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Negation of the Smooth Poincare Conjecture in Dimension 4 and Negation of the Tsirelson's Conjecture Shed Light on Quantum Gravity

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Abstract: If spacetime is a physical object, it is conceivable that it loses its integrity or is destroyed in some way as a continuum in an abrupt process initiated in spacetime itself. An example is a gravitational collapse leading to a spacetime singularity, as in the interior of a black hole. We find a conservative extension of quantum mechanics by quantum set theory over the singular domain and show that it is reconcilable with the special extension of spacetime 4-diffeomorphisms by automorphisms of Boolean models of set theory. The extension of quantum mechanics supports the random sequences of the quantum mechanical outcomes that can negate Tsirelson's conjecture, whereas the extension of 4-diffeomorphisms indicates the role of exotic smooth 4-spheres as gravitational instantons. This leads to the negation of the smooth 4-dimensional Poincaré conjecture before its final resolution by mathematicians. We also discuss the case where the Poincaré conjecture would remain true.

Keywords: Boolean models of ZFC; quantum mechanics; quantum gravity; physics of exotic smooth 4-spheres

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1. Introduction

The reconciliation of general relativity (GR) and quantum mechanics (QM) is an ongoing but unsolved project in theoretical physics. GR and QM are the main pillars of the contemporary understanding of the physical world, starting from the tiny micro-scale up to the cosmological scales. Extending QM over the biggest scales raises the question of the persistent classical character of gravity, even though the black hole singularities call for a quantum explanation. From the other side, jumping into the micro-world of elementary particles by any gravitational probe shows the practical irrelevance of gravity there. Moreover, any consistent formulation of quantum gravity theory reconciling quantum principles with gravitational understanding of spacetime fails for probably fundamental reasons which, nevertheless, are unclear at present. In this paper, we partially fill the gap.

Any successful theory of QG must shed light not only on the micro-regions but also help understand the cosmological puzzles of the physical world. This task is especially sound when one faces probably groundbreaking experimental data collected and sent to the Earth by the James Webb Space Telescope (JWST) the data that would presumably revolutionize our cosmological models so far. This paper also addresses this important issue to some extent.

In the quantum gravity (QG) limit, it is usually expected that QM remains unchanged, while GR would undergo certain modifications, which are expected at very small spatial distances or high scales of energy. One such phenomenon is the formation of black holes (BH) via gravitational collapse (the suitable amount) of matter/energy in spacetime where the densities can increase without limits, and the gravitational singularity emerges within the interior of BH. The proper description of the singularity is presumably based on the modified GR and QM. Another example comes from loop QG where spacetime in the quantum regime becomes a kind of discrete dynamical graph with quantum properties labeling it. Thus, GR must be manifestly modified in this regime since Lorentz invariance is broken. In superstring theory, GR works without limits, but the issue of the background dependence written in the theory from the very beginning shows a conflict with the rules of GR. Moreover, GR can be formulated in various dimensions, including 4, while the consistency of superstring theory requires higher dimensions (and supersymmetry, of course), breaking in this respect the original universal applicability of GR. QM seems to be valid universally in loop QG and string theory, but in this last case, there is the quantum field theory, which has to be modified again by assuming supersymmetry between particles and fields.

The approach in this paper addresses the above issue of the relation between GR and QM from the perspective of the mathematical structures and their formal description. We find room in model theory and set theory augmented by topics in differential geometry of low dimensions, which allow for the simultaneous modification of both GR and QM.

Although most physical theories are based on mathematics, which is usually formulated in the language of classical logic and set theory, notable exceptions are also known. There are certain attempts in mathematical physics, like those within the scope of topos theory, which are based on nonclassical, non-two-valued Boolean or intuitionistic logics [1,2], but there are also some other ones, starting from the celebrated work of Birkhoff and Von Neumann [3], which are based directly on quantum logic. The theories of sets, corresponding to nonclassical logics, also undergo some important modifications compared to the classical case. In the topos approaches, internal set theory is canonically governed by the structure of topoi and includes Boolean set theories in Boolean toposes, while in the regime of QM, this is a big unknown whether one can reasonably assign a set theory to the quantum regime and what would be the meaning of such a quantum set theory. Such a situation is partly the consequence of the difficulties with quantum logic as linked with quantum sets, especially the implication problem, but, as we think, it is also the lack of fundamental hints and studies devoted to set theory in the quantum regime. However, there are exceptions. One notable is the proposition by Gaisi Takeuti [4,5], developed further by Masanao Ozawa [6,7], where quantum set theory is modeled by the 'universe' $V^{\mathbb{L}}$ which is analogous to the Boolean-valued universe of sets, V^B, where V is the usual cumulative Von Neumann universe of sets and B is a complete Boolean algebra. In Takeuti's approach, B is replaced by the QM lattice of projections $\mathbb{L}(\wedge, \vee, 0, 1)$, giving rise to $V^{\mathbb{L}}$. The problem with such a 'model of the quantum universe of sets' is that it reflects much more serious difficulties than just the lack of unique implications in quantum logic. One sees that when logic encounters difficulties with implication the entire set-theory structure is ambiguous or has a non-well-defined sense as a set-theory. Maybe one should weaken the logic-sets link somehow to allow for a degree of freedom while considering sets aspects of the quantum world.

This rough idea is, in some sense, realized in this work. We allow for local 'minor' changes in logic over a spacetime manifold where set theory is also slightly modified, but the spacetime manifold undergoes decomposition in the extreme curvature limit into the local fragments. These fragments are considered here as local maps of a smooth atlas of the manifold. This follows in a reverse way to the constructability of any manifold from its smooth atlas as a colimit (over the groupoid in the category of manifolds whose objects are local patches). The fragments are no longer causally related, but the set of fragments seen in the 2-value environment is directly related to QM. From the 2-valued perspective, sets in the quantum regime are organized into the global structure but with additional relations between the spacetime fragments (replacing causality and locality). This is not that unusual when one recalls the celebrated Bohr doctrines where any oddity of QM is finally described in the 2-valued logic and set theory, and such possibility is elevated into an inherent property of the world. Here, the classical environment allows us to see the additional relations between local regions of the spacetime manifold in the quantum regime. However, such an enriched structure of the fragments is not considered to be a category-like structure, though this point requires some reflection. On the one hand, it is reasonable to think about the relations between local patches of spacetime as a kind of morphism replacing natural superset or diffeomorphism relations. On the other hand, we do not aim at taking an internal structure of such a category as modifying a set or logic structure. We rather take the 2-valued departure point as the final point, as well, from which the entire structure is to be analyzed. The modifications of logic and set theory are performed locally in the intermediate stages of the construction, and the effects of this are traced at the final classical stage.

The first part of the paper realizes the scenario of the decomposition of the spacetime regions into flat $\mathbb{R}^4$s fragments due to the symmetric collapse of matter and energy producing extra high 4-curvatures in spacetime. The domains of the extreme curvature and density of energy are modeled by 4-spheres with increasing radius—a natural geometric model for such a symmetric process up to certain approximations but still carrying the main geometric features of the process. The fragmentation of such hyper-curved spheres in spacetime is further allowed. Based on equivalence principles and some general considerations, the final stage process of such fragmented S^4s is predicted as the collection of flat patches $\mathbb{R}^4$ augmented by certain nonlocal relations between them. Extending preservation of energy and momentum over this nonclassical regime, one finds that the entire gravitational energy imprisoned in the original gravitational collapse in spacetime now is written in the relations between the flat patches, which, as being flat, have vanishing 4-curvature and no gravitational energy. To grasp properly the relations connecting flat spacetime fragments, we turn to the methods of model theory and formal set theory, and the entire scenario is developed from the very beginning by Boolean models of ZFC, which can now be seen as physical degrees of freedom (e.g., ref. [8]).

The methods in this work resemble and follow, to some extent, those in ref. [8]. However, the use of model theory and formal mathematical tools has a long tradition in physics. Especially Boolean models of set theory have emerged as a promising candidate for the study of the quantum/classical relation. This overlapping domain is not fully understood so far, especially the way in which physical spacetime emerges from the quantum realm, which in some stage should be called a 'quantized spacetime'. The Boolean models of Zermello–Fraenkel set theory with the axiom of choice and their 2-valued forcing extensions find their place in exploring the domain. The methods are strong enough to create quite a consistent understanding of the quantum gravity regime from the spacetime point of view. This is the topic of the second part of the paper.

In Section 4, the analysis of the reverse of fragmentation in space–time process is performed. The conditions for the classical limit of the fragmented domains are given. In particular, under the supposition of the negation of the smooth Poincaré conjecture in dimension four, it is found that the limit leads to exotic S^4s and that they correspond canonically to the gravitational instantons of the semiclassical Euclidean gravity in the physical dimension. In the next 'Heisenberg uncertainty and the fragmentation of S^4' section, it is shown that, indeed, the cardinality conditions imposed on the fragmented S^4s are the conclusions of the Heisenberg uncertainty relations in the original spacetime regions. In Section 4.4, the geometric representation of gravitons is given as the continuous perturbations of the original 4-metric such that it corresponds to the exotic smoothness structure on $\mathbb{R}^4$ supporting S^4s. The Section 5 'The Negation of the Tsirelson's Conjecture and V^Bs in QM' discusses the findings in the previous publication [9] of the hyperrandom sequences of QM outcomes, which are potentially falsifying Tsirelson's conjecture. For the Hilbert spaces $\mathcal{H}^{(\infty)}$, the sequences (with the suitable degree of randomness) are related to the quantum regime of gravity. We close the paper with the discussion section with future directions of the approach.

There are also two sections of the complemented materials. Appendix A contains a thorough discussion of the crucial technique of the V-classical limit of the local Boolean theories, while Appendix B is the proof of Theorem 1.

2. The Extension of GR by Homomorphisms of Models

2.1. GR in the Vicinity of Extreme Curvatures

The description of curvatures by general relativity (GR) formally does not have limits—GR can deal with arbitrary high densities of energies and values of curvatures of spacetime. Even though the singularities in GR should be defined covariantly with respect to the geodesic incompleteness [10], their formation is usually connected with abrupt processes in spacetime. Usually, certain infinities or divergent quantities characterize such processes. Some of them can be made finite just by the coordinate transformation, some others cannot, and GR fails to describe this regime. Instead, QM should be applied there. Connected with this, a quite natural assumption would be that for some sufficiently high (but finite) values of curvature, spacetime should be approached by QM rather than classically by a smooth manifold's model. The understanding of this process is roughly the aim of the present work. In this section, we want to test the limits of spacetime as a smooth manifold in the regime of extremely high curvatures by proposing a suitable formal background.

Working in a formal language like the first-order language of Zermello-Frankel set theory, $\mathcal{L}_{ZF}$, one can formulate in the language various well-formed formulas about sets. Only some subclass of them can be proved within the formal system of sets like ZF or ZFC. Some other subclass can be disproved. However, quite an important subclass is neither proved nor disproved sentences, even though they can be true statements about sets (the proofs of them are not proved in ZFC). There are also sentences that are not true in the absolute sense but rather are independent of the axioms of ZFC. That means that there are some models of set theory where the sentences become true and some other models where they are false. Recalling that in every model of ZFC, all axioms of ZFC are true, we will see that these sentences are strictly independent of ZFC. All sentences in each category above are expressible in the language $\mathcal{L}_{ZF}$ of ZF, which does not mean they are provable in ZFC. The language of ZF is a particularly strong enabling for the formulation of a large part (some say almost all) of classical mathematics. An important example is the theory of manifolds and general relativity. In fact, to express the constructions on differentiable manifolds, we need a particularly low class of ZFC formulas, the Σ_3 formulas of $\mathcal{L}_{ZF}$. The entire hierarchy of ZFC formulas is the celebrated Lévy hierarchy defined with respect to the structural complexity or the number of existential and general quantifiers respecting their order (see Appendix C).

Let $(V, \in)$ be the cumulative Von Neumann's hierarchy of all sets and M^4 a smooth spacetime 4-dimensional Lorentz manifold. M^4 is ZFC expressible and requires maximally Σ_3 formulas of the Zermello-Frankel language $\mathcal{L}_{ZF}$. Usually, it is unnecessary to refer explicitly to V or $\mathcal{L}_{ZF}$ since almost all mathematics has this property and is ZFC expressible. However, in what follows, this formal ambient environment becomes an active and dynamic player of the construction, losing its universal or fixed character. This is more or less a direct consequence of what could happen to spacetime in the regime of extreme curvatures or densities of energies as in the Planck scales. Even though spacetime is an extremely rigid physical object, we could try to understand the limits of its integrity. Einstein's equations allow for the interpretation of spacetime as an elastic fabric that undergoes deformations like bending, stretching, and the like performed by the densities of matter and energy, e.g., ref. [11]. Even though we are missing the outer perspective for such a deformed elastic medium, which is just an approximation, again, the tensor calculus enables for description of the effects internally to 4-spacetime. Moreover, the physical dimension of the Einstein tensor $G_{\mu\nu}$ is $[1/\text{m}^2]$ and of $T_{\mu\nu}$ is the same as the Young's modulus Y, i.e., $[\text{kg}/\text{m}\cdot \text{s}^{-2}]$, where the equation $F/A = Y\frac{\Delta l}{l}$ defines Y via linear deformation Δl of an elastic body of length l and the cross section A with the reacting pressure F/A. Various approximations of the Young modulus for the elastic spacetime can be further derived, which here gives us just an idea about the scale of the rigidity of spacetime. Based on the classical effects of the gravitational waves of frequency f $[1/\text{s}]$ propagating in spacetime, one finds, e.g., ref. [12]

$$Y_f = \frac{c^2}{G} f^2 \simeq 4.5 \cdot 10^{27} f^2 \, [\text{kg}/\text{m} \cdot \text{s}^{-2}].$$

Approaching the quantum regime of vacuum one finds just from the dimensional analysis (ref. [12])

$$Y_q = \frac{c^7}{h \cdot G^2} \simeq 5 \cdot 10^{113} \, [\text{kg}/\text{m} \cdot \text{s}^{-2}].$$

Both results indicate extremely high stiffness of spacetime, but still, they are finite. When the densities of mass and energy increase arbitrarily high in space–time, as can happen during a gravitational collapse, these incredibly large numbers and the corresponding densities of the energy that generates them can be formally exceeded unless a good reason is found that forbids it. The finiteness of the numbers and the ability to exceed them in physical real processes also raise different fundamental questions, like about the stability and integrity of spacetime and the perspective enabling the proper grasping of such effects. For example, if spacetime loses its integrity, the tensor calculus on a smooth manifold is non-applicable any longer. The approach we give here is the attempt to understand the limits of spacetime from the GR perspectives and physical processes causing spacetime can become a nonsmooth object.

The obstruction

Consider a family $\{\mathcal{R}_{1,\alpha\in I}, \mathcal{R}_{2,\beta\in K}\}$ of certain, to be determined, remnants of spacetime after decomposition of it as a smooth manifold M^4. We expect the following properties to be assigned to the family $\{\mathcal{R}_{1,\alpha\in I}, \mathcal{R}_{2,\beta\in K}\}$

(1) Each $\mathcal{R}_{1,\alpha}$ is a certain set of spacetime points which are still causally connected (locality).
(2) The density of gravitational energy vanishes along each $\mathcal{R}_{1,\alpha}$—it is flat if a smooth calculus in local coordinates is applicable for such domains.
(3) $\mathcal{R}_{2,\beta\in K}$ is a remnant of the gravitational energy density and curvature present in M^4 before decomposition.

Remark 1. *(1) and (2) above show that, indeed, each $\mathcal{R}_\alpha$ can be chosen to be flat since now they are not causally glued together by local diffeomorphisms. It is also reasonable that (3) above codes the curvature and gravity after the decomposition in (1) and (2).*

The obstruction in seeing spacetime as 'torn' apart into pieces by the densities of matter and energy is the lack of any external perspective to describe the pieces. Alternatively, more precisely, such a perspective should be given by QM, but still, we do not understand the process of attaining QM from the fragmented spacetime. Rather, we know that the black hole singularities have to exist. Our main observation for fixing further steps is the following correspondence:

The rigidity of spacetime is protected by the rigidity of the universe of sets V.

This is an extension of the fact that breaking causality and locality between different regions of spacetime is not an observable process with the description within spacetime itself. Special relativity theory and GR are based on the causality and locality of any transfer of information in physical processes. Even in QM, this kind of signaling theory (allowing for faster-than-light nonlocal transfer) is forbidden, though it can be considered to be some nonphysical alternative. Therefore, given a domain $U_1 \subset M^4$ and another causally disconnected domain $U_2 \subset M^4$, which means that no observer in U_1 can 'see' anything from U_2, since the regions come from fragmented spacetime and, of course, $U_1 \cap U_2 = \emptyset$. In other words, we cannot see $\tilde{M}^4$ (the fragmented part of M^4), in causally separated pieces, as the part of M^4. Alternatively, there would be no possibility to embed $\tilde{M}^4$ in M^4 since causal separation refers to spacetime regions, not to a particular future cone of any observer in spacetime. The following lemma explains the set-theory counterpart of this.

Lemma 1 (Kunen [13])**.** *There does not exist a nontrivial automorphism (a nontrivial elementary embedding) $j : V \to V$.*

Remark 2. *This property can be stated equivalently that any automorphism $j : V \to V$ is the identity and that there is no nontrivial embedding $V \to V$. Nontrivial embeddings are possible under assuming the existence of certain large cardinals.*

Thus, nontriviality here means embedding, which would not be the identity on V. Assume that given M^4, it is always $M^4 \in V$ that fixes the formal environment. Thus, given another copy of M^4 (fragmented or not), which is defined in V as well, the formal perspective enforces that the nontriviality (in this case diffeomorphism, which is not an identity) of $M^4 \to M^4$ requires the nontriviality of $V \to V$. However, this is impossible due to Lemma 1. We conclude

Corollary 1. *If $M^4 \cap M'^4 \neq \emptyset$ and $M^4 \overset{diff}{\simeq} M'^4$ then, allowing for the set-theory formal environment, $M^4 \in V, M'^4 \in V$ enforces that M^4 has to be identically diffeomorphic to M'^4 (thus excluding $M^4 \overset{Diff.\neq Id}{\simeq} M'^4$).*

Remark 3. *The formal context V of M^4 is usually neglected in mathematics, and so is its relation to itself. In such a case, the object like $\mathbb{R}^4$ (flat M^4) allows for unlimited embeddings $\mathbb{R}^4 \to \mathbb{R}^4$ onto the diffeomorphic images (submanifolds). Activating V as a valid formal context for $\mathbb{R}^4$ creates several constraints that prohibit such embeddings, and the entire procedure becomes valid in physical applications. This V assigns to $\mathbb{R}^4$ plays an analogous role in assigning certain 'quantum numbers' to physical objects; one must take care of the numbers and their preservation laws when manipulating the objects.*

Remark 4. *One can think of M^4 as formally being a pair (M^4, V) such that $(M^4, V) \to (M^4, V) \equiv (f : M^4 \to M^4, j(f) : V \to V)$ where $f \neq id \equiv j(f)$ is nontrivial.*

On the one hand, this relative to the set-theory universe V formalism forbids referring to the pieces of M'^4, but on the other hand, the formalism indicates the solution allowing for such reference. The solution is to deal with other models of ZFC than the cumulative hierarchy V, V', and to consider pairs $(M^4 \supset U, V')$.

Theorem 1. *There exist models of ZFC, V', allowing for embeddings*

$$V \to V' \to V$$

and given two such models, V'_1, V'_2, there exist nontrivial automorphisms

$$j' : V'_1 \to V'_2.$$

For the proof, see Appendix B.

Let B be a complete Boolean algebra in V, and V^B a Boolean-valued model of ZFC, then the proof of the above theorem shows

Corollary 2. $V' = V^B$.

We assign the formal neighborhoods $(U_\alpha, V^{B_\alpha})_{\alpha \in J}$ to local patches $U_\alpha, \alpha \in J$ of a (maximal) smooth atlas $\mathcal{U} = \{U_\alpha, \alpha \in J\}$ of M^4

$$\{U_\alpha \to (U_\alpha, V^{B_\alpha})\}_{\alpha \in J}, \quad U_\alpha \in \mathcal{U}(M^4)\forall_{\alpha \in J}. \tag{1}$$

Moreover, by reference to a good cover, which always exists on a smooth manifold, we can assume that $U_\alpha \overset{\text{diff}}{\simeq} \mathbb{R}^4$. For now, we are leaving unspecified the family of complete Boolean algebras and corresponding models V^Bs.

We are assuming the following heuristic rule, which would relate physics with such an enriched description of spacetime

> [H1] The presence of local Boolean models of ZFC, V^B, becomes physically relevant when sufficiently high densities of energies are attained in spacetime, hence in the limit of sufficiently big curvatures. This means that in the small densities of energies compared to the rigidity of spacetime, the physical effects are irrelevant or negligibly small.

We are leaving, for now, unspecified this sufficiently high level of densities, but intuitively, it should be related to the Planck regime, which approaches the quantum description of physical phenomena in spacetime. We will come back to that point later on.

Currently we do not have any reason to suspect that the equivalence principle (EP), which GR is based on, is not valid for physical spacetime in any regime. Thus, in the regime of extreme densities of energies in spacetime as possibly having a devastating impact on the spacetime structure or its integrity, EP still should be a guiding principle. That is to say, some variant of EP holds when spacetime is no longer a smooth object. What could that be?

A geometrical variant of EP claims that any effect of gravitational energy in spacetime can be locally eliminated by a suitable choice of the coordinate frame. This means that the local frame in which the physical process is described, on a sufficiently small scale, becomes flat $\mathbb{R}^4$ where no gravitational effects are present and, certainly, the curvature of spacetime vanishes. However, when the entire spacetime is curved as a global smooth 4-manifold, the choice of such local flat frames is not possible globally for all points in M^4 (at each point of M^4, there would be the vanishing curvature tensor; hence, it would vanish globally). That would have meant that M^4 were actually flat. However, when the domains of M^4 are causally separated and connected by generalized transition functions (not just diffeomorphisms), then it can happen

$$\text{For each } p \in M^4 \text{ there exists } U_x \in \mathcal{U} \text{ that } U_x = \text{ flat } \mathbb{R}^4.$$

and gravitational energy leaks somehow into the transitions between flat U_αs.

Thus, according to this, we have the following quite general possibility.

[H2] The final stage of the gravitational collapse of spacetime would be a family $\{U_\alpha\}$ of flat $\mathbb{R}^4$s and a family $\{\tilde{f} : \mathbb{R}^4 \to \mathbb{R}^4\}$ of nonlocal transformations between flat $\mathbb{R}^4$s.

Remark 5. *Under some conditions, the final stage of a gravitational collapse typically contains the GR part of a geometric solution of the Einstein equations, where eventually, a black hole is formed with a certain singularity, which is inaccessible by the formalism of GR. Here, we propose that a singularity contains remnants of the causally disconnected fragments of spacetime, each worth flat R^4s in V^Bs with the nonlocal symmetry between the fragments replacing diffeomorphisms of $\mathbb{R}^4$ in V.*

Understanding H2, particularly its relation to QM, is the aim of the remaining part of this paper.

2.2. Deformation of Diff($\mathbb{R}^4$)

First, note that H2 and the analysis before it realizes points (1), (2), and (3) of the previous section. To see this, it suffices to identify $\mathcal{R}_{1,\alpha}$, with $U_\alpha = $ flat $\mathbb{R}^4$, and $\mathcal{R}_{2,\gamma}$ with nonlocal $\tilde{f}_\gamma : \mathbb{R}^4 \to \mathbb{R}^4$ where $\gamma = \alpha\beta$ is such that $\tilde{f}_{\alpha\beta} : U_\alpha \to U_\beta$. Next, thinking about the extension of GR over ZFC-deformed spacetimes leads to the deformation of transition functions from diffeomorphisms $f : \mathbb{R}^4 \to \mathbb{R}^4$ to some $\tilde{f}_\gamma$. We start by determining the domain of the transformations. We expect a degree of nonlocality with respect to spacetime points. Hence, we do not call $\tilde{f}$ a function on spacetime but rather a transformation. We have already indicated above the domain of $\tilde{f}$ as U_α as is the case for diffeomorphisms f. However, based on H1 from the previous section, it should be rather U_α in a local model V^B since the deformation of M^4 in the Planck regime requires factorization through V^B for certain $B \in V$. More precisely, following H1 and H2, we have the following version of H2

[H2′] The final stage of the gravitational collapse of spacetime is a family $\{U_{B,\alpha} = R^4_B \in V^B\}_{B \in \mathcal{B}}$ of flat objects R^4_α in Boolean models V^{B_α} and a family $\{\tilde{f} : R^4_{B1} \to R^4_{B2}\}$ of nonlocal transformations.

We can be more specific about the maps $\tilde{f} : R^4_{B1} \to R^4_{B2}$ by claiming that they are determined by the maps between Boolean models and that these last maps are determined by the homomorphisms of the Boolean algebras. Using the same symbols $\tilde{f}$ for the maps of the models, $\tilde{f} : V^{B1} \to V^{B2}$, let us observe that each $\tilde{f}$ determines a map on objects $R^4_{B1} \to R^4_{B2}$ of the corresponding models. However, the origins of this $\tilde{f}$ on models lie in the homomorphisms of the Boolean algebras $B1$ and $B2$. Thus, the maps between the R^4s objects have their origins in the homomorphisms of the Boolean algebras, as in the following Lemma 2 and Corollary 3.

Lemma 2. $\tilde{f} : V^{B1} \to V^{B2}$ *comes from certain homomorphisms of Boolean algebras,* $h_f : B1 \to B2$ *in* V*.*

This lemma is a direct consequence of the behavior of Boolean-valued models of ZFC under homomorphisms of Boolean algebras, e.g., ref. [14].

Now we see that it follows that indeed

Corollary 3. $\tilde{f} : R^4_{B1} \to R^4_{B2}$.

This follows from the fact that $R^4_{B2} = R^4$ in $V^{h_f(B1)}$ which is R^4_{B2} in V^{B2}.

Remark 6. *The nonlocal character of $\tilde{f}$ in R^4_B reflects its derivation from the homomorphism of B and thus from the transformation of V^B, which is the homomorphism of the formal neighborhood of R^4 rather than any local map of R^4 itself.*

Please note that the local diffeomorphisms of $\mathbb{R}^4$ do not spoil the action of Hom V^B since any $f \in \text{Diff}\,\mathbb{R}^4$ sends open neighborhoods to open neighborhoods of $\mathbb{R}^4$ and R^4_Bs additionally refine them. Similarly, Diff M^4 does not interfere with Hom V^B. Hence, we have the following modification of the symmetry of GR

$$\text{Diff}\,M^4 \oplus \text{Hom}\,V^B \tag{2}$$

which should be understood as follows. In the deep Planck regime, the dominant way of changing the coordinate frames is by homomorphisms of V^B since M^4 can lose its integrity, while in the macroscales, where M^4 survive as a smooth manifold, there effectively dominate diffeomorphisms of $\mathbb{R}^4$. When the smoothness of M^4 is preserved (or dominates) the action of $\tilde{f} \in \text{Hom}\,V^B$ on opens of M^4 is replaced by diffeomorphisms of M^4 (there are no nonlocal transformations) while when the integrity loss of M^4 occurs, then Hom V^B enters the stage and dominate. Thus, until the density of energy and deformation of spacetime would not thorn it to the pieces, there are GR and diffeomorphisms governing its behavior. This loss of the integrity of spacetime is a natural threshold for preventing quantum phenomena of spacetime.

In the remainder of this paper, we show that the extension of Diff M^4 by Hom V^B is precisely what is needed for the formal reconciliation of QM and GR in this setting.

3. Extension of QM and GR

In the Introduction, we have mentioned the relationship between QM and the set-theoretic constructions inherently following the QM formalism. Here, we focus on this relation as the fundamental problem of the extension of QM such that the set theory of QM determines its extension. To properly approach the set theory of QM, Takeuti proposed to refer to the lattice of projections $\mathbb{L}(\mathcal{H})$ as generalizing the Boolean algebras—the maximal complete Boolean algebras of projections in $\mathbb{L}$. Then, one refers to $V^{\mathbb{L}}$ as the universe of set theory for QM. $V^{\mathbb{L}}$ directly generalizes the Boolean-valued model of ordinary sets, V^B, where this last can be seen as more general than the 2-valued cumulative class of sets $V^2 \simeq V$, which is a class-like, classical model of ZFC (here $2 = \{0,1\}$ − the 2-valued Boolean algebra). Keeping in mind that $\mathbb{L}$ can be considered to be the logic of QM and that the algebras 2 and B represent classical and Boolean logics, respectively, the construction $V^{\mathbb{L}}$ seems to be a canonical object for quantum sets. However, this approach has many intrinsic difficulties, e.g., [5,6]. Anyway, the richness and complication of the structure $V^{\mathbb{L}}$ is always locally reduced to the Boolean models $V^{B_\alpha}, \alpha \in I$ for maximal complete Boolean subalgebras, B_α, of projections in $\mathbb{L}(\mathcal{H})$. The special role of B_α for QM is that

Lemma 3 (Takeuti [4])**.** *For any family of pairwise commuting self-adjoint operators $\{A_c, c \in L\}$ on $\mathcal{H}$ there always exists a maximal complete Boolean algebra of projections B_α containing all A_c.*

Remark 7. *We say that B_α contains A_c when A_C has the spectral decomposition*

$$A_c = \int_{\sigma_c} \lambda dE_\lambda,\ \sigma_c \text{ is the spectrum of } A_c,$$

and the spectral family dE_λ has its values in B_α.

Thus, B_α determines the local Boolean logic of QM and the local set theory V_α^B of QM, but also the set of commuting observables $\{A_c, c \in L\}^{com}, A_c \in B_\alpha$ on $\mathcal{H}$. Furthermore, there exists a very special relation between the model of ZFC, V^B, and the set of observables $A_c \in B_\alpha$. Namely

Lemma 4 (Takeuti [4])**.** *There exists 1:1 correspondence between the set of Dedekind's real numbers in the ZFC Boolean model V^{B_α} and the set of self-adjoint commuting operators $A_c \in B_\alpha$.*

In the internal Boolean world V^{B_α} of sets, the set of reals (Dedekind's) is the elevated set of all commuting observables on $\mathcal{H}$, to more general Boolean logic. If there is a model of sets (in a generalized sense) with a similar property as in Lemma 4 but for *all*, also noncommuting self-adjoint operators on $\mathcal{H}$, then such a model would serve as direct counterparts for QM. In fact, we have [7,15]

Lemma 5. *For $V^{\mathbb{L}}$, there is 1:1 correspondence between Dedekind's real numbers and all self-adjoint operators on $\mathcal{H}$.*

However, the resulting set theory of $V^{\mathbb{L}}$ is not canonically a set theory. Thus, the maximal natural interpretation of sets, which preserves the quantum observables, is V^{B_α} where the observables are reduced to the set of commuting ones according to Remark 7. Still, one can glue together these local 'commutative' contexts V^{B_α}s to cover the entire lattice $\mathbb{L}(\mathcal{H})$.

Remark 8. *Please note that even though the set of $\{V^{B_\alpha}\}$ for all maximal complete Boolean algebras B_α in $\mathbb{L}$, covers the entire lattice and generates all self-adjoint operators on $\mathcal{H}$, one cannot reduce $\{V^{B_\alpha}\}$ to the single model V^{B_α} — each V^{B_α} generates only commutative family of operators and B_α is distributive while $\mathbb{L}$ is not.*

This local-to-global relation of $\{B_\alpha\}$ and $\mathbb{L}$ is the particular task of this section. Let $\mathrm{Bor}(\mathbb{R})/Null$ be the quotient algebra of all Borel subsets of $\mathbb{R}$ modulo the ideal of Lebesgue measure zero subsets.

Lemma 6 ([16], Th. 9.4.1)**.** *For complex infinite-dimensional, separable, Hilbert spaces of quantum states $\mathcal{H}^\infty$ the following are true*

i. $B_\alpha = B_{\alpha,a} \oplus B$,

ii. $B = Bor(\mathbb{R})/Null$.

iii. *B is an atomless Boolean algebra, the same for all αs, and $B_{\alpha,a}$ is an atomic part of B_α which can differ with α.*

Corollary 4. *If $\dim \mathcal{H} < +\infty$, then $B_\alpha = B_{\alpha,a}$ contains only the atomic part.*

The universality of $B = \mathrm{Bor}(\mathbb{R})/Null$ for $\mathcal{H}^\infty$ allows us to show the following result.

Corollary 5. *Let $\dim \mathcal{H} = +\infty$ and $B_\alpha = B$ be as above. Then*

$$\text{For any } h_{12} \in Aut\,(B) \text{ there corresponds some } f_{12} : V^{B1} \to V^{B2},$$

i.e., for any pair $B1, B2$ of maximal complete Boolean algebras of projections in $\mathbb{L}$ there exists $h_{12} \in Aut\,(B)$ such that $V^{h_{12}(B1)} = V^{B2}$.

The above corollary states the direct fact that any $h \in \mathrm{Aut}\,(B)$ corresponds to the isomorphism of the models $V^B \simeq V^{h(B)}$. This holds for any pair $B1, B2$ of maximal complete Boolean algebras of projections from $\mathbb{L}$ since they are isomorphic to B. This proves Corollary 5.

One can consider the extension of QM given by the set-theoretic component above, which also extends the group $\mathrm{Diff}\,\mathbb{R}^4$ from GR as in (2) from the previous section

$$\mathrm{Diff}\, M^4 \oplus \mathrm{Aut}\, V^B \tag{3}$$

where now, certainly, $B = \mathrm{Bor}(\mathbb{R})/Null$. Then, the main result follows.

Theorem 2. *Let B be the universal maximal Boolean algebra of projections as in Lemma 6 and Corollary 4. The symmetry of the structure resulting from the extension of GR (the equivalence principle extended over causally disconnected flat open fragments of M^4 internal to V^B) coincides with the symmetry of the QM projection lattice in $\mathcal{H}^\infty$ that preserves local set-theory Boolean contexts.*

Proof of Theorem 2. The resulting symmetry after the extension of GR is determined by diffeomorphisms of M^4, which are enriched along $h : V^{B1} \to V^{B2}$. This h is determined by a homomorphism $h : B1 \to B2$ (using the same symbol h). In particular, $h \in \mathrm{Aut}\,(B)$, leads to $h : V^B \to V^B$ and to generalized maps $h : R^4_{B1} \to R^4_{B2}$.

In QM on $\mathcal{H}^\infty$ and for vanishing the atomic part of B_α as in Lemma 6 and Corollary 5 one has $B = \mathrm{Bor}(\mathbb{R})/Null$ and thus $h_A \in \mathrm{Aut}\,(B)$ is the change in the local context in QM according to the following.

Let B'_A, B_D be two maximal complete Boolean algebras of projections from $\mathcal{H}$ such that A, D, $[A, D] \neq 0$ are two noncommuting self-adjoint operators in B_A, B_D correspondingly (see Remark 7). This means that there exists a projection $b_D \in B_D$, noncommuting with A.

Then $e^{i\gamma A} b_D e^{-i\gamma A}$ sends b_D to a projection in certain maximal algebra B_A for any $\gamma \in \mathbb{R}$.

It can be directly checked that

$$B_A \simeq e^{i\gamma A} B_D e^{-i\gamma A} \tag{4}$$

establishes a homomorphism of the algebras B_A and B_D.

Under the supposition of the theorem $\mathcal{H} = \mathcal{H}^\infty$ and $B_\alpha = B$, one has

$$B_A \overset{\mathrm{iso}}{\simeq} B \text{ and } B_D \overset{\mathrm{iso}}{\simeq} B$$

so the correspondence (4) establishes the isomorphism h_A of B_A and B_D. Since $B_A \simeq B$ and $B_D \simeq B$ then $h_A \in \mathrm{Aut}\,(B)$. Finally, this gives rise to the isomorphism of local models of set theories for QM, $V^{B_A} \simeq V^{B_D} \simeq V^B$. Finally, taking $R^4_B \in V^B$, there corresponds to the change in the generalized coordinates

$$h_A : R^4_B \to R^4_B$$

for the extension of GR as at the beginning of the proof. This finishes the proof of Theorem 2. □

Let us assume explicitly, as is already stated in Theorem 2, that each 4-region R^4_α of the fragmented spacetime in various V^{B_α}, is flat R^4s, then the gravitational energy densities are no longer carried by local diffeomorphisms.

Remark 9. *One consequence of this theorem is that the gravitational energy densities in the deep Planck regime are carried by* $Aut\,(B)$ *rather than by* $Diff\,M^4$ *as is the case in the subplanckian classical regime of (nonextended) GR.*

Remark 10. *The isomorphism (4) is an example of the Fourier transform between, e.g., momentum* P *and position* Q *operators in QM based on the spectral theorem. Here, we have the isomorphism of the Boolean algebras* B_P *and* B_Q *and* $\dim \mathcal{H} = +\infty$ *due to the uncertainty principle.*

4. Euclidean Quantum Gravity

4.1. S^4s and the Fragmentation of Spacetime

In this and the following subsections, we develop the formalism for gravitational energy densities close to, or in, a singularity where quantum description should dominate. We make some simplifications so that the construction presented here can be considered to be the application of a general procedure. First, let us work with manifolds with Riemannian metrics so spacetime will be a smooth 4-dimensional Riemannian M^4. Let (S^4, g_{S^4}) be a smooth 4-sphere with the standard round metric g_{S^n}. The sectional curvature of an n-sphere of radius r is $K_{S^n} = 1/r^2$ and its scalar curvature is $Sc_{S^n} = n(n-1)/r^2$ so thus $Sc_{S^4} = \frac{12}{r^2}$ gives $Sc_{S^4} = 12$ for radius $= 1$. Next, let the scalar curvature of M^4 in the vicinity of a gravitational singularity, such as in BH, be a parameter diverging to $+\infty$, and this is a true singularity, not just a coordinate one. In addition, the increase in the scalar curvature $K \to +\infty$ will be modeled by $(S^4, r), r \to 0$. The last seems to be a true limitation since S^4 has a constant scalar curvature, which in the vicinity of quantum effects is fluctuating, and S^4 is very regular and symmetric to reflect a general situation. However, the following property in Theorem 3 shows that S^4 is a quite generic model and, moreover, (S^4, r) approaches arbitrary big curvatures with a smaller and smaller radius. The metric fluctuations can be further considered on such S^4s. Given a smooth Riemannian 4-manifold (M^4, g) the map $f : M^4 \to S^4$ is called 'distance non-increasing', when

$$d_g(p,q) \leq d_{g_{S^4}}(f(p), f(q)) \text{ for any } p,q \in M^4.$$

Theorem 3 (ref. [17], Theorem A)**.** *Let* (M, g) *be a four-dimensional closed connected oriented Riemannian manifold with* $Sc_g(p) \geq 12$*. If* $f : (M, g) \to (S^4, g_{S^4})$ *is a smooth, distance non-increasing map of non-zero degree, then f is an isometry.*

Remark 11. *Given* M *with the 'big' local scalar curvature* $Sc_g(p), p \in M$*, we always find* (S^4, r) *such that* $Sc_{S^4} > Sc_g(p)$ *for all* $p \in M$ *and refer to this* S^4 *instead* M *with the distance non-increasing property for* f*. The local fluctuations of the metric can now be considered to be on* S^4*.*

Recall that the connected sum # of two smooth manifolds with boundary, M_1, M_2, is a smooth manifold $M_1 \# M_2$ obtained by cutting out the open balls from both boundaries of the manifolds and gluing smoothly the remaining manifolds along the boundaries with suitably reversed orientation.

Remark 12. *In fact, we model high-curvature regions of spacetime by a part of* S^4 *since, at this stage,* M^4 *is connected. Thus,* $S^4 \setminus \overline{D^4}$ *would be glued to* $M^4 \setminus \overline{D^4}$*, resulting in the connected sum* $M^4 \# S^4$*. Please note that (topologically and smoothly)* $S^4 \setminus \overline{D^4} \simeq S^3 \times \mathbb{R}$ *leads to the* $(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R}^3 \times \mathbb{R}) = (\mathbb{R}^4, \mathbb{R}^4)$ *topological (or standard smooth) cover.*

There is yet another clue for the $S^3 \times \mathbb{R}$ local geometry emerging in the context of the Lorentz metric of 4-spacetime for large density perturbations and for the standard ΛCDM model. In the semiclassical approach to the Lorentzian GR in this cosmological model, for the perturbations on a much smaller scale than the size of the modes, the 'separate universe' results, which may carry the geometry of the standard cosmological model but with different values of the background density or curvature. In other words, a spherically symmetric perturbation in an FLRW background behaves like a separate FLRW universe with different matter density and curvature [18]. Topologically $S^3 \times \mathbb{R} \simeq S^4 \setminus \overline{D^4}$ so in the limit of losing the integrity with spacetime (breaking the topological connected sum), and switching to the Riemannian metrics, the resulting smooth S^4 (with eventual exotic metric) can represent the gravitational instanton (see the discussion below in this section). This semiclassical gravitational description of such hyperdense regions as separate universes also refers to black holes since an observer (in the vacuum region surrounding the hyperdensity) would see the separate universe as a black hole [18]. We argue that losing the integrity with spacetime gives even more insight into the quantum interior of black holes as the semiclassical approximation reduces the description merely to the instanton's effects.

Following Remark 12 with the Euclidean metric, the integrity of spacetime is preserved. However, for certain high values of energy density and the corresponding scalar curvature of S^4, the disintegrating processes could dominate. Let us try to understand this step in our setup. Let $\rho_p, p \in M^4$ be the density of gravitational energy at $p \in M^4$, and we do not present any physical process behind this value. Let us assume the existence of a singular region S over which geodesics cannot be smoothly extended. Thus, the radius r of (S^4, r) approximates the 4-curvature of spacetime locally as $\frac{12}{r^2}$ where close to the singularity it is formally diverging due to $r \to 0$. We have the simplified or naive model of the spacetime singularity with the parameter r.

Now let us apply the procedure of Section 2.1, in particular H2 and H2$'$.

i. Let $\mathcal{U} = \{U_i, \alpha = 1, 2, 3, 4, \dots\}$ be some smooth cover of S^4 where $U_i \simeq \mathbb{R}^4$. For the standard smooth S^4, there exists a 2-element cover $\{U_1, U_2\}$; however, the maximal smooth atlas (the smoothness structure) still contains infinitely many elements.
ii. There exists r_0 (possibly of the order of the Planck length r_P) where spacetime becomes locally internal in models V^Bs.
iii. In the same regime after reaching the Planck length r_P and the local scalar curvature

$$\frac{12}{26{,}121 \cdot 10^{-70}[\mathrm{m}^2]} \simeq 4594 \cdot 10^{70}[\mathrm{m}^{-2}] \tag{5}$$

then the fragmentation of local spacetime occurs (H2$'$) which is the fragmentation of (S^4, r)

$$\overline{\mathcal{U}} = \{\overline{U}_i = R^4_{B_i} \in V^{B_i}, R^4_{B_i} = R_{B_i} \times R_{B_i} \times R_{B_i} \times R_{B_i} \text{ flat in } V^{B_i}, i = 1, 2, \dots\}$$

where each $\overline{U}_i$ is the 'local' internalization of $U_i \in \mathcal{U}_{(S^4,r)}$ to V_i^B.

Remark 13. *As far as the connected sum of (S^4, r) and M^4 is preserved then (see Remark 12) the interior int$(\overline{D^4}) = R^4_2 = \mathbb{R}^4$ removed from S^4 where $2 = B$ is the Boolean algebra and $V^2 = V$. However, the complete fragmentation of S^4 does not necessarily support such connectedness.*

The fragmentation of S^4 and its separation from M^4 is presumably not an instantaneous process but spreads in time when the connected sum $S^4 \# M^4$ lasts. The process starts with it and the final stage would be complete separation and fragmentation, i.e.,

$$M^4 \to M^4 \# S^4 \to M^4 \setminus \overline{D^4} \tilde{\cup} \{R^4_{B_i}\} \tilde{\cup} \{f_{ij} : V^{B_i} \to V^{B_j}\}. \tag{6}$$

Here, the enriched set-sum symbol $\tilde{\cup}$ is not only the set-sum of the corresponding families of functions or topological spaces but additionally, the enriched set-sum preserves the topological relations of the members of the corresponding sets. In the above expression, the right-hand side represents the part of spacetime $M^4 \setminus \overline{D^4}$ sum up with the set of the resulting fragments of S^4 and sum up with the relations between them represented by the maps of Boolean models V^{B_i}.

Following the discussion around H2′, we have, in terms of generalized 'transition functions', the following schema

$$M^4 \to M^4 \# S^4 \to M^4 \setminus \overline{D^4} \tilde{\cup} \{R^4_{B_i}\} \tilde{\cup} \{\tilde{f}_{ij} : R^4_{B_i} \to R^4_{B_j}\}. \tag{7}$$

$M^4 \overset{\text{Diff.}}{\simeq} M^4 \# S^4$ and the extremely high curvature of S^4 can be erased by a diffeomorphism that simply reflects the fact that (S^4, r_1) and (S^4, r_2) are diffeomorphic and the Riemann tensor (and scalar curvature) are not in general invariant under 4-diffeomorphisms.

Thus, the symbol $\tilde{\cup}$ in (6) and (7) is just the set operation of summation which results, e.g., in a set of points of a topological space $M^4 \setminus \overline{D^4}$ (with its default topological structure) $\cup$ a set of $R^4_{B_i}$ objects in categories V^{B_i}, $\cup$ the set of relations between the categories V^{B_i}. This is a nonhomogeneous object from the topological point of view but still has a well-defined set-theoretic meaning on which one can perform certain operations. More precisely, the forgetful operation on the topological space $M^4 \setminus \overline{D^4}$ leads to the set of points, which gives rise to a set-theoretic sum with the sets of certain objects.

Corollary 6. *If* $\dim \mathcal{H} = +\infty$, *then the final stage of fragmentation of* S^4 *reads*

$$M^4 \setminus \overline{D^4} \tilde{\cup} \{R^4_B\} \tilde{\cup} \{f_{ij} : V^B \to V^B\} \text{ or } M^4 \setminus \overline{D^4} \tilde{\cup} \{R^4_B\} \tilde{\cup} \{\tilde{f}_{ij} : R^4_B \to R^4_B\}$$

where $f_{ij} \in Aut\,(B)$ *and* $B = Bor(\mathbb{R})/Null$.

In the next sections, we will be interested in the reverse process for fragmentation of S^4 in the smooth V-limit, which is the gluing of fragments of S^4 and obtaining (if possible) a smooth S^4. Thus, our assumption here is that this gluing process, under certain conditions, leads to S^4. More formally, the gluing can be seen as the construction of a manifold M^n from its atlas $\mathcal{U} = \{U_\alpha\}$ where the set $\mathcal{U}$ is structured by the automorphisms of B.

Remark 14. *For any open cover* $\mathcal{U}$ *of a Hausdorff paracompact manifold* M^n *one assigns a cover groupoid the space of objects being the disjoint union* $\mathcal{G}_{\mathcal{U}} = \coprod_\alpha U_\alpha$ *and the space of arrows is the fiber product* $\mathcal{U} \times_{M^n} \mathcal{U}$. *Then the manifold* M^n *is the colimit of* $\mathcal{U} \times_{M^n} \mathcal{U} \rightrightarrows \mathcal{U}$ *in the category of smooth manifolds and maps.*

In the quantum regime, the set $\mathcal{U}$ is structured by introducing the additional relations between U_αs that correspond to the automorphisms of B. In the classical limit (V-smooth limit see Appendix A), there remains just the disjoint union of local patches, and the way to S^4 or M^4 follows Remark 14.

Assuming the preservation of energy in the transition process from the classical sector with S^4 to quantum with fragmented S^4, there has to be a transfer of the curvature of spacetime of S^4 and the density of gravitational energy into the algebraic family of automorphisms $\{\tilde{f}_{ij}\}$, since the geometric flat R^4_Bs all have vanishing curvature and thus the density of gravitational energy.

Accordingly, one expects similar preservation for momentum and angular momentum, i.e., the transition from classical to quantum does not spoil the main preservation laws. There could exist relative momenta (angular momenta) between local charts which would lead to the loss of net momentum (angular momentum) after losing the causal connectivity of spacetime. However, after the rebirth of the smooth S^4 and taking a connected sum with spacetime, the energy and momentum loss would be retrieved.

4.2. Towards Gravitational Instantons and Quantum Gravity

Consider two isomorphic copies of V^B, i.e., $V^{B_1} \simeq V^{B_2} \simeq V^B$ where $B_1 = B_2 = B = Bor(\mathbb{R})/Null$ is such that $B_1 = f_{21}(B_2)$ for certain $f_{21} \in \mathrm{Aut}(B)$. Let S^n_i be n-sphere in V^{B_i} (which is a Boolean model of ZFC). We want to identify smooth S^4 that becomes fragmented due to the process f of changing the models V^B but before the final flattening of the local charts as in the family $\{R^4_B\}$s. Let us choose the standard 2-element atlas $\{U_1, U_2\}$ where U_1, U_2 are still diffeomorphic to $\mathbb{R}^4$ but are rather two hemispheres of S^4 where the metric $g_{\mu\nu}$ on S^4 becomes truncated to $g^{(1)}_{\mu\nu}$ on U_1 and $g^{(2)}_{\mu\nu}$ on U_2. U_1, U_2 can be both considered to be the interiors of the closed 4-disks $\overline{D^4}_{1,2}$ with the S^3 boundaries, $\partial\overline{D^4}_{1,2} = S^3_{1,2}$ correspondingly. Now, gluing two of these closed hemispheres by diffeomorphisms can be seen as gluing along the diffeomorphism of their boundaries $S^3_1 \to S^3_2$.

Let S^n be n-sphere and $\pi : S^{n-1} \to S^{n-1}$ some diffeomorphism of the boundaries of the closed hemispheres of S^n. Let $g^{(n-1)}_{\mu\nu} = g^{(1)}_{\mu\nu}$ be an Euclidean metric on $\mathbb{R}^{n-1}$ and $g^{\pi,(n-1)}_{\mu\nu} = g^{(2)}_{\mu\nu}$ its transformation by π. Then the following is the metric on $\mathbb{R}^n$

$$ds^2 = dt^2 + \left[(1 - \lambda(t))g^{(1)}_{\mu\nu} + \lambda(t)g^{(2)}_{\mu\nu}\right]dx^\mu dx^\nu \tag{8}$$

where $\lambda : \mathbb{R} \to \mathbb{R}$ is a strictly increasing smooth function of the time t such that $\lim_{t\to-\infty} s(t) = 0$ and $\lim_{t\to+\infty} s(t) = 1$. The usual one-point compactification of $\mathbb{R}^n$ with this metric results in a smooth n-sphere (see ref. [19]). However, it can happen that diffeomorphisms $\pi : S^{n-1} \to S^{n-1}$ gluing both spheres at the equator of S^n are not continuously connected with the identity on S^{n-1}. Then the resulting smooth S^n, $\tilde{S^n}$, is nondiffeomorphic with the standard smooth S^n, i.e., $S^n \neq \tilde{S^n}$, but still $\tilde{S^n}$ is smooth and homeomorphic with S^n. More is true, all classes of smoothings (smoothness structures) on S^n are in 1:1 correspondence with the classes of 'large' (not connected with identity) diffeomorphisms of S^{n-1}, ref. [19]. However, a crucial exception is the case of dimension 4, i.e., S^4. J. Cearf [20] proved in 1968 that every diffeomorphism of the 3-sphere preserving the orientation is isotopic to the identity. This result leads to the conclusion that any diffeomorphism of S^3 extends over a diffeomorphism of the 4-ball for which S^3 is the boundary, i.e., $\Gamma_4 = \mathrm{Diff}(S^3)/\rho(\mathrm{Diff}(D^4))$ is a trivial group, where ρ is the truncation of diffeomorphisms of D^4 over its boundary S^3. Thus, in dimension 4, there is no room for exotic S^4, which would follow the construction of the 4-metric from diffeomorphisms π. Does it mean that there are no exotic 4-spheres? No, we do not know, and many researchers expect that one day, the proof of the existence of exotic S^4 will be completed, but certainly by methods different from gluing 3-diffeomorphisms. Until now, every attempt to construct exotic S^4 has failed by any means. Thus, the celebrated smooth 4-dimensional Poincaré conjecture remains still unresolved.

Reference to n-spheres, especially exotic ones, seems to be crucial for the QG program. It follows from the breakthrough analysis of gravitational anomalies by Edward Witten in ref. [19] in higher-dimensional theories of gravity like superstring theory. According to Witten, exotic S^n as construed in (8), are 'the best' candidates for gravitational instantons since to every $J = \tilde{S}^n_\pi$ there exists $J^{-1} = \tilde{S}^n_{\pi^{-1}}$ such that the connected sum $J + J^{-1} = \tilde{S}^n_{\pi\circ\pi^{-1}} =$ standard S^n which by the one-point decompactification of S^n corresponds to the flat $\mathbb{R}^n$. Such a flat n-space gives the maximal contribution to a gravitational Euclidean action S_E since $S_E[\delta_{\mu\nu}] = 0$ thus

$$\langle g_2 | g_1 \rangle = \int Dg \cdot e^{-S_E[g]} \tag{9}$$

has dominating contributions coming from the zero action in the saddle point. Witten observed that only such pairs of instanton and anti-instanton have to be included in the path integral due to the cluster decomposition effect: J has the same effect as the pair $J + J^{-1}$ (consider a very big separation of both instantons causing neglecting the action of the very distant one). Thus, still, whenever $J + J^{-1}$ gives the maximal contribution to the path integral, it remains maximal also for J alone because of the existence of J^{-1}. For gravitational instantons that do not have a well-defined anti-instanton state, there is no clear reason for including them in the path integral (there are presumably other terms with dominating contributions in the path integral).

In general, gravitational instantons are semiclassical gravitational solutions that might play a significant role between the classical spacetime and quantum regimes. In our approach, it is the state of spacetime just before its fragmentation in the fully quantum realm. However, in dimension 4, it is likely that exotic S^4s do not exist, and the entire approach based on exotic spheres would fail in this physical dimension. Nevertheless, the approach allows for skipping these 'classical' instantons and switching to more quantum analogs. To this end, let us turn to tunneling processes connected with instantons in general but also present in Formula (8) as gluing of the 3-metrics. Tunneling takes place between two metrics on $\mathbb{R}^{n-1}$, $g_{\mu\nu}$ and $g^{\pi}_{\mu\nu}$, which are related by nonlocal, large diffeomorphism π of S^{n-1} the one-point compactification of $\mathbb{R}^{n-1}$. This nonlocality is expressed in the inability to glue the two metrics by a diffeomorphic coordinate change but requires action on the whole S^{n-1}. This action is again a diffeomorphism, but it cannot be continuously connected with any local coordinate change. Rather, π belongs to another class of large diffeomorphisms of S^{n-1}. Thus, tunneling is nontrivial, and the resulting n-sphere has to be exotic, i.e., nondiffeomorphic with the standard smooth S^n. We do not know whether exotic 4-spheres exist, but there is also another reason for skipping the construction in (8) while considering the tunneling phenomena of gravity in dimension 4. It is the Cerf result that there is no large diffeomorphism of S^3 (see the discussion before in this section), and hence, there is no tunneling in the sense of (8). We will show how our approach allows for addressing such tunneling questions and leads to generalized exotic smooth S^4 or instantons in the quantum domain. The direct proposal would be a modification of (8), i.e.,

$$ds^2 = dt^2 + \underbrace{(1-\lambda(t))g^{(1)}_{\mu\nu}dx^\mu dx^\nu}_{\text{in } V^{B_1}} + \underbrace{\lambda(t)g^{(2)}_{\mu\nu}dx^\mu dx^\nu}_{\text{in } V^{B_2}} \tag{10}$$

where $g^{(2)}_{\mu\nu}$ would be $g_{\mu\nu}$ in $V^{B_2} = f(V^{B_1}), f \in \operatorname{Aut} B$ and the '+' sign in the bracket is adding the contributions of both terms in V. The last requires a more careful explanation. A general question here is: What are the contributions in V derived from different Boolean models V^{B_1}, V^{B_2}? Let $T\mathbb{R}^4$ be the tangent vector space to $\mathbb{R}^4$ then it holds (see Remark 7 for $A \in B$ where A is a self-adjoint operator and B the Boolean algebra of projections)

Lemma 7. *Let $B_1 = B_2 = B$ and $A_1 \in B_1$ and $A_2 \in B_2$ be two self-adjoint noncommuting operators on $\mathcal{H}^\infty$, then In V: If flat $\mathbb{R}^4$ corresponds to flat $R^4_B \in V^{B_1}$ then $T\mathbb{R}^4$ corresponds to $R^4_B \in V^{B_2}$.*

For the proof of this lemma, let us turn to Section 3 and note that Theorem 2 and Remark 10 give the result in Lemma 7.

Remark 15. *Flat* $\mathbb{R}^4 \overset{iso}{\simeq} T\mathbb{R}^4$ *as vector spaces and Lemma 7 refers to the constitution of local frames of a manifold, say* M^4*, as* $\mathbb{R}^4$ *or* $T\mathbb{R}^4$ *on the set-theoretic level. Thus, even though there are isomorphic copies of* R^4*s that model local* U_α*s in the smooth atlas of* M^4*, still taking into account the origins of the set theory, they can differ subtly. The difference is apparent when set theory degrees of freedom are referred to in atlases of a smooth manifold. The change in the perspective of the set theory from trivial* V *to local* V^B *is responsible for the effect.*

One can also think about this discrepancy between different R^4_B detected in V as a kind of curvature: taking a closed path in M^4 going through different $\mathbb{R}^4$ local regions then, at the beginning, the data at the initial point can differ at the end after taking the closed path. This difference is the jump or gluing operation in (10). Although flat $\mathbb{R}^4$ and $T\mathbb{R}^4$ are isomorphic, the content of the set theory distinguishes them.

There is one problem with the approach in (10), the classical spacetime limit of it is necessary the standard S^4 after 1-point compactification of this geometry, and this conclusion holds even in the hypothetical case of the existence of exotic 4-spheres. The reason for this is the triviality of the diffeomorphism classes of S^3. So, in the smooth limit, the jump operation of (10) omits the exotic spheres. We would like to have a different situation: the jump operation in the quantum regime, in the process of its reduction to the classical smooth regime, should go through the exotic S^4 if they existed. Certainly, if they do not exist, it is impossible, but the construction could still not have excluded exotic S^4s in principle. This is because alleged exotic S^4s are well-suited for being the gravitational instantons, and even if exotic 4-spheres do not exist, the jump operation, not excluding their existence, would also be opened for another kind of instanton phenomenon. We need to modify the jump operation in (10) so that it does not exclude exotic 4-spheres. The simplest proposal is to take two 4-dimensional hemispheres of S^4 and glue them by the jump operator $\tilde{f} : R^4_{B_1} \to R^4_{B_2}$ such that $g^{(1)}_{\mu\nu}$ is the 4-metric on one open hemisphere in V^{B_1} and $g^{(2)}_{\mu\nu}$ on the other in V^{B_2} where $\tilde{f} : R^4_{B_1} \to R^4_{B_2}$ and $f : B_1 \to B_2$ is the jump or gluing operator.

$$\mathfrak{f} : \underbrace{g^{(1)}_{\mu\nu} dx^\mu dx^\nu}_{\text{on } R^4_{B_1} \text{ in } V^{B_1}} \to \underbrace{g^{(2)}_{\mu\nu} dx^\mu dx^\nu}_{\text{on } R^4_{B_2} \text{ in } V^{B_2}} . \tag{11}$$

The above formula can have the smooth S^4 limit in V. This follows from the internal construction of V^B in V and, from the other side, the canonical embedding $V \hookrightarrow V^B$ so that $V \hookrightarrow V^B \hookrightarrow V$ (see Theorem 1). From this it follows that the V-limit of $\mathfrak{f}$ in (11) is $f : \mathbb{R}^4 \to \mathbb{R}^4$ which in the smooth limit should give rise to a gluing diffeomorphism $\mathbb{R}^4 \to \mathbb{R}^4$. The following immediately follows.

Lemma 8. *The smooth* V*-limit of* (11) *cannot be any smooth exotic* S^4*.*

The reason for this is that any such gluing by diffeomorphisms of $\mathbb{R}^4$ has to be factorized through the gluing of S^3, which is a diffeomorphism of S^3. If there were nontrivial classes of these diffeomorphisms, then exotic S^4 would result. However, the Cerf result discussed before forbids this possibility. This closes the proof of Lemma 8.

In general, given a smooth manifold M^n in V with a smooth atlas $\mathcal{U} = \{U_i \simeq \mathbb{R}^n, f_{ij} : \mathbb{R}^n \to \mathbb{R}^n, i,j = 1,2,\ldots\}$ there corresponds a generalized atlas $\mathcal{U}^* = \{U_i \simeq R^n_{B_i}, \tilde{f}_{ij} : R^n_{B_i} \to R^n_{B_j}, i,j = 1,2,\ldots\}$. Conversely, we call the manifold $(M^n, \mathcal{U})$ in V a smooth V-limit of the generalized manifold $(\tilde{M}^n, \mathcal{U}^*)$. Switching between a generalized manifold and the corresponding manifold in V, thus taking the V limit, is based on Theorem 1.

To allow for exotic S^4 and gravitational instantons in dimension 4, we should further modify (11). Observe that the standard S^4 can be characterized as such a smooth 4-manifold $\tilde{S}^4$ homeomorphic to S^4 such that every smooth atlas $\tilde{\mathcal{U}}$ of $\tilde{S}^4$ is incompatible as a smooth atlas with the 2-element good atlas $\mathcal{U}_2$ as in (11). This means that indeed $\tilde{S}^4$ would not be diffeomorphic with the standard S^4. Gluing just two standard open hemispheres as in (11) results in the standard smooth S^4 and also if there is a smooth open cover $\tilde{\mathcal{U}}$ of $\tilde{S}^4$ smoothly equivalent to $\mathcal{U}_2$, then $\tilde{S}^4$ would have to be the standard one. The compatibility of atlases or smooth covers we understand as in the maximal smooth atlas for a manifold M^n, which is the smoothness structure of M^n.

Lemma 9. *Let $\tilde{\mathcal{U}}$ be a smooth, good open atlas of $\tilde{S}^4$. If there does not exist any $\tilde{\mathcal{U}}$ such that $|\tilde{\mathcal{U}}| = 2$, then $\tilde{S}^4$ has to be exotic S^4.*

Remark 16. *It can be stated equivalently as: if a maximal smooth atlas on S^4 is not compatible with any two-element good cover of it, then S^4 has to be exotic.*

Now, the modification of (11) is obvious. Let $\mathcal{U}^* = \{U_i \simeq R^4_{B_i}, \tilde{f}_{ij} : R^4_{B_i} \to R^4_{B_j}, i,j = 1,2,\dots\}$ be a generalized cover of S^4 with the jump operators $\{\tilde{f}_{ij}\}$. Let a V-smooth limit of $\mathcal{U}^*$ be $\mathcal{U} = \{U_i \simeq \mathbb{R}^4, f_{ij} : \mathbb{R}^4 \to \mathbb{R}^4, i,j = 1,2,\dots\}$ and $\mathcal{U}^* = \{U_i, i = 1,2,\dots\}$, then

Corollary 7. *If smooth V-limit of $\mathcal{U}^*$ exists as a smooth manifold $(S^4, \mathcal{U})$ in V and if*

$$\forall_{\mathcal{U} \text{ an open cover of } S^4} |\mathcal{U}^*| > 2$$

then S^4 has to be an exotic smooth 4-sphere.

We have a simple criterion for detecting exotic S^4, but it has serious drawbacks. Up to now, no single cover of S^4 has been detected as being unable to be reduced to a 2-element standard one. Even more, no one knows whether such covers exist at all (e.g., ref. [21]). On the other hand, if they exist and are not reducible to the 2-element standard cover, we should check the property for any such good cover. The criterion is a theoretical tool. Nevertheless, it can be useful in understanding quantum-classical regimes. In the following, we apply the criterion to distinguish the alleged exotic S^4 in the 4-dimensional semiclassical Euclidean QG.

- [QM$_p(\kappa)$] We call a manifold M^n locally QM-supported of degree κ when for each open $\mathcal{U}$ of M^n and for the local scalar curvature $\kappa_p < \kappa$ at $p \in M^n$, it contains among its local charts $\{U_1, U_2, U_3\}$ such that $U_1 \simeq \mathbb{R}^4$, $p \in U_2 \cap U_3$, and U_2, U_3, are smooth V-limits of $R^4_{B_1}, R^4_{B_2}$ in V^{B_1}, V^{B_2} correspondingly, and for $\kappa_p > \kappa$ the charts $\{U_1, U_2, U_3\}$ are $\{\mathbb{R}^4, R^4_{B_1}, R^4_{B_2}\}$ correspondingly, where B_1, B_2 are two maximal incompatible local contexts in $\mathbb{L}$ (different maximal local Boolean algebras of projections).
- [fQM$_p(\kappa)$] We call a manifold M^n faithfully locally QM-supported of degree κ if it is locally QM-supported of degree κ and different $R^4_{B_j}$ lead to different $\{U_j\}$ in the smooth V-limit in every open cover of M^n.
- [QM(κ)] We call a spacetime manifold M^n completely QM-fragmented of degree κ if it is faithfully locally QM-supported of degree κ and for each $p \in M^n$ $\kappa_p > \kappa$.

Remark 17. *The conditions may seem artificial; however, the fragmentation of spacetime for $\kappa > \kappa_0$ is performed such that a patch, say $U_\alpha \subset S^4$ is thrown into (at least) two pieces since we require there should exist $R^4_{B_i}$ and $R^4_{B_j}$ nontrivially related by certain automorphism of B. The relation*

$\mathbb{R}^4 \to R^4_B$ is always trivial as is $\mathbb{R}^4 \to R^4_{B_i} \to \mathbb{R}^4$. Without any nontrivial phase between $R^4_{B_i}$ and $R^4_{B_j}$, the patches can be reglued identically (see Appendix A for the explanation of this important point) and are not the separated fragments of spacetime. That is why we are taking in $QM_p(\kappa)$ and $fQM_p(\kappa)$ the three fragments as the minimal number of them.

Theorem 4. *If exotic S^4s exist, then the smooth V-limit of a completely QM-fragmented 4-sphere is exotic S^4, and the spheres are gravitational instantons in dimension 4 (as gravitational solutions of Euclidean GR).*

If exotic S^4s do not exist, then the smooth V-limit is not any smooth manifold, and it is a pair of families $(\{\mathbb{R}^4_j\}, \{f_{ij} : \mathbb{R}^4_i \to \mathbb{R}^4_j\})$ in V where f_{ij} are diffeomorphisms.

Proof. First we need to show that the smooth V-limit of QM(κ) S^4 is a pair of families $(\{\mathbb{R}^4_j\}, \{f_{ij}\})$ as in the theorem. However, this follows directly from the definition of a smooth V limit and the discussion of it given in Appendix A. More precisely, the fragmentation of S^4 in V is performed, and the family of $R^4_{B_i}$s in V^{B_i}s with some new relative phases coming from $Aut(B)$ is added. These new nontrivial phases lead to the nonidentity diffeomorphisms emerging in the V-limit (see Appendix A). The resulting families of $(\{\mathbb{R}^4_{i'}\}, \{f_{i'j'}\}$ in V can still be a smooth cover of the initial smooth S^4 or can be a smooth cover of another smooth manifold $\tilde{S}^4$ in V. The conditions in the theorem indicate an exotic S^4 (if there exists any) as $\tilde{S}^4$ since the condition for the completely QM-fragmented 4-sphere assumes that there is a certain level κ on which it holds true; hence, [QM(κ)] holds true, $\kappa > 0$. This fulfills the condition for Corollary 7 to be true. The nonsmooth (nonexotic) case is given by the construction stage in which the automorphisms of B result in diffeomorphisms f_{ij} in the limit, but no smooth S^4 supports this. This finishes the proof of Theorem 4. □

In general, we have two cases; in one, there exists a smooth manifold in V as a smooth limit of QM(κ) S^4 and in the second, the limit is not a smooth manifold, even though each $R^4_{B_i}$ gives rise to the smooth copy of $\mathbb{R}^4$ (see Appendix A). In condition QM$_p(\kappa)$ above, we have directly referred to the minimal number of local patches in any smooth atlas. This can be better understood by considering $\mathbb{R}^4 \simeq S^4 \setminus \{\text{pt.}\} \simeq S^4 \setminus \overline{D^4}$ and requiring that there be at least two incompatible local patches in any smooth atlas of $\mathbb{R}^4$. Incompatible patches mean that one is the smooth V-limit of $R^4_{B_1}$ and the other $R^4_{B_2}$ where B_1, B_2 are two maximal Boolean algebras of projections from $\mathbb{L}(\mathcal{H})$ containing different projections (irreducible to the single one, thus incompatible). Thus, such a smooth $\mathbb{R}^4$ interprets the QM incompatible patches cannot be the standard smooth $\mathbb{R}^4$ since its maximal smooth atlas (the smoothness structure) is not compatible with any one-patch standard cover. Currently, mathematicians recognized two families of continuum infinitely many exotic smooth R^4s, large R^4s, and small R^4s [22]. Large are distinguished by the property that they are not embeddable smoothly into the standard $\mathbb{R}^4$, while small can all be embedded in $\mathbb{R}^4$. Now, if the exotic R^4 obtained here were any large or small known exotic R^4s, then adding a point in infinity gives the standard S^4 [22].

Lemma 10 ([22]). *The one-point compactification of any large or small exotic R^4 is the standard smooth S^4.*

It follows that if exotic S^4 exists, then removing a point leads to exotic R^4, which is not any known existing exotic R^4. In particular, there would exist an exotic $\overline{D^4} \simeq \overline{S^4 \setminus \overline{D^4}}$ such that $D^4 \simeq \mathbb{R}^4$ is not compatible with a smooth D^4 with a single element atlas. Thus, requiring both, D^4 has at least two elements in each smooth atlas (i.e., the atlas compatible with the maximal smooth atlas) or S^4 has at least three elements in every smooth atlas, leads to the same conclusion that such S^4 has to be exotic.

Remark 18. *The approach favors the existence of exotic S^4s, in which case gravitational instantons are naturally represented. If such S^4s do not exist, the smooth V-limits were not any smooth manifold, which indicates on singular nonsmooth description that could have been experimentally distinguished from the smooth case.*

Consider an operator $\mathcal{F} : (S^4, \mathcal{U}_{S^4}) \to \{\mathbb{R}^4_i \text{ flat}, i = 1, 2, \ldots\}$ where $\mathcal{U}_{S^4} = \{U_i, i = 1, 2, \ldots\}$ is a smooth good cover of S^4 with $U_i \simeq \mathbb{R}^4$ flat, $i = 1, 2, \ldots$ and $\mathcal{F}$ is a kind of forgetful operator that completely erases the curvature of S^4. Thus, gluing diffeomorphisms $f_{ij} : U_i \cap U_j \to U_i \cap U_j$ on S^4 are incompatible with the flat $\mathbb{R}^4$ system, which supports a flat merely global $\mathbb{R}^4$.

4.3. Heisenberg Uncertainty and the Fragmentation of S^4

We have found a general criterion ensuring that the smooth S^4 reglued from the local R^4_B patches is exotic or not (Lemma 9, Remark 16, Corollary 7). However, it appears that something is missing to ensure that the fragmentation of S^4 due to the high curvature and density of energy leads to more than two pieces rather than just two. The intuition from physics where the disintegration of spherical bodies due to the growing internal pressure in real experiments most probably leads to such many fragments cannot be applied directly here, i.e., it could be, but it is not enough. This important feature has a more fundamental explanation which is already indirectly present in the previous sections. The traces of the behavior of spacetime in the extreme gravitational fields can already be seen in the Heisenberg uncertainty relation (HU), which needs infinite-dimensional Hilbert spaces. For the position $\hat{x}$ and momentum $\hat{p}$ operators, HU states

$$[\hat{x}, \hat{p}]\psi = i\hbar\psi,\ \psi \in \mathcal{H}; \quad \text{or in terms of the standard deviations } \sigma_x \cdot \sigma_p \geq \frac{\hbar}{2}$$

where x, p are three-dimensional objects (3-component vectors). Including t-E uncertainty, it can be written down in 4-vector notation and the Minkowski metric as

$$\sigma^\mu \cdot \sigma_\nu \geq \frac{\hbar}{2}\delta^\mu_\nu, \mu, \nu \in \{0, 1, 2, 3\},\ \sigma_\mu = (t, x), \sigma_\nu = (E, p).$$

Now let us consider growing the curvature of a 4-sphere due to the growing of the gravitational energy density and reaching the Planck scale (both with size and energy) with the resulting disintegration. There are two factors governing this process: one is the classical growth of the gravitational density of energy, and the second is the quantum uncertainty relation. The classical indicates, in particular that to measure the position of a form of matter trapped in the smaller and smaller sphere requires bigger and bigger energy, i.e., the infiltrating waves have to have very short lengths. QM places a limit on momenta, which makes them highly undetermined, according to HU. We propose to go a step further and consider the HU as an indication that position space and momentum space become fundamentally independent in the Planck regime. The recipe is given in Lemma 7 (see also Corollary 6 and Remark 15):

> [FOLIATION] In the singular regime of spacetime (the Planck regime), S^4 is fragmented into fragments which become foliated farther into $\mathbb{R}^4$, $T\mathbb{R}^4$ as local leaves. Still, the leaves remain connected by the QM Fourier transform of the operators Q and P.

Let us assume that the 4-sphere is (in the Planck regime) represented by just two fragments—local flat $\mathbb{R}^4$s—with additional Aut V^B carrying the gravitational energy. According to FOLIATION above, the fragment $\mathbb{R}^4$ of S^4 is further foliated into the pair $(\mathbb{R}^4, T\mathbb{R}^4)$ which for the flat standard case is isometric to $(\mathbb{R}^4, \mathbb{R}^4)$. Thus, we conclude

The minimal number of local patches of any fragmented 4-sphere is never smaller than 3.

The fragments of spacetime lose their causal connections and become separated but can still be described by different QM contexts (such as Q and P operator contexts). The proper perspective on such fragmented spacetime is thus the product of QM contexts $\prod_\alpha \{\text{context in } \mathcal{H}^{(\infty)}\}_\alpha$ which in the classical V-limit gives rise to

$$\prod_\alpha \{\text{context in } \mathcal{H}^{(\infty)}\}_{B_\alpha} \longrightarrow \prod_\beta U_{B_\beta} \overset{V\text{-limit}}{\longrightarrow} \prod_\beta U_\beta \text{ where } U_{B_\beta} = R^4_{B_\beta} \text{ in } V^{B_\beta} \text{ and } U_\beta \simeq \mathbb{R}^4 \text{ in } V$$

and these last local patches describe the smooth spacetime manifold locally.

4.4. Gravitons and Smooth 4-Spheres

From the point of view of topology, 4-spheres naturally represent the universal medium connecting any topological manifold M^4 and N^4. This is expressed in the following homeomorphism [23] (Lemma 2.1)

$$M^4 \# k S^4 \simeq N^4 \# l S^4, k, l \in \mathbb{N}$$

of which the special generic case is

$$M^4 \# S^4 \simeq M^4.$$

Here # is the connected sum of two manifolds. Reading from right to left, this means the possibility of creating a 4-sphere from M^4 but following the opposite direction, i.e., the absorption of S^4 by M^4. This suggests that the interchanging 4-spheres might be a carrier for some interactions. In fact, the formalism developed in this paper indicates that smooth 4-spheres might indeed be the geometric counterpart of gravitons in QG.

The above relations are extended over smooth cases, and if exotic S^4s exist, there would be essentially new phenomena. Let us again consider the fragmentation of S^4 as emerging from spacetime. If the final stage of the fragmentation were a family of flat $\mathbb{R}^4$ s, the curvature, and hence the density of gravitational energy, would disappear. However, given the family V^Bs of ZFC models and the corresponding family of objects, R^4_Bs the relative phases from Aut$B$ between the models V^Bs can carry the energy loss. That is why we propose that gravitons (the regime of the fragmented spacetime is certainly outside of GR) are responsible for the deficit of gravitational density. Preserving the energy is not just a classical phenomenon; it is extended over the quantum regime, as is also the case with the preservation of angular momentum. So far, our model allows for partial fragmentation of spacetime and then the total fragmentation of S^4 at the Planckian regime of densities. This fragmentation is due to the emission of gravitons. The inverse process of retrieving smooth S^4s from flat $\mathbb{R}^4$ pieces is due to the absorption of (a certain number) of gravitons. A simple setting like that is possible due to the simultaneous extension of GR and QM, as presented in the previous sections. A graviton corresponds to the collection $\{h_\alpha\} \subset \text{Aut}\, B$. Based on Lemma A1 in Appendix A, we can build the example illustrating a simple (linear approximation of the) spectrum of gravitons in terms of the relative jumps in the scalar curvature of S^4_t. Therefore, there is a family $\{S^4_r, r \in \mathbb{R}_{>0}\}$ of standard S^4 with radii r and $S^4_{r_1} \simeq S^4_{r_2}$ are diffeomorphic for any $r_1, r_2 > 0$ and $f_{r_1 r_2} : S^4_{r_1} \setminus \overline{D^4_{r_1}} \to S^4_{r_2} \setminus \overline{D^4_{r_2}}$ are the induced diffeomorphisms of $\mathbb{R}^4$, i.e., $f_{r_1 r_2} \in \text{Diff}\, \mathbb{R}^4$. Let

$G^D_t(A) = \{e^{itA}De^{-itA} \in \mathrm{Aut}\, B, t \in \mathbb{R}_{>0}\}$ be a family of gravitons given by the 1-parameter family of automorphisms of B (see (A4)). Let us take $k = \sqrt{12/t^2} = \frac{2\sqrt{3}}{t}$ and assume that A, D do not commute, $[A, D] \neq 0$. Then it should hold.

$$G^D_t(A)(\mathbb{R}^4_{k'}) = \begin{cases} \mathbb{R}^4_{k'}, \text{ for } k = k' \\ \mathbb{R}^4_t, \text{ for } k > k' \end{cases}$$

so that $G^D_k(A)(\mathbb{R}^4_k) = \mathbb{R}^4_k$.

Here $\mathbb{R}^4_k \subset S^4_k$ so that they both have the same constant scalar curvature. Analogous actions in 4-spheres read

$$G^D_k(A)(S^4_{k'}) = S^4_{k'}, \text{ for } k = k' \text{ and } G^D_k(A)(S^4_{k'}) = S^4_k, \text{ for } k > k'.$$

This example also shows what the connection of geometry with the quantum gravity regime might be like. However, the example is a kind of 'first-order linear approximation' of the graviton interaction with geometry with the observed scalar curvature: the graviton can change the scalar curvature of $\mathbb{R}^4$ (S^4) globally, leading to new constant scalar curvatures. As we will see shortly, working with the constant scalar curvatures can be considered to be a certain approximation to the full-fledged description of interacting gravitons. This is analogous to a tree approximation in the terminology of quantum field theory.

In order to understand a fully fledged description of gravitons in this setup, we have some tools which we have already dealt with. These are basically the appearance of exotic S^4s as in Theorem 4 and the discussion of exotic S^4s as gravitational instantons in Euclidean QG. Certainly, provided that such spheres exist at all. If they do not, we are left with the hybridized presentation of instantons as containing the diffeomorphic local patches to $\mathbb{R}^4$ and transition functions which cannot be fully translated into a smooth structure on a 4-manifold (like S^4) in V. Thus, in this case, an irreducible automorphism of B has to be present, which would correspond to a graviton. In this way, a graviton would be related to a gravitational instanton and a certain automorphism of B. However, there is yet another role assigned to gravitons in this setup. They represent the Fourier transform of the observables (self-adjoint operators) on $\mathcal{H}^{(\infty)}$ a separable, complex, infinite-dimensional Hilbert space, i.e., they represent the change in the quantum local context in $\mathcal{H}^{(\infty)}$. The Lemma 11 below explains this.

Let $[A, D] \neq 0$ where A, D are self-adjoint operators on $\mathcal{H}^{(\infty)}$, and let B_A, B_D are two measure algebras of projections with spectral resolutions of A and B correspondingly. Recall that we say then that $A \in B_A$ and $D \in B_D$. Since B_A, B_D are complete maximal Boolean algebras of projections, they comprise all commuting with A and all commuting with D self-adjoint operators (Lemma 3). Then, given a transform: $h_{AD} : B_A \to B_D$, we have the corresponding transform on the commuting algebras of the operators in B_A and B_D. Let us denote it $h_A(D)$.

Lemma 11. *Let $[A, D] \neq 0$ where A, D are self-adjoint operators on $\mathcal{H}^{(\infty)}$. There exists $h_{A'}(t, D)$ which for each $t \in \mathbb{R}_>$ defines the isomorphic transform of the algebra of operators commuting with A onto the algebra of operators commuting with D. The operator A' is a certain self-adjoint operator in $B_{A'}$.*

This lemma follows directly from the proof of Theorem 2, which is rephrased here. Let us turn to Lemma 3, which shows that any family of self-adjoint commuting operators $\{A^{(1)}_\alpha\}$ is such that $\exists_\alpha A^{(1)}_\alpha = A$ and that the family determines an algebra $B'_{(1)}$ comprising all projections that appear in the spectral families of all $A^{(1)}_\alpha$. Let B_1 be a complete maximal Boolean algebra of projections in $\mathbb{L}(\mathcal{H}^{(\infty)})$ that extends $B'_{(1)}$. Similarly, let $\{A^{(2)}_\beta\}$ be such that it contains D and determines $B'_{(2)}$ that extends to B_2 in $\mathbb{L}(\mathcal{H}^{(\infty)})$. Please note that B_2 is an isomorphic copy of the previous maximal complete Boolean algebra B_1 (see Lemma 6 and Corollary 5). Now, it follows that the self-adjoint operator spaces, determined by such isomorphic algebras of projections, are isomorphic. In fact, there exists a self-adjoint operator $A' \in B_{A'}$, such that

$$A(D') = e^{itA'} D' e^{-itA'}, D' \in B_D \text{ and } A' \in B_{A'} \text{ and } A(D') \in B_A.$$

This should be read that while D' spans the operators from B_D and for fixed $A' \in B_{A'}$, this spanning generates the operators $A(D')$ from B_A. The change in the parameter $t \in \mathbb{R}_{>0}$ gives rise to the automorphisms of the operator algebra of the operators from B_A. Thus, $A(D) = h_{A'}(t, D)$ is the family of the isomorphisms as in the statement of the lemma, which finishes the proof of Lemma 11.

Finally, we turn to yet another property of gravitons, namely to the construction which disturbs 4-metrics

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}$$

where $g^0_{\mu\nu}$ is the undisturbed metric and $h_{\mu\nu}$ the disturbance due to gravitons. This is quite important, at least in the semiclassical limit of the theory, since gravitons should correspond to the particles of spin 2. The approach developed here deals with the absorption of gravitons by the 4-spacetime region in the 4-disk D^4 (open but involved in the bounded domain), and if it is flat, it results in the curving of this disk. Eventually, subsequent absorption of gravitons curves the disk into extremely high values, which leads to its fragmentation. The fragmentation of the disk into (at least) two subdomains with the relative phase from the Aut B (see the previous sections and Appendix A). These fragments, together with the additional fragment of spacetime, constitute S^4. Thus, equivalently, we see this process as fragmentation of the 4-sphere such that the nontrivial phase corresponds to a graviton. If the graviton, i.e., $G_D(A, t)$, would be emitted by this system, this leads to the inverse process of gluing the local patches back into S^4. However, gluing cannot result in the standard smooth S^4 (see Theorem 4). If there exist exotic S^4s, they are perfect candidates for being the final stage of fragmentation and then re-gluing. Therefore, assume that exotic S^4s exist. Then, they correspond to the gravitational instantons in dimension 4 and describe the dominant part of the contribution in the semiclassical gravitational Euclidean path integral (see the discussion of (9)). Now, the initial open 4-disk, after re-gluing it into spacetime, has to be exotic R^4-this is the interior of the (still hypothetic) 4-disk embedded into spacetime.

Thus, the interaction of 4-spacetime with gravitons leads, in the smooth V limit (semiclassical approximation), to disturbing the smooth metric of spacetime by the metrics of nested 4-disks, provided that exotic S^4s exist.

Remark 19. *Again, we should emphasize that the existence of exotic 4-disks is equivalent to the existence of exotic 4-spheres, and the exotic R^4s included in such disks, or S^4s, are completely different (unknown so far) than any known exotic R^4s (see ref. [22] and Lemma 10).*

The important feature emerges (still provided that exotic S^4_es do exist). The disturbed metric cannot be eliminated by any local diffeomorphism connected to identity. The exotic S^4 is not diffeomorphic to the standard 4-sphere, but also the disturbed exotic $R^4 \subset S^4_e$ cannot become the standard $\mathbb{R}^4$ by any local diffeomorphism.

The interactions of gravitons with spacetime in the classical limit result in the change in the smoothness structure of the local 4-domain in spacetime (the interior of the exotic disk). This modified smoothness of spacetime cannot be removed by any local diffeomorphism of spacetime.

Our concept of a graviton is a multifaceted object. Its interaction with spacetime, and probably the matter content of it, remains the specific imprint in the local smoothness structure of spacetime. The fluctuation of metric due to this is the source of gravitational waves. As we commented above, the gluing process of the fragments of S^4 is due to the emission of graviton carrying the nontrivial automorphism of B.

Before we take a closer look into interactions with the presence of gravitons, let us discuss the basic property of exotic spheres in higher than 4 dimensions, which, however, is not known whether it holds for S^4 (even supposing the existence of exotic S^4). This is the crucial fact that given an exotic sphere, $S^n, n > 6$, there always exists the inverse exotic sphere, $\tilde{S}^n_e$ (in the same dimension) such that $S^n_e \# \tilde{S}^n \simeq S^n$ where the last n-sphere is the standard smooth sphere. This is necessary if we want to consider exotic spheres as gravitational instantons (cluster decomposition; see (9) and the discussion around it, and ref. [19]).

However, even though exotic S^4s may not exist, the construction via Boolean models of ZFC ensures that smooth V-limit is given by a hybridized collection of $R^4_{B_\alpha}$ and nontrivial automorphisms of V^B. Consider a situation where we have just one $f_{ij} \in \mathrm{Aut}\, B$. Since $\mathrm{Aut}\, B$ is a group, and there exists an inverse automorphism f^{-1}_{ij} that ensures that this smooth hybridized limit V has the property required for instantons. Whatever the family of automorphisms in the smooth V-limit is, there always exists the corresponding family of inverse automorphisms. If exotic S^4 exists, the smooth V limit of fragmented 4-sphere would be one of the exotic S^4. These two cases are addressed in Theorem 4 in the previous section. If it happens, then it is still possible that exotic S^4s do exist, but the instanton property would not hold since $\tilde{S}^4 = (S^4_e)^{-1}$ does not exist. Also, in this case, the hybridized smooth V-limit can stop just before reaching exotic smooth S^4, and such truncated prelimits always have inverse counterparts.

In the remaining part of this section, we are assuming that exotic S^4s exist and they have the instanton's inverse property. Let us analyze a way in which gravitons could interact with spacetime and instantons (which are exotic 4-spheres). So far, we were approaching gravitons in the classical limit as exotic S^4s (or hybridized versions of it), which come from smooth V-limit of fragmented S^4 and 4-spacetime. We think that a more accurate geometric representation for gravitons would be exotic disks (open) sitting in S^4_es: $D^4_e \simeq R^4 \subset S^4_e$ which reflects the fact that $D^4_e \cup \{pt.\} \simeq S^4_e$. Please note that none of the known exotic R^4 could be such (interior) of the exotic $\overline{D}^4$ (see Lemma 10 and Remark 19). Figure 1 shows how the interactions of gravitons and the local geometry of spacetime could be like in the presence of gravitational instantons. In Figure 1a, there is a simplified representation of spacetime manifold M^4 and the standard S^4, which are represented by the disconnected sum of both. However, their fragmentation is performed due to the super-high density of gravitational energy and curvature. The fragments do not just independently exist in some abstract space—they acquire the relative phases in $\mathrm{Aut}\, B$ such that after taking the smooth V-limit of S^4 becomes exotic S^4. Further gluing is the connected sum of this exotic S^4 and M^4. The seed for the exoticness of S^4 lies in the exotic 4-disk. In the figure, this is represented by the rectangle area in a) engulfing two standard disks. Since they have a relatively nontrivial phase, their smooth V-limit results in exotic 4-disk as being a part of exotic S^4, as in Figure 1b. Finally, in Figure 1c, there is the connected sum of the exotic S^4 (an instanton) and M^4. The role of the exotic disk is emphasized since this represents a graviton. We can briefly summarize by saying that the connected sum of spacetime and an instanton is by interchanging a graviton. This exotic connected sum is possible but can also not be formed. Possibly a kind of equilibrium should be there between instantons forming the connected sum and separated. If the instanton remains separated off spacetime, then from the point of view of a spacetime observer, the effects of instantons are not detectable, and the only their contribution would be to the physical gravitational Euclidean path integral (9). This corresponds to the left-hand side of Figure 1b). The exotic connected sum is in Figure 1c). Thus, taking the disconnected sum of M^4 and S^4 (the standard one) as $M^4 \coprod S^4$ we can represent the process in Figure 1 as

$$M^4 \coprod S^4 \xrightarrow{G} M^4_{eloc} \# S^4 \ \text{ or } \ M^4 \coprod S^4 + G \longrightarrow M^4_{eloc} \# S^4$$

where $+G$ means just the presence and the absorption of graviton and M^4_{eloc} is a spacetime manifold with locally embedded the interior of an exotic 4-disk. While the process terminated on the disconnected instanton, S_e leaves M^4 with a hole after removing the disk. The boundary of it is a non-traversable boundary of spacetime, i.e., a singularity.

The possibility of approaching a graviton G in our approach by metric fluctuations in local coordinates builds a link between G and spin-2 fields and thus partially justifies the appearance of such G in the quantum regime of gravity. Let $g^{S^4}_{\mu\nu}$ be one of the smooth standard metrics on S^4. Then, in the local coordinates, we can express the fluctuation of the metric due to the presence of a graviton since any graviton is represented by an alleged so far exotic disk D^4 such that its interior is an exotic R^4 (still alleged), which 1-point compactification would lead to exotic S^4 (if there exist D^4 or R^4 as above). Let $\tilde{g}^{S^4_e}_{\mu\nu}(x)$ be a smooth exotic metric on exotic S^4_e in the local coordinates on S^4_e. From the point of view of the standard smoothness on S^4, the metric $\tilde{g}^{S^4_e}_{\mu\nu}(x)$ is a continuous function on S^4 although in the exotic smooth structure on S^4_e it is a (exotic) smooth function. Assuming that a graviton G interacts as geometric objects with the standard S^4 say S^4 absorbs G, we can find the resulting geometry as exotic S^4_e. Thus, the direct coordinate representation of G as the fluctuation $H_{\mu\nu}$ of the metric would follow

$$\tilde{g}_{\mu\nu}(x)^{S^4_e}(x) = g^{S^4}_{\mu\nu} + H_{\mu\nu}(x) \tag{12}$$

where the fluctuation $H_{\mu\nu}$ is a continuous function on S^4. In this way, any matter or energy field can interact with gravitons via the background metric on spacetime.

Now, we can address the universality of graviton, i.e., its ability to interact with any energy or matter. The description follows the classical and quantum regimes. Let $\phi^{\overline{\mu}}$ be some physical field in spacetime. We do not precisely specify its spin or nature; it is just defined on a region of 4-spacetime. $\phi^{\overline{\mu}}$ couples to the metric $g_{\mu\nu}$ in the usual way, such as lowering or lifting the indices also in curved spacetime. Let us assume that the spacetime is Euclidean. Then the interaction of graviton $H_{\mu\nu}$ with $\phi^{\overline{\mu}}$ is by disturbing the base metric of spacetime according to (12) so thus after the interaction, the field $\tilde{\phi}^{\overline{\mu}}$ is written down in the modified exotic metric. More precisely, there exists a bounded 4-domain containing a 4-disk D^4 on which the field $\phi^{\overline{\mu}}(x)$ is defined, $x \in \mathbb{R}^4 \subset D^4$. After the interaction with graviton, the disk becomes smooth exotic D^4_e, and the field is written down on exotic $R^4 \subset D^4_e$.

This may seem like a model for a classical interaction of the graviton with a particle; however, the crucial factor is the exotic smoothness carried by the graviton and the resulting exotic smoothness of the interaction domain of spacetime. Such a change in the smoothness structure is forbidden within GR since it is like changing the spacetime manifold to another nondiffeomorphic with the first. To understand that it is a quantum process, let us turn to the model of gravitational instanton as the tunneling process. In this approach, the geometry of an instanton is represented by an exotic 4-sphere, which results from the interaction of the standard open 4-disk with the exotic one (the graviton, see Figure 1); thus, the resulting instanton corresponds to the tunneling process between the two drastically different 4-geometries, exotic and the standard one.

The field $H_{\mu\nu}(x)$ in (12) is continuous (nonsmooth) in the exotic smoothness structure or is continuous in the standard smoothness structure of the field. The exotic structure is localized on the disk, which can be taken of the Planck size, and such disks can be localized smoothly in the spacetime manifold. The resulting fluctuations of the metric that are outside of GR have quantum gravitational origins, which might also have cosmological implications. If exotic 4-spheres do not exist, the fluctuations cannot be seen as smooth processes and this would have a more abrupt action on spacetime structure. One might be tempted to try to experimentally distinguish between these two scenarios, one supporting the existence of exotic S^4s and the other denying their existence. However, then such verification would be an indication of whether (if yes) Nature is referring to exotic S^4s even before finding their mathematical proof. As we have already noted, the change in the smoothness on $\mathbb{R}^4$ or S^4 cannot be described within GR, and thus, the semiclassical gravity with instantons seems more appropriate. However, the semiclassical approach here is the smooth V limit of a deeper QM-based description (see Appendix A).

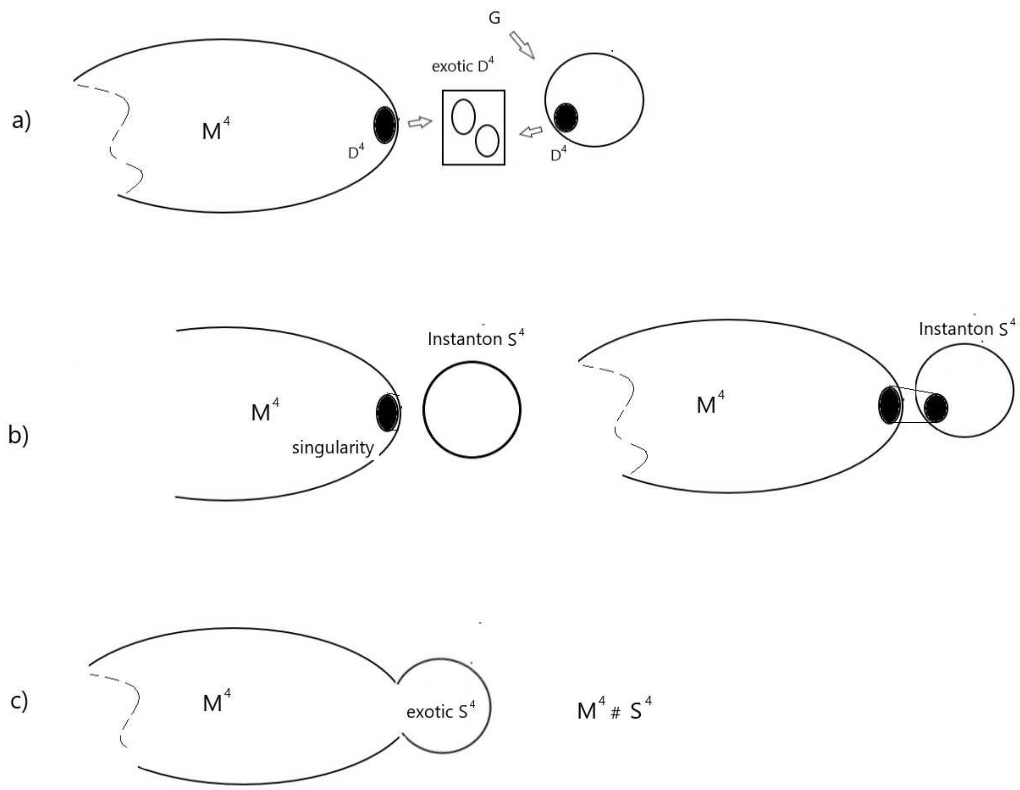

Figure 1. A schema for the interaction of a graviton with the local geometry of spacetime. (**a**) The interaction of a graviton (with suitable energy) with spacetime enables the separation of the standard fragment of S^4, which after absorption of the graviton (gluing the exotic open 4-disk) becomes exotic S^4—an instanton. Exotic 4-disk is present in the middle rectangular domain and represents a graviton. The minimal number, i.e., 2, of local patches of D^4s is schematically shown. The graviton (exotic open 4-disk) is glued into the remaining part of the standard S^4, which gains the exotic smoothness structure on S^4. (**b**) While the instanton S^4 emerges, the spacetime regions gains a non-traversable border—the singularity is formed. (**c**) The instanton is glued to spacetime, giving rise to the exotic 4-region, which corresponds to the quantum gravitational fluctuations.

5. The Negation of the Tsirelson's Conjecture and V^Bs in QM

In the first part of the paper, we have found a strong relation between QM and GR based on models of ZFC, especially Boolean-valued models V^B. However, the picture of retrieving the 2-valued level from V^B is far from complete. In the case of nonatomic Boolean algebras such as B (the measure algebra), the way to a 2-valued model from V^B goes through the nontrivial forcing procedure. This particular passage is represented by $f^{\downarrow}$ in Appendix A ((A1), (A2), (A5), (A6)). Forcing is a purely set-theory technique but also a tool for distinguishing finite- and infinite-dimensional Hilbert spaces. For finite-dimensional $\mathcal{H}$ the forcing would be trivial, i.e., nothing essentially new is added in the extended model $M[G]$ since $M[G] \simeq M$ where M is the base model (which can be V as well). This is so because the maximal Boolean algebras of the projections are atomic for $\dim(\mathcal{H}) < \infty$. In the approach to QM presented here on the ∞-dimensional $\mathcal{H}^{(\infty)}$, the model we started with is V and the maximal complete Boolean algebra of projections B is the atomless measure algebra, so thus the forcing $V[G] \simeq V^B/Ult$ is the random forcing which is nontrivial (for $\mathcal{H}^{(\infty)}$). This last means that certain sets, not present in V, are added to V as new ones (this is in the sense that these new sets extend the universe V with the Boolean value 1). In ref. [9], we studied the problem of whether these new sets might be physical in the context of QM for infinite-dimensional Hilbert spaces. We found there (see also refs. [24,25]) that these new sets could carry a new kind of quantum random behavior, and forcing procedure is strongly connected with the negation of Tsirelson's conjecture, ¬TC, the negation of which has been recently proved to be true in QM on $\mathcal{H}^{(\infty)}$. Currently, we know that ¬TC can hold only for infinite-dimensional Hilbert spaces: for any finite-dimensional spaces and their factorizations, TC is true, and ¬TC is false.

Remark 20. *Forcing in set theory is a procedure for proving the dependence or independence of certain sentences or formulas in the ZF language from the axioms of ZF or ZFC. The proofs are usually relative in the sense that given a model M of ZF(C) and a property ϕ (ϕ is a sentence in the first-order language of ZF) and finding the validity of ϕ in M, $M \vdash \phi$. Then, one builds the extended model $M[G]$ where the property does not hold, $M[G] \not\vdash \phi$. Since both M and $M[G]$ are models of ZFC (all theorems of ZFC are valid in both models), the property ϕ is not provable in ZFC but rather ϕ is logically independent on the axioms of ZFC (ϕ and $\neg\phi$ both are compatible with ZFC or ZF). This forcing procedure was invented by Paul Cohen in 1963 when he showed the independence of the continuum hypothesis on the ZFC axioms and the axiom of choice on ZF axioms. Cohen, in particular, builds a forcing extension $M[G]$ of a certain countable transitive model (CTM) of ZFC, M, such that $M \vdash CH$ and $M[G] \not\vdash CH$. Over the years, forcing techniques have been developed significantly and become a separate branch of mathematics with a huge number of applications in many mathematical disciplines.*

The application of forcing to physics, especially to QM, is based on genericity, which is complementary to logical independence. Let $M[G]$ be a certain forcing extension of a model M. Genericity can be seen as the real or physical process of the extension of the universes of sets by generic sets or ultrafilters G with the property that $G \notin M$ but $G \in M[G]$. Forcing represents a natural mechanism for the changes in ZFC models, the dynamics of which can be included in physical theories. Usually, there are very rich families of sets not in M which are 'added by forcing' and appear in the extended model $M[G]$ along with G (like a perfect tree with many reals added into $M[G]$ alone with a single Cohen real). However, there is an obvious difficulty with adding new sets to V: the universe V contains 'all' sets, so adding a set G to V which was not previously in V, seems pointless, as seems $V \subsetneq V[G]$. The problem has a solution that is based on Boolean-valued models V^B construed entirely in V, and the extensions of V are now possible with Boolean value 1. The appearance of V^Bs in V explains the gravitational fluctuations in spacetime in the present paper but also allows the building of random sequences of the QM outcomes, which show different behavior (correlations) for the product case $\mathcal{H}_a \times \mathcal{H}_b$ than for the nonproduct case $\mathcal{H}^{(\infty)}$. Thus, the correlations generated by the sequences falsify Tsirelson's conjecture [9].

Remark 21. *Tsirelson's problem was originally a conjecture (TC) that the sets of finitely many correlations of independent measurements of commuting observables on a Hilbert space $\mathcal{H}$ of a quantum system are always reproducible by the topologically complemented set of all finitely many independent measurements on the joint system with the product Hilbert space $\mathcal{H}_a \times \mathcal{H}_b$. This conjecture has been shown to be false [24], i.e., the product case does not reproduce all correlations which can appear on the entire, necessarily infinite-dimensional, Hilbert space $\mathcal{H}^{(\infty)}$. Tsirelson's conjecture is known to be equivalent to the Connes embedding problem for operator algebras, already stated in 1976 by Alain Connes [26,27].*

A. Sequences that falsify TC have been 'detected' in theory by V^B random forcing. Now, forcing emerges in the context of the quantum approach to gravity and spacetime (e.g., Equation (A6) presented here and Appendix A). We think that this is not just a pure coincidence but rather an important feature of the relationship between QM and GR. ¬TC shows that there is a vast QM domain, connected with infinite dimension of the Hilbert spaces $\mathcal{H}^\infty$, which presumably is not well-explored, both theoretically and, certainly, experimentally. ¬TC holds true necessarily only on infinite-dimensional Hilbert spaces, which is out of reach of the current experimental effort. The situation resembles this known from the attempt to detect directly the infinite dimension of $\mathcal{H}^{(\infty)}$ in finitely many correlations in any quantum experiment. Although theoretically, this has been shown to be possible, the experiments showing this are currently not known [28]. Therefore, the infinite dimensions of $\mathcal{H}$ opens a rich new realm for QM studies, but it has to await any experimental verification. Nevertheless, researchers have all the right to develop theoretical scenarios regarding that matter at least in order to understand its importance.
B. However, another important aspect of infinity in QM is the way how we construct infinite tensor products (ITP) of Hilbert spaces or operators. This is certainly a way to infinite-dimensional Hilbert spaces, and this belongs to 'classical', i.e., developed already by John von Neumann, the domain of QM [29]. Currently, we know many different ITPs that can also lead to separable $\mathcal{H}^{(\infty)}$ instead of nonseparable as construed by Von Neumann himself. Also, in proving ¬TC we have construed certain separable $\mathcal{H}^{(\infty)}$ by the use of Boolean models technique [9].
C. Connected to this is the characterization of the generic sequences of QM outcomes used to prove ¬TC from the point of view of quantum randomness. It appeared that the sequences carry enhanced randomness compared to the known from the theory of algorithmic randomness notions of $1, 2, \ldots, n, \ldots$- or ω-randomness. This new strong kind of randomness is hypothesized to be an upper limit for quantum computers generating random numbers, similarly as the true randomness becomes a limit for currently working classical computers, which generate quasi-random numbers and sequences.
D. When studying the relation of GR and QM, it is a natural approach to deal with increasing the number of degrees of freedom in QM or the number of particles such that in the limit, quantum field theory becomes closer. In the approach presented here and in the context of ¬TC, we should again readdress the procedure of attaining infinities from QM, and the indication could be the constructions of ITPs or $\mathcal{H}^{(\infty)}$ which were successfully applied in the TC problem. Therefore, the V^Bs technique and the relation to V would be an indication of how to proceed.

The points A–D above and the TC problem are not in the main line of issues of the paper, and we do not elaborate on them in more detail; however, these topics naturally correspond to the constructions here and show where the presented approach to QG might be situated. This also shows the relevance of future efforts in these directions for the QG program. Let us close this section by the following hypothesis (see ref. [8]).

> Experimental verification of hyper-randomness of the sequences negating the TC can be as difficult as experimental observing effects due to QG.

Still, it does not mean it is impossible at all. As is often the case in the history of science, breakthroughs and inventions can change the picture quite rapidly. Any progress in either of the two fields, QG or hyper-randomness in QM, would improve our understanding of the other. Anyway, we do not have a good understanding of QG, and we eagerly need new ideas, even highly theoretical ones. The way in which infinitely many dimensions or degrees of freedom are attained in (the extension of) QM is one such valuable indication, which we have learned from the negation of TC.

6. Discussion

GR and QM are extremely successful physical theories in the domains of their applicability. Any modification of each of them is thus highly constrained experimentally. On the other side, there is a vast unexplored region of 'infinite constructions' in QM and thus in GR. The notable examples are Tsirelson's conjecture, recently disproved, the infinite dimension of Hilbert spaces detected in a finite set of correlations [28] or the infinite tensor products introduced already by Von Neumann [9,29,30]. All these instances of infinity come from the heart of QM and, as we emphasized in this paper, show the way how to extend QM by the set theory suitable for the quantum regime and GR by elements that respect the equivalence principle in the extreme physical conditions where even spacetime description is inapplicable any longer. The modified smooth spacetime manifold refers to the exotic smoothness of S^4 (with the possibility not to be discovered at all, however) where such exotic S^4 would represent gravitational instantons transcending the classical regime of gravity. Thus, the paper shows a meeting point in between the extension of QM by certain infinite constructions and GR on a new kind of smooth 4-manifolds. The extension of QM refers to Turing's incomputable infinite sequences (the negation of the Tsirelson problem in QM), while the smooth manifolds (exotic 4-spheres) negate the 4-dimensional Poincarè conjecture. From the point of view of mathematics, this is not that surprising that the issue of infinity in QM is addressed by set theory since one of the crucial reasons behind the development of set theory was infinity, but this, which is truly surprising, is the existence of a natural extension of QM formalism by (quantum) set theory. In the opinion of the authors the formal possibility as above opens a narrow door toward the quantum regime of gravity.

Thus, in this work, we change the formal perspective from logical to quantum set-theoretical, which uncovers the room for the Solovay ZFC randomness. This was essentially performed in the previous publications, e.g., refs. [9,25] and referred to as the "ZFC twist", emphasizing the fact that the procedure is rather the change in the formal point of view than the proper extension of QM. It is based on the following correspondences [9]

$$\{\text{logical local contexts of QM}\} \longleftrightarrow \{\text{Boolean measure algebras, } B\}$$
$$\{\text{maximal algebras of commuting observables in QM}\} \longleftrightarrow \{\text{Boolean measure algebras}\}$$
$$\{\text{set-theoretic local contexts of QM}\} \longleftrightarrow \{\text{Boolean-valued models of ZFC, } V^B\}$$
$$\text{global set theory of QM} \longleftrightarrow \text{ universe of quantum sets } V^{\mathbb{L}} \text{ (non-Boolean, non-Heyting)}$$

where this last correspondence follows from the fact that $B \subset \mathbb{L}$ leads to $V^B \subset V$, which is a ZFC submodel of the quantum set-theory universe $V^{\mathbb{L}}$. The second correspondence says that for any family of commuting self-adjoint observables there always exists a maximal Boolean algebra B of projections determining all the operators in the family, e.g., ref. [4].

Thus, the set universe $V^{\mathbb{L}}$ is approximated by the family $\{V^B\}$ of ZFC universes valued by Boolean as local contexts. The ZFC randomness of infinite sequences of QM outcomes is, in fact, formally present in the Born probability of a single measurement (see ref. [31]). However, when the ZFC twist is performed and the dimension of $\mathcal{H}$ is infinite, a hyper-random behavior emerges of certain QM infinite sequences, which are 1. non-reducible to the n-Turing incomputable infinite sequences, $n \in \mathbb{N}$, and 2. they can distinguish TC from $\neg$TC [9].

We show that the ZFC twist of QM enables the reconciliation of QM with the extension of GR outside the limits of physical spacetime. Classical GR is deeply rooted in differential geometry and topology, but the mathematics of GR can be applied to an arbitrary dimension n. Suppose, as is the returning theme in this work, that the negation of the smooth Poincaré conjecture in dimension 4 holds true and exotic 4-spheres exist, then the extended GR distinguishes exotic 4-spheres as special geometries compatible with the semiclassical regime of Euclidean gravity. In particular, exotic 4-spheres carry the geometry of gravitational instantons. Thus, the negation of the 4-dimensional smooth Poincaré conjecture and the negation of the Tsirelson conjecture meet here in the physical dimension 4, shedding new light on the relation of GR and QM.

There exists an approach in physics according to which exotic smoothness structures on $\mathbb{R}^4$ or $S^3 \times \mathbb{R}$ are responsible for various effects in cosmology, QM or quantum field theory and particle physics (e.g., [32–36]) which has been elevated within the years into the quite powerful and independent methods of physical explanation. However, such exotic smoothness structures on $\mathbb{R}^4$ are completely different from those discussed in this paper in the context of exotic S^4. First, these are dubious at the moment, whereas the previous ones are already mathematical facts. Second, none of these former can appear as supporting any exotic 4-sphere which was explained in the course of the paper in more detail.

One more 'subtlety' of the formalism relates the step from Boolean many-valued contexts to the 2-valued standard ones, which would correspond to $V^B \to V$. However, for $\mathcal{H}^{(\infty)}$ the resulting model is the extended 2-valued model $V[G] \supsetneq V$. This has been referred to frequently in this paper and corresponds to the nontrivial forcing extension of the models (see the previous section for a more detailed presentation and ref. [9]). The problem of genericity in QM touches the fundamental questions in set theory like the multiverse vs. universe foundations for sets (this set-theory multiverse has nothing to do, but the same name, with the multiverses considered in cosmology). If random QM sequences distinguishing $\neg$TC from TC become experimentally verified realities, this, according to the set-theory approach, would indicate that the multiverse approach prevails, or nature has chosen the one at least for QM phenomena.

There are also some experimental challenges with respect to the methods in this paper. One is the experimental discriminating TC and $\neg$TC. Another, connected with the first, is to propose and perform experiments that support the ZFC genericity (e.g., ref. [9]). However, even the experimental verification of the ∞ dimension of Hilbert spaces, such as those generated in the Fock space formalism or quantum harmonic oscillators, which can be seen in the finite set of quantum correlations, is out of reach at present [28]. These topics are also related to quantum computation techniques and building advanced quantum random number generators certified by, e.g., 'Solovay genericity'. One big theoretical challenge is to understand infinite constructions present from the very beginning in quantum field theories from a QM point of view in a proper way. One major unsolved mathematical problem that awaits its resolution and is important for the analysis here is certainly the smooth Poincaré conjecture and the existence of exotic S^4s. The case that exotic S^4s do not exist is also addressed, but then the theory becomes not as canonical as with the presence of the exotic 4-spheres. We could say that the internal consistency of the approach here favors the negation of the smooth 4-dimensional Poincaré conjecture.

The Lorentzian vs Euclidean signatures of the spacetime metrics should be commented on. In the text, we have quite freely switched between the two non-equivalent cases. In particular, exotic S^4 and exotic $\mathbb{R}^4$ require Euclidean underlying metrics while physical spacetime is the Lorentzian one. However, these are not totally independent or conflicting. We consider the approach that the smoothness structure on a manifold is fundamental data allowing for further analysis. Even though a 4-manifold representing spacetime is Lorentzian, a topological manifold underlies it, and this can be considered to be exotic smooth (e.g., $\mathbb{R}^4$ or S^4), and this deep layer of smoothness can sometimes have a physical impact (e.g., gravitational instantons). A quite fundamental and important case of the global existence of exotic metrics on open 4-manifolds that solve the Einstein equations has been studied, among others, by Etesi, who also gave many physical insights [36] (see also ref. [33]). Even though the exoticness we refer to in this paper is dubious R^4 supporting exotic S^4, one can expect that they can also be related to the Lorentzian physical case. If they do not exist, the methods in the paper could serve as the departure for the experimental discrimination of the two cases.

Anyway, the tremendous effort toward building QG theory and the inability so far to achieve this goal become a good reason for exploring the QM and GR formalisms from yet unexplored points of view. The collision of the infinite constructions in QM and these from QFT may be nontrivial and not necessarily agree on the classes of their Turing uncomputability. The road from QM to QFT contains more intricacies than usually can be thought.

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Appendix A. Smooth V-Limit

In this section, we will explain the smooth V-limit, which has been used in many arguments in the text. Given R^4_B in the Boolean ZFC model V^B, we would like $\mathbb{R}^4$ with its standard smoothness structure in the smooth V limit of R^4_B. However, also conversely, any local flat $U_\alpha \simeq \mathbb{R}^4$ in spacetime is derivable as a smooth V-limit of certain $R^4_{B_\alpha}$. The correspondence is complementary in the density of gravitational energy, or in the local scalar curvature parameter κ in spacetime, in the sense that there exist $\kappa_0, \tilde{\kappa}_0 \in [0, \infty)$ such that $U_\alpha \simeq \mathbb{R}^4$ for $\kappa < \kappa_0$ and $U_\alpha \simeq R^4_{B_\alpha}$ for $\kappa \geq \tilde{\kappa}_0$. It can happen that $\tilde{\kappa}_0 < \kappa_0$ so that there is a nontrivial interval $(\tilde{\kappa}_0, \kappa_0)$ where both descriptions are still valid and complementary. For $\tilde{\kappa}_0 > \kappa_0$ there is a gap $(\kappa_0, \tilde{\kappa}_0)$ with yet another description not specified explicitly here, and the case $\tilde{\kappa}_0 = \kappa_0$ represents the usual understanding of the hyper-Planck or below-Planck regimes.

In general, it is straightforward to require that the smooth V limit of $R^4_B \in V^B$ be (isomorphic copy of) $\mathbb{R}^4$ in V. However, allowing for different models V^{B_α} and the corresponding objects R^4_α there is a problem of their intersections like $R^4_\alpha \cap R^4_\beta$. The overlap domain for the parameter κ, $(\tilde{\kappa}_0, \kappa_0)$ allows switching between $\mathbb{R}^4$ and R^4_B and thus between the descriptions based on V and based on V^B. This possibility is fundamental in set theory, though often neglected by working globally in V or V^B or simply in ZFC. Therefore, the point is to work locally with different universes of sets and to understand the eventual impact of their interchanges from the V point of view. Of particular interest would be

$$V \hookrightarrow V^B \hookrightarrow V$$

and its impact on the smooth manifold model in V. On the overlapping $U_\alpha \cap U_\beta \simeq U_{\alpha\beta} \simeq \mathbb{R}^4 \cap \mathbb{R}^4$ in V one can lift it to R^4_B

$$\mathbb{R}^4 \to R^4_B \to \mathbb{R}^4. \tag{A1}$$

It would seem that after neglecting the intermediate V^B stage, one is left with the $\mathbb{R}^4 \to \mathbb{R}^4$, which can be represented by a diffeomorphism in V or even identity. However, we want to emphasize that when there is a 'nontrivial' action of the V^B stage, the resulting diffeomorphism is not an identity on the overlapping $\mathbb{R}^4$s in V.

Following Theorem 3 and the discussion below, we represent the highly curved 4-spacetime regions $\simeq \mathbb{R}^4$ as (part of) 4-spheres with relation to $\mathrm{Aut}\, B$. We want to interpret (subgroups $\mathcal{A}$ of) $\mathrm{Aut}\, B$ in $\mathrm{Diff}\, \mathbb{R}^4$ through $p : \mathrm{Aut}\, B \to \mathrm{Diff}\, \mathbb{R}^4$ such that:

i. $\mathrm{Aut}\, B \ni Id_B \leftrightarrow Id \in \mathrm{Diff}\, \mathbb{R}^4$, i.e., $p(Id_B) = Id$ and $p^{-1}(Id) = Id_B$.
ii. $p : \mathcal{A} \to \mathrm{Diff}\, \mathbb{R}^4$ is a homeomorphism.

So, p is an isomorphism of $\mathcal{A}$ on $p(\mathcal{A}) \subset \mathrm{Diff}\, \mathbb{R}^4$. Given $h \in \mathcal{A}$ corresponds to the diffeomorphism $p(h) \in \mathrm{Diff}\, \mathbb{R}^4$ and conversely, given $f \in \mathrm{Diff}\, \mathbb{R}^4$ there are the corresponding $p^{-1}(f) \in \mathcal{A}$. Any change in the maximal Boolean algebra of projections B, hB, by a nontrivial automorphism h results in a nontrivial diffeomorphism $f \in \mathrm{Diff}\, \mathbb{R}^4$. So thus, we interpret the nontrivial stage of (A1) as

$$\mathbb{R}^4 \xrightarrow{f^\uparrow} R^4 \in V^B \xrightarrow{h} R^4 \in V^{AB} \xrightarrow{f^\downarrow} \mathbb{R}^4 \tag{A2}$$

which gives rise to the diagram

$$\begin{array}{ccccc} & & R^4 \in V^B & \longrightarrow & R^4 \in V^{AB} \\ & {}^{f^\uparrow}\nearrow & \big\downarrow{\scriptstyle f^\downarrow} & & \big\downarrow{\scriptstyle f^\downarrow} \\ \mathbb{R}^4 & \xrightarrow{Id} & \mathbb{R}^4 & \xrightarrow{\quad\Phi\quad} & \mathbb{R}^4 \end{array} \tag{A3}$$

where $f^\uparrow$ is the interpretation of $\mathbb{R}^4$ as (the isomorphic copy of) R^4_B in V^B and $f^\downarrow$ the inverse assignment of the (isomorphic copy of) $\mathbb{R}^4$ to $R^4_{h_A(t,D)B}$ in $V^{h_A(t,D)B}$. $f^\uparrow$ is the lift of the ZFC models $V \to V^B$ and $f^\downarrow$ the reduction of V^B to V resulting in nontrivial set-theory random forcing.

When we apply the assignments $f^\downarrow$ to each member of a collection $\{R^4_{B_\alpha}, \alpha \in I\}$, then the result is the collection $\{\mathbb{R}^4_\alpha\}$ in V together with a net of diffeomorphisms (gluing) $\{f_{\alpha\beta} : \mathbb{R}^4_\alpha \to \mathbb{R}^4_\beta\}$.

> If there exists a smooth 4-manifold, M^4, such that $(\{\mathbb{R}^4_\alpha\}, \{f_{\alpha\beta}\})$ is a smooth cover of it, then we say that M^4 is the smooth V-limit of $(\{R^4_{B_\alpha}\}, \{f_A : R^4_{B_\alpha} \to R^4_{B_\beta}\}, \alpha, \beta \in I, A : V^{B_\alpha} \to V^{B_\beta})$.
>
> In the special case $(\mathbb{R}^4, Id \in Diff\mathbb{R}^4)$, it is the smooth V-limit of any $(R^4_{B_\alpha}, Id \in \mathrm{Aut}\, B)$. They are isomorphic copies of $\mathbb{R}^4$ for different αs.

One can also start with a smooth manifold M^4 in V and then interpret its local $U_\alpha \simeq \mathbb{R}^4$ in V^{B_α} and consider relative phases, resulting from the automorphisms of B, between the models V^{B_α}. Finally, the phases give an interpretation as diffeomorphisms in V, and one can ask the question about the existence of a smooth manifold $\tilde{M}^4$ in V that supports such locally modified collection of $\mathbb{R}^4_\alpha$ (initially it being a smooth cover of M^4). If such a smooth $\tilde{M}^4$ exists in V it is again called the smooth V-limit of the modified collection of $R^4_{B_\alpha}$ in the family of models V^{B_α} (see ref. [25] for more details).

Now let $\mathcal{A} = \{h_A(t, D) : t \in \mathbb{R}^4_>\}$ be a 1-parameter family of automorphisms in Aut B given by (see the proof of Theorem 2)

$$e^{itA}De^{-itA}, t \in \mathbb{R}_{>0} \tag{A4}$$

where D is a real in V^B and A and $e^{itA}De^{-itA}$ are reals in $V^{h_A(t,D)B}$ (the reals in V^B corresponds uniquely to the commuting self-adjoint operators in B—see Lemma 4). Again, any change in the maximal Boolean algebra of projections B by a nontrivial automorphism $h_A(t, D)B$ results in a nontrivial diffeomorphism $f \in \mathrm{Diff}\,\mathbb{R}^4$. The nontrivial stage in (A1) reads

$$\mathbb{R}^4 \xrightarrow{f^{\uparrow}} R^4 \in V^B \xrightarrow{h_A(t,D)} R^4 \in V^{h_A(t,D)B} \xrightarrow{f^{\downarrow}} \mathbb{R}^4 \tag{A5}$$

and the corresponding diagram follows

$$\begin{array}{ccccc} & & R^4 \in V^B & \xrightarrow{h_A(t,D)} & R^4 \in V^{h_A(t,D)B} \\ & \nearrow^{f^{\uparrow}} & \downarrow f^{\downarrow} & & \downarrow f^{\downarrow} \\ \mathbb{R}^4 & \xrightarrow{Id} & \mathbb{R}^4 & \xrightarrow{\Phi} & \mathbb{R}^4 \end{array} \tag{A6}$$

Please note that if $t = 0$, then the stage V^B acts trivially, and Φ is the identity diffeomorphism.

The above example is different from the initial general case, as it leads to a change in 4-manifolds S^4_t rather than the change in coordinate local patches in a fixed manifold.

This correspondence can be paired with the 4-curvature of 4-spheres. Let $\{S^4_r, r \in \mathbb{R}_{>0}\}$ be a family of standard S^4s with the radii r. Certainly $S^4_{r_1} \simeq S^4_{r_2}$ are diffeomorphic for any $r_1, r_2 > 0$ and $f_{r_1 r_2} : S^4_{r_1} \setminus \overline{D^4_{r_1}} \to S^4_{r_2} \setminus \overline{D^4_{r_2}}$ are the induced diffeomorphisms of $\mathbb{R}^4$, i.e., $f_{r_1 r_2} \in \mathrm{Diff}\,\mathbb{R}^4$.

Lemma A1. *Any automorphism of B which is not identity results in a non-vanishing change in the scalar curvature of* $\mathbb{R}^4 \subset S^4$.

The proof of this lemma relies on the interpretation of the parameter t in (A4) as the scalar curvature of S^4 and the construction of the proper families. Let $h_A(t, D) = Id + a(t, D)$ and $a(t, D) = 0$ for $t = 0$ so that $h_A(0, D) = Id$ (on the algebra B). Please note that $e^{i0A}De^{-i0A} = Id(D) = D$. Let $t = r \in \mathbb{R}_>$ result in the family $\{S^4_t\}$. However, the constant scalar curvature of S^4_t is $Sc_{S^4_t} = 12/t^2$. This completes the proof of Lemma A1 since $\mathbb{R}^4 \subset S^4_t$ has the same scalar curvature depending on t.

Appendix B. Proof of Theorem 1

The embeddings $V \xrightarrow{1.} V' \xrightarrow{2.} V$ come from the construction of the Boolean-valued models V^B as classes in V for case 2, while for 1., from the natural isomorphic embedding of V into V^B. These two steps naturally lead to $V \subset_1 V^B \subset_2 V$ as classes, which has been referred to frequently in the main text. Thus, the proof is to show that $V' = V^B$ has the desired properties of their embeddings as in Theorem 1. First, starting with an arbitrary Boolean model, V^B shows that there is a canonical embedding $1. : V \to V^B$ [37,38]. The image $\check{V} = 1.(V)$ in V^B is construed inductively as the class of check names $\check{x}$ of sets $x \in V$, i.e., to any $x \in V$ it is assigned recursively $\check{x} := \{\langle \check{y}, 1\rangle | y \in x\}$

and $\check{V}$ becomes a class name $\check{V} := \{\check{x} | x \in V\}$ that extends the symbols of the language by the predicate $\check{V}$ for the ground model. The Boolean value in B of $\tau \in \check{V}$ is given by $[\![\tau \in \check{V}]\!] = \bigvee_{x \in V} [\![\tau = \check{x}]\!]$ (τ is a B name). Then for any set $x \in V$ we have $[\![\check{x} \in \check{V}]\!] = 1$ and $[\![\check{V}$ is a transitive class in V^B containing all ordinals $]\!] = 1$ (ref. [37] (Lemma 6)). It follows that $\check{V}$ is a 2-valued Boolean model isomorphic to V [37] and $1.(V) = \check{V}$ corresponds to embedding 1.

To show the existence of an embedding 2. follows by the construction of V^B in V as a class for any Boolean-valued model V^B. Let B be a complete Boolean algebra in V. Then V^B is construed by recursion in V as a class of B-names, i.e., τ is a B-name if $\tau = \langle \sigma, b \rangle$, where σ is a B-name and $b \in B$. The initial data for the recursion are the Boolean values of the atomic formulas built by the double recursion [37]:

$$[\![\sigma_1 \in \sigma_2]\!] = \bigvee_{\langle \kappa, b \rangle \in \sigma_2} [\![\sigma_1 = \kappa]\!] \wedge b$$

$$[\![\sigma_1 = \sigma_2]\!] = [\![\sigma_1 \subseteq \sigma_2]\!] \wedge [\![\sigma_2 \subseteq \sigma_1]\!]$$

$$[\![\sigma_1 \subset \sigma_2]\!] = \bigwedge_{\kappa \in \text{ dom } (\sigma_1)} ([\![\kappa \in \sigma_1]\!] \to [\![\kappa \in \sigma_2]\!])$$

here, certainly, $\to$, $\wedge$, $\vee$ are operations in the Boolean algebra lattice. The above recursions are performed entirely in V, leading to the class V^B of names in V and the Boolean values assigned to all assertions of set theory. Thus, given the class $V^B \in V$, it is a model of set theory. One shows that V^B is a full model, and for any complete Boolean algebra, B, the Boolean values of any axiom of ZFC have value 1. This finishes the proof of the statement that V^B is the class-like model of ZFC construed in V and 2. reflects the construction above for the specific Boolean measure algebra B.

It remains to show that there are nontrivial automorphisms $j : V^B \to V^B$. However, this was precisely explained during the construction of the proof of Theorem 2. This observation completes the proof of Theorem 1.

Appendix C. The Lévy Hierarchy

The formal language $\mathcal{L}_{ZF}$ of Zermello-Frankel set theory is the one sorted (sets), a first-order language with a single predicate symbol 'being an element of', $\in$.

A formula ϕ in $\mathcal{L}_{ZF}$ is called complex of the class (level):

Σ_{i+1}, if ϕ is equivalent to $\exists x_1 ... \exists x_n B$ in ZFC, where B is Π_i

Π_{i+1}, if ϕ is equivalent to $\forall x_1 ... \forall x_n B$ in ZFC, where B is Σ_i

Δ_i, if ϕ is *provably* both, Σ_i and Π_i.

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