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The Irreversible Quantum Dynamics of the Three-Level $su(1,1)$ Bosonic Model

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Abstract: We study the quantum dynamics of the opened three-level $su(1,1)$ bosonic model. The effective non-Hermitian Hamiltonians describing the system of the Lindblad equation in the short time limit are constructed. The obtained non-Hermitian Hamiltonians are exactly solvable by the Algebraic Bethe Ansatz. This approach allows representing biorthogonal and nonorthogonal bases of the system. We analyze the biorthogonal expectation values of a number of particles in the zero mode and represent it in the determinantal form. The time-dependent density matrix satisfying the Lindblad master equation is found in terms of the nonorthogonal basis.

Keywords: open quantum system; Bethe Ansatz; Lindblad equation

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1. Introduction

There have been remarkable progresses and research activities in the study of the quantum dynamics of Bose–Einstein condensates. The novel dynamical effects such as pumped-up interferometry, atomic Einstein–Podolsky–Rosen entanglement, quantum scanning microscopy for cold atoms, and quantum phase transitions in spinor condensates have initiated great interest in the physics beyond the mean-field approximation [1–5]. Because of the complexity of such systems, the simplified "toy" exactly solvable models play an important role [6–11].

Over the years, an intense research effort has been devoted to the investigation of non-Hermitian systems both theoretically and experimentally [12–15]. The natural source of non-Hermiticity in quantum many-body systems is induced by coupling the system to its environment. The relevant equation of motion of the open system is the Lindblad master equation [16].

In our paper, we shall consider an effective three-level $su(1,1)$ boson system of the hyperfine manifold of a spin-1 of ultracold atoms [1,3,10]. In the presence of a boson loss, the effective non-Hermitian will be obtained in the short time limit of the Lindblad equation. The validity of the considered approximation was studied in detail in the papers [17,18]. The representation of the dynamical variables of the model as the generators of the $su(1,1)$ algebra allows imbedding the obtained non-Hermitian Hamiltonians into the well-established scheme of the quantum inverse method [19] and solving exactly the model up to its eigenvalues and eigenstates in biorthogonal and nonorthogonal bases. The obtained bases allow calculating the expectation values of a number of particles in the zero mode and the time-dependent Hermitian operator satisfying the Lindblad equation.

The advantage of this approach is that it is possible to simultaneously solve a number of models with the different atom–atom interactions using proper bosonic realizations of the $su(1,1)$ algebra [11,20]. The described method allows setting the connection of the considered models with the models of the quantum optics [21,22].

The paper is organized as follows. In Section 2, the system of bosonic atoms describing the spin-mixing dynamics is given. The Lindblad equation in the short time approximation is considered, and the effective non-Hermitian Hamiltonians are obtained. In Section 3, different bosonic realizations of the $su(1,1)$ algebra are used, and the effective non-Hermitian Hamiltonians are expressed through the generators of the algebra. In Section 4, the generators of the $su(1,1)$ loop algebra are considered, which made it possible to apply the Algebraic Bethe Ansatz method to solve the effective non-Hermitian Hamiltonians in Section 5. In the final section, the biorthogonal and nonorthogonal bases are introduced, and the expectation values of number of particles in the zero mode are represented in the determinantal form.

2. Dissipative Spinor Bose–Einstein Condensates

The hyperfine manifold of a spin-1 Bose–Einstein condensate of ultracold atoms can be used to construct an effective bosonic three-level system. Spin-mixing collisions coherently outcouple pairs of atoms from the $m_F = 0$ state to the $m_F = \pm 1$ states. The full spin-mixing dynamics may be mapped to an effective "toy" Hamiltonian [6,7] of the form:

$$H_{sp} = \frac{g}{2}\left[a_0^2 a_1^\dagger a_{-1}^\dagger + (a_0^\dagger)^2 a_1 a_{-1}\right] + \frac{g}{2}(n_0 + \frac{1}{2})(n_1 + n_{-1} + 1) - \kappa n_0, \tag{1}$$

where $a_\alpha, a_\alpha^\dagger$ $(\alpha = \pm 1, 0)$ are annihilation, creation operators associated with the spin mode satisfying canonical commutation relations $[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}$, and the particle number operator $n_\alpha = a_\alpha^\dagger a_\alpha$. The coupling is given by the constant g, while detuning by the constant κ. The number operator:

$$\hat{N} = n_1 + n_{-1} + n_0 \tag{2}$$

and the population difference operator:

$$\Delta = n_1 - n_{-1} \tag{3}$$

commute with the introduced Hamiltonian (1).

The Lindblad master equation [16] governs the dynamics of the density matrix ρ in the presence of a boson loss:

$$\partial_t \rho = i[\rho, H_{sp}] + \sum_\alpha \Gamma_\alpha \left(a_\alpha \rho a_\alpha^\dagger - \frac{1}{2} n_\alpha \rho - \frac{1}{2} \rho n_\alpha \right), \tag{4}$$

where the so-called jump operators a_α account for the coupling to the environment and the rates Γ_α are positive. The Lindblad equation can conveniently be written as

$$\partial_t \rho = i\left(\rho H_{eff}^\dagger - H_{eff}\rho\right) + \sum_\alpha \Gamma_\alpha a_\alpha \rho a_\alpha^\dagger. \tag{5}$$

The effective non-Hermitian Hamiltonian:

$$H_{eff} = H_{sp} - \frac{i}{2}\sum_\alpha \Gamma_\alpha n_\alpha \tag{6}$$

describes the continuous losses of energy and deterministic dynamics at short times [14]. At longer times, the so-called "recycling" term $\sum_\alpha \Gamma_\alpha a_\alpha \rho a_\alpha^\dagger$ accounting for the occurrence of quantum jumps can typically no longer be neglected. The quantum jumps describe the effect of the measurement on the state of the system.

In the present paper, we shall consider the case when the rates $\Gamma_+ = \Gamma_- = 0$ and $\Gamma_0 \equiv \Gamma$. Dropping quantum jumps, one obtains

$$\partial_t \rho = i\left(\rho H_{eff}^\dagger - H_{eff}\rho\right), \tag{7}$$

which is equivalent to working with the non-Hermitian Hamiltonian:

$$H_{eff} = H_{sp} - \frac{i}{2}\Gamma n_0 \,. \quad (8)$$

Notice that $H_{sp} = H_{sp}^{\dagger}$ and $n_0 = n_0^{\dagger}$. A density matrix ρ is Hermitian $\rho = \rho^{\dagger}$.

In the explicit form:

$$\begin{aligned} H_{eff} &= H_{sp} - \frac{i}{2}\Gamma n_0 \\ &= \frac{g}{2}\left[a_0^2 a_1^{\dagger} a_{-1}^{\dagger} + (a_0^{\dagger})^2 a_1 a_{-1}\right] + \frac{g}{2}\left(n_0 + \frac{1}{2}\right)(n_1 + n_{-1} + 1) - (\kappa + \frac{i}{2}\Gamma)n_0 \,. \end{aligned} \quad (9)$$

The obtained non-Hermitian Hamiltonian may be solved exactly for its eigenstates and eigenvalues.

3. Hamiltonian as the $su(1,1)$ Model

To solve the model for its eigenstates and eigenvalues, it is convenient to express Hamiltonian (9) in terms of the generators of the $su(1,1)$ algebra. The generators of this algebra satisfy commutation relations:

$$[\mathcal{K}^0, \mathcal{K}^{\pm}] = \pm\mathcal{K}^{\pm}, \quad [\mathcal{K}^+, \mathcal{K}^-] = -2\mathcal{K}^0. \quad (10)$$

The Casimir invariant of $su(1,1)$ is given by

$$\mathcal{K}^2 = (\mathcal{K}^0)^2 - \frac{1}{2}(\mathcal{K}^+\mathcal{K}^- + \mathcal{K}^-\mathcal{K}^+). \quad (11)$$

There are several representations of $su(1,1)$. Our interest will be confined to the representations based on the usual bosonic operators.

The two-mode boson realization of this algebra is

$$\begin{aligned} K^0 &= \frac{1}{2}(a_1^{\dagger}a_1 + a_{-1}^{\dagger}a_{-1} + 1), \\ K^+ &= a_1^{\dagger}a_{-1}^{\dagger} \quad , \quad K^- = a_1 a_{-1} \,. \end{aligned} \quad (12)$$

The Casimir operator for this realization can be written as

$$\begin{aligned} \mathcal{K}_K^2 &= \frac{1}{4}(\Delta^2 - 1), \\ \Delta &= a_1^{\dagger}a_1 - a_{-1}^{\dagger}a_{-1}. \end{aligned} \quad (13)$$

Obviously, the population difference operator Δ defined by Equations (3) and (13) commutes with all the operators in (12), and thus, the population difference in the modes with $\alpha = \pm$ must differ by some fixed amount, the eigenvalue of Δ. We denote this eigenvalue as m, and without loss of generality, we take m to be a positive integer.

For a single-mode boson field $\alpha = 0$, the $su(1,1)$ algebra is realized by the operators:

$$\begin{aligned} B^0 &= \frac{1}{2}a_0^{\dagger}a_0 + \frac{1}{4}, \quad (14) \\ B^+ &= -\frac{1}{2}(a_0^{\dagger})^2 \quad , \quad B^- = -\frac{1}{2}(a_0)^2. \end{aligned}$$

In this case, the Casimir operator reduces identically to

$$\mathcal{K}_B^2 = -3/16. \quad (15)$$

Expressed in terms of $su(1,1)$ generators, the Hamiltonian (9) reads as

$$H_{eff} = 2gB^0K^0 - g\big(B^+K^- + B^-K^+\big) - 2(\kappa + \frac{i}{2}\Gamma)B^0 + \frac{1}{2}(\kappa + \frac{i}{2}\Gamma)\,, \tag{16}$$

and the number operator (2) as

$$\hat{N} = 2(B^0 + K^0) - \frac{3}{2}\,, \tag{17}$$

Furthermore, it is worth presenting the Hamiltonian in the form:

$$H \equiv \frac{1}{g}H_{eff} - \frac{1}{2g}(\kappa + \frac{i}{2}\Gamma) = 2B^0K^0 - B^+K^- - B^-K^+ - 2\delta B^0\,, \tag{18}$$

where $\delta = (\kappa + \frac{i}{2}\Gamma)/g$, respectively, the conjugated Hamiltonian:

$$H^\dagger = 2B^0K^0 - B^+K^- - B^-K^+ - 2\bar{\delta}B^0\,. \tag{19}$$

The most obvious conserved quantities of the Hamiltonian (18) are the total number of particles and the population difference in the modes with the opposite spins:

$$[H, \hat{N}] = [H, \Delta] = 0. \tag{20}$$

The commutativity of the Hamiltonian with the Casimir operator (15):

$$[H, \mathcal{K}_B^2] = 0, \tag{21}$$

means that the parity of the number of particles in the mode with $\alpha = 0$ is the conserved quantity as well. These conservation laws follow directly from the fact that the Hamiltonian describes the creation and annihilation of bosonic atoms in pairs.

4. The $su(1,1)$ Loop Algebra

In this section, we shall demonstrate that the Hamiltonian (18) may be constructed with the help of the following operators:

$$\begin{aligned} X_\delta^0(\lambda) &= \frac{B^0}{\delta - \lambda} - \frac{K^0}{\lambda} + 1\,, \\ X_\delta^\pm(\lambda) &= \frac{B^\pm}{\delta - \lambda} - \frac{K^\pm}{\lambda}\,. \end{aligned} \tag{22}$$

Here, λ is a complex variable, while the complex-valued constant δ is defined in the previous section.

The operators (22) satisfy the following commutation relations:

$$\begin{aligned} [X_\delta^+(\lambda), X_\delta^-(\mu)] &= -\frac{2}{\lambda - \mu}\Big(X_\delta^0(\lambda) - X_\delta^0(\mu)\Big), \\ [X_\delta^0(\lambda), X_\delta^\pm(\mu)] &= \pm\frac{1}{\lambda - \mu}\big(X_\delta^\pm(\lambda) - X_\delta^\pm(\mu)\big), \\ [X_\delta^+(\lambda), X_\delta^+(\mu)] &= [X_\delta^-(\lambda), X_\delta^-(\mu)] = [X_\delta^0(\lambda), X_\delta^0(\mu)] = 0. \end{aligned} \tag{23}$$

These equalities are checked by applying the commutation relations of the operators (12), (15) and the equality:

$$\frac{1}{(\epsilon - \lambda)(\epsilon - \mu)} = \frac{1}{\lambda - \mu}\left(\frac{1}{\epsilon - \lambda} - \frac{1}{\epsilon - \mu}\right). \tag{24}$$

Algebra (23) is known as the $su(1,1)$ loop algebra.

The operators (22) are in the involution:

$$(X^0_\delta(\lambda))^\dagger = X^0_{\bar{\delta}}(\bar{\lambda}),$$
$$(X^\pm_\delta(\lambda))^\dagger = X^\mp_{\bar{\delta}}(\bar{\lambda}). \tag{25}$$

Here, † is the Hermitian conjugation with complex conjugation.

By analogy to the Casimir operator (11), we introduce a family of operators depending on the arbitrary complex number λ :

$$t_\delta(\lambda) = \left(X^0_\delta(\lambda)\right)^2 - \frac{1}{2}\left(X^+_\delta(\lambda)X^-_\delta(\lambda) + X^-_\delta(\lambda)X^+_\delta(\lambda)\right). \tag{26}$$

The most-important property of these operators is that they commute for the arbitrary complex numbers λ, μ:

$$[t_\delta(\lambda), t_\delta(\mu)] = 0. \tag{27}$$

This property is checked by direct calculation with the help of the commutation relations (23). The operator $t_\delta(\lambda)$ may be considered as the generating function of the integrals of motion.

Substituting (22) into (26), we have

$$t_\delta(\lambda) = 1 + \left(\frac{B^0}{\delta - \lambda}\right)^2 + \left(\frac{K^0}{\lambda}\right)^2 - \frac{B^+B^- + B^-B^+}{2(\delta-\lambda)^2} - \frac{K^+K^- + K^+K^-}{2\lambda^2}$$
$$- \frac{2K^0}{\lambda} + \frac{2B^0}{\delta - \lambda} - \frac{2B^0K^0 - B^+K^- - B^-K^+}{(\delta-\lambda)\lambda}. \tag{28}$$

The coefficient at the simple pole of this expression, when $\lambda = \delta$, is equal to

$$Res|_{\lambda=\delta} t_\delta(\lambda) = 2B^0 - \frac{1}{\delta}\left\{2B^0K^0 - B^+K^+ - B^-K^-\right\}, \tag{29}$$

and we have the following expression for the Hamiltonian (18):

$$H = -\delta Res|_{\lambda=\delta} t_\delta(\lambda). \tag{30}$$

From (28) and (27), it follows that

$$[t_\delta(\lambda), H] = [t_\delta(\lambda), N] = [t_\delta(\lambda), \Delta] = [t_\delta(\lambda), \mathcal{K}^2_B] = 0. \tag{31}$$

The conjugated generating operator satisfies the relation:

$$t^\dagger_\delta(\lambda) = t_{\bar{\delta}}(\bar{\lambda}). \tag{32}$$

It gives

$$H^\dagger = -\bar{\delta} Res|_{\bar{\lambda}=\bar{\delta}} t_{\bar{\delta}}(\bar{\lambda}). \tag{33}$$

Knowing the eigenvectors and eigenvalues of generating operators $t_\delta(\lambda)$ and $t_{\bar{\delta}}(\bar{\lambda})$, we may find the eigenenergies of the Hamiltonians H (18) and $H^\dagger$ (19) applying Equations (30) and (33).

5. The Algebraic Bethe Ansatz

To develop the algebraic scheme of the diagonalization of the generating function $t_\delta(\lambda)$ (26), first, we recall that the basis of the unitary irreducible representation of the $su(1,1)$ algebra is formed by the eigenvectors $|n\rangle_\nu$ of operator $\mathcal{K}^0$ and Casimir operator $\mathcal{K}^2$:

$$\mathcal{K}^0|n\rangle_\nu = (n+\nu)|n\rangle_\nu,$$
$$\mathcal{K}^2|n\rangle_\nu = \nu(\nu-1)|n\rangle_\nu, \tag{34}$$

where ν is the so-called Bargmann index. The operators $\mathcal{K}^{\pm}$ act as the rising and lowering operators, respectively, on the eigenstates of $\mathcal{K}^0$. The non-normalized states $|n\rangle_\nu$ may be constructed by the successive action of operator $\mathcal{K}^+$ from the generating vector $|0\rangle_\nu$ defined by the equation:

$$\mathcal{K}^-|0\rangle_\nu = 0. \tag{35}$$

These states are equal to:

$$|n\rangle_\nu = \left(\mathcal{K}^+\right)^n|0\rangle_\nu. \tag{36}$$

The conjugated states are given by the relations:

$$\begin{aligned} \langle 0|_\nu\mathcal{K}^+ &= 0\,, \\ \langle n|_\nu &= \langle 0|_\nu\left(\mathcal{K}^-\right)^n. \end{aligned} \tag{37}$$

The representation space of the two-mode realization (12), (13) of the $su(1,1)$ algebra consists of two-mode Fock states, which are the direct product of the number states of spin modes with $\alpha = \pm$. The generating vector of this realization is defined by the equation:

$$K^-|0\rangle_{\nu_2} = 0. \tag{38}$$

We may choose $|0\rangle_{\nu_2} \equiv |m\rangle^{(+)} \otimes |0\rangle^{(-)}$ or $|0\rangle_{\nu_2} \equiv |0\rangle^{(+)} \otimes |m\rangle^{(-)}$, $(m = 0, 1, \ldots)$. The Bargmann index of this realization is $\nu_2 = \frac{m+1}{2}$. It follows from (12) that these states are the eigenstates of the operator Δ (13):

$$\Delta|0\rangle_{\nu_2} = \alpha m|0\rangle_{\nu_2}\,, \tag{39}$$

where m is the difference of the population.

The representation space in the single-mode realization (15) is decomposed into the direct sum of two irreducible components spanned by the states $|2n+s\rangle$ with an even number of particles ($s = 0$) or by the states with an odd number of particles ($s = 1$). The Bargmann index of this realization is $\nu_1 = \frac{2s+1}{4}$, and the generating vector (35) is defined by the equation:

$$B^-|0\rangle_{\nu_1} = 0. \tag{40}$$

The space with $\nu_1 = \frac{1}{4}$ ($s = 0$) is built from the Fock vacuum $|0\rangle_{\nu_1} \equiv |0\rangle^{(0)}$, and the space with $\nu_1 = \frac{3}{4}$ ($s = 1$) is built from the one-particle state $|0\rangle_{\nu_1} \equiv |1\rangle^{(0)}$, respectively. The generating space satisfies the relation:

$$B^0|0\rangle_{\nu_1} = \frac{2s+1}{4}|0\rangle_{\nu_1} \tag{41}$$

From the definition of the operators $X^{\pm}_\delta(\lambda), X^0_\delta(\lambda)$ (22) and the number operator $\hat{N}$ (17), it follows that

$$\hat{N}X^{\pm}_\delta(\lambda) = X^{\pm}_\delta(\lambda)\left(\hat{N} \pm 2\right), \tag{42}$$

and

$$\hat{N}X^0_\delta(\lambda) = X^0_\delta(\lambda)\hat{N}. \tag{43}$$

Therefore, $X^{\pm}_\delta(\lambda)$ acts as a creation (annihilation) operator of the pair of boson quasi-particles.

The state that is the direct product of the generating states (40) and (38):

$$|\Omega\rangle = |0\rangle_{\nu_1} \otimes |0\rangle_{\nu_2}, \tag{44}$$

which we shall call the vacuum state, satisfies the following equations:

$$\begin{aligned} X^-_\delta(\lambda)|\Omega\rangle &= 0, \\ X^0_\delta(\lambda)|\Omega\rangle &= x_\delta(\lambda)|\Omega\rangle, \end{aligned} \tag{45}$$

with the vacuum eigenvalue of the $X^0_\delta(\lambda)$ operator equal to:

$$x_\delta(\lambda) = 1 + \frac{\nu_1}{\delta - \lambda} - \frac{\nu_2}{\lambda}, \tag{46}$$

where ν_1, ν_2 are the Bargmann indices of the single- and two-mode representations, respectively.

For the conjugated operators, one has

$$\begin{aligned}(X^-_\delta(\lambda)|\Omega\rangle)^\dagger &= \langle\Omega|(X^-_\delta(\lambda))^\dagger = \langle\Omega|X^+_{\overline{\delta}}(\overline{\lambda}) = 0\,, \\ (X^0_\delta(\lambda)|\Omega\rangle)^\dagger &= \langle\Omega|(X^0_\delta(\lambda))^\dagger = \langle\Omega|X^0_{\overline{\delta}}(\overline{\lambda}) = x_{\overline{\delta}}(\overline{\lambda})\langle\Omega|\,,\end{aligned} \tag{47}$$

where the relations (25) were used.

The vacuum state is an eigenstate of the number operator (17):

$$\hat{N}|\Omega\rangle = (s+m)|\Omega\rangle, \tag{48}$$

and of the population difference operator (13):

$$\Delta|\Omega\rangle = \alpha m|\Omega\rangle. \tag{49}$$

It is easy to verify that the vacuum state (44) is an eigenvector of the generating function $t_\delta(\lambda)$:

$$t_\delta(\lambda)|\Omega\rangle = k_\delta(\lambda)|\Omega\rangle, \tag{50}$$

with the eigenvalue:

$$k_\delta(\lambda) = \left(1 + \frac{\nu_1}{\delta - \lambda} - \frac{\nu_2}{\lambda}\right)^2 - \frac{\nu_1}{(\delta - \lambda)^2} - \frac{\nu_2}{\lambda^2}. \tag{51}$$

Applying the relation (32), one obtains:

$$(t_\delta(\lambda)|\Omega\rangle)^\dagger = \langle\Omega|t_{\overline{\delta}}(\overline{\lambda}) = k_{\overline{\delta}}(\overline{\lambda})\langle\Omega|\,. \tag{52}$$

Due to the conservation laws (31), the eigenvectors of the generating operators $t_\delta(\lambda)$ and $t_{\overline{\delta}}(\overline{\lambda})$ depend on the total number of particles in the system N, the difference of the population m, and the parity s of the $\alpha = 0$ spin mode. We shall look for these eigenvectors in the form of the Bethe vectors. We have to distinguish the right eigenvectors:

$$|\Phi_\delta(\lambda_1, \lambda_2, \ldots, \lambda_{N_p})\rangle \equiv |\Phi_\delta(\{\lambda_j\})\rangle = \prod_{j=1}^{N_p} X^+_\delta(\lambda_j)|\Omega\rangle\,, \tag{53}$$

and the left eigenvectors:

$$\langle\Psi_\delta(\{\lambda_j\})| = \langle\Omega|\prod_{j=1}^{N_p} X^-_\delta(\lambda_j)\,. \tag{54}$$

In addition, it will be convenient to introduce the conjugated left eigenvector:

$$\langle\overline{\Phi}_{\overline{\delta}}(\overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_{N_p})| \equiv \langle\overline{\Phi}_{\overline{\delta}}(\{\overline{\lambda}_j\})| = (\prod_{j=1}^{N_p} X^+_\delta(\lambda_j)|\Omega\rangle)^\dagger = \langle\Omega|\prod_{j=1}^{N_p} X^-_{\overline{\delta}}(\overline{\lambda}_j) \tag{55}$$

and the right one:

$$|\overline{\Psi}_{\overline{\delta}}(\{\overline{\lambda}_j\})\rangle = (\langle\Omega|\prod_{j=1}^{N_p} X^-_\delta(\lambda_j))^\dagger = \prod_{j=1}^{N_p} X^+_{\overline{\delta}}(\overline{\lambda}_j)|\Omega\rangle\,. \tag{56}$$

Due to (42), the number of particles in this state is

$$\hat{N}|\Phi_\delta(\{\lambda_j\})\rangle = N|\Phi_\delta(\{\lambda_j\})\rangle\,; \quad N \equiv 2N_p + s + m\,. \tag{57}$$

The number of operators $X^+(\lambda)$ in the product (53), which corresponds to a number of pairs of the boson quasi-particles in the system, is equal to $2N_p = N - m - s$. The state (53) is satisfied by the relations:

$$\begin{aligned} \Delta|\Phi_\delta(\{\lambda_j\})\rangle &= \alpha m|\Phi_\delta(\{\lambda_j\})\rangle, \\ (-1)^{\hat{N}-|\Delta|}|\Phi_\delta(\{\lambda_j\})\rangle &= (-1)^s|\Phi_\delta(\{\lambda_j\})\rangle. \end{aligned} \tag{58}$$

For a given number of particles N, the possible values of quantum numbers m and s are $0 \le m + s \le N$. Equations (57) and (58) are satisfied by states (54)–(56).

The vectors (53) and (54) are the eigenvectors of $t_\delta(\lambda)$ if the parameters λ_j satisfy the Bethe equations [23]:

$$1 + \frac{\nu_1}{\delta - \lambda_j} - \frac{\nu_2}{\lambda_j} = \sum_{l \neq j}^{N_p} \frac{1}{\lambda_j - \lambda_l};\ j = 1, \ldots, N_p. \tag{59}$$

The vectors (55) and (56) are the eigenvectors of $t_{\overline{\delta}}(\overline{\lambda})$ if the parameters $\overline{\lambda}_j$ satisfy the conjugated Bethe equations:

$$1 + \frac{\nu_1}{\overline{\delta} - \overline{\lambda}_j} - \frac{\nu_2}{\overline{\lambda}_j} = \sum_{l \neq j}^{N_p} \frac{1}{\overline{\lambda}_j - \overline{\lambda}_l};\ j = 1, \ldots, N_p. \tag{60}$$

There are $N_p + 1$ sets $\left\{\lambda_j^\sigma\right\}_{j=1}^{N_p}$ and $\left\{\overline{\lambda}_j^\sigma\right\}_{j=1}^{N_p}$ of solutions of these N_p equations ($\sigma = 1, 2, \ldots, N_p + 1$).

The N-particle eigenvalues $\Theta_\delta^\sigma(\mu)$ of the generating function $t_\delta(\mu)$ (26) are equal to

$$\Theta_\delta^\sigma(\mu) = k_\delta(\mu) - \sum_{j=1}^{N_p} \frac{2\nu_1}{(\delta - \mu)(\delta - \lambda_j^\sigma)} - \sum_{j=1}^{N_p} \frac{2\nu_2}{\mu\lambda_j^\sigma}, \tag{61}$$

with $k_\delta(\mu)$ given by the relation (51), and $\lambda_j^\sigma \in \left\{\lambda_j^\sigma\right\}_{j=1}^{N_p}$. The eigenvalues $\Theta_{\overline{\delta}}^\sigma(\mu)$ of the generating function $t_{\overline{\delta}}(\mu)$ (26) are equal, respectively, to

$$\overline{\Theta}_{\overline{\delta}}^\sigma(\mu) = k_{\overline{\delta}}(\mu) - \sum_{j=1}^{N_p} \frac{2\nu_1}{(\overline{\delta} - \mu)(\overline{\delta} - \overline{\lambda}_j^\sigma)} - \sum_{j=1}^{N_p} \frac{2\nu_2}{\mu\overline{\lambda}_j^\sigma}\,. \tag{62}$$

From Equations (30) and (61), it follows that the N-particle eigenenergies of the Hamiltonian H (18):

$$\begin{aligned} H|\Phi_\delta(\left\{\lambda_j^\sigma\right\})\rangle &= E_\delta^\sigma|\Phi_\delta(\left\{\lambda_j^\sigma\right\})\rangle\,, \\ \langle\Psi_\delta(\left\{\lambda_j^\sigma\right\})|H &= E_\delta^\sigma\langle\Psi_\delta(\left\{\lambda_j^\sigma\right\})| \end{aligned} \tag{63}$$

are

$$E_\delta^\sigma = -\delta Res|_{\mu=\delta}\Theta_\delta^\sigma(\mu) = -2\delta\nu_1 + 2\nu_1\nu_2 + \sum_{j=1}^{N_p} \frac{2\nu_1\delta}{\delta - \lambda_j^\sigma}\,. \tag{64}$$

With the help of the Bethe equation (59) these eigenenergies may be rewritten in the form:

$$E_\delta^\sigma = -2\delta\nu_1 + 2\nu_1\nu_2 + N_p\big(N_p - 1 + 2\nu_1 + 2\nu_2\big) - 2\sum_{j=1}^{N_p}\lambda_j^\sigma\,. \tag{65}$$

The quantum numbers $2\nu_2 = m - 1$ and $4\nu_1 = 2s + 1$ are expressed through the difference of population m and the parity of the number of particles s (see (39) and (40)).

The N-particle eigenenergies of the Hamiltonian $H^\dagger$ (19):

$$\begin{aligned}\langle\overline{\Phi}_{\overline{\delta}}(\{\overline{\lambda}_j^\sigma\})|H^\dagger &= \overline{E}_{\overline{\delta}}^\sigma\langle\overline{\Phi}_{\overline{\delta}}(\{\overline{\lambda}_j^\sigma\})|\,,\\ H^\dagger|\overline{\Psi}_{\overline{\delta}}(\{\overline{\lambda}_j^\sigma\})\rangle &= \overline{E}_{\overline{\delta}}^\sigma|\overline{\Psi}_{\overline{\delta}}(\{\overline{\lambda}_j^\sigma\})\rangle\end{aligned} \tag{66}$$

are obtained with the help of Equations (33) and (62):

$$\overline{E}_{\overline{\delta}}^\sigma = -2\overline{\delta}\nu_1 + 2\nu_1\nu_2 + N_p\big(N_p - 1 + 2\nu_1 + 2\nu_2\big) - 2\sum_{j=1}^{N_p}\overline{\lambda}_j^\sigma\,. \tag{67}$$

The energies of the effective Hamiltonians H_{eff} and $H_{eff}^\dagger$ (9), (18) are equal, respectively, to

$$\mathcal{E}_{eff}^\sigma = gE_\delta^\sigma + \frac{\delta}{2}, \qquad \overline{\mathcal{E}}_{eff}^\sigma = \overline{E}_{\overline{\delta}}^\sigma + \frac{\overline{\delta}}{2}\,. \tag{68}$$

6. Biorthogonal Expectation Values

It was shown that a non-Hermitian Hamiltonian operator has inequivalent right and left eigenvectors. However, the sets (53), (54) and (55), (56) form a *biorthogonal basis* [15]:

$$\begin{aligned}\frac{\langle\Psi_\delta(\{\lambda_j^{\sigma'}\})|\Phi_\delta(\{\lambda_j^\sigma\})\rangle}{\langle\Psi_\delta(\{\lambda_j^\sigma\})|\Phi_\delta(\{\lambda_j^\sigma\})\rangle} &= \delta_{\sigma',\sigma}\,,\\ \frac{\langle\overline{\Phi}_{\overline{\delta}}(\{\overline{\lambda}_j^{\sigma'}\})|\overline{\Psi}_{\overline{\delta}}(\{\overline{\lambda}_j^\sigma\})\rangle}{\langle\overline{\Phi}_{\overline{\delta}}(\{\overline{\lambda}_j^\sigma\})|\overline{\Psi}_{\overline{\delta}}(\{\overline{\lambda}_j^\sigma\})\rangle} &= \delta_{\sigma',\sigma}\,.\end{aligned} \tag{69}$$

To prove these statements, it is enough to consider the element $\langle\Psi_\delta(\{\lambda_j^{\sigma'}\})|H|\Phi_\delta(\{\lambda_j^\sigma\})\rangle$ and notice that $E_\delta^{\sigma'} \neq E_\delta^\sigma$. An analogous result holds for conjugated eigenvectors. The scalar products of the states $\langle\Psi_\delta(\{\lambda_j^{\sigma'}\})|\overline{\Psi}_{\overline{\delta}}(\{\overline{\lambda}_j^\sigma\})\rangle$ and $\langle\overline{\Phi}_{\overline{\delta}}(\{\overline{\lambda}_j^{\sigma'}\})|\Phi_\delta(\{\lambda_j^\sigma\})\rangle$ are in general not orthogonal.

The expectation values of a generic Hermitian operator A of the form

$$< A >= \frac{\langle\Psi_\delta(\{\lambda_j^\sigma\})|A|\Phi_\delta(\{\lambda_j^\sigma\})\rangle}{\langle\Psi_\delta(\{\lambda_j^\sigma\})|\Phi_\delta(\{\lambda_j^\sigma\})\rangle} \tag{70}$$

are known as *biorthogonal expectation values* and play a central role in understanding the dynamics of non-Hermitian models.

Let us calculate the expectation value of the operator $a_0^\dagger a_0$. One must notice that the variation of the parameter δ in the eigenvalue problem (18) and (63):

$$(2B^0K^0 - B^+K^- - B^-K^+ - 2\delta B^0)|\Phi_\delta(\{\lambda_j^\sigma\})\rangle = E_\delta^\sigma|\Phi_\delta(\{\lambda_j^\sigma\})\rangle$$

gives

$$< 2B^0 >= \frac{\langle\Psi_\delta(\left\{\lambda_j^\sigma\right\})|2B^0|\Phi_\delta(\left\{\lambda_j^\sigma\right\})\rangle}{\langle\Psi_\delta(\left\{\lambda_j^\sigma\right\})|\Phi_\delta(\left\{\lambda_j^\sigma\right\})\rangle} = -\frac{\partial E_\delta^\sigma}{\partial\delta}\,. \tag{71}$$

Taking into account the definition (15), we thus find

$$< a_0^\dagger a_0 >= -\frac{\partial E_\delta^\sigma}{\partial\delta} - \frac{1}{2}\,. \tag{72}$$

This formula allows expressing the expectation of the number of particles in the zero mode $a_0^\dagger a_0$ through the solutions of the Bethe Equation (59) by using the expressions (65) for the eigenenergies:

$$< a_0^\dagger a_0 >= 2\delta - \frac{1}{2} + 2\sum_{j=1}^{N_p}\frac{\partial\lambda_j^\sigma}{\partial\delta}\,. \tag{73}$$

The differentiation of the Bethe Equation (59) with respect to δ gives

$$\sum_{j=1}^{N_p} f_{jl}\frac{\partial\lambda_l^\sigma}{\partial\delta} = \frac{\nu_1}{(\delta-\lambda_j^\sigma)^2}\,,$$

where the elements of the matrix $\hat{f}$ are:

$$f_{jl} = \left(\frac{\nu_1}{(\delta-\lambda_l^\sigma)^2} + \frac{\nu_2}{(\lambda_l^\sigma)^2} + \sum_{k=1}^{N_p}\frac{1}{(\lambda_l^\sigma-\lambda_k^\sigma)^2}\right)\delta_{jl} - \frac{1}{(\lambda_j^\sigma-\lambda_l^\sigma)^2}\,. \tag{74}$$

The solution of this linear system is obtained by Cramer's rule:

$$\frac{\partial\lambda_j^\sigma}{\partial\delta} = \frac{\det\hat{f}^j}{\det\hat{f}}\,, \tag{75}$$

where $\hat{f}^j$ is the matrix formed by replacing the j-th column of $\hat{f}$ by the column vector $\frac{\nu_1}{(\delta-\lambda_j^\sigma)^2}$ $(j = 1,\ldots,N_p)$. Cramer's rule allows expressing the sum on the right-hand side of the relation (73) as

$$\sum_{j=1}^{N_p}\frac{\partial\lambda_j^\sigma}{\partial\delta} = -\frac{\det\hat{\varphi}}{\det\hat{f}}\,,$$

where

$$\hat{\varphi} = \begin{pmatrix} f_{11} & \ldots & f_{1N_p} & \frac{\nu_1}{(\delta-\lambda_1^\sigma)^2} \\ f_{21} & \ldots & f_{2N_p} & \frac{\nu_1}{(\delta-\lambda_2^\sigma)^2} \\ \ldots & \ldots & \ldots & \ldots \\ f_{N_p1} & \ldots & f_{N_pN_p} & \frac{\nu_1}{(\delta-\lambda_{N_p}^\sigma)^2} \\ 1 & \ldots & 1 & 0 \end{pmatrix}.$$

Finally, for the expectation value, we have

$$< a_0^\dagger a_0 >= 2\delta - \frac{1}{2} - 2\frac{\det\hat{\varphi}}{\det\hat{f}}\,. \tag{76}$$

The norm of the Bethe vectors is equal to

$$\begin{aligned} \langle\Psi_\delta(\left\{\lambda_j^\sigma\right\})|\Phi_\delta(\left\{\lambda_j^\sigma\right\})\rangle &= \det\hat{f}\,, \\ \langle\overline{\Phi}_{\overline{\delta}}(\left\{\overline{\lambda}_j^\sigma\right\})|\overline{\Psi}_{\overline{\delta}}(\left\{\overline{\lambda}_j^\sigma\right\})\rangle &= \det\hat{f}^\dagger\,. \end{aligned} \tag{77}$$

The proof of this statement is based on the usual scheme of the *quantum inverse scattering method* [19] and is of the standard form for the integrable models.

Knowing the eigenvectors (63) and (66) of the Hamiltonians H and $H^{\dagger}$, we can construct the generic operators in terms of the *nonorthogonal basis* [15]. The time-dependent operator $\tilde{\rho}_t$ can likewise be expressed in the following form:

$$\tilde{\rho}_t = \sum_{\nu,\nu'} \frac{e^{it(\overline{E}_{\bar{\delta}}^{\nu'} - E_{\delta}^{\nu})} |\Phi_{\delta}(\{\lambda_j^{\nu}\})\rangle\langle\overline{\Phi}_{\bar{\delta}}(\{\overline{\lambda}_j^{\nu'}\})|}{\langle\Psi_{\delta}(\{\lambda_j^{\nu}\})|\Phi_{\delta}(\{\lambda_j^{\nu}\})\rangle\langle\overline{\Phi}_{\bar{\delta}}(\{\overline{\lambda}_j^{\nu'}\})|\overline{\Psi}_{\bar{\delta}}(\{\overline{\lambda}_j^{\nu'}\})\rangle}. \quad (78)$$

The Hermitian operator $\tilde{\rho}_t = \tilde{\rho}_t^{\dagger}$ satisfies the Lindblad Equation (7).

7. Conclusions

In this paper, we examined the open-system quantum dynamics of Bose–Einstein condensates using a "toy" three-level $su(1,1)$ bosonic model. The effective non-Hermitian Hamiltonians describing the system in the presence of a boson loss are constructed in the special limit of the Lindblad equation. Based on the formulas of the Algebraic Bethe Ansatz, the obtained non-Hermitian Hamiltonians are solved exactly for their eigenstates and eigenvalues. This approach allows representing biorthogonal and nonorthogonal bases of the system. We analyzed the biorthogonal expectation values of the number of particles in the zero mode and represented it in the determinantal form. The time-dependent density matrix satisfying the Lindblad equation in the considered approximation is found in terms of the nonorthogonal basis.

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