Laplace transform of generalized functions

Bogoljub Stanković
Institute of Mathematics,
University of Novi Sad, 21000 Novi Sad, Yugoslavia

1. Introduction

At the present time the theory of generalized functions has found substantial applications in the theoretical physics and became a workaday tool especially in the quantum physics (cf. for example [23], [25]). Laplace transform (in short LT) of generalized functions has an important place in this sense.

In order to see that the LT of generalized functions has some advantages of the classical LT we give first well-known facts on classical LT.

Laplace transform was originally employed to justify the Heaviside operational calculus [10]. Many papers have been published in this sense. Let us mention only some of them [5], [6], [8],... Later on the Laplace transform has been elaborated as a powerful mathematical theory (cf. [8] and [25]), very useful in practice and many times applied in solving mathematical models but also used to give correct theoretical ground for some phenomena in physics, especially in the quantum field theory (cf. [3], [4], [23]).

Let f be locally integrable function defined on $[0,\infty)$ which satisfies the inequality

$$|f(x)| \le Ce^{Hx}, \quad x > x_0 > 0, \tag{1}$$

with constants C and H (f *being of exponential type*). The classical LT of

f is defined by the integral

$$(2) \qquad \int_0^\infty e^{-st} f(t)dt = \widehat{f}(s) = \mathcal{L}(f)(x), \ Re\, s > H.$$

In despite of the popularity, LT has the following three theoretical shortcomings (cf. [12]):

1. The function f must be of exponential type. In short, applications of the LT call for some growth conditions of the originals (cf. (1)).

2. No simple characterisation of the set of images by the LT is known. So we do not know to establish in an easy way whether or not a solution $\widehat{u}$ to an equation $P(\widehat{u}) = 0$, obtained applying LT to $Q(u) = 0$, is the LT of a solution to $Q(u) = 0$.

3. The expression

$$(3) \qquad f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sx} \widehat{f}(s) ds, \quad c > H,$$

is the inversion formula for the LT only if we know that $\widehat{f}(s)$ is the LT of f. Also, it converges in general case only as Cauchy's principal value. It does not in general, converge absolutely. In short, we have not an easily applicable inversion formula.

To overcome these difficulties mathematicians invented different foundations and theories of Heaviside calculus. We can divide them in two groups by their approach: analytic or algebraic. To the first group belong those theories in which the LT is defined on a subspace of generalized functions, continuous functionals on appropriate test function space (cf. [12], [19], [24], [25], [27]), or other analytic approaches (cf. [2], [7]). The second group contains theories which use algebraic approaches (cf. [16], [17]). Of this group the most popular in applications has been Mikusinski's operator calculus.

We shall treat the LT of generalized functions because it turned out to be a useful tool in modern mathematics and also there are new important results and ideas which are not known sufficiently by the general public. It is a question of LT of Laplace hyperfunctions elaborated by H.Komatsu (cf. [12], [13], [15]). His approach successfully overcomes all the three shortcomings of the classical LT. Also it is a natural generalization of the classical LT. Namely, if $F(x),\ x \geq 0$, is a measurable function which satisfies the exponential type condition (1), then it can be naturally embedded in the space of Laplace hyperfunctions and the classical LT of f and the LT of the Laplace hyperfunction defined by f, give the same function. Consequently, in this case, one can use the table for the classical LT to realize the LT of Laplace hyperfunctions.

2. LT of tempered distributions

We repeat some specific distributions and facts related to the space $\mathcal{S}'$ of tempered distributions and to the Fourier transform of them (cf. [25]).

Let Γ be a closed convex acute cone in $\mathbb{R}^n, \Gamma^* = \{t; tx = t_1x_1 + ... + t_nx_n \geq 0,\ \forall x \in \Gamma\}$ and $C = int\Gamma^*$. Let K be a compact set in $\mathbb{R}^n$.

By $\mathcal{S}'(\Gamma + K)$ is denoted the space of tempered distributions with supports in the closed set $\Gamma + K \subset \mathbb{R}^n$. Then $\mathcal{S}'(\Gamma+)$ is

$$\mathcal{S}'(\Gamma+) = \bigcup_{K \in \mathbb{R}^n} \mathcal{S}'(\Gamma + K). \tag{4}$$

The set $\mathcal{S}'(\Gamma+)$ forms an algebra that is associative and commutative if for the operation of multiplication one takes the convolution, denoted by $*$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then the Fourier transform $\mathcal{F}$ of φ is

$$\mathcal{F}[\varphi](\xi) = \int_{\mathbb{R}^n} e^{i\xi x} \varphi(x) dx.$$

If $f \in \mathcal{S}'(\mathbb{R}^n)$, then, as we know,

$$(5) \qquad \langle \mathcal{F}[f](\xi), \varphi(\xi)\rangle = \langle f(x), \mathcal{F}[\varphi](x)\rangle.$$

Now we can use the Fourier transform to define the LT. If $f \in \mathcal{S}'(\Gamma+)$, then $f \in \mathcal{S}'(\Gamma + K)$ for a K. The LT of f is by definition

$$(6) \qquad \mathcal{L}(f)(z) = \mathcal{F}[f(\xi)e^{-x\xi}](-y),\ y \in \mathbb{R}^n,\ \text{for each } x \in C.$$

Let U_ε be the ball $U_\varepsilon = \{x; \|x\| < \varepsilon\}$. We introduce the function $\eta(x) \in \mathcal{C}^\infty(\mathbb{R}^n) : \eta(x) = 1,\ x \in \Gamma + K + U_\varepsilon$ and $\eta(x) = 0,\ x \notin \Gamma + K + U_{2\varepsilon}$, for an $\varepsilon > 0$. Thus, for each $x \in C$ and every $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}\langle \mathcal{L}(f)(x+iy), \varphi(y)\rangle &= \langle \mathcal{F}[f(\xi)e^{-x\xi}](-y), \varphi(y)\rangle \\ &= \langle \eta(\xi)f(\xi)e^{-x\xi}, \int_{\mathbb{R}^n} e^{-i\xi y}\varphi(y)dy \\ &= \int_{\mathbb{R}^n} \varphi(y)\langle f(\xi), \eta(\xi)e^{-z\xi}\rangle dy \\ &\langle\langle f(\xi), \eta(\xi)e^{-(x+iy)\xi}\rangle, \varphi(y)\rangle.\end{aligned}$$

Since this is true for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$, it follows that

$$(7) \qquad \mathcal{L}(f)(z) = \langle f(\xi), \eta(\xi)e^{-z\xi}\rangle,\ \ z \in \mathbb{C} + i\mathbb{R}^n.$$

If $\Gamma + K$ is also convex, as it is many times the case, then

$$(8) \qquad \mathcal{L}(f)(z) = \langle f(\xi), e^{-z\xi}\rangle,\ \ z \in \mathbb{C} + i\mathbb{R}^n.$$

If is easily seen that $\mathcal{L}(f)(z)$ in (7) does not depend on η with the looked - for properties. Suppose that we have two such η, η_1 and η_2. Then

$$\begin{aligned}\langle f(\xi),\ \ &\eta_1(\xi)e^{-z\xi}\rangle - \langle f(\xi), \eta_2(\xi)e^{-z\xi}\rangle \\ &= \langle f(\xi), (\eta_1(\xi) - \eta_2(\xi))e^{-z\xi}\rangle = 0,\ z \in C + i\mathbb{R}^n,\end{aligned}$$

because $(\eta_1(\xi) - \eta_2(\xi))e^{-zz\xi} = 0$ for $\xi \in \Gamma + K + U_\varepsilon$, where $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$, but $\text{supp} f \subset \Gamma + K$.

From all properties of $\mathcal{L}(f)$ we shall prove only its holomorphisity.

Proposition 2.1. *$\mathcal{L}(f)(z)$ is holomorphic in $C + i\mathbb{R}^n$.*

Proof. It suffices to establish, by virtue of the Hartogs theorem, the existence of all derivatives

$$\frac{\partial \mathcal{L}(f)(z)}{\partial z_j}, \quad j = 1, ..., n.$$

Let us do it for $j = 1$.

It is easily seen that in $\mathcal{S}(\mathbb{R}^n)$

$$\lim_{h\to 0} \frac{\eta(\xi)}{h}\left(e^{-(z+he_1)\xi} - e^{-z\xi}\right) = \eta(\xi)\xi e^{-z\xi} \tag{9}$$

for each $z \in C + i\mathbb{R}^n$, where $e_1 = (1, 0, ..., 0)$.

Now, for each $z \in C + i\mathbb{R}^n$, because of (9),

$$\begin{aligned}
&\lim_{h\to 0} \quad (\mathcal{L}(f)(z + he_1) - \mathcal{L}(f)(z))/h \\
&= \lim_{h\to 0}(\langle f(\xi), \eta(\xi)e^{-(z+he_1)\xi}\rangle - \langle f(\xi), \eta(\xi)e^{-z\xi}\rangle)/h \\
&= \lim_{h\to 0}\langle f(\xi), \frac{\eta(\xi)}{h}(e^{-(z+he_1)\xi} - e^{-z\xi})\rangle \\
&= \langle(-\xi_1)f(\xi), e^{-z\xi}\rangle.
\end{aligned}$$

The proof for any other j is just the same.

The space $\mathcal{S}'(\mathbb{R}^n)$ is a restricted subspace of distributions. For example, $e^t,\ t \in \mathbb{R}$ does not define an element of $\mathcal{S}'(\mathbb{R})$. Therefore we have to go to a larger subspace of distributions.

3. The space $\mathcal{D}'(\Gamma+, a)$ and the LT of their elements

We can enlarge the definition of the LT to a wider subspace of distributions which contains tempered distributions, as well (cf. [19], [24]).

By $\mathcal{D}'(\Gamma+)$ is denoted the set of distributions with support in $\Gamma + K,\ K$ any compact set in $\mathbb{R}^n$. $\mathcal{D}'(\Gamma+)$ is an algebra with composition as the multiplicative operation. Then $\mathcal{D}'(\Gamma + K, a)$ denotes the set of distributions $f \in \mathcal{D}'(\Gamma + K)$ such that

$$e^{-\sigma t} f(t) \in \mathcal{S}'(\Gamma + K),\ \sigma > a\ (\sigma_i > a_i,\ i = 1, ..., n) \tag{10}$$

and

$$\mathcal{D}'(\Gamma+, a) = \bigcup_{K \subset \mathbb{R}^n} \mathcal{D}'(\Gamma + K, a),\ K \text{ compact set.} \tag{11}$$

We give some properties of the spaces $\mathcal{D}'(\Gamma + K, a),\ a \in \mathbb{R}^n$.

Proposition 3.1. *1. If $a \leq b$, then $\mathcal{D}'(\Gamma + K, a) \subset \mathcal{D}'(\Gamma + K, b)$. Consequently $\mathcal{S}'(\Gamma + K) \subset \mathcal{D}'(\Gamma + K, a),\ a \geq 0$.*

2. If $f \in \mathcal{D}'(\Gamma+, a)$ and $g \in \mathcal{D}'(\Gamma+, b),\ a \leq b$, then:

a) $f \in \mathcal{D}'(\Gamma+, b)$, as well;

b) also $f * g \in \mathcal{D}'(\Gamma+, b)$ and

c) $fe^{-\sigma t} * ge^{-\sigma t} = (f * g)e^{-\sigma t},\ \sigma > b$.

3. $\mathcal{D}'(\Gamma+, a)$ is a convolution algebra.

4. If $f \in \mathcal{D}'(\Gamma + K, a),\ g = \frac{\partial}{\partial t_i}\delta(t),\ i \in \mathbb{N}$, then

$$f * \frac{\partial}{\partial t_i}\delta(t) = \frac{\partial}{\partial t_i} f \in \mathcal{D}'(\Gamma + K, a).$$

Therefore, partial derivatives are inner operations in $\mathcal{D}'(\Gamma + K, a)$.

Proof. Properties 1 and 2 a) are obvious. We start with the proof of 2 b) and 2 c). Suppose that $\text{supp} f \subset \Gamma + K_1$ and $\text{supp} g \subset \Gamma + K_2,\ K_1$ and K_2 are compact sets. Let $\eta_i,\ i = 1, 2$, be two functions belonging to $\mathcal{C}^\infty(\mathbb{R}^n)$, $\eta_i(x) = 1,\ x \in \Gamma + K_i + U_{\varepsilon_i}$ and $\eta_i(x) = 0,\ x \notin \Gamma + K_i + U_{2\varepsilon_i}$ (see Section 2).

By definition of the convolution in $\mathcal{S}'(\mathbb{R}^n)$

$$\begin{aligned}\langle f e^{-\sigma t} * g e^{-\sigma t}, \varphi\rangle &= \langle \eta_1 f e^{-\sigma t} * \eta_2 g e^{-\sigma t}, \varphi\rangle \\ &= \langle f(x) * g(y),\ \eta_1(x)\eta_2(y) e^{-\sigma(x+y)} \varphi(x+y)\rangle \\ &= \langle (f * g)(t) e^{-\sigma t},\ \varphi(t)\rangle.\end{aligned}$$

Since it is true for every $\varphi \in \mathcal{S}'(\mathbb{R}^n)$, we have 2 b) and 2 c). Also, by the property of convolution $\text{supp}(f * g) \subset \Gamma + K_1 + K_2,\ K_1 + K_2$ is compact in $\mathbb{R}^n$.

3 is a consequence of 2. Also 4 is a consequence of 2 and the property of the δ distribution.

Now, we can define LT of $f \in \mathcal{D}'(\Gamma+, a)$:

$$\mathcal{L}(f)(z) = \mathcal{F}[f(t) e^{-xt}](-y),\ x \in C + a,\ y \in \mathbb{R}^n. \tag{12}$$

Since $f(t)e^{-xt} \in \mathcal{S}'(\Gamma + K)$ for a compact set K and $\eta(t) e^{-((x-\sigma)+iy)t} \in \mathcal{S}(\mathbb{R}^n),\ \mathcal{L}(f)(z)$ can be written as

$$\mathcal{L}(f)(z) = \langle f(t) e^{-\sigma t},\ \eta(t) e^{-((x-\sigma)+iy)t}\rangle,\ x \in C + \sigma,\ \sigma > a. \tag{13}$$

The basic properties of the LT of elements of $\mathcal{D}'(\Gamma+, a)$ have been given by

Proposition 3.2. *Let $\Gamma + K = \overline{\mathbb{R}}^n_+$, what we have often in applications, then $f \in \mathcal{D}'(\overline{\mathbb{R}}^n_+, a)$ if and only if $\mathcal{L}(f)(z)$ is holomorphic function in $z \in \mathbb{R}^n_+ + \sigma + i\mathbb{R}^n$ for $\sigma > a$ and*

$$|\mathcal{L}(f)(z)| \leq M(\varepsilon, \sigma) e^{\sigma x}(1 + \|z\|^m),\ x > \sigma > a, \tag{14}$$

for every $\varepsilon > 0,\ x > \sigma > a,\ m = m(\sigma) \geq 0,$ *where* $M(\varepsilon, \sigma)$ *depends only on* ε *and* σ.

We have also the inverse formula for the LT:

$$f(t) = (2\pi)^{-n} e^{\sigma t} \mathcal{F}[\widehat{f}(\sigma + i\omega)](t), \ \textit{for any } \sigma > a. \tag{15}$$

Proof. The holomorphicity follows directly from Proposition 2.1. For the inequality (14) see [24]. To prove (15) let us start with (6):

$$\begin{aligned} \widehat{f}(z) &= \mathcal{F}[f(\xi)e^{-x\xi}](-y) \\ &= (2\pi)^n \mathcal{F}^{-1}[\eta(\xi)f(\xi)e^{-x\xi}](y),\ y \in \mathbb{R}^n,\ x \in \mathbb{R}^n_+, \end{aligned}$$

where η is given in Section 2. Applying, now, the Fourier transform, we have

$$\eta(x)f(x) = (2\pi)^{-n} e^{x\sigma} \mathcal{F}[\widehat{f}(\sigma + i\omega)](x),\ \sigma > a. \tag{16}$$

Since (16) does not depend on the chosen η, supp$f \in \overline{\mathbb{R}}^n_+$.

Proposition 3.2. can be proved not only for $\mathcal{D}'(\overline{\mathbb{R}}^n_+, a)$ but for the general case $\mathcal{D}'(\Gamma+, a)$, as well.

Proposition 3.3. *If* $f \in \mathcal{D}'(\Gamma+, a)$, *then:*

1. $\mathcal{L}\Big(\dfrac{\partial^{k_1+\ldots+k_n}}{\partial t_1^{k_1}\ldots\partial t_n^{k_n}} f\Big)(z) = z_1^{k_1}\ldots z_n^{k_n}\mathcal{L}(f)(z),$

   $k_1, ..., k_n \in \mathbb{N},\ \ z \in C + a + i\mathbb{R}^n.$

2. $\mathcal{L}(f(t)e^{\lambda t})(z) = \mathcal{L}(f)(z - \lambda),\ z \in C + a + Re\lambda + i\mathbb{R}^n.$

3. *If also* $g \in \mathcal{D}'(\Gamma+, a)$, *then* $\mathcal{L}(f * g)(z) = \mathcal{L}(f)(z)\mathcal{L}(g)(z),\ z \in C + a + i\mathbb{R}^n.$

4. $\mathcal{L}(f(t - \beta))(z) = e^{-\beta z}\mathcal{L}(f)(z),\ \beta \geq 0,\ z \in C + a + i\mathbb{R}^n.$

5. $\mathcal{L}(\delta(t - \beta))(z) = e^{-\beta z},\ z \in C + i\mathbb{R}^n,\ \beta \geq 0.$

Proof. 1. follows by 4. in Proposition 3.1. and 5. follows from (13) with $\sigma = 0$. For the proof of 3. see [24]. It remains to prove only 2. and 4.

By (12)

$$\mathcal{L}(fe^{\lambda t})(z) = \mathcal{F}[fe^{-(x-\lambda)t}](-y) = \mathcal{L}(f)(z-\lambda)$$

which gives 2.

We know that if $f \in \mathcal{D}'(\Gamma+, a)$, then $f(t-\beta) \in \mathcal{D}'(\Gamma+, a)$, too. Also

$$f(t-\beta) = f * \delta(t-\beta).$$

If we apply LT to this equality, we have by 3. and 5.

$$\mathcal{L}(f(t-\beta))(z) = e^{-\beta z}\mathcal{L}(f)(z), \quad x > a.$$

This completes the proof of Proposition 3.3.

With the space $\mathcal{D}'(\Gamma+, a)$ we have also restriction on the growth of elements. For example, e^{t^2}, $t \in \mathbb{R}$, can not define an element of $\mathcal{D}'(\overline{\mathbb{R}}_+, a)$ for any $a \in \mathbb{R}$.

4. Hyperfunctions and the LT of hyperfunctions

We have seen, that the LT of functions and distributions have always called for some growth conditions. This fact restricts the effectiveness of the LT in applications. In this section we review the results of H.Komatsu (cf. [12], [13], [15]) which do not require any growth condition.

Let ω be an open subset of $\mathbb{R}$ and U be a *complex neighbourhood of* ω *in* $\mathbb{C}$ (ω is a closed subset of U, U an open set in $\mathbb{C}$). Let $\mathcal{O}(U)$ denotes the space of holomorphic functions defined on U.

Definition 4.1. *([18]). The quotient space*

$$\mathcal{B}(\omega) = \mathcal{O}(U \setminus \omega)/\mathcal{O}(U) \tag{17}$$

defines the space of hyperfunctions defined on ω.

If $f \in \mathcal{B}(\omega)$, then there exists a function $F \in \mathcal{O}(U \setminus \omega)$ such that f is defined by the class $[F]$; F_1 belongs to the class $[F]$ if and only if $F_1 \in \mathcal{O}(U \setminus \omega)$ and $F - F_1 \in \mathcal{O}(U)$. F *is called a defining function of* f.

In the sense of vector space isomorphism, the definition of $\mathcal{B}(\omega)$ do not depend on the chosen complex neighbourhood U of ω.

Definition 4.2.

$$\mathcal{B}_{[a,\infty)} = \mathcal{O}(\mathbb{C} \setminus [a,\infty))/\mathcal{O}(\mathbb{C}) \tag{18}$$

is the space of hyperfunctions with support in $[a,\infty)$.

If $f \in \mathcal{B}_{[a,\infty)}$, then there exists an $F \in \mathcal{O}(\mathbb{C} \setminus [a,\infty))$ which defines a class $[F]$ such that $f = [F]$ and f is also written in the form

$$f = F_+(x + i0) - F_-(x - i0),$$

F is a defining function of f. The space of real analytic functions $\mathcal{A}(\mathbb{R})$, the space of continuous functions $\mathcal{C}(\mathbb{R})$, the space of locally integrable functions $\mathcal{L}_{loc}(\mathbb{R})$ and distributions $\mathcal{D}'(\mathbb{R})$ are, in the sense of algebraic isomorphism, subspaces of $\mathcal{B}(\mathbb{R})$. One can find in [11], Chapter 1, how to determine a defining function F for an f which belongs to $\mathcal{A}(\mathbb{R})$, $\mathcal{C}(\mathbb{R}), \mathcal{L}_{loc}(\mathbb{R})$ or $\mathcal{D}'(\mathbb{R})$.

Conversely, if we have a defining function of a hyperfunction f, then we can characterize the subspace of hyperfunctions to which it belongs (see Theorem 2.4. in [15]).

Specially the following theorem is often used

Theorem 4.1. *(Theorem 1.3.2 in [11]). Let f be a continuous function. Let F denotes a defining function of the hyperfunction lf defined by f. Then $F_+(x+i\varepsilon) - F_-(x-i\varepsilon)$ converges locally uniformly to f as $\varepsilon \to 0$.*

If $f \in \mathcal{L}_{loc}(0,\infty)$, then $F_+(x+i\varepsilon) - F_-(x-i\varepsilon)$ converges for almost all x to f, when $\varepsilon \to 0$.

To define Laplace hyperfunctions we need some definitions. Let $\mathcal{O}$ be the radial compactification of $\mathbb{C}$

$$\mathbf{O} = \mathbb{C} \cup \mathcal{S}^1_\infty = \{re^{i\beta};\ 0 \le r \le \infty,\ 0 \le \beta < 2\pi\} \tag{19}$$

equipped with the natural topology.

We define now the space $\mathcal{O}^{\exp}$. For each open set V in $\mathbf{O}$ the space $\mathcal{O}^{\exp}(V)$ is defined to be the space of all the holomorphic functions $F(x)$ on $V \cap \mathbb{C}$ such that on each closed sector

$$\Sigma = \{z \in \mathbb{C} : \alpha \le arg(z-b) \le \beta\}$$

whose closure $\overline{\Sigma}_0$ in $\mathbf{O}$ is included in V, we have

$$|F(z)| \le ce^{H|z|}, \qquad z \in \Sigma,$$

with constants H and c.

Definition 4.3. *([13]).*

$$\mathcal{B}^{\exp}_{[a,\infty]} = \mathcal{O}^{\exp}(\mathbf{O} \setminus [a,\infty])/\mathcal{O}^{\exp}(\mathbf{O}) \tag{20}$$

is the space of Laplace hyperfunctions with support in $[a,\infty]$.

A Laplace hyperfunction f with support in $[a,\infty]$, $f \in \mathcal{B}^{\exp}_{[a,\infty]}$, is represented by the class $[F]$, where F is a holomorphic function, $F \in \mathcal{O}^{\exp}(\mathbf{O} \setminus [a,\infty])$, or by

$$f(x) = F_+(x+i0) - F_-(x-i0).$$

The next theorem gives an interesting relation between $\mathcal{B}^{\exp}_{[a,\infty]}$ and $\mathcal{B}_{[a,\infty)}$.

Theorem 4.2. *(Theorem 2 in [13]). The restriction mapping:* $\mathcal{O}^{\exp}(\mathbf{O} \setminus [a,\infty]) \to \mathcal{O}(\mathbb{C} \setminus [a,\infty))$ *induces a natural mapping* ρ :

$$\rho : \mathcal{B}^{\exp}_{[a,\infty]} \to \mathcal{B}_{[a,\infty)}. \tag{21}$$

The mapping ρ *is surjective but not injective. Its kernel equals* $\mathcal{B}^{\exp}_{[\infty]}$*. Hence we have the natural isomorphism:*

$$\mathcal{B}_{[a,\infty)} \cong \mathcal{B}^{\exp}_{[a,\infty]}/\mathcal{B}^{\exp}_{[\infty]}. \tag{22}$$

More generally we can consider the space

$$\mathcal{B}_{[a,b)} = \mathcal{O}(U \setminus [a,b))/\mathcal{O}(U), \tag{23}$$

U is a complex neighbourhood of $[a,b)$, of hyperfunctions with support in $[a,b)$, $-\infty < a < b \leq \infty$. The restriction mapping: $\mathcal{O}^{\exp}(\mathbf{O} \setminus [a,\infty]) \to \mathcal{O}(U \setminus [a,b))$ induces a natural mapping ρ

$$\rho : \mathcal{B}^{\exp}_{[a,\infty]} \to \mathcal{B}_{[a,b)}.$$

Theorem 4.3. *The natural mapping* ρ *is a surjective mapping whose kernel is identified with* $\mathcal{B}^{\exp}_{[b,\infty]}$*. Hence we have the natural isomorphism*

$$\mathcal{B}_{[a,b)} \cong \mathcal{B}^{\exp}_{[a,\infty]}/\mathcal{B}^{\exp}_{[b,\infty]}. \tag{24}$$

Remark 1. It follows from Theorem 4.2 and Theorem 4.3 that every locally integrable function, every distribution defined on $[a,b)$, $-\infty < a < b \leq \infty$, can be extended to a Laplace hyperfunction with support in $[a,\infty]$.

First we define the LT of Laplace hyperfunctions.

Definition 4.4. *([13]). Laplace transform* $\widehat{f} \equiv \mathcal{L}f$ *of an* $f = [F] \in \mathcal{B}^{exp}_{[a,\infty]}$ *is defined by*

$$\hat{f}(s) = \int_C e^{-sz} F(z) dz, \quad s \in \Omega, \tag{25}$$

where C *is a path of integration composed of a ray from* ${}_\infty e^{i\alpha}$ *to a point* $c < a$ *and a ray from* c *to* ${}_\infty e^{i\beta}$ *with* $-\pi/2 < \alpha < 0 < \beta < \pi/2$.

We note that the domain Ω of $\widehat{f}$ depends on the choice of the defining function F. Every such Ω is a complex neighbourhood of the set $B \equiv \{{}_\infty e^{i\beta};\ |\beta| < \pi/2\}$ in $\mathbf{O}$.

For an $f \in \mathcal{B}^{\exp}_{[a,\infty)}$ all its Ω shrink, containing the set B. Then the function $\widehat{f}$ should be regarded as the set of holomorphic functions such that if $\Omega_1 \subset \Omega_2 \subset \Omega$, then $\widehat{f}(\Omega_1)$ can be analytically continuable to Ω_2 and this continuation is just $\widehat{f}(\Omega_2)$.

Next theorem characterizes the LT and the space $\mathcal{L}\mathcal{B}^{\exp}_{[a,\infty]}$.

Theorem 4.4. *(Theorem 1 in [13]).* $\mathcal{L}$ *is an isomorphism*

$$\mathcal{L} : \mathcal{B}^{\exp}_{[0,\infty]} \to \mathcal{L}\mathcal{B}^{\exp}_{[a,\infty]}, \tag{26}$$

where $\mathcal{L}\mathbf{B}^{\exp}_{[a,\infty]}$ *is the space of holomorphic functions* $\hat{f}$ *of exponential type defined on a neighbourhood* Ω *of the semi-circle* $\mathcal{S} = \{{}_\infty e^{i\theta}; |\theta| < \pi/2\}$ *in* $\mathbf{O}$ *such that*

$$\overline{\lim}_{\rho\to\infty} \frac{\log|\widehat{f}(\rho e^{i\theta})|}{\rho} \le -a\cos\theta, |\theta| < \pi/2. \tag{27}$$

If $\widehat{f} \in \mathcal{L}\mathcal{B}^{\exp}_{[a,\infty]}$, *then a defining function* F *of its inverse image is given by*

$$F(z) = \frac{1}{2\pi i} \int_u^\infty e^{zs} \widehat{f}(s) ds, \quad z \in \mathbb{C} \setminus [a, \infty), \tag{28}$$

where u is a fixed point in Ω and the path of integration is a convex curve in Ω.

F belongs to $\mathcal{O}^{\exp}(\mathbb{C} \setminus [a,\infty))$ and

$$f(x) = F_+(x + i0) - F_-(x - i0).$$

In connection with (22) we have that $\mathcal{L}$ induces the isomorphisms:

$$\mathcal{L}\mathcal{B}_{[a,\infty)} \cong \mathcal{L}\mathcal{B}^{\exp}_{[a,\infty]}/\mathcal{L}\mathcal{B}^{\exp}_{[\infty]} \quad \text{or } \mathcal{L}\mathcal{B}_{[a,b)} \cong \mathcal{L}\mathcal{B}^{\exp}_{[a,\infty]}/\mathcal{L}\mathcal{B}^{\exp}_{[b,\infty]}. \tag{29}$$

Since every continuous function or locally integrable function on $[a,\infty)$ is indentified with a hyperfunction in $\mathcal{B}_{[a,\infty)}$, its LT makes sense as a class of holomorphic functions. But some classes of functions can be directly imbedded into $\mathcal{B}^{\exp}_{[a,\infty]}$ using LT. Such a class is $\mathcal{C}^{\exp}([0,\infty))$, the space of all continuous functions f on $[0,\infty)$ satisfying

$$|f(x)| \leq Ce^{Hx}, \quad x \geq 0. \tag{30}$$

If $G \in \mathcal{C}^{\exp}([0,\infty))$, then its Laplace transform

$$\widehat{g}(s) = \int_0^\infty e^{-sx} G(x)dx, \quad Res > H,$$

represents a holomorphic function which satisfies the estimate

$$|\widehat{g}(s)| \leq \frac{C}{Res - H}, \quad Res > H.$$

Because of (27) $\widehat{g}$ belongs to $\mathcal{L}\mathcal{B}^{\exp}_{[a,\infty]}$. Hence, the inverse image of $\widehat{g}$ gives by (28) the defining function F of $f \in \mathcal{B}^{\exp}_{[a,\infty]}$ and by Theorem 4.1 we obtain that f extends G on $[0,\infty]$. In this way the space $\mathcal{C}^{\exp}([0,\infty))$ is naturally imbedded in $\mathcal{B}^{\exp}_{[a,\infty]}$. If $G \in \mathcal{C}^{\exp}([0,\infty))$, then we denote by θG the corresponding Laplace hyperfunction. (θ stands for the Heaviside function). Similarly we can imbed measurable functions satisfying exponential type condition (30). A direct consequence is that the classical LT of an $G \in \mathcal{C}^{\exp}_{([0,\infty))}$ and the LT of $\theta G \in \mathcal{B}^{\exp}_{[a,\infty]}$ coincide. This makes possible the use of well elaborated classical theory of LT ([8], [26]).

We give some properties of the LT of Laplace hyperfunctions (see [13]). For $f \in \mathcal{B}^{\exp}_{[a,\infty]}$

$$\mathcal{L}(x^n e^{ax} f(x))(s) = \Big(-\frac{d}{ds}\Big)^n \mathcal{L}f(s-a),\ n \in \mathbb{N} \cup \{0\},\ a \in \mathbb{C}. \tag{31}$$

$$\mathcal{L}\Big(\frac{d^n}{dx^n} f(x+c))(s) = s^n e^{cs} \mathcal{L}f(s),\ n \in \mathbb{N} \cup \{0\},\ c \in \mathbb{R}. \tag{32}$$

$$\mathcal{L}(\delta^{(\alpha)})(s) = s^\alpha,\ \text{where } \delta^{(\alpha)} = \delta^{(\alpha)}(x),\ \alpha = 0, 1, 2, ... \text{ and} \tag{33}$$

$$\delta^{(\alpha)} = x_+^{-\alpha-1}/\Gamma(-\alpha), \quad \alpha \neq 0, 1, , 2, ..., \alpha \in \mathbb{C}. \tag{34}$$

We can define the convolution of Laplace hyperfunctions of two elements f, g belonging to $\mathcal{B}^{\exp}_{[a,\infty]}$ without any restriction. Since $\widehat{f} \cdot \widehat{g} \in \mathcal{L}\mathcal{B}^{\exp}_{[a,\infty]}$, the convolution is defined by

$$\mathcal{L}(f * g)(x) = \widehat{f}(s)\widehat{g}(s). \tag{35}$$

Hence, $f * g \in \mathcal{B}^{\exp}_{[a,\infty]}$.

If $\theta f, \theta g \in \theta \mathcal{C}^{\exp}_{[0,\infty)}$, then

$$\theta f * \theta g = \theta \int_0^t f(t-\tau) g(\tau) d\tau. \tag{36}$$

5. A suggestion to develop applications of the LT

Komatsu's results, as we have seen, overcome all the shortcomings of the LT defined in different spaces. But he uses a very abstract space, hyperfunctions, and only in one dimension. Such an abstractness, from the experience, can not be easily accepted by the great part of people working in applications.

Komatsu's basic idea can be used to elaborate LT of an larger space than $\mathcal{D}'(\Gamma+, a),\ a \geq 0$ but not so large as hyperfunctions. We give an idea to construct such a space, noted by $\mathcal{D}'_*(P, a)$.

Let P be the set

$$P = \prod_{i=1}^{n} [p_i, q_i),\ p_i, q_i \in \mathbb{R},\ p_i < q_i,\ i = 1, ..., n.$$

$\overline{P}$ is compact. Since $\overline{\mathbb{R}}^n_+$ is a closed convex and acute cone, $\mathcal{D}'(\overline{\mathbb{R}}^n_+ + \overline{P}, a),\ a \geq 0$ is well defined.

Let $\mathcal{A}$ be the vector space

$$\mathbf{A} = \{T \in \mathcal{D}'(\overline{\mathbb{R}}^n_+ + \overline{P}, a);\ \text{supp}T \subset ((\overline{\mathbb{R}}^n_+ + \overline{P}) \setminus P)\}.$$

Now, we can define an equivalence relation in $\mathcal{D}'(\overline{\mathbb{R}}^n_+ + \overline{P}, a);\ f \sim g \Longleftrightarrow f - g \in \mathbf{A}$. Let us denote by $\mathbf{B}$ the quotient space

$$\mathbf{B} = \mathcal{D}'(\overline{\mathbb{R}}^n_+ + \overline{P}, a)/A$$

and by

$$\mathcal{D}'_*(P, a) = \{T \in \mathcal{D}'(P);\ \exists\ \overline{T} \in \mathcal{D}'(\overline{\mathbb{R}}^n_+ + \overline{P}, a),\ \overline{T}|_P = T\},$$

where $\overline{T}|_P$ is the restriction of $\overline{T}$ on P. Since $\mathcal{D}'$ is not a flabby sheaf, $\mathcal{D}'_*(P, a) \neq \mathcal{D}'(P)$.

It can be prove that $\mathcal{D}'_*(P,a)$ is algebraically isomorphic to $\mathbf{B}$. Now, the procedure to define the LT and to use this LT is just the same as for the LT of hyperfunctions. In this way, for example, every $f \in \mathcal{L}_{loc}[0,\infty)$, should have LT, without any growth condition.

6. Applications

The LT of hyperfunctions can be successfully applied to linear equations with derivatives, partial derivatives, fractional derivatives and convolutions to find classical and generalized solutions, global or localized. We illustrate Section 4. applying the LT to an equation which describes the dynamics of a rod made of generalized Kelvin-Voight viscoelastic material (cf. [1], [21] and [22]) in which the force $F = \omega + A\delta(t-t_0),\ t_0 > 0$, where ω is a constant and δ is the Dirac distribution, $\gamma \in \mathbb{R}$ and $Q \in \mathcal{L}_{loc}(\mathbb{R})$.

The quation is the following

$$(37)\quad T^{(2)}(t)+\gamma T^{(\beta)}(t)+\omega T(t)+A\delta(t-t_0)T(t)=Q(t),\ t>0,\ 0<\beta<1.$$

With initial condition: $T(0)=T_0,\ T^{(1)}(0)=T'_0$.

Since δ can be treated as a measure, (37) has a meaning if $T \in \mathcal{C}((0,\infty))$ because $t_0 > 0$. Therefore we look for a solution to (37) belonging to $\mathcal{C}([0,\infty))$, where $\delta(t-t_0)T(t) = T(t_0)\delta(t-t_0)$.

In (37) we have a singular coefficient $A\delta(t-t_0)$ therefore we localize the solution on the interval $(0,t_0)$ supposing that there exists a solution $T \in \mathcal{C}^2((0,t_0)\cap\mathcal{C}^1([0,t_0)),\ T^{(2)} \in \mathcal{L}^1([0,t_0))$.

The first step will be to find the correspondent equation in $\mathcal{B}_{[0,t_0)}$. Let H be the function $H(t)=1,\ t\in[0,t_0)$ and $H(t)=0,\ t\in(-\infty,0)\cup[t_0,\infty)$.

Multiplying (37) by H,

$$HT^{(2)}(t) + \gamma HT^{(\beta)}(t) + \omega HT(t) = HQ(t),\ t \in \mathbb{R}. \tag{38}$$

Let $f = \{HT\}$ be the hyperfunction which corresponds to the continuous function $HT(t),\ t \geq 0$, and $\{T^{(2)},\ 0 < t < t_0\}$ (in short $\{T^{(2)}\}$) the hyperfunction which correspond to the function $T^{(2)}(t),\ 0 < t < t_0$. The Green formula gives the relation between $D^2\{HT\}$ and $\{T^{(2)}\}$ ($D^2\{HT\}$ is the second derivative in the sense of hyperfunctions):

$$\begin{aligned} D^2\{HT\} &= \{T^{(2)}\} + T^{(1)}(0)\delta(t) + T(0)\delta^{(1)}(t) \\ &= \{T^{(2)}\} + T_0'\delta(t) + T_0\delta^{(1)}(t). \end{aligned} \tag{39}$$

With regards to $T^{(\beta)}$, we have by definition:

$$T^{(\beta)}(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t T(t-u)u^{-\beta}du,\ 0 < t,\ 0 < \beta < 1.$$

We can use (34), (36), (39) and the supposition $T \in \mathcal{C}([0.t_0))$ to prove:

$$\begin{aligned} \{D^\beta T\} &= \Big\{\frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t T(t-u)u^{-\beta}du,\ 0 < t < t_0\Big\} \\ &= \frac{1}{\Gamma(1-\beta)}\Big\{\frac{d}{dt}H(t)\int_0^t T(t-u)u^{-\beta}du,\ 0 < t < t_0\Big\} \\ &= \frac{1}{\Gamma(1-\beta)}\Big\{\frac{d}{dt}\int_0^t H(u)T(u)(t-u)^{-\beta}du, 0 < t < t_0\Big\} \\ &= \frac{1}{\Gamma(1-\beta)}D\{(HT * u^{-\beta})\} = \delta^{(\beta)} * \{HT\} = D^\beta\{HT\}. \end{aligned} \tag{40}$$

In (39) and (40) we use the supposed properties of the solution T.

In view of (39) and (40), to (38) with the initial condition $T(0) = T_0,\ T^{(1)}(0) = T_0'$ it corresponds in $\mathcal{B}_{[0,\infty)}$

$$D^2\{HT\} + \gamma D^\beta\{HT\} + \omega\{HT\} = \{HQ\} + T_0'\delta(t) + T_0\delta^{(1)}(t). \tag{41}$$

Now, we can apply LT to (41)

$$(s^2 + \gamma s^\beta + \omega)\mathcal{L}(\{\overline{HT}\})(s) = \mathcal{L}(\{\overline{HQ}\})(s) + T_0' + T_0 s + \mathcal{L}(W)(s), \tag{42}$$

where $\{\overline{HT}\}$ is an element of $\mathcal{B}^{\exp}_{[a,\infty]}$ which correspond to $\{HT\} \in \mathcal{B}_{[0,t_0)}$ and $W \in \mathcal{B}^{\exp}_{[t_0,\infty]}$.

Thus,

$$\mathcal{L}(\{\overline{HT}\})(s) = \frac{1}{s^2 + \gamma s^\beta + \omega}(\mathcal{L}(\{HQ\})(s) + T_0' + T_0 s + \mathcal{L}(W)(s)) \tag{43}$$

We can prove that

$$g(s) = \frac{1}{s^2 + \gamma s^\beta + \omega}$$

has the following properties:

1. There exists a function $G \in C^{\exp}([0,\infty))$ such that $\mathcal{L}(G)(s) = g(s)$ in the sense of the classical LT.

2. $g(s)$ also satisfies conditions in Theorem 4.3. Consequently there exists $f \in \mathcal{B}^{\exp}_{[0,\infty]}$ such that $\mathcal{L}(f)(s) = g(s)$. Thus f is defined by the function G.

Now, (43) gives

$$\{\overline{HT}\} = \theta G * \{\overline{HT}\} + \theta G T_0' + T_0\{G^{(1)}\} + \theta G * W.$$

Since $\theta G * W \in \mathcal{B}^{\exp}_{[t_0,\infty]}$,

$$T(t) = (G * Q)(t) + T_0' G(t) + T_0 G^{(1)}(t),\ 0 < t < t_0.$$

It can be proved that $T \in \mathcal{C}^2((0,,t_0)) \cap \mathcal{C}^1([0,t_0))$ (cf. [21]).

If we consider equation (37) with $t > 0$, then we can prove that it has no classical solutions. Namely, the solution T belongs to $\mathcal{C}([0,\infty))$, but it has two first derivatives only in the sense of distributions because of the singular coefficient in (37).

Let us analyze this case too.

Let $\{\theta T\}$ be the hyperfunction belonging to $\mathcal{B}_{[0,\infty)}$ which corresponds to the continuous function $\theta(t)T(t),\ t \in \mathbb{R}$. ($\theta$ is the Heaviside function). The function $(\theta T)_{t_0}(t) = \theta(t)T(t),\ t \in \mathbb{R},\ t \neq t_0$, is a locally integrable function which defines the same hyperfunction $\{\theta T\}$. Then

$$\{\theta T\} = \{(\theta T)_{t_0}\} = \{(\theta T)_1\} + \{T_2\}, \quad \text{where } (\theta T)_1(t) = \theta(t)T(t)|_{t\in[0,t_0)}$$

(the restriction of $\theta T(t)$ on $[0,t_0)$) and

$$(\theta T)_1(t) = 0,\ t \in (-\infty, 0) \cup (t_0, \infty);\ T_2(t) = T(t), t \in (t_0, \infty)$$

and

$$T_2(t) = 0,\ t \in (-\infty, t_0).$$

(cf. Chapter I, §3 in [11]). By the results on (37) when $0 < t < t_0$, we can suppose that $(\theta T)_1 \in \mathcal{C}^2((0,t_0)) \cap \mathcal{C}^1([0,t_0))$ and (39), (40) can be applied to $(\theta T)_1$. Hence,

$$\begin{aligned} D^2\{\theta T\} &= D^2\{(\theta T)_1\} + D^2\{T_2\} \\ &= \{T^{(2)}, 0 < t < t_0\} + D^2\{T_2\} + T_0'\delta(t) + T_0\delta^{(1)}(t) \\ &= D^2(\{T, 0 < t < t_0\} + \{T_2\} + T_0'\delta(t) + T_0\delta^{(1)}(t) \\ &= D^2\{(\theta T)_{t_0}, t > 0\} + T_0'\delta(t) + T_0\delta^{(1)}(t). \end{aligned}$$

To (37) it corresponds now in $\mathcal{B}_{[0,\infty)}$

$$\begin{aligned} D^2\{\theta T\} + \gamma D^\beta\{\theta T\} + \omega\{\theta T\} &= T_0\delta^{(1)}(t) + T_0'\delta(t) \\ &\quad -AT(t_0)\delta(t-t_0) + \{\theta Q\}. \end{aligned}$$

Applying LT to the last equation, we have instead of (43)

$$\mathcal{L}(\{\overline{\theta T}\})(s) \quad = \frac{1}{s^2+\gamma s^\beta+\omega}(\mathcal{L}(\{\overline{\theta T}\})(s)+T_0 s+T_0'$$

$$-AT(t_0)e^{-t_0 s}+\mathcal{L}(W)(s)).$$

So that the sought solution with the supposed properties is

$$T(t) \quad = \int_0^\infty G(t-u)Q(u)du+T_0 G^{(1)}(t)+T_0' G(t)$$

$$-AT(t_0)\theta(t-t_0)G(t-t_0), \quad t>0.$$

References

[1] Atanacković T.M., Stanković B., Dynamics of a functional derivative type of a viscoelastic rod, ZAMM (to be published).

[2] Berg L., Asymptotische Auffasung der Operatorenrechnung, Studia Math. XX (1962), 215-229.

[3] Bogolubov N.N., Vladinirov V.S. and Tavkelidze A.N., On automodel asymptotics in quantum field theory, Proc. Steklov Inst. Math. Issue 1 (1978), 27-54.

[4] Бръичков Ю. Широков Ю ., Об асимтотическом поведении преобраований Фуръие. ТМФ 4, Но 3, (1970), 301-309.

[5] Bromwich T.J.A., Normal coordinates in dynamical systems, Proc. London Math. Soc. 15, (1916), 401-448.

[6] Carson J.R., The Heaviside operational calculus, Bull. Amer. Math. Soc. 32, (1926), 43-68.

[7] Ditkin V.A., Operator calculus, Uspehi Mat. Nauk, T. II, Vip. 6 (22) (1947), 72-158 (in Russian).

[8] Doetsch G., Handbuch der Laplace-Transformation, I-III Basel - Stuttgart, 1950-1956.

[9] Erdélyi A., Higher transcedental functions, New York, Toronto, London, 1955.

[10] Heaviside O., Electromagnetic Theory, Vols. I, II, Electrician, London, 1983.

[11] Kaneko A., Introduction to hyperfunctions, Kluwer Ac. Publishers, Dordrecht 1988.

[12] Komatsu H., Laplace transform of hyperfunctions - A new fondation of the Heaviside calculus, J. Fac. Sci. Univ. Tokyo, IA, 34 (1987), 805-820.

[13] Komatsu H., Solution of differential equations by means of Laplace hyperfunctions, Structure of solutions of differential equations (Katata / Kyoto, 1995), World Sci. Publishing, River Edge, NJ (1996), 227-252.

[14] Komatsu H., Ultradistrubutions, I, Structure theorems and a characterization, J. Fac. Sci. Univ. Tokyo, Sec. IA Vol. 20 (1973), 25-107.

[15] Komatsu H., An introduction to the theory of hyperfunctions, Lecture Notes in Math., 287, (1973), 3-40.

[16] Mikusinski J., Sur les fondaments du calcul opératoire, Studia Math. XI, (1950), 41-70.

[17] Rosinger E.E., Non-Linear Partial Differential Equations, An Algebraic View of Generalized Solutions, North - Holland, Amsterdam, 1990.

[18] Sato M., Theory of hyperfunctions, J. Fac. Sci., Univ. Tokyo, Sec. I, 8 (1959), 139-193.

[19] Schwartz L., Théorie des Distributions, 3 éd., Hermann, Paris, 1966.

[20] Stanković B., Differential equation with fractional derivative and nonconstant coefficients, Integral Transforms and Special Functions, (to be published).

[21] Stanković B. and Atanacković T., Dynamics of a rod made of generalized Kelvin+Voigt visco-elastic material, Journal of Math. Anal. and Appl. (to be published).

[22] Stanković B. and Atanacković T., On one model of a viscoelastic rod, Fractional Calculus and Applied Analysis (to be published).

[23] Владимиров В.С., Завъялов Б.И., Тауберовы теоремы в квантовой теории поля, Итоги науки и техники, Винити, 1980, Т. 25, 95-130.

[24] Владимиров В.С., Уравнения математицеской физики, Москва "Наука", 1988.

[25] Widder D.V., Laplace transform, Princeton Univ. Press, Princeton, 1941.

[26] Zharinov V.V., Laplace transform of Fourier hyperfunctions and other close classes of analytic functionals, Teor. Math. Phys., 33:3. (1977), 291-309 (in Russian).