cup Products

We assume again more generally that $\mathcal{R} = B, c$ or vacuous. Let $k, q \geq 0$, let $\alpha \in C_{\mathcal{R}}^k(G, A)$ and $\beta \in C_{\mathcal{R}}^q(G, A)$. The *cup product* is defined as

$$\begin{aligned} \cup : H_{\mathcal{R}}^k(G, A) \times H_{\mathcal{R}}^q(G, A) &\longrightarrow H_{\mathcal{R}}^{k+q}(G, A), \\ ([\alpha], [\beta]) &\longmapsto [\alpha] \cup [\beta], \end{aligned}$$

where $[\alpha] \cup [\beta]$ is represented by the cocycle $\alpha \cup \beta \in C_{\mathcal{R}}^{k+q}(G, \mathbb{R})^G$ given by

$$\alpha \cup \beta : (g_0, \dots, g_{k+q}) \longmapsto \alpha(g_0, \dots, g_k) \cdot \beta(g_k, \dots, g_{k+q}).$$

This map then induces a well-defined map in the respective bounded cochains, this gives also cup products in bounded cohomology

$$\cup : H_{\mathcal{R}b}^k(G, \mathbb{R}) \times H_{\mathcal{R}b}^q(G, \mathbb{R}) \longrightarrow H_{\mathcal{R}b}^{k+q}(G, \mathbb{R}).$$

In particular, all cohomology groups equipped with the cup product structure form cohomology rings. Further, the following naturality property with respect to group homomorphisms holds: If $f : H \rightarrow G$ is a group homomorphism, then for the pullback map f^* in (bounded) cohomology one has that

$$f^*([\alpha] \cup [\beta]) = f^*([\alpha]) \cup f^*([\beta])$$

where $[\alpha] \in H_{\mathcal{R}(b)}^k(G, A)$, $[\beta] \in H_{\mathcal{R}(b)}^q(G, A)$. In the discrete case, we refer to [Bro82, Ch. 5, §3] for a detailed treatment. The construction and properties extend naturally to the continuous and Borel measurable (bounded) settings, where the cup product maintains the appropriate regularity conditions and is compatible with pullbacks induced by group homomorphisms.

Remark. To follow up on the previous remark: Cup products of bounded cohomological classes have also been considered in the study of the bounded cohomology of the non-abelian free group, where merely examples of trivial cup products have been exhibited so far [BM18],[Fou20],[Heu20],[AB22].

Long Exact Sequences in Borel Cohomologies

We now set $\mathcal{R} = B$ to be Borel measurable cohomology and consider the short exact sequence of trivial coefficients

$$\mathbb{Z} \xrightarrow{i} \mathbb{R} \xrightarrow{p} \mathbb{R}/\mathbb{Z}$$

which induces the following short exact sequence on the cochain level

$$C_B^k(G, \mathbb{Z})^G \xrightarrow{i_*} C_B^k(G, \mathbb{R})^G \xrightarrow{p_*} C_B^k(G, \mathbb{R}/\mathbb{Z})^G.$$

This gives rise to the *Bockstein connecting homomorphism* $\delta^k : H_B^k(G, \mathbb{R}/\mathbb{Z}) \rightarrow H_B^{k+1}(G, \mathbb{Z})$, which we now construct explicitly.

First, we fix the canonical Borel section $\sigma : \mathbb{R}/\mathbb{Z} \rightarrow [0, 1) \subset \mathbb{R}$, such that $\sigma([t])$ is the unique element of $[0, 1)$ representing $[t]$, so that $p \circ \sigma = \text{id}_{\mathbb{R}/\mathbb{Z}}$. For a cocycle $c \in C_B^k(G, \mathbb{R}/\mathbb{Z})^G$ we take its canonical lift $\tilde{c} = \sigma \circ c \in C_B^k(G, \mathbb{R})^G$, so that $p_*(\tilde{c}) = c$. Since $d^k(c) = 0$ in $C_B^{k+1}(G, \mathbb{R}/\mathbb{Z})^G$, by commutativity of the bottom square in the following diagram

$$\begin{array}{ccccc} & & C_B^{k+1}(G, \mathbb{Z})^G & \xrightarrow{d^{k+1}} & C_B^{k+2}(G, \mathbb{Z})^G \\ & & \downarrow i_* & & \downarrow i_* \\ C_B^k(G, \mathbb{R})^G & \xrightarrow{d^k} & C_B^{k+1}(G, \mathbb{R})^G & \xrightarrow{d^{k+1}} & C_B^{k+2}(G, \mathbb{R})^G \\ p_* \downarrow \Downarrow & & \downarrow p_* & & \\ C_B^k(G, \mathbb{R}/\mathbb{Z})^G & \xrightarrow{d^k} & C_B^{k+1}(G, \mathbb{R}/\mathbb{Z})^G & & \end{array} \quad (\text{II.4})$$

we have that $p_*(d^k \tilde{c}) = 0$, and hence $d^k \tilde{c} \in \text{Ker}(p_*) = \text{Im}(i_*)$. By injectivity of i_* , there exists a

unique $z \in C_B^{k+1}(G, \mathbb{Z})^G$ with $i_*(z) = d^k \tilde{c}$. Using the commutativity of the top square in Diagram (II.4), we then have

$$i_*(d^{k+1}z) = d^{k+1} \circ i_*(z) = d^{k+1} \circ d^k \tilde{c} = 0.$$

and injectivity of i_* implies that $d^{k+1}z = 0$, proving that z is a cocycle. We may then define homomorphism δ^k on the cohomological level by

$$\delta^k : H_B^k(G, \mathbb{R}/\mathbb{Z}) \longrightarrow H_B^k(G, \mathbb{Z}), \quad [c] \longmapsto [z].$$

This map is well-defined: Any other cocycle of representing $[c]$ has the form

$$c' = c + d^{k-1}c_0,$$

where $c_0 \in C_B^{k-1}(G, \mathbb{R}/\mathbb{Z})^G$ is some cocycle. We further take the canonical lifts $\tilde{c}' = \sigma \circ c'$ and $\tilde{c}_0 = \sigma \circ c_0$ and observe that since $\sigma(a+b) - \sigma(a) - \sigma(b) \in \mathbb{Z}$ for all $a, b \in \mathbb{R}/\mathbb{Z}$, there exists $t \in C_B^k(G, \mathbb{Z})^G$ such that

$$\tilde{c}' = \tilde{c} + d^{k-1}\tilde{c}_0 + i_*(t).$$

Applying d^k , we see that $d^k \tilde{c}' = d^k \tilde{c} + i_*(d^k t)$. For $z' \in C_B^{k+1}(G, \mathbb{Z})^G$ such that $i^*(z') = d^k \tilde{c}'$, we then obtain that

$$i_*(z') = i_*(z + d^k t),$$

and injectivity of i_* yields $z' = z + d^k t$. That is z' and z differ by a coboundary in $C_B^{k+1}(G, \mathbb{Z})^G$, so that $[z'] = [z] \in H_B^{k+1}(G, \mathbb{Z})$.

As a result, we have produced the Bockstein connecting homomorphism δ^k , with a slight abuse of notation denoted by δ from now and on, that gives rise to the following long exact sequence in Borel cohomology

$$\cdots \longrightarrow H_B^k(G, \mathbb{Z}) \xrightarrow{i^*} H_B^k(G, \mathbb{R}) \xrightarrow{p^*} H_B^k(G, \mathbb{R}/\mathbb{Z}) \xrightarrow{\delta} H_B^{k+1}(G, \mathbb{Z}) \longrightarrow \cdots .$$

In fact, this construction can be upgraded to produce a connecting homomorphism also in the Borel *bounded* cohomology: Since i is an isometric embedding (preserves the norm) and p a bounded surjection (norm non-increasing), the short exact sequence of coefficients admits a bounded (in fact norm-preserving) linear section, which allows to obtain the induced short exact sequences in Borel bounded cohomology

$$C_{Bb}^k(G, \mathbb{Z})^G \xleftarrow{i_*} C_{Bb}^k(G, \mathbb{R})^G \xrightarrow{p_*} C_{Bb}^k(G, \mathbb{R}/\mathbb{Z})^G.$$

Let now $c_b \in C_{Bb}^k(G, \mathbb{R}/\mathbb{Z})^G$ be a bounded Borel k -cochain and we take the canonical lift $\tilde{c}_b = \sigma \circ c_b$. Then the cochain \tilde{c}_b is bounded, since it takes values in the compact set $[0, 1)$. By the formula

$$(d^k \tilde{c}_b)(x_0, \dots, x_{k+1}) = \sum_{i=0}^{k+1} (-1)^i \tilde{c}_b(x_0, \dots, \widehat{x}_i, \dots, x_{k+1}),$$

each summand lies in $[0, 1)$, hence the sum is contained in $[-(k+2), k+2)$. Analogous to the unbounded case, we have that $p_*(d^k \tilde{c}_b) = 0$ implies $d^k \tilde{c}_b \in \text{Im}(i_*) = \text{Ker}(p_*)$, so that it

is in particular integer-valued. An integer-valued function that takes values in the interval $[-(k+2), k+2]$ is uniformly bounded, so $d^k \tilde{c}_b$ is also a bounded cochain. By injectivity of i_* , there exists a unique cocycle $z_b \in C_{Bb}^{k+1}(G, \mathbb{R})^G$ such that $i_*(z_b) = d^k \tilde{c}_b$. Then $\delta^k([c_b]) := [z_b]$ is also well-defined, for which the argument straightforwardly translates from the unbounded case, yielding the *bounded* Bockstein connecting homomorphism δ^k , that we shall from now on simply denote by δ^b , in the long exact sequence

$$\dots \longrightarrow H_{Bb}^k(G, \mathbb{Z}) \xrightarrow{i^*} H_{Bb}^k(G, \mathbb{R}) \xrightarrow{p^*} H_B^k(G, \mathbb{R}/\mathbb{Z}) \xrightarrow{\delta^b} H_{Bb}^{k+1}(G, \mathbb{Z}) \longrightarrow \dots$$

where we used that $H_{Bb}^k(G, \mathbb{R}/\mathbb{Z}) \cong H_B^k(G, \mathbb{R}/\mathbb{Z})$.

Since comparison maps (II.3) commute with homomorphisms, these fit as vertical maps of the following commutative diagram

$$\begin{array}{ccccccccc} \dots & \longrightarrow & H_{Bb}^k(G, \mathbb{R}) & \longrightarrow & H_{Bb}^k(G, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\delta^b} & H_{Bb}^{k+1}(G, \mathbb{Z}) & \longrightarrow & H_{Bb}^{k+1}(G, \mathbb{R}) & \longrightarrow & \dots \\ & & \downarrow c_{\mathbb{R}} & & \parallel & & \downarrow c_{\mathbb{Z}} & & \downarrow c_{\mathbb{R}} & & \\ \dots & \longrightarrow & H_B^k(G, \mathbb{R}) & \longrightarrow & H_B^k(G, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\delta} & H_B^{k+1}(G, \mathbb{Z}) & \longrightarrow & H_B^{k+1}(G, \mathbb{R}) & \longrightarrow & \dots \end{array}$$

(II.5)

This commutative diagram will be our central method of computation. We showcase with the following example on compact Lie groups:

Example II.4. For a compact Lie group K due to [Gui80, Ch. III, §2.1, Corollary 2.1] and Theorem II.3, we have that

$$H_B^{*>0}(K, \mathbb{R}) \cong H_{Bb}^{*>0}(K, \mathbb{R}) = 0,$$

we immediately deduce from Diagram (II.5) that

$$H_{Bb}^{*>0}(K, \mathbb{Z}) \cong H_B^{*>0}(K, \mathbb{R}/\mathbb{Z}) \cong H_{Bb}^{*>0}(K, \mathbb{Z}).$$

This example and similar applications of Diagram (II.5) will be shown in Section III.4 in particular.

II.2 Cohomology for Manifolds

We discuss de Rham cohomology, singular homology as well as cohomology, followed by important features, including long exact sequences, Poincaré duality and straightening maps.

II.2.1 De Rham Cohomology

For this subsection we recall Definition I.16 based on which, we formalize several types of de Rham cohomologies; we refer to [Lee12].

Standard Variant

For X be a smooth manifold, we recall Definition I.16, that $(\Omega^*(X), d^*)$ with the exterior derivative $d^k : \Omega^k(X) \rightarrow \Omega^{k+1}(X)$ satisfying $d^k \circ d^{k-1} = 0$ defines a cochain complex, called *de Rham complex*, so that it defines a cohomology

$$\mathbb{H}_{dR}^k(X) = \text{Ker}(d^k) / \text{Im}(d^{k-1}),$$

as the quotient of closed k -forms by exact k -forms, called *k -th de Rham cohomology* of X .

Relative Variant

For X be a smooth manifold and $Y \subset X$ a smooth closed submanifold we define

$$\Omega^k(X, Y) = \{\alpha \in \Omega^k(X) \mid \alpha|_Y = 0\}.$$

The exterior derivatives restrict to this set and it still holds that $d^k \circ d^{k-1} = 0$. As such $(\Omega^*(X, Y), d^*)$ is a subcomplex of $(\Omega^*(X), d^*)$. We denote the cohomology of this subcomplex as $\mathbb{H}_{dR}^k(X, Y)$, called the *k -th relative de Rham cohomology*. Clearly, whenever $Y = \emptyset$, the relative de Rham cohomology coincides with the standard one.

Invariant Variant

If G is a Lie group acting smoothly and properly on a smooth manifold X , then the subspace of G -invariant differential forms,

$$\Omega^k(X)^G := \{\alpha \in \Omega^k(X) \mid g^* \alpha = \alpha \text{ for all } g \in G\},$$

is preserved by the exterior derivative d , and again $d^k \circ d^{k-1} = 0$ continues to hold. Hence $(\Omega^*(X)^G, d^*)$ forms a subcomplex of the de Rham complex, which defined *k -th G -invariant de Rham cohomology* $\mathbb{H}_{dR}^k(X)^G$.

Clearly, the wedge product $\wedge : \Omega^\ell(X) \times \Omega^k(X) \rightarrow \Omega^{k+\ell}(X)$ induces a product on the standard de Rham cohomology, which is then inherited by its other versions, thereby endowing all of these cohomologies with ring structures.

II.2.2 Singular Cohomology

For the sake of completeness, we also remind of singular cohomologies, its relative versions and bounded counterparts. We then state Poincaré duality in order to make sense of integration on the cohomological level.

Singular Homology

Let X be a topological space and let Δ^k be the *standard oriented k -simplex*, given by the convex hull of $k + 1$ affinely independent points $\{p_0, \dots, p_k\}$ in \mathbb{R}^n , i.e.

$$\Delta^k = [p_0, \dots, p_k] = \left\{ \sum_{i=0}^k \lambda_i p_i \mid \sum_{i=0}^k \lambda_i = 1, \lambda_i \geq 0 \right\}.$$

The points $\{p_0, \dots, p_k\}$ are called *vertices* of the simplex, and the convex hull of k of these vertices is called a *face*. A *singular k -simplex* in X is a continuous map $\sigma : \Delta^k \rightarrow X$.

Given an abelian group A , the group of *singular k -chains with coefficients in A* is the free A -module $S_k(X, A)$ generated by all singular k -simplices in X , i.e., an element $s \in S_k(X, A)$ can be written as

$$s = \sum_{\sigma: \Delta^k \rightarrow X} a_\sigma \cdot \sigma, \quad a_\sigma \in A.$$

The *boundary map* $\partial_k : S_k(X, A) \rightarrow S_{k-1}(X, A)$ is defined by

$$\partial_k \sigma = \sum_{i=0}^k (-1)^i \sigma \circ \delta_i,$$

where $\delta_i : \Delta^{k-1} \hookrightarrow \Delta^k$ is the inclusion of the i -th face, obtained by omitting the vertex p_i and preserving the ordering of the remaining vertices.

It is a classical calculation to check that $\partial_k \circ \partial_{k+1} = 0$, so one obtains the *chain complex* $(S_*(X, A), \partial_*)$ and the k -th *singular homology group with coefficients in A* is defined by

$$H_k(X, A) = \text{Ker}(\partial_k) / \text{Im}(\partial_{k+1}).$$

Singular Cohomology: Standard Variant

The group of *singular k -cochains with coefficients in A* is given by

$$S^k(X, A) = \text{Hom}_{\mathbb{Z}}(S_k(X, \mathbb{Z}), A)$$

and the coboundary map as $d^k : S^k(X, A) \rightarrow S^{k+1}(X, A)$ is given by

$$d^k(\phi)(\sigma) := \phi(\partial^{k+1} \sigma)$$

and satisfies $d^{k+1} \circ d^k = 0$. Then k -th *singular cohomology with coefficients in A* is given by

$$H^k(X, A) := \text{Ker}(d^k) / \text{Im}(d^{k-1}).$$

Relative Variant

We now consider a pair (X, Y) such that $Y \subset X$ is closed subspace of a topological space X . The *relative singular chain group* is defined as

$$S_k(X, Y, A) = S_k(X, A) / S_k(Y, A)$$

with the induced boundary maps ∂_k satisfying $\partial_k \circ \partial_{k+1} = 0$ and thus forming relative singular chain complex $(S_k(X, Y), \partial_k)$. We obtain the *relative singular homology group* as

$$H_k(X, Y, A) := \text{Ker}(\partial_k) / \text{Im}(\partial_{k+1}).$$

The *relative k -cochains* are defined as

$$S^k(X, Y, A) = \{\phi \in S^k(X, A) \mid \phi|_{S^k(Y, A)} = 0\},$$

with induced boundary maps d^k satisfying $d^k \circ d^{k-1} = 0$ so that we obtain the relative k -cochain complex $(S^k(X, Y, A), d^k)$ allowing to define *relative singular cohomology* analogously as

$$H^k(X, Y, A) = \text{Ker}(d^k) / \text{Im}(d^{k-1}).$$

Bounded Variant

We define the bounded singular k -cochains by

$$S_b^k(X, A) = \{\alpha \in S^k(X, A) \mid \|\alpha\|_\infty < \infty\},$$

where $\|\alpha\|_\infty = \sup\{|\alpha(\sigma)| : \sigma \text{ singular } k\text{-simplex}\}$. The coboundary operator d^k restricts to $S_b^k(X, A)$ still satisfying $d^k \circ d^{k-1} = 0$, so that $(S_b^k(X, A), d^k)$ forms a subcomplex of $(S^k(X, A), d^k)$ and its k -th cohomology $H_b^k(X, A)$ is called *k -th bounded singular cohomology*. Finally, the bounded relative singular k -cochains

$$S_b^k(X, Y, A) = S_b^k(X, A) \cap S^k(X, Y, A),$$

with the restricted boundary maps d^k satisfying $d^k \circ d^{k-1} = 0$ give rise to the subcomplex $(S_b^k(X, Y, A), d^k)$ of $(S^k(X, Y, A), d^k)$, from which we obtain the *k -th bounded relative singular cohomology* $H_b^k(X, Y, A)$.

Clearly, whenever $Y = \emptyset$ the (bounded) relative cohomology coincides with the (bounded) ordinary cohomology

II.2.3 Properties

We state several analogous features to the ones we have seen in group cohomology, relevant for the present work.

Comparison Map

Let Y be a closed subspace of a topological space X . The inclusion of complexes $(S_b^k(X, Y, A), d^k) \subset (S^k(X, Y, A), d^k)$ induces in degree k the comparison map

$$c_{(X, Y)} : H_b^k(X, Y, A) \rightarrow H^k(X, Y, A).$$

For $Y = \emptyset$ this reduces to a comparison map c_X in ordinary singular cohomology.

Long Exact Sequences

In straight analogy to the group cohomological case in Subsection II.1.2, we see that the short exact sequence of trivial coefficients

$$\mathbb{Z} \xrightarrow{i} \mathbb{R} \xrightarrow{p} \mathbb{R}/\mathbb{Z}$$

induces for any topological space X the following long exact sequence in (bounded) cohomology

$$\cdots \longrightarrow H_{(b)}^*(X, \mathbb{Z}) \xrightarrow{i^*} H_{(b)}^*(X, \mathbb{R}) \xrightarrow{p^*} H_{(b)}^*(X, \mathbb{R}/\mathbb{Z}) \xrightarrow{\delta^{(b)}} H_{(b)}^{*+1}(X, \mathbb{Z}) \longrightarrow \cdots ,$$

for which one can then similarly consider Diagram (II.5). There is yet another long exact sequence of interest for us: Let A be an arbitrary abelian coefficient group. For Y now a closed subspace of X we consider the inclusion of pairs $(X, \emptyset) \subset (X, Y)$, which induces the short exact sequence:

$$S_{(b)}^*(X, Y, A) \hookrightarrow S_{(b)}^*(X, A) \twoheadrightarrow S_{(b)}^*(Y, A) ,$$

where the right-hand map is understood as extension by zero, namely, each cochain on Y is viewed as a cochain on X that vanishes on all simplices not contained in Y . Similarly to before, we obtain the following long exact sequence in (bounded) cohomology

$$\cdots \longrightarrow H_{(b)}^*(Y, A) \longrightarrow H_{(b)}^*(X, A) \xrightarrow{j} H_{(b)}^*(X, Y, A) \xrightarrow{\delta^{(b)}} H_{(b)}^{*+1}(Y, A) \longrightarrow \cdots$$

where δ^* is the connecting homomorphism.

We shall see an application of the long exact sequence in bounded cohomology in a geometric setting relevant for us in Example II.10. Later in Chapter IV, we will use a combination of both long exact sequences to prove a key congruence relation in Theorem IV.5.

Straightening Procedure for Hadamard Manifolds

We assume now for this subsection that X is a Hadamard m -manifold and consider \mathbb{R} as coefficient group. Let $1 \leq k \leq m$ and denote by $S_k^\infty(X, \mathbb{R})$ the complex of smooth singular simplices, i.e. $s \in S_k(X, \mathbb{R})$ such that s is a smooth singular simplex. To obtain smooth singular simplices from arbitrary singular simplices, we introduce a procedure called *simplex straightening*, which exploits the particular feature of Hadamard manifolds, that any two points can be joined by a unique geodesic. We follow [Fri17, Section 8.4] (cf. also [Dup76]).

Definition II.5. For vertices $x_0, \dots, x_k \in X$, we define the *straight simplex*

$$\text{str}[x_0, \dots, x_k] : \Delta^k \rightarrow X$$

inductively by joining points with unique geodesics:

- For $k = 0$, set $\text{tr}[x_0]$ to be the constant simplex at x_0 , i.e the map $\text{str}[x_0]$ is the constant map to x_0 .

- Assuming $\text{str}[x_0, \dots, x_{k-1}]$ is defined, one constructs $\text{str}[x_0, \dots, x_k]$ as follows: for each point $z \in \Delta^{k-1} \subset \Delta^k$, we set

$$\text{str}[x_0, \dots, x_k]|_{[z, x_k]}(t)$$

to be the constant-speed parametrization of the unique geodesic segment $[z, x_k]$ from z to x_k .

This defines a smooth simplex $\text{str}[x_0, \dots, x_k] \in S_k^\infty(X)$ with x_k being its k -th vertex .

The straightening procedure induces a chain map $\text{Str}: S_*(X, \mathbb{R}) \rightarrow S_*^\infty(X, \mathbb{R})$, which is homotopic to the identity on $S_*(X, \mathbb{R})$, so that the natural inclusion $S_*^\infty(X, \mathbb{R}) \hookrightarrow S_*(X, \mathbb{R})$ induces an isomorphism in homology. That is, the smooth singular homology of X coincides with the ordinary singular homology.

Poincaré Duality

Let M be a connected, smooth m -manifold and A an abelian coefficient group. First, we observe that if M is moreover closed, then there exists a canonical evaluation pairing

$$\langle \ , \ \rangle : H^m(M, A) \times H_m(M, A) \rightarrow A$$

which takes a cohomology class in $H^m(M, A)$ and evaluates on some cycle representing the homology class in $H_m(M, A)$; this is in particular independent of the choice of such representative. If M is compact with boundary ∂M , the analogous pairing then also holds for the relative case.

The point of Poincaré dualities in top-degree is that under certain conditions the canonical evaluation pairing induces an isomorphism of the top cohomology on its coefficient group. First, as a preparation, we state the homological perspective of *orientation* equivalent Definition I.18 from Chapter I, that we gave in terms of differential m -forms (cf. [Hat02, pp. 233-236]):

Definition II.6. A (*local*) *orientation* on a connected smooth m -manifold M is a choice, for each point $p \in M$, of a generator

$$[M]_p \in H_m(M, M \setminus \{p\}, \mathbb{Z}) \cong \mathbb{Z},$$

such that for all $p, p' \in M$ there exists a neighbourhood U containing both p and p' for which the image $[M]_p$ and $[M]_{p'}$ coincide in $H_m(M, M \setminus U, \mathbb{Z})$ under the maps induced by the inclusions of pairs

$$(M, M \setminus U) \subset (M, M \setminus \{p\}) \quad \text{and} \quad (M, M \setminus U) \subset (M, M \setminus \{p'\}).$$

Then if M is oriented in this sense and closed, then there exists a unique class $[M] \in H_m(M, \mathbb{Z})$ called the *fundamental class* of M , that satisfies the property that for every $p \in X$, the image of $[M]$ in $H_m(M, M \setminus \{p\}, \mathbb{Z}) \cong \mathbb{Z}$ corresponds to the chosen local orientation at p .

If M is compact with boundary ∂M , and its interior is oriented, then similarly the *relative fundamental class* is the unique class

$$[M, \partial M] \in H_m(M, \partial M; \mathbb{Z})$$

whose image in $H_m(M, M \setminus \{p\}, \mathbb{Z}) \cong \mathbb{Z}$ corresponds to the chosen local orientation at p .

We note that for any abelian coefficient group A , the canonical homomorphism

$$\iota_A : \mathbb{Z} \longrightarrow A, \quad n \longmapsto n \cdot 1_A,$$

induces a map on (relative) homology

$$\begin{aligned} H_m(M, \mathbb{Z}) &\longrightarrow H_m(M, A), & [M] &\longmapsto [M]_A \quad \text{and} \\ H_m(M, \partial M, \mathbb{Z}) &\longrightarrow H_m(M, \partial M, A), & [M, \partial M] &\longmapsto [M, \partial M]_A. \end{aligned}$$

In particular, this applies to $A = \mathbb{R}$ and $A = \mathbb{R}/\mathbb{Z}$. By a slight abuse of notation, we shall simply write $[M]$ and $[M, \partial M]$ for the A -valued fundamental classes whenever the coefficient group is clear from the context.

We are now in a position to state Poincaré duality for the case when M is closed, and Poincaré-Lefschetz duality for the case when M is compact with boundary ∂M , in the context of top-degree cohomology:

Theorem II.7. *Let M be a connected, oriented, smooth m -manifold and let $A \in \{\mathbb{Z}, \mathbb{R}, \mathbb{R}/\mathbb{Z}\}$. We further assume that $H_m(M; \mathbb{Z}) \cong \mathbb{Z}$, and if $A = \mathbb{Z}$ assume additionally that $H_{m-1}(M; \mathbb{Z})$ is torsion-free.*

(i) *If M is closed, then the canonical evaluation pairing $\langle \cdot, \cdot \rangle : H^m(M, A) \times H_m(M, A) \rightarrow A$ induces an isomorphism*

$$H^m(M, A) \xrightarrow{\cong} \text{Hom}_{\mathbb{Z}}(H_m(M, A), A),$$

under which the fundamental class $[M] \in H_m(M, A)$ corresponds to the evaluation map $\alpha \mapsto \langle \alpha, [M] \rangle$. In particular, evaluation on the fundamental class gives an isomorphism

$$H^m(M, A) \xrightarrow{\cong} A.$$

(ii) *If M is compact with boundary ∂M , then evaluation on the relative fundamental class*

$$\langle \cdot, [M, \partial M] \rangle : H^m(M, \partial M, A) \rightarrow A$$

is an isomorphism.

Proof. For a proof we refer to [May99, Ch. 20] or [Hat02, §3]. □

Example II.8. Let M be an oriented closed Riemannian manifold. We recall Definition I.22, namely that M comes with the unique volume form $\text{vol}_M \in \Omega^m(M)$ that can be integrated over M . By slight abuse of notation, we denote $\text{vol}_M := [\text{vol}_M] \in H_{dR}^m(M)$, and observe that

$$\text{Vol}(M) = \int_M \text{vol}_M = \langle \text{vol}_M, [M] \rangle,$$

which can be seen as the de Rham cohomological instance of Poincaré duality. In analogy, we will see that the Euler characteristic can also be expressed via Poincaré duality (see Section

III.2.1).

Remark. In some contexts, Poincaré duality can be substituted by transfer maps, a more algebraic formalism [BG99]. In fact, bounded transfer maps will be the focus of Chapter V.

II.3 Interplay of Cohomology Theories

We now outline the relationships among the cohomologies discussed above, referring to most of the classical isomorphisms by the names of the original authors. This is followed by a brief introduction to Kähler and volume classes.

II.3.1 Isomorphisms between Cohomology Theories

Classically, singular cohomology can be related to the (bounded) group cohomology for an arbitrary abelian coefficient group A , in the following setting:

Theorem II.9. *Let M be a Riemannian manifold of non-positive curvature and let A be an abelian group with trivial $\pi_1(M)$ -action. Then it holds that*

$$H^*(M, A) \cong H^*(\pi_1(M), A) \quad \text{and} \quad H_b^*(M, A) \cong H_b^*(\pi_1(M), A)$$

Proof. The proof of the first statement, which holds more generally for M aspherical, i.e. $\pi_n(M) = 0$ for $n > 1$, can be found in [Bro82, Ch. I, §5]. whereas for the second statement, also in wider generality, we refer to [Gro82, §2.3, §3.3] and [Iva87, Theorem 4.1]. \square

We are now in position to present the following example that plays a central role in this thesis:

Example II.10. Let M be a non-compact Riemannian manifold of finite volume with pinched negative curvature. Using the thick-thin decomposition (cf. Section I.2.4, in particular Figure I.1), M can be decomposed into a compact core N with smooth boundary ∂N and its complement $M \setminus N$, which is a disjoint union of finitely many ℓ cusps, $\mathcal{C}_j \cong \mathcal{S}_j \times [0, \infty)$ with \mathcal{S}_j the corresponding cusp cross sections for $1 \leq j \leq \ell$. In particular, we have the homotopy equivalences

$$M \simeq N \quad \text{and} \quad M \setminus N \simeq \partial N.$$

and so the inclusions of pairs $(M, \emptyset) \subset (M, M \setminus N)$ and $(N, \emptyset) \subset (N, \partial N)$ induce isomorphic relative long exact sequences in singular cohomology. Now, if M has moreover the property that all of its cusps are modelled by amenable groups, we can deduce from the bounded cohomological sequence in real coefficients

$$\cdots \longrightarrow H_b^*(\partial N, A) \longrightarrow H_b^*(N, A) \xrightarrow{j} H_b^*(N, \partial N, A) \xrightarrow{\delta^b} H_b^{*+1}(\partial N, A) \longrightarrow \cdots,$$

that the middle map j is an isomorphism¹, since we have the following vanishing statement:

$$H_b^{*>0}(\partial N, \mathbb{R}) \cong H_b^{*>0}(\sqcup_{j=1}^{\ell} \mathcal{S}_j, \mathbb{R}) \cong H_b^{*>0}(\sqcup_{j=1}^{\ell} \mathcal{C}_j, \mathbb{R}) \cong \bigoplus_{j=1}^{\ell} H_b^{*>0}(\pi_1(\mathcal{C}_j), \mathbb{R}) = 0,$$

where for the second isomorphism we used that cusps deformation retract to the corresponding cusp cross sections, for the third - Theorem II.9, to pass to bounded group cohomology, and the vanishing result is due to Theorem II.3.

Next, setting the coefficient group to $A = \mathbb{Z}$, we encounter a relation between Borel and singular cohomology of classifying spaces; for this introduce:

Definition II.11. For G be a second-countable locally compact group, denote by EG any weakly contractible space (i.e. all of its homotopy groups $\pi_n(EG)$ vanish for $n \geq 0$) with a proper free G -action. Then one may take the quotient $BG := EG/G$ known as the *classifying space* of G .

As the classifying space BG can be modelled by a CW-complex, we may consider its singular cohomology and we state the following:

Theorem II.12 (Wigner isomorphism [Wig73]). *For any second-countable locally compact group G , it holds that*

$$H_B^*(G, \mathbb{Z}) \cong H^*(BG, \mathbb{Z}).$$

We develop our formalism of Chern classes in the framework of bounded group cohomology in Section III.3, essentially building upon the isomorphism.

For the remaining part, we fix for the coefficient group $A = \mathbb{R}$. First, a modern result by Austin-Moore, linking continuous and Borel measurable cohomology, is the essential bridge that allows to execute most of our work in Borel cohomology; implicitly used throughout the remaining chapters:

Theorem II.13 (Austin-Moore isomorphism [AM13, Theorem A]). *Let G be a second countable locally compact group. Then it holds that*

$$H_c^*(G, \mathbb{R}) \cong H_B^*(G, \mathbb{R}).$$

Then, continuous group cohomology is further linked to the algebra of invariant differential forms; it will play a role for defining the Kähler and volume class in Section II.4:

Theorem II.14 (Van-Est isomorphism [Est53],[HM62, Theorem 6.1]). *Let G a Lie group and X a G - homogeneous space. Then we have that*

$$H_c^*(G, \mathbb{R}) \cong H_{dR}^*(X)^G.$$

If X is a symmetric space G/K , then every G -invariant differential form on G/K is closed and we have the Van Est isomorphism

$$H_c^*(G, \mathbb{R}) \cong \Omega^*(X)^G.$$

¹It is in fact an isometric isomorphism due to [Buc+14].

Moreover, we give a link between continuous group cohomology and singular cohomology, which can be derived from the Van-Est isomorphism (e.g. [Sta78, §5] and [HO12, §2.1]), for the following setting:

Theorem II.15. *For G a non-compact semi-simple Lie group with finite center and let X_u be its compact dual, we have that*

$$H_c^*(G, \mathbb{R}) \cong H^*(X_u, \mathbb{R}).$$

In the context of this work, it most importantly allows to determine that the top-degree continuous cohomology of $\mathrm{PU}(n, 1)$ is one-dimensional:

Example II.16. Using that the complex projective space $\mathbb{C}\mathbb{P}^n$ is the compact dual of $\mathrm{PU}(n, 1)$ (cf. Table I.50), we straightforwardly deduce that

$$H_c^*(\mathrm{PU}(n, 1), \mathbb{R}) = H^*(\mathbb{C}\mathbb{P}^n, \mathbb{R}) = \begin{cases} \mathbb{R}, & \text{if } * \leq 2n \text{ is even,} \\ 0, & \text{if } * \text{ is odd.} \end{cases}$$

Last but not least, we also have a relation between de Rham and singular cohomology:

Theorem II.17 (De Rham isomorphism). *Let M be a closed smooth manifold. Then it holds that*

$$H_{\mathrm{dR}}^*(M) \cong H^*(M; \mathbb{R}).$$

If M is a compact smooth manifold with boundary ∂M , then it holds that

$$H_{\mathrm{dR}}^*(M, \partial M) \cong H^*(M, \partial M; \mathbb{R}).$$

We note that later in Chapter V, we will see explicit cochain maps realizing the Van-Est and the de Rham isomorphisms.

II.4 (Bounded) Kähler and Volume Classes

For this section we fix G to be a simple non-compact Hermitian Lie group with finite center. We introduce here natural real-valued classes in $H_c^*(G, \mathbb{R})$ corresponding to distinguished differential forms via the Van-Est isomorphism in Theorem II.14 and then formalize their continuous bounded counterparts as classes in $H_{cb}^*(G, \mathbb{R})$.

II.4.1 Kähler Classes and Dupont Cocycle

Let X be an irreducible Hermitian symmetric space of non-positive curvature associated to G with the minimal holomorphic curvature normalized everywhere to -1 . We recall that X admits a unique G -invariant Kähler form ω_X , in fact more specifically the Bergman-Kähler form (cf. Subsection I.3.2), which gives rise to the following definition:

Definition II.18. The continuous class $\bar{\omega}_2 \in H_c^2(G, \mathbb{R})$ is called the *Kähler class* if it corresponds via the Van-Est isomorphism to the Bergman-Kähler form $\omega_X \in \Omega^2(X)^G$.

Since $H_c^2(G, \mathbb{R}) \cong \mathbb{R}$ (see [Gui80, Ch. III, §7.5]), the Kähler class $\bar{\omega}_2$ can be considered as a canonical generator of this one-dimensional group (in Subsection III.3.2, we will encounter an alternative choice of generator). Then, since by Theorem II.1, the comparison map $c_{\mathbb{R}} : H_{cb}^2(G, \mathbb{R}) \rightarrow H_c^2(G, \mathbb{R})$ is an isomorphism, this allows to define:

Definition II.19. The bounded class $\bar{\omega}_2^b := c_{\mathbb{R}}(\bar{\omega}_2) \in H_{cb}^2(G, \mathbb{R})$ is called the *bounded Kähler class*.

It is instructive to show a more explicit and geometric way to introduce both Kähler classes through its representatives (cf. [BIW07, §1], [BI04, §5.1]). The Bergman-Kähler form ω_X gives rise to a *Dupont cocycle* Dup_G given by

$$\text{Dup}_G(g_1, g_2, g_3) := \int_{\text{str}[g_1x_0, g_2x_0, g_3x_0]} \omega_X,$$

for any choice of basepoint $x_0 \in X$ and $\text{str}[-]$ is the straightening of a 2-simplex with vertices in the 3-tuple (cf. Definition II.5). Since the Dupont cocycle Dup_G is G -invariant and continuous [Dup79] as well as bounded [CØ03; DT87], it represents both Kähler classes $\bar{\omega}_2$ and $\bar{\omega}_2^b$ we defined above.

It will be important to distinguish the following notation in our thesis: in the case where $G = \text{PU}(n, 1)$, we denote the corresponding (bounded) Kähler classes by omitting the bar in the notation

$$\omega_2^{(b)} \in H_{c^{(b)}}^2(\text{PU}(n, 1), \mathbb{R}).$$

Remarkably, in this case the Dupont cocycle is related to *Cartan's angular invariant* [Car32], which assigns to any triple of points in $\partial\mathbb{H}_{\mathbb{C}}^n$ an analogue of the cross-ratio from real hyperbolic geometry. Specifically, the Dupont cocycle coincides with the Cartan invariant up to a scalar multiple when evaluated on ideal triangles in $\mathbb{H}_{\mathbb{C}}^n$, and in particular detects totally geodesic subspaces in $\mathbb{H}_{\mathbb{C}}^n$: Cartan's invariant takes the zero value whenever a triple in $\partial\mathbb{H}_{\mathbb{C}}^n$ lies in a totally real Lagrangian subspace of $\mathbb{H}_{\mathbb{C}}^n$ and it attains its maximum value $\pm\pi/2$ whenever such a triple lies in a complex geodesic (cf. [Gol99, §7.1]).²

II.4.2 Volume Classes

We now focus on the group $G = \text{PU}(n, 1)$ with its associated Hermitian symmetric space $\mathbb{H}_{\mathbb{C}}^n$ and introduce natural classes in top-degree continuous cohomology $H_c^{2n}(\text{PU}(n, 1), \mathbb{R})$, in the same spirit as above.

Definition II.20. The continuous class $\omega_{2n} \in H_c^{2n}(\text{PU}(n, 1), \mathbb{R})$ is called the *volume class* if it corresponds via the Van-Est isomorphism to the top volume form $\text{vol}_{\mathbb{H}_{\mathbb{C}}^n} \in \Omega^{2n}(\mathbb{H}_{\mathbb{C}}^n)^{\text{PU}(n, 1)}$.

According to Example II.16, one has $H_c^{2n}(\text{PU}(n, 1), \mathbb{R}) \cong \mathbb{R}$, so that ω_{2n} can be chosen as the canonical generator of this one-dimensional group; also here an alternative generator is provided in Subsection III.3.2. Recalling now Theorem II.2 due to Hartnick and Ott, the comparison map $c_{\mathbb{R}} : H_{cb}^{2n}(\text{PU}(n, 1), \mathbb{R}) \rightarrow H_c^{2n}(\text{PU}(n, 1), \mathbb{R})$ is only known to be surjective, which nevertheless allows to introduce the following terminology:

²Notably, Toledo's viewpoint of the Cartan's invariant as bounded cohomological representative, led to his rigidity results on representations of closed surface groups.

Definition II.21. Any continuous bounded class in the preimage $c_{\mathbb{R}}^{-1}(\omega_{2n}) \in H_{cb}^{2n}(\mathrm{PU}(n, 1), \mathbb{R})$ is called a *bounded volume class*.

Mirroring the discussion on Dupont cocycles for the Kähler class, a natural choice of bounded volume class comes from top-dimensional Dupont cocycle [Dup76]: The volume form $\mathrm{vol}_{\mathbb{H}_{\mathbb{C}}^n}$ gives rise to the *continuous volume cocycle* $\mathrm{Vol}_{\mathrm{PU}(n,1)}$ defined as

$$\mathrm{Vol}_{\mathrm{PU}(n,1)}(g_1, \dots, g_{n+1}) := \int_{\mathrm{str}[g_1 x_0, \dots, g_{n+1} x_0]} \mathrm{vol}_{\mathbb{H}_{\mathbb{C}}^n},$$

for any choice of basepoint $x_0 \in X$ and where $\mathrm{str}[_]$ is the straightening n -simplex with vertices in the $(n + 1)$ -tuple (see Definition II.5). Since $\mathrm{Vol}_{\mathrm{PU}(n,1)}$ is $\mathrm{PU}(n, 1)$ -invariant, continuous and bounded (see [Dup76; Gro82]), it defines a cocycle in $C_{cb}^{2n}(\mathrm{PU}(n, 1), \mathbb{R})^{\mathrm{PU}(n,1)}$ and thus represents both the volume class ω_{2n} and a bounded class in $c_{\mathbb{R}}(\omega_{2n})$. This is regarded as a canonical choice of bounded volume class.

We note that the Kähler class $\bar{\omega}_2$ will be used directly in Subsection IV.6.1. On the other hand, in this thesis, we will not be considering the classical choice for the bounded volume class $[\mathrm{Vol}_{\mathrm{PU}(n,1)}]$ introduced here, but rather a different bounded volume class in $H_{cb}^{2n}(\mathrm{PU}(n, 1), \mathbb{R})$, adapted to our setup in the next chapter (this will lead to a volume invariant of representations different from the classical one).

Chapter III

Chern Classes and Cohomological Properties of $\mathrm{PU}(n, 1)$

The first goal of this chapter is to introduce Chern classes in various cohomology theories that we previously discussed in Chapter II, and explain how these are related. This will require first some preparation on principal bundles. The second goal is to discuss cohomologies of $\mathrm{PU}(n, 1)$ and its point-stabilizer subgroups $\mathrm{U}(n)$ and $\mathrm{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$. Our primary focus is on establishing cohomological properties of the minimal parabolic group $\mathrm{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$ and its subgroups. Following this we examine certain key properties of Chern classes within this framework. These results play a direct role in the proofs of our central theorems presented in Chapter IV.

III.1 Principal G -Bundles

We introduce the main object of this section:

Definition III.1. Let G be a Lie group. A *principal G -bundle* over M is a smooth manifold P with a smooth surjective map $\pi : P \rightarrow M$ and a smooth right action $P \times G \rightarrow P$, such that:

- $\pi(p \cdot g) = \pi(p)$ for all $p \in P, g \in G$,
- G acts freely and transitively on each fiber $\pi^{-1}(x)$,
- for each $x \in M$, there exists a neighborhood $U \subseteq M$ and a G -equivariant diffeomorphism $\pi^{-1}(U) \cong U \times G$.

We observe that this is a group-theoretic perspective of vector bundles we have seen in Chapter I (see Definition I.11). In fact, there is an implicit aspect of vector bundles that we have not yet emphasized: Specifically, let M be a smooth manifold and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We recall that, by definition, any rank n \mathbb{K} -vector bundle $\pi : E \rightarrow M$ admits a system of local trivializations

$$\varphi_{\alpha} : \pi^{-1}(U_{\alpha}) \rightarrow U_{\alpha} \times \mathbb{K}^n,$$

where $\{U_{\alpha}\}$ is an open cover of M . On overlaps $U_{\alpha} \cap U_{\beta}$, transition functions given

$$g_{\alpha\beta} : U_{\alpha} \cap U_{\beta} \rightarrow \mathrm{GL}_n(\mathbb{K}), \quad g_{\alpha\beta}(x) = \varphi_{\alpha} \circ \varphi_{\beta}^{-1}(x, \cdot),$$

describe how the local trivializations are glued together. These functions satisfy the cocycle conditions

$$g_{\alpha\alpha} = \mathrm{id}, \quad g_{\alpha\beta} = g_{\beta\alpha}^{-1}, \quad g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$$

on triple overlaps. We say that the vector bundle E has *structure group*¹ G if the local trivializations can be chosen so that all transition functions $g_{\alpha\beta}$ take values in G . The Lie subgroup $G \leq \mathrm{GL}_n(\mathbb{K})$ in which the transition functions take their values, is called the *structure group*. This allows to state the following classical correspondance:

Theorem III.2. *Let $G \leq \mathrm{GL}_n(\mathbb{K})$ be a Lie subgroup, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then there is a natural bijection*

$$\left\{ \begin{array}{l} \text{principal} \\ G\text{-bundles} \\ \text{over } M \end{array} \right\} / \sim_{iso} \longleftrightarrow \left\{ \begin{array}{l} \text{rank-}n \text{ } \mathbb{K}\text{-vector bundles} \\ \text{with structure group } G \\ \text{over } M \end{array} \right\} / \sim_{iso}$$

between the respective isomorphism classes.

Proof. For a proof we refer to [Coh25]. □

Our primary case of interest in this section will be the unitary group $G = \mathrm{U}(n)$. As a structure group it preserves a Hermitian inner product on \mathbb{C}^n , yielding *Hermitian vector bundles*. Formally, this corresponds to a *reduction* of the structure group $\mathrm{GL}_n(\mathbb{C})$ to the subgroup $\mathrm{U}(n)$; different choices of such reductions correspond to distinct Hermitian metrics preserved by the transition functions.

A non-trivial example of a principal $\mathrm{U}(n)$ -bundle, is the tangent bundle over $\mathbb{C}\mathbb{P}^n$. As a prototype of principal $\mathrm{U}(1)$ -bundles (also referred to as *line bundles*), can be considered the following example:

Example III.3. The *tautological line bundle* γ_1 over $\mathbb{C}\mathbb{P}^n$ is given by

$$\gamma_1 = \{([v], w) \in \mathbb{C}\mathbb{P}^n \times \mathbb{C}^{n+1} \mid w \in [v]\},$$

with the projection map $\pi : \gamma_1 \rightarrow \mathbb{C}\mathbb{P}^n$ given by $\pi([v], w) = [v]$.

Other classical examples, include determinant and canonical line bundles (cf. Subsection I.2.1).

We note that principal G -bundles can also be encoded by other types of data, such as representations, and classifying maps, as we shall see in the following subsection.

III.1.1 Principal G -Bundle associated to a Representation

The classification problem of representations discussed in Section I.1.3 further admits the following classical bundle-theoretic perspective:

¹The structure group is not uniquely determined: the transition functions generate a smallest (not necessarily closed) subgroup of $\mathrm{GL}_n(\mathbb{K})$, but one usually fixes a Lie subgroup G encoding additional geometric structure (e.g. $\mathrm{O}(n)$, $\mathrm{U}(n)$).

Proposition III.4. *Let M be a connected smooth compact manifold, let G be a Lie group. Then there is the following bijection,*

$$\left\{ \begin{array}{l} \text{Flat principal} \\ \text{\(G\)-bundles over } M \end{array} \right\} /_{\sim_{iso}} \longleftrightarrow \text{Hom}(\pi_1(M), G) / G,$$

between the isomorphism classes² of principal G -bundles with flat connection and conjugation classes respectively.

Proof. For a proof we refer to [Hos13]. □

As will be addressed in Definition III.7, *characteristic classes* are topological invariants of isomorphism classes of principal G -bundles, independently of any choice of connection. Via the correspondance in the previous proposition, the conjugation class of representations $\pi_1(M) \rightarrow G$ determines a principal G -bundle, and hence give rise to characteristic classes as invariants of such representations. For an excellent introductory survey on this subject, with a handful of examples, we refer to [Mor88].

We shall focus more specifically on *Chern classes* in this Chapter and *Chern numbers* in Section IV.1 to motivate the constructions of our bounded-cohomological invariants associated to representations, also capturing complete non-compact manifolds of finite volume.

III.1.2 Universal Principal G -Bundle

We introduce now a universal example of principal bundles, on which much of this chapter will be based: Recall from Definition II.11 that for any Lie group G its classifying space is the quotient $BG := EG/G$, where EG is a weakly contractible space with a proper free G -action. By construction the natural projection $EG \rightarrow BG$ is a principal G -bundle, called the *universal bundle*. We now set $G = U(n)$, the case of interest to us.

Example III.5. There is a homotopy equivalence $BU(1) \simeq \mathbb{C}P^\infty$ with the infinite-dimensional complex projective space. More generally, $BU(n)$ is homotopy equivalent to the infinite-dimensional complex Grassmannian $Gr_n(\mathbb{C}^\infty)$.

Notably, we have the following correspondence:

Proposition III.6. *Let M be a smooth manifold. Then the following bijection holds*

$$\left\{ \begin{array}{l} \text{continuous} \\ \text{maps} \\ M \rightarrow BU(n) \end{array} \right\} /_{\sim_{homotopy}} \longleftrightarrow \left\{ \begin{array}{l} \text{principal} \\ U(n)\text{-bundles} \\ \text{over } M \end{array} \right\} /_{\sim_{iso}}$$

between homotopy classes and isomorphism classes respectively.

Proof. For a proof we refer to [MS74, §5]. □

²Here, by a *flat connection* on a principal G -bundle, we mean a connection with vanishing curvature. Two flat principal bundles (P, ∇) and (P', ∇') are isomorphic if there exists a principal bundle isomorphism $\phi : P \rightarrow P'$ such that $\phi^*(\nabla') = \nabla$.

In particular, the following *universality property* holds: Any principal G -bundle over a smooth manifold M is isomorphic to the pullback along some continuous map $f : M \rightarrow \mathrm{BU}(n)$, called *classifying map* that fits into the following diagram

$$\begin{array}{ccc} f^* \mathrm{EU}(n) & \longrightarrow & \mathrm{EU}(n) \\ \pi^* f \downarrow & & \pi \downarrow \\ M & \xrightarrow{f} & \mathrm{BU}(n) \end{array}$$

For example, the tautological line bundle γ_1 over $\mathbb{C}\mathbb{P}^n$ is classified by the inclusion $\mathbb{C}\mathbb{P}^n \hookrightarrow \mathbb{C}\mathbb{P}^\infty \simeq \mathrm{BU}(1)$, i.e. γ_1 is the pullback of the universal tautological line bundle.

III.2 Chern Classes

We first define a general type of invariants to classify principal bundles:

Definition III.7. Let M be a smooth manifold and G a Lie group. A *characteristic class* c is an assignment

$$c : \left\{ \begin{array}{l} \text{principal } G\text{-bundles} \\ \text{on manifold } M \end{array} \right\} \longrightarrow \mathrm{H}^*(M, \mathbb{Z})$$

satisfying the *naturality condition*: For any principal G -bundle $\pi : P \rightarrow M$ and $f : N \rightarrow M$ a continuous between smooth manifolds, that fit into the following commutative diagram

$$\begin{array}{ccc} f^* P & \longrightarrow & P \\ f^* \pi \downarrow & & \pi \downarrow \\ N & \xrightarrow{f} & M, \end{array}$$

the characteristic class satisfies

$$c(f^* P) = f^*(c(P)) \in \mathrm{H}^*(N, \mathbb{Z}).$$

For specific choices of G such as the orthogonal group or unitary group, one can refine the definition to classical characteristic classes: Euler, Stiefel–Whitney, and Chern classes. Focussing on the latter, we give two equivalent viewpoints.

III.2.1 Axiomatic Definition of Chern Classes

We first give the axiomatic definition for Chern classes of complex vector bundles; we refer to [Hir95, §4.2, p.58].

Definition III.8. Let $E \rightarrow M$ be a complex vector bundle of rank n over a smooth manifold M . The *total Chern class* of $c(E)$ is defined by the sum

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_n(E) \in \mathrm{H}^{2*}(M, \mathbb{Z}),$$

where each $c_k(E) \in \mathrm{H}^{2k}(M, \mathbb{Z})$ is called the *k-th Chern class*, which is a characteristic class satisfying moreover the following axioms:

- *Normalization:* For the tautological line bundle γ_1 over $\mathbb{C}\mathbb{P}^n$ (cf. Example III.3)

$$c_1(\gamma_1) = -x,$$

where $x \in H^2(\mathbb{C}\mathbb{P}^n, \mathbb{Z})$ is a generator.

- *Whitney sum formula:* For two complex vector bundles E_1, E_2 over M , one has that

$$c_k(E_1 \oplus E_2) = \sum_{i+j=k} c_i(E_1) \cup c_j(E_2).$$

- *Normal degree:* $c_0(E) = 1$, and $c_k(E) = 0$ for $k > n$.

Notably, the top Chern class of the tangent bundle of any closed oriented manifold M evaluates via Poincaré duality to the *Euler characteristic* of M - classically defined as

$$\chi(M) = \sum_{k \geq 0} (-1)^k \cdot \text{rk}_{\mathbb{R}} H_k(M, \mathbb{R}).$$

This statement is also known as the *cohomological Gauß-Bonnet theorem*, more precisely:

Theorem III.9. *Let M be a closed complex n -manifold. Then it holds that:*

$$\chi(M) = \langle c_n(TM), [M] \rangle,$$

where $c_n(TM) \in H^{2n}(M, \mathbb{Z})$ is the top Chern class of the holomorphic tangent bundle TM .

Proof. For a proof we refer to [BT82, §20, (20.10.6)] and [BT82, Proposition 11.24]. □

We now turn to a specific case central to this thesis. We recall that $\mathbb{C}\mathbb{P}^n$ serves as the compact dual of $\mathbb{H}_{\mathbb{C}}^n$ (cf. Subsection I.3.1). Hirzebruch's celebrated proportionality principle relates Chern classes of $\mathbb{C}\mathbb{P}^n$ to those of compact quotients of $\text{PU}(n, 1)$. The principle holds in more generality, however we limit ourselves to this case relevant to this thesis:

Theorem III.10 ([Hir58, p.137], [Hir95, §22.3]). *Let M be a closed complex hyperbolic n -manifold. Then it holds that*

$$c_{(\lambda)}(TM) = (-1)^n \cdot c_{(\lambda)}(T\mathbb{C}\mathbb{P}^n),$$

where $c_{(\lambda)} = c_{\lambda_1} \cup \dots \cup c_{\lambda_r}$ denotes cup products of Chern classes and $(\lambda) = (\lambda_1, \dots, \lambda_r)$ is a partition of weight n .

Next, we note that since for the canonical generator $x \in H^2(\mathbb{C}\mathbb{P}^n, \mathbb{Z})$ and for all $0 \leq k \leq n$ classically the relation $c_k(T\mathbb{C}\mathbb{P}^n) = \binom{n+1}{k} \cdot x^k$ holds, this implies that

$$(n+1)^{n-1} \cdot c_n(T\mathbb{C}\mathbb{P}^n) = c_1(T\mathbb{C}\mathbb{P}^n)^n. \tag{III.1}$$

This relation together Hirzebruch's proportionality principle above, allows to deduce the following straightforwardly:

Corollary III.11. *Let M be a closed complex hyperbolic n -manifold. Then it holds that:*

$$(n+1)^{n-1} \cdot c_n(TM) = c_1(TM)^n.$$

We will also require the following instance of Hirzebruch's proportionality relation in terms of the Euler characteristic and the volume (see Example II.8):

Theorem III.12 ([Hir58, p.136],[Hir57]). *Let M be closed complex hyperbolic n -manifold. Then it holds that*

$$\frac{\chi(M)}{\mathrm{Vol}(M)} = (-1)^n \cdot \frac{\chi(\mathbb{C}\mathbb{P}^n)}{\mathrm{Vol}(\mathbb{C}\mathbb{P}^n)} = (-1)^n \cdot \frac{(n+1)!}{(4\pi)^n}.$$

We shall denote this fraction by $h_n := \frac{(n+1)!}{(-4\pi)^n}$ and refer to it as the *Hirzebruch proportionality constant* (see also [HP96, p.720]).

III.2.2 Universal Chern Classes

We define the *universal k -th Chern class* for all $1 \leq k \leq n$ by

$$c_k^{\mathrm{univ}} := c_k(\mathrm{EU}(n)),$$

where $c_k(\mathrm{EU}(n))$ denotes the k -th Chern class of the universal bundle $\mathrm{EU}(n) \rightarrow \mathrm{BU}(n)$. By the universality property of $\mathrm{BU}(n)$ in Proposition III.6 together with the naturality of characteristic classes (see Definition III.7), the Chern class of any principal $\mathrm{U}(n)$ -bundle is obtained as the pullback of the universal Chern class along its classifying map.

In particular, Chern classes of complex vector bundles (see Definition III.8) can be obtained this way: Recall that every rank- n complex vector bundle $E \rightarrow M$ admits a reduction of its structure group from $\mathrm{GL}_n(\mathbb{C})$ to $\mathrm{U}(n)$, corresponding to the choice of a Hermitian metric on the fibers. Importantly, this reduction does not change the isomorphism class of the underlying complex vector bundle and hence neither its Chern classes $c_k(E)$ for all $1 \leq k \leq n$. Consequently, the Chern classes $c_k(E)$ correspond to the pullback of the universal Chern classes c_k^{univ} along a suitable classifying map $M \rightarrow \mathrm{BU}(n)$ for $1 \leq k \leq n$.

Further, the integral cohomology ring of $\mathrm{BU}(n)$ is freely generated by these Chern classes, which both justifies their existence and highlights their central role for our discussions:

Proposition III.13. *The integral cohomology ring of $\mathrm{BU}(n)$ is isomorphic to the polynomial ring generated by the corresponding Chern classes $\{c_k^{\mathrm{univ}}\}_{k=1}^n$, with each c_k^{univ} of degree $2k$:*

$$\mathrm{H}^*(\mathrm{BU}(n), \mathbb{Z}) \cong \mathbb{Z}[c_1^{\mathrm{univ}}, \dots, c_n^{\mathrm{univ}}].$$

Proof. For a proof we refer to [Koc96, Theorem 2.3.1(a)] □

In this cohomology ring, universal Chern classes satisfy the following stability property:

Proposition III.14. *Let*

$$\begin{aligned} i: \mathrm{U}(n-1) &\longrightarrow \mathrm{U}(n) \\ A &\longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

be the natural inclusion map and consider the induced classifying map $Bi : \text{BU}(n-1) \rightarrow \text{BU}(n)$. Then it holds that

$$Bi^* : H^*(\text{BU}(n), \mathbb{Z}) \longrightarrow H^*(\text{BU}(n-1), \mathbb{Z}),$$

$$c_k^{\text{univ}} \longmapsto \begin{cases} c_k^{\text{univ}}, & \text{if } k < n. \\ 0, & \text{if } k \geq n. \end{cases}$$

In particular, for any natural inclusion $U(n_1) \rightarrow U(n_2)$ for $n_1 < n_2$ this property holds.

Proof. For a proof, we refer to [Koc96, Theorem 2.3.1(b)] □

We deduce a simple corollary (we require in Proposition III.29):

Corollary III.15. *The determinant $\det : U(n) \rightarrow U(1)$ induces an isomorphism*

$$\det^* : H^2(\text{BU}(1), \mathbb{Z}) \longrightarrow H^2(\text{BU}(n), \mathbb{Z})$$

in degree 2. In particular, $c_1^{\text{univ}} = \det^*(c_1^{\text{univ}})$.

Proof. The canonical inclusion

$$i : U(1) \longrightarrow U(n)$$

$$z \longmapsto \begin{pmatrix} z & 0 \\ 0 & \text{Id}_{n-1} \end{pmatrix}$$

is a right inverse to the determinant, $\det \circ i = \text{Id}_{U(1)}$. In particular, the composition

$$H^2(\text{BU}(1), \mathbb{Z}) \xrightarrow{\det^*} H^2(\text{BU}(n), \mathbb{Z}) \xrightarrow{i^*} H^2(\text{BU}(1), \mathbb{Z})$$

is the identity. But in degree 2, both cohomology groups are isomorphic to \mathbb{Z} , generated by their respective Chern classes c_1^{univ} , with i^* sending $c_1^{\text{univ}} \in H^2(\text{BU}(n), \mathbb{Z})$ to $c_1^{\text{univ}} \in H^2(\text{BU}(1), \mathbb{Z})$ by the stability in Proposition III.14, and the lemma follows. □

III.3 Chern Classes in Group Cohomology

The goal of this section is to make sense of Chern classes in integral and real-valued group cohomology in light of Wigner's isomorphism from Theorem II.12, which we recall asserts that for any Lie group G , there is an isomorphism

$$H^*(BG, \mathbb{Z}) \cong H_B^*(G, \mathbb{Z}).$$

This means that any characteristic class may be interpreted as a class in integral Borel cohomology. We obtain the real valued counterparts as the images of the natural map

$$H_B^*(G, \mathbb{Z}) \longrightarrow H_B^*(G, \mathbb{R}),$$

induced by the change of coefficients $\mathbb{Z} \mapsto \mathbb{R}$.

In the following subsections, we first formalize the *first* Chern class in integral and real group cohomology for a simple non-compact Hermitian Lie group with finite center and then deal with the special case $\mathrm{PU}(n, 1)$, where we introduce all Chern classes. To keep track of these distinctions, we introduce separate notation for the integral and real cases.

Our strategy for obtaining integral Chern classes relies on the observation that the canonical inclusion $K \hookrightarrow G$ of a maximal compact group induces a homotopy equivalence on its classifying spaces $BK \hookrightarrow BG$ and consequently by Wigner's isomorphism

$$H_B^*(K, \mathbb{Z}) \cong H_B^*(G, \mathbb{Z}).$$

This allows to first define our desired classes on K .

III.3.1 First Chern Class for Hermitian Lie Group

We first treat the general case where G is a simple non-compact Hermitian Lie group with finite center and let K be a maximal compact subgroup. We recall from Lemma I.56 in Chapter I, that

$$K/[K, K] \cong \mathrm{U}(1)$$

which allows to identify the natural projection $K \rightarrow K/[K, K]$ with a homomorphism³

$$\det : K \longrightarrow \mathrm{U}(1).$$

Then, using its induced pullback map \det^* in degree 2 integral Borel cohomology, we give the following definitions:

Definition III.16. Let G be a simple non-compact Hermitian Lie group with finite center and K a maximal compact subgroup.

- The *integral first Chern class* \overline{C}_1 associated to G , is the image of the class $c_1^{\mathrm{univ}} \in H^2(\mathrm{BU}(1), \mathbb{Z})$ under the maps

$$H^2(\mathrm{BU}(1), \mathbb{Z}) \xrightarrow{\cong} H_B^2(\mathrm{U}(1), \mathbb{Z}) \xrightarrow{\det^*} H_B^2(K, \mathbb{Z}) \xrightarrow{\cong} H_B^2(G, \mathbb{Z}),$$

where the outer-most maps are isomorphisms due to Wigner's isomorphism.

- The *real first Chern class* \overline{C}_1 associated to G , is the image of the integral first Chern class \overline{C}_1 under the map induced through change of coefficients $\mathbb{Z} \hookrightarrow \mathbb{R}$, namely

$$H_B^2(G, \mathbb{Z}) \longrightarrow H_B^2(G, \mathbb{R}).$$

We justify that both the integral and real first Chern classes in the definition are non-trivial classes:

Lemma III.17. *Let G be a simple non-compact Hermitian Lie group with finite center and K a maximal compact subgroup. The following statements hold:*

³We shall see in Subsection III.3.4 that this is indeed a generalization of the classical determinant map $\det : \mathrm{U}(n) \rightarrow \mathrm{U}(1)$.

(i) The map $\det : K \rightarrow \mathrm{U}(1)$ induces a non-zero map $\det^* : \mathrm{H}_B^2(\mathrm{U}(1), \mathbb{Z}) \rightarrow \mathrm{H}_B^2(K, \mathbb{Z})$.

(ii) The change of coefficients $\mathbb{Z} \hookrightarrow \mathbb{R}$ induces in Borel cohomology an injective homomorphism

$$\mathrm{H}_B^2(G, \mathbb{Z}) \hookrightarrow \mathrm{H}_B^2(G, \mathbb{R})$$

Proof. Ad (i): First, we recall that for any locally compact group H , the second Borel cohomology group $\mathrm{H}_B^2(H, \mathbb{Z})$ is naturally identified with the set of equivalence classes of central extensions of H by \mathbb{Z} (see [Bro82, Ch. IV, §2, Theorem 3.12]). Under this correspondence, the zero cohomology class corresponds precisely to split extensions.

The group $\mathrm{H}_B^2(\mathrm{U}(1), \mathbb{Z}) \cong \mathbb{Z}$ is generated by the integral first Chern class C_1 , which is represented by the standard central extension

$$\mathbb{Z} \hookrightarrow \mathbb{R} \xrightarrow{\exp} \mathrm{U}(1),$$

where $\exp(t) = e^{2\pi it}$. We claim that the pullback class $\det^*(C_1) \in \mathrm{H}_B^2(K, \mathbb{Z})$ is non-zero. For this, we consider the pullback central extension

$$\det^*(\mathbb{R}) = \{(k, t) \in K \times \mathbb{R} \mid \det(k) = e^{2\pi it}\},$$

which fits into the commutative diagram

$$\begin{array}{ccccc} \mathbb{Z} & \hookrightarrow & \det^*(\mathbb{R}) & \xrightarrow{\pi} & K \\ \parallel & & \downarrow \text{pr} & & \downarrow \det \\ \mathbb{Z} & \hookrightarrow & \mathbb{R} & \xrightarrow{\exp} & \mathrm{U}(1). \end{array} \quad (\text{III.2})$$

Here $\pi(k, t) = k$ and $\text{pr}(k, t) = t$. The top row is a central extension of K by \mathbb{Z} , and it represents the cohomology class $\det^*(C_1)$. We suppose for contradiction that this extension splits. Then there exists a continuous group homomorphism

$$\sigma : K \longrightarrow \det^*(\mathbb{R}) \quad \text{such that} \quad \pi \circ \sigma = \text{id}_K.$$

Composing σ with the projection pr yields a continuous group homomorphism $\tilde{\sigma} := \text{pr} \circ \sigma$ satisfying $\exp \circ \tilde{\sigma} = \det$. Since K is compact, the image $\tilde{\sigma}(K) \subset \mathbb{R}$ is a compact subgroup of \mathbb{R} . The only compact subgroup of \mathbb{R} is $\{0\}$, hence $\tilde{\sigma}$ is trivial. It follows that $\exp \circ \tilde{\sigma}$ is the trivial homomorphism, contradicting the fact that $\det : K \rightarrow \mathrm{U}(1)$ is surjective and therefore non-trivial. This contradiction shows that the extension in the top row of Diagram (III.2) does not split. Consequently, $\det^*(C_1) \neq 0$ in $\mathrm{H}_B^2(K, \mathbb{Z})$, which proves (i).

Ad (ii): We first prove that a simple non-compact Hermitian Lie group with finite center G is perfect, i.e. $G = [G, G]$.

The commutator subgroup $[G, G]$ is closed and normal. We suppose for contradiction that $[G, G] \neq G$. Then the quotient $G/[G, G]$ is a non-trivial abelian Lie group. Since G is connected, $G/[G, G]$ is connected as well, hence it has positive dimension and is therefore infinite. On the

other hand, since G is a connected simple Lie group with finite center, any proper normal subgroup is discrete and contained in the center. Such a subgroup is of infinite index unless it is open. But a connected Lie group admits no proper open subgroup. Hence, G admits no proper normal subgroup of finite index. This contradicts that $[G, G] < G$ is a finite index subgroup. Hence $[G, G] = G$, so G is perfect.

Next, we consider the short exact sequence of coefficients

$$\mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z},$$

which induces the long exact sequence in Borel cohomology (cf. Subsection II.1.2):

$$\dots \longrightarrow H_B^1(G, \mathbb{R}/\mathbb{Z}) \longrightarrow H_B^2(G, \mathbb{Z}) \longrightarrow H_B^2(G, \mathbb{R}) \longrightarrow \dots$$

Since G is a locally compact group that is perfect, one has

$$H_B^1(G, \mathbb{R}/\mathbb{Z}) \cong \mathrm{Hom}_c(G, \mathbb{R}/\mathbb{Z}) \cong \mathrm{Hom}_c(G/[G, G], \mathbb{R}/\mathbb{Z}) = 0$$

Exactness of the long exact sequence implies that then the homomorphism $H_B^2(G, \mathbb{Z}) \rightarrow H_B^2(G, \mathbb{R})$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ is injective. □

In particular, as mentioned previously in Subsection II.4.1, as a generator for the group $H_c^2(G, \mathbb{R}) \cong \mathbb{R}$, one may either choose the Kähler class $\bar{\omega}_2$ or the first Chern class \bar{C}_1 .

We remark that the injective map $H_B^2(G, \mathbb{Z}) \rightarrow H_B^2(G, \mathbb{R})$ in Lemma III.17 may be understood through the diagram in the following proposition that relies on work by Hartnick and Ott [HO12].

Proposition III.18. *Let G be a simple non-compact Hermitian Lie group with finite center, K its maximal compact subgroup and X_u its compact dual. If $f : X_u \rightarrow \mathrm{BU}(1)$ is the classifying map of the determinant line bundle $\det(TX_u)$ then the following diagram*

$$\begin{array}{ccc}
 H_B^2(G, \mathbb{Z}) & \longrightarrow & H_B^2(G, \mathbb{R}) \\
 \cong \downarrow & & \uparrow \\
 H_B^2(K, \mathbb{Z}) & & \\
 \cong \downarrow \text{Wigner} & & \\
 H^2(\mathrm{BK}, \mathbb{Z}) & & \cong \\
 (B \det)^* \uparrow & & \\
 H^2(\mathrm{BU}(1), \mathbb{Z}) & & \\
 f^* \downarrow & & \\
 H^2(X_u, \mathbb{Z}) & \longrightarrow & H^2(X_u, \mathbb{R})
 \end{array} \tag{III.3}$$

commutes up to a sign. In particular, the class $c_1(TX_u)$ is mapped to $-\bar{C}_1$ under the right vertical map.

Proof. Let $\sigma_G : H^2(\mathrm{BG}, \mathbb{R}) \rightarrow H_c^2(G, \mathbb{R})$ be the universal map in the sense of Bott, as described

by Hartnick and Ott in [HO12, Proposition 1.2]. Then the following diagram commutes

$$\begin{array}{ccc}
 \mathrm{H}^2(BG, \mathbb{Z}) & \longrightarrow & \mathrm{H}^2(BG, \mathbb{R}) \xrightarrow{-\sigma_G} \mathrm{H}_c^2(G, \mathbb{R}) \\
 \cong \uparrow \text{Wigner} & & \cong \uparrow \text{Austin-Moore} \\
 \mathrm{H}_B^2(G, \mathbb{Z}) & \longrightarrow & \mathrm{H}_B^2(G, \mathbb{R})
 \end{array} \tag{III.4}$$

where the top-left and bottom horizontal maps are induced by the change of coefficients $\mathbb{Z} \hookrightarrow \mathbb{R}$.

Next, let $f : X_u \rightarrow \mathrm{BU}(1)$ be the classifying map of the determinant line bundle $\det(TX_u)$ of the compact dual $X_u = G_u/K$. Then there exists a unique classifying map $\tilde{f} : X_u \rightarrow BK$, up to homotopy, such that

$$f \simeq B \det \circ \tilde{f}. \tag{III.5}$$

In particular, here \tilde{f} classifies the canonical principal K -bundle $G_u \rightarrow X_u$, and the homotopy equivalence follows since $\det(TX_u)$ is the line bundle associated to this bundle via the character $\det : K \rightarrow \mathrm{U}(1)$ (see Subsection III.3.1).

Further, by [HO12, §4.1, Eq. (10)], the following diagram commutes

$$\begin{array}{ccc}
 \mathrm{H}^2(BG, \mathbb{R}) & \xrightarrow{-\sigma_G} & \mathrm{H}_c^2(G, \mathbb{R}) \\
 \cong \uparrow \text{Wigner} & & \cong \uparrow \\
 \mathrm{H}^2(BK, \mathbb{R}) & \xrightarrow{\tilde{f}^*} & \mathrm{H}^2(X_u, \mathbb{R})
 \end{array} ,$$

where the bottom horizontal map is induced by the classifying map $\tilde{f} : X_u \rightarrow BK$. The commutativity up to sign of Diagram (III.3) follows by combining this diagram with (III.4) and the homotopy relation (III.5). By construction, the right vertical map sends $c_1(TX_u)$ to the class $-\overline{C}_1$. \square

III.3.2 Chern Classes for $\mathrm{PU}(n, 1)$

Since there is a homotopy equivalence $\mathrm{BPU}(n, 1) \simeq \mathrm{BU}(n)$, it follows from Theorem III.13 that

$$\mathrm{H}^*(\mathrm{BPU}(n, 1)(n), \mathbb{Z}) = \mathrm{H}^*(\mathrm{BU}(n), \mathbb{Z}) \cong \mathbb{Z}[c_1^{\mathrm{univ}}, \dots, c_n^{\mathrm{univ}}],$$

and so to fix our notation we introduce:

Definition III.19. Let $1 \leq k \leq n$. Under Wigner's isomorphism, the universal k -th Chern class

$$c_k^{\mathrm{univ}} \in \mathrm{H}^{2k}(\mathrm{BPU}(n, 1), \mathbb{Z})$$

corresponds to a unique class we denote by $C_k \in \mathrm{H}_B^{2k}(\mathrm{PU}(n, 1), \mathbb{Z})$, which we call the *integral k -th Chern classes*.

We note that the first Chern class $\overline{C}_1 \in \mathrm{H}^2(G, \mathbb{Z})$ in Definition III.13 coincides with $C_1 \in \mathrm{H}^2(\mathrm{PU}(n, 1), \mathbb{Z})$: Here the homomorphism \det reduces to the classical determinant map $\det : \mathrm{U}(n) \rightarrow \mathrm{U}(1)$, $A \mapsto \det(A)$ with $\mathrm{Ker}(\det) = \mathrm{SU}(n) = [\mathrm{U}(n), \mathrm{U}(n)]$, whose pullback

$$\det^* : \mathrm{H}^2(\mathrm{BU}(1), \mathbb{Z}) \longrightarrow \mathrm{H}^2(\mathrm{BU}(n), \mathbb{Z}),$$

preserves the universal first Chern class due to Corollary III.15.

In this thesis, we will only be concerned with the real-valued counterparts of the first and top Chern classes. Again, we denote these classes in boldface throughout:

Definition III.20. Under the change of coefficients $\mathbb{Z} \hookrightarrow \mathbb{R}$, the images of C_1 and C_n under the induced map for $k = 1, n$, namely

$$\mathrm{H}_B^{2k}(\mathrm{PU}(n, 1), \mathbb{Z}) \longrightarrow \mathrm{H}_B^{2k}(\mathrm{PU}(n, 1), \mathbb{R}),$$

are referred to as the *real first Chern class* \mathbf{C}_1 and *real top Chern class* \mathbf{C}_n respectively.

The classes \mathbf{C}_n and \mathbf{C}_1 are normalizations of the volume class ω_{2n} and Kähler class ω_2 respectively (cf. Section II.4) and are related to each other via a multiplicative formula:

Proposition III.21. *It holds that*

$$\mathbf{C}_n = h_n \cdot \omega_{2n} \quad \text{and} \quad \mathbf{C}_1 = -\frac{n+1}{4\pi} \cdot \omega_2,$$

where $h_n = \frac{(n+1)!}{(-4\pi)^n}$ is the Hirzebruch proportionality constant. Moreover, it holds that

$$(n+1)^{n-1} \cdot \mathbf{C}_n = \mathbf{C}_1^n.$$

We remark that the first relation is the analogue of [BBI21, §3] in the real-hyperbolic case.

Proof. Let M' be an auxiliary compact manifold such that $\pi_1(M') < \mathrm{PU}(n, 1)$ is a uniform torsion-free lattice. We denote its canonical inclusion by $\iota : \pi_1(M') \hookrightarrow \mathrm{PU}(n, 1)$ and further note that it induces a continuous classifying map

$$B\iota : M' \longrightarrow \mathrm{BPU}(n, 1) \simeq \mathrm{BU}(n) \tag{III.6}$$

that classifies the holomorphic tangent bundle TM' .

Now, we observe that the change of coefficients $\mathbb{Z} \hookrightarrow \mathbb{R}$ induces the horizontal maps in the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}_B^{2n}(\mathrm{PU}(n, 1), \mathbb{Z}) & \longrightarrow & \mathrm{H}_B^{2n}(\mathrm{PU}(n, 1), \mathbb{R}) \\ \iota^* \downarrow & & \iota^* \downarrow \\ \mathrm{H}^{2n}(\pi_1(M'), \mathbb{Z}) & \longrightarrow & \mathrm{H}^{2n}(\pi_1(M'), \mathbb{R}) \end{array} .$$

By Theorem II.9 and Poincaré duality, it holds that

$$\mathrm{H}^{2n}(\pi_1(M'), A) \cong \mathrm{H}^{2n}(M', A) \cong A$$

for $A = \mathbb{Z}, \mathbb{R}$. So, using the same notation for the images of cohomological classes under this isomorphism, we see that taking the top Chern class C_n under the left vertical map, we get:

$$\langle \iota^*(C_n), [M'] \rangle = \langle (B\iota)^*(c_n^{\mathrm{univ}}), [M'] \rangle = \langle c_n(TM'), [M'] \rangle = \chi(M'),$$

where $(B\iota)^* : H^{2n}(\mathrm{BU}(n), \mathbb{Z}) \rightarrow H^{2n}(M', \mathbb{Z})$ is induced by the classifying map $B\iota$, with the target deformation retracted to $\mathrm{BU}(n)$ as in (III.6). Commutativity of the square-diagram further implies that $\langle \iota^*(\mathbf{C}_n), [M'] \rangle = \chi(M')$. Taking now the volume class ω_{2n} (see Definition II.20) under the right vertical map, we get

$$\langle \iota^*(\omega_{2n}), [M'] \rangle = \langle \mathrm{vol}_{M'}, [M'] \rangle = \mathrm{Vol}(M').$$

Finally, since by Theorem III.12 it holds that $\chi(M') = h_n \cdot \mathrm{Vol}(M')$, this implies that $c_n(TM') = h_n \cdot \mathrm{vol}_{M'}$ and further $\mathbf{C}_n = h_n \cdot \omega_{2n}$, which proves the first relation of the proposition.

Then, due to commutativity of cup products with homomorphisms and by Corollary III.11, we have that

$$\iota^*(\mathbf{C}_1^n) = c_1(TM')^n = (n+1)^{n-1} \cdot c_n(TM') = (n+1)^{n-1} \cdot \iota^*(\mathbf{C}_n),$$

so that the multiplicative relation $(n+1)^{n-1} \cdot \mathbf{C}_n = \mathbf{C}_1^n$ indeed holds. Finally, using additionally Wirtinger's formula $n! \cdot \omega_{2n} = \omega_2^n$ (see Proposition I.30), we calculate that

$$\mathbf{C}_1^n = (n+1)^{n-1} \cdot \mathbf{C}_n = (n+1)^{n-1} \cdot h_n \cdot \omega_{2n} = \frac{(n+1)^{n-1}}{n!} \cdot \frac{(n+1)!}{(-4\pi)^n} \cdot \omega_2^n = \left(\frac{n+1}{-4\pi} \right)^n \cdot \omega_2^n,$$

which proves the second statement of the proposition. \square

In particular, both real Chern classes \mathbf{C}_1 and \mathbf{C}_n are non-zero and are as such properly defined objects (in fact, for the former this is already established in Lemma III.17). The real top Chern class \mathbf{C}_n can be considered as yet another choice of generator for the one-dimensional group $H_c^{2n}(\mathrm{PU}(n, 1), \mathbb{R})$.

In the spirit of Proposition III.18, the map $H_B^{2n}(\mathrm{PU}(n, 1), \mathbb{Z}) \rightarrow H_B^{2n}(\mathrm{PU}(n, 1), \mathbb{R})$ can be understood through the diagram in the following proposition that relies on work by Hartnick and Ott [HO12]:

Proposition III.22. *Let $f : \mathbb{C}\mathbb{P}^n \rightarrow \mathrm{BU}(n)$ denote the classifying map of the holomorphic tangent bundle $T\mathbb{C}\mathbb{P}^n$. Then the following diagram*

$$\begin{array}{ccc} H_B^{2n}(\mathrm{PU}(n, 1), \mathbb{Z}) & \longrightarrow & H_B^{2n}(\mathrm{PU}(n, 1), \mathbb{R}) \\ \cong \downarrow & & \uparrow \\ H_B^{2n}(\mathrm{U}(n), \mathbb{Z}) & & \\ \cong \downarrow \text{Wigner} & & \\ H^{2n}(\mathrm{BU}(n), \mathbb{Z}) & & \\ f^* \downarrow & & \cong \\ H^{2n}(\mathbb{C}\mathbb{P}^n, \mathbb{Z}) & \longrightarrow & H^{2n}(\mathbb{C}\mathbb{P}^n, \mathbb{R}) \end{array} \quad (\text{III.7})$$

commutes up to the factor $(-1)^n$. In particular, the class $c_n(T\mathbb{C}\mathbb{P}^n)$ is mapped to $-\mathbf{C}_n$ under the right vertical map.

Proof. The proof is similar to the one of Proposition III.18: On the one hand, due to [HO12,

Proposition 1.2] commutativity of Diagram (III.4) also holds in top degree. On the other hand, by [HO12, §4.1, Eq. (10)] the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}^{2n}(\mathrm{BPU}(n, 1), \mathbb{R}) & \xrightarrow{(-1)^n \cdot \sigma_G} & \mathrm{H}_c^{2n}(\mathrm{PU}(n, 1), \mathbb{R}) \\ \cong \uparrow \text{Wigner} & & \cong \uparrow \\ \mathrm{H}^{2n}(\mathrm{BU}(n), \mathbb{R}) & \xrightarrow{f^*} & \mathrm{H}^{2n}(X_u, \mathbb{R}) \end{array},$$

where the horizontal map is induced by the classifying map $f : \mathbb{C}\mathbb{P}^n \rightarrow \mathrm{BU}(n)$ of the tangent bundle $T\mathbb{C}\mathbb{P}^n$. We conclude that Diagram (III.7) commutes up to factor $(-1)^n$. By construction, the right vertical map sends $c_n(T\mathbb{C}\mathbb{P}^n)$ to the class $-\mathcal{C}_n$. \square

We note that more generally, dby [HO12, §4.1, Eq. (10)], Diagram (III.7) commutes up to the factor $(-1)^k$ in cohomological degree $2k$ with $1 \leq k \leq n$.

III.3.3 Bounded Chern Classes

We fix G to be simple non-compact Hermitian Lie group with finite center and we consider the following statement:

Lemma III.23. *For $1 \leq k \leq n$, the comparison map $c_{\mathbb{Z}} : \mathrm{H}_{Bb}^{2k}(G, \mathbb{Z}) \rightarrow \mathrm{H}_B^{2k}(G, \mathbb{Z})$ is surjective. Furthermore, if $k = 1$, then $c_{\mathbb{Z}}$ is an isomorphism.*

This proof along the lines of [BBI21, Lemma 4.1]. We also note that we implicitly use Austin-Moore's Theorem II.13, which implies that the results stated for continuous (bounded) cohomology in Chapter II also hold for Borel (bounded) cohomology.

Proof. We first prove surjectivity for $1 \leq k \leq n$: We consider Diagram (II.5) on page 63, which we display here again for convenience

$$\begin{array}{ccccccc} \mathrm{H}_B^{2k-1}(G, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\delta} & \mathrm{H}_B^{2k}(G, \mathbb{Z}) & \longrightarrow & \mathrm{H}_B^{2k}(G, \mathbb{R}) & \longrightarrow & \mathrm{H}_B^{2k}(G, \mathbb{R}/\mathbb{Z}) \\ \parallel & & c_{\mathbb{Z}} \uparrow & & c_{\mathbb{R}} \uparrow & & \parallel \\ \mathrm{H}_{Bb}^{2k-1}(G, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\delta^b} & \mathrm{H}_{Bb}^{2k}(G, \mathbb{Z}) & \longrightarrow & \mathrm{H}_{Bb}^{2k}(G, \mathbb{R}) & \longrightarrow & \mathrm{H}_{Bb}^{2k}(G, \mathbb{R}/\mathbb{Z}) \end{array},$$

where the vertical maps are comparison maps in the respective coefficients. Since the top and bottom horizontal rows are exact, $c_{\mathbb{R}}$ is surjective by Theorem II.2 and the outer-most comparison maps $c_{\mathbb{R}/\mathbb{Z}}$ are isomorphisms, we may apply an instance of the Four-Lemma (an instance of the Five-Lemma, see e.g. [Hat02, p. 129, The Five-Lemma (b)]) to conclude that also $c_{\mathbb{Z}}$ is surjective.

We now prove injectivity in case $k = 1$. Let $\alpha_{\mathbb{Z}}^b \in \mathrm{H}_{Bb}^2(G, \mathbb{Z})$ be a non-zero class such that $c_{\mathbb{Z}}(\alpha_{\mathbb{Z}}^b) = 0$. Then, since by Theorem II.1 the comparison map $c_{\mathbb{R}}$ is an isomorphism, by commutativity of the middle square in the above diagram, this implies that the image of $\alpha_{\mathbb{Z}}^b$ under the bottom horizontal map vanishes in $\mathrm{H}_{Bb}^2(G, \mathbb{R})$. But by Lemma III.17, the map $\mathrm{H}_B^2(G, \mathbb{Z}) \rightarrow \mathrm{H}_B^2(G, \mathbb{R})$ is injective, so that $\alpha_{\mathbb{Z}}^b = 0$. \square

In light of the preceding lemma, we may now formalize the notion of bounded Chern classes:

Definition III.24. Let G be a simple non-compact Hermitian Lie group with finite center. We refer to the unique bounded classes

$$\overline{C}_1^b = c_{\mathbb{Z}}^{-1}(\overline{C}_1) \in H_{Bb}^2(G, \mathbb{Z}) \quad \text{and} \quad \overline{C}_1^b = c_{\mathbb{R}}^{-1}(\overline{C}_1) \in H_{Bb}^2(G, \mathbb{R}),$$

as the *integral bounded and real bounded first Chern class*, respectively.

When $G = \text{PU}(n, 1)$, we consider the top-degree Borel bounded cohomology and fix choices of an *integral bounded top Chern class* $C_n^b \in H_{Bb}^{2n}(\text{PU}(n, 1), \mathbb{Z})$ and *real bounded top Chern class* $C_n^b \in H_{Bb}^{2n}(\text{PU}(n, 1), \mathbb{R})$, namely

$$C_n^b \in c_{\mathbb{Z}}^{-1}(C_n) \quad \text{and} \quad C_n^b \in c_{\mathbb{R}}^{-1}(C_n),$$

such that the former is mapped to the latter under the map $H_{Bb}^{2n}(\text{PU}(n, 1), \mathbb{Z}) \rightarrow H_{Bb}^{2n}(\text{PU}(n, 1), \mathbb{R})$ induced by the coefficients' inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$.

III.3.4 (Bounded) Chern classes for Subgroups of $\text{PU}(n, 1)$.

Let $H < \text{PU}(n, 1)$ be an arbitrary subgroup and $i : H \hookrightarrow \text{PU}(n, 1)$ the canonical inclusion. Denote for all $k \in \mathbb{N}_{>0}$ under the induced map

$$i^* : H^{2k}(\text{PU}(n, 1), \mathbb{Z}) \longrightarrow H^{2k}(H, \mathbb{Z}),$$

the restriction of C_k by $C_{k|H}$. In particular, the inclusion i induces the vertical maps in the following commutative diagram

$$\begin{array}{ccc} H_B^*(\text{PU}(n, 1), \mathbb{Z}) & \longrightarrow & H_B^*(\text{PU}(n, 1), \mathbb{R}) \\ i^* \downarrow & & i^* \downarrow \\ H_B^*(H, \mathbb{Z}) & \longrightarrow & H_B^*(H, \mathbb{R}) \end{array}$$

where the horizontal maps are induced by the change of coefficients $\mathbb{Z} \hookrightarrow \mathbb{R}$. That is $i^*(C_{k|H})$ is mapped to the restriction of C_k , which we denote by $C_{k|H}$.

We have the same commutative diagram also for *bounded* Borel cohomology, for which we denote by analogy

$$i^*(C_k^b) = C_{k|H}^b \quad \text{and} \quad i^*(C_k^b) = C_{k|H}^b.$$

III.4 Cohomologies and Chern Classes for $\text{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$ -Subgroups

The goal of this section is to describe Borel cohomologies in both \mathbb{Z} and \mathbb{R} -coefficients of $\text{PU}(n, 1)$ and its point-stabilizing subgroups that are conjugate to $U(n)$ and $\text{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$. We will require the unbounded, but also the bounded version occasionally.

For $\text{PU}(n, 1)$ and its maximal compact subgroup $U(n)$, we summarize the following known facts:

- In \mathbb{Z} -coefficients, due to Example II.4 we have the isomorphism

$$H_B^*(U(n), \mathbb{Z}) \cong H_{Bb}^*(U(n), \mathbb{Z})$$

and further due to the homotopy equivalence $\mathrm{PU}(n, 1) \simeq U(n)$ combined with Wigner's isomorphism, as well as Proposition III.13, the following isomorphisms hold:

$$H_B^*(\mathrm{PU}(n, 1), \mathbb{Z}) \cong H_B^*(U(n), \mathbb{Z}) \cong \mathbb{Z}[C_1, \dots, C_n].$$

- In \mathbb{R} -coefficients, we have

$$H_B^*(\mathrm{PU}(n, 1), \mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{for even } * \leq n, \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad H_B^{*>0}(U(n), \mathbb{R}) \cong H_{Bb}^{*>0}(U(n), \mathbb{R}) = 0,$$

where the left part is due to Proposition II.16 and the right due to Example II.4. The real-valued continuous bounded cohomology of $\mathrm{PU}(n, 1)$ is unknown for degrees greater than 3. In particular, we recall that the comparison map $c_{\mathbb{R}}$ is an isomorphism in degree 2 due to Theorem II.1, which is also the case in degree 3 due to a recent result by Bucher and Savini in [BS25]).

We shall implicitly use Austin-Moore's Theorem II.13, which implies that the results stated for continuous (bounded) cohomology in Chapter II also hold for Borel (bounded) cohomology.

III.4.1 Cohomologies of $\mathrm{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$ and its Subgroups

The present work moreover requires an extensive description for the Borel cohomology of the minimal parabolic group $P = \mathrm{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$. For this it is useful to recall from Theorem I.65, that

$$P = \mathcal{H}_{\mathbb{R}}^{2n-1} \rtimes (U(n-1) \times \mathbb{R}_{>0}),$$

in order to prove the following proposition:

Proposition III.25. *The Borel cohomologies of $P = \mathrm{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$ in \mathbb{R} -coefficients are given by*

$$H_{Bb}^*(P, \mathbb{R}) = \begin{cases} \mathbb{R}, & \text{if } * = 0, \\ 0, & \text{if } * \geq 1, \end{cases} \quad \text{and} \quad H_B^*(P, \mathbb{R}) = \begin{cases} \mathbb{R}, & \text{if } * = 0, 1 \\ 0, & \text{if } * \geq 2. \end{cases},$$

and in \mathbb{Z} -coefficients the Borel cohomologies are given by:

$$\begin{aligned} H_B^1(P, \mathbb{Z}) &= H_{Bb}^1(P, \mathbb{Z}) = 0, \\ H_B^2(P, \mathbb{Z}) &\cong \mathbb{Z}, \quad H_{Bb}^2(P, \mathbb{Z}) \cong \mathbb{R} \times \mathbb{Z}, \\ H_B^k(P, \mathbb{Z}) &\cong H_B^{k-1}(P, \mathbb{R}/\mathbb{Z}) \cong H_{Bb}^k(P, \mathbb{Z}), \quad \forall k > 2. \end{aligned}$$

Proof. In case of real coefficients, we have the vanishing of the Borel bounded cohomology by amenability of P (see Theorem II.3), while its unbounded cohomology is isomorphic to the

cohomology of the maximal split torus $A = \mathbb{R}_{>0}$ [Mon22, Proposition 3.2], so that $H_B^k(P, \mathbb{R}) \cong \mathbb{R}$ if $k = 0, 1$ and 0 for $k \geq 2$.

For \mathbb{Z} -coefficients we proceed as follows. Since $U(n-1)$ is a maximal compact subgroup of P , then Wigner's isomorphism in Theorem II.12 implies that

$$H_B^*(P, \mathbb{Z}) \cong H_B^*(U(n-1), \mathbb{Z}),$$

and the cohomology of $U(n-1)$ is a polynomial ring on its Chern classes, which yields the statements for the cohomologies in degrees 1 and 2 in the unbounded case. On the bounded side for $k = 1$, we have $H_{Bb}^1(P, \mathbb{Z}) = 0$, which is valid for any group, whereas for $k = 2$, we apply Diagram (II.5) on page 63, to obtain the commutative diagram

$$\begin{array}{ccc} H_{Bb}^1(P, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\cong} & H_{Bb}^2(P, \mathbb{Z}) \\ \parallel & & \downarrow \\ \mathbb{R} \hookrightarrow H_B^1(P, \mathbb{R}/\mathbb{Z}) & \longrightarrow & \mathbb{Z} \end{array} .$$

From the bottom horizontal short exact sequence, we deduce that

$$H_B^1(P, \mathbb{R}/\mathbb{Z}) = H_{Bb}^1(P, \mathbb{R}/\mathbb{Z}) = \mathbb{R} \times \mathbb{Z},$$

which via the top isomorphism implies then also $H_{Bb}^2(P, \mathbb{Z}) = \mathbb{R} \times \mathbb{Z}$.

□

We will require additional cohomological properties that relate the group $P = \text{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$ to its subgroups. More specifically, it will be useful to reinterpret its semi-direct product structure in terms of the short exact sequence:

$$\mathcal{H}_{\mathbb{R}}^{2n-1} \hookrightarrow P \xrightarrow{\pi} U(n-1) \times \mathbb{R}_{>0},$$

to establish properties of the corresponding pullback maps in degree 2 integral Borel bounded cohomology:

Proposition III.26. *The natural projection $\pi : P = \text{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1}) \rightarrow U(n-1) \times \mathbb{R}_{>0}$ induces an isomorphism*

$$\pi^* : H_{Bb}^2(U(n-1) \times \mathbb{R}_{>0}, \mathbb{Z}) \longrightarrow H_{Bb}^2(P, \mathbb{Z}).$$

Proof. Since both $P = \text{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$ and $U(n-1) \times \mathbb{R}_{>0}$ are amenable, their bounded continuous cohomology with real coefficients vanishes. Therefore, by the exactness of the top row in (II.5), the proposition reduces to showing that π induces an isomorphism

$$\pi^* : H_B^1(U(n-1) \times \mathbb{R}_{>0}, \mathbb{R}/\mathbb{Z}) \longrightarrow H_B^1(P, \mathbb{R}/\mathbb{Z}).$$

For this, consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{H}_B^1(P, \mathbb{R}) & \hookrightarrow & \mathrm{H}_B^1(P, \mathbb{R}/\mathbb{Z}) & \twoheadrightarrow & \mathrm{H}_B^2(P, \mathbb{Z}) \\ \pi^* \uparrow & & \pi^* \uparrow & & \pi^* \uparrow \\ \mathrm{H}_B^1(\mathrm{U}(n-1) \times \mathbb{R}_{>0}, \mathbb{R}) & \hookrightarrow & \mathrm{H}_B^1(\mathrm{U}(n-1) \times \mathbb{R}_{>0}, \mathbb{R}/\mathbb{Z}) & \twoheadrightarrow & \mathrm{H}_B^2(\mathrm{U}(n-1) \times \mathbb{R}_{>0}, \mathbb{Z}). \end{array}$$

We observe that the two horizontal maps on the left are indeed injective since both P and $\mathrm{U}(n-1) \times \mathbb{R}_{>0}$ are homotopically equivalent to their maximal compact subgroup $\mathrm{U}(n-1)$ and $\mathrm{H}_B^1(\mathrm{U}(n-1), \mathbb{Z}) = 0$. The surjectivity of the two horizontal maps on the right follows from the vanishing of the real valued cohomology in degree 2: for P this was observed above in Proposition III.25 and for $\mathrm{U}(n-1) \times \mathbb{R}_{>0}$ it follows from the Künneth formula that

$$\mathrm{H}_B^2(\mathrm{U}(n-1) \times \mathbb{R}_{>0}, \mathbb{R}) \cong \mathrm{H}_B^2(\mathrm{U}(n-1), \mathbb{R}) \oplus \mathrm{H}_B^2(\mathbb{R}_{>0}, \mathbb{R}) = 0 \oplus 0 = 0.$$

For the right vertical map, since both groups in question are homotopy equivalent to $\mathrm{U}(n-1)$, the corresponding cohomology groups are isomorphic to $\mathrm{H}_B^2(\mathrm{U}(n-1), \mathbb{Z}) \cong \mathbb{Z}$. Furthermore, these isomorphisms are induced by natural inclusions of the subgroup $\mathrm{U}(n-1)$, that fit into the following commuting diagram

$$\begin{array}{ccc} P & \xrightarrow{\pi} \twoheadrightarrow & \mathrm{U}(n-1) \times \mathbb{R}_{>0} \\ \uparrow & \nearrow & \\ \mathrm{U}(n-1) & & \end{array}, \tag{III.8}$$

and thus proves that $\pi^* : \mathrm{H}_B^2(\mathrm{U}(n-1) \times \mathbb{R}_{>0}, \mathbb{Z}) \rightarrow \mathrm{H}_B^2(P, \mathbb{Z})$ is an isomorphism.

For the left vertical map we argue likewise with the natural inclusion of $\mathbb{R}_{>0}$ in both groups,

$$\begin{array}{ccc} P & \xrightarrow{\pi} \twoheadrightarrow & \mathrm{U}(n-1) \times \mathbb{R}_{>0} \\ \uparrow & \nearrow & \\ \mathbb{R}_{>0} & & \end{array}.$$

Indeed, the inclusions induce isomorphism on the first real valued cohomology for both groups: for P we can invoke [Mon22, Proposition 3.2] to see that $\mathrm{H}_B^1(P, \mathbb{R}) \cong \mathrm{H}_B^1(\mathbb{R}_{>0}, \mathbb{R})$; whereas for the product $\mathrm{U}(n-1) \times \mathbb{R}_{>0}$ it is clear since there are no homomorphisms from $\mathrm{U}(n-1)$ to \mathbb{R} , which using Künneth's theorem implies

$$\mathrm{H}_B^1(\mathrm{U}(n-1) \times \mathbb{R}_{>0}, \mathbb{R}) \cong \mathrm{H}_B^1(\mathrm{U}(n-1), \mathbb{R}) \oplus \mathrm{H}_B^1(\mathbb{R}_{>0}, \mathbb{R}) \cong \mathrm{H}_B^1(\mathbb{R}_{>0}, \mathbb{R}).$$

In conclusion, by the Four Lemma (instance of Five lemma, see e.g. [Hat02, p. 129, The Five-Lemma (b)]) we have thus established that

$$\pi^* : \mathrm{H}_B^1(\mathrm{U}(n-1) \times \mathbb{R}_{>0}, \mathbb{R}) \longrightarrow \mathrm{H}_B^1(P, \mathbb{R})$$

is an isomorphism, which concludes the proof of the proposition. \square

We conclude this section with a consequence of Proposition III.26, which will be a decisive step of Theorem B in Chapter IV:

Corollary III.27. *The inclusion $i : \mathcal{H}_{\mathbb{R}}^{2n-1} \rightarrow P = \text{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$ induces the zero map*

$$i^* : H_{Bb}^2(P, \mathbb{Z}) \longrightarrow H_{Bb}^2(\mathcal{H}_{\mathbb{R}}^{2n-1}, \mathbb{Z}).$$

Proof. Given that we have the short exact sequence

$$\mathcal{H}_{\mathbb{R}}^{2n-1} \xleftarrow{i} P \xrightarrow{\pi} \mathbb{U}(n-1) \times \mathbb{R}_{>0},$$

it holds that $\pi \circ i$ is trivial. Therefore, for the induced map in degree 2 integral Borel bounded cohomology we have that $i^* \circ \pi^* = 0$. Consequently, since by Proposition III.26 the map π^* is an isomorphism, we conclude that

$$i^* = i^* \circ \text{id}_{H_{Bb}^2(P, \mathbb{Z})} = i^* \circ \pi^* \circ (\pi^*)^{-1} = 0.$$

□

III.4.2 Chern Classes of $\text{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$ and its Subgroups

The restriction of the bounded *top* Chern class versus bounded *first* Chern class to the parabolic group $P = \text{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$, exhibit contrasting behaviour. To demonstrate this, we shall exploit the following commutative diagram of canonical inclusions

$$\begin{array}{ccc} \text{PU}(n, 1) & \longleftarrow & \text{U}(n) \\ \uparrow & & \uparrow \\ P & \longleftarrow & \text{U}(n-1), \end{array}$$

which induces the right commutative square in integral Borel cohomology

$$\begin{array}{ccccc} H_{Bb}^*(\text{PU}(n, 1), \mathbb{Z}) & \xrightarrow{c} & H_B^*(\text{PU}(n, 1), \mathbb{Z}) & \xrightarrow{\cong} & H_B^*(\text{U}(n), \mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ H_{Bb}^*(P, \mathbb{Z}) & \xrightarrow{c} & H_B^*(P, \mathbb{Z}) & \xrightarrow{\cong} & H_B^*(\text{U}(n-1), \mathbb{Z}) \end{array} \quad (\text{III.9})$$

while the left square comes from the inclusion of P in $\text{PU}(n, 1)$ in bounded and unbounded cohomology respectively.

On the one hand, this allows to obtain the following statement (directly used in the proof of Theorem A in Chapter IV):

Proposition III.28. *Let $n \geq 2$. The restriction of the top bounded Chern class $C_b^n \in H_{Bb}^{2n}(\text{PU}(n, 1), \mathbb{Z})$ to P vanishes.*

Proof. Consider the Diagram (III.9) in top cohomology and follow the top bounded Chern class $C_b^n \in H_{Bb}^{2n}(\text{PU}(n, 1), \mathbb{Z})$ horizontally: it is by definition mapped to $C_n \in H_B^{2n}(\text{U}(n), \mathbb{Z})$, whose restriction to $\text{U}(n-1)$ vanishes. Since from considerations in Proposition III.25 all bottom horizontal maps are isomorphisms, this proves the proposition. □

On the other hand, the restriction of $C_1^b \in H_b^2(\mathrm{PU}(n, 1), \mathbb{Z})$ to P is non-zero. We denote this class by $C_{1|P}^b$, for which we require the following specific statement (directly used in the proof of [Theorem E](#) in Chapter IV):

Proposition III.29. *Let $\pi : P \rightarrow \mathrm{U}(n-1) \times \mathbb{R}_{>0}$ be the canonical projection and*

$$(\det, \mathrm{id}) : \mathrm{U}(n-1) \times \mathbb{R}_{>0} \rightarrow \mathrm{U}(1) \times \mathbb{R}_{>0}$$

be the product map. Then the induced pullback of $(\det, \mathrm{id}) \circ \pi$ in degree 2 bounded Borel cohomology

$$\pi^* \circ (\det, \mathrm{id})^* : H_{Bb}^2(\mathrm{U}(n-1) \times \mathbb{R}_{>0}, \mathbb{Z}) \longrightarrow H_{Bb}^2(P, \mathbb{Z}).$$

satisfies

$$\pi^* \circ (\det, \mathrm{id})^*(\alpha_b) = C_{1|P}^b.$$

for some bounded class $\alpha_b \in H_{Bb}^2(\mathrm{U}(1) \times \mathbb{R}_{>0}, \mathbb{Z})$.

Proof. We consider the diagram induced by the composition $\pi \circ (\det, \mathrm{id})$ in bounded and unbounded Borel cohomology:

$$\begin{array}{ccc} H_{Bb}^2(P, \mathbb{Z}) & \xrightarrow{c_Z} & H_B^2(P, \mathbb{Z}) \\ \pi^* \circ (\det, \mathrm{id})^* \uparrow & & \uparrow \pi^* \circ (\det, \mathrm{id})^* \\ H_{Bb}^2(\mathrm{U}(1) \times \mathbb{R}_{>0}, \mathbb{Z}) & \xrightarrow{c_Z} & H_B^2(\mathrm{U}(1) \times \mathbb{R}_{>0}, \mathbb{Z}). \end{array}$$

By the square's commutativity, it suffices to find a class $\alpha \in c_Z(H_B^2(\mathrm{U}(1) \times \mathbb{R}_{>0}, \mathbb{Z}))$ such that

$$\pi^* \circ (\det, \mathrm{id})^*(\alpha) = C_{1|P}^b.$$

For this, we consider the following commutative diagram

$$\begin{array}{ccccc} & & H_B^2(P, \mathbb{Z}) & & \\ & \nearrow \pi^* & & \searrow \cong & \\ H_B^2(\mathrm{U}(n-1) \times \mathbb{R}_{>0}, \mathbb{Z}) & \xrightarrow{\cong} & & \xrightarrow{\cong} & H_B^2(\mathrm{U}(n-1), \mathbb{Z}) \\ (\det, \mathrm{id})^* \uparrow & & & & \uparrow \det^* \cong \\ H_B^2(\mathrm{U}(1) \times \mathbb{R}_{>0}, \mathbb{Z}) & \xrightarrow{\cong} & & \xrightarrow{\cong} & H_B^2(\mathrm{U}(1), \mathbb{Z}), \end{array}$$

where the upper triangle comes from [Diagram \(III.8\)](#) and the square is induced by the determinant map $\det : \mathrm{U}(n-1) \rightarrow \mathrm{U}(1)$ as well as the inclusions $i : \mathrm{U}(k) \hookrightarrow \mathrm{U}(k) \times \mathbb{R}_{>0}$ for $k = 1, n-1$. All marked isomorphisms have been determined previously in this section. Commutativity then shows that both π^* and $(\mathrm{id}, \det)^*$ are isomorphisms, so we take α to be the preimage of $C_{1|P}^b$ under the left vertical maps. But then, α corresponds to the first Chern class $C_{1|\mathrm{U}(1)}$ under the horizontal isomorphism and hence (by [Example II.4](#)) to

$$C_{1|\mathrm{U}(1)}^b \in H_{Bb}^2(\mathrm{U}(1), \mathbb{Z}).$$

Since $U(1)$ is a deformation retract of $U(1) \times \mathbb{R}_{>0}$ the following map in bounded Borel cohomology

$$i^* : H_{Bb}^2(U(1) \times \mathbb{R}_{>0}, \mathbb{Z}) \longrightarrow H_{Bb}^2(U(1), \mathbb{Z}),$$

is surjective. Finally, commutativity of i^* with comparison maps implies that there exists a bounded class $\alpha_b \in H_{Bb}^2(U(1) \times \mathbb{R}_{>0}, \mathbb{Z})$ such that $c_{\mathbb{Z}}(\alpha_b) = \alpha$.

□

Chapter IV

Integrality of Invariants for Representations

With the previous chapters, we have laid the groundwork to study representations of complex hyperbolic lattices. Our focus is on *non-uniform* torsion-free lattices in particular, to which we will associate two main invariants, namely the Euler number $E(\cdot)$ and the degree $\text{Deg}(\cdot)$. The primary goal of this chapter is to prove that both of these real-valued invariants are in fact integral.

For compact manifolds, these invariants are characteristic numbers, which classically take integer values. However, for non-compact manifolds with cusps, the characteristic numbers vanish and thus fail to provide meaningful information. To define useful analogues in this setting, one instead turns to a construction based on bounded cohomology, relying on its vanishing on amenable cusps. This approach yields an evaluation on a relative fundamental class of a compact core whose boundary consists of amenable cusp cross-sections.

From this point, establishing integrality demands intricate considerations, in contrast to the compact case that arises from standard theory, since bounded-cohomological counterparts for characteristic classes exhibit different behavior and moreover much about bounded cohomology is still not well understood. This can be overcome by passing to other cohomology theories using the relations we established in Chapter II.

Essentially, the idea is to employ the key congruence relation from [BIW10; BBI21] relating our \mathbb{R} -valued invariants to certain \mathbb{R}/\mathbb{Z} -valued invariants associated to the manifold's cusps. This allows to reduce proving integrality to establishing vanishing results on the cusps. We then proceed with different strategies for the Euler number $E(\cdot)$ compared later to the degree $\text{Deg}(\cdot)$.

In the last section, we deduce integrality from $\text{Deg}(\cdot)$ for a related invariant common in modern literature, the Toledo invariant $\text{Tol}(\cdot)$, and also for a volume-type invariant $\text{Vol}(\cdot)$. We then consider invariants associated to mixed representations, motivated by Anti-de-Sitter geometry, and also prove integrality results for these.

IV.1 Invariants for Uniform Lattices

Let $\Gamma < \mathrm{PU}(n, 1)$ be a torsion-free uniform lattice. Then $M = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$ is a connected compact complex hyperbolic n -manifold (cf. Subsection I.4.2). We begin with a classical construction - considered folklore - for obtaining real-valued invariants of uniform lattices. The change of coefficients $\mathbb{Z} \hookrightarrow \mathbb{R}$ induces the following commutative diagram in singular cohomology

$$\begin{array}{ccc} \mathrm{H}^{2n}(M, \mathbb{R}) & \xrightarrow{\langle \cdot, [M] \rangle} & \mathbb{R} \\ \uparrow & & \uparrow \\ \mathrm{H}^{2n}(M, \mathbb{Z}) & \xrightarrow{\langle \cdot, [M] \rangle} & \mathbb{Z}, \end{array} \quad (\text{IV.1})$$

where the horizontal maps are evaluations on the fundamental class via Poincaré duality (see Theorem II.7). This is just to say that any class in $\mathrm{H}^{2n}(M, \mathbb{R})$ arising from integral cohomology evaluates to an integer. In particular, this holds for Chern classes and cup products thereof; their evaluations on fundamental classes are called *Chern numbers*.

For example, the top Chern class of the holomorphic tangent bundle over M , namely $c_n(TM) \in \mathrm{H}^{2n}(M, \mathbb{R})$ evaluates to the integer-valued Euler characteristic (see Theorem III.9)

$$\chi(M) = \langle c_n(TM), [M] \rangle \in \mathbb{Z}. \quad (\text{IV.2})$$

Crucially, in case $\Gamma < G$ is a torsion-free *non-uniform* torsion-free lattice, then M is a *non-compact* manifold so that $\mathrm{H}^{2n}(M, A) = 0$ for $A = \mathbb{Z}, \mathbb{R}$ and the Chern numbers vanish. One way around this issue is to use *bounded group* cohomology.

To motivate, we go back to $\Gamma < \mathrm{PU}(n, 1)$ being a uniform torsion-free lattice and introduce the following group-cohomological invariant that captures Chern numbers in particular: We take an arbitrary class $\alpha_{\mathbb{R}} \in \mathrm{H}^{2n}(\Gamma, \mathbb{R})$ under the isomorphism $\Phi : \mathrm{H}^{2n}(\Gamma, \mathbb{R}) \rightarrow \mathrm{H}^{2n}(M, \mathbb{R})$ (cf. Theorem II.9) and evaluate via Poincaré duality to obtain

$$I(\alpha_{\mathbb{R}}) := \langle \Phi(\alpha_{\mathbb{R}}), [M] \rangle. \quad (\text{IV.3})$$

We note that if $\Phi(\alpha_{\mathbb{R}})$ is in particular a Chern class, then since M is compact, one has that $I(\alpha_{\mathbb{R}}) \in \mathbb{Z}$. For example, in case of (IV.2) this gives us

$$\chi(M) = I(i^*(\mathbf{C}_n)), \quad (\text{IV.4})$$

where $\mathbf{C}_n \in \mathrm{H}_c^2(\mathrm{PU}(n, 1), \mathbb{R})$ is the real top Chern class (see Definition III.19) and the map $i^* : \mathrm{H}_c^*(\mathrm{PU}(n, 1), \mathbb{R}) \rightarrow \mathrm{H}^*(\Gamma, \mathbb{R})$ is induced by the canonical inclusion $i : \Gamma \hookrightarrow \mathrm{PU}(n, 1)$.

By analogy, we extend this example to arbitrary representations of complex hyperbolic lattices, namely $\rho : \Gamma \rightarrow G$ into its ambient Lie group $\mathrm{PU}(n, 1)$ or more generally a simple non-compact Hermitian Lie group with finite center as target. In particular, the invariant $I(\cdot)$ is a template, that we use to introduce our two main \mathbb{R} -valued invariants individually and deduce their integrality.

IV.1.1 Euler Number of Representations.

We take the pullback in top-degree continuous cohomology of the representation $\rho : \Gamma \rightarrow \mathrm{PU}(n, 1)$, which is given by

$$\rho^* : H_c^{2n}(\mathrm{PU}(n, 1), \mathbb{R}) \longrightarrow H^{2n}(\Gamma, \mathbb{R}),$$

to introduce the following invariant:

Definition IV.1. The *Euler number* $E(\cdot)$ associated to an arbitrary representation $\rho : \Gamma \rightarrow \mathrm{PU}(n, 1)$ of a uniform torsion-free lattice $\Gamma < \mathrm{PU}(n, 1)$, is given by

$$E(\rho) := I(\rho^*(\mathbf{C}_n)) = \langle \Phi(\rho^*(\mathbf{C}_n)), [M] \rangle,$$

where $\mathbf{C}_n \in H_c^{2n}(\mathrm{PU}(n, 1), \mathbb{R})$ is the real top Chern class (see Definition III.19).

In particular, for the canonical inclusion $i : \Gamma \hookrightarrow \mathrm{PU}(n, 1)$, one has that $E(i) = \chi(M)$ due to (IV.2) and (IV.4). In fact, the classical *Milnor-Wood type inequality* due to [BCG07] tells us that $|E(\rho)| \leq \chi(M)$. Clearly, we have that $E(\rho) \in \mathbb{Z}$, since $\Phi^*(\rho^*(\mathbf{C}_n))$ corresponds to the pullback $B\rho^*(c_n^{\mathrm{univ}})$ of the universal top Chern class, which by naturality in Definition III.7 is a Chern class in its own right.

IV.1.2 Degree of Representations.

We assume now more generally that G is simple non-compact Hermitian Lie group with finite center. Again, let $\Gamma < \mathrm{PU}(n, 1)$ be a uniform torsion-free lattice, denote the canonical inclusion by $i : \Gamma \hookrightarrow \mathrm{PU}(n, 1)$, and let $\rho : \Gamma \rightarrow G$ be an arbitrary representation. We consider the degree-2 pullbacks in continuous cohomology of ρ and i , namely

$$\rho^* : H_c^2(G, \mathbb{R}) \longrightarrow H^2(\Gamma, \mathbb{R}) \quad \text{and} \quad i^* : H_c^2(\mathrm{PU}(n, 1), \mathbb{R}) \longrightarrow H^2(\Gamma, \mathbb{R}),$$

and define the following:

Definition IV.2. The *degree* $\mathrm{Deg}(\cdot)$ of an arbitrary representation $\rho : \Gamma \rightarrow G$ of a uniform torsion-free lattice $\Gamma < \mathrm{PU}(n, 1)$ into a simple non-compact Hermitian Lie group with finite center, is defined by

$$\mathrm{Deg}(\rho) = I(\alpha_{\mathbb{R}}) \quad \text{for} \quad \alpha_{\mathbb{R}} = \rho^*(\overline{\mathbf{C}}_1) \cup i^*(\mathbf{C}_1)^{n-1} \in H^{2n}(\Gamma, \mathbb{R}),$$

where $\overline{\mathbf{C}}_1 \in H_c^2(G, \mathbb{R})$ and $\mathbf{C}_1 \in H_c^2(\mathrm{PU}(n, 1), \mathbb{R})$ are the first real Chern classes (cf. Definition III.16).

Similarly as for the Euler number, one observes that $\Phi(\rho^*(\overline{\mathbf{C}}_1) \cup i^*(\mathbf{C}_1)^{n-1})$ is a Chern class, so that $\mathrm{Deg}(\rho) \in \mathbb{Z}$.

IV.2 Invariants for Non-Uniform Lattices

We assume from now on that $\Gamma < \mathrm{PU}(n, 1)$ is a *non-uniform* torsion-free lattice, so that $M = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$ is a non-compact complex hyperbolic n -manifold with cusps (cf. Subsection I.4.2).

As motivated previously, we modify and generalize the cohomological invariant $I(\cdot)$ to a bounded-cohomological one, to adapt to this new setting.

IV.2.1 Template for Invariants

We first give the procedure for assigning a real number to a bounded class $\alpha_{\mathbb{R}}^b \in H_b^{2n}(\Gamma, \mathbb{R})$: Upon decomposing M into a homotopy-equivalent compact core N with smooth boundary ∂N and cusp regions $M \setminus N$ (cf. thin-thick decomposition in Theorem I.40), we consider the following sequence of isomorphisms in top-degree cohomology:

$$H_b^{2n}(\Gamma, \mathbb{R}) \cong H_b^{2n}(M, \mathbb{R}) \cong H_b^{2n}(N, \mathbb{R}), \quad (\text{IV.5})$$

where the first isomorphism is due to Theorem II.9. Then, we recall from Example II.10, that inclusion of pairs $(N, \emptyset) \hookrightarrow (N, \partial N)$ induces the following long exact sequence in relative bounded group cohomology:

$$\dots \longrightarrow H_b^{2n-1}(\partial N, \mathbb{R}) \longrightarrow H_b^{2n}(N, \partial N, \mathbb{R}) \xrightarrow{j} H_b^{2n}(N, \mathbb{R}) \longrightarrow H_b^{2n}(\partial N, \mathbb{R}) \longrightarrow \dots, \quad (\text{IV.6})$$

and since each connected component of ∂N is an infranilmanifold, which is modelled by an amenable group (cf. Subsection I.4.2), we have that $H^{*>0}(\partial N, \mathbb{R}) = 0$, and consequently the map

$$j : H_b^{2n}(N, \partial N, \mathbb{R}) \longrightarrow H_b^{2n}(N, \mathbb{R})$$

is an isomorphism. We denote the composition of the isomorphisms (IV.5) with the inverse j^{-1} by Φ_b for short. By further post-composing with the comparison map

$$c_{(N, \partial N)} : H_b^{2n}(N, \partial N, \mathbb{R}) \longrightarrow H^{2n}(N, \partial N, \mathbb{R}),$$

we may subsequently use relative Poincaré duality, to define

$$I(\alpha_{\mathbb{R}}^b) := \langle c_{(N, \partial N)} \circ \Phi_b(\alpha_{\mathbb{R}}^b), [N, \partial N] \rangle \in \mathbb{R}. \quad (\text{IV.7})$$

We present the generalizations of the Euler number $E(\cdot)$ and degree $\text{Deg}(\cdot)$ for the non-uniform lattice-case, relying on our previously given definitions of bounded Chern classes in Subsection III.3.3.

IV.2.2 Euler number of Representations

For $\Gamma < \text{PU}(n, 1)$ a non-uniform torsion-free lattice and $\rho : \Gamma \rightarrow \text{PU}(n, 1)$ an arbitrary representation, consider the pullback map in top-degree *continuous bounded* cohomology

$$\rho^* : H_{cb}^{2n}(\text{PU}(n, 1), \mathbb{R}) \longrightarrow H_b^{2n}(\Gamma, \mathbb{R}).$$

The following definition is in analogy to the invariant considered in [BBI13; BBI21] for real-hyperbolic lattices represented into $\text{SO}^+(n, 1)$ (for a closer version, as the *volume*, see Subsection IV.6.1):

Definition IV.3. The *Euler number* of $\rho : \Gamma \rightarrow \mathrm{PU}(n, 1)$, with $\Gamma < \mathrm{PU}(n, 1)$ a non-uniform lattice, is given by

$$E(\rho) = I(\alpha_b^{\mathbb{R}}) \quad \text{for} \quad \alpha_b^{\mathbb{R}} = \rho^*(C_n^b),$$

where $C_n^b \in H_{cb}^{2n}(\mathrm{PU}(n, 1), \mathbb{R})$ is the real bounded top Chern class (cf. Subsection III.3.3).

IV.2.3 Degree of Representations

Again, let $\Gamma = \pi_1(M) < \mathrm{PU}(n, 1)$ be a non-uniform torsion-free lattice and $i : \Gamma \hookrightarrow \mathrm{PU}(n, 1)$ the canonical inclusion. Further, let $\rho : \Gamma \rightarrow G$ be a representation into a simple non-compact Hermitian Lie group with finite center G . In analogy to the uniform case, we consider the degree 2 pullbacks in continuous bounded cohomology of ρ and i , namely

$$\rho^* : H_{cb}^2(G, \mathbb{R}) \longrightarrow H_b^2(\Gamma, \mathbb{R}) \quad \text{and} \quad i^* : H_{cb}^2(\mathrm{PU}(n, 1), \mathbb{R}) \longrightarrow H_b^2(\Gamma, \mathbb{R}),$$

and formalize the following:

Definition IV.4. The *degree* of a representation $\rho : \Gamma \rightarrow G$, from $\Gamma < \mathrm{PU}(n, 1)$ a non-uniform torsion-free lattice into a simple non-compact Lie group G with finite center, is defined as

$$\mathrm{Deg}(\rho) = I(\alpha_{\mathbb{R}}^b) \quad \text{for} \quad \alpha_{\mathbb{R}}^b = \rho^*(\overline{C}_1^b) \cup i^*(C_1^b)^{n-1},$$

where $\overline{C}_1^b \in H_{cb}^2(G, \mathbb{R})$ and $C_1^b \in H_{cb}^2(\mathrm{PU}(n, 1), \mathbb{R})$ are the real bounded first Chern classes (cf. Definition III.24).

Remark. (i) All invariants can be defined in terms of real-valued Borel (bounded) cohomology instead of continuous cohomology (due to Austin-Moore's isomorphism in Theorem II.13). In fact, this is the appropriate choice in order to pass to the framework of integral Borel (bounded) cohomology later in Subsection IV.3.3.

(ii) When taking uniform-lattices, the bounded-cohomological invariants $E(\cdot)$ and $\mathrm{Deg}(\cdot)$ recover the Definitions IV.1 and IV.2 respectively. Moreover, clearly $E(\cdot)$ and $\mathrm{Deg}(\cdot)$ coincide for the case $n = 1$, that is, whenever $\rho : \Gamma \rightarrow \mathrm{PU}(1, 1)$ and $\Gamma < \mathrm{PU}(1, 1)$ a torsion-free lattice. Moreover, both invariants satisfy Milnor-Woods type inequalities (cf. Section V.2)

(iii) We notice that if Γ is *not* torsion-free, then according to [BBI13, p.50], one may set

$$E(\rho) := \frac{E(\rho|_{\Lambda})}{[\Gamma : \Lambda]} \quad \text{and} \quad \mathrm{Deg}(\rho) := \frac{\mathrm{Deg}(\rho|_{\Lambda})}{[\Gamma : \Lambda]},$$

where $\Lambda < \Gamma$ is a torsion-free subgroup of finite index (cf. [Sel60] in Subsection I.1.2).

IV.3 \mathbb{R}/\mathbb{Z} -valued Invariants of Cusps

As previously observed, integrality of Chern numbers follows naturally by passing to integral cohomology. For our bounded-cohomological invariants, the idea is more intricate:

IV.3.1 Definition

Let $\alpha_{\mathbb{Z}}^b \in H_b^{2n}(\Gamma, \mathbb{Z})$ and $\alpha_{\mathbb{R}}^b \in H_b^{2n}(\Gamma, \mathbb{R})$ be bounded cohomology classes such that $\alpha_{\mathbb{R}}^b$ lies in the image of $\alpha_{\mathbb{Z}}^b$ under the coefficient inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$. The strategy is then to associate a number

$$J(\alpha_{\mathbb{Z}}^b) \in \mathbb{R}/\mathbb{Z}$$

and establish a congruence relation to $I(\alpha_{\mathbb{R}}^b)$. The crux is that if $J(\alpha_{\mathbb{Z}}^b)$ vanishes, this implies integrality of $I(\alpha_{\mathbb{R}}^b)$. The precise construction is as follows.

To define $J(\cdot)$, we first consider the sequence of maps in top-degree integral bounded cohomology

$$H_b^{2n}(\Gamma, \mathbb{Z}) \cong H_b^{2n}(M, \mathbb{Z}) \cong H_b^{2n}(N, \mathbb{Z}) \rightarrow H_b^{2n}(\partial N, \mathbb{Z}), \quad (\text{IV.8})$$

where the right-most map is the restriction map to the boundary ∂N induced by the inclusion $\partial N \hookrightarrow N$. Then, the short exact sequence of coefficients

$$\mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z}$$

induces the long exact sequence in singular bounded cohomology of ∂N , namely

$$\dots \rightarrow H_b^{2n-1}(\partial N, \mathbb{R}) \rightarrow H_b^{2n-1}(\partial N, \mathbb{R}/\mathbb{Z}) \xrightarrow{\delta^b} H_b^{2n}(\partial N, \mathbb{Z}) \rightarrow H_b^{2n}(\partial N, \mathbb{R}) \rightarrow \dots \quad (\text{IV.9})$$

Again, since by amenability of cusps, we have that $H_b^{* >}(\partial N, \mathbb{R}) = 0$, this implies that the connecting homomorphism

$$\delta^b : H_b^{2n-1}(\partial N, \mathbb{R}/\mathbb{Z}) \rightarrow H_b^{2n}(\partial N, \mathbb{Z})$$

is in fact an isomorphism. We denote the composition of the maps in (IV.8) with the inverse $(\delta^b)^{-1}$ and the isomorphic comparison map $c_{\mathbb{R}/\mathbb{Z}}$ by Ψ for short. As cusp cross-sections are oriented manifolds of real codimension one, we may evaluate through Poincaré duality to obtain the number

$$J(\alpha_{\mathbb{Z}}^b) := \langle \Psi(\alpha_{\mathbb{Z}}^b), [\partial N] \rangle \in \mathbb{R}/\mathbb{Z}.$$

Remark. For $n = 1$, due to [BBI21, §7], the number $J(\rho^*(C_1))$ is in fact a generalization of the *classical rotation number* of orientation-preserving homomorphisms of the circle [Ghy87].

IV.3.2 Congruence Relation

Notably, the numbers $J(\alpha_{\mathbb{Z}}^b)$ and $I(\alpha_{\mathbb{R}}^b)$ are related through the following congruence relation:

Theorem IV.5 ([BIW10, §8.1], [BBI21, §4.2]). *Let $\alpha_{\mathbb{Z}}^b \in H_b^{2n}(\Gamma, \mathbb{Z})$ and $\alpha_{\mathbb{R}}^b \in H_b^{2n}(\Gamma, \mathbb{R})$ be bounded cohomology classes such that $\alpha_{\mathbb{R}}^b$ lies in the image of $\alpha_{\mathbb{Z}}^b$ under the coefficient inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$. Then it holds that*

$$I(\alpha_{\mathbb{R}}^b) \equiv -J(\alpha_{\mathbb{Z}}^b) \pmod{\mathbb{Z}}.$$

Clearly, the number $I(\alpha_{\mathbb{R}}^b)$ is an integer, if $J(\alpha_{\mathbb{Z}}^b)$ vanishes.

Proof. For the sake of completeness, we restate here the proof, for which we follow closely [BBI21, §4.2]. Let z be a bounded singular \mathbb{Z} -valued cochain on N representing $\alpha_{\mathbb{Z}}^b$, i.e. $[z]_{\mathbb{Z}} = \alpha_{\mathbb{Z}}^b$. The goal is to construct using z an explicit representative of

$$j^{-1}(\alpha_{\mathbb{R}}^b) \in H_b^{2n}(N, \partial N, \mathbb{R}),$$

where $j : H_b^{2n}(N, \partial N, \mathbb{R}) \rightarrow H_b^{2n}(N, \mathbb{Z})$ is the isomorphism in (IV.6), in order to evaluate $\langle j^{-1}(\alpha_{\mathbb{R}}^b), [N, \partial N] \rangle \bmod \mathbb{Z}$. The calculation will yield the statement of the theorem.

First, using the restriction map $H_b^{2n}(N, \mathbb{Z}) \rightarrow H_b^{2n}(\partial N, \mathbb{Z})$, we have that

$$\alpha_{\mathbb{Z}|\partial N}^b = [z|_{\partial N}]_{\mathbb{Z}}.$$

We recall from (IV.9) that the long exact sequence in bounded cohomology induced by the short exact sequence $\mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z}$ gives

$$\dots \longrightarrow \underbrace{H_b^{2n-1}(\partial N, \mathbb{R})}_{=0} \longrightarrow H_b^{2n-1}(\partial N, \mathbb{R}/\mathbb{Z}) \xrightarrow{\delta^b} H_b^{2n}(\partial N, \mathbb{Z}) \longrightarrow \underbrace{H_b^{2n}(\partial N, \mathbb{R})}_{=0} \longrightarrow \dots$$

Hence, viewing $z|_{\partial N}$ as an \mathbb{R} -valued bounded cochain, we have $[z|_{\partial N}]_{\mathbb{R}} = [0]$, so it must be a coboundary in $S_b^{2n-1}(\partial N, \mathbb{R})$. Therefore, there exists a bounded cochain $b \in S_b^{2n-1}(\partial N, \mathbb{R})$ such that

$$db = z|_{\partial N}. \quad (\text{IV.10})$$

Since $z|_{\partial N}$ is \mathbb{Z} -valued, we have $[db]_{\mathbb{Z}} \in H_b^{2n}(\partial N, \mathbb{Z})$. Thus $(b \bmod \mathbb{Z})$ is an \mathbb{R}/\mathbb{Z} -valued cocycle, and we can write

$$[b \bmod \mathbb{Z}] = [b]_{\mathbb{R}/\mathbb{Z}} \in H_b^{2n-1}(\partial N, \mathbb{R}/\mathbb{Z}).$$

Applying the bounded Bockstein connecting homomorphism δ^b , we obtain

$$\alpha_{\mathbb{Z}|\partial N}^b = [z|_{\partial N}]_{\mathbb{Z}} = \delta^b([b]_{\mathbb{R}/\mathbb{Z}}) \in H_b^{2n}(\partial N, \mathbb{Z}) \quad (\text{IV.11})$$

Next, extend b to an \mathbb{R} -valued bounded cochain \bar{b} on N by

$$\bar{b}(\sigma) = \begin{cases} b(\sigma) & \text{if } \sigma \subset \partial N, \\ 0 & \text{otherwise.} \end{cases}$$

Viewing z also as an \mathbb{R} -valued cochain, we have $[z - d\bar{b}]_{\mathbb{R}} = [z]_{\mathbb{R}} \in H_b^{2n}(N, \mathbb{R})$. Moreover, by (IV.10),

$$(z - d\bar{b})|_{\partial N} = z|_{\partial N} - db = 0$$

and therefore $z - d\bar{b}$ represents a class in the relative bounded cohomology group $H_b^{2n}(N, \partial N, \mathbb{R})$, i.e.

$$[z - d\bar{b}]_{\mathbb{R}} = j^{-1}(\alpha_{\mathbb{R}}^b) \in H_b^{2n}(N, \partial N, \mathbb{R}). \quad (\text{IV.12})$$

Let now t be a singular chain representing the relative fundamental class $[N, \partial N]$, so that ∂t represents $[\partial N]$.

Then, using (IV.12) and the integrality of z , we have

$$\langle j^{-1}(\alpha_{\mathbb{R}}^b), [N, \partial N] \rangle \bmod \mathbb{Z} \equiv \langle z - d\bar{b}, t \rangle \equiv \underbrace{\langle z, t \rangle}_{\in \mathbb{Z}} - \langle d\bar{b}, t \rangle \equiv -\langle d\bar{b}, t \rangle.$$

Finally we calculate

$$\langle d\bar{b}, t \rangle = \langle \bar{b}, \partial t \rangle = \langle b \bmod \mathbb{Z}, [\partial N] \rangle = \langle (\delta^b)^{-1}(\alpha_{\mathbb{Z}}^b) |_{\partial N}, [\partial N] \rangle,$$

where the last equality follows from (IV.11). This yields

$$\langle j^{-1}(\alpha_{\mathbb{R}}^b), [N, \partial N] \rangle \bmod \mathbb{Z} \equiv -\langle (\delta^b)^{-1}(\alpha_{\mathbb{Z}}^b) |_{\partial N}, [\partial N] \rangle,$$

which establishes the desired congruence. □

IV.3.3 Vanishing Results and Methods towards Integrality

The strategy is to prove vanishing of the number $J(\alpha_{\mathbb{Z}}^b)$ solely for certain types of cusps, which results in additional constants depending on the cusp structures. This is formalized in the following lemma (cf. [BBI21, p. 232]):

Lemma IV.6. *Let M be a complete complex hyperbolic n -manifold with ℓ cusps $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ so that each admits a finite m_j -cover $\tilde{\mathcal{C}}_j$ for $1 \leq j \leq \ell$. Further, let $\Psi|_{\mathcal{C}_j}$ be the sequence of maps*

$$H_b^{2n}(\Gamma, \mathbb{Z}) \cong H_b^{2n}(M, \mathbb{Z}) \cong H_b^{2n}(N, \mathbb{Z}) \longrightarrow H_b^{2n}(\mathcal{C}_j, \mathbb{Z}),$$

where the latter map is induced by the inclusion $\mathcal{C}_j \hookrightarrow N$ (cf. sequence (IV.8)), and suppose that

$$\langle \Psi|_{\tilde{\mathcal{C}}_j}(\alpha_{\mathbb{Z}}^b), [\tilde{\mathcal{C}}_j] \rangle = 0.$$

Then this implies that

$$\text{lcm}(m_1, \dots, m_\ell) \cdot I(\alpha_{\mathbb{R}}^b) \in \mathbb{Z}.$$

Proof. For every cusp \mathcal{C}_j where $1 \leq j \leq \ell$, let $\pi^{(j)} : \tilde{\mathcal{C}}_j \rightarrow \mathcal{C}_j$ be the finite m_j -cover projection. Then the following holds:

$$\begin{aligned} m_j \cdot \langle (\Psi|_{\mathcal{C}_j})^*(\alpha_{\mathbb{Z}}^b), [\mathcal{C}_j] \rangle &= \langle (\Psi|_{\mathcal{C}_j})^*(\alpha_{\mathbb{Z}}^b), m_j \cdot [\mathcal{C}_j] \rangle \\ &= \langle (\Psi|_{\mathcal{C}_j})^*(\alpha_{\mathbb{Z}}^b), \pi_*^{(j)}[\tilde{\mathcal{C}}_j] \rangle \\ &= \langle (\Psi|_{\tilde{\mathcal{C}}_j})^*(\alpha_{\mathbb{Z}}^b), [\tilde{\mathcal{C}}_j] \rangle = 0, \end{aligned}$$

where we applied the vanishing hypothesis in the lemma. This implies that

$$\text{lcm}(m_1, \dots, m_\ell) \cdot J(\alpha_{\mathbb{Z}}^b) = \sum_{j=1}^{\ell} \text{lcm}(m_1, \dots, m_\ell) \cdot \langle (\Psi|_{\mathcal{C}_j})^*(\alpha_{\mathbb{Z}}^b), [\mathcal{C}_j] \rangle = 0.$$

Finally, by the congruence relation in Theorem IV.5 one obtains that

$$\text{lcm}(m_1, \dots, m_\ell) \cdot I(\alpha_{\mathbb{R}}^b) \in \mathbb{Z}$$

which yields the result. \square

Next, we detail two simple but essential methods towards vanishing results of $J(\alpha_{\mathbb{Z}}^b)$.

- (i) **Restriction to cusps.** When starting with a representation $\rho : \pi_1(M) \rightarrow G$ into simple non-compact Hermitian Lie group with finite center, we may in fact directly consider it as the representation $\rho : \pi_1(N) \rightarrow G$, with N a compact core of M . Commutativity of the left diagram implies the commutativity of the induced diagram in integral Borel bounded cohomology on the right:

$$\begin{array}{ccc} \pi_1(N) & \xrightarrow{\rho} & G \\ & \searrow & \uparrow \rho|_{\pi_1(\partial N)} \\ & & \pi_1(\partial N) \end{array} \qquad \begin{array}{ccc} H_{Bb}^*(\pi_1(N), \mathbb{Z}) & \xleftarrow{\rho^*} & H_{Bb}^*(G, \mathbb{Z}) \\ & \searrow & \downarrow (\rho|_{\pi_1(\partial N)})^* \\ & & H_b^*(\pi_1(\partial N), \mathbb{Z}) \end{array},$$

using that $\partial N \hookrightarrow N$ is the natural embedding of the boundary. That is, for $\alpha_{\mathbb{Z}}^b \in H_{Bb}^*(G, \mathbb{Z})$ we have that

$$\rho^*(\alpha_{\mathbb{Z}}^b)|_{\pi_1(\partial N)} = (\rho|_{\pi_1(\partial N)})^*(\alpha_{\mathbb{Z}}^b).$$

Upshot : We may restrict directly to the representation $\rho : \pi_1(\partial N) \rightarrow G$ for our considerations, and for simplicity still use the analogous short-cut notation $J(\cdot)$.

- (ii) **Subgroups containing $\text{Im}(\rho)$.** In view of the upshot above, let us consider directly a representation $\rho : \pi_1(\partial N) \rightarrow \text{PU}(n, 1)$. Another useful strategy is to identify specific subgroups $A \leq \text{PU}(n, 1)$ that contain the image of the representation up to conjugation. In this case, since the invariants $E(\rho)$ and $\text{Deg}(\rho)$ are invariant under conjugation, we can, without loss of generality, assume that the image of ρ is actually contained in A .

This allows us to reduce the situation to consider the canonical inclusion $i : A \hookrightarrow \text{PU}(n, 1)$, together with the following commutative diagram on the left, which in turn induces a corresponding commutative diagram in integral Borel bounded cohomology on the right:

$$\begin{array}{ccc} \pi_1(\partial N) & \xrightarrow{\rho} & \text{PU}(n, 1) \\ & \searrow \rho & \uparrow i \\ & & A \end{array} \qquad \begin{array}{ccc} H_b^*(\pi_1(\partial N), \mathbb{Z}) & \xleftarrow{\rho^*} & H_{Bb}^*(\text{PU}(n, 1), \mathbb{Z}) \\ & \searrow \rho^* & \downarrow i^* \\ & & H_{Bb}^*(A, \mathbb{Z}). \end{array}$$

That is, for $\alpha_{\mathbb{Z}}^b \in H_{Bb}^*(\text{PU}(n, 1), \mathbb{Z})$ and its image under the restriction map i^* denoted by $\alpha_{\mathbb{Z}|A}^b$, we have that

$$\rho^*(\alpha_{\mathbb{Z}}^b) = \rho^*(\alpha_{\mathbb{Z}|A}^b).$$

Hence, for the invariants $E(\rho)$ and $\text{Deg}(\rho)$, one may restrict attention to bounded Chern

classes in $H_{Bb}^{2n}(\mathrm{PU}(n, 1), \mathbb{Z})$ and their restrictions to subgroups of $\mathrm{PU}(n, 1)$ (cf. Section III.3.4).

IV.4 Euler Number and Integrality

The goal of this section is to prove this first major result of the thesis.

Theorem A. *Let $n \geq 2$ and $\Gamma < \mathrm{PU}(n, 1)$ a non-uniform torsion-free lattice. Then for any representation $\rho : \Gamma \rightarrow \mathrm{PU}(n, 1)$, we have in case $n > 2$ that*

$$\mathrm{lcm}(m_1, \dots, m_k) \cdot E(\rho) \in \mathbb{Z},$$

and in case $n = 2$ that

$$\mathrm{lcm}(m_1, \dots, m_k) \cdot E(\rho) \in \frac{1}{2}\mathbb{Z},$$

where $m_j := [E_j : \tilde{\Delta}^{(j)}] < \infty$ for every cusp C_j of M modelled by E_j and $\tilde{\Delta}^{(j)}$ is a certain finite index subgroup of $E_j \cap \mathcal{H}_{\mathbb{R}}^{2n-1}$ for $1 \leq j \leq \ell$.

Remark. To obtain the statement in **Main Theorem (A)** of the introduction we observe the we may set

$$\mathrm{Cst}(\Gamma) := \mathrm{lcm}(m_1, \dots, m_\ell).$$

We first outline the proof and then go into more details in the sections that follow.

Proof outline of Theorem A. We consider the cusp C_1 modelled by an almost-Bieberbach group E_1 that contains the finite-index subgroup $\Delta_1 = E_1 \cap \mathcal{H}_{\mathbb{R}}^{2n-1}$, up to conjugation. In light of method (i) in the previous section, we may consider the representation $\rho : \Delta_1 \rightarrow \mathrm{PU}(n, 1)$. We further use the bounded integral top Chern class $C_n^b \in H_{Bb}^{2n}(\mathrm{PU}(n, 1), \mathbb{Z})$ in what follows. First, by Proposition IV.7, the image of ρ is either contained in $U(n)$ or P up to conjugation. For the case of P , we conclude from Proposition III.28 that $J(C_n^b)$ vanishes.

For the case of $U(n)$, Lemma IV.8 yields that the image of a finite-index subgroup of Δ_1 under ρ is contained in $U(1)^n$ if $n > 2$, up to conjugation. Exploiting the fact that $U(1)^n$ is an abelian group, allows to conclude from Proposition IV.9 that $J(C_n^b)$ vanishes up to a constant. Then for $n = 2$, by Corollary IV.10, a similar statement holds for $2 \cdot J(C_2^b)$.

After applying the same argument to all the other cusps, we can then conclude using the congruence relation in Theorem IV.5 and Lemma IV.6 the statement of the theorem. \square

IV.4.1 Representations of Lattices Modelling Nilmanifolds

By the generalized Bieberbach theorem in Theorem I.68 in Chapter I, any cusp of the non-compact complex hyperbolic manifold M of finite volume is modelled by an almost-Bieberbach group E that contains the uniform torsion-free sublattice $\Delta = E \cap \mathcal{H}_{\mathbb{R}}^{2n-1}$ of finite index, up to conjugation. We recall its explicit presentation from Theorem I.69, as we shall use the notation

for our proofs:

$$\Delta = \left\langle \begin{array}{l} x_1, \dots, x_{n-1}, \\ y_1, \dots, y_{n-1}, z \end{array} \left| \begin{array}{l} [x_i, y_i] = z^{k_i} \quad \text{for } i = 1, \dots, n-1, \\ [x_i, x_j] = [y_i, y_j] = [x_i, y_j] = e \quad \text{for } i \neq j, \\ [z, x_i] = [z, y_i] = e \quad \text{for all } i \end{array} \right. \right\rangle \quad (\text{IV.13})$$

such that $k_i = dr_i$ where $r_1 = 1$, and $r_i \mid r_{i+1}$ for all $i = 1, \dots, n-2$, and $d \in \mathbb{N}^*$ is the index of $[\Delta, \Delta]$ in the center $Z(\Delta)$.

We first proceed similarly to [BBI21, Lemma 5.2]: Following strategy (i) outlined in the previous section, we may reduce to considering the restricted representation $\rho: \Delta \rightarrow \text{PU}(n, 1)$.

Proposition IV.7. *Let $\Delta < \mathcal{H}_{\mathbb{R}}^{2n-1}$ be a uniform torsion-free lattice and let $\rho: \Delta \rightarrow \text{PU}(n, 1)$ be a representation. Then the image $\rho(\Delta)$ lies either in the maximal compact subgroup $\text{U}(n)$ or in the minimal parabolic subgroup $\text{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$ up to conjugation.*

Proof. Let $\Delta < \mathcal{H}_{\mathbb{R}}^{2n-1}$ be a uniform torsion-free lattice. Then Δ is a discrete nilpotent subgroup of $\text{PU}(n, 1)$ and hence its image $\rho(\Delta) < \text{PU}(n, 1)$ is also nilpotent. Then [Bow95, Proposition 3.11] implies that any nilpotent subgroup of $\text{Isom}(\mathbb{H}_{\mathbb{C}}^n)$ is an elementary¹ group (see Definition I.38), that is the limit set $\Lambda_{\rho(\Delta)} \subset \partial\mathbb{H}_{\mathbb{C}}^n$ has cardinality at most 2.

If $\Lambda_{\rho(\Delta)} = \emptyset$, then $\rho(\Delta)$ fixes a point in $\mathbb{H}_{\mathbb{C}}^n$ and is conjugate into $\text{U}(n)$. If $\Lambda_{\rho(\Delta)}$ consists of a single point in $\partial\mathbb{H}_{\mathbb{C}}^n$, then $\rho(\Delta)$ is conjugate into $\text{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$.

We now assume that $\Lambda_{\rho(\Delta)} = \{\xi_-, \xi_+\}$ consists of exactly two points. Then $\rho(\Delta)$ preserves the geodesic joining ξ_- and ξ_+ , and hence acts by isometries on it. In particular, we may consider the restriction of ρ to this geodesic, so that we may restrict ρ to a representation from Δ straight into $\text{Isom}(\mathbb{R})$. Since both endpoints belong to the limit set, the image contains a non-trivial translation γ . For the central element z of Δ (cf. Presentation (IV.13)), we observe that $\rho(z)$ commutes with γ , and therefore $\rho(z)$ must act as a (possibly trivial) translation of \mathbb{R} . If $\rho(z)$ is trivial, then the image of Δ is abelian and since it contains a non-trivial translation, it consists entirely of translations. If $\rho(z)$ is a non-trivial translation, then for any $h \in \Delta$, the commutation relation $[h, z] = e$ implies that $\rho(h)$ is also a translation. In either case, $\rho(\Delta)$ acts on the preserved geodesic by translations only, and thus fixes both ξ_- and ξ_+ . Consequently, $\rho(\Delta)$ is conjugate into $\text{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$. □

IV.4.2 Case Distinction between Possible Targets

The previous proposition allows to reduce the consideration to two cases: For the first case we may suppose that

$$\rho: \Delta \longrightarrow \text{Sim} \mathcal{H}_{\mathbb{R}}^{2n-1}.$$

Due to method (ii) from the previous section, we may compose ρ with the canonical inclusion map $\text{Sim} \mathcal{H}_{\mathbb{R}}^{2n-1} \hookrightarrow \text{PU}(n, 1)$ in order to reduce our argument to proving that via the induced

¹This may be viewed as a manifestation of the general principle that amenable groups acting on rank-one Hadamard manifolds are elementary: since $\rho(\Delta)$ is amenable, Theorem I.43 implies that then the action of $\rho(\Delta)$ on the rank 1 Hadamard manifold $\mathbb{H}_{\mathbb{C}}^n$ (cf. Definition I.27), fixes a point on $\partial\mathbb{H}_{\mathbb{C}}^n$ or preserves a totally geodesic subspace of $\mathbb{H}_{\mathbb{C}}^n$ isometric to \mathbb{R}^k for $k = 0, 1$. Equivalently, $\rho(\Delta)$ is elementary.

sequence in integral bounded Borel cohomology

$$H_{Bb}^{2n}(\mathrm{PU}(n, 1), \mathbb{Z}) \longrightarrow H_{Bb}^{2n}(\mathrm{Sim} \mathcal{H}_{\mathbb{R}}^{2n-1}, \mathbb{Z}) \xrightarrow{\rho^*} H_b^{2n}(\Delta, \mathbb{Z}),$$

the image of the integral top Chern class C_n^b is the zero class. But this follows directly from Proposition III.28 in Chapter III, where we established that already the restriction of C_n^b to $\mathrm{Sim} \mathcal{H}_{\mathbb{R}}^{2n-1}$ vanishes.

We may now treat the second case when

$$\rho: \Delta \longrightarrow \mathrm{U}(n)$$

and first show that we may restrict even further to more suitable subgroups of $\mathrm{U}(n)$. For this, whenever $n > 2$, we shall replace Δ in (IV.13) by its finite-index group given by

$$\Delta' = \left\langle \begin{array}{l} x_1^{m_1}, \dots, x_{n-1}^{m_{n-1}}, \\ y_1, \dots, y_{n-1}, z \end{array} \left| \begin{array}{l} [x_i^{m_i}, y_i] = z^K \quad \text{for } i = 1, \dots, n-1, \\ [x_i^{m_i}, x_j^{m_j}] = [y_i, y_j] = [x_i^{m_i}, y_j] = e \quad \text{for } i \neq j, \\ [z, x_i^{m_i}] = [z, y_i] = e \quad \text{for all } i \end{array} \right. \right\rangle \quad (\text{IV.14})$$

such that $K := k_{n-1} = dr_{n-1}$ and $m_i = \frac{r_{n-1}}{r_i}$ for $1 \leq i \leq n-1$, which is necessarily a positive integer since $r_{n-1} = \mathrm{lcm}(r_1, \dots, r_{n-1})$. In particular, the index is given by $[\Delta: \Delta'] = \prod_{i=1}^{n-1} \frac{r_{n-1}}{r_i}$.

Lemma IV.8. *Let $\Delta < \mathcal{H}_{\mathbb{R}}^3$ be lattice with Presentation (IV.13) and let $\rho: \Delta \rightarrow \mathrm{U}(2)$ be a unitary representation. Then the image of the generators x_1, y_1, z under ρ is given up to conjugation by*

$$\rho(x_1) = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix}, \quad \rho(y_1) = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}, \quad \text{and} \quad \rho(z) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where $u, b, c \in \mathrm{U}(1)$ and $\lambda_i \in \mathbb{C}$ with $|\lambda_i| = 1$, $\lambda_i^{k_1} = -1$ for $i = 1, 2$. For $n \geq 3$, let $\Delta' < \mathcal{H}_{\mathbb{R}}^{2n-1}$ be a lattice with Presentation IV.14. Then the image of an arbitrary unitary representation $\rho: \Delta' \rightarrow \mathrm{U}(n)$ is contained in a maximal torus.

The essential idea of the proof is to decompose unitary representations into irreducible summands, use Schur's lemma to control the action of central elements, and then apply simultaneous diagonalisation of commuting unitary matrices.

Proof. We first treat the case $n = 2$ and consider a representation $\rho: \Delta \rightarrow \mathrm{U}(2)$. Since unitary representations are completely reducible, we may decompose the representation ρ into irreducible subrepresentations. There are two possibilities.

- **ρ is reducible:** In this case ρ splits as a direct sum of one-dimensional unitary characters. After a choice of basis, one-dimensional characters are one-dimensional matrices, so that the image of ρ is contained in a maximal torus $\mathrm{U}(1)^2$.
- **ρ is irreducible of dimension 2:** We apply Schur's lemma and see that the center of Δ , generated by z (cf. Presentation IV.13), acts by scalars on the two-dimensional irreducible representation, so that $\rho(z) = \omega \cdot \mathrm{Id}_2$ for some $\omega \in \mathrm{U}(1)$. Since $[x_1, y_1] = z^{k_1}$ and

$\rho(z^{k_1}) = \omega^{k_1} \cdot \text{Id}_2$, we get that

$$\rho(x_1)\rho(y_1) = \omega^{k_1}\rho(y_1)\rho(x_1). \quad (\text{IV.15})$$

If $\omega^{k_1} = 1$, then $\rho(x_1)$ and $\rho(y_1)$ commute; since they are unitary matrices (hence normal), they are simultaneously diagonalizable, contradicting irreducibility. Hence $\omega^{k_1} \neq 1$. For this case, we may suppose that

$$\rho(x_1) = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}, \quad \rho(y_1) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{and} \quad \rho(z^{k_1}) = \begin{pmatrix} \omega^{k_1} & 0 \\ 0 & \omega^{k_1} \end{pmatrix}.$$

Then Equation (IV.15) can be expressed coordinatewise by

$$\begin{pmatrix} u_1 a & u_1 b \\ u_2 c & u_2 d \end{pmatrix} = \begin{pmatrix} \omega^{k_1} u_1 a & \omega^{k_1} u_2 b \\ \omega^{k_1} u_1 c & \omega^{k_1} u_2 d \end{pmatrix}$$

and is thus equivalent to the system of equations

$$\begin{cases} (\omega^{k_1} - 1)a = 0, \\ (\omega^{k_1} u_2 - u_1)b = 0, \\ (\omega^{k_1} u_1 - u_2)c = 0, \\ (\omega^{k_1} - 1)d = 0. \end{cases}$$

Since $\omega^{k_1} \neq 1$, then $a = d = 0$, which implies that $c, b \in \text{U}(1)$ and the two equations

$$u_1 = \omega^{k_1} u_2 \quad \text{and} \quad u_2 = \omega^{k_1} u_1$$

readily imply that $\omega^{2k_1} = 1$ and hence $\omega^{k_1} = -1$. We conclude that

$$\rho(x_1) = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix}, \quad \rho(y_1) = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}, \quad \text{and} \quad \rho(z^{k_1}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

with $u, b, c \in \text{U}(1)$, which yields the first statement in our theorem.

We now assume that $n \geq 3$ and consider an arbitrary representation $\rho : \Delta' \rightarrow \text{U}(n)$, where the presentation of Δ' is given in (IV.14). Since ρ is unitary, it decomposes as an orthogonal direct sum of irreducible subrepresentations

$$\mathbb{C}^n = V_1 \oplus \cdots \oplus V_r,$$

and by Schur's lemma the central element z acts on each V_j by a scalar $\omega_j \in \text{U}(1)$ for $1 \leq j \leq r$. It therefore suffices to show that on each irreducible summand V_j of dimension $q \geq 2$, all generators are simultaneously diagonalizable. To simplify notation, we fix such a summand and denote it by V with $\rho(z)|_V = \omega \cdot \text{Id}_q$ for some $\omega \in \text{U}(1)$. Further, inside this block, for $1 \leq i \leq n-1$, we write

$$X_i := \rho(x_i^{m_i})|_V, \quad Y_i := \rho(y_i)|_V \quad \text{and} \quad Z := \rho(z)|_V$$

Since the X_i commute and are normal, we may conjugate so that each X_i is diagonal, say $X_i = \text{diag}(u_{i1}, \dots, u_{iq})$. We deduce from Presentation (IV.14) that

$$[X_i, Y_i] = Z^K = \omega^K \text{Id}_q,$$

which reads entrywise as $u_{ia}(Y_i)_{ab} = \omega^K (Y_i)_{ab} u_{ib}$ for all $1 \leq a, b \leq q$. Hence if $a = b$ we obtain $(1 - \omega^K)(Y_i)_{aa} = 0$, so either $\omega^K = 1$ or all diagonal entries of Y_i vanish. Suppose for contradiction that $\omega^K \neq 1$. Then every Y_i has zero diagonal entries, and whenever $(Y_i)_{ab} \neq 0$ with $a \neq b$ we must have

$$u_{ia} = \omega^K u_{ib}. \tag{IV.16}$$

We now define the *support* of each Y_i as

$$S_i := \{(a, b) : a \neq b, (Y_i)_{ab} \neq 0\}.$$

Then, for $i \neq j$, the commutation relation $[X_i, Y_j] = \text{Id}_q$ implies $u_{ia}(Y_j)_{ab} = (Y_j)_{ab} u_{ib}$, so that $(Y_j)_{ab} \neq 0$ forces $u_{ia} = u_{ib}$, which combined with Equality (IV.16) shows that no position can simultaneously be non-zero in both Y_i and Y_j . This means that all of the supports are disjoint: $S_i \cap S_j = \emptyset$ for $i \neq j$. We consider the following two cases:

- If $q \leq n - 1$, we consider the first column of Y_1 whose first entry $(Y_1)_{11}$ is necessarily zero. By disjointness of supports, every other entry of Y_1 in its first column is also zero because all corresponding entry-positions in Y_2, \dots, Y_q are already occupied by a non-zero entries. Thus the entire first column of Y_1 must be zero, contradicting unitarity. We conclude that Y_1 must be diagonal, which in turn forces $Z^K = [X_1, Y_1] = \text{Id}_q$, so that all remaining Y_i with $2 \leq i \leq q$ are also diagonal. Hence $\omega^K = 1$ in this case.
- If $q = n$, then disjointness of supports forces each Y_i to have exactly one non-zero off-diagonal entry in each row and column, and the supports S_1, \dots, S_{n-1} partition all off-diagonal entries. We now fix distinct $a, b \in \{1, \dots, n\}$. Then the pair (a, b) lies in the support of some Y_i and none other. If $i \geq 2$, the commutation relation $[X_1, Y_i] = \text{Id}_n$ yields $u_{1a} = u_{1b}$. If instead (a, b) lies in the support of Y_1 , then since $n \geq 3$, we may choose $c \in \{1, \dots, n\} \setminus \{a, b\}$ to consider the pairs (a, c) and (b, c) . Then clearly (a, c) is not in the support of Y_1 but of some Y_j with $1 < j \leq n$, so that the relation $[X_1, Y_j] = \text{Id}_n$ implies that $u_{1a} = u_{1c}$. By a similar argument, we also obtain that $u_{1b} = u_{1c}$. This shows that all diagonal entries of X_1 must be equal, and therefore $Z^K = [X_1, Y_1] = \text{Id}_n$, contradicting $\omega^K \neq 1$.

Hence $\omega^K = 1$ in every irreducible block, and consequently all X_i and Y_i commute and, being normal, are simultaneously diagonalizable. Therefore the image of ρ is contained in a maximal torus of $U(n)$. □

IV.4.3 Abelianization Trick

For the final step, we first deal with the case $n > 2$. The strategy now is to take a certain finite-index subgroup $\tilde{\Delta} < \Delta'$ with its canonical inclusion map, and in view of Lemma IV.8, we

consider the following maps

$$\begin{array}{ccc} \tilde{\Delta} & \longleftrightarrow & \Delta' \xrightarrow{\rho} \mathrm{U}(1)^n \\ & \searrow \text{---} & \uparrow \\ & & \rho|_{\tilde{\Delta}} \end{array}$$

We then compose the restriction $\rho|_{\tilde{\Delta}}$ with the canonical inclusion maps $\mathrm{U}(1)^n \hookrightarrow \mathrm{U}(n) \hookrightarrow \mathrm{PU}(n, 1)$ and prove that via the induced sequence in integral bounded Borel cohomology

$$\mathrm{H}_{Bb}^{2n}(\mathrm{PU}(n, 1), \mathbb{Z}) \longrightarrow \mathrm{H}_{Bb}^{2n}(\mathrm{U}(n), \mathbb{Z}) \longrightarrow \mathrm{H}_{Bb}^{2n}(\mathrm{U}(1)^n, \mathbb{Z}) \xrightarrow{\rho|_{\tilde{\Delta}}^*} \mathrm{H}_b^{2n}(\tilde{\Delta}, \mathbb{Z}) \quad (\text{IV.17})$$

the top integral Chern class $C_n^b \in \mathrm{H}_{Bb}^{2n}(\mathrm{PU}(n, 1), \mathbb{Z})$ is mapped to the zero class. More precisely, we prove the following statement:

Proposition IV.9. *Let $n > 2$, let $\Delta' < \mathcal{H}_{\mathbb{R}}^{2n-1}$ be the lattice with Presentation IV.14 and let $\rho: \Delta' \rightarrow \mathrm{U}(1)^n$ be a representation. Then there exists a finite-index subgroup $\tilde{\Delta} < \Delta'$ such that*

$$(\rho|_{\tilde{\Delta}})^* (C_{n|\mathrm{U}(1)^n}^b) = 0 \in \mathrm{H}_b^{2n}(\tilde{\Delta}, \mathbb{Z}),$$

where $C_{n|\mathrm{U}(1)^n}^b \in \mathrm{H}_b^{2n}(\mathrm{U}(1)^n, \mathbb{Z})$ denotes the restriction of $C_n^b \in \mathrm{H}_{Bb}^{2n}(\mathrm{PU}(n, 1), \mathbb{Z})$ in (IV.17).

Proof. Since $\mathrm{U}(1)^n$ is an abelian group, the representation $\rho: \Delta' \rightarrow \mathrm{U}(1)^n$ factors through the abelianisation of Δ' , which gives rise to the following commutative diagram

$$\begin{array}{ccc} & & \mathrm{U}(1)^n \\ & \nearrow \rho & \uparrow \\ \Delta' & \xrightarrow{\pi_{\mathrm{ab}}} & \Delta' / [\Delta', \Delta'] \end{array}$$

From Presentation (IV.14), we deduce that

$$\Delta' / [\Delta', \Delta'] \cong \mathbb{Z}_d \times \mathbb{Z}^{2n-2},$$

and it clearly contains a finite-index copy of the abelian group \mathbb{Z}^{2n-2} . But then the preimage $\tilde{\Delta} := \pi_{\mathrm{ab}}^{-1}(\mathbb{Z}^{2n-2})$ is a finite-index subgroup of Δ' in its own right. Thus the above diagram extends to the following commutative diagram:

$$\begin{array}{ccc} & & \mathrm{U}(1)^n \\ & \nearrow \rho & \uparrow \\ \Delta' & \xrightarrow{\pi_{\mathrm{ab}}} & \mathbb{Z}_d \times \mathbb{Z}^{2n-2} \\ \uparrow & & \uparrow \\ \tilde{\Delta} & \xrightarrow{\pi_{\mathrm{ab}}} & \mathbb{Z}^{2n-2} \end{array}$$

where the vertical maps in the square are canonical subgroup inclusions. Then we obtain the

following induced diagram in integral bounded (Borel) cohomology

$$\begin{array}{ccc}
 & & \mathrm{H}_{Bb}^{2n}(\mathrm{U}(1)^n, \mathbb{Z}) \\
 & \swarrow \rho^* & \downarrow \\
 \mathrm{H}_b^{2n}(\Delta', \mathbb{Z}) & \xleftarrow{\pi_{ab}^*} & \mathrm{H}_b^{2n}(\mathbb{Z}_d \times \mathbb{Z}^{2n-2}, \mathbb{Z}) \\
 \downarrow & & \downarrow \\
 \mathrm{H}_b^{2n}(\tilde{\Delta}, \mathbb{Z}) & \xleftarrow{\pi_{ab}^*} & \mathrm{H}_b^{2n}(\mathbb{Z}^{2n-2}, \mathbb{Z}).
 \end{array} \tag{IV.18}$$

Further, we observe that the following holds:

$$\mathrm{H}_b^{2n}(\mathbb{Z}^{2n-2}, \mathbb{Z}) \cong \mathrm{H}^{2n-1}(\mathbb{Z}^{2n-2}, \mathbb{R}/\mathbb{Z}) = 0,$$

so that by commutativity of the diagram, we have that the composition of ρ^* with the left vertical map is the zero map. In particular, the image of the class $C_{n|\mathrm{U}(1)^n}^b \in \mathrm{H}_{Bb}^{2n}(\mathrm{U}(1)^n, \mathbb{Z})$ of this composition, which is precisely $(\rho|_{\tilde{\Delta}})^*$, vanishes. This yields the proof of the proposition. \square

To deal with the case $n = 2$, we may use a similar strategy as in the previous proposition to conclude:

Corollary IV.10. *Let $\Delta < \mathcal{H}_{\mathbb{R}}^3$ be a uniform torsion-free lattice and let $\rho : \Delta \rightarrow \mathrm{U}(2)$ be a representation. Then there exists a subgroup $\tilde{\Delta} < \Delta$ of **even** finite index, such that*

$$2 \cdot \rho|_{\tilde{\Delta}}^* \left(C_{2|\mathrm{U}(1)^2}^b \right) = 0 \in \mathrm{H}_b^4(\tilde{\Delta}, \mathbb{Z}),$$

where $C_{2|\mathrm{U}(1)^2}^b \in \mathrm{H}_b^4(\mathrm{U}(1)^2, \mathbb{Z})$ denotes the restriction of $C_2^b \in \mathrm{H}_{Bb}^4(\mathrm{PU}(2, 1), \mathbb{Z})$ in (IV.17).

Proof. For x_1, y_1 and z as in Lemma IV.8, consider the subgroup

$$\Lambda = \langle x_1, y_1^2, z^{2k_1} \mid [x_1, y_1^2] = z^{2k_1}, [x_1, z] = [y_1^2, z] = e \rangle,$$

of index 2 in Δ . We observe that the representation ρ restricted to Λ has image in the commutative diagonal subgroup $\mathrm{U}(1)^2$ of $\mathrm{U}(2)$, so that we can argue as in Proposition IV.9, by using now that the abelianization of Λ is given by $\mathbb{Z}_{2k_1} \times \mathbb{Z}^2$ and setting $\tilde{\Delta} := \pi_{ab}^{-1}(\mathbb{Z}^2)$ to conclude that $2 \cdot (\rho|_{\tilde{\Delta}})^*$ is the zero map. This proves the corollary. \square

IV.5 Degree and Integrality

We now prove another central result of the thesis.

Theorem B. *Let $n \geq 2$ and $\Gamma < \mathrm{PU}(n, 1)$ a non-uniform lattice. Then for any representation $\rho : \Gamma \rightarrow G$ with G being a simple non-compact Hermitian Lie group with finite center, we have that*

$$\mathrm{lcm}(m'_1, \dots, m'_\ell) \cdot \mathrm{Deg}(\rho) \in \mathbb{Z},$$

where $m'_j := [E_j : \Delta^{(j)}] < \infty$ for every cusp \mathcal{C}_j of M modelled by E_j and $\Delta^{(j)} = E_j \cap \mathcal{H}_{\mathbb{R}}^{2n-1}$ for $1 \leq j \leq \ell$.

Remark. To obtain the statement in [Main Theorem \(B\)](#) of the introduction we observe the following: Since by [Theorem I.68](#) in Chapter I, the indices $m'_j = [E : \Delta^{(j)}]$ have a universal upper bound $I_n \leq 2 \cdot (6\pi)^{\frac{1}{2}n(n-1)}$, we may set

$$\mathrm{Cst}(n) := \mathrm{lcm}(1, \dots, 2(6\pi)^{(2n-1)(n-1)}).$$

In particular, for $n = 2$ it was established in [\[HP96\]](#) that $I_2 = 6$ and hence we may set $\mathrm{Cst}(2) = 6! = 720$.

Proof of Theorem B. Again, we recall that by the generalized Bieberbach theorem in [Theorem I.68](#) in Chapter I, any cusp of a complete non-compact complex hyperbolic manifold M of finite volume is modelled by an almost-Bieberbach group E that contains the uniform torsion-free sublattice $\Delta = E \cap \mathcal{H}_{\mathbb{R}}^{2n-1}$ of finite index, up to conjugation.

We consider an arbitrary representation $\rho : \pi_1(M) \rightarrow G$ into a simple non-compact Hermitian Lie group with finite center. By the congruence relation in [Theorem IV.5](#) we have that

$$\mathrm{Deg}(\rho) \equiv -\mathrm{J}(\alpha_{\mathbb{Z}}^b) \pmod{\mathbb{Z}} \quad \text{for} \quad \alpha_{\mathbb{Z}}^b = \rho^*(\overline{C}_1^b) \cup i^*(C_1^b)^{n-1},$$

where for the right-hand side we may restrict to $\rho : \Delta \rightarrow G$ and the canonical inclusion $i : \Delta \hookrightarrow \mathrm{PU}(n, 1)$. We observe that in order to show that $\mathrm{J}(\alpha_{\mathbb{Z}}^b) = 0$, it suffices to prove that

$$i^*(C_1^b) = [0] \in \mathrm{H}_b^2(\Delta, \mathbb{Z}). \tag{IV.19}$$

For this consider the sequence of canonical inclusions

$$\Delta \hookrightarrow \mathcal{H}_{\mathbb{R}}^{2n-1} \hookrightarrow P \hookrightarrow \mathrm{PU}(n, 1)$$

which induces in degree 2 integral Borel bounded cohomology

$$\mathrm{H}_{Bb}^2(\mathrm{PU}(n, 1), \mathbb{Z}) \longrightarrow \mathrm{H}_{Bb}^2(P, \mathbb{Z}) \longrightarrow \mathrm{H}_b^2(\mathcal{H}_{\mathbb{R}}^{2n-1}, \mathbb{Z}) \longrightarrow \mathrm{H}_b^2(\Delta, \mathbb{Z})$$

But in [Corollary III.27](#) in Chapter III, we proved that the middle map $\mathrm{H}_{Bb}^2(P, \mathbb{Z}) \rightarrow \mathrm{H}_{Bb}^2(\mathcal{H}_{\mathbb{R}}^{2n-1}, \mathbb{Z})$ is the zero map. We conclude that the image of $C_1^b \in \mathrm{H}_b^2(\mathrm{PU}(n, 1), \mathbb{Z})$ under this sequence of

pullback maps is indeed the zero class. After repeating the same argument for all cusps of M , Lemma IV.6 then yields the proof. \square

IV.6 Integrality of Related and New Invariants

The Euler number $E(\cdot)$ and the degree $\text{Deg}(\cdot)$ considered up to this point, are normalizations of a volume-type invariant and the well-known *Toledo invariant* (cf. Chapter V), respectively. We first deduce integrality up to rescaling for these invariants. Subsequently, we demonstrate that our methods developed for the integrality proofs in the previous sections, extend to invariants in a different context, motivated by Anti-de-Sitter geometry.

IV.6.1 A Volume-Type Invariant and the Toledo Invariant

We return to our template-invariant given in (IV.7) by

$$I(\alpha_{\mathbb{R}}^b) = \langle c \circ \Phi_b(\alpha_{\mathbb{R}}^b), [N, \partial N] \rangle \quad \text{with} \quad \alpha_{\mathbb{R}}^b \in H_c^{2n}(\Gamma, \mathbb{R}).$$

For $\alpha_{\mathbb{R}}^b$, we considered classes that originate from integral Borel bounded cohomology via the cohomological map induced through change of coefficients $\mathbb{Z} \hookrightarrow \mathbb{R}$. We now take a different perspective by considering a class originating from distinguished differential forms; we recall shortly from Subsection II.4: Let G be a simple non-compact Hermitian Lie group with finite center and we consider the bounded Kähler classes $\bar{\omega}_2^b \in H_{cb}^2(G, \mathbb{R})$ and $\omega_2^b \in H_{cb}^2(\text{PU}(n, 1), \mathbb{R})$ (see Definition II.18). Furthermore, we recall that the comparison map $c_{\mathbb{R}} : H_{cb}^{2n}(\text{PU}(n, 1), \mathbb{R}) \rightarrow H_c^{2n}(\text{PU}(n, 1), \mathbb{R})$ is only known to be surjective by Theorem II.2 and we make the choice of the following bounded volume class

$$\omega_{2n}^b := \frac{C_n^b}{h_n},$$

where $h_n = (-1)^n \frac{\chi(\mathbb{C}\mathbb{P}^n)}{\text{Vol}(\mathbb{C}\mathbb{P}^n)}$ is the Hirzebruch proportionality constant. This is in accordance with Definition II.21, since we have that $c_{\mathbb{R}}(\omega_{2n}^b) = \frac{1}{h_n} \cdot c_{\mathbb{R}}(C_n^b) = \frac{1}{h_n} \cdot C_n = \omega_{2n}$ due to Proposition III.21.

We are now in position to introduce the following invariants:

Definition IV.11. Let $\Gamma < \text{PU}(n, 1)$ be a non-uniform torsion-free lattice.

- For any representation $\rho : \Gamma \rightarrow \text{PU}(n, 1)$ we define the *volume-type invariant* $\text{Vol}(\cdot)$ of ρ as the invariant

$$\text{Vol}(\rho) := I(\alpha_{\mathbb{R}}^b) \quad \text{for} \quad \alpha_{\mathbb{R}}^b = \rho^*(\omega_{2n}^b).$$

- For any representation $\rho : \Gamma \rightarrow G$ into a simple non-compact Hermitian Lie group with finite center we define the *Toledo invariant* $\text{Tol}(\cdot)$ of ρ as the invariant

$$\text{Tol}(\rho) := I(\alpha_{\mathbb{R}}^b) \quad \text{for} \quad \alpha_{\mathbb{R}}^b = \rho^*(\bar{\omega}_2^b) \cup i^*(\omega_2^b)^{n-1}.$$

By comparing the definitions of the Volume with the Euler number in Definition IV.3, we

deduce immediately that

$$\text{Vol}(\rho) = \text{I}(\rho^*(\omega_{2n}^b)) = \text{I}(\rho^*(h_n^{-1} \cdot \mathbf{C}_n^b)) = h_n^{-1} \cdot \text{I}(\rho^*(\mathbf{C}_n^b)) = h_n^{-1} \cdot \text{E}(\rho).$$

Whereas for the Toledo invariant and the degree in Definition IV.4, we observe that:

$$\text{Tol}(\rho) = \text{I}(\rho^*(\bar{\omega}_2^b) \cup i^*(\omega_2^b)^{n-1}) = \text{I}(\rho^*(\bar{\lambda} \cdot \bar{\mathbf{C}}_1^b) \cup i^*(\lambda \cdot \mathbf{C}_1^b)^{n-1}) = \bar{\lambda} \cdot \lambda^{n-1} \cdot \text{I}(\rho^*(\bar{\mathbf{C}}_1) \cup i^*(\mathbf{C}_1^b)^{n-1}),$$

where $\bar{\lambda}, \lambda \in \mathbb{R}$ are certain numbers that satisfy the relations

$$\bar{\omega}_2 = \bar{\lambda} \cdot \bar{\mathbf{C}}_1 \in \text{H}_c^2(G, \mathbb{R}) \cong \mathbb{R} \quad \text{and} \quad \omega_2 = \lambda \cdot \mathbf{C}_1 \in \text{H}_c^2(\text{PU}(n, 1), \mathbb{R}) \cong \mathbb{R},$$

so that we further deduce immediately that $\text{Tol}(\rho) = \bar{\lambda} \cdot \lambda^{n-1} \cdot \text{Deg}(\rho)$.

Remark. As established in Proposition III.21, we have in fact that $\lambda = -\frac{4\pi}{n+1}$. In last chapter, we give an extended discussion on the values $\bar{\lambda}$ depending on the simple non-compact Hermitian Lie group with finite center G , namely in Proposition V.11 on page 136.

Finally, we obtain the following immediate corollary of Theorems A and B, thereby completing the statement of our Main Theorem from the introduction:

Corollary C. *Let $\Gamma := \pi_1(M) < \text{PU}(n, 1)$ be a non-uniform torsion-free lattice.*

(A) *For any representation $\rho : \Gamma \rightarrow \text{PU}(n, 1)$, it holds that*

$$\text{Vol}(\rho) \in \frac{1}{h_n \cdot \text{lcm}(m_1, \dots, m_k)} \cdot \mathbb{Z},$$

where $h_n = (-1)^n \frac{\chi(\mathbb{C}\mathbb{P}^n)}{\text{Vol}(\mathbb{C}\mathbb{P}^n)}$ and $m_j := [E_j : \tilde{\Delta}^{(j)}] < \infty$ for every cusp \mathcal{C}_j of M modelled by E_j and $\tilde{\Delta}^{(j)}$ is a certain finite index subgroup of $E_j \cap \mathcal{H}_{\mathbb{R}}^{2n-1}$ for $1 \leq j \leq \ell$.

(B) *For any representation $\rho : \Gamma \rightarrow G$ into a simple non-compact Hermitian Lie group with finite center, we have that*

$$\text{Tol}(\rho) \in \frac{\bar{\lambda} \cdot \lambda^{n-1}}{\text{lcm}(m'_1, \dots, m'_\ell)} \cdot \mathbb{Z}$$

where $\lambda = -\frac{4\pi}{n+1}$, $\bar{\lambda} \in \mathbb{R} \setminus \{0\}$ is such that $\bar{\omega}_2 = \bar{\lambda} \cdot \bar{\mathbf{C}}_1$ and $m'_j := [E_j : \Delta^{(j)}] < \infty$ for every cusp \mathcal{C}_j of M modelled by E_j and $\Delta^{(j)} = E_j \cap \mathcal{H}_{\mathbb{R}}^{2n-1}$ for $1 \leq j \leq \ell$.

Remark. (i) We note for any representation $\rho : \Gamma \rightarrow \text{PU}(n, 1)$ of a non-uniform torsion-free lattice, the canonical volume $\text{Vol}_{\text{Dup}}(\rho) := \text{I}(\rho^*([\text{Vol}_{\text{PU}(n,1)}]))$ should conjecturally coincide with $\text{Vol}(\rho)$, since the comparison map $c_{\mathbb{R}}$ is expected to be an isomorphism so that $\omega_{2n}^b = [\text{Vol}_{\text{PU}(n,1)}]$ (cf. Subsection II.4.2).

(ii) Moreover, part (A) is a similar statement as for $\text{Vol}_{\text{Dup}}(\cdot)$ in [BBI21] for non-uniform torsion-free lattices in $\text{SO}^+(n, 1)$ for n even, represented into their ambient Lie group.

The volume $\text{Vol}_{\text{Dup}}(\cdot)$ has been primarily considered for representations of non-uniform real-hyperbolic lattices, under equivalent definitions in [BBI13] (bounded-cohomological), [Dun99; Fra04; FK06] (via pseudo-developing maps) and [KK14] (ℓ_1 -and bounded cohomological) to show volume rigidity recovering Mostow-Prasad rigidity. The equivalence of the definitions in a more general setting was proven in [Kim16].

- (iii) Our results do not hold for non-uniform torsion-free lattices in $\text{PU}(1, 1)$ as first observed in [BIW10], where $\text{Tol}(\cdot) = \text{Vol}(\cdot)$ take values continuously in a closed interval (continuity in this case is recovered in Theorem V.12).

IV.6.2 New Integral Invariants

We show that it is natural to extend the integrality statements beyond invariants associated to single representations, to include invariants arising from multiple, possibly non-conjugate, representations.

A natural geometric motivation for this comes from Anti-de-Sitter geometry: Up to finite cover, any closed Anti-de-Sitter 3-manifold is a quotient of $\text{PSL}_2(\mathbb{R})$ by a discrete subgroup as the image of

$$j \times \rho : \Gamma \longrightarrow \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R}),$$

where Γ is a fundamental group of a closed oriented surface, j a discrete and faithful representation, and ρ a representation strictly dominated² by j ; the group $j \times \rho(\Gamma)$ acts by left and right multiplication on $\text{PSL}_2(\mathbb{R})$ (cf. work by Kulkarny-Raymond [KR85] and Klingler [Kli96]). Analogous characterizations of such lattices were extended to the case of $\text{SO}^+(n, 1)$ in [Kas08; Kob93] and then more generally to Lie groups of rank 1 in [Gué+17, §1.6].

Now, Tholozan studied in [Tho18] the volume associated with such pairs of representations (j, ρ) of real hyperbolic lattices into $\text{SO}^+(n, 1)$, including the non-uniform case for $n = 2$, and proved its proportionality to the sum of top Chern classes of j and ρ . The nature and realizability of these volume values are addressed in [Tho18, §5.2]. In subsequent work, Tholozan also derived a closed formula for the volume in the case of uniform complex hyperbolic lattices [Tho16, Theorem 4]. The extension of this framework to the *non-uniform* complex hyperbolic case, however, remains open and serves as one of our motivations here. Building on this perspective, we consider invariants arising from more general products of distinct representations of non-uniform torsion-free complex hyperbolic lattices. In fact, our invariant $\text{Deg}(\rho)$ in Definition IV.4 is one such instance, as it can be viewed as an invariant associated with the representation

$$\rho \times i^{(n-1)} : \Gamma \longrightarrow G \times \underbrace{\text{PU}(n, 1) \times \cdots \times \text{PU}(n, 1)}_{n-1},$$

where $\Gamma < \text{PU}(n, 1)$ is a torsion-free lattice, $i : \Gamma \hookrightarrow \text{PU}(n, 1)$ is the canonical inclusion and $\rho : \Gamma \rightarrow G$ is an arbitrary representation into a simple non-compact Hermitian Lie group with finite center. In particular, we observe that due to naturality of the cup product and functoriality of cohomology under group homomorphisms (cf. Subsection II.1.2), the bounded cohomological

²There exists a (j, ρ) -equivariant map $\mathbb{H}_{\mathbb{R}}^2 \rightarrow \mathbb{H}_{\mathbb{R}}^2$ that is λ -Lipschitz for some $\lambda < 1$.

map $\rho^* \cup (i^*)^{n-1}$, that gives rise to Definition IV.4, factors as follows

$$\begin{array}{ccc} \mathrm{H}_{cb}^2(G, \mathbb{R}) \times \mathrm{H}_{cb}^{2(n-1)}(\mathrm{PU}(n, 1), \mathbb{R}) & \xrightarrow{\rho^* \cup (i^*)^{n-1}} & \mathrm{H}_b^{2n}(\Gamma, \mathbb{R}) \\ \downarrow \cup^n & \nearrow & \\ \mathrm{H}_{cb}^{2n}(G \times \mathrm{PU}(n, 1)^{n-1}, \mathbb{R}), & & (\rho \times i^{(n-1)})^* \end{array}$$

where the vertical map takes the n -fold product of the cohomological factors, composed with the cohomological pullback induced by the product $\rho \times i^{n-1}$. Considering products where more than one factor varies—while at least one remains the canonical inclusion—leads directly to the following generalization of the construction above as an immediate consequence of **Theorem B**:

Corollary D. *Let $n \geq 2$, let $\Gamma < \mathrm{PU}(n, 1)$ be a non-uniform torsion-free lattice and $i : \Gamma \hookrightarrow \mathrm{PU}(n, 1)$ the canonical inclusion. Further, let $\rho_i : \Gamma \rightarrow G$ be representations with target a simple non-compact Hermitian Lie group with finite center for $1 \leq i \leq n-1$. Then*

$$\mathrm{lcm}(m'_1, \dots, m'_k) \cdot \mathrm{I}(\alpha_{\mathbb{R}}^b) \in \mathbb{Z} \quad \text{for} \quad \alpha_{\mathbb{R}}^b = \rho_1^*(\overline{\mathcal{C}}_1^b) \cup \dots \cup \rho_{n-1}^*(\overline{\mathcal{C}}_1^b) \cup i^*(\mathcal{C}_1^b)$$

where $m'_j := [E_j : \Delta^{(j)}] < \infty$ for every cusp \mathcal{C}_j of M modelled by E_j and $\Delta^{(j)} = E_j \cap \mathcal{H}_{\mathbb{R}}^{2n-1}$ for $1 \leq j \leq \ell$.

Proof. Following the proof of **Theorem B**, an analogous application of the congruence relation in **Theorem IV.5** and **Observation (IV.19)** on the vanishing of $i^*(\mathcal{C}_1^b)$ in $\mathrm{H}_b^2(\Delta^{(j)}, \mathbb{Z})$ for all $1 \leq j \leq \ell$ yield the proof. \square

Having established integrality when one factor is the canonical inclusion, it is natural to ask whether such integrality persists for arbitrary products of representations. The following theorem answers this affirmatively, which together with the last corollary completes the proof of **Complementary Main Theorem**:

Theorem E. *Let $n \geq 2$ and $\Gamma < \mathrm{PU}(n, 1)$ be a non-uniform torsion-free lattice. Let $\rho_i : \Gamma \rightarrow \mathrm{PU}(n, 1)$ be arbitrary representations for $1 \leq i \leq n$. Then*

$$\mathrm{lcm}(m''_1, \dots, m''_k) \cdot \mathrm{I}(\alpha_{\mathbb{R}}^b) \in \mathbb{Z} \quad \text{for} \quad \alpha_{\mathbb{R}}^b = \rho_1^*(\mathcal{C}_1^b) \cup \dots \cup \rho_n^*(\mathcal{C}_1^b),$$

where $m''_j := [E_j : \tilde{\Delta}^{(j)}] < \infty$ for every cusp \mathcal{C}_j of M modelled by E_j and $\tilde{\Delta}^{(j)}$ is a certain finite index subgroup of $E_j \cap \mathcal{H}_{\mathbb{R}}^{2n-1}$ for $1 \leq j \leq \ell$.

Proof. For any cusp of M modelled by an almost Bieberbach group E , we consider $\Delta := E \cap \mathcal{H}_{\mathbb{R}}^{2n-1}$ consider a representation $\rho : \Delta \rightarrow \mathrm{PU}(n, 1)$. By **Proposition IV.7**, we may suppose the image of ρ_i to be contained either in $\mathrm{U}(n)$ or $\mathrm{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$ up to conjugation. In both cases, we draw inspiration from the strategy employed in **Proposition IV.9**. For this, we denote again by $\tilde{\Delta}$ the preimage of \mathbb{Z}^{2n-1} under the projection map $\pi : \Delta \rightarrow \mathbb{Z}_d \times \mathbb{Z}^{2n-2}$ onto the abelianisation of Δ for a suitable d .

- If $\text{Im}(\rho_i)$ is contained in the maximal compact group $U(n)$ up to conjugation, we postcompose ρ_i with the determinant map $\det : U(n) \rightarrow U(1)$ to obtain a homomorphism into an abelian group and we factor through the abelianisation of Δ . This ensures that the left diagram commutes (compare with Diagram (IV.18)) and induces the commutative diagram on the right:

$$\begin{array}{ccc}
 U(n) & \xrightarrow{\det} & U(1) \\
 \rho_i \uparrow & & \uparrow \exists! \psi_i \\
 \Delta & \xrightarrow{\pi} & \mathbb{Z}_d \times \mathbb{Z}^{2n-2} \\
 \uparrow & & \uparrow \\
 \tilde{\Delta} & \xrightarrow{\pi} & \mathbb{Z}^{2n-2}
 \end{array}
 \qquad
 \begin{array}{ccc}
 H_b^2(U(n), \mathbb{Z}) & \xleftarrow{\det^*} & H_b^2(U(1), \mathbb{Z}) \\
 \rho_i^* \downarrow & & \downarrow \psi_i^* \\
 H_b^2(\Delta, \mathbb{Z}) & \xleftarrow{\pi^*} & H_b^2(\mathbb{Z}_d \times \mathbb{Z}^{2n-2}, \mathbb{Z}) \\
 \downarrow & & \downarrow \\
 H_b^2(\tilde{\Delta}, \mathbb{Z}) & \xleftarrow{\pi^*} & H_b^2(\mathbb{Z}^{2n-2}, \mathbb{Z})
 \end{array}$$

Further, in the cohomological diagram, it is crucial to observe that the first Chern class in the left top corner is in the image of the Chern class in the right top corner: Let

$$C_{1|U(n)}^b \in H_{Bb}^2(U(n), \mathbb{Z})$$

be the restriction of $C_1^b \in H_{Bb}^2(\text{PU}(n, 1), \mathbb{Z})$. Due to Corollary III.15 from Chapter III, the determinant map \det^* is an isomorphism, and in particular because for compact groups bounded and unbounded integral Borel cohomology coincides, we have that

$$\det^*(C_{1|U(1)}^b) = C_{1|U(n)}^b. \quad (\text{IV.20})$$

- If $\text{Im}(\rho_i)$ is contained in the minimal parabolic group $\text{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1})$ up to conjugation, we may postcompose ρ_i with the projection

$$\pi : \text{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1}) \longrightarrow U(n-1) \times \mathbb{R}_{>0} \quad \text{and} \quad (\det, \text{id}) : U(n-1) \times \mathbb{R}_{>0} \longrightarrow U(1) \times \mathbb{R}_{>0}$$

to again obtain as the target an abelian group. So the following diagram commutes (again compare with Diagram (IV.18))

$$\begin{array}{ccc}
 P & \xrightarrow{\pi} & U(n-1) \times \mathbb{R}_{>0} & \xrightarrow{(\det, \text{id})} & U(1) \times \mathbb{R}_{>0} \\
 \rho_i \uparrow & & & & \uparrow \exists! \psi_i \\
 \Delta & \xrightarrow{\pi_{\text{ab}}} & \mathbb{Z}_d \times \mathbb{Z}^{2n-2} \\
 \uparrow & & \uparrow \\
 \tilde{\Delta} & \xrightarrow{\pi_{\text{ab}}} & \mathbb{Z}^{2n-1}
 \end{array}$$

and induces the following diagram in integral bounded Borel cohomology

$$\begin{array}{ccc}
 \mathrm{H}_{Bb}^2(P, \mathbb{Z}) & \xleftarrow{\pi^*} & \mathrm{H}_{Bb}^2(\mathrm{U}(n-1) \times \mathbb{R}_{>0}, \mathbb{Z}) \xleftarrow{(\det, \mathrm{id})^*} \mathrm{H}_{Bb}^2(\mathrm{U}(1) \times \mathbb{R}_{>0}, \mathbb{Z}) \\
 \rho_i^* \downarrow & & \downarrow \psi_i^* \\
 \mathrm{H}_b^2(\Delta, \mathbb{Z}) & \xleftarrow{\pi_{ab}} & \mathrm{H}_b^2(\mathbb{Z}_d \times \mathbb{Z}^{2n-2}, \mathbb{Z}) \\
 \downarrow & & \downarrow \\
 \mathrm{H}_b^2(\tilde{\Delta}, \mathbb{Z}) & \xleftarrow{\pi_{ab}} & \mathrm{H}_b^2(\mathbb{Z}^{2n-2}, \mathbb{Z})
 \end{array}$$

In a similar spirit as in the previous case, we observe: Let

$$C_{1|P}^b \in \mathrm{H}_{Bb}^2(P, \mathbb{Z})$$

be the restriction of $C_1^b \in \mathrm{H}_{Bb}^{2n}(\mathrm{PU}(n, 1), \mathbb{Z})$. By Proposition III.29 in Chapter III, there exists a bounded class $\alpha \in \mathrm{H}_{Bb}^2(\mathrm{U}(1) \times \mathbb{R}_{>0}, \mathbb{Z})$ such that

$$\pi^* \circ (\det, \mathrm{id})^*(\alpha) = C_{1|P}^b. \quad (\text{IV.21})$$

Now, we set

$$A_i = \begin{cases} \mathrm{U}(n), & \text{if } \mathrm{Im}(\rho_i^*) \text{ is contained in } \mathrm{U}(n) \text{ up to conjugation,} \\ \mathrm{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1}), & \text{if } \mathrm{Im}(\rho_i^*) \text{ is contained in } \mathrm{Sim}(\mathcal{H}_{\mathbb{R}}^{2n-1}) \text{ up to conjugation.} \end{cases}$$

Consider the factor-wise canonical inclusion

$$\prod_{i=1}^n A_i \hookrightarrow \mathrm{PU}(n, 1)^{(n)}$$

which induces the left vertical map in the commutative diagram

$$\begin{array}{ccc}
 \mathrm{H}_{Bb}^{2n}(\mathrm{PU}(n, 1)^{(n)}, \mathbb{Z}) & \xrightarrow{(\rho_1 \times \cdots \times \rho_n)^*} & \mathrm{H}_b^{2n}(\Delta, \mathbb{Z}) \\
 \downarrow & \nearrow & \\
 \mathrm{H}_{Bb}^{2n}(\prod_{i=1}^n A_i, \mathbb{Z}) & &
 \end{array} \quad (\text{IV.22})$$

In particular, the n -fold cup product

$$C_b^1 \cup \cdots \cup C_b^1 \in \mathrm{H}_{Bb}^{2n}(\mathrm{PU}(n, 1)^{(n)}, \mathbb{Z}),$$

is mapped (by naturality of cup products with respect to homomorphisms) vertically to

$$C_{b|A_1}^1 \cup \cdots \cup C_{b|A_n}^1 \in \mathrm{H}_{Bb}^{2n}(\prod_{i=1}^n A_i, \mathbb{Z}).$$

Next, we set

$$(\varphi_i, B_i) = \begin{cases} (\det, \mathrm{U}(1)), & \text{if } A_i = \mathrm{U}(n), \\ ((\det, \mathrm{id}) \circ \pi, \mathrm{U}(1) \times \mathbb{R}_{>0}), & \text{if } A_i = P. \end{cases}$$

Since the product $B_1 \times \cdots \times B_n$ is again an abelian group, then the following diagram also commutes

$$\begin{array}{ccc}
 \prod_{i=1}^n A_i & \xrightarrow{(\varphi_1, \dots, \varphi_n)} & \prod_{i=1}^n B_i \\
 \rho_1 \times \cdots \times \rho_n \uparrow & & \uparrow \psi_1 \times \cdots \times \psi_n \\
 \Delta & \xrightarrow{\pi_{ab}} & \mathbb{Z}_d \times \mathbb{Z}^{2n-2} \\
 \downarrow & & \downarrow \\
 \tilde{\Delta} & \xrightarrow{\pi_{ab}} & \mathbb{Z}^{2n-2}
 \end{array} .$$

and induces the following commutative diagram in the cohomological top degree

$$\begin{array}{ccc}
 H_{Bb}^{2n}(\prod_{i=1}^n A_i, \mathbb{Z}) & \xleftarrow{(\varphi_1, \dots, \varphi_n)^*} & H_{Bb}^{2n}(\prod_{i=1}^n B_i, \mathbb{Z}) \\
 (\rho_1 \times \cdots \times \rho_n)^* \downarrow & & \downarrow (\psi_1 \times \cdots \times \psi_n)^* \\
 H_b^{2n}(\Delta, \mathbb{Z}) & \xleftarrow{\pi_{ab}^*} & H_b^{2n}(\mathbb{Z}_d \times \mathbb{Z}^{2n-2}, \mathbb{Z}) \\
 \downarrow & & \downarrow \\
 H_b^{2n}(\tilde{\Delta}, \mathbb{Z}) & \xleftarrow{\pi_{ab}^*} & H_b^{2n}(\mathbb{Z}^{2n-2}, \mathbb{Z}).
 \end{array}$$

Since $H_b^{2n}(\mathbb{Z}^{2n-2}, \mathbb{Z}) \cong H^{2n-1}(\mathbb{Z}^{2n-2}, \mathbb{R}/\mathbb{Z}) = 0$, by commutativity of the diagram this implies that the composition of $(\varphi_1, \dots, \varphi_n)^*$ with the left two vertical maps is the zero map.

But then, due to (IV.20) and (IV.21), as well as the naturality of cup products with respect to homomorphisms, there exists a bounded class $\beta \in H_{Bb}^{2n}(\prod_{i=1}^n A_i, \mathbb{Z})$, which is in fact a n -fold cup product of bounded classes, such that

$$(\varphi_1, \dots, \varphi_n)^*(\beta) = C_{b|A_1}^1 \cup \cdots \cup C_{b|A_n}^1,$$

whose image under the vertical left maps is the zero class. Finally, by commutativity of Diagram (IV.22), we conclude that

$$(\rho_1 \times \cdots \times \rho_n)_{|\tilde{\Delta}}^*(C_1^b \cup \cdots \cup C_1^b) = [0].$$

and the congruence relation in Theorem IV.5 together with Lemma IV.6, yield the proof of the theorem. \square

Chapter V

Transfer-Theoretic Invariants

In the present chapter we characterize the invariants $E(\cdot)$, $\text{Vol}(\cdot)$ as well as $\text{Tol}(\cdot)$, $\text{Deg}(\cdot)$ introduced in Chapter IV, in terms of bounded transfer maps. While this is yet another way to uniformize the definitions, it also has the following purposes: Since the invariant $\text{Tol}(\cdot)$ does not seem to appear in literature under the form in Definition IV.11, our first goal is to establish its equivalence with the existing Toledo invariants, thereby justifying our terminology. In the generality we adopt, the existing Toledo invariants have essentially two equivalent definitions: an algebraic perspective via transfer maps [Poz15, p. 1299], [BI07a] and a differential-geometric perspective via the L_2 -cohomology [BI07b; BI00].

The second goal is to prove that all the invariants in question are continuous. This allows to conclude that $E(\cdot)$, $\text{Vol}(\cdot)$ and $\text{Tol}(\cdot)$, $\text{Deg}(\cdot)$ label unions of connected components of the respective character varieties:

$$\text{Hom}(\Gamma, \text{PU}(n, 1)) // \text{PU}(n, 1) \quad \text{and} \quad \text{Hom}(\Gamma, G) // G,$$

where $\Gamma < \text{PU}(n, 1)$ is a non-uniform torsion-free lattice and G a simple non-compact Hermitian Lie group with finite center (cf. Section I.1.3).

V.1 Transfer Maps

We first introduce the classical transfer maps in continuous bounded cohomology, in terms of which we introduce two invariants $e(\cdot)$ and $d(\cdot)$. We then give geometrically flavoured transfer maps following the machinery established in [BBI13], which will enable us to relate the transfer-theoretic invariants to their counterparts $E(\cdot)$, $\text{Vol}(\cdot)$, $\text{Deg}(\cdot)$ and $\text{Tol}(\cdot)$ obtained via Poincaré duality.

V.1.1 Geometrically Flavoured Cochains

As a follow-up to the fundamentals on cohomology theories in Chapter II, we summarize here additional preparatory material for this chapter; this concerns alternative cochains that realize cohomology theories we have already introduced in the more specific setting of the complex hyperbolic space.

We fix the following notation for this chapter: For a non-uniform torsion-free lattice $\Gamma < \mathrm{PU}(n, 1)$, we denote the associated manifold by $M = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$. In particular it comes with universal cover $\widetilde{M} \cong \mathbb{H}_{\mathbb{C}}^n$ and the universal covering map $\pi : \mathbb{H}_{\mathbb{C}}^n \rightarrow M$. As in Chapter IV, via the thin-thick decomposition, we choose a compact core N such that the complement $M \setminus N$ consists of finitely many cusp regions $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ of M and we set

$$U := \pi^{-1}(M \setminus N) = \{\pi^{-1}(\mathcal{C}_1), \dots, \pi^{-1}(\mathcal{C}_\ell)\},$$

which is in particular a disjoint union of horoballs in $\mathbb{H}_{\mathbb{C}}^n$.

Table V.1 summarizes all the alternative geometrically flavoured (bounded) cochain complexes, in terms of which the corresponding cohomologies from Subsection II.1.1 can be computed. For instance, the cochain complex $C_c^*(\mathbb{H}_{\mathbb{C}}^n, \mathbb{R})^{\mathrm{PU}(n,1)}$ realizes $H_c^*(\mathrm{PU}(n, 1), \mathbb{R})$ due to [Gui80, Chapter III, Proposition 2.3] and the bounded cochain complex $C_{cb}^*(\mathbb{H}_{\mathbb{C}}^n, \mathbb{R})^{\mathrm{PU}(n,1)}$ realizes isometrically $H_{c(b)}^*(\mathrm{PU}(n, 1), \mathbb{R})$ due to [Mon01, Corollary 7.5.9].

Cohomology Type	Computed via (Bounded) Cochains
$H_{c(b)}^*(\mathrm{PU}(n, 1), \mathbb{R})$	$C_{c(b)}^*(\mathbb{H}_{\mathbb{C}}^n, \mathbb{R})^{\mathrm{PU}(n,1)}$
$H_{(b)}^*(\Gamma, \mathbb{R})$	$C_{c(b)}^{*+1}(\mathbb{H}_{\mathbb{C}}^n, \mathbb{R})^\Gamma$
$H_{(b)}^*(M, \mathbb{R})$	$S_{(b)}^*(\mathbb{H}_{\mathbb{C}}^n, \mathbb{R})^\Gamma$
$H_{dR}^*(M, \mathbb{R})$	$\Omega^*(\mathbb{H}_{\mathbb{C}}^n)^\Gamma$
$H_{(b)}^*(M, M \setminus N, \mathbb{R})$	$S_{(b)}^*(\mathbb{H}_{\mathbb{C}}^n, U)^\Gamma$
$H_{dR}^*(M, M \setminus N)$	$\Omega^*(\mathbb{H}_{\mathbb{C}}^n, U)^\Gamma$

Table V.1: Real-valued cohomologies and associated cochain complexes

We note that the relative cochains $S_{(b)}^*(\mathbb{H}_{\mathbb{C}}^n, U)^\Gamma$ and $\Omega^*(\mathbb{H}_{\mathbb{C}}^n, U)^\Gamma$ are defined by Γ -invariant cochains that vanish on U , providing models for the corresponding (bounded) relative singular and relative de Rham cohomology of the pair $(M, M \setminus N)$ (cf. notation in the real hyperbolic setting used in [BBI13, §3.1]).

V.1.2 Transfer-Theoretic Definitions

We first define transfer maps in bounded cohomology; for classical sources we refer to [Mon01, §8.6] and [Fri17, §8.6]. Let L be a locally compact group and we assume it admits a torsion-free lattice Γ . Further, let $p : L \rightarrow \Gamma \backslash L$ be a canonical projection and we fix $s : \Gamma \backslash L \rightarrow \Gamma$ a Borel section, i.e. $p \circ s = \mathrm{id}$, so that

$$\mathcal{D} := s(\Gamma \backslash L) \tag{V.1}$$

is a Borel measurable fundamental domain for the action of Γ on L (see Definition I.7). For each $1 \leq k \leq n$, the *bounded transfer map* on the cochain level is defined by

$$\mathrm{tr}_{2k}^b : C_{cb}^{2k}(L, \mathbb{R})^\Gamma \longrightarrow C_{cb}^{2k}(L, \mathbb{R})^L \tag{V.2}$$

given by

$$\mathrm{tr}_{2k}^b(\alpha)(x_0, \dots, x_{2k}) := \int_{\mathcal{D}} \alpha(gx_0, \dots, gx_{2k}) d\mu(g),$$

where μ is an invariant measure on \mathcal{D} normalized by $\mu(\mathcal{D}) = 1$. The definition is independent on the choice of fundamental domain in (V.1) by Γ -invariance of α . A direct calculation shows that tr_{2k}^b is indeed continuous and bounded. Moreover, it commutes with the coboundary operator, while the cochain complexes $C_{cb}^{2k}(L, \mathbb{R})^\Gamma$ and $C_{cb}^{2k}(L, \mathbb{R})^L$ realize isometrically the bounded cohomologies $H_b^{2k}(\Gamma, \mathbb{R})$ and $H_{cb}^{2k}(L, \mathbb{R})$, respectively (see [Mon01, Corollary 7.4.10 and Proposition 8.6.2]), so that the map tr_{2k}^b induces the cohomological *bounded transfer map*

$$\mathrm{tr}_{2k}^b : H_b^{2k}(\Gamma, \mathbb{R}) \longrightarrow H_{cb}^{2k}(L, \mathbb{R}),$$

denoted by the same symbol by abuse of notation. This induced map enjoys two key properties: It is norm non-increasing and it is a left-inverse of i^* , induced by the canonical inclusion $i : \Gamma \hookrightarrow L$ i.e. $\mathrm{tr}_{2k}^b \circ i^* = \mathrm{id}$. We remark that in case Γ is a *uniform* torsion-free lattice, one may define transfer maps on unbounded cochains and cohomology which then coincide with the bounded transfer maps. We are now in position to define transfer-theoretic invariants central to this chapter. For this fix now Γ to be a non-uniform torsion-free lattice in $\mathrm{PU}(n, 1)$ and G to be a simple non-compact Hermitian Lie group with finite center.

Invariant $e(\cdot)$

For any representation $\rho : \Gamma \rightarrow \mathrm{PU}(n, 1)$ consider its degree $2n$ pullback, post-composed by the bounded transfer map tr_{2n}^b and subsequently the comparison map:

$$H_{cb}^{2n}(\mathrm{PU}(n, 1), \mathbb{R}) \xrightarrow{\rho^*} H_b^{2n}(\Gamma, \mathbb{R}) \xrightarrow{\mathrm{tr}_{2n}^b} H_{cb}^{2n}(\mathrm{PU}(n, 1), \mathbb{R}) \xrightarrow{c_{\mathbb{R}}} H_c^{2n}(\mathrm{PU}(n, 1), \mathbb{R}).$$

We recall that the real top Chern class C_n can be viewed as a generator of $H_c^{2n}(\mathrm{PU}(n, 1), \mathbb{R}) \cong \mathbb{R}$ and we chose the real bounded top Chern class $C_n^b \in c_{\mathbb{R}}^{-1}(C_n)$ (see Definition III.24).

Definition V.1. For a representation $\rho : \Gamma \rightarrow \mathrm{PU}(n, 1)$, we define $e(\rho)$ as the number satisfying the relation

$$\mathrm{tr}_{2n}^b \circ \rho^*(C_n^b) = e(\rho) \cdot C_n \in \mathbb{R}.$$

This definition is analogous the one in the real hyperbolic case [BBI13, Proposition 3.3].

Invariants $d(\cdot)$ and $t(\cdot)$

For any representation $\rho : \Gamma \rightarrow G$ consider its degree 2 pullback, post-composed by the bounded transfer map tr_2^b :

$$H_{cb}^2(G, \mathbb{R}) \xrightarrow{\rho^*} H_b^2(\Gamma, \mathbb{R}) \xrightarrow{\mathrm{tr}_2^b} H_{cb}^2(\mathrm{PU}(n, 1), \mathbb{R}).$$

Both the first bounded Chern class \mathbf{C}_1^b and the bounded Kähler class ω_2^b can be considered a generator of $H_{cb}^2(\mathrm{PU}(n, 1), \mathbb{R}) \cong H_c^2(\mathrm{PU}(n, 1), \mathbb{R}) \cong \mathbb{R}$ (see Definitions III.24 and II.19). We use these generators to define the following quantities:

Definition V.2. For any representation $\rho: \Gamma \rightarrow G$, we define $d(\rho)$ and $t(\rho)$ as the numbers that satisfies the relations

$$\mathrm{tr}_2^b \circ \rho^*(\overline{\mathbf{C}}_1^b) = d(\rho) \cdot \mathbf{C}_1^b \in \mathbb{R}. \quad \text{and} \quad \mathrm{tr}_2^b \circ \rho^*(\overline{\omega}_2^b) = t(\rho) \cdot \omega_2^b$$

We note that $t(\rho)$ is precisely the Toledo invariant considered in [Poz15, p. 1299] and [BI07a].

V.1.3 Transfer Maps on every Level

For the present section, we essentially follow work by Bucher, Burger and Iozzi [BBI13] and refer to Krifka's master thesis [Kri15] on this paper as a complementary source, given his detailed treatment of the technical components. Although the arguments in [BBI13] are developed in the real-hyperbolic setting and with twisted coefficients in cohomology, they translate verbatim to our complex-hyperbolic case with trivial coefficients, so that no essential changes in the reasoning are required. We give an outline of the machinery developed in [BBI13, §3] for the sake of completeness and to set the ground for the remaining chapter.

But first we give a short preview of this section: Consider the sequence of homomorphisms appearing in the definition of the invariant $I(\cdot)$ in Section IV.2, which is the sequence of left vertical maps of the squares ① and ② in the following diagram:

$$\begin{array}{ccc}
 H_b^{2n}(\Gamma, \mathbb{R}) & \xrightarrow{\mathrm{tr}_{2n}^b} & H_{cb}^{2n}(\mathrm{PU}(n, 1), \mathbb{R}) \\
 \cong \uparrow & & \parallel \\
 H_b^{2n}(M, \mathbb{R}) & & \\
 \cong \downarrow & & \\
 H_b^{2n}(N, \mathbb{R}) & \text{①} & \\
 \cong \uparrow j & & \\
 H_b^{2n}(N, \partial N, \mathbb{R}) & \xrightarrow{\mathrm{tr}_{N, \partial N}^b} & H_{cb}^{2n}(\mathrm{PU}(n, 1), \mathbb{R}) \\
 \downarrow c_{N, \partial N} & \text{②} & \downarrow c \\
 H^{2n}(N, \partial N, \mathbb{R}) & \xrightarrow[\tau_{dR}]{\mathrm{tr}_{N, \partial N}} & H_c^{2n}(\mathrm{PU}(n, 1), \mathbb{R}) \\
 \cong \uparrow & \text{③} & \cong \uparrow \\
 H_{dR}^{2n}(N, \partial N, \mathbb{R}) & \xrightarrow[\cong]{\mathrm{tr}_{dR}} & \Omega^{2n}(\mathbb{H}_{\mathbb{C}}^n)^{\mathrm{PU}(n, 1)}
 \end{array} \tag{V.3}$$

We notice that the top horizontal map is induced by the bounded transfer (V.2). Our aim is to define the remaining horizontal maps individually as transfer maps adapted to the domain

in question and prove the commutativity of each of the squares ①, ② and ③ and ultimately the commutativity of the entire diagram. The constructions of these geometrically flavoured transfer maps are organized in the subsection that follow:

V.1.4 We first construct for unbounded cohomology the *relative transfer map* $\text{tr}_{N,\partial N}$. This naturally restricts to the *relative bounded transfer map* $\text{tr}_{N,\partial N}^b$. The squares ① and ② will then commute by construction.

V.1.5 We then define the *de Rham-theoretical transfer map* tr_{dR} corresponding to the bottom horizontal map of the the diagram. We will see that it will naturally induce τ_{dR} making the square ③ commutes as well.

V.1.6 Finally, it turns out that $\text{tr}_{N,\partial N}$ and τ_{dR} are the same map, which allows to conclude the commutativity of the entire diagram.

This ladder diagram is the core idea when it comes to comparing our definitions $E(\cdot), \text{Vol}(\cdot)$ and $\text{Deg}(\cdot), \text{Tol}(\cdot)$ to transfer-theoretic invariants $e(\cdot)$ and $d(\cdot)$ in the next section.

V.1.4 Transfer maps on cochain level tr_β and tr_β^b

To prove commutativity of ① and ② in the ladder diagram (V.3), let us construct the transfer maps $\text{tr}_{N,\partial N}$ and $\text{tr}_{N,\partial N}^b$ by using suitable cochains (cf. Section V.1.1), as maps

$$S_{(b)}^{2n}(\mathbb{H}_{\mathbb{C}}^n, U)^\Gamma \longrightarrow C_{c(b)}^{2n}(\mathbb{H}_{\mathbb{C}}^n, \mathbb{R})^{\text{PU}(n,1)}.$$

mimicking Definition (V.2). This requires an integration for which the integrand must satisfy sufficient regularity and the simplices over which ones integrates must be smooth. For this reason, we proceed by constructing an auxiliary regularizing map to Borel measurable cochains

$$\beta^* : S_b^{2n}(\mathbb{H}_{\mathbb{C}}^n, U)^\Gamma \longrightarrow C_{Bb}^{2n}(\mathbb{H}_{\mathbb{C}}^n, \mathbb{R})^\Gamma,$$

which will induce the isomorphism $H_b^{2n}(N, \partial N, \mathbb{R}) \cong H_b^{2n}(\Gamma, \mathbb{R})$, and will be part of the following commutative diagram, which induces square ① in (V.3):

$$\begin{array}{ccc} C_{Bb}^{2n}(\mathbb{H}_{\mathbb{C}}^n, \mathbb{R})^\Gamma & \xrightarrow{\text{tr}_{2n}^b} & C_{cb}^{2n}(\mathbb{H}_{\mathbb{C}}^n, \mathbb{R})^{\text{PU}(n,1)} \\ \uparrow \beta^* & & \parallel \\ S_b^{2n}(\mathbb{H}_{\mathbb{C}}^n, U)^\Gamma & \xrightarrow{\text{tr}_b^\beta} & C_{cb}^{2n}(\mathbb{H}_{\mathbb{C}}^n, \mathbb{R})^{\text{PU}(n,1)} \end{array}$$

- where tr_{2n}^b is the map (V.2) in terms of Borel measurable bounded cochains, which actually ranges in continuous bounded cochains,
- and the dashed arrow given by tr_b^β is the induced map from the composition $\text{tr}_{2n}^b \circ \beta^*$.

The construction of the regularizing map β^* involves several steps, following [BBI13, §3.2.4.]. We start by picking a basepoint b_0 in the interior of the compact core N of M . For each $1 \leq i \leq \ell$,

we further pick a basepoint b_i in the cusp region \mathcal{C}_i of M . We first define the collapse map

$$\beta' : M \longrightarrow \{b_0, \dots, b_\ell\}$$

as the measurable map that sends N to b_0 and each cusp \mathcal{C}_i to b_i .

We make a choice of lifts $\tilde{b}_0, \dots, \tilde{b}_\ell$ of b_0, \dots, b_ℓ respectively under the covering map π and define

$$\begin{aligned} \beta : \mathbb{H}_{\mathbb{C}}^n &\longrightarrow \pi^{-1}(\{b_0, b_1, \dots, b_\ell\}) \\ \gamma D_i &\longmapsto \gamma \tilde{b}_i, \end{aligned}$$

where we choose D_0 as a Borel fundamental domain under the Γ -action on $\pi^{-1}(N)$ that contains \tilde{b}_0 ; for each $1 \leq i \leq \ell$, we choose D_i as a Borel fundamental domain under the Γ -action on $\pi^{-1}(\mathcal{C}_i)$ (see Definition I.9). An illustration of the map β can be seen in Figure V.1.

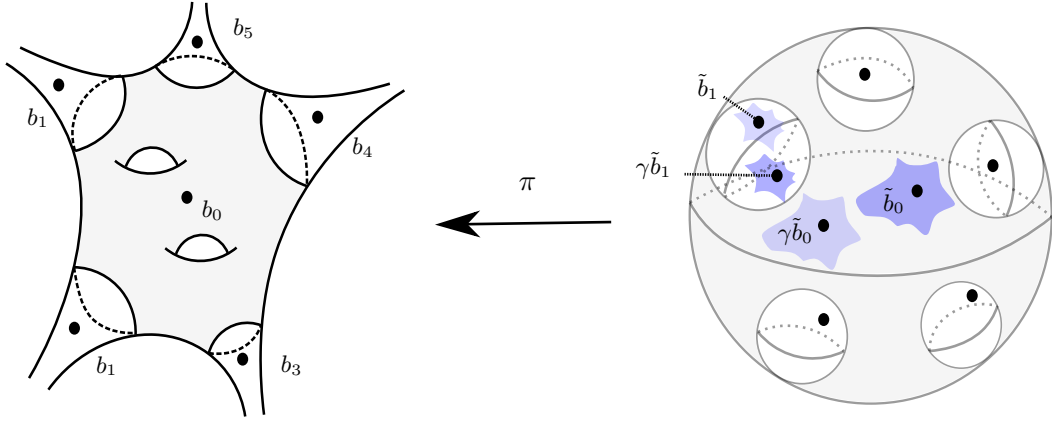


Figure V.1: Illustration of the map β through the ball model of $\mathbb{H}_{\mathbb{C}}^n$ where horoballs are Euclidean balls tangent to the boundary $\partial\mathbb{H}_{\mathbb{C}}^n$.

For any relative singular Γ -invariant cochain $c \in S^{2n}(\mathbb{H}_{\mathbb{C}}^n, U)^\Gamma$, we define the map

$$\begin{aligned} \beta^*(c) : \mathbb{H}_{\mathbb{C}}^{n(2n+1)} &\longrightarrow \mathbb{R}, \\ (x_0, \dots, x_{2n}) &\longmapsto c(\beta(x_0), \dots, \beta(x_{2n})), \end{aligned}$$

which satisfies the following property:

Lemma V.3. *For any cochain $c \in S^{2n}(\mathbb{H}_{\mathbb{C}}^n, U)^\Gamma$, one has that $\beta^*(c) \in C_B^{2n}(\mathbb{H}_{\mathbb{C}}^n)^\Gamma$.*

Proof. By construction, the map $\beta : \mathbb{H}_{\mathbb{C}}^n \rightarrow \Gamma \cdot \{\tilde{b}_0, \dots, \tilde{b}_\ell\}$ is Borel measurable (as it is defined via Borel fundamental domains), so that the induced map

$$\beta^{2n+1} : (x_0, \dots, x_{2n}) \longmapsto (\beta(x_0), \dots, \beta(x_{2n}))$$

is also measurable. Because β takes values in the countable set $\{\gamma \cdot \tilde{b}_i \mid \gamma \in \Gamma, 0 \leq i \leq \ell\} \subset \mathbb{H}_{\mathbb{C}}^n$, the restriction of c to the image $\text{Im}(\beta^{2n+1})$ is clearly measurable. So, the composition $\beta^*(c) = c \circ \beta^{2n+1}$

is measurable. Moreover, due to the Γ -equivariance of β and the Γ -invariance of c , for any $\gamma \in \Gamma$ we have that

$$\begin{aligned} \beta^*(c)(\gamma x_0, \dots, \gamma x_{2n}) &= c(\beta(\gamma x_0), \dots, \beta(\gamma x_{2n})) \\ &= c(\gamma\beta(x_0), \dots, \gamma\beta(x_{2n})) \\ &= c(\beta(x_0), \dots, \beta(x_{2n})) = \beta^*(c)(x_0, \dots, x_{2n}), \end{aligned}$$

so that $\beta^*(c)$ is also Γ -invariant. □

Remark. In our construction, $\beta^*(c)$ is independent of the chosen lift β' , but it does depend on the chosen basepoints b_0, \dots, b_ℓ .

Therefore, we may now integrate the measurable cochain $\beta^*(c) \in C_B^{2n}(\mathbb{H}_\mathbb{C}^n)^\Gamma$ on the fundamental domain of $\Gamma \backslash \text{PU}(n, 1)$:

Proposition V.4. *Let $c \in S^{2n}(\mathbb{H}_\mathbb{C}^n, U)^\Gamma$ and let $\text{tr}_\beta(c) : \mathbb{H}_\mathbb{C}^{n(2n+1)} \rightarrow \mathbb{R}$ be given by*

$$\text{tr}_\beta(c)(x_0, \dots, x_{2n}) := \int_{\mathcal{D}} \beta^*(c)(gx_0, \dots, gx_{2n}) d\mu(g),$$

where \mathcal{D} is defined as in (V.1) and μ is an invariant measure such that $\mu(\mathcal{D}) = 1$. Then $\text{tr}_\beta(c)$ is well-defined as a integral, in the sense that $\beta^*(c)$ is integrable. Moreover $\text{tr}_\beta(c) \in C_c^{2n}(\mathbb{H}_\mathbb{C}^n, \mathbb{R})^{\text{PU}(n,1)}$.

Proof. For any $c \in S^{2n}(\mathbb{H}_\mathbb{C}^n, U)^\Gamma$, in order to prove that $\beta^*(c)$ is integrable, we first fix a tuple $(x_0, \dots, x_{2n}) \in \mathbb{H}_\mathbb{C}^{n(2n+1)}$ and set

$$A := \max_{1 \leq i \leq 2n} \text{dist}(x_0, x_i).$$

Let $N_A \subset M$ be the closed A -neighborhood of the compact core $N \subset M$. Let $g \in \mathcal{D}$ so that $gx_0 \notin \pi^{-1}(N_A)$ and in particular $gx_i \in \pi^{-1}(M \setminus N)$ for all $1 \leq i \leq \ell$. Since β maps each cusp region $\pi^{-1}(\mathcal{C}_i)$ to itself, all tuple entries of $(\beta(gx_0), \dots, \beta(gx_{2n}))$ lie in $U = \pi^{-1}(M \setminus N)$. As c is a relative cochain with respect to U , we have

$$\beta^*(c)(gx_0, \dots, gx_{2n}) = 0,$$

that is, the integrand $\beta^*(c)$ has support in a compact set of \mathcal{D} . Moreover, as a compact set, $\pi^{-1}(N_A) \subset \mathbb{H}_\mathbb{C}^n$ intersects finitely many fundamental domains γD_i . So restricted to $\pi^{-1}(N_A)$, the map β takes values in a finite set, and hence so does the integrand $\beta^*(c)(gx_0, \dots, gx_{2n})$. We conclude that $\beta^*(c)$ is indeed integrable and hence the integral

$$\int_{\mathcal{D}} \beta^*(c)(gx_0, \dots, gx_{2n}) d\mu(g)$$

converges.

Next, to prove that $\text{tr}_\beta(c)$ is continuous, let $(x_0^{(j)}, \dots, x_{2n}^{(j)}) \rightarrow (x_0, \dots, x_{2n})$ be a pointwise convergent sequence in $\mathbb{H}_\mathbb{C}^{n(2n+1)}$ and denote

$$F_j(g) := \beta^*(c)(gx_0^{(j)}, \dots, gx_{2n}^{(j)}), \quad F(g) := \beta^*(c)(gx_0, \dots, gx_{2n}).$$

We now define that set

$$\Omega := \{g \in \mathcal{D} \mid gx_i, gx_i^{(j)} \in \pi^{-1}(M \setminus \partial N) \text{ for all } 0 \leq i \leq 2n \text{ and } j \in \mathbb{N}\}.$$

Since $\partial N \subset M$ is a finite union of codimension 1 submanifolds, it has measure zero, and thus so does its preimage under the projection

$$\mathcal{D} \longrightarrow M, \quad g \longmapsto \pi(gz) \quad \text{for any } z \in \pi^{-1}(\partial N)$$

Hence, Ω has full measure and $F_j(g) \rightarrow F(g)$ pointwise for almost every $g \in \mathcal{D}$.

We now show that $|F_j(\dot{g})|$ is uniformly bounded in j and \dot{g} , which uses a similar strategy as for showing well-definedness above [Kri15, Lemma III.2.8]: we define

$$A := \max \left\{ \sup_j \max_{1 \leq i \leq 2n} \text{dist}(x_0^{(j)}, x_i^{(j)}), \max_{1 \leq i \leq 2n} \text{dist}(x_0, x_i) \right\}.$$

Let $N_A \subset M$ be the closed A -neighborhood of the compact core $N \subset M$. If $gx_0^{(j)} \notin \pi^{-1}(N_A)$, then all $gx_i^{(j)} \in \pi^{-1}(M \setminus N)$, and by construction $\beta^*(c)$ vanishes on such tuples. Therefore, for each j , the support of F_j is contained in a compact subset of \mathcal{D} .

Since β takes only finitely many values, the images $\beta(gx_i^{(j)})$ lie in a finite set $\mathbb{H}_{\mathbb{C}}^n$. Hence, each F_j takes only finitely many values on a compact set, and so there exists a uniform bound C such that $|F_j(g)| \leq C$ for all j and g . By Lebesgue's dominated convergence theorem, it follows that

$$\text{tr}_{\beta}(c)(x_0^{(j)}, \dots, x_{2n}^{(j)}) \longmapsto \text{tr}_{\beta}(c)(x_0, \dots, x_{2n}),$$

so $\text{tr}_{\beta}(c)$ is continuous.

Finally, due to Γ -invariance of c and $\beta^*(c)$, and $\text{PU}(n, 1)$ -invariance of μ , we have that $\text{PU}(n, 1)$ -invariance of $\text{tr}_{\beta}(c)$. This concludes that $\text{tr}_{\beta}(c) \in C_c^{2n}(\mathbb{H}_{\mathbb{C}}^n, \mathbb{R})^{\text{PU}(n, 1)}$. \square

In conclusion, we obtain the cochain map $\text{tr}_{\beta} : S^{2n}(\mathbb{H}_{\mathbb{C}}^n, U)^{\Gamma} \rightarrow C_c^{2n}(\mathbb{H}_{\mathbb{C}}^n, \mathbb{R})^{\text{PU}(n, 1)}$. Moreover, for any bounded relative singular cochain $c \in S_b^{2n}(\mathbb{H}_{\mathbb{C}}^n, U)^{\Gamma}$, then verifies immediately from the proof of Proposition V.4 that the regularization map β^* preserves boundedness and that also $\text{tr}_{\beta}(c)$ is bounded. Hence these maps restrict to

$$\beta^* : S_b^{2n}(\mathbb{H}_{\mathbb{C}}^n, \mathbb{R})^{\Gamma} \longrightarrow C_{Bb}^{2n}(\mathbb{H}_{\mathbb{C}}^n, \mathbb{R})^{\Gamma}, \quad \text{and} \quad \text{tr}_{\beta}^b : S_b^{2n}(\mathbb{H}_{\mathbb{C}}^n, U)^{\Gamma} \longrightarrow C_{cb}^{2n}(\mathbb{H}_{\mathbb{C}}^n, \mathbb{R})^{\text{PU}(n, 1)}.$$

Since both maps tr_{β} and its restriction tr_{β}^b commute with coboundary operators, they induce the corresponding cohomological maps

$$\text{tr}_{N, \partial N} : H^{2n}(N, \partial N, \mathbb{R}) \rightarrow H_c^{2n}(\text{PU}(n, 1), \mathbb{R}) \quad \text{and} \quad \text{tr}_{N, \partial N}^b : H_b^{2n}(N, \partial N, \mathbb{R}) \rightarrow H_{cb}^{2n}(\text{PU}(n, 1), \mathbb{R}).$$

Therefore, since $\text{tr}_{N, \partial N}^b = \text{tr}_{2n}^b \circ \beta^*$, the square ① commutes and by construction since $c_{N, \partial N} \circ \text{tr}_{N, \partial N}^b = \text{tr}_{N, \partial N} \circ c_{\mathbb{R}}$, then ② commutes as well.

V.1.5 Transfer Maps tr_{dR} and τ_{dR}

We briefly explain now commutativity of the square (3). We now introduce a transfer map on the level of invariant differential forms. We define the map

$$\text{tr}_{dR} : \Omega^{2n}(\mathbb{H}_{\mathbb{C}}^n, U)^\Gamma \longrightarrow \Omega^{2n}(\mathbb{H}_{\mathbb{C}}^n)^{\text{PU}(n,1)},$$

by

$$\text{tr}_{dR}(\alpha)(v_1, \dots, v_{2n}) := \int_{\mathcal{D}} \alpha(gv_1, \dots, gv_{2n}) d\mu(g),$$

where again \mathcal{D} is as in (V.1) with μ an invariant measure such that $\mu(\mathcal{D}) = 1$ and $v_1, \dots, v_{2n} \in T_p \mathbb{H}_{\mathbb{C}}^n$ for an arbitrary but fixed basepoint $p \in \mathbb{H}_{\mathbb{C}}^n$. Importantly, it enjoys the following property:

Lemma V.5. *The cochain map tr_{dR} induces an isomorphism*

$$\text{tr}_{dR} : H_{dR}^{2n}(M, M \setminus N) \longrightarrow \Omega^{2n}(\mathbb{H}_{\mathbb{C}}^n)^{\text{PU}(n,1)},$$

which satisfies:

$$\text{tr}_{dR}(\text{vol}_{N, \partial N}) = \text{vol}_{\mathbb{H}_{\mathbb{C}}^n}, \quad \text{and} \quad \text{tr}_{dR}(\text{eu}_{N, \partial N}) = h_n \cdot \text{vol}_{\mathbb{H}_{\mathbb{C}}^n},$$

where $h_n = (-1)^n \frac{\chi(\mathbb{C}\mathbb{P}^n)}{\text{Vol}(\mathbb{C}\mathbb{P}^n)}$ is the Hirzebruch proportionality constant and $\text{vol}_{N, \partial N}, \text{eu}_{N, \partial N} \in H_{dR}^{2n}(M, M \setminus N)$ are unique classes such that

$$\langle \text{vol}_{N, \partial N}, [N, \partial N] \rangle = \text{Vol}(M) \quad \text{and} \quad \langle \text{eu}_{N, \partial N}, [N, \partial N] \rangle = \chi(M).$$

Proof. A straightforward calculation shows that the the cochain map tr_{dR} clearly commutes with the differential coboundary operator and hence induces the desired cohomology map. Detailed calculations for a proof of the second statement can be followed in [Kri15, Proposition III.2.5]. \square

Moreover, the cochain map tr_{dR} can be considered as the bottom map of the following diagram

$$\begin{array}{ccc} S^{2n}(\mathbb{H}_{\mathbb{C}}^n, U)^\Gamma & \overset{\tau_{dR}}{\dashrightarrow} & C_c^{2n}(\mathbb{H}_{\mathbb{C}}^n, \mathbb{R})^{\text{PU}(n,1)} \\ \Xi \uparrow \cong & & \Theta \uparrow \cong \\ \Omega^{2n}(\mathbb{H}_{\mathbb{C}}^n, U)^\Gamma & \xrightarrow{\text{tr}_{dR}} & \Omega^{2n}(\mathbb{H}_{\mathbb{C}}^n)^{\text{PU}(n,1)} \end{array}$$

where Θ and Ξ are cochain maps that induce the Van-Est and de-Rham isomorphisms respectively, and the dashed arrow for τ_{dR} is the induced map. In particular, the Van-Est isomorphism is realized on the cochain-level by the map

$$\Theta : \Omega^{2n}(\mathbb{H}_{\mathbb{C}}^n)^{\text{PU}(n,1)} \longrightarrow C_B^{2n}(\mathbb{H}_{\mathbb{C}}^n, \mathbb{R})^{\text{PU}(n,1)},$$

given by

$$\Theta(\alpha) = \int_{\text{str}[x_0, \dots, x_{2n}]} \alpha,$$

where we evaluate the differential form on a straightened simplices in the sense of Definition

II.5. In the same spirit, the de Rham isomorphism is realized on the cochain-level by the map

$$\Xi : \Omega^{2n}(\mathbb{H}_{\mathbb{C}}^n, U)^{\Gamma} \rightarrow S^{2n}(\mathbb{H}_{\mathbb{C}}^n, U)^{\Gamma}$$

given by

$$\Xi(\alpha)(\sigma) = \int_{\pi_*(\text{str}[x_0, \dots, x_{2n}])} \alpha,$$

where σ is a singular simplex and x_i are some lifts of σ 's vertices under the universal projection π . We notice that this map is well-defined, since whenever σ is contained in U , then so is its straightened simplex, since all horoballs in U are convex by Proposition I.37.

V.1.6 τ_{dR} and $\text{tr}_{N, \partial N}$ are Identical

In order to conclude the commutativity of the ladder diagram (V.3), it suffices to observe:

Lemma V.6. *For any cocycle $c \in S^{2n}(\mathbb{H}_{\mathbb{C}}^n, U)^{\Gamma}$, the cochains $\tau_{dR}(c)$ and $\text{tr}_{\beta}(c)$ in $C_c^{2n}(\mathbb{H}_{\mathbb{C}}^n, \mathbb{R})^{\text{PU}(n,1)}$ differ by a bounded coboundary. In particular the induced maps of τ_{dR} and tr_{β} in cohomology coincide.*

Sketch of the proof. Since Ξ induces an isomorphism in cohomology and one has that $\Theta \circ \text{tr}_{dR} = \tau_{dR} \circ \Xi$, it suffices to prove that $\Theta \circ \text{tr}_{dR} = \text{tr}_{\beta} \circ \Xi$. For this we fix $\alpha \in \Omega^{2n}(\mathbb{H}_{\mathbb{C}}^n)^{\text{PU}(n,1)}$. Using the corresponding definitions, it is straightforward to calculate:

$$\begin{aligned} \text{tr}_{\beta} \circ \Xi(\alpha)(x_0, \dots, x_{2n}) &= \int_{\mathcal{D}} \Xi(\alpha)(\beta(gx_0), \dots, \beta(gx_{2n})) d\mu(g) \\ &= \int_{\mathcal{D}} \left(\int_{\text{str}[\beta(gx_0), \dots, \beta(gx_{2n})]} \alpha \right) d\mu(g) \end{aligned}$$

as well as

$$\Theta \circ \text{tr}_{dR}(\alpha)(x_0, \dots, x_q) = \int_{\mathcal{D}} \left(\int_{\text{str}[gx_0, \dots, gx_{2n}] } \alpha \right) d\mu(g).$$

If now α is a cocycle representing a cohomology class, i.e. $d\alpha = 0$, then the following \mathbb{R} -valued function defined for any tuple $(x_0, \dots, x_{2n}) \in \mathbb{H}_{\mathbb{C}}^{n(2n+1)}$ and given by

$$f(x_0, \dots, x_{2n-1}) = \sum_{i=0}^{2n-1} (-1)^i \int_{\mathcal{D}} \left(\int_{\text{str}[\beta(gx_0), \dots, \beta(gx_i), gx_i, \dots, gx_{2n-1}]} \alpha \right) d\mu(g)$$

is a cochain in $C^{2n}(\mathbb{H}_{\mathbb{C}}^n, \mathbb{R})^{\text{PU}(n,1)}$ and moreover its coboundary df satisfies

$$df = \Theta \circ \text{tr}_{dR}(\alpha) - \text{tr}_{\beta} \circ \Xi(\alpha).$$

A long but straightforward calculation to check this can be found in [Kri15, Lemma III.2.9]. \square

V.2 Equivalence of Definitions

We now prove a simple lemma in the general setting where the bounded transfer map (V.2) was defined. Combined with the big diagram (V.3), this will allow us to relate the the transfer theoretic definition $d(\cdot)$ to those using Poincaré duality $\text{Deg}(\cdot)$ and $\text{Tol}(\cdot)$ in the previous chapter.

Lemma V.7. *Let $k, q \in \mathbb{N}$. Let L be a locally compact group that admits a non-uniform torsion-free lattice Γ , denote the canonical inclusion by $i : \Gamma \hookrightarrow L$ and consider its induced map in continuous bounded cohomology*

$$i^* : H_{cb}^q(L, \mathbb{R}) \longrightarrow H_b^q(\Gamma, \mathbb{R}).$$

Then for arbitrary $\alpha \in H_b^k(\Gamma, \mathbb{R})$, $\beta \in H_{cb}^q(L, \mathbb{R})$, it holds that

$$\mathrm{tr}_{k+q}^b(\alpha \cup i^*(\beta)) = \mathrm{tr}_k^b(\alpha) \cup \beta$$

Proof. We recall that the bounded cochain complexes $C_b^k(L, \mathbb{R})^\Gamma$ and $C_b^q(L, \mathbb{R})^L$ realize the bounded cohomologies $H_b^k(\Gamma, \mathbb{R})$ and $H_{cb}^q(L, \mathbb{R})$ respectively (see [Mon01, Proposition 8.6.2]), and we consider the cocycles $\alpha \in C_b^k(L, \mathbb{R})^\Gamma$ and $\beta \in C_b^q(L, \mathbb{R})^L$.

By definition of transfer maps and cup products on the cochain-level, we have that

$$\mathrm{tr}_{k+q}^b(\alpha \cup i^*(\beta))(x_0, \dots, x_{k+q}) = \int_{\mathcal{D}} \alpha(gx_0, \dots, gx_k) \cdot i^*(\beta)(gx_k, \dots, gx_{k+q}) d\mu(g). \quad (\text{V.4})$$

It is a straightforward observation that $i : \Gamma \hookrightarrow L$ induces on cochain level the inclusion

$$i^* : C_b^q(L, \mathbb{R})^L \subset C_b^q(L, \mathbb{R})^\Gamma,$$

as any L -invariant map is Γ -invariant, so that $i^*(\beta) = \beta$. Then for every term of the integral in (V.4), due to L -invariance of β , we observe that

$$i^*(\beta)(gx_k, \dots, gx_{k+q}) = \beta(x_k, \dots, x_{k+q}).$$

Finally, we obtain that

$$\begin{aligned} \mathrm{tr}_{k+q}^b(\alpha \cup i^*(\beta))(x_0, \dots, x_{k+q}) &= \int_{\mathcal{D}} \alpha(gx_0, \dots, gx_k) \cdot \beta(x_k, \dots, x_{k+q}) d\mu(g) \\ &= \int_{\mathcal{D}} \alpha(gx_0, \dots, gx_k) d\mu(g) \cdot \beta(x_k, \dots, x_{k+q}) = \mathrm{tr}_k^b(\alpha) \cup \beta, \end{aligned}$$

which concludes the proof. □

We are now in position to prove the following theorem, where (ii) proves the **Main Lemma** from the introduction:

Theorem V.8. *Let $n \geq 1$ and let $\Gamma < \mathrm{PU}(n, 1)$ be a torsion-free lattice and G a simple non-compact Hermitian Lie group with finite center. Then we have that*

(i) for any representation $\rho : \Gamma \rightarrow \mathrm{PU}(n, 1)$ it holds that

$$e(\rho) = \frac{E(\rho)}{\chi(M)} = \frac{\mathrm{Vol}(\rho)}{\mathrm{Vol}(M)}.$$

(ii) for any representation $\rho: \Gamma \rightarrow G$ it holds that

$$d(\rho) = \frac{1}{(n+1)^{n-1}} \cdot \frac{\text{Deg}(\rho)}{\chi(M)} \quad \text{and} \quad t(\rho) = \frac{1}{n!} \cdot \frac{\text{Tol}(\rho)}{\text{Vol}(M)}.$$

In particular, it holds that $t(\rho) = -\frac{\bar{\lambda} \cdot (n+1)}{4\pi} \cdot d(\rho)$, where $\bar{\lambda} \in \mathbb{R} \setminus \{0\}$ is such that $\bar{\omega}_2 = \bar{\lambda} \cdot \bar{\mathcal{C}}_1$.

Proof of Theorem V.8. In both cases, the proof relies on the commutativity of the ladder diagram (V.3).

Ad (i): Consider $\rho^*(\mathcal{C}_n^b) \in H_b^{2n}(\Gamma, \mathbb{R})$ in the top left corner of the ladder diagram (V.3). On the one hand, taking this class under the horizontal tr_{2n}^b and then mapping it vertically under the comparison map $c_{\mathbb{R}}$, we get by Definition V.1 the class $e(\rho) \cdot \mathcal{C}_n$. Due to Lemma V.5, we have that

$$e(\rho) \cdot \mathcal{C}_n = e(\rho) \cdot \tau_{dR}(\text{eu}_{N, \partial N}) \in H_c^{2n}(\text{PU}(n, 1), \mathbb{R}). \quad (\text{V.5})$$

On the other hand, if we take $\rho^*(\mathcal{C}_n^b)$ vertically downwards in the ladder diagram (V.3), we obtain the class

$$\kappa \cdot \text{eu}_{N, \partial N} \in H^{2n}(N, \partial N, \mathbb{R}),$$

that evaluates via the relative Poincaré duality to $\chi(\rho) = \kappa \cdot \chi(M)$, for some $\kappa \in \mathbb{R} \setminus \{0\}$. By commutativity of the ladder diagram (V.3) and the fact that τ_{dR} is an isomorphism, we see that

$$\tau_{dR}(\kappa \cdot \text{eu}_{N, \partial N}) = \kappa \cdot \tau_{dR}(\text{eu}_{N, \partial N}) = \kappa \cdot \mathcal{C}_n$$

and hence using (V.5) we deduce that $e(\rho) = \frac{E(\rho)}{\chi(M)}$. The Hirzebruch proportionality relation in Theorem III.12 immediatly yields $e(\rho) = \frac{\text{Vol}(\rho)}{\text{Vol}(M)}$.

Ad (ii): We consider now in the top left corner of the ladder diagram (V.3) the class

$$\rho^*(\bar{\mathcal{C}}_1^b) \cup i^*(\mathcal{C}_1^b)^{n-1} \in H_b^{2n}(\Gamma, \mathbb{R})$$

and take it horizontally under the map tr_{2n}^b . By applying Lemma V.7 iteratively, we get that

$$\text{tr}_{2n}^b \left(\rho^*(\bar{\mathcal{C}}_1^b) \cup i^*(\mathcal{C}_1^b)^{n-1} \right) = \text{tr}_2^b \circ \rho^*(\bar{\mathcal{C}}_1^b) \cup (\mathcal{C}_1^b)^{n-1}.$$

Since by Definition V.2 we have that

$$\text{tr}_2^b \circ \rho^*(\bar{\mathcal{C}}_1^b) = d(\rho) \cdot \mathcal{C}_1^b,$$

this implies that

$$\text{tr}_2^b \circ \rho^*(\bar{\mathcal{C}}_1^b) \cup (\mathcal{C}_1^b)^{n-1} = d(\rho) \cdot (\mathcal{C}_1^b)^n.$$

In the ladder diagram (V.3) we apply vertically the map $c_{\mathbb{R}}: H_{cb}^{2n}(\text{PU}(n, 1), \mathbb{R}) \rightarrow H_c^{2n}(\text{PU}(n, 1), \mathbb{R})$. By naturality of cup products, the comparison map $c_{\mathbb{R}}$ being an isomorphism in degree 2, as well as the fact that $(n+1)^{n-1} \cdot \mathcal{C}_n = \mathcal{C}_1^n$ (see Proposition III.21), we obtain:

$$c_{\mathbb{R}} \left(\text{tr}_2^b \circ \rho^*(\bar{\mathcal{C}}_1^b) \cup (\mathcal{C}_1^b)^{n-1} \right) = d(\rho) \cdot (n+1)^{n-1} \cdot \mathcal{C}_n.$$

By Lemma V.5, we have that

$$d(\rho) \cdot (n+1)^{n-1} \cdot \mathbf{C}_n = d(\rho) \cdot (n+1)^{n-1} \cdot \tau_{dR}(\text{eu}_{N,\partial N}) \in H_c^{2n}(\text{PU}(n,1), \mathbb{R}). \quad (\text{V.6})$$

On the other hand, analogous to the previous case, taking $\rho^*(\overline{\mathbf{C}}_1^b) \cup i^*(\mathbf{C}_1^b)^{n-1}$ down the left vertical maps of the ladder diagram (V.3), we have that

$$\kappa' \cdot \text{eu}_{N,\partial N} \in H^{2n}(N, \partial N, \mathbb{R}),$$

which via relative Poincaré duality evaluates to $\text{Tol}(\rho) = \kappa' \cdot \text{Vol}(M)$ for some $\kappa' \in \mathbb{R}$. Therefore commutativity of the ladder diagram (V.3) and by (V.6) implies that

$$\tau_{dR}(\kappa' \cdot \text{eu}_{N,\partial N}) = \kappa' \cdot \tau_{dR}(\text{eu}_{N,\partial N}) = d(\rho) \cdot (n+1)^{n-1} \cdot \mathbf{C}_n$$

so that $\frac{\text{Deg}(\rho)}{\chi(M)} = d(\rho) \cdot (n+1)^{n-1}$. Now, with a similar diagram chase for the class $\rho^*(\overline{\omega}_2^b) \cup i^*(\omega_2^b)^{n-1}$, and an application of Wirtinger's formula $n! \cdot \omega_{2n} = \omega_2^n$, we get on the one hand that

$$t(\rho) \cdot n! \cdot \omega_{2n} = t(\rho) \cdot n! \cdot \tau_{dR}(\text{vol}_{N,\partial N}),$$

while on the other hand, that $\text{Tol}(\rho) = \kappa'' \cdot \text{Vol}(M)$ and $\tau_{dR}(\kappa'' \cdot \text{vol}_{N,\partial N}) = t(\rho) \cdot n! \cdot \omega_{2n}$. Therefore, analogously one concludes that $\frac{\text{Tol}(\rho)}{\text{Vol}(M)} = n! \cdot t(\rho)$. Finally, we may apply Hirzebruch's proportionality principle from Theorem III.21 to $\text{Vol}(M)$ as well as the relation

$$\text{Tol}(\rho) = \bar{\lambda} \cdot \left(\frac{-4\pi}{n+1} \right)^{n-1} \cdot \text{Deg}(\rho)$$

to compute

$$t(\rho) = \frac{1}{n!} \cdot \frac{\text{Tol}(\rho)}{\text{Vol}(M)} = \frac{\bar{\lambda} \cdot (-4\pi)^{n-1} \cdot (n+1)!}{n! \cdot (n+1)^{n-1} \cdot (-4\pi)^n} \cdot \frac{\text{Deg}(\rho)}{\chi(M)} = -\frac{\bar{\lambda} \cdot (n+1)}{4\pi} \cdot \frac{1}{(n+1)^{n-1}} \cdot \frac{\text{Deg}(\rho)}{\chi(M)}.$$

Due to the expression of $d(\rho)$ established previously, we immediately conclude the relation $t(\rho) = -\frac{\bar{\lambda} \cdot (n+1)}{4\pi} \cdot d(\rho)$. \square

Remark. We note that if $G = \text{PU}(n,1)$, then the invariants $t(\cdot)$ and $d(\cdot)$ coincide. From the proof, we see that for the canonical inclusion $i : \Gamma \hookrightarrow \text{PU}(n,1)$, it holds that $E(i) = \chi(M)$ and in particular, $e(i) = 1$. Furthermore, we also have:

$$\text{Deg}(i) = (n+1)^{n-1} \cdot \chi(M), \quad \text{Tol}(i) = n! \cdot \text{Vol}(M),$$

so that $d(i) = t(i) = 1$ as well. In fact, these values are extremal: Considering the supremum norm of $e(\rho)$, we have:

$$|e(\rho)| = \frac{\|\text{tr}_{2n}^b \circ \rho^*(\mathbf{C}_n^b)\|_\infty}{\|\mathbf{C}_n^b\|_\infty},$$

and since the maps involved are norm-increasing, it follows that $|e(\rho)| \leq 1$ (cf. [BBI13, Proposition 3.3]). Similarly, we obtain the bound $|d(\rho)| = |t(\rho)| \leq 1$. This recovers the Milnor-Wood type inequalities for both invariants from [KK14] and [BI07b]. More generally, for any represen-

tation $\rho : \Gamma \rightarrow G$ into a semisimple non-compact Hermitian Lie group with finite center, one has that

$$|\mathfrak{t}(\rho)| \leq \text{rk}(X),$$

where X is the symmetric space associated to G (cf. [BI07b, Lemma 5.1]).

Burger and Iozzi prove in [BI07b] that their L_2 -cohomological invariant in formula [BI07b, (1.2)] is equivalent to the transfer-theoretic definition $\mathfrak{t}(\cdot)$ in [BI07b, Lemma 5.3]. This in turn is a normalization of our Toledo invariant $\text{Tol}(\cdot)$, as stated in our formula in Theorem V.8. On the other hand, since then L_2 -cohomology on compact manifolds reduces to de Rham cohomology, one obtains immediately the differential-geometric formula for $\text{Tol}(\cdot)$ for representations of *uniform* torsion-free lattices (cf. [HO11, Proposition 4.1]). In particular this proves the following corollary:

Corollary V.9. *Let $\Gamma < \text{PU}(n, 1)$ be uniform torsion-free lattice and let $M = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$. Further let G be a simple non-compact Hermitian Lie group with finite center and X the associated Hermitian symmetric space. Let $\rho : \Gamma \rightarrow G$ be a representation and $f : \mathbb{H}_{\mathbb{C}}^n \rightarrow X$ a smooth ρ -equivariant map and consider the induced pullback map*

$$f^* : \Omega^2(X)^G \longrightarrow \Omega^2(\mathbb{H}_{\mathbb{C}}^n)^{\Gamma} \cong \Omega^2(M).$$

Then it holds that

$$\text{Tol}(\rho) = \int_M f^* \omega_X \wedge \omega_M^{n-1}, \quad (\text{V.7})$$

where $\omega_X \in \Omega^2(X)^G$ and $\omega_M \in \Omega^2(M)$ denote the respective Bergman-Kähler forms.

Alternatively to the previous paragraph, we provide the following more direct proof.

Proof. Let $\Gamma < \text{PU}(n, 1)$ be a uniform torsion-free lattice and $M = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$. We recall that the Van-Est isomorphism identifies continuous cohomology with invariant differential forms: $H_c^2(G, \mathbb{R}) \cong \Omega^2(X)^G$, and similarly for $\text{PU}(n, 1)$.

By Definition II.18, the continuous Kähler class $\bar{\omega}_2 \in H_c^2(G, \mathbb{R})$ corresponds to the invariant Kähler form ω_X . Since $f : \mathbb{H}_{\mathbb{C}}^n \rightarrow X$ is ρ -equivariant, the form $f^* \omega_X$ is Γ -invariant and closed, hence descends to a closed 2-form on M . By naturality of the Van-Est isomorphism with respect to group homomorphisms and equivariant maps, the form $f^* \omega_X$ represents the cohomology class $\rho^*(\bar{\omega}_2) \in H^2(\Gamma, \mathbb{R}) \cong H_{dR}^2(M)$. Applying the same argument to the inclusion $i : \Gamma \hookrightarrow \text{PU}(n, 1)$, the class $i^*(\bar{\omega}_2)$ is represented by the Kähler form ω_M .

Since the Van-Est isomorphism is multiplicative and cup products correspond to wedge products under the de Rham isomorphism, pairing with the fundamental class of M yields

$$\text{Tol}(\rho) = \langle \rho^*(\bar{\omega}_2) \cup i^*(\bar{\omega}_2)^{n-1}, [M] \rangle = \int_M f^* \omega_X \wedge \omega_M^{n-1}.$$

□

Moreover, up to a sign, our invariant $\text{Deg}(\cdot)$ coincides with the algebro-geometric degree on *uniform* torsion-free lattices as defined in [KM10, p. 212]. This identification relies on work by Hartnick and Ott in [HO12] and Proposition III.18 on page 84:

Proposition V.10. *Let $\Gamma < \text{PU}(n, 1)$ be uniform torsion-free lattice and let $M = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$. Further let be G a simple non-compact Hermitian Lie group with finite center and X the associated Hermitian symmetric space. Let $\rho : \Gamma \rightarrow G$ be a representation and $f : \mathbb{H}_{\mathbb{C}}^n \rightarrow X$ a smooth ρ -equivariant map. Then it holds that*

$$\text{Deg}(\rho) = (-1)^n \cdot \int_M f^* c_1(K_X) \wedge c_1(K_M)^{n-1},$$

where K_X and K_M denote the holomorphic canonical line bundles. Here, $c_1(K_X)$ denotes the G -invariant Chern-Weil curvature form of the Chern connection on K_X , associated to the canonical Hermitian metric on X ; similar for $c_1(K_M)$. The form $f^* c_1(K_X)$ is Γ -invariant on $\mathbb{H}_{\mathbb{C}}^n$ and hence descends to a closed 2-form on M .

Proof. We first determine the image of $\overline{\mathcal{C}}_1$ under the Van-Est isomorphism $H_c^2(G, \mathbb{R}) \cong \Omega^2(X)^G$. For this we follow [HO12]: Let $X = G/K$ be the Hermitian symmetric space associated to G and $X_u = G_u/K$ its compact dual. We write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ for the corresponding Cartan decompositions. The linear map $\iota : \mathfrak{p} \rightarrow i\mathfrak{p}$, $X \mapsto iX$ induces an isomorphism

$$\iota^* : \left(\bigwedge^2 (i\mathfrak{p})^* \right)^K \longrightarrow \left(\bigwedge^2 \mathfrak{p}^* \right)^K$$

such that $\iota^*(\alpha)(X_1, X_2) = \alpha(iX_1, iX_2)$. Using the standard identifications obtained by restricting invariant forms to the base point eK ,

$$\Omega^2(X)^G \cong \left(\bigwedge^2 \mathfrak{p}^* \right)^K \quad \text{and} \quad \Omega^2(X_u)^{G_u} \cong \left(\bigwedge^2 (i\mathfrak{p})^* \right)^K,$$

the map ι^* yields (by abuse of notation) an isomorphism $\iota^* : \Omega^2(X_u)^{G_u} \rightarrow \Omega^2(X)^G$ (cf. [HO12, pp. 2074–2075]).

Let now F_A and F_{A_u} be curvature 2-forms of the connection 1-forms A and A_u on the principal K -bundles $G \rightarrow X$ and $G_u \rightarrow X_u$ respectively, where A and A_u are obtained by projecting the Maurer-Cartan 1-forms on the \mathfrak{k} -factor of the Cartan decompositions (cf. [HO12, p.2080]). For any K -invariant symmetric function $P \in I^2(\mathfrak{k}^*)$, [HO12, Claim on p. 2081] shows that

$$\iota^*(P(F_{A_u}, F_{A_u})) = -P(F_A, F_A).$$

Now choose $P(X) = \frac{i}{2\pi} \text{Tr}(X)$. By Chern-Weil theory (cf. [HO12, §3.1]), the invariant form

$$P(F_{A_u}, F_{A_u}) \in \Omega^2(X_u)^{G_u}$$

represents the first Chern class $c_1(TX_u) \in H_{dR}^2(X_u, \mathbb{R})$. By Proposition III.18, under the isomorphism $H^2(X_u, \mathbb{R}) \cong H_c^2(G, \mathbb{R})$ the class $c_1(TX_u)$ corresponds $-\overline{\mathcal{C}}_1$. Since the following diagram commutes up to sign

$$\begin{array}{ccc} \Omega^2(X)^G & \xrightarrow{\cong} & H_c^2(G, \mathbb{R}) \\ \iota^* \uparrow \cong & & \cong \uparrow \\ \Omega^2(X_u)^{G_u} & \xrightarrow{\cong} & H_c^2(X_u, \mathbb{R}) \end{array},$$

we conclude that the top horizontal map takes the closed form $c_1(TX) := P(F_A, F_A)$ to our first

Chern class $\overline{\mathbf{C}}_1$.

Let now $\Gamma < \mathrm{PU}(n, 1)$ be a uniform torsion-free lattice, $M = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$, and let $\rho : \Gamma \rightarrow G$ be a representation with associated ρ -equivariant smooth map $f : \mathbb{H}_{\mathbb{C}}^n \rightarrow X$. By naturality of the Van–Est isomorphism with respect to group homomorphisms and equivariant maps, the closed 2-form

$$f^* c_1(TX) = -f^* c_1(K_X)$$

represents the class $\rho^*(\overline{\mathbf{C}}_1) \in H^2(\Gamma, \mathbb{R}) \cong H_{dR}^2(M)$. Similarly, for the inclusion $i : \Gamma \hookrightarrow \mathrm{PU}(n, 1)$, $i^*(\mathbf{C}_1)$ is represented by $-c_1(K_M)$.

Using the compatibility of cup products with wedge products under the de Rham isomorphism and pairing with the fundamental class $[M]$, we obtain

$$\mathrm{Deg}(\rho) = \langle \rho^*(\overline{\mathbf{C}}_1) \cup i^*(\mathbf{C}_1)^{n-1}, [M] \rangle = (-1)^n \int_M c_1(f^* K_X) \wedge c_1(K_M)^{n-1},$$

which concludes the proof. □

Finally, we determine explicitly the constant relating our Toledo invariant $\mathrm{Tol}(\cdot)$ and the degree $\mathrm{Deg}(\cdot)$:

Proposition V.11. *Let $\Gamma < \mathrm{PU}(n, 1)$ be a torsion-free lattice and G a simple non-compact Hermitian Lie group. Then for any representation $\rho : \Gamma \rightarrow G$, it holds that*

$$\mathrm{Tol}(\rho) = (-1)^n \cdot \frac{2^{2n} \cdot \pi^n}{c_{G/K} \cdot (n+1)^{n-1}} \cdot \mathrm{Deg}(\rho),$$

where, if G is a classical¹ simple non-compact Hermitian Lie group, the constant $c_{G/K} \in \mathbb{R}$ is given by Table V.2:

$c_{G/K}$	G	K
$p + q$	$\mathrm{SU}(p, q)$	$\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$
p	$\mathrm{SO}_0(p, 2)$	$\mathrm{SO}(p) \times \mathrm{SO}(2)$
$n + 1$	$\mathrm{Sp}(n, \mathbb{R})$	$\mathrm{U}(n)$
$2(n - 1)$	$\mathrm{SO}^*(2n)$	$\mathrm{U}(n)$

Table V.2: Values of $c_{G/K}$ for classical simple non-compact Hermitian Lie groups with finite center, up to finite cover.

Proof. Under the Van–Est isomorphism $H_c^2(G, \mathbb{R}) \cong \Omega^2(X)^G$, the bounded Kähler class $\overline{\omega}_2$ corresponds to the G -invariant Kähler form ω_X on X , while the real first Chern class $\overline{\mathbf{C}}_1$ corresponds to the Chern–Weil representative of $c_1(TX)$ (cf. the proof of Proposition V.10). As before, we employ the same notation for cohomology classes and their chosen invariant differential-form representatives. Consequently, the relation $\overline{\omega}_2 = \overline{\lambda} \cdot \overline{\mathbf{C}}_1$ in $H_c^2(G, \mathbb{R})$ with $\overline{\lambda} \in \mathbb{R}$ (cf. Definition IV.11)

¹Here *classical* refers to simple non-compact Hermitian Lie groups whose complexified Lie algebra is of classical type in the Cartan–Killing classification. The only simple non-compact Hermitian Lie groups not of classical type are the exceptional groups $E_{6(-14)}$ and $E_{7(-25)}$.

corresponds, under the Van–Est isomorphism, to

$$\omega_X = \bar{\lambda} \cdot c_1(TX)$$

at the level of invariant differential forms². Koziarz and Maubon prove in [KM10, pp. 212–217] that in fact

$$\bar{\lambda} = -\frac{4\pi}{c_{G/K}},$$

where the values for $c_{G/K}$ are deduced in [KM10, p. 216] and given in Table V.2. We note that in case $G = \text{PU}(n, 1)$, we have previously computed that $\bar{\lambda} = \lambda = -\frac{4\pi}{n+1}$ in Proposition III.21. Applying Definitions IV.4 and IV.11, then gives that

$$\text{Tol}(\rho) = \bar{\lambda} \cdot \lambda^{n-1} \cdot \text{Deg}(\rho),$$

which yields the statement. □

V.3 Continuity

We build on Subsection I.1.3 and recall, for any finitely generated group Γ and Lie group G , that the set $\text{Hom}(\Gamma, G)$ can be endowed with the compact-open topology. On this topological space, we establish continuity of the invariants $e(\cdot)$ and $d(\cdot)$ in their respective settings. This, in turn, implies continuity of the related invariants in Theorem V.8.

Theorem V.12. *Let $n \geq 1$, let $\Gamma < \text{PU}(n, 1)$ be a non-uniform torsion-free lattice and G a simple non-compact Hermitian Lie group with finite center. Then the functions*

$$\begin{array}{ccc} \text{Hom}(\Gamma, \text{PU}(n, 1)) & \longrightarrow & \mathbb{R}, & \text{Hom}(\Gamma, G) & \longrightarrow & \mathbb{R} \\ \rho & \longmapsto & e(\rho), & \rho & \longmapsto & d(\rho) \end{array}$$

are continuous.

Proof. The strategy is to express $e(\cdot)$ and $d(\cdot)$ as evaluations of cohomology classes obtained via continuous maps at the cochain level, verifying continuity at each step by considering appropriate topologies.

For $\Gamma < \text{PU}(n, 1)$ a non-uniform torsion-free lattice and G a simple non-compact Hermitian Lie group with finite center, we consider continuous bounded cocycles

$$\varphi_n \in C_{cb}^{2n}(\text{PU}(n, 1), \mathbb{R})^{\text{PU}(n, 1)}, \quad \text{and} \quad \varphi_1 \in C_{cb}^2(\text{PU}(n, 1), \mathbb{R})^{\text{PU}(n, 1)}, \quad \bar{\varphi}_1 \in C_{cb}^2(G, \mathbb{R})^G,$$

that represent our choice of real bounded top Chern class \mathcal{C}_n^b as well as the bounded real first Chern classes \mathcal{C}_1^b and $\bar{\mathcal{C}}_1^b$ respectively. In particular for $\bar{\varphi}_2$ we could more explicitly take the multiple $\frac{1}{\lambda} \cdot \text{Dup}_G$ of the Dupont cocycle (cf. Subsection II.4). Let $k \geq 0$ be an integer. We equip the following sets with suitable topologies:

- $\text{Hom}(\Gamma, G)$ with compact-open topology (cf. Subsection I.1.3),

²We remark that under the isomorphism $(\iota^*)^{-1}$ (see proof of Proposition V.10), this then corresponds to the relation $\omega_{X_u} = -\bar{\lambda} \cdot c_1(TX_u)$.

- $C_{cb}^k(G, \mathbb{R})^G$ with the topology of uniform convergence,
- $C_{cb}^k(\Gamma, \mathbb{R})^\Gamma$ with the topology of pointwise convergence with control of norms, i.e. $\alpha_j \rightarrow \alpha$ converges pointwise and $\sup_j \|\alpha_j\|_\infty < \infty$.

With these choices the following pullback map is continuous

$$\begin{aligned} \text{Hom}(\Gamma, \text{PU}(n, 1)) \times C_{cb}^{2n}(\text{PU}(n, 1), \mathbb{R})^{\text{PU}(n, 1)} &\longrightarrow C_{cb}^{2n}(\Gamma)^\Gamma \\ (\rho, \varphi_n) &\longmapsto \rho^*(\varphi_n), \end{aligned} \quad (\text{V.8})$$

as is the following map:

$$\begin{aligned} \text{Hom}(\Gamma, G) \times C_{cb}^2(G, \mathbb{R})^G &\longrightarrow C_{cb}^2(\Gamma)^\Gamma \\ (\rho, \bar{\varphi}_1) &\longmapsto \rho^*(\bar{\varphi}_1), \end{aligned}$$

and continuity is preserved under cup products:

$$((\rho, i, \dots, i), (\bar{\varphi}_1, \varphi_1, \dots, \varphi_1)) \mapsto \rho^*(\bar{\varphi}_1) \cup i^*(\varphi_1)^{n-1}. \quad (\text{V.9})$$

Next, we define on the cochain level the bounded transfer map

$$T : C_{cb}^{2n}(\Gamma, \mathbb{R})^\Gamma \rightarrow C_{cb}^{2n}(\text{PU}(n, 1), \mathbb{R})^{\text{PU}(n, 1)}$$

given by

$$(T\alpha)(x_0, \dots, x_{2n}) := \int_{\mathcal{D}} \alpha(r(gx_0), \dots, r(gx_{2n})) d\mu(g),$$

where $r(g) \in \Gamma$ satisfies $g = r(g) \cdot s(p(g))$ for a Borel measurable section $s : \Gamma \backslash G \rightarrow G$ (see (V.1)). Indeed, this map is well-defined by [BBI21, Proposition A.2 (1)] and clearly induces the cohomological bounded transfer tr_{2n}^b from Subsection V.1.2. Moreover, [BBI21, Proposition A.2 (2)] proves that the cochain map T is continuous with respect to the chosen topologies: If $\alpha_j \rightarrow \alpha$ converges pointwise in $C_{cb}^{2n}(\Gamma, \mathbb{R})^\Gamma$ with control of norms, then $T\alpha_j \rightarrow T\alpha$ converges uniformly on compact subsets of $\text{PU}(n, 1)^{(2n+1)}$. Since $\text{PU}(n, 1)$ is locally compact and the cochains $T\alpha_j$ are bounded, this implies uniform convergence on all $\text{PU}(n, 1)^{(2n+1)}$, i.e. $C_{cb}^{2n}(\text{PU}(n, 1), \mathbb{R})^{\text{PU}(n, 1)}$.

In light of this, we now show that the numerical quantities $e(\rho)$ and $d(\rho)$ can be obtained by passing continuously to singular cochains of a compact manifold and then evaluating on its fundamental class. For this, we fix an auxiliary uniform torsion-free lattice $\Gamma' < \text{PU}(n, 1)$ and then consider the following sequence of maps:

$$\begin{array}{ccc} C_{cb}^{2n}(\Gamma, \mathbb{R})^\Gamma & \xrightarrow{T} & C_{cb}^{2n}(\text{PU}(n, 1), \mathbb{R})^{\text{PU}(n, 1)} \\ & & \downarrow c_{\mathbb{R}} \\ & & C_c^{2n}(\text{PU}(n, 1), \mathbb{R})^{\text{PU}(n, 1)} \xrightarrow{\iota^*} C_c^{2n}(\Gamma', \mathbb{R})^{\Gamma'}, \end{array}$$

where $c_{\mathbb{R}}$ denotes the comparison map and the right horizontal map is induced by the canonical inclusion $\iota : \Gamma' \hookrightarrow \text{PU}(n, 1)$. Both maps are continuous for $C_c^{2n}(\text{PU}(n, 1), \mathbb{R})^{\text{PU}(n, 1)}$ and $C_c^{2n}(\Gamma', \mathbb{R})^{\Gamma'}$ each equipped with compact-open topologies; for the latter, since Γ' is discrete,

this coincides topology of pointwise convergence.

Now setting $M' = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$, there is a canonical isomorphism

$$\Phi : C_c^{2n}(\Gamma', \mathbb{R})^{\Gamma'} \longrightarrow S^{2n}(M', \mathbb{R}), \quad \Phi(c)(\sigma) = c(\gamma_0, \dots, \gamma_{2n})$$

where $\sigma : \Delta^{2n} \rightarrow M'$ is a singular simplex, $\tilde{\sigma}$ any lift to $\mathbb{H}_{\mathbb{C}}^n$, and $\gamma_i \in \Gamma'$ is the unique element such that $\tilde{\sigma}(v_i) \in \gamma_i D$ for a fixed Borel fundamental domain $D \subset \mathbb{H}_{\mathbb{C}}^n$ and v_i the i -th vertex of the simplex Δ^{2n} for all $1 \leq i \leq 2n$. We then equip $S^{2n}(M', \mathbb{R})$ with the topology of pointwise convergence, for which Φ is continuous: evaluation of $\Phi(c)$ on a simplex σ depends solely on the value of c on the finite tuple $(\gamma_0, \dots, \gamma_{2n}) \in \Gamma^{2n+1}$ determined by σ and the fundamental domain D . Consequently, for fixed σ , convergence on c in the topology of pointwise convergence implies convergence of $\Phi(c)(\sigma)$.

Finally, the evaluation map

$$\langle \cdot, [M'] \rangle : S^{2n}(M', \mathbb{R}) \longrightarrow \mathbb{R}$$

is continuous for the topology of pointwise convergence, since the fundamental class $[M'] \in H_{2n}(M', \mathbb{R})$ admits a singular cycle representative supported on finitely many simplices, making the evaluation a finite linear combination of coordinate projections.

Therefore, denoting for $i = 1, 2$

$$\alpha_{i,\rho} = \begin{cases} \rho^*(\varphi_n) & \text{if } i = 1 \text{ and } \rho : \Gamma \rightarrow \text{PU}(n, 1), \\ \rho^*(\overline{\varphi}_1) \cup i^*(\varphi_1)^{n-1} & \text{if } i = 2 \text{ and } \rho : \Gamma \rightarrow G. \end{cases},$$

we conclude that the map

$$\rho \longmapsto \langle [\Phi \circ \iota^* \circ c_{\mathbb{R}} \circ T(\alpha_{i,\rho})], [M'] \rangle$$

is continuous. Finally, we observe that if $i = 1$, keeping the same notation for induced maps in cohomology and applying the transfer-theoretic definition of $e(\rho)$, we obtain

$$\begin{aligned} \langle [\Phi \circ \iota^* \circ c_{\mathbb{R}} \circ T(\alpha_{1,\rho})], [M'] \rangle &= \langle \Phi \circ \iota^* \circ c_{\mathbb{R}} \circ \text{tr}_b^{2n} \circ \rho^*(\mathbf{C}_n^b), [M'] \rangle \\ &= e(\rho) \cdot \langle \Phi \circ \iota^*(\mathbf{C}_n), [M'] \rangle \\ &= e(\rho) \cdot \chi(M') \end{aligned}$$

and if $i = 2$, applying the transfer-theoretic definition for $d(\rho)$ gives

$$\begin{aligned} \langle [\Phi \circ \iota^* \circ c_{\mathbb{R}} \circ T(\alpha_{2,\rho})], [M'] \rangle &= \langle \Phi \circ \iota^* \circ c_{\mathbb{R}} \circ \text{tr}_b^{2n} \circ (\rho^*(\overline{\mathbf{C}}_1^b) \cup i^*(\mathbf{C}_1^b)^{n-1}), [M'] \rangle \\ &= (n+1)^{n-1} \cdot d(\rho) \cdot \langle \Phi \circ \iota^*(\mathbf{C}_n), [M'] \rangle \\ &= (n+1)^{n-1} \cdot d(\rho) \cdot \chi(M'). \end{aligned}$$

Since n and M' are fixed, we conclude that $e(\rho)$ and $d(\rho)$ are continuous, completing the proof. \square

Remark. We observe that our continuity proof relies strongly on [BBI21, Appendix A], although we used a different strategy to conclude. In the real hyperbolic case [BBI21, pp. 240-241], the continuity of the volume $\text{Vol}_{\text{Dup}}(\cdot)$ is achieved through the absence of coboundaries among $\text{Isom}(\mathbb{H}_{\mathbb{R}}^n)$ -invariant continuous bounded cochains on the real hyperbolic n -space $\mathbb{H}_{\mathbb{R}}^n$ in

$$H_{cb}^n(\text{Isom}^+(\mathbb{H}_{\mathbb{R}}^n), \mathbb{R}) \cong H_{cb}^n(\text{Isom}(\mathbb{H}_{\mathbb{R}}^n), \mathbb{R}_{\varepsilon}) \cong H_c^n(\text{Isom}(\mathbb{H}_{\mathbb{R}}^n), \mathbb{R}_{\varepsilon}) \cong \mathbb{R},$$

where $\varepsilon : \text{Isom}(\mathbb{H}_{\mathbb{R}}^n) \rightarrow \{\pm 1\}$ is a homomorphism that sends orientation-preserving isometries to 1 and all others to -1 . In particular, the integral of $T(\rho^*(\text{Vol}_{\text{Dup}}))$ over a compact maximal subgroup of $\text{Isom}(\mathbb{H}_{\mathbb{R}}^n)$ is continuous with respect to the representation ρ of a non-uniform real hyperbolic lattice into its ambient Lie group, and coincides with the cocycle

$$\frac{\text{Vol}_{\text{Dup}}(\rho)}{\text{Vol}(M)} \cdot \text{Vol}_{\text{Dup}},$$

with Vol_{Dup} being the Dupont volume cocycle (cf. Subsection II.4.2).

By contrast, in our complex hyperbolic case, we do not work with twisted coefficients and it is not known whether such coboundaries in top degree continuous bounded cohomology occur, so that one cannot hope for a strict identification of the cocycle as above. Instead, we passed to the cohomology of uniform complex hyperbolic lattices. As we have seen, continuity of our invariants then follows from functoriality and continuity of this evaluation process.

We conclude the thesis by proving the **Main Corollary** from the introduction:

Corollary V.13. *For $n \geq 2$, let $\Gamma < \text{PU}(n, 1)$ be a non-uniform torsion-free lattice and G a non-compact simple Hermitian Lie group with finite center. Then*

- *all of the invariants $e(\cdot)$, $E(\cdot)$ and $\text{Vol}(\cdot)$ are constant on connected components of the character variety*

$$\text{Hom}(\Gamma, \text{PU}(n, 1)) // \text{PU}(n, 1),$$

- *all of the invariants $d(\cdot)$, $t(\rho)$, $\text{Deg}(\cdot)$ and $\text{Tol}(\cdot)$ are constant on connected components of the character variety*

$$\text{Hom}(\Gamma, G) // G.$$

Proof. Let $n \geq 2$ and $\Gamma < \text{PU}(n, 1)$ a non-uniform torsion-free lattice and G a simple non-compact Hermitian Lie group with finite center. Since the invariants

$$e(\rho), E(\rho), \text{Vol}(\rho) \quad \text{of} \quad \rho : \Gamma \longrightarrow G \quad \text{with} \quad G = \text{PU}(n, 1)$$

and the invariants

$$d(\rho), t(\rho), \text{Deg}(\rho), \text{Tol}(\rho) \quad \text{of} \quad \rho : \Gamma \longrightarrow G$$

are integral by Theorems A, B, and are moreover continuous on $\text{Hom}(\Gamma, G)$ accordingly, by Theorem V.12, this implies their constancy on the path-connected components $\pi_0(\text{Hom}(\Gamma, G))$.

In view of the discussion in Section I.1.3 from Chapter I, we observe that the following

bijections of path-connected components hold (cf. [Mar25, §6.1]):

$$\pi_0(\mathrm{Hom}(\Gamma, G)) \cong \pi_0(\mathrm{Hom}(\Gamma, G) / \mathrm{Inn}(G)),$$

as G acting on the connected topological group continuously $\mathrm{Hom}(\Gamma, G)$ by conjugation. Moreover, the Hausdorffification induces the following isomorphism

$$\pi_0(\mathrm{Hom}(\Gamma, G) / \mathrm{Inn}(G)) \cong \pi_0(\mathrm{Haus}(\mathrm{Hom}(\Gamma, G) / \mathrm{Inn}(G)))$$

Therefore, using the more compact notation for the set in consideration on the right-hand side, we have that

$$\pi_0(\mathrm{Hom}(\Gamma, G)) \cong \pi_0(\mathrm{Hom}(\Gamma, G) // G),$$

which shows the statement of the corollary. □

V.4 Open Questions

Finally, we identify several open questions that emerge both in a direct and more nuanced manner from this thesis.

V.4.1 On the Invariant $J(\cdot)$

In light of [Theorem A](#) and [Section IV.4](#), we pose the following question:

Open Question 1. *Let $n \geq 2$, let $\Gamma < \mathrm{PU}(n, 1)$ be a non-uniform torsion-free lattice, and let $\rho : \Gamma \rightarrow \mathrm{PU}(n, 1)$ a representation. If all cusps of $M = \Gamma \backslash \mathbb{H}_{\mathbb{C}}^n$ are nilmanifolds, does $J(\rho^*(C_n))$ vanish?*

An affirmative answer would provide a direct analogy to the integrality result of [BBI21] in the real hyperbolic setting. We ask a question paralleling the one raised in [BBI21, p. 212]:

Open Question 2. *Let $n \geq 2$, let $\Gamma < \mathrm{PU}(n, 1)$ be a non-uniform torsion-free lattice and $i : \Gamma \hookrightarrow \mathrm{PU}(n, 1)$ the canonical inclusion. For arbitrary representations $\rho : \Gamma \rightarrow \mathrm{PU}(n, 1)$ and $\bar{\rho} : \Gamma \rightarrow G$ into G a simple non-compact Hermitian Lie group with finite center, do the invariants*

$$J(\rho^*(C_n)) \quad \text{and} \quad J(\bar{\rho}^*(\bar{C}_1) \cup i^*(C_1^n))$$

vanish on all infranilmanifolds?

If the answer is negative, it would be interesting to investigate whether $J(\cdot)$ admits further interpretations, beyond its role as a generalized rotation number in [BIW10] - for instance, as a potential obstruction invariant to an infranilmanifold occurring as a cusp of a complete non-compact finite-volume complex hyperbolic manifold.

In a different context, namely for representations of complete non-compact oriented real hyperbolic surfaces of finite volume, Kim, Pansu and Wang [KPW25] established a relation between the Toledo invariant, the signature and the so-called ρ -invariant, with the latter being a reminiscent of the celebrated η -invariant from the Atiyah-Patodi Singer index theory. As

shown in [LR00], the η -invariant obstructs certain orientable flat 3-manifolds from occurring as cusps of complete finite-volume real hyperbolic 4-manifolds (cf. Subsection I.4.2).

For complex hyperbolic surfaces, another candidate to compare $J(\cdot)$ to, is the Burns-Epstein invariant [BE88; Kam07].

V.4.2 On the Invariant $I(\cdot)$

In [GKL01], Goldman, Kapovich and Leeb studied representations $\rho : \Gamma \rightarrow \mathrm{PU}(2, 1)$, where $\Gamma < \mathrm{PU}(1, 1)$ is a uniform torsion-free lattice. They showed that every possible value of the Toledo invariant is realized by some representation, and moreover identified certain values realized by convex-cocompact discrete and faithful representations. We therefore ask:

Open Question 3. *Is every value obtained in our [Main Theorem](#) and lying the interval prescribed by the corresponding Milnor-Wood type inequalities realized by some representation?*

Clearly zero is realized by the trivial representation and the maximal values by lattice embeddings. Another question one could pose is the following one:

Open Question 4. *Is there a closed formula relating the degree $\mathrm{Deg}(\rho)$ and the Euler number $E(\rho)$ for any representation $\rho : \Gamma \rightarrow \mathrm{PU}(n, 1)$ with $\Gamma < \mathrm{PU}(n, 1)$ a torsion-free lattice?*

This would give further restrictions on the values the invariants could take. It is natural to ask for further generalizations of the Euler number and Toledo invariant in the context of one-rank symmetric spaces:

Open Question 5. *Let $n \geq 1$, let $\Gamma < \mathrm{PSp}(n, 1)$ be a non-uniform torsion-free lattice and let $\rho : \Gamma \rightarrow \mathrm{PSp}(n, 1)$ be a representation. What values does $E(\rho)$ take?*

As moreover, there have been recent developments in the study of the *quaternionic Toledo invariant*, an analogous question would be interesting as well in this case. Next, we raise the following question regarding the definition of the volume:

Open Question 6. *Let $\Gamma < \mathrm{PU}(n, 1)$ be non-uniform torsion-free lattice. Do the definitions $\mathrm{Vol}(\cdot)$ and $\mathrm{Vol}_{\mathrm{Dup}}(\cdot)$ coincide? Is the volume invariant of any representation $\rho : \Gamma \rightarrow \mathrm{PU}(n, 1)$ independent of the choice of bounded volume class, in the sense of [Definition II.21](#)?*

Since the comparison map $c_{\mathbb{R}} : H_{cb}^{2n}(\mathrm{PU}(n, 1), \mathbb{R}) \rightarrow H_c^{2n}(\mathrm{PU}(n, 1), \mathbb{R})$ is conjectured to be an isomorphism, the answer is expected to be affirmative.

Furthermore, there are the numerous proofs using different methods of the *volume rigidity theorem* for the volume $\mathrm{Vol}_{\mathrm{Dup}}(\cdot)$ in [BBI13; Fra04; FK06; Dun99; KK16] in the real hyperbolic setting, which imply the *Mostow-Prasad rigidity theorem*, stating that if any two real hyperbolic lattices isomorphic, then they are conjugate to each other in $\mathrm{Isom}^+(\mathbb{H}_{\mathbb{R}}^n)$. We therefore ask:

Open Question 7. *Let $\Gamma < \mathrm{PU}(n, 1)$ be non-uniform torsion free lattice. Is any representation $\rho : \Gamma \rightarrow \mathrm{PU}(n, 1)$ conjugate to the canonical embedding $i : \Gamma \hookrightarrow \mathrm{PU}(n, 1)$ in $\mathrm{Isom}^{\circ}(\mathbb{H}_{\mathbb{C}}^n)$, if and only if, it holds that*

$$\mathrm{Vol}_{\mathrm{Dup}}(\rho) = |\mathrm{Vol}_{\mathrm{Dup}}(i)|?$$

We remark that this statement was proved for the Toledo invariant $\mathrm{Tol}(\cdot)$ in [KM08a] using harmonic maps, and in [BI09] with bounded cohomological methods.

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