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Nonlinear interactions in gravitational and electromagnetic waves

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Abstract

Plane waves are a recurring concept in various branches of physics. Although they are, to some extent, idealizations, they provide the groundwork for describing a multitude of radiative processes. The great strength of this concept lies in the structural simplicity of plane waves, a simplicity that allows us to analytically tackle various problems and derive an intuitive and computable framework that would otherwise be obscured by mathematical complexity. Here, we will leverage these characteristics to study the nonlinear behavior of particles moving in strong electromagnetic fields and nonlinear gravitational waves. We will show how the large number of symmetries possessed by plane waves enables the exact calculation of the interaction between a moving charge and the field it generates, an effect known as radiation-reaction. We will also investigate the gravitational emission of moving particles in an electromagnetic wave and demonstrate that the resulting amplitude can be proved to be in a simple proportionality relation with its electromagnetic counterpart, both from the classical and quantum perspectives. Additionally, we will explore the dynamics in plane wave spacetimes that generalize the usual treatment in the weak-field approximation, showing how this can be connected to known results for the motion in an electromagnetic wave within flat spacetime. Finally, we will study the process of photon emission by an electron in the aforementioned curved spacetime.

Zusammenfassung

Ebene Wellen sind ein wiederkehrendes Konzept in verschiedenen Zweigen der Physik. Obwohl sie bis zu einem gewissen Grad Idealvorstellungen sind, bieten sie die Grundlage für die Beschreibung einer Vielzahl von Strahlungsprozessen. Die große Stärke dieses Konzepts liegt in der strukturellen Einfachheit der ebenen Wellen, einer Einfachheit, die es ermöglicht, verschiedene Probleme analytisch zu behandeln und einen intuitiven und berechenbaren Rahmen abzuleiten, der sonst durch mathematische Komplexität verschleiert wäre. Hier werden wir diese Eigenschaften nutzen, um das nichtlineare Verhalten von Partikeln in starken elektromagnetischen Feldern und nichtlinearen Gravitationswellen zu untersuchen. Wir werden zeigen, wie die große Anzahl von Symmetrien, die ebenen Wellen besitzen, die exakte Berechnung der Wechselwirkung zwischen einer sich bewegenden Ladung und dem Feld, das sie erzeugt, ermöglicht – ein Effekt, der als "Radiation-Reaction" bekannt ist. Desweiteren werden wir beweisen, dass die resultierende Amplitude, sowohl aus klassischer als auch aus quantenmechanischer Perspektive, in einer einfachen Proportionalitätsbeziehung zu ihrem elektromagentischen Gegenstück steht. Zusätzlich werden wir die Dynamik in ebenen Wellen-Raumzeiten erforschen, die die übliche Behandlung im Schwachfeldansatz verallgemeinern, und zeigen, wie dies mit bekannten Ergebnissen für die Bewegung in einer elektromagnetischen Welle innerhalb der flachen Raumzeit verbunden werden kann. Schließlich werden wir den Prozess der Photonenausstrahlung durch ein Elektron in der zuvor erwähnten gekrümmten Raumzeit untersuchen.

List of publications

[1] ***"Analytical spectrum of nonlinear Thomson scattering including radiation reaction"***,

Antonino Di Piazza and Giulio Audagnotto,
Phys.Rev.D, **104** (2021),
doi: 10.1103/PhysRevD.104.016007

[2] ***"Proportionality of gravitational and electromagnetic radiation by an electron in an intense plane wave"***,

Giulio Audagnotto, Christoph H. Keitel and Antonino Di Piazza,
Phys.Rev.D, **106** (2022),
doi: 10.1103/PhysRevD.106.076009

[3] ***"Dynamics, quantum states and Compton scattering in nonlinear gravitational waves"***,

Giulio Audagnotto and Antonino Di Piazza,
JHEP **06** (2024),
doi: 10.1007/JHEP06(2024)023

In preparation:

[4] ***"Radiation reaction effects in Redmond field configurations"***,

Giulio Audagnotto and Antonino Di Piazza.

Contents

Preface

This thesis is divided into six Chapters, which are intended to be read in pairs. The first two Chapters cover the topics of classical motion and radiation by a charge moving in an electromagnetic plane wave, including the self-interaction of the charge with itself. In Chapter 1 we discuss the classical dynamics in such a background and we study the radiation-reaction effects in this context. In Chapter 2 we present the analytical expression of the radiated energy spectrum from a charge driven by a strong plane wave [62].

The second pair of Chapters is devoted to the study of the interactions between QED in intense electromagnetic fields and linearized gravity. In Chapter 3 we develop the fundamental framework necessary to explore the topic. In Chapter 4 we show how the electromagnetic and gravitational emissions in strong electromagnetic waves can be related by a simple proportionality, both at the classical level and the quantum leading order [20].

Finally, the last two Chapters are devoted to the study of exact (or nonlinear) gravitational waves in general relativity. In Chapter 5 we discuss how to define gravitational waves as solutions of Einstein's equations and investigate the dynamics in such spacetimes. In Chapter 6 we examine the construction of quantum states in these curved spacetimes and we provide the squared S-matrix element for the photon emission in a nonlinear gravitational wave background [21].

At the end of Chapters 2, 4 and 6 the reader will find a Support Material section, which can be useful to elucidate small details or calculations appearing in the main text. These sections are not essential to understand this work and they can be skipped without altering the comprehension of the central content.

Due to the structure of this thesis, we believe it is more sensible to discuss the results at the end of every pair of Chapters. The reader will thus find discussion sections at the end of Chs. 2, 4 and 6.

Notation

The signature of the metric chosen in this work is -2 and we assume $\hbar = c = \varepsilon_0 = 1$, $\kappa = \sqrt{32\pi G}$. Referring to the Misner-Thorne-Wheeler systematization [131] within general relativity, we adopt the convention $[-,-,-]$. While the first $-$ refers to the metric signature, the other two correspond to the conventions

$$
\begin{aligned}
&R^{\nu}{}_{\mu\rho\alpha} = \partial_\alpha \Gamma^{\nu}_{\mu\rho} - \partial_\rho \Gamma^{\nu}_{\mu\alpha} + \Gamma^{\beta}_{\mu\rho}\Gamma^{\nu}_{\alpha\beta} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\nu}_{\rho\beta},\\
&R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}.
\end{aligned}
\tag{1}
$$

Plane waves propagating along $\boldsymbol{z}$ depend exclusively on the variable $t - z$. For this reason we will employ very often the light-cone coordinates $\{x^+, x^i, x^-\}$ defined as $x^- = t - z \equiv \phi$, $x^+ = \frac{1}{2}(t+z)$, $x^i = \{x, y\}$ (see App. A.1 for further details). Latin indices will be used throughout this paper referring to the two transverse coordinates $\{x, y\}$. Whenever a vector (or a matrix) has a transverse index, this can be substituted with a four-dimensional index in order to lighten the notation. For example, given v^i we will assume the convention

$$
v^\alpha = \delta^\alpha_i v^i. \tag{2}
$$

Certainly, v^i will be properly defined before such a convention is applied, in order to avoid confusion. From Chapter 5 on, letters belonging to the first half of the Greek alphabet α, β, ... will refer to the flat spacetime metric (unless otherwise stated), whereas μ, ν, ... will be used as curved spacetime indices.

Recurring symbols and conventions	
Misner-Thorne-Wheeler convention	$[-,-,-]$
metric convention	$(+,-,-,-)$
natural units (can be restored when needed)	$\epsilon_0 = c = \hbar = 1$
fine structure constant	$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{1}{137}$
gravitational coupling	$\kappa = \sqrt{32\pi G}$
electron mass	m
Minkowski scalar product	$v \cdot w = v^\alpha w^\beta \eta_{\alpha\beta}$
three-dimensional scalar product	$\boldsymbol{v} \cdot \boldsymbol{w} = v^i w^j \delta_{ij}$
plane wave wavevector	$k^\alpha = \omega n^\alpha = \omega(1,0,0,1)$
emitted radiation wavevector	$l^\alpha = \omega' n'^\alpha$
light-cone basis	$\{n^\alpha, \eta^{i\alpha}, \tilde{n}^\alpha\}$
light-cone coordinates	$\{x^-, x^i, x^+\}$
plane wave phase	$\varphi = k \cdot x$
dimensional phase	$\phi = n \cdot x = x^-$
plane wave structure tensor	$f_i^{\alpha\beta} = n^\alpha \delta_i^\beta - n^\beta \delta_i^\alpha$
classical background field	A^α
quantum field	$\mathcal{A}^\alpha$
classical nonlinearity parameter	ξ_0
quantum nonlinearity parameter	$\chi_0 = \eta_0 \xi_0$
radiation reaction parameter	$\tau_e = \frac{2\alpha}{3m}$
vierbein, inverse vierbein	$e^\alpha{}_\mu,\ e_\alpha{}^\mu$
flat-; curved-spacetime fields	$\mathit{\Phi}, \mathit{\Psi};\ \Phi, \Psi$
Dirac matrices	$\bar{\gamma}^\alpha$
positive-, negative-energy Volkov states	$\mathcal{U}_p, \mathcal{V}_p$
symmetrization	$T_{(\mu_1..\mu_n)} = \frac{1}{n!}\sum_{\sigma\in S_n} T_{\mu_{\sigma(1)}\cdots\mu_{\sigma(n)}}$
antisymmetrization	$T_{[\mu_1..\mu_n]} = \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) T_{\mu_{\sigma(1)}\cdots\mu_{\sigma(n)}}$
Chirped Pulse Amplification	CPA
Radiation Reaction	RR
Lorentz-Abraham-Dirac equation	LAD
Landau-Lifshitz equation	LL
Classical Radiation Dominated Regime	CRDR
Strong-Field QED	SFQED
Ordinary Differential Equation	ODE

Introduction

"Love, despite what they tell you,
does not conquer all,
nor does it even usually last.
In the end the romantic aspirations
of our youth are reduced to,
whatever works."

Boris Yellnikoff

*L*ight *a*mplification by *s*timulated *e*mission of *r*adiation: laser. This is the instrument which enabled the research in strong electromagnetic fields to develop and advance. It thus warrants a brief exploration of its history. The full name is by itself quite descriptive, or at least it contains the fundamental characteristics of a laser, which is indeed an amplifier of electromagnetic radiation. The very first ancestor of this instrument was designed to amplify microwaves and was therefore historically referred to as a "maser". However, scientists quickly recognized that utilizing optical light would be significantly more advantageous. The utility of a laser is not limited to the amplification, indeed with it we can produce extremely directional and coherent light. By coherence we mean that the emitted waves keep a constant phase difference, a property due to the process underlying the amplification process known as "stimulated emission". As often happens discussing the physics of the twentieth century, the seminal idea about this mechanism belongs to Einstein, who proposed it in a paper dated 1917 about "The quantum theory of radiation" [77]. The idea is simple: a photon can induce an atomic transition from one quantum state E_1 to another with lower energy E_0, resulting in two emitted photons (see the last picture in Fig. 1).

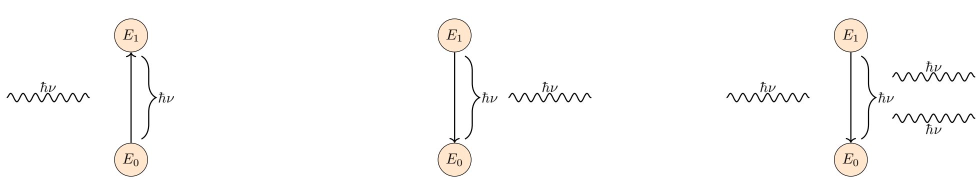


Figure 1: From left to right: photon absorption, spontaneous emission and stimulated emission.

If a photon goes through a medium, it can surely be absorbed bringing an atom or molecule from the lower to the higher energy state $E_0 \to E_1$. This process populates E_1 while the stimulated and spontaneous emissions do the opposite. Summing the contributions we can describe the evolution of the system as

$$\frac{dN_1}{dt} = \rho(\nu)(B_{0\to1}N_0 - B_{1\to0}N_1) - AN_1, \tag{3}$$

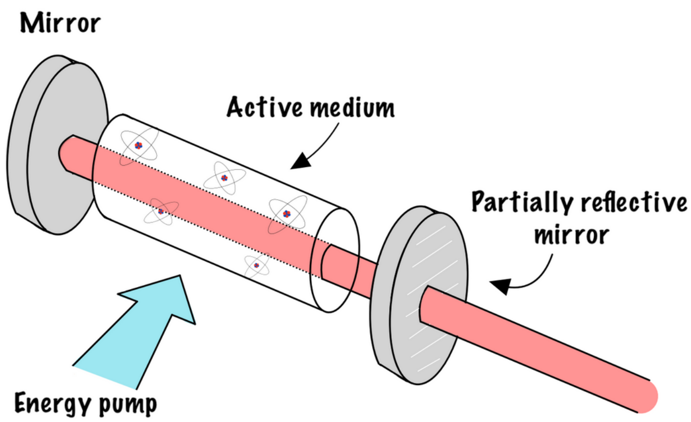


Figure 2: Schematic representation of a laser.

where $\rho(\nu)$ is the photon distribution at the frequency ν and $B_{0\to1}, B_{1\to0}, A$ are transition coefficients. If we consider only the photon-stimulated emission we can set aside the spontaneous relaxation and consider the B terms only. Einstein recognized [77, 164] that the coefficients should not depend on the direction of the transition: $B_{0\to1}$ has to be equal to $B_{1\to0}$. Including this observation, the high energy state population evolves as

$$\frac{dN_1}{dt} = \rho(\nu)B(N_0 - N_1). \tag{4}$$

This means that in order to have a great number of stimulated emissions we need $N_1 \gg N_0$, which is not the case in a spontaneous equilibrium where $N_1/N_0 = \exp\left(-\frac{\Delta E}{k_B T}\right)$. We thus need to pump energy into the system and induce a population inversion. The resulting machinery is an instrument which exploits an external source of energy to amplify the original input signal. It is worth highlighting that unlike spontaneous emission, the stimulated process generates radiation with the same direction and characteristics of the incoming one. This is what allows a coherent amplification, enabling to reach high intensities and tightly focused beams. If we put two mirrors at the extremities of the laser medium we can build a resonator, where now the light gets amplified multiple times before exiting the device as a beam (see Fig. 2)

The active medium can be chosen among a large variety of materials, the most powerful optical lasers today work with titanium-sapphire crystals (Ti:Sa) [133]. On 16 May 1960 Theodore Maiman switched on the first ever working laser in Malibu, California. An interesting fact: the *Physical Review Letters* editors at the time turned down Maiman report, probably tired of receiving too many maser papers. This accident did not stop the work to be published elsewhere, as it did not stop the public opinion speculations. After the announcement to the news media, the first newspaper article was titled "Death rays possibilities probed by scientists". Newspapers have consistently found science fiction more engaging than physics. Despite not being weapons belonging to "*The war of the worlds*", lasers gave a unique boost to the theoretical investigation of interactions between particles and strong electromagnetic fields. Among the pioneers who first worked on the topic in the

60s it is worth mentioning Howard Reiss [154], Ritus and Nikishov [139], Brown and Kibble [49] and Eberly [75]. A large number of people contributed to the development of these seminal works and we refer to [66, 81] for a comprehensive report. It should be mentioned that the solution of the Dirac equation in an electromagnetic plane wave, an ubiquitous element in the literature about these topics, was obtained by Volkov in 1935 [173], thirty years before it could be properly exploited. In research, patience is indispensable. In the same years in which laser research was born, Peter Higgs was working on what he first thought being "completely useless", a keystone of the Sandard Model, experimentally proved only 50 years later.

The research on strong-field interactions grew over the following 20 years but soon laser technology encountered an obstacle. Indeed, at intensities of the order of GW/cm^2 the light starts damaging the laser active medium during the amplification process, making impossible to reach higher intensities. In 1985 the two researcher at Rochester university Donna Strickland and Gerard Mourou introduced a procedure to solve this problem: the "Chirped Pulse Amplification" (CPA) [168]. The idea is simple, in order to further amplify a laser pulse avoiding the aforementioned problem we can stretch it before the amplification, drastically diminishing its intensity, and then recompress it afterwords. This technology today allows to generate pulses with the incredible intensities of 10^{23}W/cm^2 [1, 2]. Nowadays, thanks to numerous good people, very strong lasers are available. In the following we will describe in detail what can be explored with them. Before proceeding let us mention that modern technology also allows to produce ultra-short laser pulses in the attosecond (10^{-18}s) regime. This allows to investigate the very florid research branch known as "ultra-fast physics", which despite being very fascinating will not be treated in this work. Here we will focus on the physics at very high intensities. Thus, what happens when a charged particle interacts with a very intense coherent pulse? The physics emerging in these conditions can be divided into three macro-categories: *classical nonlinearity*, *quantum nonlinearity* and *radiation reaction*. In the following chapters we will discuss these topics extensively, let us thus introduce them concisely here.

Firstly, classical nonlinearity. In this context the adjective "nonlinear" refers to the equations of motion, namely the Lorentz equation for a charged particle. If we consider an electron this reads [115]

$$\frac{d\boldsymbol{u}}{dt} = \frac{e}{m}(\boldsymbol{E} + \gamma^{-1}\boldsymbol{u} \times \boldsymbol{B}), \tag{5}$$

where $\boldsymbol{u} = \gamma \boldsymbol{v}$ is the relativistic velocity of the particle, m is the electron mass and $e < 0$. The presence of $\boldsymbol{u}$ in the r.h.s. makes the differential equation nonlinear and this is directly related to the magnitude of the magnetic term in the force. For a plane wave $|\boldsymbol{E}| = |\boldsymbol{B}|$ in natural units and this equation can be easily solved analytically [122], showing that the nonlinearity in this case is reflected in the nonlinear dependence of the momentum on the wave field (see Sec. 1.1.1).

The same is true for nonlinear quantum effects. Let us consider for example a scattering process taking place in a electromagnetic field background A^α. If the latter is sufficiently intense we cannot treat it perturbatively, instead we have to solve the dynamics exactly in A^α, using the deriving "dressed" Feynman rules to calculate the process [66]. This is another way to say that we have to sum all the possible contributions of the background field to the fermion lines. This procedure is explained in Sec. 3.1.

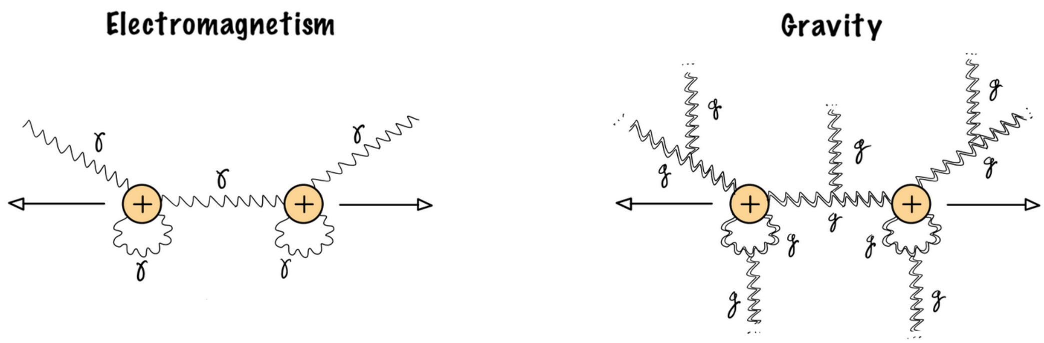


Figure 3: Electromagnetic and gravitational interactions between two repelling particles.

Finally, the radiation-reaction (RR). Since the advent of a proper theory of electrodynamics in the early 1900s, discussions on RR effects have been ongoing. The point is very clear: the Lorentz equation for a charged particle cannot be complete because it does not take into account the self-interaction of the particle on itself mediated by the field it generates [122]. Requiring momentum conservation leads to a modified equation of motion known as LAD equation, from Lorentz, Abraham, and Dirac (see Sec. 1.2.2). This equation is not Newtonian as it involves derivatives of the acceleration. As such, it allows analytical solutions which cannot be considered physical like "runaway" motions, in which the electron acquires a diverging acceleration even in absence of external fields [66]. These problems can be avoided employing a an approximation procedure introduced by Landau and Lifshitz [122] as we will discuss in Sec. 1.2.3 . Despite being such an old problem, RR is still debated due to the measurement difficulties it entails. In fact, in most of the situations the RR forces are extremely small compared to the Lorentz force.

Let us now shift our focus and discuss a topic that might seem unrelated to what we have said to this point: gravity. Despite its undeniable beauty and the simplicity of its founding principles, Einstein theory of gravity is practically a complicated machinery. Indeed, it is an intrinsically nonlinear theory. The reason for this is very simple: everything is a source for the gravitational field. At least everything with mass or energy. It is evident that this feature drastically complicates things, although nature tends to be nonlinear much more often than we would like to admit. Let us think about the aforementioned RR effects: also in this case a charge becomes source of the electromagnetic field itself, which then gets involved in the equations of motion. This generates the recursive self-interaction. Now, the major difference between gravity and electrodynamics in this sense does not lie in the matter but in the fields. Let me clarify. Suppose we have a system made of two positive charges. If we let them free to move they will surely repel each other and accelerate in opposite directions (Fig. 3) Due to their motion they will generate non-Coulombian electromagnetic and gravitational fields. A part of the electromagnetic field will be radiated and another part will interact with both of the charges. The same is true for the gravitational field but in

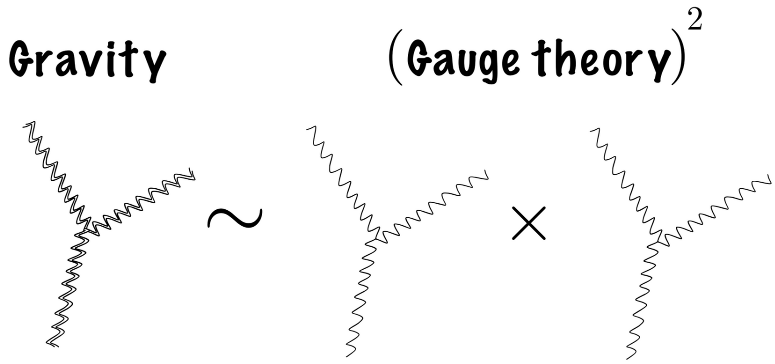


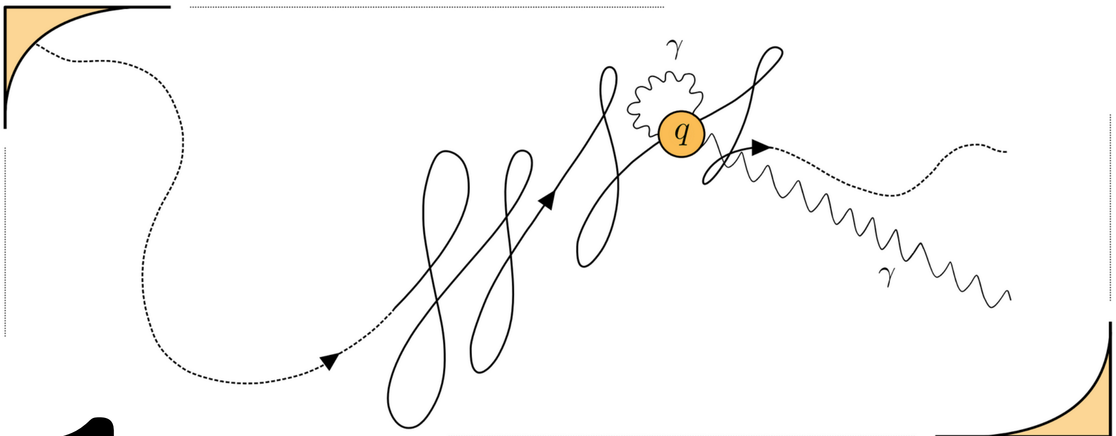
Figure 4: In some respects, gravity can be seen as a double copy of a gauge theory.

this case it is not the end of the story, both the generated fields will turn into gravitational sources themselves!. In the language of quantum field theory we would say that at leading order photons only couple to charges but never to themselves, while gravitons couple to everything. This of course corresponds to the fact that Quantum Electrodynamics (QED) is an abelian gauge theory and gravity is not.

The microscopic world is ruled by two other fundamental forces and none of them is an abelian gauge theory. Indeed, both weak and strong interactions admit multiple-bosons vertices. Let us take into account Quantum Chromodynamics (QCD) for example. In this theory the gauge group is SU(3), which permits the existence of three- and four-gluons vertices. This feature makes QCD, in some respects, closer to gravity. Clearly there are many differences between the two theories, but when we treat them perturbatively we notice that at the semiclassical level (three-level) the connections between them are more than what we expected. These relations have been widely investigated by a numerous group of people and today constitutes a branch of research which goes under the name of "double copy theory"[31, 30]. Here "double copy" refers to the fact that amplitudes involving gravitons can be written as proper products of amplitudes in gauge theories (Fig. 4). This sort of relations find their roots in a string theory paper dated 1985 [116] and have been shown to be very valuable in the last thirty years. The practical utility of their use stems from the empirical observation that calculating amplitudes involving gravitons is significantly more complex than calculating amplitudes involving gluons. Thus, if we can derive the former from the latter, we gain a clear advantage.

In some specific cases, similar relations can be found for processes involving only gravity and electrodynamics [110, 36, 37, 13, 27, 12], an example is the proportionality between Compton scattering and gaviton photoproduction at tree level [56, 111]. These will be the subject of the second part of this work (Chs. 3, 4). In particular we will show how these can be extended including the electromagnetic nonlinearity in the system through background plane waves [20]. Indeed, even though electromagnetic and gravitational interactions are profoundly different, the high number of symmetries belonging to plane waves allows to find connections between the two theories which are not limited to the first order calculations.

In the last part of this work we will study the physics emerging in nonlinear gravitational waves (Chs. 5, 6). We will find out that even in this context it is possible to find interesting connections and similarities with the electromagnetic case, both in the classical and quantum regimes. We will show, for example, how to obtain the fermion quantum states in a gravitational wave from a direct analogy with the solutions that Volkov found ninety years ago for strong electromagnetic fields [173]. After the first direct observation by LIGO-VIRGO [3] in 2015, gravitational waves have attracted a lot of attention. However, due to the weakness of perturbations reaching the Earth, the work done on extending the theoretical analysis to nonlinear gravitational waves has been relatively moderate. Here we will argue that if we dear to do that, forgetting about the experimental complications for a moment, we can actually discover some intriguing physics and explore QFT in a highly symmetrical curved spacetime.



1

Classical dynamics

1.1 Dynamics neglecting the particle self-interaction

1.1.1 Motion in a plane wave and nonlinearity parameters

Before we dive deep into strong-field electromagnetism it is useful to get a sense of the physical quantities that control the processes in this context. Let us consider the relativistic Lorentz equation for an electron

$$\frac{d\gamma \boldsymbol{v}}{dt} = \frac{e}{m}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}), \tag{1.1}$$

where m is the electron mass and $e < 0$. As we anticipated in the introduction, if the magnetic interaction is not negligible then this equation becomes *nonlinear*, in the sense that the unknown variable appears in the interaction as well. In general we cannot find a specific condition regarding the motion such that $|\boldsymbol{v} \times \boldsymbol{B}| \sim |\boldsymbol{E}|$ without fixing the relation between the electric and magnetic fields. However, if we choose a plane wave as a background this relation is always defined to be $\boldsymbol{B} = \boldsymbol{n} \times \boldsymbol{E}$, where $\boldsymbol{n}$ is the wave propagation direction.

This means that in this case the nonlinearity condition will be satisfied when the particle becomes highly relativistic $v \lesssim 1$ [66]. Now, the motion in a plane wave is periodic, so we need the particle to become relativistic in a period. Thus, we need the energy transferred to the particle in a period to be of the same order of its rest energy $eEc/\omega \sim mc^2$. These considerations suggest the introduction of the classical nonlinearity parameter ξ as

$$\xi = \frac{|e|\,E}{m\omega c}, \tag{1.2}$$

such that nonlinear effects emerge for $\xi \gtrsim 1$. For a laser this can be expressed in terms of the peak field E_0 and we will denote it by $\xi_0 = \frac{eE_0}{m\omega c}$. In order to have a clearer point of view about this parameter in a plane wave background we can directly solve the equations of motion, a surprisingly easy task for such a field. The Lorentz equation for the momentum $\pi^\alpha = mu^\alpha$ in covariant form reads

$$\frac{d\pi^\alpha}{d\tau} = \frac{e}{m} F^{\alpha\beta}\pi_\beta. \tag{1.3}$$

Plane waves propagating along $\boldsymbol{z}$ depend on the variable $t-z$ only. Moreover the corresponding field $A^\alpha(\phi)$ has only components in the directions transverse to $\boldsymbol{z}$, if we choose a proper gauge [66]. Indeed we can choose to work in the Lorenz gauge $\partial_\mu A^\mu(\phi) = n \cdot \dot{A}(\phi) = 0$, where $k^\alpha = \omega n^\alpha$ is the background wave vector, with the additional condition $A^0(\phi) = 0$. Here and below the overdot stands for a derivative with respect to ϕ. By assuming that $\lim_{\phi\to\pm\infty} \boldsymbol{A}(\phi) = \boldsymbol{0}$, then the Lorenz-gauge condition implies $\boldsymbol{n} \cdot \boldsymbol{A}(\phi) = 0$, such that the field has indeed only transverse components. For these reasons we will choose to work with light-cone coordinates $\{x^+, x^i, x^-\}$ defined as (see App. A.1)

$$\text{Light-cone coordinates} \quad \Rightarrow \quad \begin{cases} x^- = n \cdot x = t - z \equiv \phi \\ x^+ = \tilde{n} \cdot x = \frac{1}{2}(t+z) \\ x^i = \{x, y\} \end{cases} . \tag{1.4}$$

The Latin indices will be used throughout this work referring to the two transverse coordinates. Here we can also introduce the light-cone basis $\{n^\mu, \delta^\mu_i, \tilde{n}^\mu\}$ with $n^\mu = (1,0,0,1)$ being proportional to the wave vector and $\tilde{n}^\mu = \frac{1}{2}(1,0,0,-1)$, such that any vector v^μ can be decomposed as

$$v^\mu = v^- \tilde{n}^\mu + v^+ n^\mu + v^i \delta^\mu_i \tag{1.5}$$

with $v^- = n \cdot v$, and $v^+ = \tilde{n} \cdot v$. We can thus utilize ϕ as an evolution parameter in place of the proper time and rewrite the Lorentz equation as

$$\dot{\pi}^\alpha_p(\phi) = \frac{e}{p^-} F^{\alpha\beta}(\phi)\pi_{p,\beta}(\phi), \tag{1.6}$$

where with the subscript p we indicates the momentum initial condition $p^\alpha = mu^\alpha_0$. The last step to solve this equation is to observe that the plane wave Maxwell tensor

$$F^{\alpha\beta}(\phi) = \dot{A}^i(\phi)(n^\alpha \delta^\beta_i - n^\beta \delta^\alpha_i) \equiv \dot{A}^i(\phi) f^{\alpha\beta}_i \tag{1.7}$$

commutes with itself at different phases, due to the structure of the tensor $f_i^{\alpha\beta}$. This means that Eq. (1.6) can be directly solved by exponentiation, the time ordering being needless

$$\pi_p^\alpha(\phi) = \exp\left(\frac{e}{p^-}\int^\phi d\tilde{\phi} F(\tilde{\phi})\right)^{\alpha\beta} p_\beta. \tag{1.8}$$

We can extract the dimensional parameters from the Maxwell tensor rewriting $F^{\alpha\beta}(\phi) = \frac{E_0}{\omega}\dot{\psi}^i(\phi) f_i^{\alpha\beta}$, where now the functions ψ^i are dimensionless. If we now name $\xi_\perp^i(\phi) = \xi_0\psi^i(\phi)$ we can write the solution in a compact form, where the dependence on the nonlinearity parameter is manifest and clear

$$\begin{aligned}\pi_p^\alpha(\phi) &= \exp\left(\frac{m}{p^-}\xi_\perp^i(\phi) f_i\right)^{\alpha\beta} p_\beta = \\ &= p^\alpha - m\xi_\perp^\alpha(\phi) + \frac{m}{p^-}\left[p\cdot\xi_\perp(\phi) - \frac{m}{2}\xi_\perp(\phi)\cdot\xi_\perp(\phi)\right] n^\alpha.\end{aligned} \tag{1.9}$$

The exponential expansion stops at the second order due to the property $f_i{}^\alpha{}_\beta f_j{}^\beta{}_\gamma f_k{}^\gamma{}_\delta = 0$. Here we can fully appreciate the fact that the quadratic term in the background field is proportional to ξ_0^2, as anticipated in the introduction. This parameter is then a proper indicator of the motion nonlinearity.

There are other ways to solve the motion in a plane wave, besides the direct exponentiation. Surely the most straightforward one is to observe that the three coordinates x^i, x^+ are cyclic in the system Lagrangian. From this follow three conserved quantities $\pi^i + m\xi^i = p^i, \pi^- = p^-$. Adding to these the on-shell condition, the motion is completely solved. While we may not reach six or seven explanations as Feynman recommended, having various ways to convey the same point is still worthwhile. In order to visualize the motion we can pick a monochromatic wave for simplicity and describe the trajectory from the so-called average rest frame (see App. A.3). In this frame there is no drift velocity along the wave propagation direction and the motion in the $\{x, z\}$ and $\{y, z\}$ planes traces a characteristic figure-eight pattern [159] (Fig. 1.1).

It is useful to have a numerical expression for ξ_0 involving only the laser features [66], recalling that the peak intensity is defined as $I_0 = c\epsilon_0 E_0^2$ we have

$$\xi_0 = 7.5\sqrt{I_0[10^{20}\mathrm{W/cm^2}]}/\omega[\mathrm{eV}] = 6\sqrt{I_0[10^{20}\mathrm{W/cm^2}]}\lambda[\mu\mathrm{m}]. \tag{1.10}$$

To have an idea of the numbers involved it is enough to remember that super-strong lasers usually work at optical frequencies $\lambda \sim \mu\mathrm{m}\,(\omega \sim \mathrm{eV})$, such that for these sorts of facilities $\xi_0 \sim \sqrt{I_0[10^{18}\mathrm{W/cm^2}]}$. Modern PW-lasers can reach peak intensities of the order of $I_0 \sim 10^{23}\mathrm{W/cm^2}$ at the focus, thus the parameter space for which $\xi_0 \gtrsim 1$ is already available [1, 5].

The reader could object that Eq. (1.2) is not a manifestly Lorentz- and gauge-invariant definition and be skeptical about its utility. In order to clarify this issue let us notice that such a definition exists and reads [107]

$$\langle\xi\rangle = \frac{|e|}{m}\sqrt{\frac{-\langle(F^{\alpha\beta}p_\beta)^2\rangle}{(k\cdot p)^2}}, \tag{1.11}$$

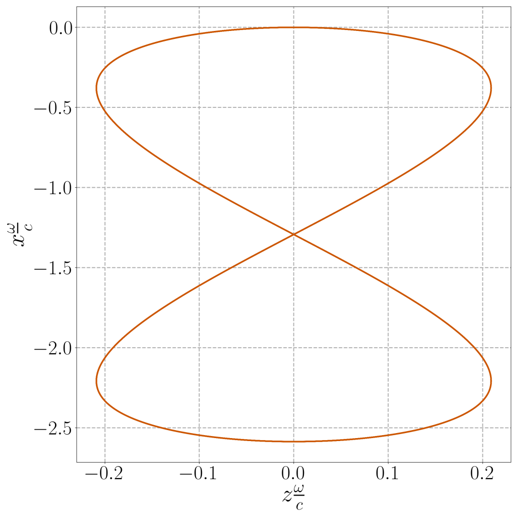


Figure 1.1: Trajectory in the $\{z, x\}$-plane of an electron in a plane wave propagating along $\boldsymbol{z}$ with $\xi_0 = 3.19$ as seen in the average rest frame.

where p is the initial particle momentum, k^α is the laser wave vector and $\langle ... \rangle$ indicates the average over a phase period. The averaged quantity in this expression is proportional to $T^{\alpha\beta}p_\alpha p_\beta$, where $T^{\alpha\beta}$ is the energy-momentum tensor of the electromagnetic wave. This is clearly Lorentz- and gauge-invariant and when we take its square root and divide it by m it represents the energy density seen by the charge in its rest frame. The other invariant is the denominator $k \cdot p = \omega p^-$. The dynamical factor p^- represents the laser frequency Doppler shift in the electron reference frame. In fact, if the particle is propagating (or counter-propagating) along the wave direction we have that

$$p^- = mc\sqrt{\frac{1-\beta}{1+\beta}}. \tag{1.12}$$

Thus $\omega p^-/mc$ is the laser frequency seen by the moving electron. If the latter is propagating along the wave direction then $p^-/mc < 1$ and we have a redshift. Clearly a blueshift occurs in the counter-propagating case.

If there is any wavelength λ characterizing an electromagnetic field we could say that the classical nonlinearity parameter is the work done by the field on a charge in a field wavelength λ, ratio the particle rest energy. Again, this is because the motion becomes nonlinear if the energy gained by the charge in a characteristic length of the background field is of the same order of its mass. Now, if the work done by the field in the particle rest frame on a Compton wavelength λ_C is of the order of its rest energy, then *quantum nonlinearity effects* emerge, indeed this energy is enough to produce an e^-e^+ pair. The correspondent quantum nonlinearity parameter regulating the emergence of such effects can be written in a covariant form as [66]

$$\chi = \frac{e\hbar\sqrt{-(F^{\alpha\beta}p_\beta)^2}}{m^3c^4}. \tag{1.13}$$

For a laser field $\chi_0 = \eta_0\xi_0$, with $\eta_0 = \hbar\frac{k \cdot p}{m^2c^2}$. The parameter η_0 makes the quantum nonlinearity dependent on the particle initial motion. Indeed, recalling that $k \cdot p$ represents the laser frequency Doppler shifted in the electron reference frame, we see that we can enhance χ_0 shooting the electron in a head-on collision with the wave, blueshifting the laser frequency. Another way to write χ_0 involves the Schwinger critical field $F_{cr} = m^2c^3/e\hbar = 1.3 \times 10^{16}\text{V/cm} = 4.4 \times 10^{13}\text{G}$

$$\chi_0 = \frac{p^-}{mc}\frac{F_0}{F_{cr}}, \tag{1.14}$$

where F_0 is the plane wave peak electric (or magnetic) field. It is important to underline that in order to observe quantum nonlinear effects we do not only need very strong fields, but also very fast electrons. Of course if we could reach the critical field it would be enough to produce pairs out of vacuum, but this is far beyond the actual technical possibilities. Also in this case, it is worth having an operative numerical expression for χ_0, depending only on the setup specifics. For a fast electron initially counterpropagating with respect to the plane wave this reads [66]

$$\chi_0 = 5.9 \times 10^{-2}\varepsilon_0[\text{GeV}]\sqrt{I_0[10^{20}\text{W/cm}^2]}, \tag{1.15}$$

where here ε_0 is the electron initial energy.

1.1.2 Motion in a Redmond field

In this section we will briefly study the dynamics of a particle in a field composed of a plane wave and a constant magnetic field aligned along the wave propagation direction. This configuration is known as Redmond field [153] in the literature. The interesting aspect of this field is that it allows a particle moving under its influence to enter in a resonant regime.. This happens when the synchrotron frequency and the wave frequency in the electron frame are the same, such that the particle receives energy from both in a constructive way.

The Lorentz equation in this configuration reads

$$\dot{\pi}^\alpha(\phi) = \frac{e}{p^-}\left(F^{\alpha\beta}(\phi) + B^{\alpha\beta}\right)\pi_\beta(\phi), \tag{1.16}$$

where $B^{\alpha\beta}$ is the Maxwell tensor associated to the constant magnetic field aligned to $\hat{\boldsymbol{z}}$ and with magnitude H. We can generally write this tensor as [153]

$$B_{\alpha\beta} = iH(\varepsilon_\alpha\varepsilon^*_\beta - \varepsilon^*_\alpha\varepsilon_\beta), \tag{1.17}$$

with $\varepsilon^\alpha = 1/\sqrt{2}(0,1,i,0)$ such that $\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}^* = -1$ and $\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} = 0$. One way to solve the equation is to observe that the components $\varepsilon\cdot\pi$ and $\varepsilon^*\cdot\pi$ decouple [153]. However, here we will solve it by exponentiation as we did in the previous section for the single plane wave, indeed this procedure abbreviates the calculations and can be more suitable for further generalizations. The magnetic field tensor is constant, thus we can collect its dependence and write

$$\pi^\alpha_p(\phi) = \exp\left(\frac{e}{p^-}B\phi\right)^{\alpha\beta}\Pi_{p,\beta}, \tag{1.18}$$

such that the Lorentz equation can be written in terms of Π^α_p as

$$\dot{\Pi}^\alpha_p(\phi) = \frac{e}{p^-}\exp\left(-\frac{e}{p^-}B\phi\right)^{\alpha\beta}F_{\beta\gamma}\exp\left(\frac{e}{p^-}B\phi\right)^{\gamma\delta}\Pi_{p,\delta} \equiv \frac{e}{p^-}\tilde{F}^{\alpha\beta}\Pi_{p,\beta}. \tag{1.19}$$

Now, the important thing to observe is that the tensor $\tilde{F}_{\alpha\beta}$ is structurally similar to the original plane wave tensor, indeed it satisfies $(\tilde{F}^2)^{\alpha\beta} \propto n^\alpha n^\beta$. This means that the equation can now be directly solved by exponentiation as in Sec. 1.1.1

$$\pi^\alpha_p(\phi) = \exp\left(\frac{e}{p^-}B\phi\right)^{\alpha\beta} \exp\left(\frac{e}{p^-}\int^\phi d\tilde{\phi}\tilde{F}(\tilde{\phi})\right)_\beta{}^\gamma p_\gamma. \tag{1.20}$$

It is useful to introduce the following notation

$$\begin{aligned} \mathscr{A}^i &= \int^\phi d\tilde{\phi}\left(\dot{A}^1\cos\Omega\tilde{\phi} + \dot{A}^2\sin\Omega\tilde{\phi}, \dot{A}^2\cos\Omega\tilde{\phi} - \dot{A}^1\sin\Omega\tilde{\phi}\right) \\ &= \int^\phi d\tilde{\phi} R^i{}_j(-\Omega\tilde{\phi})\dot{A}^j(\tilde{\phi}), \end{aligned} \tag{1.21}$$

where $\Omega = \frac{eH}{p^-}$ is the Doppler-shifter cyclotron frequency and $R^i{}_j(\varphi)$ is the counterclockwise rotation matrix

$$R^i{}_j(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}. \tag{1.22}$$

The solution of Eq. (1.20) can now be expanded as

$$\pi^\alpha_p = p^+ n^\alpha + p^-\tilde{n}^\alpha + R^\alpha{}_j(\Omega\phi)[p^j - e\mathscr{A}^j(\phi)] + \frac{e}{p^-}\left(p\cdot\mathscr{A} - \frac{e}{2}\mathscr{A}\cdot\mathscr{A}\right)n^\alpha. \tag{1.23}$$

If we turn off the magnetic field then $\mathscr{A}^\alpha \overset{H\to 0}{\to} A^\alpha$. It is thus straightforward to find the two limits of pure plane wave and magnetic field

$$\pi^\alpha_p \overset{H=0}{=} p^\alpha - eA^\alpha + \frac{e}{p^-}\left(p\cdot A - \frac{e}{2}A^2\right)n^\alpha \quad , \quad \pi^\alpha_p \overset{A=0}{=} p^+ n^\alpha + p^-\tilde{n}^\alpha + R^\alpha{}_j(\Omega\phi)p^j, \tag{1.24}$$

in agreement with Eq. (1.9). Let us look at the field $\mathscr{A}^\alpha$, which is the key of the dynamics in this configuration. We immediately see that if there are components in the plane wave oscillating with a frequency equal or close to Ω, then $\mathscr{A}^\alpha$ will grow with ϕ indefinitely rather than oscillating (see Fig. 1.2). This resonance condition can be explicitly written as

$$\frac{\Omega}{\omega} = \sqrt{\frac{1-\boldsymbol{v}_0^2}{(1-v_0^z)^2}}\xi_0\frac{H}{H_0} \sim 1, \tag{1.25}$$

where ω is the plane wave frequency and H_0 its peak magnetic field. If the electron is initially counterpropagating with respect to the wave, as in most of the experimental setups involving strong lasers, then the velocity-dependent prefactor is ≤ 1. The maximum pulsed magnetic field available is of the order of $H \sim 10^6$G and it lasts few ms. Thus, in this configuration is technically unlikely to obtain such a resonance with optical strong lasers because the required magnetic field would exceed the technical possibilities. In contrast, if the electron is propagating along the wave direction, then for ultrarelativistic initial velocities $u_0^- \sim \gamma^{-1}$ and the resonance condition is in principle achievable. However, in this case other experimental complications emerge, indeed it is very difficult to keep the electron in the laser focus for a sufficiently long time.

This being said, the study of the Redmond configuration for lower frequency fields and different pulse shapes can unveil interesting insights. In Sec. 1.2.6 we will briefly discuss the emergence of radiation-reaction phenomena in this configuration.

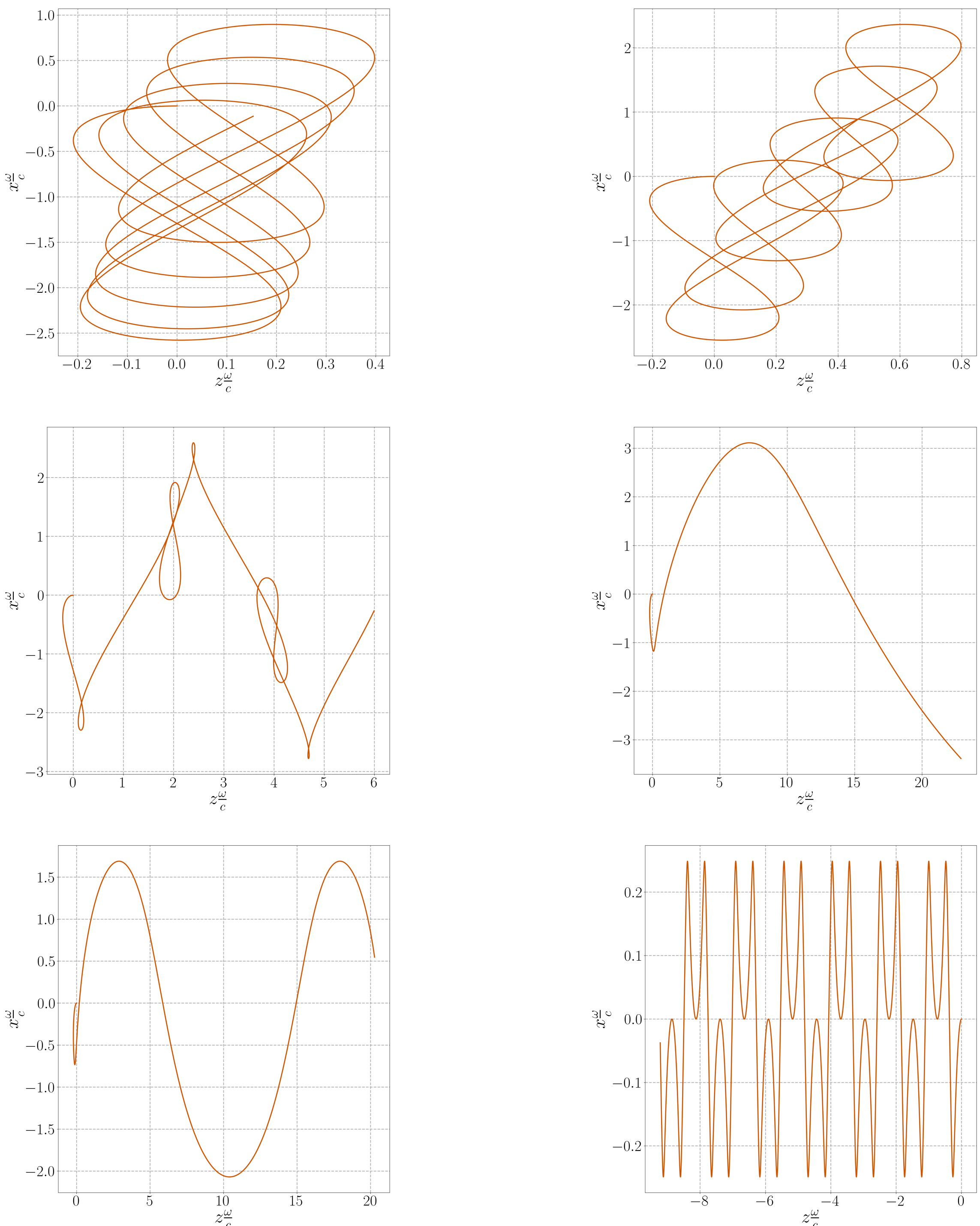


Figure 1.2: Trajectories in the $\{z, x\}$-plane of an electron in a Redmond field with $\xi_0 = 3.19$ and different values of Ω/ω. From top to bottom and left to right, this ratio takes the values: 0.05, 0.1, 0.3, 1, 1.5, 3. In the fourth figure the field satisfies the resonance condition.

1.2 Radiation reaction

1.2.1 Overview

Maxwell's and Lorentz equations allow in principle to describe self-consistently the classical dynamics of electric charges and their electromagnetic fields. However, even in the case of a single elementary charge, an electron for definiteness, the solution of the self-consistent problem of the electron dynamics and of that of its own electromagnetic field is plagued by physical inconsistencies, which ultimately are related to the divergent self energy of a point-like charge. In fact, the inclusion of the "reaction" of the self electromagnetic field on the electron dynamics (known as radiation reaction) implies an unavoidable Coulomb-like divergence when one evaluates the self field at the electron position [115, 122, 26, 158]. However, this divergence can be reabsorbed via a redefinition of the electron mass, which ultimately leads to one of the most controversial equations in physics, the Lorentz-Abraham-Dirac (LAD) equation (see Sec. 1.2.2) [4, 125, 71]. In the case of interest here, where the external force is also electromagnetic, the LAD equation can be derived by eliminating from the Maxwell-Lorentz system of equations the electromagnetic field generated by the electron. In this respect, solving the LAD equation amounts to solving exactly the electron dynamics in the external electromagnetic field and plugging the resulting solution into the Liénard-Wiechert potentials amounts to determining the corresponding exact electromagnetic field.

Even after the absorption of the divergent electron self-energy via the classical mass renormalization, the LAD equation remains problematic as it allows for so-called runaway solutions, where the electron's acceleration may exponentially increase with time even if the external field, for example, vanishes identically [115, 122, 26, 158]. The origin of the existence of the runaway solutions is precisely a term in the radiation-reaction force, known as Schott term, which depends on the time-derivative of the electron acceleration, thus rendering the LAD equation a third-order differential equation with non-Newtonian features.

Landau and Lifshitz realized that within the realm of classical electrodynamics, i.e., if quantum effects are negligible, the radiation-reaction force in the instantaneous rest frame of the electron is always much weaker than the Lorentz force [122]. This allows one to replace the electron four-acceleration in the radiation-reaction four-force with its leading-order expression, i.e., by the Lorentz four-force divided by the electron mass (see Se. 1.2.3) [122]. It is important to stress that the "reduction of order" proposed by Landau and Lifshitz is such that neglected quantities are much smaller than corrections induced by quantum effects, which are already ignored classically. The resulting equation is known as Landau-Lifshitz (LL) equation and it is free of the physical inconsistencies of the LAD equation [165]. The equivalence between the LL equation and the LAD equation within the realm of classical electrodynamics has to be intended as these equations differ by terms much smaller than quantum corrections (see Refs. [120, 51] for numerical tests about this equivalence). Presently the LL equation, as well as the problem of radiation reaction in general, are being investigated by several groups both theoretically [174, 39, 169, 124, 106, 185, 53, 175, 70, 69, 102, 155, 136, 137] and experimentally [182, 57, 149] (see also the recent reviews [99, 66, 52, 38] for previous publications).

1.2.2 The Lorentz-Abraham-Dirac equation

The Lorentz equation for an electron $\frac{du^\alpha}{d\tau} = \frac{e}{m}F^{\alpha\beta}u_\beta$ is not complete, indeed it does not take into account the field produced by the charge itself. Lorentz [125] first introduced an additional term in the nonrelativistic regime starting from the well known Larmor formula for the radiated power $P = \frac{2}{3}\frac{e^2a^2}{4\pi}$, arguing that a damping force $\boldsymbol{F}_R = \frac{2}{3}\frac{e^2}{4\pi}\frac{d\boldsymbol{a}}{dt}$ should be included to take into account the energy loss. This was then generalized to its relativistic form by Abraham [4] and Dirac [71], leading to the Lorentz-Abraham-Dirac (LAD) equation

$$\frac{du^\alpha}{d\tau} = \frac{e}{m}F^{\alpha\beta}u_\beta + \frac{2}{3}\frac{\alpha}{m}\left(\frac{d^2u^\alpha}{d\tau^2} + \frac{du^\beta}{d\tau}\frac{du_\beta}{d\tau}u^\alpha\right), \tag{1.26}$$

where units $\hbar = c = \epsilon_0 = 1$ are employed, such that $\alpha = e^2/4\pi$. The problems of this equation are well known [66] and they are mostly due to the non-Newtonian term proportional to the derivative of the acceleration, also known as Schott term.

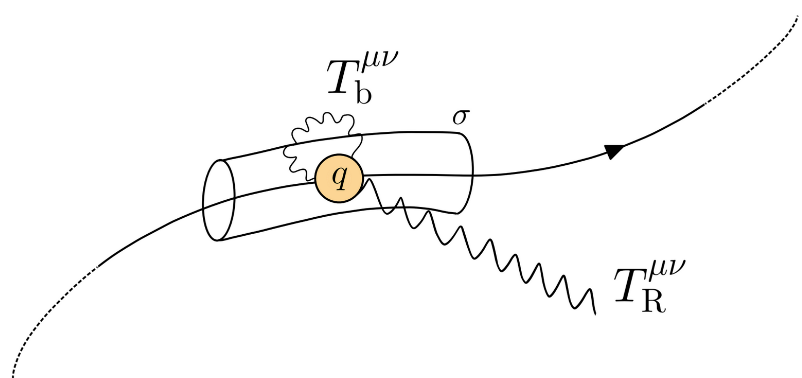


Figure 1.3: Bound and radiated energy of a moving charge.

It is instructive to sketch the derivation of the LAD equation from the energy-momentum conservation. Let us consider a particle of charge q and mass m_q moving along a trajectory $z^\mu(\tau)$ as depicted in Fig. 1.3. The electromagnetic tensor produced by the charge can be directly calculated from the Lienard-Wiechert potential and it can be divided into two parts (see, e.g., [115])

$$F_{RR}^{\mu\nu}(x) = \frac{2q}{4\pi\rho}\left\{(a\cdot t)\,u^{[\mu}t^{\nu]} + a^{[\mu}t^{\nu]}\right\}\Big|_{\tau_r} + \frac{2q}{4\pi\rho^2}\,u^{[\mu}t^{\nu]}\Big|_{\tau_r} = F_1^{\mu\nu}(x) + F_2^{\mu\nu}(x), \tag{1.27}$$

where τ_r is the retarded time defined by $[x - z(\tau_r)]^2 = 0$ with $t \geq z^0(\tau_r)$ and we have introduced the invariant distance $\rho(\tau_r) = -u^\mu(\tau_r)[x_\mu - z_\mu(\tau_r)]$ and the four-vector $t^\mu = \rho^{-1}[x^\mu - z^\mu(\tau_r)]$. Here and below the subscripts stand for the power of ρ^{-1} appearing in the tensors. The first part of this tensor depends on the acceleration of the charge and scales with the first power of ρ, both characteristics lead us to suppose that this tensor is associated with the electromagnetic radiation. The energy momentum tensor, being proportional to the square of $F_{\alpha\beta}$, takes the form [170]:

$$T_{\rm RR}^{\mu\nu} = T_2^{\mu\nu} + T_3^{\mu\nu} + T_4^{\mu\nu}. \tag{1.28}$$

One can directly verify that the first tensor and the sum of the last two are separately conserved outside the worldline:

$$\partial_\mu T_2^{\mu\nu}(x) = 0 \quad , \quad \partial_\mu[T_3^{\mu\nu}(x) + T_4^{\mu\nu}(x)] = 0 \qquad x \notin z^\mu(\tau). \tag{1.29}$$

This property suggests that we consider these two components separately and, to make the notation more convenient, we define

$$T_2^{\mu\nu} = T_{\rm R}^{\mu\nu} \quad , \quad T_3^{\mu\nu} + T_4^{\mu\nu} = T_{\rm b}^{\mu\nu}, \tag{1.30}$$

where R, b stand for "radiated" and "bound" respectively. The tensor $T_{\rm R}^{\mu\nu}$ is indeed responsible for the electromagnetic radiation, as it is proportional to $t^\mu t^\nu$ and the flux through a closed surface that contains the charge is independent of the choice of the surface. The calculation of the impulse transferred by this tensor through a spacelike surface σ with normal vector λ^α shows that [170]

$$\frac{dP_{\rm R}^\mu(\tau)}{d\tau} = -\frac{d}{d\tau}\int_{\sigma(\tau)} T_{\rm R}^{\mu\nu}\lambda_\nu\, d^3\sigma = \frac{2}{3}\frac{q^2}{4\pi}a^2(\tau)\, u^\mu(\tau), \tag{1.31}$$

where one immediately recognizes the Larmor formula for the emitted radiation. It is worth observing that the radiated impulse $P_{\rm R}^\mu$ is not a function of state. By this, we mean that it cannot be written in terms of the trajectory $z^\mu(\tau)$ and its derivatives, as it must always be considered at the retarded time and therefore depends on the history of the particle. This, although it is entirely logical here, is worth being noticed in opposition to the result that is obtained for the bound momentum $P_{\rm b}^\mu$, which is indeed a function of state. Let us now deal with the bound tensor, which is decidedly less clear compared to the radiation contribution. The flux of $T_{\rm R}^{\mu\nu}$ is well-defined. Indeed it goes like ρ^{-2}, while the surfaces grow like ρ^2 such that it possible to take the limit for small surfaces at will, maintaining consistency. In the case of $T_{\rm b}^{\mu\nu}$, this is no longer true and the integral itself is not defined on the worldline. However, the authors of [170] noted an interesting property, namely

$$T_{\rm b}^{\mu\nu} = \partial_\alpha K_2^{\mu\nu\alpha}, \tag{1.32}$$

where $K_2^{\mu\nu\alpha}$ is an antisymmetric tensor in the last two indices. It is worth spending a few words to explain the importance of this property. As we have seen, we cannot go on the worldline in the integration of the bound tensor, so the associated momentum will have a form of the type

$$P_{\rm b}^\mu(\tau) = -\lim_{\varepsilon\to 0}\lim_{R\to\infty}\int d^4x T_{\rm b}^{\mu\nu}\lambda_\nu\, \theta(\rho-\varepsilon)\theta(R-\rho)\delta(\lambda\cdot r) \tag{1.33}$$

where $\delta(\lambda \cdot r) = \delta[\lambda\cdot(x - z(\tau))]$ and ε, R are the radii of two world tubes around the worldline, the inner one very close to it while the outer one tends to infinity. The property (1.32) allows us to perform an integration by parts and write

$$\begin{aligned} P_{\rm b}^\mu(\tau) = &\lim_{R\to\infty}\int d^4x K_2^{\mu\nu\alpha}\partial_\nu\rho\, n_\alpha\, \delta(R-\rho)\delta(\lambda\cdot r) \\ &- \lim_{\varepsilon\to 0}\int d^4x K_2^{\mu\nu\alpha}\partial_\nu\rho\, n_\alpha\, \delta(\rho-\varepsilon)\delta(\lambda\cdot r). \end{aligned} \tag{1.34}$$

One can show that the integral on the outer surface is zero and the result for the total momentum is [170]

$$P^\mu_{\rm b}(\tau) = \frac{q^2}{4\pi}\left[\frac{u^\mu}{2\varepsilon} + \frac{1}{6\varepsilon}\frac{(\eta^{\mu\nu}+u^\mu u^\nu)\lambda_\nu}{\lambda\cdot u} - \frac{2}{3}a^\mu\right]. \tag{1.35}$$

Interestingly, this momentum is a function of state. This is a peculiarity, as the bound tensor depends on the history of the particle through the retarded time. The fact that it is a function of state is of great importance as it allows us to perform a procedure of renormalization of the divergences. The bound momentum does not depend on the history of the particle due to the possibility of writing $T^{\mu\nu}_{\rm b}$ as a divergence through Eq. (1.32).

We will now try to give an interpretation of these results. We can identify the first divergent term with the infinite self-energy of the particle due to Coulombian repulsion and the second, which depends on the orientation of the surface, with a term of self-stress. The renormalization process described by Teitelboim [170] consists in absorbing the self-energy into the mass

$$m_q = m_0 + \frac{q^2}{8\pi\varepsilon}, \tag{1.36}$$

while to cancel the ε-dependent self-stress term we modify ad hoc the energy momentum tensor of the particle $T_{\rm P}$ by simply adding to it this term with opposite sign. Certainly, one could argue about the legitimacy of this procedure. However, divergences in the description of point particles in classical electromagnetism are intrinsic and unavoidable, therefore one has to deal with them in a way or another. The sum of the particle and bound momenta after the renormalization is independent of the surfaces of integration and reads

$$P^\mu_{\rm P} + P^\mu_{\rm b} = \pi^\mu - \frac{2}{3}\frac{q^2}{4\pi}a^\mu, \tag{1.37}$$

where $\pi^\mu = m_q u^\mu$. This is the momentum we would expect to find in order to keep into account the energy loss due to the electromagnetic radiation. We can now derive the equations of motion. Recalling Eq. (1.31) and assuming an external Lorentz force we find

$$\frac{d}{d\tau}\left(\pi^\mu - \frac{2}{3}\frac{q^2}{4\pi}a^\mu\right) = \frac{e}{m_q}F^{\mu\nu}\pi_\nu + \frac{2}{3}\frac{q^2}{4\pi}a^2u^\mu, \tag{1.38}$$

which reproduces the LAD equation

$$\frac{du^\alpha}{d\tau} = \frac{e}{m_q}F^{\alpha\beta}u_\beta + \frac{2}{3}\frac{q^2}{4\pi m_q}\left(\frac{d^2u^\alpha}{d\tau^2} + \frac{du^\beta}{d\tau}\frac{du_\beta}{d\tau}u^\alpha\right). \tag{1.39}$$

Being the result of momentum conservation, this equation represents the most convincing way to include the radiation reaction into the particle motion. It is not completely convincing due to the arguable classical renormalization involved in its derivation, nonetheless is the best we have when starting from first principles. However, being non-Newtonian, it allows solutions which are intrinsically not physical. An example are runaway solutions in which an electron accelerates exponentially, even in absence of an external field [158]. For this reason we will prefer to employ one of its approximations, which can be shown to be completely reliable in the regime of classical physics [64]. This approximation was introduced by Landau and Lifshitz [122] and it is the topic of the next section.

1.2.3 The Landau-Lifshitz equation

An order reduction of the LAD equation due to Landau and Lifshitz is possible if the RR force in the particle rest frame is significantly smaller than the Lorentz force. As anticipated in the introduction to this section, this requirement is always satisfied if quantum effects can be neglected. We will now provide a more comprehensive clarification of this statement. Let us consider the frame momentarily comoving with the particle. In this system the equation restores the non-relativistic limit $m\boldsymbol{a} = \boldsymbol{F}_L + \frac{2}{3}\alpha\dot{\boldsymbol{a}}$, where the external force $\boldsymbol{F}_L$ is the classical Lorentz expression and the overdot stands for a proper time derivative. Now, deriving this expression we see that $m\dot{\boldsymbol{a}} = \dot{\boldsymbol{F}}_L + \mathcal{O}(\alpha)$. This means that the radiation damping term is small compared to the Lorentz one in this frame if

$$|\boldsymbol{F}_L| \gg \frac{\alpha}{m}\left|\dot{\boldsymbol{F}}_L\right|. \tag{1.40}$$

If we now substitute the velocity derivative to the order α in this expression we get the following condition

$$|\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}| \gg \frac{\alpha}{m}|\dot{\boldsymbol{E}} + \boldsymbol{v} \times \dot{\boldsymbol{B}} + \frac{e}{m}\boldsymbol{E} \times \boldsymbol{B}|. \tag{1.41}$$

Let us introduce the variation scale of the field as λ, such that $\dot{\boldsymbol{E}} \sim \boldsymbol{E}/\lambda$. The above condition corresponds then to the two requirements [66, 122]

$$\frac{\alpha}{m\lambda} \ll 1 \quad \text{or} \quad \lambda \gg \alpha\lambda_C \qquad , \qquad \frac{e\alpha F}{m^2} \ll 1 \quad \text{or} \quad F \ll \frac{F_{cr}}{\alpha}. \tag{1.42}$$

The first of these conditions constrains the field variation scale, while the second its intensity [122]. Now, if these conditions are fulfilled we can operate the order reduction and obtain the Landau-Lifshitz (LL) equation (see App. A.4 for the explicit form of the LL eq. in terms of three-vectors)

$$\frac{du^\alpha}{d\tau} = \frac{e}{m}F^{\alpha\beta}u_\beta + \frac{2e}{3m^2}\alpha\left[\dot{F}^{\alpha\beta}u_\beta + \frac{e}{m}F^{\alpha\beta}F_\beta{}^\gamma u_\gamma - \frac{e}{m}u_\delta F^{\delta\beta}F_\beta{}^\gamma u_\gamma u^\alpha\right]. \tag{1.43}$$

It is important to observe that the two aforementioned conditions Eq. (1.42) are always satisfied if quantum effects can be neglected, in fact in order to treat electrodynamics classically the two weaker conditions $\lambda \gg \lambda_C$ and $F \ll F_{cr}$ have to be fulfilled in the particle rest frame. The fact that the radiation-reaction force in the electron rest frame is assumed to be much smaller than the Lorentz force does not prevent the two forces to be of the same order in the laboratory frame. Indeed this happens when [122, 66, 161]

$$\gamma^2 F \sim \frac{F_{cr}}{\alpha}, \tag{1.44}$$

as one can check comparing the last term in Eq. (1.43) with the Lorentz force. Thus, if the particle is fast enough, classical radiation reaction can be observed and it is consistently described by the LL equation. The above formula Eq. (1.44) is a rough estimation of the ratio $\boldsymbol{F}_R/\boldsymbol{F}_L$, where $\boldsymbol{F}_R$ is the radiation reaction force term. Of course this ratio also depends on the specific form of the Maxwell tensor involved and the estimation may be partially misleading. In order to dispel any doubt, in the next section we will briefly present the analytical solution of the LL in a plane wave background, first found in 2008 by A. Di Piazza [59] and here derived in a slightly different fashion.

1.2.4 Analytical solution of the Landau-Lifshitz equation in a plane wave

Let us consider a plane wave $F^{\alpha\beta} = \dot{A}^i(\phi) f_i^{\alpha\beta}$, with $f_i^{\alpha\beta} = 2n^{[\alpha}\delta_i^{\beta]}$ and where the overdot is a derivative with respect to ϕ. We recall that in this case the Maxwell tensor satisfies the identities $(F^2)^{\alpha\beta} = -(\dot{A}^i\dot{A}_i)n^\alpha n^\beta$, $F^{\alpha\beta}n_\beta = 0$. The first step to solve the LL equation (1.43) is to use the phase as the evolution parameter. Observing that $\frac{d\phi}{d\tau} = u^-$ we have

$$u^-\dot{u}^\alpha = \frac{e}{m}F^{\alpha\beta}u_\beta + \frac{2e}{3m^2}\alpha\left[u^-\dot{F}^{\alpha\beta}u_\beta + \frac{e}{m}F^{\alpha\beta}F_\beta{}^\gamma u_\gamma - \frac{e}{m}u_\delta F^{\delta\beta}F_\beta{}^\gamma u_\gamma u^\alpha\right]. \tag{1.45}$$

One of the main reasons why this equation is analytically solvable is that we can explicitly find u^- as function of the wave field. Indeed, despite not being constant as in absence of RR, this variable here follows the differential equation

$$\dot{u}^- = -\frac{2e^2\alpha}{3m^3}u_\delta F^{\delta\beta}F_\beta{}^\gamma u_\gamma u^\alpha = \frac{2e^2\alpha}{3m^3}\dot{A}^i\dot{A}_i(u^-)^2, \tag{1.46}$$

as it can be easily verified contracting by n_α both sides of the LL. This equation is easily solved as $u^-(\phi) = u_0^-/h(\phi)$ with [59]

$$h(\phi) = 1 - \frac{2\alpha}{3\omega}\eta_0\xi_0^2\int_{\phi_0}^{\phi} d\tilde{\phi}\,\dot{\psi}^i(\tilde{\phi})\dot{\psi}_i(\tilde{\phi}), \tag{1.47}$$

where $\dot{A}^i(\phi) = \frac{E_0}{\omega}\dot{\psi}^i(\phi)$. The phase ϕ is dimensional so $\dot{\psi}^i(\phi)$ have the dimension of ω. In the following we will often use the dimensionless phase $\varphi = k\cdot x$ in order to properly extract the physical parameters, however here it suffices to observe that the second term in h is proportional to the parameter

$$R = \alpha\eta_0\xi_0^2. \tag{1.48}$$

This is the parameter controlling the magnitude of the RR effects in a plane wave [59], we will further discuss its role in the following. Once u^- is found we can write the equation in terms of the rescaled momentum $\tilde{\pi}^\alpha = h\pi^\alpha$. The derivative of the function h cancels out the last term in the LL equation, leaving us with the more compact

$$\dot{\tilde{\pi}}^\alpha = \frac{e}{p^-}hF^{\alpha\beta}\tilde{\pi}_\beta + \frac{2e}{3m^2}\alpha\left(\dot{F}^{\alpha\beta}\tilde{\pi}_\beta + \frac{e}{p^-}hF^{\alpha\beta}F_\beta{}^\gamma\tilde{\pi}_\gamma\right). \tag{1.49}$$

Once again, we can exploit the structure of the plane wave Maxwell tensor to solve this equation in few steps. Choosing an initial phase ϕ_0 and inserting the formal solution

$$\tilde{\pi}_p^\alpha = p^\alpha + \int_{\phi_0}^{\phi} d\tilde{\phi}\left[\frac{e}{p^-}hF^{\alpha\beta}\tilde{\pi}_\beta + \frac{2e}{3m^2}\alpha\left(\dot{F}^{\alpha\beta}\tilde{\pi}_\beta + \frac{e}{p^-}hF^{\alpha\beta}F_\beta{}^\gamma\tilde{\pi}_\gamma\right)\right] \tag{1.50}$$

repeatedly in the r.h.s. of Eq. (1.49), one easily observes that due to the algebraic properties of $F^{\alpha\beta}$

$$\dot{\tilde{\pi}}_p^\alpha(\phi) = \frac{d}{d\phi}\exp\left[\frac{e}{p^-}\int_{\phi_0}^{\phi} d\tilde{\phi}\left(hF + \frac{2\alpha p^-}{3m^2}\dot{F}\right)\right]^{\alpha\beta} p_\beta + \frac{2\alpha e^2}{3m^2p^-}hF^{\alpha\beta}F_\beta{}^\gamma p_\gamma. \tag{1.51}$$

As a last step we notice that the second term in the r.h.s. of this equation can be written

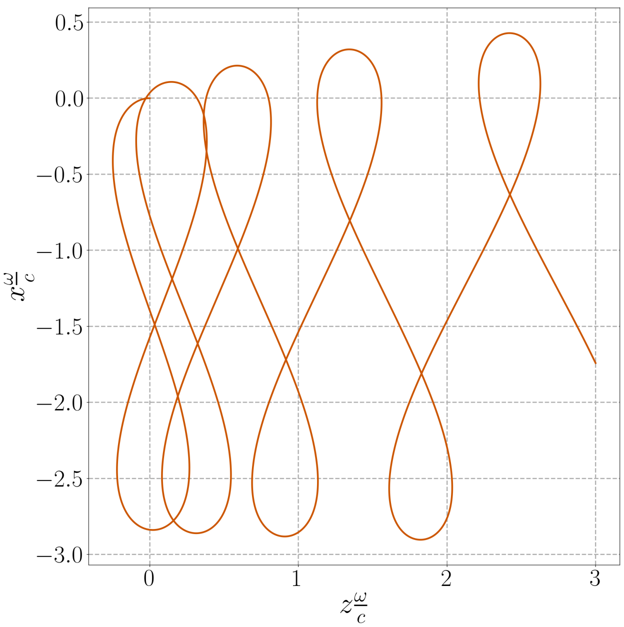


Figure 1.4: Trajectory in the $\{z, x\}$-plane of an electron in a plane wave propagating along $\boldsymbol{z}$ including the RR effects. The typical eight-figure is shifted and lost with the cumulation of RR contributions.

as

$$\frac{2\alpha e^2}{3m^2 p^-} h F^{\alpha\beta} F_\beta{}^\gamma p_\gamma = \frac{m^2}{2p^-}\frac{dh^2}{d\phi} n^\alpha. \tag{1.52}$$

After a trivial integration the solution is found to be

$$\tilde{\pi}_p^\alpha(\phi) = \exp\left(\frac{e}{p^-} I^i(\phi) f_i\right)^{\alpha\beta} p_\beta + \frac{m^2}{2p^-}[h^2(\phi) - 1] n^\alpha, \tag{1.53}$$

where we introduced the notation

$$I^i(\phi) = \int_{\phi_0}^{\phi} d\tilde{\phi}\left[h(\tilde{\phi})\dot{A}^i(\tilde{\phi}) + \frac{2\alpha p^-}{3m^2}\ddot{A}^i(\tilde{\phi})\right]. \tag{1.54}$$

If the RR is neglected then $I^i = A^i$ and $h = 1$, such that the solution (1.53) reduces to Eq. (1.8) previously discussed in absence of RR. For standard optical lasers, being rapidly oscillating fields, we can usually assume that $|\dot{A}^i| \sim |\ddot{A}^i|$. This means that the second term in the previous expression can generally be neglected in agreement with the LL reduction of order [122, 62](see Ref. [69] for a situation where this term cannot be ignored). We will thus assume that

$$I^i(\phi) \simeq \int_{\phi_0}^{\phi} d\tilde{\phi} h(\tilde{\phi})\dot{A}^i(\tilde{\phi}) \equiv \frac{m}{e}\mathcal{F}_\perp^i(\phi). \tag{1.55}$$

For the sake of clarity let us expand the exponential and write the solution explicitly in terms of the field $\mathcal{F}_\perp^i$ [59]

$$\pi_p^\alpha(\phi) = \frac{1}{h(\phi)}\left\{p^\alpha - m\mathcal{F}_\perp^\alpha(\phi) + \frac{m}{p^-}\left[p\cdot\mathcal{F}_\perp(\phi) - \frac{m}{2}\mathcal{F}_\perp(\phi)\cdot\mathcal{F}_\perp(\phi) + \frac{m}{2}\left(h^2(\phi) - 1\right)\right] n^\alpha\right\}. \tag{1.56}$$

In this expression we clearly recognize the solution in absence of RR Eq. (1.9) with the substitution $\xi^i_\perp \to \mathcal{F}^i_\perp$, plus an additional term in n^α which is not present in the solution of the Lorentz equation.

1.2.5 Magnitude of RR effects

In a plane wave field one can show that for an ultrarelativistic electron the radiative term in the LL proportional to $(Fu)^2$ gives a larger contribution to the RR compared to the Schott term [59]. As discussed in Sec. 1.2.4, the parameter controlling the magnitude of the RR effects in this field is

$$R = \alpha\eta_0\xi_0^2 = \alpha\xi_0\chi_0, \tag{1.57}$$

where $\chi_0 \ll 1$ is required in order to treat the problem in the classical domain. Asking $R \sim 1$ and $\chi_0 \ll 1$ defines the Classical Radiation Dominated Regime (CRDR) [66] and imposes some very stringent constrains on the parameter space. To give an example let us fix $\chi \sim 10^{-1}$ such that we need $\xi_0 \sim 10^3$ in order to have $R \sim 1$. For optical lasers $\omega \sim$ eV this corresponds to an intensity of $I_0 \sim 10^{24}\text{W/cm}^2$, an incredibly large intensity. Similar intensities are not completely unimaginable but are at the boundary of what is expected to be achieved in the next years, they are just five orders of magnitude below the Schwinger critical intensity. Moreover, it is worth observing that under these assumptions, if the electrons are initially counterpropagating with respect to the wave they cannot be very fast, otherwise quantum effects emerge. This is one of the main problems at the experimental level: measuring RR effects is very difficult, measuring classical RR effects is generally even more challenging.

So far, the best experimental tests of RR came from experiments in crystals [182] and only very recently high significance observations of RR in strong fields were published [126]. A comment is important here: RR effects are cumulative, they depends on the history of the particle. For a plane wave this is manifested in the decreasing of the otherwise constant u^-, which including RR reads $u^- = u_0^-/h(\phi)$. Let us recall Eq. (1.47) [59]

$$h = 1 + \frac{2}{3}\frac{R}{\omega}\int_{\phi_0}^{\phi} d\tilde{\phi}\dot{\psi}^2(\tilde{\phi}). \tag{1.58}$$

For a monochromatic, circularly polarized wave this is $h = 1 + \frac{2}{3}R\omega\Delta\phi$, where $\Delta\phi$ is the phase duration of the interaction. This means that in principle we could enhance the RR contributions by increasing the interaction time. However, strong Gaussian laser pulses are focused on a longitudinal length of the order of the Rayleigh length, which is usually very small, especially for a relativistic electron. Despite this problem, an enhancement can be in principle obtained if other pulses are considered. The interesting case of Flying-Focus Pulses has been recently studied in [84].

A system which behaves in a somehow complementary way regarding RR effects is the one in which a constant magnetic field is chosen as a background. The radiation reaction for the motion in this field has been considered in several papers (see, e.g., [161, 162, 184, 19, 166]). Studying the motion, it is possible to show that in the ultrarelativistic limit the radius of the spiral followed by the particle in its transverse motion is halved when $\gamma_0\frac{\omega_H^2}{\omega_e}t \sim 1$ (see, e.g. [166] eq. 7.25 and App. A.2 for further details), where $\omega_H = \frac{eH}{m}$ is the cyclotron frequency in the particle rest frame and $\omega_e = \frac{3m}{2\alpha} \simeq 1.6 \times 10^{23}$Hz is proportional to the electron fundamental frequency [161]. On the other hand, the Lorentz force which generates the circular motion operates on a scale defined by $\omega_H/\gamma_0 t \sim 1$. It follows that the proper RR parameter in this scenario is [161]

$$R = \gamma_0^2 \frac{\omega_H}{\omega_e}. \tag{1.59}$$

The quantum effects here becomes important when the emitted photons have the same energy of the electron, namely when [161] $\gamma_0\omega_H/\alpha\omega_e \sim 1$. This means that in order to stay in the CRDR and treat the motion classically having observable RR effects we need very fast electrons $\gamma_0 \gg 137$ and not extremely strong magnetic fields $\omega \ll \omega_e$. In this sense this situation is complementary to the head-on collision of a particle with a strong laser field [59]. The Redmond configuration merges these two fields and could thus be an interesting background for studying RR effects.

1.2.6 Considerations about the Landau-Lifshitz equation in a Redmond field

Let us discuss the various parameters involved in motion within a Redmond field configuration. As discussed before, the parameter quantifying the pure magnetic RR is $R^{(B)} = \gamma^2 \frac{\omega_H}{\omega_e}$. On the other hand, in a plane wave these effects are regulated by the parameter $R^{(A)} = \alpha\xi_0\chi_0$. If we consider optical lasers then $\omega \gg \omega_H$ and there is no practical possibility to produce constant fields with an eV($\sim 10^{14}$Hz) cyclotron frequency in a laboratory at the moment, this would require magnetic fields of the order $\sim 10^7$G. Let us now assume that we are working in a neighborhood of the resonance for the RR-free case, such that $\omega_H/u^- \sim \omega$. If the particle is counterpropagating with respect to the wave and it is ultrarelativistic then this relation becomes

$$\omega_H \sim \gamma\omega, \tag{1.60}$$

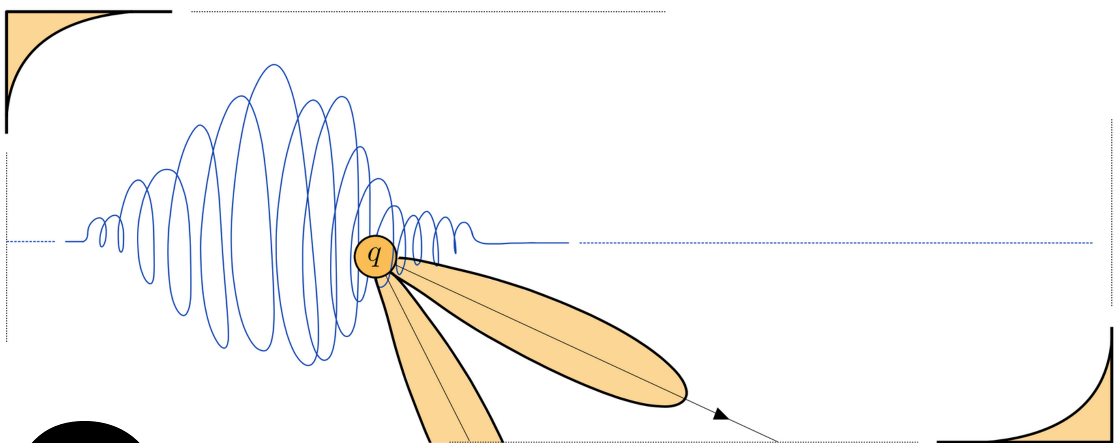
due to the blueshift of the laser frequency in the particle frame. In this case the magnetic parameter $R^{(B)} \sim \gamma^3 \frac{\omega}{\omega_e} \sim 10^{-9}\gamma^3$. We can thus work in the CRDR for the laser field neglecting the magnetic field RR. In fact, if we set $\xi_0 \sim 10^3$, $\chi_0 \sim 10^{-1}$ for optical frequencies and a $\gamma \sim 10$ for the electron, the condition $R^{(B)} \ll 1$ is satisfied within the laser CRDR. With these considerations in mind one can neglect the RR terms involving the magnetic field in the LL equation and find an approximate solution in the form

$$\begin{aligned} \pi_p^\alpha = p^+ n^\alpha + p^- \tilde{n}^\alpha + R^\alpha{}_j(\Omega\Psi(\phi))[p^j - e\mathscr{A}^j(\phi)] \\ + \frac{e}{p^-}\left(p \cdot \mathscr{A}(\phi) - \frac{e}{2}\mathscr{A}(\phi)\cdot\mathscr{A}(\phi)\right) n^\alpha + \frac{m^2}{2p^-}(h^2(\phi) - 1)n^\alpha, \end{aligned} \tag{1.61}$$

where now

$$\mathscr{A}^i = \int_{\phi_0}^{\phi} d\tilde{\phi} h(\tilde{\phi}) R^i{}_j\left(-\Omega\Psi(\tilde{\phi})\right)\dot{A}^j(\tilde{\phi}) \quad \text{and} \quad \Psi(\phi) = \int_{\phi_0}^{\tilde{\phi}} d\phi' h(\phi'). \tag{1.62}$$

We refer to Sec. 1.1.2 for the rest of the notation. Here we clearly see the RR contribution in the modification of $\mathscr{A}$ from Eq. (1.21). The resonance condition in this case cannot really be satisfied for an extended period. Indeed, as one would expect, keeping into account the energy loss due to RR the synchrotron frequency seen by the charge changes continuously. This is represented by the presence of Ψ instead of the phase in the rotation matrix. However, this feature could be an interesting resource for studying classical RR effects in such a field configuration if small displacement from the resonance condition could be precisely measured. At the moment these are just heuristic opinions, a detailed study of the possible experimental applications is left for future work.



2

Nonlinear Thomson scattering

2.1 Overview

In this chapter, we present analytical expressions of the energy emission spectrum of an electron driven by an external intense plane wave (nonlinear Thomson scattering) found in [62] by taking into account radiation-reaction effects via the LL equation. To achieve this goal, we use the analytical solution of the LL equation in an arbitrary plane wave [59] and we derive the angularly-resolved and the angularly-integrated energy spectra as double integrals over the phase of the plane wave. In this respect, we complete the analytical determination of the self-consistent classical dynamics of the electron and its electromagnetic field in the sense mentioned above: additional classical corrections, which would be brought about, i.e., by using the LAD equation would be smaller than already ignored quantum corrections. Finally, the corresponding expressions within the so-called locally-constant field approximation (LCFA) are derived as single phase-integrals [157, 23, 66]. These results obtained here also complement the ones obtained in Ref. [60], where the analytical expression of the infrared limit of the emission spectrum including radiation-reaction effects was presented. Units with $\hbar = c = \epsilon_0 = 1$ are employed throughout.

2.2 Analytical spectrum of nonlinear Thomson scattering

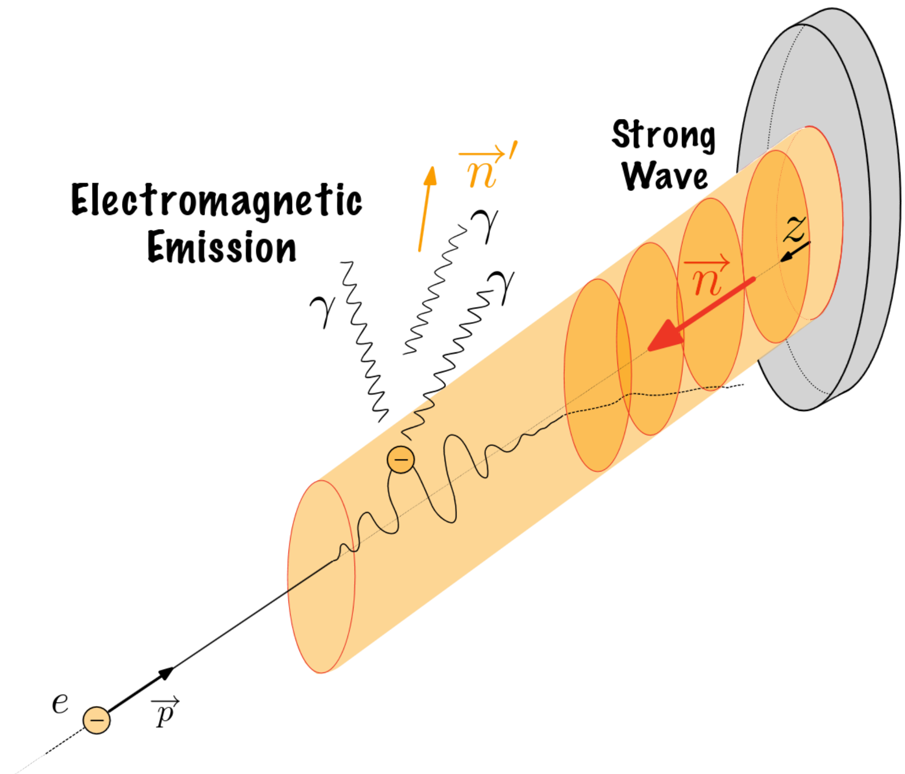


Figure 2.1: Schematic representation of Thomson scattering in a plane wave.

Let us consider an electron (charge $e < 0$ and mass m, respectively), whose trajectory is characterized by the instantaneous position $\boldsymbol{x}(t)$ and the instantaneous velocity $\boldsymbol{v}(t) = d\boldsymbol{x}(t)/dt$. The electromagnetic energy $\mathcal{E}$ radiated by the electron per unit of angular frequency ω' and along the direction $\boldsymbol{n}' = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$ within a solid angle $d\Omega = \sin\vartheta d\vartheta d\varphi$ is given by [see, e.g., Eq. (14.67) in Ref. [115]]

$$\frac{d\mathcal{E}}{d\omega' d\Omega} = \frac{\alpha\omega'^2}{4\pi^2}\left|\int_{-\infty}^{\infty} dt\, \boldsymbol{n}' \times (\boldsymbol{n}' \times \boldsymbol{v}(t))e^{i\omega'(t-\boldsymbol{n}'\cdot\boldsymbol{x}(t))}\right|^2, \tag{2.1}$$

and we stress that this expression of the emitted energy is valid for an arbitrary trajectory of the electron. Now, we assume that the electron moves in the presence of a plane-wave background field, described by the four-vector potential $A^\mu(\phi) = (A^0(\phi), \boldsymbol{A}(\phi))$, where $\phi = n \cdot x = t - \boldsymbol{n} \cdot \boldsymbol{x}$, with $n^\mu = (1, \boldsymbol{n})$ and the unit vector $\boldsymbol{n}$ identifying the propagation direction of the plane wave itself. As explained in Sec. 1.1.1, if we assume $\lim_{\phi\to\pm\infty} \boldsymbol{A}(\phi) = \boldsymbol{0}$ and work in the Lorenz gauge $\partial_\mu A^\mu(\phi) = n \cdot \dot{A}(\phi) = 0$ with the additional condition $A^0(\phi) = 0$, then $A^\mu(\phi)$ is left with only transverse components.

It is convenient first to express the emitted energy $d\mathcal{E}/d\omega' d\Omega$ as an integral over the laser dimensionless phase $\varphi = \omega\phi$, where ω is the central angular frequency of the plane wave (or, more in general, an arbitrary frequency scale describing the time dependence of the plane wave). This is easily done because $d\phi(t)/dt = 1 - \boldsymbol{n}\cdot\boldsymbol{v}(t)$ along the electron trajectory and one obtains

$$\frac{d\mathcal{E}}{d\omega' d\Omega} = \frac{\alpha}{4\pi^2}\frac{\omega'^2}{\omega^2}\left|\int_{-\infty}^{\infty} d\varphi\, \frac{\boldsymbol{n}'\times(\boldsymbol{n}'\times\boldsymbol{\pi}(\varphi))}{\pi^-(\varphi)} e^{i\frac{\omega'}{\omega}\int_{-\infty}^{\varphi} d\varphi' \frac{\varepsilon(\varphi')-\boldsymbol{n}'\cdot\boldsymbol{\pi}(\varphi')}{\pi^-(\varphi')}}\right|^2, \tag{2.2}$$

where $\pi^\mu(\varphi) = (\varepsilon(\varphi), \boldsymbol{\pi}(\varphi)) = \varepsilon(\varphi)(1, \boldsymbol{v}(\varphi))$, with $\varepsilon(\varphi) = m/\sqrt{1-\boldsymbol{v}^2(\varphi)}$, is the electron four-momentum and $\pi^-(\varphi) = n\cdot\pi(\varphi)$. Now, we recall that the LL equation in an external electromagnetic field $F^{\mu\nu} = F^{\mu\nu}(x)$ reads [122] [see Eq. (1.43)]

$$\frac{du^\alpha}{d\tau} = \frac{e}{m}F^{\alpha\beta}u_\beta + \tau_e\frac{e}{m}\left[(\partial_\gamma F^{\alpha\beta})u^\gamma u_\beta + \frac{e}{m}F^{\alpha\beta}{F_\beta}^\gamma u_\gamma - \frac{e}{m}u_\delta F^{\delta\beta}{F_\beta}^\gamma u_\gamma u^\alpha\right]. \tag{2.3}$$

where τ is the electron proper time, $\tau_e = \frac{2\alpha}{3m}$ and $u^\mu(s) = \pi^\mu(\tau)/m$ is the electron four-velocity. As we have extensively discussed in Sec. 1.2.4, in the case of a plane wave background this equation is analytically solvable in a compact form [59]. By indicating as $p^\mu = (p^0, \boldsymbol{p})$, with $p^0 = \sqrt{m^2+\boldsymbol{p}^2}$, the initial four-momentum of the electron, i.e., $\lim_{\varphi\to-\infty}\pi^\mu(\varphi) = p^\mu$, the four-momentum $\pi^\mu(\varphi)$ at the generic phase φ is given by Eq. (1.56) [59]

$$\pi^\alpha(\phi) = \frac{1}{h(\phi)}\left\{p^\alpha - m\mathcal{F}^\alpha_\perp(\phi) + \frac{m}{p^-}\left[p\cdot\mathcal{F}_\perp(\phi) - \frac{m}{2}\mathcal{F}_\perp(\phi)\cdot\mathcal{F}_\perp(\phi) + \frac{m}{2}\left(h^2(\phi)-1\right)\right]n^\alpha\right\}. \tag{2.4}$$

Here we have dropped the momentum subscript p to lighten the notation. π^α will always refer to this initial condition in this section, ensuring that no confusion arises. We recall the definitions introduced in this expression, which we rewrite here in terms of $\boldsymbol{\xi}_\perp(\varphi) = (e/m)\boldsymbol{A}'_\perp(\varphi)$ and $\xi^{\mu\nu}(\varphi) = (e/m)[n^\mu A'^\nu(\varphi) - n^\nu A'^\mu(\varphi)]$

$$h(\varphi) = 1 + \frac{2}{3}\alpha\eta_0\int_{-\infty}^{\varphi} d\tilde{\varphi}\,\boldsymbol{\xi}^2_\perp(\tilde{\varphi}), \tag{2.5}$$

$$\mathcal{F}^{\mu\nu}(\varphi) = \int_{-\infty}^{\varphi} d\tilde{\varphi}\left[h(\tilde{\varphi})\xi^{\mu\nu}(\tilde{\varphi}) + \frac{2}{3}\alpha\eta_0\xi'^{\mu\nu}(\tilde{\varphi})\right]. \tag{2.6}$$

Here and below the prime indicates the derivative with respect to the dimensionless phase φ. Note that, assuming that $|\xi^{\mu\nu}(\varphi)| \sim |\xi'^{\mu\nu}(\varphi)|$ as it is typically the case for standard laser fields, the term proportional to $\xi'^{\mu\nu}(\varphi)$ in $\mathcal{F}^{\mu\nu}(\varphi)$ can be neglected according to Landau and Lifshitz reduction of order [122] (see Sec. 1.2.4). For this reason we assume

$$\mathcal{F}^{\mu\nu}(\varphi) = \int_{-\infty}^{\varphi} d\tilde{\varphi}\, h(\tilde{\varphi})\xi^{\mu\nu}(\tilde{\varphi}) \tag{2.7}$$

and we use this expression below. For the sake of later convenience, we also report here the light-cone components of the four-momentum of the electron in the plane wave including radiation reaction:

$$\pi^-(\varphi) = \frac{p^-}{h(\varphi)}, \tag{2.8}$$

$$\boldsymbol{\pi}_\perp(\varphi) = \frac{1}{h(\varphi)}[\boldsymbol{p}_\perp - m\boldsymbol{\mathcal{F}}_\perp(\varphi)], \tag{2.9}$$

$$\pi^+(\varphi) = \frac{m^2 + \boldsymbol{\pi}_\perp^2(\varphi)}{2\pi^-(\varphi)} = \frac{1}{h(\varphi)}\frac{m^2h^2(\varphi) + [\boldsymbol{p}_\perp - m\boldsymbol{\mathcal{F}}_\perp(\varphi)]^2}{2p^-}, \tag{2.10}$$

where $\boldsymbol{\mathcal{F}}_\perp(\varphi) = \int_{-\infty}^{\varphi} d\tilde{\varphi}\, h(\tilde{\varphi})\boldsymbol{\xi}_\perp(\tilde{\varphi})$ [see Eq. (2.7)] as well as the corresponding longitudinal momentum [$\pi_\|(\varphi) = \boldsymbol{n}\cdot\boldsymbol{\pi}(\varphi)$] and the energy:

$$\pi_\|(\varphi) = \pi^+(\varphi) - \frac{\pi^-(\varphi)}{2} = \frac{p^-}{2h(\varphi)}\left\{\frac{m^2h^2(\varphi) + [\boldsymbol{p}_\perp - m\boldsymbol{\mathcal{F}}_\perp(\varphi)]^2}{(p^-)^2} - 1\right\}, \tag{2.11}$$

$$\varepsilon(\varphi) = \pi^+(\varphi) + \frac{\pi^-(\varphi)}{2} = \frac{p^-}{2h(\varphi)}\left\{\frac{m^2h^2(\varphi) + [\boldsymbol{p}_\perp - m\boldsymbol{\mathcal{F}}_\perp(\varphi)]^2}{(p^-)^2} + 1\right\}. \tag{2.12}$$

Before replacing Eq. (2.4) [or equivalently Eqs. (2.8)-(2.10)] in Eq. (2.2), it is convenient to write the latter equation in the form

$$\frac{d\mathcal{E}}{d\boldsymbol{l}} = -\frac{\alpha}{4\pi^2}\int d\varphi d\varphi' \frac{\pi(\varphi)\cdot\pi(\varphi')}{(k\cdot\pi(\varphi))(k\cdot\pi(\varphi'))} e^{i\int_{\varphi'}^{\varphi} d\tilde{\varphi}\frac{l\cdot\pi(\tilde{\varphi})}{k\cdot\pi(\tilde{\varphi})}}, \tag{2.13}$$

where we have introduced the wavevector of the emitted radiation $l^\mu = (\omega', \boldsymbol{l}) = \omega'(1, \boldsymbol{n}')$. This identity follows from the hypothesis that contour terms do not contribute to the integral (see also Refs. [115, 23] on this), and it can be proved in few simple steps. First we write

$$\begin{aligned} \frac{d\mathcal{E}}{d\omega' d\Omega} &= \frac{\alpha}{4\pi^2}\frac{\omega'^2}{\omega^2}\left|\int_{-\infty}^{\infty} d\varphi\, \frac{\boldsymbol{n}'\times(\boldsymbol{n}'\times\boldsymbol{\pi}(\varphi))}{\pi^-(\varphi)} e^{i\frac{\omega'}{\omega}\int_{-\infty}^{\varphi} d\varphi' \frac{\varepsilon(\varphi') - \boldsymbol{n}'\cdot\boldsymbol{\pi}(\varphi')}{\pi^-(\varphi')}}\right|^2 = \\ &= \frac{\alpha}{4\pi^2}\left|\int_{-\infty}^{\infty} d\varphi\, \frac{\boldsymbol{n}'(\boldsymbol{l}\cdot\boldsymbol{\pi}(\varphi)) - \omega'\boldsymbol{\pi}(\varphi)}{k\cdot\pi(\varphi)} e^{i\int_{-\infty}^{\varphi} d\tilde{\varphi}\frac{l\cdot\pi(\tilde{\varphi})}{k\cdot\pi(\tilde{\varphi})}}\right|^2 \end{aligned} \tag{2.14}$$

Now, if we call the phase factor $\Phi(\varphi) = \int_{-\infty}^{\varphi} d\tilde{\varphi}\frac{l\cdot\pi(\tilde{\varphi})}{k\cdot\pi(\tilde{\varphi})}$ and we assume that the boundary contributions can be neglected we have

$$\int d\varphi \frac{d}{d\varphi}e^{i\Phi(\varphi)} = i\int d\varphi \frac{l\cdot\pi(\varphi)}{k\cdot\pi(\varphi)}e^{i\Phi(\varphi)} = i\int d\varphi \frac{\omega'\varepsilon(\varphi) - \boldsymbol{l}\cdot\boldsymbol{\pi}(\varphi)}{k\cdot\pi(\varphi)}e^{i\Phi(\varphi)} = 0. \tag{2.15}$$

It follows the useful identity

$$\int d\varphi \frac{\boldsymbol{l}\cdot\boldsymbol{\pi}(\varphi)}{k\cdot\pi(\varphi)}e^{i\Phi(\varphi)} = \int d\varphi \frac{\omega'\varepsilon(\varphi)}{k\cdot\pi(\varphi)}e^{i\Phi(\varphi)}, \tag{2.16}$$

which can be exploited to rewrite the energy distribution as

$$\begin{aligned} \frac{d\mathcal{E}}{d\omega' d\Omega} &= \frac{\alpha\omega'^2}{4\pi^2}\left|\int_{-\infty}^{\infty} d\varphi\, \frac{\varepsilon(\varphi)\boldsymbol{n}' - \boldsymbol{\pi}(\varphi)}{k\cdot\pi(\varphi)}e^{i\Phi(\varphi)}\right|^2 = \\ &= \frac{\alpha\omega'^2}{4\pi^2}\int_{-\infty}^{\infty} d\varphi d\varphi'\, \frac{[\varepsilon(\varphi)\boldsymbol{n}' - \boldsymbol{\pi}(\varphi)]\cdot[\varepsilon(\varphi')\boldsymbol{n}' - \boldsymbol{\pi}(\varphi)]}{(k\cdot\pi(\varphi))(k\cdot\pi(\varphi'))}e^{i\Delta\Phi(\varphi,\varphi')} = \\ &= -\frac{\alpha\omega'^2}{4\pi^2}\int_{-\infty}^{\infty} d\varphi d\varphi'\, \frac{\pi(\varphi)\cdot\pi(\varphi')}{(k\cdot\pi(\varphi))(k\cdot\pi(\varphi'))}e^{i\Delta\Phi(\varphi,\varphi')}, \end{aligned} \tag{2.17}$$

where we defined the phase difference $\Delta\Phi(\varphi,\varphi') = \int_{\varphi'}^{\varphi} d\tilde{\varphi} \frac{l\cdot\pi(\tilde{\varphi})}{k\cdot\pi(\tilde{\varphi})}$. Eq. (2.13) is recovered once the derivatives are expressed in terms of the spatial momentum $\boldsymbol{l}$. Equation (2.13) is especially useful if the four-dimensional scalar products are expressed in light-cone coordinates, observing that the electron four-momentum is on-shell, i.e., $\pi^2(\varphi) = m^2$. After a few straightforward manipulations, one can easily write Eq. (2.13) in a more suitable form (see. S.m. 2.6.1 for a detailed derivation)

$$\frac{d\mathcal{E}}{d\boldsymbol{l}} = -\frac{\alpha}{8\pi^2 m^2 \eta_0^2} \int d\varphi d\varphi' \, e^{i\frac{l^-}{2p^-\eta_0}\int_{\varphi'}^{\varphi} d\tilde{\varphi}\, h^2(\tilde{\varphi})[1+\hat{\boldsymbol{\pi}}_\perp^2(\tilde{\varphi})]} \left\{h^2(\varphi) + h^2(\varphi') + [\boldsymbol{\mathcal{F}}_\perp(\varphi) - \boldsymbol{\mathcal{F}}_\perp(\varphi')]^2\right\}, \tag{2.18}$$

where

$$\hat{\boldsymbol{\pi}}_\perp(\varphi) = \frac{1}{m}\left[\boldsymbol{\pi}_\perp(\varphi) - \frac{\pi^-(\varphi)}{l^-}\boldsymbol{l}_\perp\right] = \frac{1}{mh(\varphi)}\left[\boldsymbol{p}_\perp - m\boldsymbol{\mathcal{F}}_\perp(\varphi) - \frac{p^-}{l^-}\boldsymbol{l}_\perp\right]. \tag{2.19}$$

Expression (2.18) shows that the effects of radiation reaction are all encoded in the function $h(\varphi)$ [see Eq. (2.5)] and if radiation reaction is ignored, i.e., for $h(\varphi) = 1$, one obtains the classical spectrum of Thomson scattering. This, in turn, can be obtained as the classical limit of the spectrum of nonlinear Compton scattering as reported, e.g., in Ref. [67], which is accomplished by neglecting the recoil of the emitted radiation (emitted photon in the quantum language) on the electron. More precisely, we recall here that Eq. (2.18) divided by ω' corresponds to the classical limit of the average number of photons emitted by the electron per units of emitted photon momentum [93, 63].

As one can easily recognize, from Eq. (2.18) one can obtain the angularly-integrated energy emission spectrum $d\mathcal{E}/dl^-$ by using the fact that $d\boldsymbol{l} = (\omega'/l^-)dl^- d\boldsymbol{l}_\perp$ such that

$$\begin{aligned}\frac{d\mathcal{E}}{dl^-} &= -\frac{\alpha}{8\pi^2 m^2 \eta_0^2} \int d\boldsymbol{l}_\perp \frac{\omega'}{l^-} \int d\varphi d\varphi' \, e^{i\frac{l^-}{2p^-\eta_0}\int_{\varphi'}^{\varphi} d\tilde{\varphi}[h^2(\tilde{\varphi})+\hat{\boldsymbol{\pi}}_\perp^2(\tilde{\varphi})]} \\ &\quad\times \left\{h^2(\varphi) + h^2(\varphi') + [\boldsymbol{\mathcal{F}}_\perp(\varphi) - \boldsymbol{\mathcal{F}}_\perp(\varphi')]^2\right\}.\end{aligned} \tag{2.20}$$

By noticing that $\omega' = l^+ + l^-/2 = \boldsymbol{l}_\perp^2/2l^- + l^-/2$, the integral in $d\boldsymbol{l}_\perp$ is easily taken as it is Gaussian. By passing for convenience to the average and the relative phases $\varphi_+ = (\varphi+\varphi')/2$ and $\varphi_- = \varphi - \varphi'$, the resulting energy spectrum is given by (see S.m. 2.6.2 for a detailed derivation)

$$\begin{aligned}\frac{d\mathcal{E}}{dl^-} &= -\frac{i\alpha}{8\pi\eta_0}\frac{l^-}{p^-}\int \frac{d\varphi_+ d\varphi_-}{\varphi_- + i0} e^{i\frac{l^-}{2p^-\eta_0}\left\{\int_{-\varphi_-/2}^{\varphi_-/2} d\tilde{\varphi}[h^2(\varphi_++\tilde{\varphi})+\boldsymbol{\mathcal{F}}_\perp^2(\varphi_++\tilde{\varphi})] - \frac{1}{\varphi_-}\left[\int_{-\varphi_-/2}^{\varphi_-/2} d\tilde{\varphi}\,\boldsymbol{\mathcal{F}}_\perp(\varphi_++\tilde{\varphi})\right]^2\right\}} \\ &\quad\times\left\{h^2\left(\varphi_+ + \frac{\varphi_-}{2}\right) + h^2\left(\varphi_+ - \frac{\varphi_-}{2}\right) + \left[\boldsymbol{\mathcal{F}}_\perp\left(\varphi_+ + \frac{\varphi_-}{2}\right) - \boldsymbol{\mathcal{F}}_\perp\left(\varphi_+ - \frac{\varphi_-}{2}\right)\right]^2\right\} \\ &\quad\times\left(1 + \frac{m^2}{(p^-)^2}\left\{\frac{1}{\varphi_-}\int_{-\varphi_-/2}^{\varphi_-/2} d\tilde{\varphi}\left[\frac{\boldsymbol{p}_\perp}{m} - \boldsymbol{\mathcal{F}}_\perp(\varphi_+ + \tilde{\varphi})\right]\right\}^2 + \frac{2im^2\eta_0}{l^-p^-}\frac{1}{\varphi_- + i0}\right),\end{aligned} \tag{2.21}$$

where the shift of the pole at $\varphi_- = 0$ towards the negative imaginary half-plane can be understood by imposing that the Gaussian integral converges [23, 70, 67].

We observe that the structure of the exponential function in the first line of this equation allows for introducing the concept of electron dressed inside a plane wave [49, 119] also when radiation-reaction effects are important. Indeed, the phase-dependent electron square dressed mass $\tilde{m}^2(\varphi_+,\varphi_-)$ can be defined here as [see Eq. (2.21)]

$$\begin{aligned}\tilde{m}^2(\varphi_+,\varphi_-) = m^2\Bigg\{&\frac{1}{\varphi_-}\int_{-\varphi_-/2}^{\varphi_-/2} d\tilde{\varphi}h^2(\varphi_+ + \tilde{\varphi}) \\ &+\frac{1}{\varphi_-}\int_{-\varphi_-/2}^{\varphi_-/2} d\tilde{\varphi}\boldsymbol{\mathcal{F}}_\perp^2(\varphi_+ + \tilde{\varphi}) - \left[\frac{1}{\varphi_-}\int_{-\varphi_-/2}^{\varphi_-/2} d\tilde{\varphi}\,\boldsymbol{\mathcal{F}}_\perp(\varphi_+ + \tilde{\varphi})\right]^2\Bigg\}.\end{aligned} \tag{2.22}$$

This expression generalizes the phase-dependent square electron dressed mass as reported, e.g., in Refs. [49, 119, 104, 61], including radiation-reaction effects.

Equation (2.21) can be explicitly regularized. First, we integrate by parts the term proportional to $[h^2(\varphi_+ + \varphi_-/2) + h^2(\varphi_+ - \varphi_-/2)]/\varphi_-^2$ and we obtain

$$\begin{aligned}\frac{d\mathcal{E}}{dl^-} = &-\frac{i\alpha}{4\pi\eta_0}\frac{l^-}{p^-}\int \frac{d\varphi_+ d\varphi_-}{\varphi_- + i0}\, e^{i\frac{l^-}{2p^-}\frac{\tilde{m}^2(\varphi_+,\varphi_-)}{m^2\eta_0}\varphi_-} \\ &\times\Bigg(\bar{h}_2(\varphi_+,\varphi_-)\left\{1+\frac{m^2}{(p^-)^2}[\boldsymbol{p}_\perp - \langle\boldsymbol{\mathcal{F}}_\perp\rangle(\varphi_+,\varphi_-)]^2 - \frac{m^2}{(p^-)^2}\bar{h}_2(\varphi_+,\varphi_-)\right\} \\ &+\frac{im^2\eta_0}{l^-p^-}\left[h\left(\varphi_+ + \frac{\varphi_-}{2}\right)h'\left(\varphi_+ + \frac{\varphi_-}{2}\right) - h\left(\varphi_+ - \frac{\varphi_-}{2}\right)h'\left(\varphi_+ - \frac{\varphi_-}{2}\right)\right] \\ &-\frac{m^2}{p_{0,-}^2}\bar{h}_2(\varphi_+,\varphi_-)\left[\bar{\boldsymbol{\mathcal{F}}}_{2,\perp}(\varphi_+,\varphi_-) + \langle\boldsymbol{\mathcal{F}}_\perp\rangle^2(\varphi_+,\varphi_-) - 2\bar{\boldsymbol{\mathcal{F}}}_\perp(\varphi_+,\varphi_-)\cdot\langle\boldsymbol{\mathcal{F}}_\perp\rangle(\varphi_+,\varphi_-)\right] \\ &+\frac{1}{2}\left[\boldsymbol{\mathcal{F}}_\perp\left(\varphi_+ + \frac{\varphi_-}{2}\right) - \boldsymbol{\mathcal{F}}_\perp\left(\varphi_+ - \frac{\varphi_-}{2}\right)\right]^2 \\ &\times\left\{1+\frac{m^2}{(p^-)^2}[\boldsymbol{p}_\perp - \langle\boldsymbol{\mathcal{F}}_\perp\rangle(\varphi_+,\varphi_-)]^2 + \frac{2im^2\eta_0}{l^-p^-}\frac{1}{\varphi_-}\right\}\Bigg),\end{aligned} \tag{2.23}$$

where we have introduced the notation

$$\bar{h}_2(\varphi_+,\varphi_-) = \frac{1}{2}\left[h^2\left(\varphi_+ + \frac{\varphi_-}{2}\right) + h^2\left(\varphi_+ - \frac{\varphi_-}{2}\right)\right], \tag{2.24}$$

$$\langle h^2\rangle(\varphi_+,\varphi_-) = \frac{1}{\varphi_-}\int_{-\varphi_-/2}^{\varphi_-/2} d\tilde{\varphi}\, h^2(\varphi_+ + \tilde{\varphi}), \tag{2.25}$$

$$\bar{\boldsymbol{\mathcal{F}}}_\perp(\varphi_+,\varphi_-) = \frac{1}{2}\left[\boldsymbol{\mathcal{F}}_\perp\left(\varphi_+ + \frac{\varphi_-}{2}\right) + \boldsymbol{\mathcal{F}}_\perp\left(\varphi_+ - \frac{\varphi_-}{2}\right)\right], \tag{2.26}$$

$$\bar{\boldsymbol{\mathcal{F}}}_{2,\perp}(\varphi_+,\varphi_-) = \frac{1}{2}\left[\boldsymbol{\mathcal{F}}_\perp^2\left(\varphi_+ + \frac{\varphi_-}{2}\right) + \boldsymbol{\mathcal{F}}_\perp^2\left(\varphi_+ - \frac{\varphi_-}{2}\right)\right], \tag{2.27}$$

$$\langle\boldsymbol{\mathcal{F}}_\perp\rangle(\varphi_+,\varphi_-) = \frac{1}{\varphi_-}\int_{-\varphi_-/2}^{\varphi_-/2} d\tilde{\varphi}\,\boldsymbol{\mathcal{F}}_\perp(\varphi_+ + \tilde{\varphi}), \tag{2.28}$$

$$\langle\boldsymbol{\mathcal{F}}_\perp^2\rangle(\varphi_+,\varphi_-) = \frac{1}{\varphi_-}\int_{-\varphi_-/2}^{\varphi_-/2} d\tilde{\varphi}\,\boldsymbol{\mathcal{F}}_\perp^2(\varphi_+ + \tilde{\varphi}). \tag{2.29}$$

Note that with these definitions, the square of the electron dressed mass can be simply written as

$$\tilde{m}^2(\varphi_+,\varphi_-) = m^2\left[\langle h^2\rangle(\varphi_+,\varphi_-) + \langle \boldsymbol{\mathcal{F}}_\perp^2\rangle(\varphi_+,\varphi_-) - \langle \boldsymbol{\mathcal{F}}_\perp\rangle^2(\varphi_+,\varphi_-)\right]. \tag{2.30}$$

At this point only the terms in the second line of Eq. (2.23) need an explicit regularization. In the absence of radiation reaction, this is achieved by imposing that the emission spectrum has to vanish in the absence of the external field [23, 70, 67]. Here, due to the effect of radiation reaction, we need a slightly more complicated regularization procedure. To this end, we introduce the function

$$H_2(\varphi_+,\varphi_-) = \varphi_-\langle h^2\rangle(\varphi_+,\varphi_-) = \int_{-\varphi_-/2}^{\varphi_-/2} d\tilde{\varphi}\, h^2(\varphi_+ + \tilde{\varphi}) \tag{2.31}$$

and notice that

$$\frac{\partial H_2(\varphi_+,\varphi_-)}{\partial\varphi_-} = \bar{h}_2(\varphi_+,\varphi_-) > 0 \tag{2.32}$$

for any φ_+. Now, for any positive real number a, it is

$$\int_{-\infty}^{\infty}\frac{dH_2}{H_2+i0}e^{iaH_2} = 0. \tag{2.33}$$

We have indicated the integration variable as H_2 here because, by exploiting the result in Eq. (2.32), we change variable to φ_- and obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{d\varphi_-}{H_2(\varphi_+,\varphi_-)+i0}\frac{\partial H_2(\varphi_+,\varphi_-)}{\partial\varphi_-}e^{iaH_2(\varphi_+,\varphi_-)}\\ &\quad=\int_{-\infty}^{\infty}\frac{d\varphi_-}{H_2(\varphi_+,\varphi_-)+i0}\bar{h}_2(\varphi_+,\varphi_-)e^{iaH_2(\varphi_+,\varphi_-)} = 0.\end{aligned} \tag{2.34}$$

This result shows that we can formally regularize the remaining terms of Eq. (2.23) by subtracting the vanishing quantity

$$\begin{aligned}&h^2(\varphi_+)\left[1+\frac{m^2}{(p^-)^2}\{[\boldsymbol{p}_\perp - \boldsymbol{\mathcal{F}}_\perp(\varphi_+)]^2 - h^2(\varphi_+)\}\right]\\ &\quad\times\int_{-\infty}^{\infty}\frac{d\varphi_-}{H_2(\varphi_+,\varphi_-)+i0}\bar{h}_2(\varphi_+,\varphi_-)e^{i\frac{l^-}{2p^-\eta_0}H_2(\varphi_+,\varphi_-)}\end{aligned} \tag{2.35}$$

inside the integral in φ_+. As it will be clear below, the additional front factor $h^2(\varphi_+)$ is included because for $|\varphi_-| \ll 1$ it is $H_2(\varphi_+,\varphi_-) \approx h^2(\varphi_+)\varphi_-$. The resulting regularized expression of the energy spectrum reads

$$
\begin{aligned}
\frac{d\mathcal{E}}{dl^-} &= -\frac{i\alpha}{4\pi\eta_0}\frac{l^-}{p^-}\int d\varphi_+ d\varphi_-\, e^{i\frac{l^-}{2p^-}\frac{\tilde{m}^2(\varphi_+,\varphi_-)}{m^2\eta_0}\varphi_-} \\
&\times \Bigg(\bar{h}_2(\varphi_+,\varphi_-)\Bigg\{\frac{1+\frac{m^2}{(p^-)^2}[\boldsymbol{p}_\perp - \langle\boldsymbol{\mathcal{F}}_\perp\rangle(\varphi_+,\varphi_-)]^2 - \frac{m^2}{(p^-)^2}\bar{h}_2(\varphi_+,\varphi_-)}{\varphi_-} \\
&- \frac{1+\frac{m^2}{(p^-)^2}[\boldsymbol{p}_\perp-\boldsymbol{\mathcal{F}}_\perp(\varphi_+)]^2 - \frac{m^2}{(p^-)^2}h^2(\varphi_+)}{H_2(\varphi_+,\varphi_-)/h^2(\varphi_+)}\, e^{-i\frac{l^-}{2p^-\eta_0}\varphi_-\left[\langle\boldsymbol{\mathcal{F}}^2_\perp\rangle(\varphi_+,\varphi_-)-\langle\boldsymbol{\mathcal{F}}_\perp\rangle^2(\varphi_+,\varphi_-)\right]}\Bigg\} \\
&+ \frac{im^2\eta_0}{l^-p^-}\frac{1}{\varphi_-}\left[h\left(\varphi_+ + \frac{\varphi_-}{2}\right)h'\left(\varphi_+ + \frac{\varphi_-}{2}\right) - h\left(\varphi_+ - \frac{\varphi_-}{2}\right)h'\left(\varphi_+ - \frac{\varphi_-}{2}\right)\right] \\
&- \frac{m^2}{p^2_{0,-}}\frac{\bar{h}_2(\varphi_+,\varphi_-)}{\varphi_-}\left[\bar{\boldsymbol{\mathcal{F}}}_{2,\perp}(\varphi_+,\varphi_-) + \langle\boldsymbol{\mathcal{F}}_\perp\rangle^2(\varphi_+,\varphi_-) - 2\bar{\boldsymbol{\mathcal{F}}}_\perp(\varphi_+,\varphi_-)\cdot\langle\boldsymbol{\mathcal{F}}_\perp\rangle(\varphi_+,\varphi_-)\right] \\
&+ \frac{1}{2\varphi_-}\left[\boldsymbol{\mathcal{F}}_\perp\left(\varphi_+ + \frac{\varphi_-}{2}\right) - \boldsymbol{\mathcal{F}}_\perp\left(\varphi_+ - \frac{\varphi_-}{2}\right)\right]^2 \\
&\times \left\{1 + \frac{m^2}{(p^-)^2}[\boldsymbol{p}_\perp - \langle\boldsymbol{\mathcal{F}}_\perp\rangle(\varphi_+,\varphi_-)]^2 + \frac{2im^2\eta_0}{l^-p^-}\frac{1}{\varphi_-}\right\}\Bigg),
\end{aligned}
\tag{2.36}
$$

where we have removed the now unnecessary shift $+i0$ of the pole. Notice that the above regularization prescription reduces to the known one in the absence of radiation reaction, which guarantees that the energy spectrum $d\mathcal{E}/dl^-$ vanishes if the external plane wave vanishes.

2.3 The Locally Constant Field Approximation (LCFA)

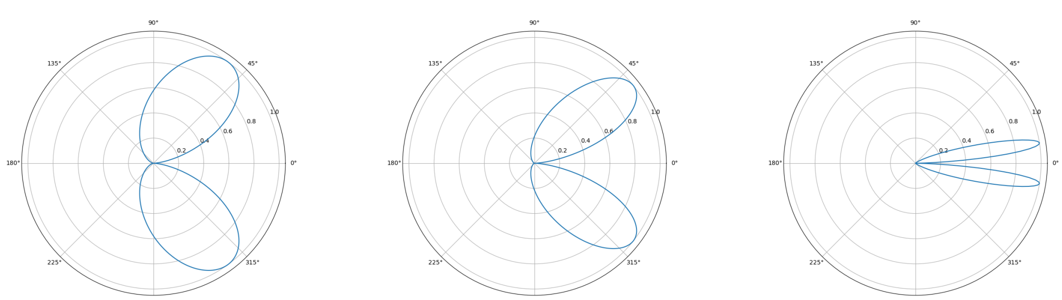


Figure 2.2: Plot of the power emitted by a charge in linear motion, properly normalized. From left to right we have considered velocities equal (in natural units) to $0.3, 0.5, 0.95$, respectively.

The power radiated by an accelerated charge is easily calculated through the expression of the Liénard-Wiechert potential and the definition of Poynting vector (see, e.g. [115])

$$\frac{dP}{d\Omega} = \frac{\alpha}{4\pi} \frac{|\boldsymbol{n}' \times [(\boldsymbol{n}' - \boldsymbol{v}) \times \boldsymbol{a}]|^2}{(1 - \boldsymbol{v} \cdot \boldsymbol{n}')^5}. \tag{2.37}$$

Now, if we consider for simplicity the case in which velocity and acceleration are parallel and along $\boldsymbol{z}$, this expression takes the simple form

$$\frac{dP}{d\Omega} = \frac{\alpha a^2}{4\pi} \frac{\sin^2 \vartheta}{(1 - v\cos\vartheta)^5}, \tag{2.38}$$

where as before ϑ is the angle measured from the $\boldsymbol{z}$ axis. Looking at the denominator of the r.h.s. we clearly see that for high velocities $v \sim 1$ the peak of the emitted power occurs at small angles (see Fig. 2.2). To be more precise, a simple derivative shows that the angle at which the emission is maximal is

$$\vartheta_{max} = \arccos\left\{\frac{1}{3v}\left[\sqrt{1 + 15v^2} - 1\right]\right\} \overset{v \sim 1}{\approx} \frac{1}{2\gamma}. \tag{2.39}$$

This is a very general future of electromagnetic radiation: for relativistic particles it is pinned in a cone with an opening angle of the order of $\frac{1}{\gamma}$. It can also be seen as a simple consequence of Lorentz contraction, the faster is the particle the more its Liénard-Wiechert field look like an electromagnetic wave propagating along the particle velocity. This fact can be extremely useful in order to efficiently approximate spectra formulas, resulting in very compact expressions which can be easily plotted.

Indeed, let us consider a particle moving along a trajectory with a large deflection angle compared to the emission cone angle $\frac{1}{\gamma}$. Under this hypothesis, the portion of trajectory responsible for the emission along a chosen direction $\boldsymbol{n}'$ is very small and of the order of

$$l_f = \frac{R}{\gamma}, \tag{2.40}$$

where R is the curvature radius of the particle trajectory and l_f is known as formation length [23](see Fig. 2.3). If the formation length associated to a system is very small, then we can assume as a good approximation the background field to be constant over the scale defined by l_f. This means that the emission spectrum, under this approximation, will not depend anymore on the value of the external field but just on the local values of velocity and acceleration. In other words, we can adopt a Locally Constant Field Approximation (LCFA). One can prove that for plane wave backgrounds, applying the LCFA corresponds to keeping only the third order in φ_- in the phase and the first order in the pre-exponential terms of the spectrum formula Eq. (2.13), once it is expressed in terms of $\varphi_\pm$ (see, e.g., [23]). Indeed, we can interpret φ_- as the coordinate spanning a length corresponding to a trajectory segment of the order of the formation length. In this interpretation, the integral over φ_+ corresponds to the evolution parameter and at any fixed φ_+, the integration over φ_- represents the contributions in a formation length around φ_+. For larger values of φ_- the radiation is suppressed by destructive interference [66].

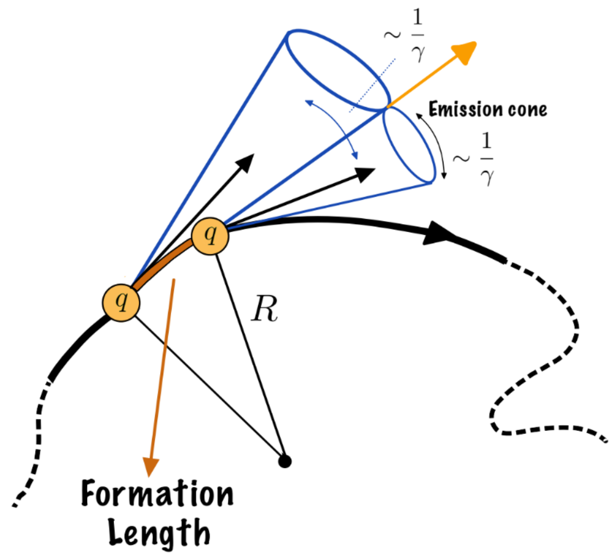


Figure 2.3: Formation length for charge moving in an electromagnetic field.

2.4 The emission spectrum within the LCFA

In order to implement the LCFA, we use the same strategy as in Ref. [67] by expanding Eqs. (2.18) and (2.21) for small values of $|\varphi_-|$ [recall that within the LCFA the problematic term proportional to $1/(\varphi_- + i0)$ can be integrated analytically, see, e.g., Refs. [23, 67], whereas it is easier to perform the integration by parts of the terms proportional to $1/(\varphi_- + i0)^2$ after the expansion for $|\varphi_-| \ll 1$].

It is interesting to notice that the regime where the LCFA applies well overlaps with the regime where classical radiation-reaction effects are large (CRDR). In fact, the LCFA is typically applicable at large values of the classical nonlinearity parameter $\xi_0 = |e|E_0/m\omega$ [157, 23, 66], where E_0 is the amplitude of the external plane wave (see Refs. [22, 118, 65, 181, 103, 70, 67, 40, 16, 68, 113, 150, 112, 152] for investigations about the limitations of the LCFA). Moreover, in the realm of classical electrodynamics one has to assume that the quantum nonlinearity parameter $\chi_0 = \eta_0\xi_0$ is much smaller than unity [157, 23, 66]. Under these conditions the LCFA is expected to be very accurate except possibly for extremely small emitted radiation frequencies, which we do not consider here [67, 68, 113]. This indeed well overlaps with the regime where classical radiation-reaction effects are typically large because, apart from long laser pulses, radiation-reaction effects become large for $\xi_0 \gg 1$ [see Eq. (2.5)] but still with $\chi_0 \ll 1$, to be able to neglect quantum corrections.

Under the above assumptions, as discussed in Sec. 2.3, one has to expand the phases in Eqs. (2.18) and (2.21) up to the third order in φ_-, whereas the leading-order expansion is sufficient for the pre-exponential functions. The resulting angularly-resolved and angularly-integrated energy spectra within the LCFA can be written as [see S.m. 2.6.3 for details on the more involved derivation of Eq. (2.42)]

$$\begin{aligned}\frac{d\mathcal{E}_{\text{LCFA}}}{d\boldsymbol{l}} &= \frac{\alpha}{\sqrt{3}\pi^2}\frac{1}{m^2\eta_0}\int d\varphi_+\frac{h^2(\varphi_+)}{\chi(\varphi_+)}\sqrt{1+\hat{\boldsymbol{\pi}}_\perp^2(\varphi_+)}\\ &\quad\times\left[1+2\hat{\boldsymbol{\pi}}_\perp^2(\varphi_+)\right]\mathrm{K}_{1/3}\left(\frac{2}{3}\frac{l^-}{p^-}\frac{h^2(\varphi_+)}{\chi(\varphi_+)}\left[1+\hat{\boldsymbol{\pi}}_\perp^2(\varphi_+)\right]^{3/2}\right),\end{aligned}\tag{2.41}$$

$$\begin{aligned}\frac{d\mathcal{E}_{\text{LCFA}}}{dl^-} &= \frac{2\alpha}{\sqrt{3}\pi}\frac{l^-}{p^-}\int d\varphi_+\frac{\varepsilon(\varphi_+)}{\pi^-(\varphi_+)}\frac{h^2(\varphi_+)}{\eta_0}\\ &\quad\times\left[\mathrm{K}_{2/3}\left(\frac{2}{3}\frac{l^-}{p^-}\frac{h^2(\varphi_+)}{\chi(\varphi_+)}\right)-\frac{1}{2}\mathrm{IK}_{1/3}\left(\frac{2}{3}\frac{l^-}{p^-}\frac{h^2(\varphi_+)}{\chi(\varphi_+)}\right)\right].\end{aligned}\tag{2.42}$$

Here, we have introduced the local quantum nonlinearity parameter $\chi(\varphi) = \eta_0|\boldsymbol{\xi}(\varphi)|$ (this equality holds in our units where $\hbar = 1$ and it is easily checked that the above formulas do not explicitly contain $\hbar$), the modified Bessel function $\mathrm{K}_\nu(z)$ of order ν [142] and the function

$$\mathrm{IK}_\nu(z) = \int_z^\infty dz'\mathrm{K}_\nu(z').\tag{2.43}$$

As expected from the very meaning of the LCFA, the above Eqs. (2.41)-(2.42) can be obtained from the corresponding expressions in the absence of radiation reaction by replacing the components of the electron four-momentum obtained from solving the Lorentz equation in the plane wave with the corresponding expressions obtained from solving the LL equation [see Eqs. (2.8)-(2.12)]. In particular, one can find that in the absence of radiation reaction Eq. (2.42) has exactly the same form as the classical limit of the quantum energy emitted spectrum as computed in Ref. [157]. However, we point out that the quantities $d\mathcal{E}_{\text{LCFA}}/d\boldsymbol{l}$ and $d\mathcal{E}_{\text{LCFA}}/dl^-$ are not local in φ_+ because both the function $h(\varphi)$ [see Eq. (2.5)] and the function $\boldsymbol{\mathcal{F}}_\perp(\varphi)$ [see the definitions below Eqs. (2.6) and (2.10)] are not local in the laser phase. This is also expected from the physical meaning of radiation reaction, with one of the main physical consequences being the accumulation effects of energy-momentum loss.

As an additional remark, we notice that by taking the integral of Eq. (2.42) in dl^- one obtains that the total energy radiated is given by

$$\mathcal{E}_{\text{LCFA}} = \frac{2}{3}\alpha\eta_0\int d\varphi_+\frac{\varepsilon(\varphi_+)}{h(\varphi_+)}\boldsymbol{\xi}_\perp^2(\varphi_+).\tag{2.44}$$

This result coincides with the total energy radiated also *beyond* the LCFA, a curious circumstance, which also occurs in the absence of radiation reaction [157].

Interestingly, the total minus component

$$\frac{d\mathcal{K}_-}{d\varphi_+} = \int_0^\infty dl^-\int d\boldsymbol{l}_\perp\frac{l^-}{\omega'}\frac{d\mathcal{E}}{dl^-d\boldsymbol{l}_\perp d\varphi_+} = \int_0^\infty dl^-\int d\boldsymbol{l}_\perp\frac{d\mathcal{E}}{d\boldsymbol{l}d\varphi_+}\tag{2.45}$$

of the four-momentum radiated classically per unit of laser phase by an electron in a plane wave including radiation reaction has been recently computed within the LCFA in Ref. [105] in the different context of the so-called Ritus-Narozhny conjecture on strong-field QED [156, 134, 135, 132, 15, 82]. According to Eq. (2.45), by defining $d\mathcal{E}_{\text{LCFA}}/d\boldsymbol{l}d\varphi_+$ as the integrand in Eq. (2.41), and by performing the integral of this quantity over $d\boldsymbol{l}_\perp$ one can easily show that

$$\frac{d\mathcal{K}_{-,\mathrm{LCFA}}}{d\varphi_+} = \frac{2\alpha}{\sqrt{3}\pi}\int_0^\infty dl^- \frac{l^-}{p^-}\frac{h^2(\varphi_+)}{\eta_0}\left[\mathrm{K}_{2/3}\left(\frac{2}{3}\frac{l^-}{p^-}\frac{h^2(\varphi_+)}{\chi(\varphi_+)}\right) - \frac{1}{2}\mathrm{IK}_{1/3}\left(\frac{2}{3}\frac{l^-}{p^-}\frac{h^2(\varphi_+)}{\chi(\varphi_+)}\right)\right], \tag{2.46}$$

in agreement with the result in Ref. [105].

2.5 Discussion

To summarize, we have derived analytically the angularly-resolved and the angularly-integrated energy emission spectra of nonlinear Thomson scattering by including radiation-reaction effects. This has been accomplished by starting from the analytical solution of the LL in an arbitrary plane wave and by using the classical formulas of radiation by accelerated charges.

The spectra are obtained as double integrals over the plane-wave phase. A particular, new regularization technique has to be used in order to regularize the angularly-integrated spectrum. We point out that the resulting spectra include higher-order classical radiative corrections and can be considered as “classically exact” in the sense of the Landau and Lifshitz reduction of order, meaning that neglected classical corrections are much smaller than quantum corrections, which have been of course ignored from the beginning.

Moreover, we have obtained a phase-dependent expression of the electron dressed mass, which includes radiation-reaction effects.

Finally, the expressions of the angularly-resolved and the angularly integrated spectra within the locally-constant field approximations have been derived as well. These expressions have the property that are expressed as single integrals over the laser phase of the corresponding expressions without radiation reaction with the electron four-momentum replaced with its expression including radiation reaction. Thus, they turn out to be non-local exactly for the nature itself of radiation reaction giving rise to cumulative energy-momentum loss effects.

We will now get more into the details of Eq. (2.42) in order to size the RR effects on the emission spectrum. Let us consider a setup in which the electron is initially counterpropagating with respect to the wave, such that $\boldsymbol{p}_\perp = \mathbf{0}$. We can explicitly substitute the momentum components Eqs. (2.12), (2.8) into the LCFA spectrum to obtain

$$\frac{d\mathcal{E}_{\mathrm{LCFA}}}{dl^-}(x) = \frac{\alpha}{2\sqrt{3}\pi\eta_0}x\int d\varphi_+\left[\frac{h^2(\varphi_+) + \boldsymbol{\mathcal{F}}_\perp(\varphi_+)^2}{(u_0^-)^2} + 1\right]h^2(\varphi_+)\mathrm{IK}_{5/3}\left(\frac{2}{3}\frac{h^2(\varphi_+)}{\chi(\varphi_+)}x\right), \tag{2.47}$$

where we introduced the dimensionless parameter $x = \frac{l^-}{p^-}$ and used the following relation between Bessel functions [see, e.g., [96] pag. 929 Eq. (11)]

$$2\mathrm{K}_\nu(x) = \mathrm{IK}_{\nu+1}(x) + \mathrm{IK}_{\nu-1}(x). \tag{2.48}$$

For an ultrarelativistic electron in head-on collision with the wave $p^- \sim 2\varepsilon_0$, where ε_0 is the electron initial energy. Under these hypothesis we can thus write the total radiated energy Eq. (2.44) as

$$\mathcal{E}_{\text{LCFA}} \simeq \frac{2}{3}\varepsilon_0 R \int d\varphi_+ \left[\frac{h^2(\varphi_+) + \boldsymbol{\mathcal{F}}_\perp(\varphi_+)^2}{(u_0^-)^2} + 1\right] \frac{|\boldsymbol{\psi}|^2(\varphi_+)}{h^2(\varphi_+)}. \tag{2.49}$$

For strong plane waves $\xi_0 \gg 1$, what really makes the difference, when RR effects are included, is the denominator h^2. Indeed for strong fields $|\boldsymbol{\mathcal{F}}_\perp| \gg h$ but the factor h^2 in the denominator can differ significantly from one.

The parameter space region we are interested in here is the CRDR, defined by the requirements $R \sim 1$ and $\chi_0 \ll 1$. Indeed, we want RR effects to be enhanced but at the same time avoid any quantum effects, otherwise our classical description here presented would be useless. Recalling that for a plane wave $R = \alpha\xi_0\chi_0$ and assuming $\chi_0 \sim 0.1$, we can choose a field strength of the order $\xi_0 \sim 10^3$ so that $R \lesssim 1$. Interestingly, the emission spectra for different values of R in this conditions are clearly distinguishable (Fig. 2.4). Moreover, the total emitted energy for the largest value of R considered in Fig. 2.4 is about four times smaller than the energy emitted in the RR-free case. One could sensibly argue that at $\chi_0 \sim 0.1$ quantum effects cannot be neglected [126]. However, even for values of χ_0 one order of magnitude smaller, the distinction is neatly present (Fig. 2.5). As of today, there are not really convincing experimental results regarding classical RR effects from pure wave-electron interaction and this is due to the technical difficulty of such a precise measurement. However, we think that the analytical study of this phenomenon in more realistic laser configurations could be fruitful for future verifications of the validity of the LL equation.

As a side note, It can be instructive to examine the asymptotic limit of the spectrum for small values of x. This can be easily obtained recalling that for small x

$$\text{IK}_\nu(x) \sim \Gamma(\nu - 1)\left(\frac{2}{x}\right)^{\nu-1}. \tag{2.50}$$

It then follows that $\frac{d\mathcal{E}_{\text{LCFA}}}{dl^-}(x) \overset{x \ll 1}{\simeq} C x^{\frac{1}{3}}$, with C given by

$$C = \alpha\left(\frac{\xi_0^2}{\eta_0}\right)^{\frac{1}{3}} \frac{3^{\frac{1}{6}}\Gamma(\frac{2}{3})}{2\pi} \int d\varphi_+ \left[\frac{h^2(\varphi_+) + \boldsymbol{\mathcal{F}}_\perp(\varphi_+)^2}{(u_0^-)^2} + 1\right] \left[h(\varphi_+)\,|\boldsymbol{\psi}|(\varphi_+)\right]^{\frac{2}{3}}. \tag{2.51}$$

We see that when the electron momentum is small compared to the emitted one, the dependence is generally cubic as in absence of RR.

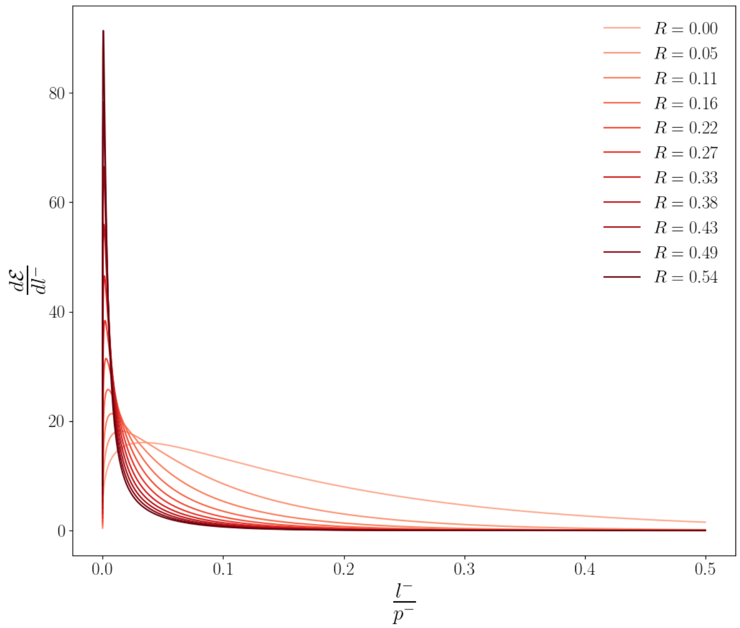


Figure 2.4: Nonlinear Thomson emission energy spectrum in the LCFA approximation for $\chi_0 = 0.11$ and different values of R.

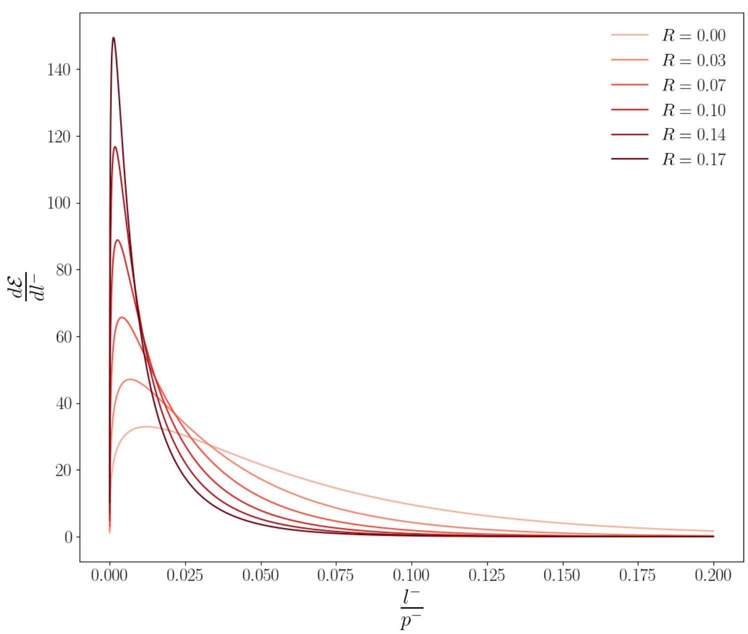


Figure 2.5: Nonlinear Thomson emission energy spectrum in the LCFA approximation for $\chi_0 = 0.036$ and different values of R.

2.6 Support material

2.6.1 Derivation of Eq. (2.13)

Let us show how to get Eq. (2.13). First we focus on the phase:

$$\Delta\Phi(\varphi,\varphi') = \int_{\varphi'}^{\varphi} d\tilde{\varphi} \frac{l\cdot\pi(\tilde{\varphi})}{k\cdot\pi(\tilde{\varphi})}. \tag{2.52}$$

From Eq. (2.4) we have $k\cdot\pi(\tilde{\varphi}) = \omega\frac{p^-}{h(\varphi)}$, moreover we can expand the numerator in light-cone coordinates

$$l\cdot\pi = l^-\pi^+ + l^+\pi^- - \boldsymbol{l}_\perp\cdot\boldsymbol{\pi}_\perp \tag{2.53}$$

and replace all the " + " terms through the on-shell relation $\pi^+ = \frac{\boldsymbol{\pi}_\perp^2+m^2}{2\pi^-}$, such that

$$l\cdot\pi = l^-\frac{\boldsymbol{\pi}_\perp^2+m^2}{2\pi^-} + \frac{\boldsymbol{l}_\perp^2}{2l^-}\pi^- - \boldsymbol{l}_\perp\cdot\boldsymbol{\pi}_\perp = \frac{l^-}{2\pi^-}\left\{m^2 + \left[\boldsymbol{\pi}_\perp - \frac{p_-}{l^-}\boldsymbol{l}_\perp\right]^2\right\} = \frac{l^-m^2}{2\pi^-}\left(1+\hat{\boldsymbol{\pi}}_\perp^2\right). \tag{2.54}$$

Now, the point here is that expanding in light-cone coordinates one can easily show that this product is quadratic in $\boldsymbol{l}_\perp$ (this is true in general, due to the light-cone expansion), our aim was to obtain a manifestly Gaussian integral. It is easy to see that the phase can be written as

$$\Delta\Phi(\varphi,\varphi') = \int_{\varphi'}^{\varphi} d\tilde{\varphi}\frac{h(\tilde{\varphi})}{\omega p^-}\frac{l^-m^2}{2\pi^-(\tilde{\varphi})}\left(1+\hat{\boldsymbol{\pi}}_\perp^2(\tilde{\varphi})\right) = \frac{l^-}{2p^-\eta_0}\int_{\varphi'}^{\varphi} d\tilde{\varphi}\, h^2(\tilde{\varphi})\left(1+\hat{\boldsymbol{\pi}}_\perp^2(\tilde{\varphi})\right). \tag{2.55}$$

Now we have dealt with the phase, let us focus on the prefactor of Eq. (2.13). Again we expand in light-cone coordinates and collect the squares of perpendicular components:

$$\begin{aligned}
&\frac{\pi(\varphi)\cdot\pi(\varphi')}{(k\cdot\pi(\varphi))(k\cdot\pi(\varphi'))} = \\
&\quad= \frac{h(\varphi)h(\varphi')}{\omega^2(p^-)^2}\left[\pi^+(\varphi)\pi^-(\varphi') + \pi^-(\varphi)\pi^+(\varphi') - \boldsymbol{\pi}_\perp(\varphi)\cdot\boldsymbol{\pi}_\perp(\varphi')\right] = \\
&\quad= \frac{h(\varphi)h(\varphi')}{\omega^2(p^-)^2}\left[\pi^-(\varphi')\frac{\boldsymbol{\pi}_\perp^2(\varphi)+m^2}{2\pi^-(\varphi)} + \pi^-(\varphi)\frac{\boldsymbol{\pi}_\perp^2(\varphi')+m^2}{2\pi^-(\varphi')} - \boldsymbol{\pi}_\perp(\varphi)\cdot\boldsymbol{\pi}_\perp(\varphi')\right] = \\
&\quad= \frac{h(\varphi)h(\varphi')}{2\omega^2(p^-)^2}\left[m^2\left(\frac{\pi^-(\varphi')}{\pi^-(\varphi)} + \frac{\pi^-(\varphi)}{\pi^-(\varphi')}\right) + \left(\sqrt{\frac{\pi^-(\varphi')}{\pi^-(\varphi)}}\boldsymbol{\pi}_\perp(\varphi) - \sqrt{\frac{\pi^-(\varphi)}{\pi^-(\varphi')}}\boldsymbol{\pi}_\perp(\varphi')\right)^2\right] = \\
&\quad= \frac{1}{2\omega^2(p^-)^2}\left[m^2\left(h^2(\varphi)+h^2(\varphi')\right) + \left(h(\varphi)\boldsymbol{\pi}_\perp(\varphi) - h(\varphi')\boldsymbol{\pi}_\perp(\varphi')\right)^2\right] = \\
&\quad= \frac{m^2}{2\omega^2(p^-)^2}\left[h^2(\varphi)+h^2(\varphi') + \left(\boldsymbol{\mathcal{F}}_\perp(\varphi) - \boldsymbol{\mathcal{F}}_\perp(\varphi')\right)^2\right].
\end{aligned} \tag{2.56}$$

Where in the last line we made use of Eq. (2.9). Using these two results, Eq. (2.18) is recovered.

2.6.2 Derivation of Eq. (2.21)

Let us show step by step how to obtain Eq. (2.21) starting from Eq. (2.20). First we observe that

$$\begin{aligned}\frac{d\mathcal{E}}{dl^-} &= -\frac{\alpha}{8\pi^2 m^2\eta_0^2}\int d\boldsymbol{l}_\perp \frac{\omega'}{l^-}\int d\varphi d\varphi' \left\{h^2(\varphi)+h^2(\varphi')+[\boldsymbol{\mathcal{F}}_\perp(\varphi)-\boldsymbol{\mathcal{F}}_\perp(\varphi')]^2\right\} e^{i\Delta\Phi(\varphi,\varphi')} = \\ &= -\frac{\alpha}{16\pi^2 m^2\eta_0^2}\int d\varphi d\varphi' d\boldsymbol{l}_\perp \left[1+\frac{\boldsymbol{l}_\perp^2}{(l^-)^2}\right] \left\{h^2(\varphi)+h^2(\varphi')+[\boldsymbol{\mathcal{F}}_\perp(\varphi)-\boldsymbol{\mathcal{F}}_\perp(\varphi')]^2\right\} e^{i\Delta\Phi(\varphi,\varphi')}.\end{aligned} \tag{2.57}$$

Now we recall that the phase has the form $\Delta\Phi(\varphi,\varphi') = \frac{l^-}{2p^-\eta_0}\int_{\varphi'}^{\varphi} d\tilde{\varphi}\, h^2(\tilde{\varphi})\left(1+\hat{\boldsymbol{\pi}}_\perp^2(\tilde{\varphi})\right)$ and it is quadratic in $\boldsymbol{l}_\perp$ (see Eq. (2.19)). We can thus write

$$h(\varphi)\hat{\boldsymbol{\pi}}_\perp = \boldsymbol{C} + D\boldsymbol{l}_\perp \qquad \text{with} \quad \boldsymbol{C} = \frac{\boldsymbol{p}_\perp - m\boldsymbol{\mathcal{F}}_\perp(\varphi)}{m} \quad , \quad D = -\frac{p^-}{ml^-}, \tag{2.58}$$

so that the phase can be written as

$$\begin{aligned}\Delta\Phi(\varphi,\varphi') &= \frac{l^-}{2p^-\eta_0}\int_{\varphi'}^{\varphi} d\tilde{\varphi}\,\left(h^2(\tilde{\varphi}) + h^2(\tilde{\varphi})\hat{\boldsymbol{\pi}}_\perp^2(\tilde{\varphi})\right) = \\ &= \frac{l^-}{2p^-\eta_0}\int_{\varphi'}^{\varphi} d\tilde{\varphi}\,\left(h^2(\tilde{\varphi}) + \boldsymbol{C}^2 + 2D\boldsymbol{C}\cdot\boldsymbol{l}_\perp + D^2\boldsymbol{l}_\perp^2)\right) = \\ &= \frac{l^-}{2p^-\eta_0}\int_{\varphi'}^{\varphi} d\tilde{\varphi}\,\left(h^2(\tilde{\varphi}) + \boldsymbol{C}^2\right) + \frac{l^- D^2}{2p^-\eta_0}\int_{\varphi'}^{\varphi} d\tilde{\varphi}\,(2\frac{\boldsymbol{C}\cdot\boldsymbol{l}_\perp}{D} + \boldsymbol{l}_\perp^2) = \\ &= \frac{l^-}{2p^-\eta_0}\int_{\varphi'}^{\varphi} d\tilde{\varphi}\,\left(h^2(\tilde{\varphi}) + \boldsymbol{C}^2\right) + \frac{l^- D^2}{2p^-\eta_0}(\varphi-\varphi')\left[\boldsymbol{l}_\perp^2 + \frac{1}{\varphi-\varphi'}\int_{\varphi'}^{\varphi} d\tilde{\varphi}\, 2\frac{\boldsymbol{C}\cdot\boldsymbol{l}_\perp}{D}\right] = \\ &= \frac{l^-}{2p^-\eta_0}\int_{\varphi'}^{\varphi} d\tilde{\varphi}\,\left(h^2(\tilde{\varphi}) + \boldsymbol{C}^2\right) + \frac{l^- D^2}{2p^-\eta_0}(\varphi-\varphi')\left[\boldsymbol{l}_\perp + \frac{1}{(\varphi-\varphi')D}\int_{\varphi'}^{\varphi} d\tilde{\varphi}\,\boldsymbol{C}\right]^2 - \\ &\quad - \frac{l^-}{2p^-\eta_0}\frac{1}{\varphi-\varphi'}\left[\int_{\varphi'}^{\varphi} d\tilde{\varphi}\,\boldsymbol{C}\right]^2.\end{aligned} \tag{2.59}$$

It is now natural to shift the perpendicular momentum and perform a Gaussian integral. Therefore, we shift the integration variable

$$\boldsymbol{l}_\perp \to \boldsymbol{l}'_\perp = \boldsymbol{l}_\perp + \frac{1}{(\varphi-\varphi')D}\int_{\varphi'}^{\varphi} d\tilde{\varphi}\,\boldsymbol{C} \tag{2.60}$$

and Eq. (2.57) becomes

$$\begin{aligned}\frac{d\mathcal{E}}{dl^-} =& -\frac{\alpha}{16\pi^2 m^2\eta_0^2}\int d\varphi d\varphi' d\boldsymbol{l}_\perp \left[1+\frac{1}{(l^-)^2}\left(\boldsymbol{l}_\perp - \frac{1}{(\varphi-\varphi')D}\int_{\varphi'}^{\varphi} d\tilde{\varphi}\,\boldsymbol{C}\right)^2\right] \\ &\times \left\{h^2(\varphi)+h^2(\varphi')+[\boldsymbol{\mathcal{F}}_\perp(\varphi)-\boldsymbol{\mathcal{F}}_\perp(\varphi')]^2\right\} \\ &\times \exp i\left\{\frac{l^-}{2p^-\eta_0}\int_{\varphi'}^{\varphi} d\tilde{\varphi}\,\left(h^2(\tilde{\varphi}) + \boldsymbol{C}^2\right) + \frac{p^-}{2m^2 l^-\eta_0}(\varphi-\varphi')\boldsymbol{l}_\perp^2 - \frac{l^-}{2p^-\eta_0}\frac{1}{\varphi-\varphi'}\left[\int_{\varphi'}^{\varphi} d\tilde{\varphi}\,\boldsymbol{C}\right]^2\right\}.\end{aligned} \tag{2.61}$$

One can now easily integrate over the perpendicular momentum $\boldsymbol{l}_\perp$, expressing the result in terms of the variables $\varphi_\pm$ Eq. (2.21) is recovered.

2.6.3 Derivation of Eq. (2.42)

We start from Eq. (2.21). In order to implement the LCFA, we expand each term of the pre-exponent up to the leading order for $|\varphi_-| \ll 1$, whereas we keep terms up to φ_-^3 in the phase (see, e.g., [67]):

$$\begin{aligned}\frac{d\mathcal{E}_{\text{LCFA}}}{dl^-} &= -\frac{i\alpha}{4\pi\eta_0}\frac{l^-}{p^-}\int d\varphi_+\, h^2(\varphi_+)\int \frac{d\varphi_-}{\varphi_- + i0}\, e^{i\frac{l^-}{2p^-\eta_0}h^2(\varphi_+)\varphi_-\left[1+\frac{1}{12}\boldsymbol{\xi}'^2_\perp(\varphi_+)\varphi_-^2\right]} \\ &\quad\times\left[1+\frac{1}{2}\boldsymbol{\xi}'^2_\perp(\varphi_+)\varphi_-^2\right]\left\{1+\frac{m^2}{(p^-)^2}\left[\frac{\boldsymbol{p}_\perp}{m}-\boldsymbol{\mathcal{F}}_\perp(\varphi_+)\right]^2+\frac{2im^2\eta_0}{l^-p^-}\frac{1}{\varphi_-+i0}\right\}.\end{aligned} \tag{2.62}$$

Now, we integrate by parts the only term containing $1/(\varphi_- + i0)^2$ in the pre-exponent and we obtain

$$\begin{aligned}\frac{d\mathcal{E}_{\text{LCFA}}}{dl^-} &= -\frac{i\alpha}{4\pi\eta_0}\frac{l^-}{p^-}\int d\varphi_+\, h^2(\varphi_+)\int \frac{d\varphi_-}{\varphi_- + i0}\, e^{i\frac{l^-}{2p^-\eta_0}h^2(\varphi_+)\varphi_-\left[1+\frac{1}{12}\boldsymbol{\xi}'^2_\perp(\varphi_+)\varphi_-^2\right]} \\ &\quad\times\Bigg(\left\{1+\frac{m^2}{(p^-)^2}\left[\frac{\boldsymbol{p}_\perp}{m}-\boldsymbol{\mathcal{F}}_\perp(\varphi_+)\right]^2\right\}\left[1+\frac{1}{2}\boldsymbol{\xi}'^2_\perp(\varphi_+)\varphi_-^2\right] \\ &\quad-\frac{m^2}{(p^-)^2}h^2(\varphi_+)\left[1+\frac{1}{4}\boldsymbol{\xi}'^2_\perp(\varphi_+)\varphi_-^2\right]+\frac{im^2\eta_0}{l^-p^-}\boldsymbol{\xi}'^2_\perp(\varphi_+)\varphi_-\Bigg).\end{aligned} \tag{2.63}$$

This equation is already regular and can been expressed in terms of modified Bessel function but, for the sake of convenience, we integrate by parts the last term

$$\begin{aligned}\frac{d\mathcal{E}_{\text{LCFA}}}{dl^-} &= -\frac{i\alpha}{4\pi\eta_0}\frac{l^-}{p^-}\int d\varphi_+\, h^2(\varphi_+)\int \frac{d\varphi_-}{\varphi_- + i0}\, e^{i\frac{l^-}{2p^-\eta_0}h^2(\varphi_+)\varphi_-\left[1+\frac{1}{12}\boldsymbol{\xi}'^2_\perp(\varphi_+)\varphi_-^2\right]} \\ &\quad\times\left[1+\frac{1}{2}\boldsymbol{\xi}'^2_\perp(\varphi_+)\varphi_-^2\right]\left\{1+\frac{m^2}{(p^-)^2}\left[\frac{\boldsymbol{p}_\perp}{m}-\boldsymbol{\mathcal{F}}_\perp(\varphi_+)\right]^2\right. \\ &\quad\left.-\frac{m^2}{(p^-)^2}h^2(\varphi_+)\left[1+\frac{1}{4}\boldsymbol{\xi}'^2_\perp(\varphi_+)\varphi_-^2\right]\right\} \\ &= -\frac{i\alpha}{4\pi\eta_0}\frac{l^-}{p^-}\int d\varphi_+\, h^2(\varphi_+)\int \frac{dy}{y+i0}\, e^{i\frac{l^-}{p^-}\frac{h^2(\varphi_+)}{\chi(\varphi_+)}y\left(1+\frac{y^2}{3}\right)} \\ &\quad\times(1+2y^2)\left\{1+\frac{m^2}{(p^-)^2}\left[\frac{\boldsymbol{p}_\perp}{m}-\boldsymbol{\mathcal{F}}_\perp(\varphi_+)\right]^2-\frac{m^2}{(p^-)^2}h^2(\varphi_+)(1+y^2)\right\},\end{aligned} \tag{2.64}$$

where we have introduced local quantum nonlinearity parameter $\chi(\varphi) = \eta_0|\boldsymbol{\xi}'(\varphi)|$(see also the main text). At this point, we observe that the main contribution to the integral in y comes from the region $|y| \lesssim 1$. Moreover, we recall that within the LCFA we are assuming that $\xi_0 \gg 1$ (see the discussion at the beginning of Sect. 2.4), which means that the largest contribution to the integral in φ_+ comes from the regions where $\mathcal{F}_\perp(\varphi_+)$ is at the largest.

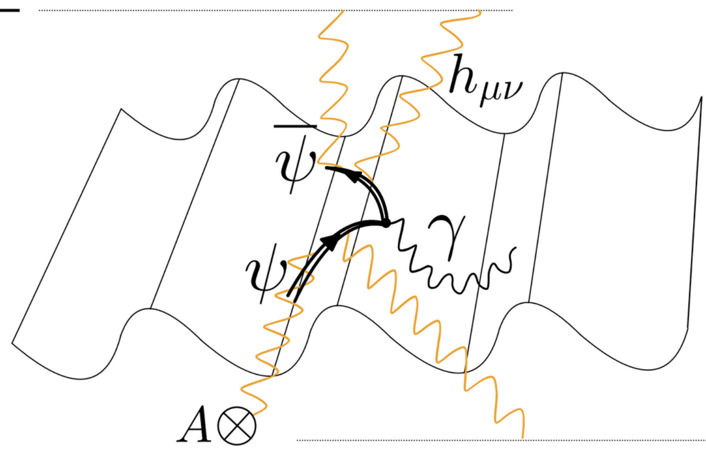
From the definitions below Eqs. (2.6) and (2.10), we obtain that

$$\begin{aligned}\boldsymbol{\mathcal{F}}_\perp(\varphi) &= \int_{-\infty}^{\varphi} d\tilde{\varphi}\, h(\tilde{\varphi})\boldsymbol{\xi}_\perp(\tilde{\varphi}) = \frac{e}{m}\int_{-\infty}^{\varphi} d\tilde{\varphi} h(\tilde{\varphi})\boldsymbol{A}'_\perp(\varphi) \\ &= \frac{e}{m}\left[h(\varphi)\boldsymbol{A}_\perp(\varphi) - \frac{2}{3}e^2\eta_0 \int_{-\infty}^{\varphi} d\tilde{\varphi}\, \boldsymbol{\xi}_\perp^2(\tilde{\varphi})\boldsymbol{A}_\perp(\tilde{\varphi})\right],\end{aligned} \tag{2.65}$$

which shows that $|\boldsymbol{\mathcal{F}}_\perp(\varphi)| \lesssim h(\varphi)\xi_0$. In conclusion, we can consistently neglect the last term in Eq. (2.64) as compared to the second-last one within the LCFA (note that we do not make any assumptions about the values of $|\boldsymbol{p}_\perp|/m$ and p^-/m as compared with ξ_0) and we finally obtain the expression in the main text:

$$\begin{aligned}\frac{d\mathcal{E}_{\text{LCFA}}}{dl^-} &= \frac{2e^2}{\sqrt{3}\pi}\frac{l^-}{p^-}\int d\varphi_+\, \frac{\varepsilon(\varphi_+)}{\pi^-(\varphi_+)}\frac{h^2(\varphi_+)}{\eta_0} \\ &\times \left[\mathrm{K}_{2/3}\left(\frac{2}{3}\frac{l^-}{p^-}\frac{h^2(\varphi_+)}{\chi(\varphi_+)}\right) - \frac{1}{2}\mathrm{IK}_{1/3}\left(\frac{2}{3}\frac{l^-}{p^-}\frac{h^2(\varphi_+)}{\chi(\varphi_+)}\right)\right],\end{aligned} \tag{2.66}$$

where we have used the integral definitions of the modified Bessel functions $K_\nu(z)$ [142] and the expression (2.12) of the energy of the electron inside the plane wave, again neglecting the term proportional to m^2 there. The approximations used and the integrations by parts carried out prevent the possibility of interpreting the integrand of Eq. (2.66) as the energy emitted per unit of laser phase and unit of l^-.



3

Strong-Field QED and graviton interactions

The aim of this chapter is to introduce the basic background needed in order to deal with SFQED and linearized gravity. These basic notions are essential for setting the stage for the next chapter, where we will investigate the process of graviton photoproduciton in SFQED and in strong classical waves. In the first part of this chapter, we briefly present how to construct quantum states and amplitudes in SFQED. In the second part we introduce few main features of general relativity, with a particular focus on the weak-field approximation and the leading-order QFT approach.

3.1 A brief introduction to Strong-Field QED

Quantum electrodynamics (QED) is described by the well-known Lagrangian

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\left(i\partial\!\!\!/ - eA\!\!\!/ - m\right)\psi. \tag{3.1}$$

To be more precise, this the QED Lagrangian in vacuum. Let us consider the case in which a classical electromagnetic field is present and serves as a background for quantum processes. We will call this field A^α to distinguish it from the quantum field $\mathcal{A}^\alpha$. We can assume it is generated by some current J_B^α and we do not really need to know the details of this source. The full Lagrangian describing the system can be easily found shifting $\mathcal{A}^\alpha \to \mathcal{A}^\alpha + A^\alpha$ in $\mathcal{L}_{\mathrm{QED}}$ and adding the coupling with J_B^α

$$\mathcal{L} = -\frac{1}{4}(F^{\mu\nu} + F_B^{\mu\nu})(F_{\mu\nu} + F_{B,\mu\nu}) + \bar{\psi}\left(i\partial\!\!\!/ - e\mathcal{A}\!\!\!/ - eA\!\!\!/ - m\right)\psi - eJ_B^\alpha(\mathcal{A}_\alpha + A_\alpha), \quad (3.2)$$

where here and below the subscript B stands for "Background". Now, all the terms involving exclusively classical quantities can be dropped, indeed they do not interact with the quantum system we want to describe. Moreover, we assume that the field A^α is connected to the classical source by the Maxwell equations $\partial_\alpha F_B^{\alpha\beta} = eJ_B^\beta$ and that the interactions between the quantum field and the classical source are negligible. This is a sensible assumptions if this source is far away from the interaction region. Under these hypothesis one can integrate by parts and eliminate the mixed term

$$-\frac{1}{2}F_B^{\mu\nu}F_{\mu\nu} = \mathcal{A}_\nu\partial_\mu F_B^{\mu\nu} = e\mathcal{A}_\nu J_B^\nu. \quad (3.3)$$

We are then left with the compact Strong-Field QED (SFQED) Lagrangian in what is known as Furry representation [88]

$$\mathcal{L}_{\mathrm{SFQED}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\left(i\partial\!\!\!/ - e\mathcal{A}\!\!\!/ - eA\!\!\!/ - m\right)\psi. \quad (3.4)$$

In the resulting theory the classical background field manifests solely as an interaction with the matter fields, allowing its dynamics to be entirely disregarded. Now, if this field is very strong we cannot treat it perturbatively and we must take into account all the contributions of A^α on the quantum states, exactly. In other words, the spinor states in the Furry picture are not the usual Dirac states but they are dressed by the background field. This corresponds to solve the Dirac equation in presence of A^α

$$\left(i\partial\!\!\!/ - eA\!\!\!/ - m\right)\psi = 0. \quad (3.5)$$

This equation can be analytically solved for a very small number of background fields, namely when A^α is either constant, Coulombian or a plane wave. Here we consider the last of these possibilities and assume $A^\alpha(\phi)$ dependent, as before, only on the variable $\phi = t - z$. As we have already discussed (see, e.g., Sec. 2.2), if we assume $\lim_{\phi\to\pm\infty}\boldsymbol{A}(\phi) = \boldsymbol{0}$ then we can choose A^α with components exclusively in the directions perpendicular to $\boldsymbol{z}$. There are various ways to solve this equation, here we present the most common and straightforward one. However, in Sec. 6.2 a more general procedure is discussed, which can be generalized to higher spin and gravitational backgrounds.

One can quickly verify that the solution of the scalar Klein-Gordon equation in presence of A^α, namely $(D_\alpha D^\alpha + m^2)\Phi = 0$ with $D_\alpha = \partial_\alpha + ieA_\alpha$, is solved by $\Phi_p = e^{iS_p(x)}$ where

$$S_p(x) = -p\cdot x - \frac{e}{p^-}\int^\phi d\tilde{\phi}\left[p\cdot A(\tilde{\phi}) - \frac{e}{2}A^2(\tilde{\phi})\right]. \quad (3.6)$$

The function S_p is the classical action of a particle moving in a plane wave. This fact is quite peculiar of plane wave backgrounds and highlights the semiclassical nature of quantum states in this context. Indeed, the WKB approximation here leads to the exact result at leading order. Being the Hamilton-Jacobi action, S_p possesses the property

$$-\partial^\alpha S_p(x) = \pi_p^\alpha(\phi) + eA^\alpha(\phi), \tag{3.7}$$

where $\pi_p^\alpha(\phi)$ is the momentum given by Eq. (1.9). With this in mind and recalling the on-shell condition $\pi_p^\alpha \pi_{p,\alpha} = m^2$, it is trivial to prove that $(D_\alpha D^\alpha + m^2)\Phi_p = 0$.

We can now look for positive-energy solutions of the Dirac equation in the form $\mathcal{U}_p(x) = e^{iS_p(x)} U_p(\phi)$ with the proper plane wave limit $\lim_{\phi\to\pm\infty} \mathcal{U}_p = e^{-ip\cdot x} u_p$. Here u_p is the usual free spinor with positive energy satisfying $(\not{p} - m)u_p = 0$.

The equation can be solved by simple quadrature, multiplying it by $(i\not{D} + m)$. Exploiting the property

$$\bar{\gamma}^\alpha \bar{\gamma}^\beta = \eta^{\alpha\beta} + 2i\Sigma^{\alpha\beta}, \tag{3.8}$$

where $\bar{\gamma}^\alpha$ are the Dirac matrices and $\Sigma^{\alpha\beta} = -\frac{i}{4}[\bar{\gamma}^\alpha, \bar{\gamma}^\beta]$ are the generators of the quadri-spinor representation $\left(\frac{1}{2}, \frac{1}{2}\right)$ of the Lorentz group, one gets

$$(D_\alpha D^\alpha + 2i\Sigma^{\alpha\beta} D_\alpha D_\beta + m^2)\mathcal{U}_p = 0. \tag{3.9}$$

Because of the antisymmetry of the generators and the gauge choice $\partial_\alpha A^\alpha = 0$, the equation reduces to

$$(D_\alpha D^\alpha - e\Sigma^{\alpha\beta} F_{\alpha\beta} + m^2)\mathcal{U}_p = 0. \tag{3.10}$$

We can now insert the ansatz $\mathcal{U}_p(x) = e^{iS_p(x)} U_p(\phi)$ and reduce the equation to a condition on the spinor matrix U_p

$$\dot{U}_p^a(\phi) = \frac{ie}{2p^-} F_{\alpha\beta}(\phi)(\Sigma^{\alpha\beta})^a{}_b U_p^b(\phi). \tag{3.11}$$

It is interesting to note that the spinor follows a Lorentz-kind equation, where the Maxwell tensor is projected onto the Lorentz group generators in the correspondent representation. This fact and its possible generalizations will be further discussed later (see Sec. 6.2). Due to the fact that the operator $F_{\alpha\beta}(\Sigma^{\alpha\beta})^a{}_b$ is nilpotent and recalling the free particle initial condition, the equation simplifies to

$$\dot{U}_p^a(x^-) = \frac{ie}{2p^-} F_{\alpha\beta}(\Sigma^{\alpha\beta})^a{}_b u_p^b, \tag{3.12}$$

which can be directly integrated to find

$$U_p^a(\phi) = \left[\delta_b^a + \frac{ie}{2p^-}\int_{-\infty}^{\phi} d\tilde{\phi} F_{\alpha\beta}(\tilde{\phi})(\Sigma^{\alpha\beta})^a{}_b\right] u_p^b = \left[\delta_b^a + e\frac{(\not{n}\not{A}(\phi))^a{}_b}{2p^-}\right] u_p^b. \tag{3.13}$$

The full solution can be compactly written as

$$\mathcal{U}_p(x) = e^{iS_p(x)}\left[1 + e\frac{\not{n}\not{A}(\phi)}{2p^-}\right] u_p \tag{3.14}$$

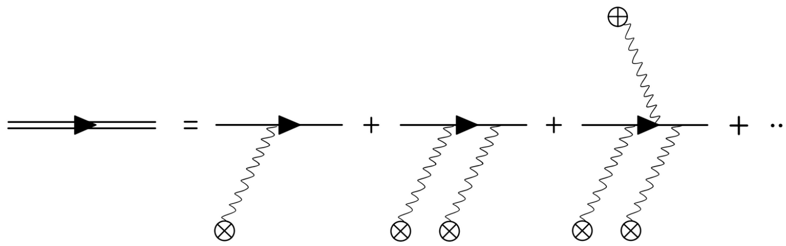


Figure 3.1: Resummation of the background contributions to the fermion propagator in the Furry picture.

and is known as positive-energy "Volkov state"[173]. This solution is one of the fundamental building blocks of Strong-Field QED. These states are often written in terms of the Ritus matrices $E_p(x)$ [157] as

$$\mathcal{U}_p(x) = E_p(x) u_p, \tag{3.15}$$

such that $E_p(x)$ encodes the full background dependence. It is easy to obtain the negative-energy Volkov states $\mathcal{V}_p(x)$ with the same procedure in the form

$$\mathcal{V}_p(x) = E_{-p}(x) v_p, \tag{3.16}$$

where now v_p is the negative-energy free spinor satisfying $(\not{p} + m)v_p = 0$

3.2 Nonlinear Compton scattering in Strong-Field QED

3.2.1 Calculation of the squared amplitude

In this section we will briefly sum up the main features of the photon emission by a charge moving in a strong electromagnetic plane wave, also known as "nonlinear Compton scattering". In Ch. 4 we will see how this process can be related to the conversion of a photon into a graviton. Nonlinear Compton scattering for a general plane wave background was first studied in [128]. Here we have only one diagram, which actually corresponds to an infinity of diagrams (see Figs. 3.1, 3.2). An incoming fermion in a plane wave with

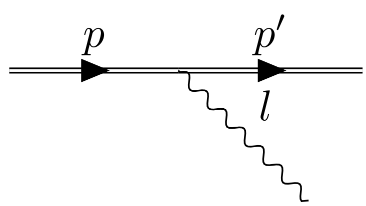


Figure 3.2: Nonlinear Compton scattering amplitude. The double line represents a dressed fermion.

wavevector k^μ is represented by a positive-energy Volkov state, as discussed in the previous section

$$\mathcal{U}_p(x) = e^{iS_p(x)} \left[1 + e\frac{\not{n}\not{A}(\phi)}{2p^-}\right] u_p. \tag{3.17}$$

Once again, we assume $\lim_{\phi\to\pm\infty} \boldsymbol{A}(\phi) = \boldsymbol{0}$, such that in the Lorenz gauge the wave field has only transverse components. The momentum conservation in the S-matrix for this process is not complete. Indeed, the background is supplying energy to the electron but it is not taken into account in the energy momentum conservation of the whole system. In other words, the plane wave contributes to the diagram as an infinite amount of photons with wavevector proportional to k^μ, thus there is no limit to the momentum it can supply along this direction. Therefore, our matrix element will involve an integral over a variable, namely the phase ϕ. This because the action in the exponential S_p is more than just $p \cdot x$ and the additional terms depends only on ϕ. If we consider an outgoing photon with polarization ε^α and momentum l^α we have

$$S^\gamma_{fi} = -ie \int d^4x e^{i\left[S_p(x)-S_{p'}(x)+l\cdot x\right]} \bar{u}_{p'} \left[1 + e\frac{\not{A}(\phi)\not{k}}{2k\cdot p'}\right] \not{\varepsilon}'^* \left[1 + e\frac{\not{k}\not{A}(\phi)}{2k\cdot p}\right] u_p, \tag{3.18}$$

where we recall that $k \cdot p = \omega p^-$. Expanding the actions in the exponential one immediately finds the momentum conservation along the three directions x^i, x^+. The matrix element can thus be written as [128]

$$S^\gamma_{fi} = (2\pi)^3 \delta(p^- - p'^- - l^-)\delta^{(2)}(\boldsymbol{p}_\perp - \boldsymbol{p'}_\perp - \boldsymbol{l}_\perp) M^\gamma_{fi}. \tag{3.19}$$

The element M^γ_{fi} takes the form

$$M^\gamma_{fi} = -ie \int d\phi e^{ig(\phi)} \bar{u}_{p'} \left[1 + e\frac{\not{A}(\phi)\not{k}}{2k\cdot p'}\right] \not{\varepsilon}'^* \left[1 + e\frac{\not{k}\not{A}(\phi)}{2k\cdot p}\right] u_p, \tag{3.20}$$

where

$$g(\phi) = -\int^\phi d\tilde{\phi} \left[\pi^+_p(\tilde{\phi}) - \pi^+_{p'}(\tilde{\phi}) - l^+\right]. \tag{3.21}$$

We can extract the field dependence assuming $A^\mu = \psi(\phi)\, a^\mu$, this leads to the form

$$M^\gamma_{fi} = ie\, \bar{u}_{p'} \left[\not{\varepsilon}'^* f_0 + \frac{e}{2}\left(\frac{\not{a}\not{k}\not{\varepsilon}'^*}{k\cdot p'} + \frac{\not{\varepsilon}'^*\not{k}\not{a}}{k\cdot p}\right) f_1 - \frac{e^2\, a^2\, k\cdot\varepsilon'^*}{2\, k\cdot p\, k\cdot p'}\not{k} f_2\right] u_p, \tag{3.22}$$

where we introduced [128]

$$f_a = \int d\phi\, e^{ig(\phi)}\, [\psi(\phi)]^a. \tag{3.23}$$

From this expressions we can easily find the unpolarized squared amplitude, averaging over the initial states and summing over the final ones we have

$$\begin{aligned}\frac{1}{2}\sum_{\text{pol., spin}} M^{\gamma\dagger}_{fi} M^\gamma_{fi} = e^2 Tr \Bigg\{ &(\not{p}' + m)\left[\bar{\gamma}^\mu f_0 + \frac{e}{2}\left(\frac{\not{a}\not{k}\bar{\gamma}^\mu}{k\cdot p'} + \frac{\bar{\gamma}^\mu\,\not{k}\not{a}}{k\cdot p}\right) f_1 + \frac{e^2\, a^2\, k^\mu}{2\, k\cdot p\, k\cdot p'}\not{k}\, f_2\right] \\ &\times(\not{p} + m)\left[\bar{\gamma}_\mu f^*_0 + \frac{e}{2}\left(\frac{\not{a}\not{k}\bar{\gamma}_\mu}{k\cdot p} + \frac{\bar{\gamma}_\mu\,\not{k}\not{a}}{k\cdot p'}\right) f^*_1 + \frac{e^2\, a^2\, k_\mu}{2\, k\cdot p\, k\cdot p'}\not{k}\, f^*_2\right]\Bigg\}.\end{aligned} \tag{3.24}$$

We are not really interested in going through a detailed calculation of this trace here. Indeed, this can be easily done with a short code in FORM like the one we listed in App. (A.11.1) as ComptonEWlinear.frm. The result is the following

$$\frac{1}{2}\sum_{\text{pol., spin}} M_{fi}^{\gamma\dagger} M_{fi}^{\gamma} = 2e^2 \Big\{ 2\,|f_0|^2\,(p\cdot p' - 2m^2) \\ -2e\Re(f_0\,f_1^*)\left[p'\cdot a\left(\frac{k\cdot p' - k\cdot p}{k\cdot p'}\right) + p\cdot a\left(\frac{k\cdot p - k\cdot p'}{k\cdot p}\right)\right] \\ +2\,e^2\,a^2\,\Re(f_0\,f_2^*) - e^2\,a^2\,|f_1|^2\left(\frac{k\cdot p}{k\cdot p'} + \frac{k\cdot p'}{k\cdot p}\right)\Big\}. \tag{3.25}$$

Moreover, the code ComptonEWlinear.frm also allows to linearize the squared amplitude recovering the usual Compton scattering in vacuum QED. Taking into account the degrees of freedom of the background, one finds

$$\frac{1}{4}\sum_{\text{pol., spin}} M_{fi}^{\gamma\dagger} M_{fi}^{\gamma} \overset{\mathcal{O}(e^4)}{=} 2e^4\left[m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot l}\right)^2 + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot l}\right) + \frac{p\cdot l}{p\cdot k} + \frac{p\cdot k}{p\cdot l}\right], \tag{3.26}$$

which is the correct result for Compton scattering at tree-level (see, e.g, [148]). If interested, one can check with the same code the results found in scalar QED, both in the linear and nonlinear cases.

In the next section we will explicitly calculate the linearization of nonlinear Compton amplitude, this is instructive in order to grasp the connection between vacuum QED and strong-field QED. Moreover, in Ch. 6 we will adapt such a limit to processes in nonlinear gravitational waves.

3.2.2 Linearization of the amplitude

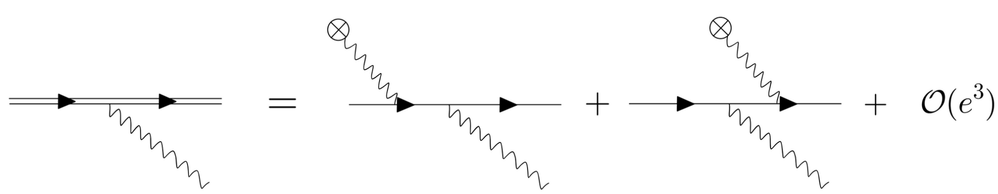


Figure 3.3: First order contribution of the background field to nonlinear Compton scattering.

In this section we will show how to treat the linear expansion of nonlinear amplitudes in order to obtain known results in vacuum QED. Let us consider a positive-energy Volkov states

$$\mathcal{U}_p(x) = e^{iS_p(x)}\left[1 + e\frac{\not{n}\not{A}(\phi)}{2p^-}\right]u_p. \tag{3.27}$$

Recalling that

$$S_p(x) = -p\cdot x - \frac{e}{p^-}\int^{\phi} d\tilde{\phi}\left[p\cdot A(\tilde{\phi}) - \frac{e}{2}A^2(\tilde{\phi})\right], \tag{3.28}$$

we find that, to the first order in the coupling e

$$\begin{aligned}\mathcal{U}_p(x) &= e^{-ip\cdot x}\left[1 - i\frac{e}{p^-}\int^{\phi} d\tilde{\phi} p\cdot A(\tilde{\phi}) + \frac{e\not{n}\not{A}(\phi)}{2p^-}\right]u_p + \mathcal{O}(e^2),\\ \bar{\mathcal{U}}_p(x) &= e^{ip\cdot x}\bar{u}_p\left[1 + i\frac{e}{p^-}\int^{\phi} d\tilde{\phi} p\cdot A(\tilde{\phi}) + \frac{e\not{A}(\phi)\not{n}}{2p^-}\right] + \mathcal{O}(e^2).\end{aligned} \tag{3.29}$$

Now, in the context of scattering amplitudes in QFT, the linearization is accompanied by the monochromatic assumption for the incoming wave $A^\mu = \varepsilon^\mu e^{-i\omega\phi} = \varepsilon^\mu e^{-ik\cdot x}$. Inserting this hypothesis in the previous expansion we get

$$\mathcal{U}_p(x) \simeq e^{-ip\cdot x}u_p + e^{-i(p+k)\cdot x}\frac{e}{2k\cdot p}\left(2p\cdot\varepsilon + \not{k}\not{\varepsilon}\right)u_p. \tag{3.30}$$

Now, the amplitude for the emission of a photon is

$$S^{\gamma}_{fi} = -ie\int d^4x \bar{\mathcal{U}}_{p'}(x)\not{\mathcal{A}}^{*'}_l(x)\mathcal{U}_p(x). \tag{3.31}$$

Inserting our expansion we find

$$\begin{aligned}S^{\gamma}_{fi} =& -ie\int d^4x e^{-i(p-p'-l)\cdot x}\\ &\times \bar{u}_{p'}\left[1 + e^{-ik\cdot x}\frac{e}{2k\cdot p'}\left(-2p'\cdot\varepsilon + \not{\varepsilon}\not{k}\right)\right]\not{\varepsilon}'^{*}\left[1 + e^{-ik\cdot x}\frac{e}{2k\cdot p}\left(2p\cdot\varepsilon + \not{k}\not{\varepsilon}\right)\right]u_p.\end{aligned} \tag{3.32}$$

The first order in e gives a null contribution because of momentum conservation, while the second order reads

$$S^{\gamma}_{fi} = -ie^2(2\pi)^4\delta(p+k-p'-l)\bar{u}_{p'}\left[\frac{1}{2k\cdot p'}\left(-2p'\cdot\varepsilon + \not{\varepsilon}\not{k}\right)\not{\varepsilon}'^{*} + \not{\varepsilon}'^{*}\frac{1}{2k\cdot p}\left(2p\cdot\varepsilon + \not{k}\not{\varepsilon}\right)\right]u_p. \tag{3.33}$$

Introducing the notation $S^{\gamma}_{fi} = (2\pi)^4\delta(p+k-p'-l)M^{\gamma}_{fi}$ we have

$$M^{\gamma}_{fi} = -ie^2\bar{u}_{p'}\left[\frac{1}{2p'\cdot k}\left(-2p'\cdot\varepsilon + \not{\varepsilon}\not{k}\right)\not{\varepsilon}'^{*} + \frac{1}{2p\cdot k}\not{\varepsilon}'^{*}\left(2p\cdot\varepsilon + \not{k}\not{\varepsilon}\right)\right]u_p, \tag{3.34}$$

which is exactly what is found for the usual tree level Compton scattering in vacuum [see, e.g., Eq. (5.74) in Ref. [148]]. The first term, coming from the dressing of the outgoing electron, is the u-channel while the second one which corresponds to the dressing of the incoming electron is the s-channel (see Fig. 3.3). The denominators $2p\cdot k, 2p'\cdot k$ are the on-shell propagators poles, here they come from the scalar classical action and the fact that the field depends only on $k\cdot x$. The term $\not{\varepsilon}'^{*}\left(2p\cdot\varepsilon + \not{k}\not{\varepsilon}\right)$ represents the numerator of the spinorial Feynman propagator with momentum $p+k$, here the contribution in k is taken into account by the correction to the spinor U_p, while the rest of the propagator comes again from the classical action.

3.3 Gravity from a field theory perspective

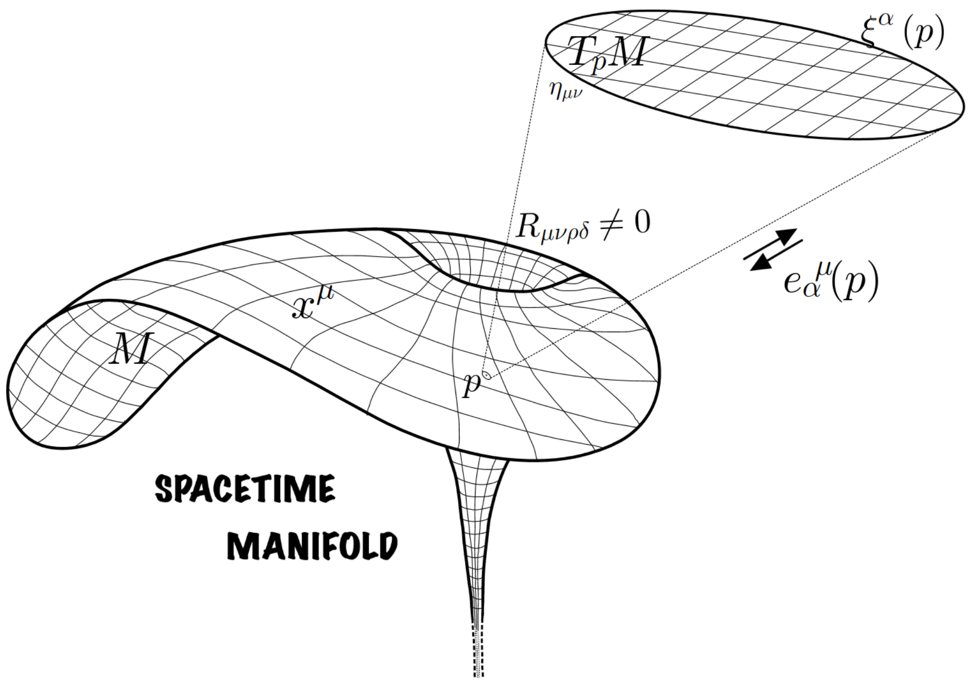


Figure 3.4: The curvature of the spacetime is described by the Riemann tensor and the scalars one can construct from it. The vierbein connects the description on the manifold to the one on the tangent space.

3.3.1 Vierbein and transformations in general relativity

General relativity can be described as a classical field theory, the field being the metric tensor $g_{\mu\nu}$. The spacetime evolution of the metric is described by the Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu}, \tag{3.35}$$

where $R_{\mu\nu}, R$ are the Ricci tensor and scalar, respectively, and $T_{\mu\nu}$ is the energy momentum tensor of the system. The spacetime curvature is described by the "second derivatives" of the metric and not by the metric itself. Indeed, it is encoded in the Riemann tensor $R_{\mu\nu\rho\delta}$, which involves products and derivatives of the Christoffel symbols. The gauge group of general relativity is very big, in fact the theory is invariant under all coordinate transformations

$$x^\mu \to x'^\mu(x), \tag{3.36}$$

which define the group of diffeomorphism in four dimensions. Under such a transformation, the metric transforms covariantly and we can find an arbitrary number of different $g_{\mu\nu}$ describing the same spacetime.

Let us be more precise and define general relativity as that theory which is invariant under general coordinate transformations and locally equivalent to special relativity. Basically this is everything one needs to construct the theory. Let us think about it geometrically: if M is the spacetime manifold, then for any $p \in M$ the tangent space T_pM is isomorphic to the Minkowski space (see Fig. 3.4). This is of course fundamental: it is not enough to say that the theory is invariant under Diff(M). The statement about the local tangent spaces, also know as "*equivalence principle*", is what really selects the correct theory. This is not something one can derive, it is a postulate, the tangent space could be described by a four-dimensional Euclidean spacetime, but it is not.

Now, let us make more explicit the definition of general relativity we have just given. Suppose we have a point p in spacetime, $p \in M$. This point is"attached" to the manifold so if we change coordinates it will be identified by different numbers but will still be there, unperturbed as the physics itself. Let us construct on the space tangent to the manifold at this point T_pM a chart $\{\xi^\alpha(p)\}$, such that in a very small neighborhood of p on the manifold

$$ds_p^2 = \eta_{\alpha\beta} d\xi^\alpha(p) d\xi^\beta(p) = g_{\mu\nu}(p) dx^\mu dx^\nu. \tag{3.37}$$

This choice is of course always possible because of the equivalence principle. From this equation we find that

$$\eta_{\alpha\beta} \frac{\partial \xi^\alpha(x)}{\partial x^\mu} \frac{\partial \xi^\beta(x)}{\partial x^\nu} = g_{\mu\nu}(x), \tag{3.38}$$

which identifies $e^\alpha{}_\mu = \frac{\partial \xi^\alpha}{\partial x^\mu}$ as the object connecting the manifold description to its tangent space (see Fig. 3.4). This matrix is known as "vierbein" and we will make an extensive use of it in the following chapters. It is worth stressing that this matrix has one flat spacetime index α and a curved spacetime index μ. For this reason it behaves as a vector with respect to both general coordinate transformations and local Lorentz transformations.

The vierbein is what really encodes the information about the gauge (coordinates) choice, because it is connected to both the manifold and the local tangent spaces. We could formulate general relativity in terms of the vierbein, than the symmetry of the theory would be simply expressed: everything has to be covariant under general transformations of coordinates and local Lorentz rotations

$$e'^\alpha{}_\mu(x') = \Lambda^\alpha{}_\beta(x) e^\beta{}_\nu(x) \frac{\partial x^\nu}{\partial x'^\mu}. \tag{3.39}$$

Another way to see this issue is to think about a general coordinate transformation as something that affects the tangent space as well, citing [143]:

"Under a general coordinate transformation $x^\mu \to x'^\mu$, there is no reason to suppose that the basis vectors for the local orthonormal frame are unaffected. A change of coordinates will be taken to induce a local Lorentz transformation of the tangent space. This means that under a general coordinate transformation, $e^\alpha{}_\mu$ must transform like a covariant vector up to a local Lorentz transformation, and A^α must transform like a set of scalars, again up to a local Lorentz transformation".

The sentence "up to a local Lorentz transformation" makes all the difference. This has to be true, because if we choose the general transformation to be a Lorentz one and the

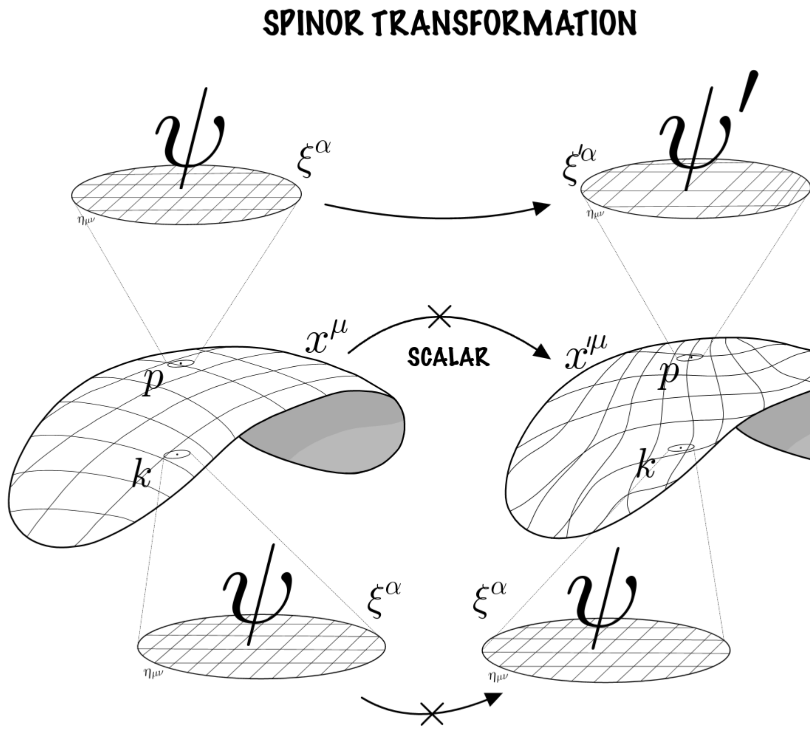


Figure 3.5: Spinors transform under local Lorentz transformations and act like scalars under general coordinate transformations on the manifold.

space is flat, than this is just equivalent to locally Lorentz transform every tangent space with the same transformation and a vector field in this case transforms covariantly (as any other field with spin $\neq 0$), it is not a scalar. Therefore, whenever in this context we say that something is a scalar with respect to general coordinate transformations, what we actually mean is that it is a scalar with respect to general coordinate transformations up to local Lorentz transformations. Now the picture is clear let us consider spinors, later we will need to include them in a curved spacetime description (see Ch. 6). In flat spacetime a spinor $\psi(x)$ is an object which transforms under infinitesimal Lorentz transformations $x'^{\alpha} = x^{\alpha} + \omega^{\alpha\beta} x_{\beta}$ as

$$\psi'(x') = e^{\frac{i}{2}\omega_{\alpha\beta}\Sigma^{\alpha\beta}}\psi(x), \tag{3.40}$$

where we recall that $\Sigma^{\alpha\beta} = -\frac{i}{4}[\bar{\gamma}^{\alpha}, \bar{\gamma}^{\beta}]$ are the generators of the Dirac representation of the Lorentz group (clearly one could do the same within the Weyl representation). While we know how to transform a vector or a tensor under GL(4), there is no finite dimensional spinor representation of this group. Therefore, when dealing with spinors in curved spacetime we are used to consider their vierbein definition, which is just called $\psi(x)$. With this in mind, the generalization of a spinor transformation to curved spacetime is natural. Let us focus on the local tangent spaces: in curved spacetime a spinor is an object which transforms under local Lorentz transformations $e'^{\alpha}_{\ \mu}(x) = \Lambda^{\alpha}_{\ \beta}(x) e^{\beta}_{\ \mu}(x)$ as

$$\psi'(x') = e^{\frac{i}{2}\omega_{\alpha\beta}(x)\Sigma^{\alpha\beta}}\psi(x), \tag{3.41}$$

where $\Lambda_{\alpha\beta}(x) \simeq \eta_{\alpha\beta}+\omega_{\alpha\beta}(x)$. Note that we do not care at all about the change of coordinate on M, but only on TM: spinors are scalars under general coordinate transformation on the manifold (see Fig. 3.5).

3.3.2 Gravitational waves in the weak-field approximation

In this section, our focus is on the regime in which the gravitational field is weak. To implement such an assumption we can simply expand the metric as a flat component plus a small perturbation (see Fig. 3.6)

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{3.42}$$

where $\kappa = \sqrt{32\pi G}$. By "small" we mean that $|\kappa h_{\mu\nu}| \ll 1$, such that we can work with at the first order in κ and neglect any higher order contribution. To be more precise, we assume there exists a reference frame (coordinate system) where the condition $|\kappa h_{\mu\nu}| \ll 1$ is satisfied in a sufficiently large region (see, e.g., [129]). Indeed, we know that the values of the metric depend on the gauge choice.

This assumption defines the weak-field approximation (see, e.g., [178]), and the resulting linearized theory is widely used in the study of gravitational waves and other scenarios where deviations from flat spacetime are small. As we discussed in the introduction, Einstein's theory is practically much more complicated than classical electromagnetism and the reason is clear: the charge of the gravitational field is any form of matter or energy, even the field itself. For this reason, linearized gravity is not only a great simplification but it allows us to intuitively understand many phenomena that would otherwise be hidden behind mathematical complications. From now on in this section, we will deal with the field $h_{\mu\nu}$ more than $g_{\mu\nu}$ and discuss how to describe it from a field theory perspective.

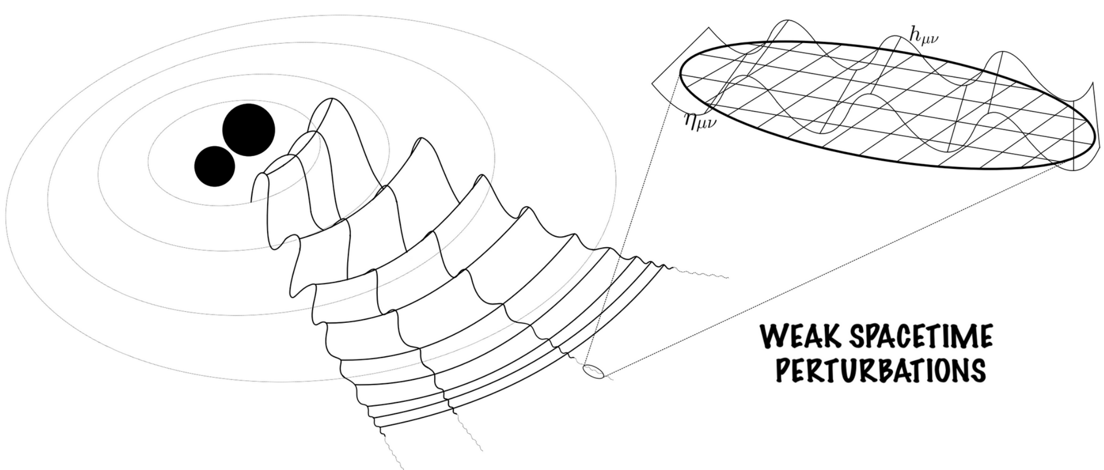


Figure 3.6: Gravitational waves produced by merging black holes can be treated in the weak-field approximation when measured at great distance from the source.

Let us insert the expansion $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ into the Einstein's Eq. (3.35) and keep only the first order in κ. The Ricci tensor is defined as

$$R_{\mu\alpha} = R^{\nu}{}_{\mu\nu\alpha} = \partial_\alpha \Gamma^{\nu}_{\mu\nu} - \partial_\nu \Gamma^{\nu}_{\mu\alpha} + \Gamma^{\beta}_{\mu\nu}\Gamma^{\nu}_{\alpha\beta} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\nu}_{\nu\beta}, \tag{3.43}$$

observing that $\Gamma \sim \mathcal{O}(\kappa)$ we have

$$R_{\mu\alpha} = \partial_\alpha \Gamma^{\nu}_{\mu\nu} - \partial_\nu \Gamma^{\nu}_{\mu\alpha} + \mathcal{O}(\kappa^2). \tag{3.44}$$

The Christoffel symbols are immediately calculated in this approximation

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\kappa}{2}\left(\partial_\mu h^{\lambda}_{\nu} + \partial_\nu h^{\lambda}_{\mu} - \partial^{\lambda} h_{\mu\nu}\right), \tag{3.45}$$

such that the Ricci tensor can be written as

$$R_{\mu\nu} = \frac{\kappa}{2}\left(\Box h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\mu\partial_\alpha h^{\alpha}_{\nu} - \partial_\nu\partial_\alpha h^{\alpha}_{\mu}\right), \tag{3.46}$$

where $\Box = \eta^{\alpha\beta}\partial_\beta\partial_\alpha$, $h = h^{\alpha}{}_{\alpha}$. It is now convenient to consider the Einstein equation in the form

$$R_{\mu\nu} = -\frac{\kappa^2}{4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \tag{3.47}$$

where $T = T^{\alpha}_{\alpha}$. We finally end up with the following linearized equation (see, e.g., [178])

$$\Box h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\mu\partial_\alpha h^{\alpha}_{\nu} - \partial_\nu\partial_\alpha h^{\alpha}_{\mu} = -\frac{\kappa}{2}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right). \tag{3.48}$$

Let us now stop for a moment and address the following question: how many physical degrees of freedom owns $h_{\mu\nu}$? In principle this is a 4×4 symmetric tensor, thus it has 10 different entries. However, we have to remember that the gauge group of general relativity is quite large. We are free to choose the reference frame, ora gauge, we like. An useful choice is the harmonic gauge, defined imposing the following condition on the Christoffel symbols

$$\Gamma^{\lambda}_{\mu\nu} g^{\mu\nu} = 0. \tag{3.49}$$

The linearization of this condition reads

$$\partial_\alpha h^{\alpha}_{\mu} = \frac{1}{2}\partial_\mu h \tag{3.50}$$

and with it the wave equation reduces to the compact form

$$\Box h_{\mu\nu} = -\frac{\kappa}{2}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right). \tag{3.51}$$

From the 10 initial variables we are now left with 6, we got rid of 4 imposing the gauge-breaking condition Eq. (3.49). However, we are not done. Indeed, under infinitesimal coordinate transformations $x^\mu \to x^\mu + \xi^\mu$ the perturbation is easily found to transform in the following way

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \tag{3.52}$$

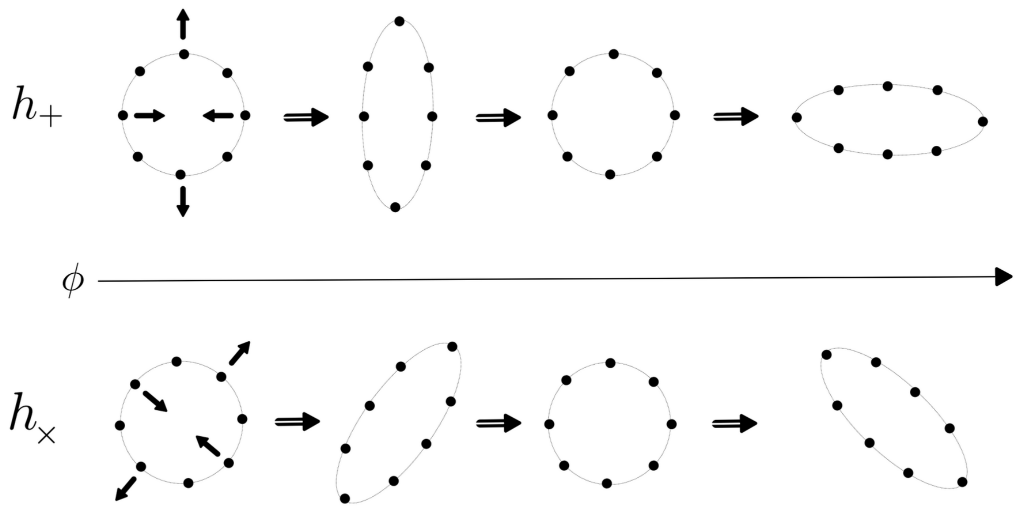


Figure 3.7: The deformation of spacetime induced by the two gravitational wave polarizations.

as one can check expanding the transformation $g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\delta}{\partial x'^\nu} g_{\rho\delta}(x)$ to the first order. Now, transforming in this way the field in Eq. (3.50) one finds

$$\partial_\alpha h^\alpha_\mu - \Box\xi_\mu = \frac{1}{2}\partial_\mu h, \tag{3.53}$$

which is invariant for any function ξ^μ satisfying $\Box\xi_\mu = 0$. Such a function can always found, therefore we can eliminate other 4 degrees of freedom, leaving $h_{\mu\nu}$ with only 2 physical polarizations. One can show that ξ^μ can be chosen to make $h_{\mu\nu}$ traceless (see, e.g. [129]). This particular choice of gauge, sum of the harmonic and traceless conditions, is known as Transverse-Traceless (TT) gauge and it is commonly chosen to present gravitational waves in the weak-field approximation. It is worth observing that this gauge can only be defined in vacuum, where $\Box\xi_\mu = 0$ can hold everywhere and it only makes sense in the weak-field approximation. As we will see, it is not easily generalized to the full nonlinear theory.

This being said, let us focus on waves in vacuum. In the TT-gauge a monochromatic solution of the wave equation $\Box h_{\mu\nu} = 0$ propagating along $\boldsymbol{z}$ has only components in the directions transverse to $\boldsymbol{z}$, namely

$$h_{ij}(x) = \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix} \cos(k \cdot x), \tag{3.54}$$

where $h_{+,\times}$ are the two polarizations and k^α is the wave vector and $i, j = \{1, 2\}$. The effects of waves with polarization $h_{+,\times}$ on a system of particles can be easily derived from the linearization of the geodesic equation [129] (see Fig. 3.7).

The polarization tensor of the wave, when expressed in terms of helicity states $h_{\mu\nu} = \varepsilon_{+,\mu\nu}e^{ik\cdot x} + \varepsilon_{-,\mu\nu}e^{-ik\cdot x}$, can be written as a product of spin-1 polarization vectors [98, 110]

$$\varepsilon^{\mu\nu}_\pm = \varepsilon^\mu_\pm \varepsilon^\nu_\pm, \tag{3.55}$$

where $\varepsilon^\mu_\pm = \frac{1}{\sqrt{2}}(0,1,\pm i,0)$ are the photon right and left polarizations. We will often employ this decomposition later in order to prove proportionality relations between graviton and photon emissions.

3.3.3 Graviton Lagrangian in the linearized theory

A natural way to obtain the action of the field $h_{\mu\nu}$ is to linearize the full gravity Einstein-Hilbert action. However, one can also follow a bottom-up approach and construct the graviton theory from scratch. If we want to take this way we need to build the theory of a spin-2 massless particle coupling with matter and energy.

This is actually quite easy, once the equations of motion of the field are known. We know that the linearized theory in the harmonic gauge is described by a simple wave equation (see Eq. (3.51))

$$\Box \bar{h}_{\mu\nu} = -\frac{\kappa}{2} T_{\mu\nu}, \tag{3.56}$$

where we defined $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$. We are looking for an action involving up to the second power of the field $h_{\mu\nu}$, such that we can extract a linear theory from it. In the free graviton sector we need to find $\Box \bar{h}_{\mu\nu} = 0$ as the equation of motion, in other words we can naturally guess the presence of a dynamical term of the form $\mathcal{L}_h \propto h^{\mu\nu}\Box\bar{h}_{\mu\nu}$. We should not forget that a metric determinant $\sqrt{g} = \sqrt{|\det g_{\mu\nu}|}$ would naturally appear in the action integral within the invariant measure. However, its weak-field expansion reds $\sqrt{g} = 1 + \frac{\kappa}{2} h + \mathcal{O}(h^2)$ and therefore when multiplied by $\mathcal{L}_h$ it contributes to the $\mathcal{O}(h^2)$ action as a unity. Our guess for the free graviton action thus reads

$$S_h^{g.f.} = C \int d^4x\, h^{\mu\nu}\Box\bar{h}_{\mu\nu} = C \int d^4x \left[h^{\mu\nu}\Box h_{\mu\nu} - \frac{1}{2} h\Box h\right], \tag{3.57}$$

where the superscript $g.f.$ stands for "gauge fixed". Adding a minimal coupling with the source is now straightforward. Observing that the free Lagrangian is quadratic in the gravitational field and therefore we get a factor of 2 deriving the equations of motion, we have

$$S_h^{g.f.} + S_{\text{int}} = C \int d^4x \left[h^{\mu\nu}\Box h_{\mu\nu} - \frac{1}{2} h\Box h + \kappa h_{\mu\nu} T^{\mu\nu}\right]. \tag{3.58}$$

The constant C is not really relevant by now, however in order to match with the energy momentum definition adopted in this text, we choose it to be $C = -\frac{1}{2}$. Thus, let us write the final form of the action as

$$S_h^{g.f.} + S_{\text{int}} = \int d^4x \left[-\frac{1}{2} h^{\mu\nu}\Box h_{\mu\nu} + \frac{1}{4} h\Box h - \frac{\kappa}{2} h_{\mu\nu} T^{\mu\nu}\right]. \tag{3.59}$$

Of course, one can verify that this is the correct action taking the linear limit of the Einstein-Hilbert one and imposing the proper gauge (see [178, 129]). It is useful to recall that the matter energy momentum tensor in general relativity can be defined as the functional derivative of the matter action with respect to the metric

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} = 2\,\frac{\partial \mathcal{L}_{\text{matter}}}{\partial g^{\mu\nu}} - g_{\mu\nu}\,\mathcal{L}_{\text{matter}}. \tag{3.60}$$

The linearization of this definition reads

$$T_{\mu\nu} = -\frac{2}{\kappa}\frac{\partial \mathcal{L}^{(1)}_{\text{matter}}}{\partial h^{\mu\nu}} - \eta_{\mu\nu}\,\mathcal{L}^{(0)}_{\text{matter}}, \tag{3.61}$$

where the superscripts refer to the order of $h_{\mu\nu}$. From this definition we see that at first order in κ the matter interaction is defined to be $\mathcal{L}^{(1)}_{\text{matter}} = -\frac{\kappa}{2}h_{\mu\nu}T^{\mu\nu}$, which checks with Eq. (3.59). This energy momentum tensor is naturally symmetrical, unlike the Noether definition commonly employed in special relativity

$$T^{\text{Noether}}_{\mu\nu} = 2\,\frac{\partial \mathcal{L}^{(0)}_{\text{matter}}}{\partial \partial^{\mu}\Psi^{a}}\partial_{\nu}\Psi^{a} - \eta_{\mu\nu}\,\mathcal{L}^{(0)}_{\text{matter}}, \tag{3.62}$$

where here with Ψ^a we refer to any matter field in the flat spacetime Lagrangian we are dealing with. These two definitions in general do not give the same result, electromagnetism is a renowned example of this eventuality. However, when integrated over space they give the same physical energy-momentum [129]

$$\int_V d^3\boldsymbol{x}T^{\text{Noether}}_{\mu\nu} = \int_V d^3\boldsymbol{x}T_{\mu\nu}. \tag{3.63}$$

The tensor $T^{\text{Noether}}_{\mu\nu}$ is not an observable quantity on its own, indeed in the electromagnetic case is not even gauge invariant. On the other hand the general definition $T_{\mu\nu}$ is well defined and preferable in most cases. If the two aforementioned definitions were exactly the same, one could be tempted to write

$$\mathcal{L}^{(1)}_{\text{matter}} \stackrel{?}{=} -\kappa\frac{\partial \mathcal{L}^{(0)}_{\text{matter}}}{\partial \partial^{\mu}\Psi^{a}}\partial_{\nu}\Psi^{a}h^{\mu\nu}. \tag{3.64}$$

However, this does not hold unless some integrations by parts are done in the corresponding actions and proper rearrangements are operated exploiting the equations of motion. An easy example one can consider is the Proca Lagrangian describing a massive spin-1 field. The r.h.s. of the previous formula involves only the derivatives of the field while the general relativistic definition produces also a term proportional to the mass $-\kappa m^2\mathcal{A}^{\alpha}\mathcal{A}^{\beta}h_{\alpha\beta}$. In the following we will use either definition based on the specific necessities.

3.4 Interactions between QED and linearized gravity

First of all, let us recall the flat QED Lagrangian

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\left(\frac{i}{2}\overleftrightarrow{\not\partial} - e\not{\mathcal{A}} - m\right)\psi, \tag{3.65}$$

where $\bar{\psi}\overleftrightarrow{\not\partial}\psi = \bar{\psi}\overrightarrow{\not\partial}\psi - \bar{\psi}\overleftarrow{\not\partial}\psi$. We now want to include gravity in this picture. As we said several times, gravitons couple to everything with mass or energy, thus the final Lagrangian will be of the form

$$\mathcal{L} = \mathcal{L}_{\text{QED}} + \mathcal{L}_h + \mathcal{L}_{h\mathcal{A}} + \mathcal{L}_{hf} + \mathcal{L}_{h\mathcal{A}f}, \tag{3.66}$$

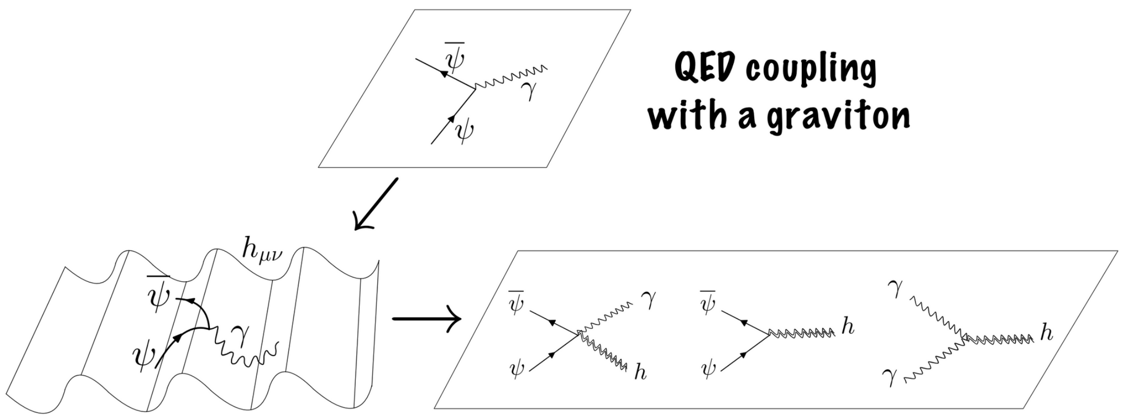


Figure 3.8: The interaction vertices involving QED and linear gravity.

where the second term in the r.h.s. is the free graviton Lagrangian, while the last three are the graviton-matter interactions. We have already derived the graviton Lagrangian in Eq. (3.59), in the harmonic gauge it reads 4.33

$$\mathcal{L}_h = -\frac{1}{2} h^{\mu\nu} \Box h_{\mu\nu} + \frac{1}{4} h \Box h. \tag{3.67}$$

From this Lagrangian we could extract the graviton propagator, however we will not need it in this section so we will refer to App. A.5 for an explicit derivation. The interaction is fully determined by the knowledge of the energy momentum tensor of the system

$$\mathcal{L}_{h\mathcal{A}} + \mathcal{L}_{hf} + \mathcal{L}_{h\mathcal{A}f} = -\frac{\kappa}{2} h_{\mu\nu} T^{\mu\nu}. \tag{3.68}$$

The tensor $T^{\mu\nu}$ is the sum of the Maxwell energy momentum tensor, the fermion one and an interaction between the two fields. Now, in order to properly treat fermions in general relativity we should employ the vierbein formulation described in sec. (3.3.1) and introduce the covariant derivative in the spin-$\frac{1}{2}$ representation. This being said, at the linear order this is not really necessary and we prefer to postpone the rigorous exposition of this topic to Ch. 6. We thus direct the reader to that chapter for more details. We can calculate the fermion energy momentum tensor through the Noether definition Eq. 3.62)

$$T_f^{\mu\nu} = \frac{\partial \mathcal{L}_f^{(0)}}{\partial \partial_\mu \psi} \partial^\nu \psi + \partial^\nu \bar{\psi} \frac{\partial \mathcal{L}_f^{(0)}}{\partial \partial_\mu \bar{\psi}} - \eta^{\mu\nu} \mathcal{L}_f^{(0)} = \frac{i}{2} \bar{\psi} \bar{\gamma}^\mu \overleftrightarrow{\partial}^\nu \psi - \eta^{\mu\nu} \bar{\psi} (\frac{i}{2} \overleftrightarrow{\not{\partial}} - m) \psi. \tag{3.69}$$

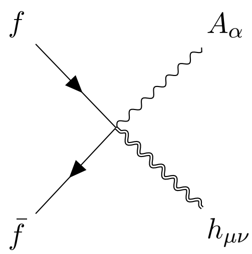
To include the interactions with the photon $\mathcal{A}^\alpha$ we can just ask this tensor to be invariant under the electromagnetic U(1) gauge symmetry. This is achieved replacing $\partial_\alpha \to D_\alpha = \partial_\alpha + ie\mathcal{A}_\alpha$, such that

$$T_f'^{\mu\nu} = \frac{i}{2} \bar{\psi} \bar{\gamma}^\mu \overleftrightarrow{D}^\nu \psi - \eta^{\mu\nu} \bar{\psi} (\frac{i}{2} \overleftrightarrow{\not{D}} - m) \psi, \tag{3.70}$$

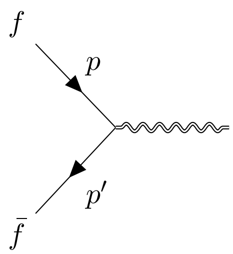
with $\bar{\psi}\overleftrightarrow{\not{D}}\psi = \bar{\psi}\overrightarrow{\not{\partial}}\psi - \bar{\psi}\overleftarrow{\not{\partial}}\psi + 2ie\bar{\psi}\not{\mathcal{A}}\psi$. We are now ready to list all the interaction terms of the Lagrangian (3.66):

$$\begin{aligned}
\mathcal{L}_{h\mathcal{A}} &= -\frac{\kappa}{2}h_{\mu\nu}\left[F^{\mu}_{\ \alpha}F^{\alpha\nu} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right], \\
\mathcal{L}_{hf} &= -\frac{\kappa}{2}\,h_{\mu\nu}\,\left[\frac{i}{2}\,\bar{\psi}\bar{\gamma}^{\mu}\overleftrightarrow{\partial}^{\nu}\,\psi - \eta^{\mu\nu}\bar{\psi}(\frac{i}{2}\overleftrightarrow{\not{\partial}} - m)\psi\right], \\
\mathcal{L}_{h\mathcal{A}f} &= -e\frac{\kappa}{2}\,h_{\mu\nu}\,\left[-\,\mathcal{A}^{\mu}\,\bar{\psi}\bar{\gamma}^{\nu}\psi + \eta^{\mu\nu}\,\mathcal{A}_{\alpha}\bar{\psi}\bar{\gamma}^{\alpha}\psi\right].
\end{aligned} \tag{3.71}$$

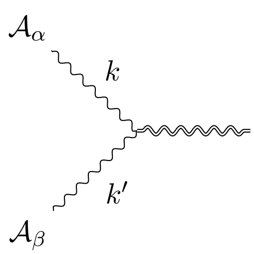
From these interactions we can extract the corresponding Feynman rules, in the usual way [179, 148, 114, 56]



$$= -ie\frac{\kappa}{4}\,(2\,\eta_{\mu\nu}\bar{\gamma}_{\alpha} - \eta_{\mu\alpha}\bar{\gamma}_{\nu} - \eta_{\nu\alpha}\bar{\gamma}_{\mu})\,,$$



$$= i\frac{\kappa}{8}\left[\bar{\gamma}_{\mu}(p-p')_{\nu} + \bar{\gamma}_{\nu}(p-p')_{\mu} - 2\eta_{\mu\nu}(\not{p} - \not{p}' - 2m)\right], \tag{3.72}$$



$$\begin{aligned}
= \quad i\frac{\kappa}{2}\,[k\cdot k'(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}) \\
+\eta_{\mu\nu}k_{\alpha}k'_{\beta} + 2\,\eta_{\alpha\beta}k_{\{\mu}k'_{\nu\}} \\
-2\eta_{\alpha\{\mu}k'_{\nu\}}k_{\beta} - 2\eta_{\beta\{\mu}k_{\nu\}}k'_{\alpha}\Big],
\end{aligned}$$

where all momenta are ingoing and $a^{\{\alpha}b^{\beta\}} = \frac{1}{2}(a^{\alpha}b^{\beta} + b^{\alpha}a^{\beta})$. With these three vertices we are ready to compute graviton photoproduction. This process and its generalization will be the main topic of the next chapter.

3.4.1 Graviton photoproduction in vacuum QED

By graviton photoproduction we refer to the emission of a graviton by an electron moving under the influence of an electromagnetic field, namely

$$e(p) + \gamma(k,\varepsilon) \rightarrow e(p') + g(l,\varepsilon'\varepsilon'). \tag{3.73}$$

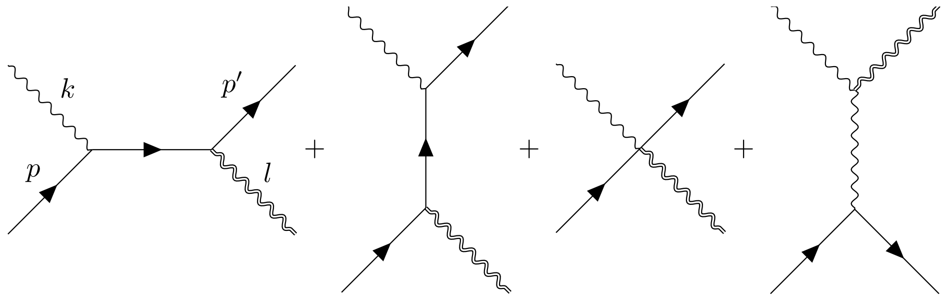


Figure 3.9: Tree-level contributions to graviton photoproduction.

From the Feynman rules we listed in the previous section (3.4) one can easily write down the diagrams contributing to this process at tree-level [Fig. 3.9].

A direct calculation shows that the amplitude reads (see, e.g., [110])

$$M^g_{fi} = -ie\frac{\kappa}{2}\bar{u}_{p'}\left[\frac{p\cdot\varepsilon'^*}{2p\cdot l}\left(-2p\cdot\varepsilon'^* + \not{\varepsilon}'^*\not{l}\right)\not{\varepsilon} + \frac{p'\cdot\varepsilon'^*}{2p'\cdot l}\not{\varepsilon}\left(2p'\cdot\varepsilon'^* + \not{l}\not{\varepsilon}'^*\right)\right.$$
$$\left. + \frac{\varepsilon'^*\cdot k}{k\cdot l}\left(\varepsilon'^*\cdot\varepsilon\not{l} - \varepsilon'^*\cdot k\not{\varepsilon} - \varepsilon\cdot l\not{\varepsilon}'^*\right)\right]u_p. \tag{3.74}$$

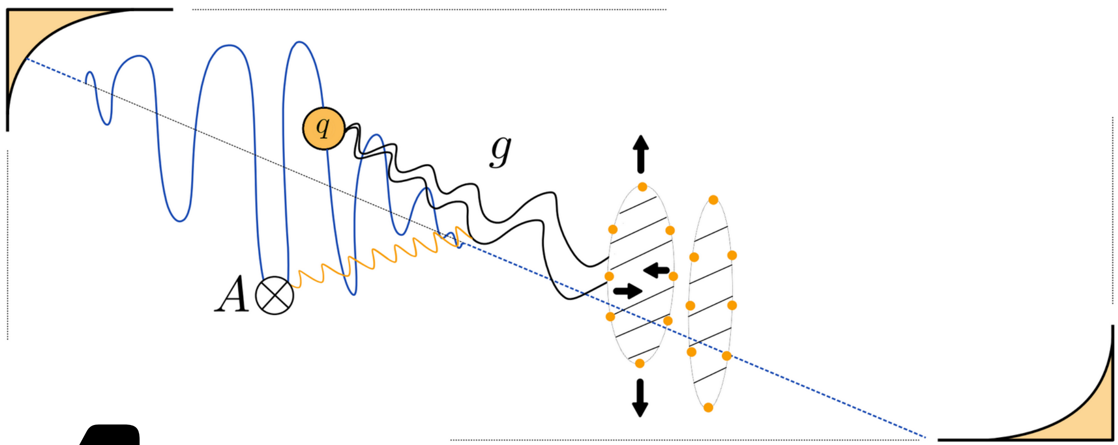
An interesting feature of this amplitude is that it can be shown to be proportional to the amplitude for Compton scattering Eq. (3.34) with the convention

$$e(p) + \gamma(k,\varepsilon) \to e(p') + \gamma(l,\varepsilon'). \tag{3.75}$$

To be more precise, with the use of some algebra, one can prove that (see Refs. [56, 110])

$$M^g_{fi} = \frac{\kappa}{2e}\left(\frac{p\cdot\varepsilon'^* l\cdot p' - p'\cdot\varepsilon'^* l\cdot p}{k\cdot l}\right)M^\gamma_{fi} = HM^\gamma_{fi}, \tag{3.76}$$

where M^γ_{fi} is exactly the expression of Eq. (3.34). The proportionality constant is usually reported in the literature as H, and we will adopt this convention here. The same proportionality can be easily derived in the context of scalar-QED, we refer to App. A.6 for further details. In the next chapter we will show how this proportionality can be extended to the regime of nonlinear electrodynamics both at the classical and quantum levels.



4

Gravitational emission by an electron in an intense plane wave

4.1 Overview

Recently, large interest has awakened in the connection between gravity and Standard Model gauge theories. There are multiple motivations driving this research area, concerning both fundamental and technical aspects [31, 46, 36, 30]. It is known that a satisfactory description of quantum gravity is not available today, nevertheless the low-energy limit of any possible model should be properly connected to the Standard Model and the classical theory of gravity, which are both experimentally well verified [148, 86, 171]. The fact that canonical quantum gravity is not strictly renormalizable does not affect these considerations and leading-order calculations are of notable interest. Indeed, it is possible to describe the dynamics of massive objects through the classical limit of scattering amplitudes [28]. By taking into account the proper Feynman diagrams, one can find corrections to the Coulomb or Newton potential [35, 34].

The relation between gravity and gauge theories manifests itself, for instance, through the Kawai-Lewellen-Tye relations [116, 30] derived in the context of string theory, which relate graviton and gauge-bosons tree-level amplitudes. These relations suggest a more fundamental connection between general relativity and gauge theories: at the semiclassical level gravity actually behaves as a double copy of a gauge theory [31, 30, 32] (see Refs. [8, 7] for studies about double copy in a background plane wave electromagnetic field). Moreover, general considerations about conservation laws [94, 55, 56] in tree-level diagrams have been exploited to relate, for example, graviton photoproduction and QED Compton scattering. These factorization properties and various other techniques [79] played a central role in calculating on-shell amplitudes involving gravitons as Compton-like scattering and photoproduction [110, 36, 37, 111, 13, 12, 27].

On a different side, the recent detection of gravitational waves has attracted a lot of attention [3]. Despite the outstanding experimental result the lack of measurable events makes it necessary to search for different sources of these perturbations. The classical interplay between gravity and electromagnetism has a long history and in this context Refs. [140, 141] are of particular interest. In these works it is proved that a proportionality exists between the electromagnetic and gravitational energy spectra of a charge driven by a monochromatic plane wave.

In a QFT perspective we can introduce the amplitudes for polarized Compton scattering and graviton photoproduction

$$\begin{aligned} &e(p_i) + \gamma(k_i, \varepsilon_i) \to e(p_f) + \gamma(k_f, \varepsilon_f) \\ &e(p_i) + \gamma(k_i, \varepsilon_i) \to e(p_f) + g(k_f, \varepsilon_f \varepsilon_f), \end{aligned} \tag{4.1}$$

where $\varepsilon_{i,f}$ are the polarizations of incoming and outgoing bosons, respectively. Note that the polarization tensor $\varepsilon_f^{\mu\nu}$ of the graviton is assumed to be written as $\varepsilon_f^{\mu\nu} = \varepsilon_f^{\mu}\varepsilon_f^{\nu}$, where ε_f^{μ} is the helicity polarization four-vector of a photon with the same four-momentum [98](see Sec. 3.3.2). The relations studied in Refs. [94, 55, 56] lead to the following result regarding the amplitudes associated to these two processes (see Sec. 3.4.1)

$$\varepsilon_{i,\alpha}\varepsilon^*_{f,\mu}\varepsilon^*_{f,\nu}M^{\alpha\mu\nu}_{e\gamma\to eg} = H\varepsilon_{i,\alpha}\varepsilon^*_{f,\mu}M^{\alpha\mu}_{e\gamma\to e\gamma}, \tag{4.2}$$

where

$$H = \frac{\kappa}{2e}\left(\frac{p_i\cdot\varepsilon^*_f k_f\cdot p_f - p_f\cdot\varepsilon^*_f k_f\cdot p_i}{k_i\cdot k_f}\right), \tag{4.3}$$

and we recall that $\kappa = \sqrt{32\pi G}$ in units where $\hbar = c = 1$, which are used throughout.

Equations (4.2) and (4.3) lead to the same proportionality between spectra found in Refs. [140, 141] but a relation like Eq. (4.2) is not expected to hold for a higher number of incoming photons [12] because the arguments based on conservation laws [94] cease to apply. Thus, an investigation of this property in the context of strong-field QED, where the effects of an electromagnetic background are taken into account exactly, is certainly relevant and timely.

In the present chapter we show both classically and quantum mechanically that the electromagnetic and the gravitational radiation amplitudes in an arbitrary plane-wave background field are proportional to each other and that the proportionality constant is the same in both cases and equal to H. Classically, this is achieved by introducing a concept of radiation amplitude in analogy with the quantum one. Quantum mechanically we work within strong-field QED in the Furry picture, which allows to take into account exactly the effects of the plane wave into the electron dynamics [29] (see Sec. 3.1, see also Ref. [89] for a similar computation in a circularly-polarized monochromatic plane wave). It is remarkable that the proportionality relies only on the symmetries of the background plane wave and on the energy-momentum conservation laws which manifest themselves in the semiclassical nature of dressed electrons in a plane wave.

4.2 Classical amplitudes proportionality

4.2.1 The source of gravitational radiation

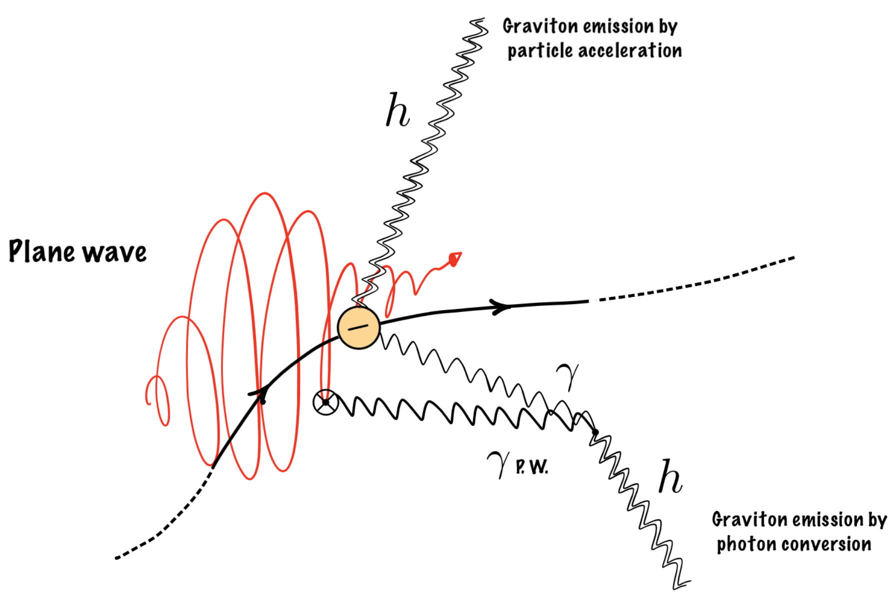


Figure 4.1: The two mechanisms of gravitational emission by a charge moving in an electromagnetic field when RR is neglected.

An electron in the presence of an intense electromagnetic plane wave radiates both light and gravitational waves. The electron is characterized by an initial four-momentum p, whereas the plane wave is described by the four-vector potential $A^\mu(\phi)$, where $\phi = n \cdot x$, with $n^\mu = (1, \boldsymbol{n})$, $\boldsymbol{n}$ being the unit vector along the propagation direction of the plane wave as usual. We assume that $A(\phi)$ satisfies the Lorenz-gauge condition and $A^0(\phi) = 0$, and that $\lim_{\phi \to \pm\infty} A^\mu(\phi) = 0$, such that the plane wave field has only transverse components.

As we observed several times, according to Einstein's equations any form of matter or energy is a source of gravity. Due to their remarkably small amplitudes in most situations, we will consider here linear gravitational waves in the first-order weak-field approximation $\mathcal{O}(\kappa)$ (see Sec. 3.3.2), where the metric is expanded as [178, 129]

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}. \tag{4.4}$$

This implies that the energy-momentum brought by the gravitational wave itself is not included in the total energy-momentum tensor $T^{\mu\nu}$, which is taken as the source of the gravitational field. Thus, the sources of gravitational waves in the system under consideration are the particle P, the background field A^μ and the field radiated by the charge A^μ_R. Moreover, the electromagnetic stress tensor $T^{\mu\nu}_{EM} = F^{\mu\alpha}F_\alpha{}^\nu + \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}$ is quadratic in the field $F^{\mu\nu} = F^{\mu\nu}_R + F^{\mu\nu}_B$, where $F^{\mu\nu}_B = 2\partial^{[\mu}A^{\nu]}$ and $F^{\mu\nu}_R = 2\partial^{[\mu}A^{\nu]}_R$, with $2a^{[\mu}b^{\nu]} = a^\mu b^\nu - a^\nu b^\mu$, and therefore a mixed term arises involving background and radiation fields. Consequently, on the whole the source for gravitational radiation reads

$$T^{\mu\nu} = T^{\mu\nu}_P + T^{\mu\nu}_R + T^{\mu\nu}_B + T^{\mu\nu}_{RB}. \tag{4.5}$$

Now, we are interested in a regime where the background plane wave can be intense, in the sense that the classical nonlinearity parameter $\xi_0 = |e|E_0/m\omega$ can be larger than unity [66, 95, 80]. Here, E_0 and ω are the peak value of the electric field of the wave and its typical angular frequency, respectively. We work in a parameter range, where radiation-reaction effects can be neglected. Classically, this implies that the parameter $\alpha\chi_0\xi_0\Phi = R\Phi$ is much smaller than unity [59], where Φ is the total phase duration of the plane-wave field. Quantum mechanically, this implies that multiple photon emissions and radiative corrections are negligible, which is the case if $\alpha\xi_0\Phi \ll 1$ at $\chi_0 \lesssim 1$ [63, 151, 85, 66, 95, 80]. Moreover, neglecting radiation-reaction effects means classically that the conservation of the total energy-momentum tensor has to be equivalent to the electron dynamics being described by the Lorentz equation [115]

$$\partial_\nu T^{\mu\nu}_P = eF^{\mu\alpha}_B J_{P,\alpha}, \tag{4.6}$$

where J_P is the particle four-current $J^\mu_P(x) = \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}(t))dx^\mu(t)/dt$ of the electron moving along the trajectory $x^\mu(t)$. This implies that the energy-momentum tensor $T^{\mu\nu}_R$ of the electromagnetic field produced by the electron can be ignored in the source of gravitational radiation $T^{\mu\nu}$.

It should be stressed that if $T^{\mu\nu}_R$ is taken into account, analytical problems arise because of its divergence on the electron trajectory. This divergence is not avoidable unless one introduces a finite size model for the electron. In this way, we have that $T^{\mu\nu} = T^{\mu\nu}_P + T^{\mu\nu}_B + T^{\mu\nu}_{RB}$ (see also Refs. [140, 141]). Let us emphasize that this is not a rough approximation, RR effects are generally small and even if they are not we will see that the main contribution to the gravitational emission comes from the non-local interaction between the radiated field and the background wave $T^{\mu\nu}_{RB}$.

At this point, one could expect the largest electromagnetic contribution to the gravitational field to come from the background term [122]

$$T^{\mu\nu}_B(\phi) = -\dot{A}^2(\phi)n^\mu n^\nu, \tag{4.7}$$

where $\dot{A}^\mu = dA^\mu/d\phi$ and $\dot{A}^2(\phi) = \dot{A}(\phi)\cdot\dot{A}(\phi)$. However, since $T^{\mu\nu}_B$ depends only on ϕ, its Fourier transform

$$T^{\mu\nu}_B(l) = (2\pi)^3\delta(l^-)\delta^{(2)}(\boldsymbol{l}_\perp)\rho(l^+)n^\mu n^\nu, \quad \text{with} \quad \rho(l^+) = -\int d\phi \exp(il^+\phi)\dot{A}^2(\phi), \tag{4.8}$$

is always zero unless $l^\mu \propto k^\mu$, where $k^\mu = \omega n^\mu$ is the background wavevector. This is absolutely natural, we are asserting that a plane wave with wavevector k^μ has only wave

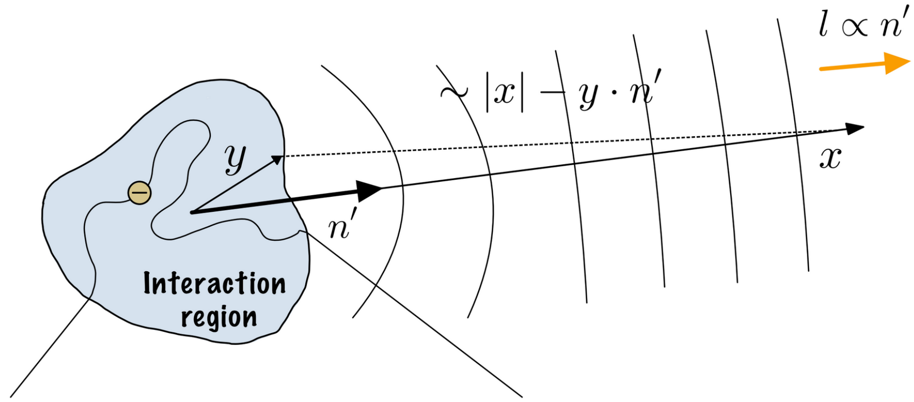


Figure 4.2: Emission from a moving source measured in the radiation zone.

components along k^μ. The consequence of this is that the $T_B^{\mu\nu}$ cannot contribute to the gravitational radiation. Indeed, when contracting with a physical polarization tensor we find

$$T_B^{\mu\nu}(l)\varepsilon_\mu^*(l)\varepsilon_\nu^*(l) \propto k^\mu k^\nu \varepsilon_\mu^*(k)\varepsilon_\nu^*(k) = 0, \tag{4.9}$$

where the last equality follows from gauge invariance. This is a general property in the weak-field approximation: an electromagnetic plane wave cannot generate a linear gravitational wave. It is worth observing that this is not true in the full nonlinear theory (see Ch. (6)).

We than conclude that [140, 141]

$$T^{\mu\nu} = T_P^{\mu\nu} + T_{RB}^{\mu\nu}. \tag{4.10}$$

4.2.2 Classical amplitudes

The classical electromagnetic emission, under certain conditions, can be expressed as the square of the source current Fourier transform (see Eq. (2.13)). This is a general feature following from the inversion of the wave equation

$$\Box A_R^\alpha = eJ_P^\alpha. \tag{4.11}$$

Let us consider the situation depicted in Fig. 4.2, where an electron is interacting with a background field within a certain region and we are measuring the radiation it produces far away from the source. One can easily prove that the electromagnetic field radiated by the charge in this region along the direction $\boldsymbol{n}'$ reads (see, e.g., [115, 178])

$$A_R^\alpha(x) = -\frac{e}{|\boldsymbol{x}|}e^{-i\omega'(t-|\boldsymbol{x}|)}\int d^3\boldsymbol{y}J_P^\alpha(\boldsymbol{y},\omega')e^{-i\omega'\boldsymbol{n}'\cdot\boldsymbol{y}}, \tag{4.12}$$

where $\boldsymbol{y}$ spans the interaction region and ω' is a Fourier frequency. This field has the form of a plane wave propagating along $\boldsymbol{n}'$, indeed when we measure the radiation far away from the source it locally looks like a plane wave. Being propagating along $\boldsymbol{n}'$ with frequency ω', its wave vector is $l^\alpha = \omega' n'^\alpha = \omega'(1, \boldsymbol{n}')$. Introducing this notation in Eq. (4.12) we have that

$$A_R^\alpha(x) = -\frac{e}{|\boldsymbol{x}|} e^{-il\cdot x} J_P^\alpha(l). \tag{4.13}$$

From this expression one clearly sees that the amplitude of the wave per unit solid angle in the direction $\boldsymbol{l}$ is proportional to $-eJ_P^\alpha(l)$, thus the energy spectrum will be proportional to the square of this quantity. The same considerations can be applied to the gravitational case, the only difference being the starting wave equation (3.51)

$$\Box h_{\mu\nu} = -\frac{\kappa}{2}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right). \tag{4.14}$$

As a result of these considerations, we can introduce the amplitudes $S_c^\gamma(l)$ and $S_c^g(l)$ for the emission of electromagnetic and gravitational radiation, respectively, in such a way that they coincide with the first-order quantum counterparts:

$$S_c^\gamma(l) = -ieJ_P^\mu(l)\varepsilon_\mu^*, \tag{4.15}$$

$$S_c^g(l) = -i\frac{\kappa}{2}T^{\mu\nu}(l)\varepsilon_\mu^*\varepsilon_\nu^*. \tag{4.16}$$

As we have mentioned in the introduction, ε^μ ($\varepsilon^\mu\varepsilon^\nu$) is the helicity polarization four-vector (tensor) of the electromagnetic (gravitational) wave such that $\varepsilon\cdot\varepsilon = l\cdot\varepsilon = 0$ [98]. Contracting the sources with the polarizations we are selecting the physical degrees of freedom, such that squaring the above amplitudes and summing over the polarizations, one finds the corresponding energy emission spectra (see Refs. [178, 115, 141]):

$$\frac{d\mathcal{E}_\gamma}{d\boldsymbol{l}} = \frac{1}{16\pi^3}\sum_{\text{pol.}} S_c^{\gamma *}(l)\, S_c^\gamma(l) = -\frac{e^2}{16\pi^3}\, J_{P,\mu}^*(l) J_P^\mu(l), \tag{4.17}$$

$$\frac{d\mathcal{E}_g}{d\boldsymbol{l}} = \frac{1}{16\pi^3}\sum_{\text{pol.}} S_c^{g*}(l) S_c^g(l) = \frac{\kappa^2}{64\pi^3}\left[T^{\mu\nu}(l)T_{\mu\nu}^*(l) - \frac{1}{2}T^\mu{}_\mu(l)T^{*\nu}{}_\nu(l)\right], \tag{4.18}$$

where we used the completeness relations [29, 172]

$$\begin{aligned}&\sum_{\text{pol.}} \varepsilon_\mu\varepsilon_\nu^* \Rightarrow -\eta_{\mu\nu},\\ &\sum_{\text{pol.}} \varepsilon_{\mu\nu}\varepsilon_{\alpha\beta}^* \Rightarrow \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right).\end{aligned} \tag{4.19}$$

These polarization sums are equivalent to the physical ones by gauge invariance.

4.2.3 Proof of the proportionality

In order to show the proportionality between $S_c^g(l)$ and $S_c^\gamma(l)$, we first consider the mixed energy-momentum tensor $T_{RB}^{\mu\nu}(l)$ in Fourier space, which can be written as

$$\begin{aligned}T_{RB}^{\mu\nu}(l) = \int \frac{d^4q}{(2\pi)^4}\Big[&\frac{1}{2}F_B^{\alpha\beta}(l-q)F_{R,\alpha\beta}(q)\eta^{\mu\nu}\\ &+ F_B^{\mu\alpha}(l-q)F_{R,\alpha}{}^\nu(q) + F_B^{\nu\alpha}(l-q)F_{R,\alpha}{}^\mu(q)\Big].\end{aligned} \tag{4.20}$$

By employing the retarded solution of the wave equation $\Box A^\mu_R = eJ^\mu_P$, the field tensor $F^{\mu\nu}_R(q)$ in momentum space is given by

$$F^{\mu\nu}_R(q) = 2\int d^4x\, e^{iq\cdot x}\partial^{[\mu}A^{\nu]}_R(x) = \frac{2ie}{q^2+i\epsilon q^0}q^{[\mu}J^{\nu]}_P(q), \tag{4.21}$$

and, since $\varepsilon\cdot\varepsilon = 0$, we can write the amplitude (4.16) as

$$S^g_c(l) = -\frac{\kappa}{2}\left[iT^{\mu\nu}_P(l) - 4e\int\frac{d^4q}{(2\pi)^4}\frac{F^{\mu\alpha}_B(l-q)q_{[\alpha}J^{\nu]}_P(q)}{q^2+i\epsilon q^0}\right]\varepsilon^*_\mu\varepsilon^*_\nu. \tag{4.22}$$

The energy-momentum tensor of the electron in Fourier space is [178]

$$T^{\mu\nu}_P(l) = (\pi^\mu J^\nu_P)(l) = \int\frac{d^4q}{(2\pi)^4}\pi^\mu(l-q)J^\nu_P(q), \tag{4.23}$$

where $\pi^\mu(\phi)$ is the electron four-momentum in the plane wave A^μ, with the initial condition $\lim_{\phi\to-\infty}\pi^\mu(\phi) = p^\mu$. Now we observe that

$$F^{\mu\alpha}_B(q) = (2\pi)^3\delta(q^-)\delta^{(2)}(\boldsymbol{q}_\perp)F^{\mu\alpha}_B(q^+), \tag{4.24}$$

and that in the chosen gauge it is $\pi_\perp(\phi) = p_\perp - eA_\perp(\phi)$, such that

$$F^{\mu\nu}_B(q^+) = \frac{2iq^+}{e}[2\pi\delta(q^+)p^{[\mu}_\perp n^{\nu]} - \pi^{[\mu}_\perp(q^+)n^{\nu]}]. \tag{4.25}$$

It is worth observing that one cannot use the identity $q^+\delta(q^+) = 0$ here because of the term $q^2+i\epsilon q^0$ in the denominator. In general, the product involving the delta-function is not associative and, by replacing Eqs. (4.24) and (4.25) in Eq. (4.22), one finds that

$$\begin{aligned} S^g_c(l) = &-\frac{i\kappa}{2l^-}\left(l\cdot np_\perp\cdot\varepsilon^* - l\cdot p_\perp n\cdot\varepsilon^*\right)J_P(l)\cdot\varepsilon^* \\ &-\frac{\kappa}{2}\left[iT^{\mu\nu}_P(l) - \frac{2i}{l^-}k_\alpha\left(\pi^{[\mu}_\perp n^{\alpha]}J^\nu_P\right)(l) + \frac{e}{l^-}\left(F^{\mu\alpha}_B J_{P,\alpha}\right)(l)n^\nu\right]\varepsilon^*_\mu\varepsilon^*_\nu, \end{aligned} \tag{4.26}$$

where we have used the fact that in Eq. (4.22) on can replace $q^\mu - l^\mu = (q^+ - l^+)n^\mu$.

At this point one can exploit the equations of motion in Fourier space

$$\partial_\nu T^{\mu\nu}_P = eF^{\mu\alpha}_B J_{P,\alpha} \;\Rightarrow\; e\left(F^{\mu\alpha}_B J_{P,\alpha}\right)(l) = -ik_\nu T^{\mu\nu}_P(l) \tag{4.27}$$

and use Eq. (4.23) to obtain an expression of $S^g_c(l)$ depending only on the electron four-current and four-momentum:

$$\begin{aligned} S^g_c(l) = &-\frac{i\kappa}{2l^-}\left(l\cdot np_\perp\cdot\varepsilon^* - l\cdot p_\perp n\cdot\varepsilon^*\right)J_P(l)\cdot\varepsilon^* \\ &-\frac{i\kappa}{2l^-}\left[-l^-\pi^\mu_\perp J^\nu_P + l\cdot\pi_\perp n^\mu J^\nu_P - l\cdot\pi n^\mu J^\nu_P + l^-\pi^\mu J^\nu_P\right](l)\varepsilon^*_\mu\varepsilon^*_\nu. \end{aligned} \tag{4.28}$$

Finally, by observing that $l\cdot\pi = l^-\pi^+ + l^+\pi^- + k_\perp\cdot\pi_\perp$ and that $\pi^- = p^-$, the following proportionality is found

$$S^g_c(l) = \frac{\kappa}{2e}\left(\frac{p\cdot\varepsilon^* l\cdot k - l\cdot pk\cdot\varepsilon^*}{l\cdot k}\right)S^\gamma_c(l), \tag{4.29}$$

where $S_c^\gamma(l) = -ieJ_P^\mu(l)\varepsilon_\mu^*$.

On the one hand, the proportionality constant in Eq. (4.29) coincides with that in Eq. (4.3), as it can be easily verified using the relations $p_i^\mu + k_i^\mu = p_f^\mu + k_f^\mu$ and $k_f \cdot \varepsilon_f^* = 0$ and by identifying $p_i^\mu = p^\mu$, $k_i^\mu = k^\mu$, $k_f^\mu = l^\mu$, and $\varepsilon_f^\mu = \varepsilon^\mu$. Thus, we indicate it also as H, i.e.,

$$S_c^g(l) = HS_c^\gamma(l). \tag{4.30}$$

On the other hand, Eq. (4.29) generalizes the results in Refs. [140, 141] as here the proportionality is shown to exist already at the level of the amplitudes and for an arbitrary plane wave, whereas in Refs. [140, 141] the proportionality was found in the energy spectra and for a monochromatic plane wave. Indeed, finding the gravitational energy spectrum is straightforward [see Eqs. (4.17) and (4.18)]:

$$\frac{d\mathcal{E}_g}{d\boldsymbol{l}} = -\frac{1}{2}\left(\frac{\partial H}{\partial \varepsilon_\mu^*}\right)^2 \frac{d\mathcal{E}_\gamma}{d\boldsymbol{l}}, \tag{4.31}$$

where

$$-\frac{1}{2}\left(\frac{\partial H}{\partial \varepsilon_\mu^*}\right)^2 = -\frac{4\pi G}{e^2}\left(m^2 - 2\frac{p\cdot l p\cdot k}{l\cdot k}\right) \tag{4.32}$$

in agreement with Ref. [141], once one takes into account that there the calculations are carried out in the average rest frame of the electron and that the authors of Ref. [141] use the opposite Minkowski metric tensor as compared to ours (see S.m. 4.5.1 for a detailed proof).

4.3 Graviton photoproduciton at tree level in strong-field QED

Now, we pass to the quantum case. By linearizing the Einstein-Hilbert action [178, 122], or with the bottom-up procedure described in Sec. 3.3.3, one obtains a field theory for the graviton $h_{\mu\nu}$ describing a spin-2 massless particle [129, 178]. Working in the harmonic (or de Donder) gauge, the Lagrangian density of the field $h_{\mu\nu}$ coupled to a generic, conserved energy-momentum tensor $T^{\mu\nu}$ is given by Eq. (3.59) [129]

$$\mathcal{L}_g = \frac{1}{2}\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} - \frac{1}{4}\partial^\mu h \partial_\mu h - \frac{\kappa}{2}h_{\mu\nu}T^{\mu\nu}, \tag{4.33}$$

where $h = h^\mu{}_\mu$. The electromagnetic sector is described by the strong-field QED Lagrangian

$$\mathcal{L}_\gamma = -\frac{1}{4}F_Q^{\mu\nu}F_{Q,\mu\nu} + \bar{\psi}\left(i\partial\!\!\!/ - eA\!\!\!/ - m\right)\psi - e\mathcal{A}^\mu J_{D,\mu}, \tag{4.34}$$

where $J_D^\mu = \bar{\psi}\bar{\gamma}^\mu\psi$ is the Dirac four-current and $\mathcal{A}^\mu$ is the photon (Quantum) field ($F_Q^{\mu\nu} = \partial^\mu\mathcal{A}^\nu - \partial^\nu\mathcal{A}^\mu$). We assume to work within the Furry picture [88] (see Sec. 3.1), where the Dirac field is quantized in the presence of the plane-wave field A^μ. Thus,we recall that the positive-energy state for an electron with four-momentum p^μ outside the plane wave is given by the Volkov spinor [173, 29]

$$\mathcal{U}_p(x) = e^{iS_p(x)}\left[1 + e\frac{n\!\!\!/ A\!\!\!/(\phi)}{2n\cdot p}\right]u_p, \tag{4.35}$$

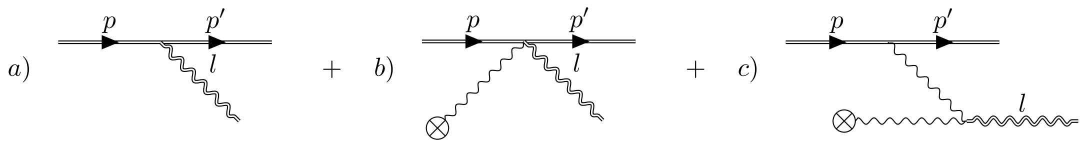


Figure 4.3: Feynman diagrams contributing to the graviton photoproduction at the order $\mathcal{O}(\kappa)$ in the presence of a strong electromagnetic plane-wave background field. The double oscillating lines represent the graviton, while the double fermion lines correspond to Volkov states, and the oscillating lines with the cross represent the background field source.

where S_p is the classical action of an electron in a plane wave (3.6) [122, 66]: $-\partial^\mu S_p(x) = \pi_p^\mu(\phi) + eA^\mu(\phi)$ and u_p is the free positive-energy spinor (for notational simplicity, the spin quantum number is not indicated). The energy momentum tensor coupled to the field $h_{\mu\nu}$ in Eq. (4.33) is $T^{\mu\nu} = T_D^{\mu\nu} + T_Q^{\mu\nu} + T_{QB}^{\mu\nu} + T_B^{\mu\nu}$, where

$$\begin{aligned} T_D^{\mu\nu} &= \bar{\psi}\left[\frac{i}{2}\bar{\gamma}^{\{\mu}\overleftrightarrow{\partial}^{\nu\}} - e\bar{\gamma}^{\{\mu}A^{\nu\}} - \right. \\ &\qquad \left. -\eta^{\mu\nu}\left(\frac{i}{2}\overleftrightarrow{\not\partial} - e\not{A} - m\right)\right]\psi, \\ T_Q^{\mu\nu} &= F_Q^{\mu\alpha}F_{Q,\alpha}{}^{\nu} + \frac{1}{4}\eta^{\mu\nu}F_Q^{\alpha\beta}F_{Q,\alpha\beta}, \\ T_{QB}^{\mu\nu} &= F_Q^{\mu\alpha}F_{B,\alpha}{}^{\nu} + F_B^{\mu\alpha}F_{Q,\alpha}{}^{\nu} + \frac{1}{2}\eta^{\mu\nu}F_Q^{\alpha\beta}F_{B,\alpha\beta}, \end{aligned} \tag{4.36}$$

with $2a^{\{\mu}b^{\nu\}} = a^\mu b^\nu + a^\nu b^\mu$ and $\bar{\psi}\overleftrightarrow{\partial}^\nu\psi = \bar{\psi}\overrightarrow{\partial}^\nu\psi - \bar{\psi}\overleftarrow{\partial}^\nu\psi$. This tensors can be easily obtained shifting $\mathcal{A}^\alpha \to \mathcal{A}^\alpha + A^\alpha$ in Eq. (3.71).

The S-matrix transition amplitude of the graviton photoproduction by an electron driven by an intense plane wave ($\xi_0 \gtrsim 1$) is given by

$$S_{fi}^g = \langle p'; l, \varepsilon_{\mu\nu}|S|p\rangle, \tag{4.37}$$

where the initial and the final electron states are Volkov states. The process $e \to e + g$ here is allowed because the background plane wave supplies the otherwise missing energy-momentum. The S-matrix is defined as $S = \mathcal{T}\exp\left[i\int d^4x\,\mathcal{L}_{\text{int}}(x)\right]$, where $\mathcal{T}$ is the time-ordering operator and

$$\mathcal{L}_{\text{int}} = -\frac{\kappa}{2}h_{\mu\nu}T^{\mu\nu} - e\mathcal{A}^\mu J_{D,\mu}. \tag{4.38}$$

At the first order in κ the process is described by the Feynman diagrams in Fig. 4.3 and the corresponding amplitude is given by

$$\begin{aligned} S_{fi}^g = &- i\frac{\kappa}{2}\mathcal{T}\langle p'|\bigg\{\int d^4x\, e^{il\cdot x}\varepsilon_\mu^*\varepsilon_\nu^*\bigg[T_D^{\mu\nu}(x) \\ &- ieT_{QB}^{\mu\nu}(x)\int d^4y\, J_{D,\alpha}(y)\mathcal{A}^\alpha(y)\bigg]\bigg\}|p\rangle. \end{aligned} \tag{4.39}$$

As in the classical case $T_B^{\mu\nu}$ cannot contribute because of gauge invariance [see Eq. (4.9)]. Moreover, it is easily seen that this matrix element has exactly the same form of the classical amplitude Eq. (4.22), the only differences being the spinorial nature of the particle and the photon propagator which now has to follow the Feynman prescription instead of the retarded one

$$S^g_{fi} = -\frac{\kappa}{2}\left[iT^{\mu\nu}_V(l) - 4e\int\frac{d^4q}{(2\pi)^4}\frac{F^{\mu\alpha}_B(l-q)q_{[\alpha}J^{\nu]}_V(q)}{q^2+i\epsilon}\right]\varepsilon^*_\mu\varepsilon^*_\nu, \tag{4.40}$$

where

$$J^\mu_V = \langle p'|J^\mu_D|p\rangle = \bar{\mathcal{U}}_{p'}\bar{\gamma}^\mu\mathcal{U}_p, \tag{4.41}$$

$$T^{\mu\nu}_V = \langle p'|T^{\mu\nu}_D|p\rangle = \bar{\mathcal{U}}_{p'}\left(\frac{i}{2}\bar{\gamma}^{\{\mu}\overleftrightarrow{\partial}^{\nu\}} - e\bar{\gamma}^{\{\mu}A^{\nu\}}\right)\mathcal{U}_p \tag{4.42}$$

are the matrix elements of the corresponding operators between Volkov states. Although the structure of the amplitude is similar to the classical one in Eq. (4.22), we stress the fact that quantum phenomena like spin effects and the recoil on the electron are taken into account in Eq. (4.40). Now, the considerations about the background field corresponding to Eqs. (4.24) - (4.25) clearly remain valid here. Thus, it is easily seen that the retarded and the Feynman prescriptions lead to the same result because the minus and the perpendicular components of q^μ are fixed by the conservation laws and the remaining term $l^+ - q^+$ in the denominator is compensated as in the classical case. Consequently, one can derive the analogous of Eq. (4.26), which now reads

$$\begin{aligned}S^g_{fi} =& -i\frac{\kappa}{2l^-}\left(l\cdot np_\perp\cdot\varepsilon^* - l\cdot p_\perp n\cdot\varepsilon^*\right)J_V(l)\cdot\varepsilon^* \\ &- \frac{\kappa}{2}\left[iT^{\mu\nu}_V(l) - \frac{2i}{l^-}l_\alpha\left(\pi^{[\mu}_\perp n^{\alpha]}J^\nu_V\right)(l) + \frac{e}{l^-}\left(F^{\mu\alpha}_B J_{V,\alpha}\right)(l)n^\nu\right]\varepsilon^*_\mu\varepsilon^*_\nu.\end{aligned} \tag{4.43}$$

Moreover, the equations of motion for the matrix elements $T^{\mu\nu}_V$ and J^μ_V are the same as for the classical quantities [see Eq. (4.27) and S.m. 4.5.2 for a detailed derivation]:

$$e\left(F^{\mu\alpha}_B J_{V,\alpha}\right)(l) = -il_\nu T^{\mu\nu}_V(l). \tag{4.44}$$

However, due to the spinorial structure of Volkov states, the quantity $T^{\mu\nu}_V$ cannot be put in the classical form as in Eq. (4.23) but it can rather be written as (see S.m. 4.5.3)

$$T^{\mu\nu}_V(l)\varepsilon^*_\mu\varepsilon^*_\nu = \left(J^\mu_V\pi^\nu_p + i\frac{e}{4p^-}\bar{\mathcal{U}}_{p'}\bar{\gamma}^\mu\not{F}_B\mathcal{U}_p n^\nu\right)(l)\varepsilon^*_\mu\varepsilon^*_\nu, \tag{4.45}$$

where $\not{F}_B = F^{\alpha\beta}_B\gamma_\alpha\gamma_\beta$. Interestingly, the spin terms do cancel in the combination $iT^{\mu\nu}_V - (i/l^-)l_\alpha T^{\mu\alpha}_V n^\nu$ in Eq. (4.43), see S.m. 4.5.4. Thus, in conclusion, the same result as in the classical treatment is found

$$S^g_{fi} = \frac{\kappa}{2e}\left(\frac{p\cdot\varepsilon^* n'\cdot n - n'\cdot pn\cdot\varepsilon^*}{n'\cdot n}\right)S^\gamma_{fi}, \tag{4.46}$$

where $S^\gamma_{fi} = -ieJ_V(l)\cdot\varepsilon^*$ is the matrix element of nonlinear Compton scattering of Eq. (3.18) (see Fig. 4.4).

Let us schematically summarize all the key steps of the proof here presented, both for the sake of clarity and future convenience.

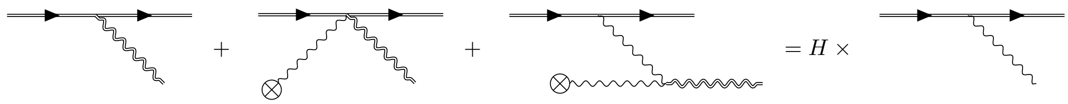


Figure 4.4: Proportionality between graviton photoproduction and photon emission in strong-field QED.

Compton scattering proportionality procedure

1. Write the amplitude S^g_{fi} in the form

$$S^g_{fi} = -\frac{\kappa}{2}\left[iT^{\mu\nu}_V(l) - 4e\int\frac{d^4q}{(2\pi)^4}\frac{F^{\mu\alpha}_B(l-q)q_{[\alpha}J^{\nu]}_V(q)}{q^2+i\epsilon}\right]\varepsilon^*_\mu\varepsilon^*_\nu, \tag{4.47}$$

where

$$J^\mu_V = \bar{\mathcal{U}}_{p'}\bar{\gamma}^\mu\mathcal{U}_p, \tag{4.48}$$

$$T^{\mu\nu}_V = \bar{\mathcal{U}}_{p'}\left(\frac{i}{2}\bar{\gamma}^{\{\mu}\overleftrightarrow{\partial}^{\nu\}} - e\bar{\gamma}^{\{\mu}A^{\nu\}}\right)\mathcal{U}_p. \tag{4.49}$$

2. Make use of the symmetries of the plane wave background write

$$\begin{aligned}S^g_{fi} =& - i\frac{\kappa}{2l^-}\left(l\cdot np_\perp\cdot\varepsilon^* - l\cdot p_\perp n\cdot\varepsilon^*\right)J_V(l)\cdot\varepsilon^*\\ &-\frac{\kappa}{2}\left[iT^{\mu\nu}_V(l) - \frac{2i}{l^-}l_\alpha\left(\pi^{[\mu}_\perp n^{\alpha]}J^\nu_V\right)(l) + \frac{e}{l^-}\left(F^{\mu\alpha}_B J_{V,\alpha}\right)(l)n^\nu\right]\varepsilon^*_\mu\varepsilon^*_\nu.\end{aligned} \tag{4.50}$$

3. Use the equations of motion to substitute

$$e\left(F^{\mu\alpha}_B J_{V,\alpha}\right)(l) = -il_\nu T^{\mu\nu}_V(l). \tag{4.51}$$

4. Exploit the explicit form of the energy momentum tensor

$$T^{\mu\nu}_V(l)\varepsilon^*_\mu\varepsilon^*_\nu = \left(J^\mu_V\pi^\nu_p + i\frac{e}{4p^-}\bar{\mathcal{U}}_{p'}\bar{\gamma}^\mu\not{F}_B\mathcal{U}_p n^\nu\right)(l)\varepsilon^*_\mu\varepsilon^*_\nu. \tag{4.52}$$

to show that the spin terms do cancel in the combination $iT^{\mu\nu}_V - (i/l^-)l_\alpha T^{\mu\alpha}_V n^\nu$.

5. Rearrange the kinematical terms to obtain

$$S^g_{fi} = \frac{\kappa}{2e}\left(\frac{p\cdot\varepsilon^* n'\cdot n - n'\cdot pn\cdot\varepsilon^*}{n'\cdot n}\right)S^\gamma_{fi}. \tag{4.53}$$

It is quite straightforward to apply this procedure to photon and graviton pair-production. Let us take into account the two processes in a strong background field

$$e^-(p)\to\gamma(l,\varepsilon^\mu_l) + e^-(p') \qquad , \qquad \gamma(-l,\varepsilon^\mu_{-l})\to e^+(-p) + e^-(p'). \tag{4.54}$$

The two amplitudes associated to these processes are related by cross symmetry, in terms of dressed states we can simply write

$$S^{\gamma}_{fi} \overset{\mathcal{U}_p \to \mathcal{V}_{-p}}{=} S^{\gamma e^+ e^-}_{fi}, \tag{4.55}$$

where $\mathcal{U}, \mathcal{V}$ are Volkov states of positive and negative energy. In fact

$$S^{\gamma}_{fi} = -ie \int d^4x e^{il\cdot x} \bar{\mathcal{U}}_{p'}(x) \not{\varepsilon}^* \mathcal{U}_p(x) \qquad \text{, while} \qquad S^{e^+e^-}_{fi} = -ie \int d^4x e^{il\cdot x} \bar{\mathcal{U}}_{p'}(x) \not{\varepsilon}^* \mathcal{V}_{-p}(x). \tag{4.56}$$

If we introduce the notation

$$\tilde{J}_V = \bar{\mathcal{U}}_{p'} \gamma^{\mu} \mathcal{V}_{-p}, \tag{4.57}$$

we can rewrite the S-matrix elements as $S^{\gamma}_{fi} = -ie\varepsilon^* \cdot J_V(l)$ and $S^{e^+e^-}_{fi} = -ie\varepsilon^* \cdot \tilde{J}_V(l)$. Now, exactly the same reasoning applies to the graviton processes

$$e^-(p) \to g(l, \varepsilon^{\mu\nu}_l) + e^-(p') \qquad ; \qquad g(-l, \varepsilon^{\mu\nu}_{-l}) \to e^+(-p) + e^-(p'), \tag{4.58}$$

indeed all the diagrams are obtained considering an incoming electron as an outgoing positron with opposite momentum. One can thus reproduce the procedure employed for Compton scattering in the context of pair production.

Pair production procedure

Now we will go through all the aforementioned steps for the amplitude of graviton pair production $S^{ge^+e^-}_{fi}$

$$S^{g}_{fi} \overset{\mathcal{U}_p \to \mathcal{V}_{-p}}{=} S^{ge^+e^-}_{fi}. \tag{4.59}$$

1. Using the fact that $S^{g}_{fi} \overset{\mathcal{U}_p \to \mathcal{V}_{-p}}{=} S^{ge^+e^-}_{fi}$ the first step now consists in writing the amplitude as

$$S^{ge^+e^-}_{fi} = -\frac{\kappa}{2} \left[i\tilde{T}^{\mu\nu}_V(l) - 4e \int \frac{d^4q}{(2\pi)^4} \frac{F^{\mu\alpha}_B(l-q) q_{[\alpha} \tilde{J}^{\nu]}_V(q)}{q^2 + i\epsilon} \right] \varepsilon^*_{\mu} \varepsilon^*_{\nu}, \tag{4.60}$$

   where

$$\tilde{J}^{\mu}_V = \bar{\mathcal{U}}_{p'} \bar{\gamma}^{\mu} \mathcal{V}_{-p}, \tag{4.61}$$

$$\tilde{T}^{\mu\nu}_V = \bar{\mathcal{U}}_{p'} \left(\frac{i}{2} \bar{\gamma}^{\{\mu} \overleftrightarrow{\partial}^{\nu\}} - e\bar{\gamma}^{\{\mu} A^{\nu\}} \right) \mathcal{V}_{-p}. \tag{4.62}$$

2. The symmetries of the plane wave are still the same, thus the second step is equal to the Compton scattering case.

3. It is easy to show that the equations of motion are still the same, in fact one can write the Volkov states as

$$\mathcal{U}_p(x) = E_p(x) u_p \qquad \text{and} \qquad \mathcal{V}_p(x) = E_{-p}(x) v_p \tag{4.63}$$

and the proof of the equations of motion is based on the spacetime dependent matrix $E_p(x)$, which is the same for $\mathcal{U}_p(x)$ and $\mathcal{V}_{-p}(x)$.

4. The form of the energy momentum tensor is the same, as showed before, with the proper crossing symmetry understood.

$$\tilde{T}_V^{\mu\nu}(l)\varepsilon^*_\mu\varepsilon^*_\nu = \left(\tilde{J}_V^\mu \pi_p^\nu + i\frac{e}{4p^-}\bar{\mathcal{U}}_{p'}\bar{\gamma}^\mu \not{F}_B \mathcal{V}_{-p} n^\nu\right)(l)\varepsilon^*_\mu\varepsilon^*_\nu. \tag{4.64}$$

The spin terms do cancel in the combination $i\tilde{T}_V^{\mu\nu} - (i/l_-) l_\alpha \tilde{T}_V^{\mu\alpha} n^\nu$.

5. Rearrange the kinematical terms to obtain

$$S_{fi}^{ge^+e^-} = H S_{fi}^{\gamma e^+e^-}. \tag{4.65}$$

4.4 Discussion

A comment is in order, which pertains to both the classical and the quantum regime. The proportionality constant H diverges as $1/\theta$ in the limit where the angle θ between the graviton and the plane-wave photons tends to zero (collinear emission). Since in the same limit, the Compton-scattering probability tends to zero linearly [29], one concludes that the graviton-emission probability diverges logarithmically [37, 141]. The same is true for the classical gravitational energy spectrum [141]. For small scattering angles the dominant contribution comes from the interaction between the particle field and the background. Classically this can be seen from the trend of the formation length which grows with the collinearity because the radiated electromagnetic field and the background interact for a longer and longer time before the gravitational conversion takes place [141]. Since the mixed electromagnetic energy-momentum tensor grows with the formation length, the emission probability increases. Quantum mechanically this corresponds to the dominance of the t-channel diagram in Fig. 4.3 (c) in this limit. Indeed, it is $t = (p - p')^2 \propto l \cdot k_B$ and when this goes to zero an infinite contribution arises from the photon propagator. It is worth noting that this diagram is dominant also in the non-relativistic range where the photon recoil is negligible and $p \to p'$ [58].

To summarize, we have shown both classically and quantum mechanically that the amplitudes of graviton and photon emission by an electron in an arbitrary plane wave are proportional to each other. Although the electron dynamics is highly nonlinear in the plane wave and quantum effects are large, the proportionality constant is classical and it does not depend on the plane-wave intensity. At the fundamental quantum level, by combining strong-field QED and quantum gravity, our proof shows that the proportionality relies only on the symmetries of the plane wave and the semiclassical nature of the motion of a quantum particle in a plane wave background. This proportionality is by itself quite interesting and appears naturally in different contexts [56]. We believe that a further investigation about its nature and its possible generalizations to higher orders could bring more interesting insights.

4.5 Support material

4.5.1 Proportionality constant in the average rest frame

It is instructive to show the agreement between the constant found in Eq. (4.32) and the one reported in ref. [141]. Since the proportionality constant is a Lorentz-invariant quantity, we can assume without loss of generality that in the laboratory frame the electron is initially at rest. Below, the subscript L (R) indicates quantities in the laboratory (average rest) frame [see App. (A.3)].

In the average rest frame the electron energy corresponds to the so-called effective mass $m_* = m\gamma_*$ [141], such that $\gamma_* = \sqrt{(1+v_d)/(1-v_d)}$ describes the relativistic Doppler effect between the two frames: $\omega_L = \gamma_* \omega_R$. In the laboratory frame it is $p_L^- = m$, such that

$$m_* = \frac{p_L^- \omega_L}{\omega_R} = \frac{k_L \cdot p_L}{\omega_R}. \tag{4.66}$$

Thus, since the quantity $k \cdot p$ is Lorentz invariant, one finds $m_* = p_R^-$. The proportionality constant introduced in Ref. [141] [see Eqs. (1.5), (1.6), and (2.23) there] can then be written as

$$C = \frac{4\pi G}{e^2} \frac{(p_R^-)^2 \boldsymbol{l}_{R,\perp}^2}{(l_R^-)^2}. \tag{4.67}$$

Finally, the equivalence of the two expressions of the proportionality constant is obtained by observing that in the average rest frame the electron is initially counterpropagating with respect to the plane wave so that $\boldsymbol{p}_{R,\perp} = \boldsymbol{0}$ and therefore

$$-\frac{4\pi G}{e^2}\left(m^2 - 2\frac{p_R \cdot l_R \, p_R \cdot k_R}{l_R \cdot k_R}\right) = \frac{4\pi G}{e^2} \frac{(p_R^-)^2 \boldsymbol{l}_{R,\perp}^2}{(l_R^-)^2}. \tag{4.68}$$

4.5.2 Derivation of the equations of motion

In this section we will prove Eq. (4.44)

$$e\,(F_B^{\mu\alpha} J_{V,\alpha})\,(l) = -il_\nu T_V^{\mu\nu}(l), \tag{4.69}$$

where

$$T_V^{\mu\nu}(l) = \frac{i}{2}\int d^4x\, e^{il\cdot x}\, \bar{\mathcal{U}}_{p'}(x) \bar{\gamma}^{\{\mu}\, D_B^{\nu\}} \mathcal{U}_p(x) \tag{4.70}$$

with $D_B^\alpha = \partial^\alpha + ieA^\alpha$. Let us start from the contraction with the first index, exploiting the Dirac equation $(i\not{D}_B - m)\mathcal{U}_p = 0$ we have

$$\begin{aligned} l_\mu \int d^4x\, e^{il\cdot x}\, \bar{\mathcal{U}}_{p'}(x)\bar{\gamma}^\mu\, D_B^\nu \mathcal{U}_p(x) &= i\int d^4x\, e^{il\cdot x}\, \partial_\mu[\bar{\mathcal{U}}_{p'}(x)\bar{\gamma}^\mu\, D_B^\nu \mathcal{U}_p(x)] \\ &= e\int d^4x\, e^{il\cdot x}\, \bar{\mathcal{U}}_{p'}(x)\,(\partial^\nu \not{A} - \not{\partial} A^\nu)\,\mathcal{U}_p(x) \\ &= e\int d^4x\, e^{il\cdot x}\, F_B^{\nu\alpha} \bar{\mathcal{U}}_{p'}(x)\bar{\gamma}_\alpha \mathcal{U}_p(x) = e\,(F_B^{\nu\alpha} J_{V,\alpha})\,(l) \end{aligned} \tag{4.71}$$

On the other hand, contracting with the second index one gets

$$l_\nu \int d^4x\, e^{il\cdot x}\, \bar{\mathcal{U}}_{p'}(x)\bar{\gamma}^\mu\, D_B^\nu \mathcal{U}_p(x) = i\int d^4x\, e^{il\cdot x}\, \partial_\nu[\bar{\mathcal{U}}_{p'}(x)\bar{\gamma}^\mu\, D_B^\nu \mathcal{U}_p(x)] \\ = -\frac{e}{4}\int d^4x\, e^{il\cdot x}\, \bar{\mathcal{U}}_{p'}(x)\left(\not{F}_B\bar{\gamma}^\mu - \bar{\gamma}^\mu \not{F}_B\right)\mathcal{U}_p(x), \tag{4.72}$$

where we employed the quadratic Dirac equations

$$\left(\Box + 2ieA_\nu\partial^\nu - e^2A^2 + m^2 + i\frac{e}{2}\not{F}_B\right)\psi = 0. \tag{4.73}$$

Noting now that

$$\not{F}_B\gamma^\mu = \bar{\gamma}^\mu\not{F}_B - 4F_B^{\mu\alpha}\bar{\gamma}_\alpha \tag{4.74}$$

we finally have

$$l_\nu \int d^4x\, e^{il\cdot x}\, \bar{\mathcal{U}}_{p'}(x)\bar{\gamma}^\mu\, D_B^\nu \mathcal{U}_p(x) = e\int d^4x\, e^{il\cdot x}\, F_B^{\mu\alpha}\bar{\mathcal{U}}_{p'}(x)\bar{\gamma}_\alpha\mathcal{U}_p(x) = e\,(F_B^{\mu\alpha}J_{V,\alpha})\,(l). \tag{4.75}$$

Collecting the two previous results we recover Eq. (4.44)

$$e\,(F_B^{\mu\alpha}J_{V,\alpha})\,(l) = -il_\nu T_V^{\mu\nu}(l). \tag{4.76}$$

4.5.3 Energy momentum contracted with external polarizations

The energy momentum contracted with external polarizations reads

$$T_V^{\mu\nu}(l)\varepsilon_\mu^*\varepsilon_\nu^* = \frac{i}{4}\int d^4x\, e^{il\cdot x}\, \bar{\mathcal{U}}_{p'}(x)\bar{\gamma}^{\{\mu}\overleftrightarrow{D}_B^{\nu\}}\mathcal{U}_p(x)\,\varepsilon_\mu^*\varepsilon_\nu^* = \\ = i\int d^4x\, e^{il\cdot x}\, \bar{\mathcal{U}}_{p'}(x)\bar{\gamma}^\mu\, D_B^\nu\mathcal{U}_p(x)\,\varepsilon_\mu^*\varepsilon_\nu^*, \tag{4.77}$$

where $D_B^\alpha = \partial^\alpha + ieA^\alpha$ is the background covariant derivative. Exploiting the relation [see Eq. (3.11)]

$$i\overrightarrow{D}_B^\mu\mathcal{U}_p = \left(\pi_p^\mu + \frac{ie\not{F}_B}{4p^-}\,n^\mu\right)\mathcal{U}_p, \tag{4.78}$$

we find

$$T_V^{\mu\nu}(l)\varepsilon_\mu^*\varepsilon_\nu^* = \left(J_V^\mu\pi_p^\nu + i\frac{e}{4p^-}\bar{\mathcal{U}}_{p'}\bar{\gamma}^\mu\not{F}_B\mathcal{U}_p n^\nu\right)(l)\varepsilon_\mu^*\varepsilon_\nu^*. \tag{4.79}$$

4.5.4 Decoupling of the spin term

It is easy to show that the spin terms disappear in the following expression

$$\left[iT_V^{\mu\nu}(l) - (i/l^-)l_\alpha T_V^{\mu\alpha}(l)n^\nu\right]\varepsilon_\mu^*\varepsilon_\nu^*, \tag{4.80}$$

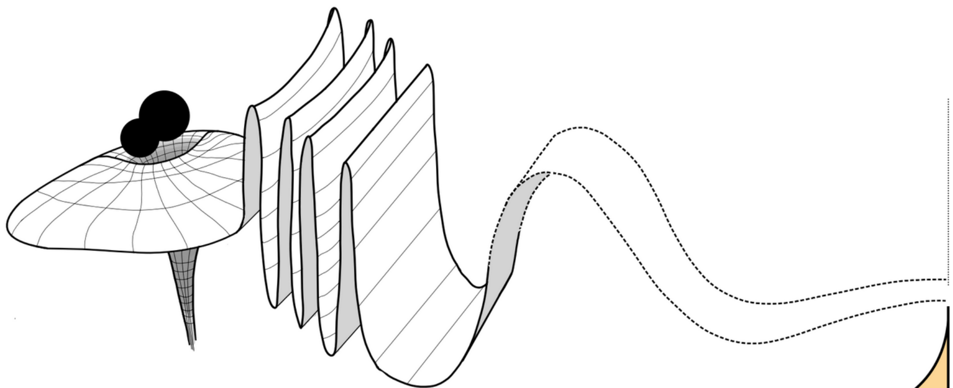
just recalling the form of the energy momentum tensor

$$T_V^{\mu\alpha}(l)\varepsilon_\mu^*\varepsilon_\alpha^* = \left(J_V^\mu\pi_p^\alpha + i\frac{e}{4p^-}\bar{\mathcal{U}}_{p'}\bar{\gamma}^\mu\not{F}_B\mathcal{U}_p n^\alpha\right)(l)\varepsilon_\mu^*\varepsilon_\alpha^*. \tag{4.81}$$

Indeed, we clearly see that

$$\frac{l_\alpha}{l^-}\bar{\mathcal{U}}_{p'}\bar{\gamma}^\mu \not{F}_B \mathcal{U}_p n^\alpha n^\nu = \bar{\mathcal{U}}_{p'}\bar{\gamma}^\mu \not{F}_B \mathcal{U}_p n^\nu, \tag{4.82}$$

such that the full spin term of the complete momentum is restored and canceled in the considered combination.



5

Gravitational plane waves in general relativity

5.1 Overview

Gravitational waves that reach the Earth are generally weak. To have an idea, the typical fractional deformation coming from astrophysical sources is of the order of $\sim 10^{-21}$ or less. This means that Earth-based detectors like LIGO, Virgo or KAGRA, which have sizes of the order of 10^3m, need the incredible displacement sensitivity of $\sim 10^{-18}$m in order to detect a gravitational perturbation [24, 83, 3]. This is the reason why gravitational waves are usually treated in the weak field approximation [178], in which the spacetime metric $g_{\mu\nu}$ is approximated as the flat metric $\eta_{\mu\nu}$ plus a small correction $\kappa h_{\mu\nu}$ and a first-order treatment of the latter is in most cases enough for any measurable prediction. Despite the weakness of the perturbation amplitudes typically measured on Earth, nonlinear gravitational plane waves, as exact solutions of Einstein's equations, can become an interesting subject for different reasons. One motivation comes from the fact that higher order corrections can grow substantially with the distance between the source and the observer, making nonlinear effects eventually not negligible. This has been pointed out, for example, in Refs. [101, 100]. This is due to the fact that the dynamics can be expressed in terms of a matrix e_{ij} satisfying the harmonic equation $\ddot{e}_{ij} = H_{ik}e^k{}_j$, where the profile H_{ij} encodes the spacetime curvature as a function of the wave phase (see Sec. 5.3). For small amplitudes one can identify $2H_{ij} = \kappa\ddot{h}_{ij}$. However, this does not necessarily imply that only the first-order term in e_{ij} has to be considered, in fact the second-order correction involves an integration of $(\dot{h}^i{}_i)^2$ and this generally grows with the phase length of the wave [101]. These large-scale effects could in principle affect pulsar timing measurements [109, 101], which very recently evidenced the presence of low-frequency background gravitational waves [11]. Nonlinear effects are interesting also because they can be of a different nature than the linear ones, position and velocity memory effects provide an example that attracted a lot of attention in the last years as well [47, 101, 100, 97, 186]. Another reason for which exact gravitational plane waves are worth to be studied comes from the so-called Penrose limit [147, 41]. Penrose proved that "any spacetime has a plane wave as a limit", namely the spacetime in a small region around a null geodesic assumes a plane wave form. This is the gravitational analog of a well known fact in electromagnetism, indeed an observer moving at ultrarelativistic velocities perceives an arbitrary electromagnetic field approximately as a plane wave and the faster the observer moves the more accurate is the similarity [115]. The generality of this property suggests that despite being idealizations, exact plane waves can provide an interesting scenario for studying limiting cases.

This chapter is organized as follows, in Sec. 5.2 we discuss how plane waves in flat spacetime can be defined through their symmetries and the role that these play in the dynamics. Section 5.3 is devoted to the description of Brinkmann and Rosen charts, which provide two somehow complementary ways of describing a nonlinear gravitational wave. In the next chapter Sec. 6.2 we examine the construction of quantum states in a plane wave spacetime for scalar, spinor, and vector particles, underlining the similarities with the flat spacetime case deriving from shared symmetries and extending the results found in Ref. [6]. Finally, in Sec. 6.3 we provide the full spin and polarization summed squared S-matrix element for photon emission scattering in a nonlinear gravitational wave background.

5.2 Plane waves and their symmetries

5.2.1 Characterizing a plane wave through its symmetries

Very often in physics, basic concepts we thought to have solidly understood turn out to be difficult to generalize, unveiling the weak points in their definitions. At first sight waves do not seem to belong to this category, however a deeper investigation will suggest us to be cautious in dealing with this concept. In the following we will focus on a specific but very useful type of wave: plane waves. The issue of giving a definition of wave in general relativity has very old roots. In 1913, two years before the correct filed equations were formulated, Einstein and Born had the following conversation in Vienna

> Born: " I should like to put to Herr Einstein a question, namely, how quickly the action of gravitation is propagated in your theory. That it happens at the speed of light does not elucidate it to me. There must be a very complicated

connection between these ideas".

Einstein: " It is extremely simple to write down the equations for the case when the perturbations that one introduces in the field are infinitely small. Then the g's differ only infinitesimally from those that would be present without the perturbation. The perturbations then propagate with the same speed of light".

Einstein is clearly referring to what we called weak-field approximation here. As we discussed in Sec. 3.3.2 one can easily find wave solutions moving at the speed of light in this context. The conversation went on

Born: " But for great perturbations things are surely very complicated?".

Einstein: " Yes it is a mathematically complicated problem. It is especially difficult to find exact solutions of the equations, as the equations are nonlinear".

Finding exact solutions to Einstein's equations is quite challenging. What is even more challenging is finding solutions that are of significant interest.

Gravity is a theory about spacetime and as such is one of the greatest manifestations of the power of symmetries in physics. We could thus be tempted, in our attempt to define waves as solutions of Einstein's equations, to try and generalize the concept of plane wave extending their symmetries to the realm of curved spacetime. This was the approach that Bondi, Pirani and Robinson followed to get the first definition of plane wave in general relativity in 1959 [45]. In order to adopt this approach we clearly have to define the symmetries of plane waves in flat spacetime first.

Thus, what is a plane wave? We are used to think about a plane wave $\Phi(\phi)$ as a field that depends only on the combination $\phi = n \cdot x = t - z$ of spacetime coordinates, with $\boldsymbol{z}$ identifying the propagation direction of the plane wave itself. As we have pointed out several times in the previous chapters, this intuitive idea unveils three symmetries. Namely, the translations along all the spacetime coordinates except ϕ, i.e. $x^+ = (t+z)/2$ and x^i, where $i = 1, 2$ refers to the coordinates transverse to $\boldsymbol{z}$. A more careful investigation reveals that plane waves feature two more symmetries χ_i belonging to the Lorentz group. These Killing vector fields correspond to the infinitesimal Lorentz transformations $\Lambda^\alpha{}_\beta = \delta^\alpha{}_\beta + \omega^\alpha{}_\beta$ leaving n^α unchanged, namely $\omega_{\alpha\beta} n^\beta = 0$. This equation defines the massless Wigner little group associated to the null vector n^α (see e.g. [180, 25, 179]). The Lorentz coefficients satisfying this equation are a linear combination of the two antisymmetric tensors $f_i^{\alpha\beta} = n^\alpha \delta_i^\beta - n^\beta \delta_i^\alpha$, which are the natural transverse tensors associated to the wave. These tensors appear for example in the Maxwell tensor of a plane wave [see (1.7)]

$$F^{\alpha\beta}(\phi) = \dot{A}^i(\phi) f_i^{\alpha\beta}. \tag{5.1}$$

Now, recalling the scalar field representation of the Lorentz generators $\Sigma_\Phi^{\alpha\beta} = 2ix^{[\alpha}\partial^{\beta]}$ we conclude that the two symmetries χ_i leaving unchanged the scalar wave $\chi_i \Phi(\phi) = 0$ are

$$\chi_i = 2i f_i^{\alpha\beta} x_\alpha \partial_\beta = 2i \left(\phi \frac{\partial}{\partial x^j} - x_j \frac{\partial}{\partial x^+} \right). \tag{5.2}$$

In terms of coordinates transformations (vector representation of the Lorentz group), a transformation generated by these vectors read (see App. A.7)

$$x'^\alpha = \left(e^{b^i f_i} \right)^\alpha{}_\beta x^\beta = x^\alpha + b^i f_i{}^\alpha{}_\beta x^\beta + \frac{1}{2} b^i f_i{}^\alpha{}_\beta b^j f_j{}^\beta{}_\gamma x^\gamma + ... \tag{5.3}$$

Observing that

$$b^i f_{i\ \beta}^{\alpha} b^j f_{j\ \gamma}^{\beta} = -b_i b^i n^\alpha n_\gamma \quad \text{and} \quad f_{i\ \beta}^{\alpha} f_{j\ \gamma}^{\beta} f_{k\ \varepsilon}^{\delta} = 0 \tag{5.4}$$

we are left with

$$x'^\alpha = x^\alpha + b^i f_{i\ \beta}^{\alpha} x^\beta - b_i b^i n^\alpha x^- . \tag{5.5}$$

Adding to these the translations along x^+ and x^i through a displacement parameter a^α, we get the full group of isometries

$$\text{Plane wave isometries group} \quad \begin{cases} x'^- = x^- \\ x'^+ = x^+ + b_i(x^i + a^i - \frac{1}{2} b^i x^-) + a^+ \\ x'^i = x^i - b^i x^- + a^i \end{cases} . \tag{5.6}$$

These Poincare' symmetries leave ∂_+ and $dx^i dx_i$ invariant, acting on constant phases hyperplanes. This group can also be understood as a partially broken Carrol group (see App. A.8).

The five Killing vector fields $(\partial_+, \partial_i, \chi_j)$ have been reported by Bondi, Pirani and Robinson in one of the first treatments of exact gravitational waves [45], along with the coordinate isometries they generate. Above, we considered a scalar wave but the same symmetries preserve electromagnetic waves as well, i.e., the vector field $A^\alpha(\phi)$. To be more precise, the Lorentz transformations χ_i in this case perform a gauge transformation on $A^\alpha(\phi)$ [121, 50] and therefore $F_{\alpha\beta}$ is precisely invariant. We will discuss this in more detail in the following. It has been observed that these generators define a Heisenberg algebra (see e.g. [8], Eq. 2.3), which is substantially the same as the one satisfied by space and momentum operators in quantum mechanics but with ∂_+ replacing the identity.

One way to extend plane waves to curved spacetime is to require the latter to exhibit at least five Killing vectors. This is what has been accomplished in Ref. [45], where it is observed that such a metric can always be locally put in the form $ds^2 = 2dx^+ d\phi + \gamma_{ij}(\phi) dx^i dx^j$ [44, 78]. This being said, we can try to guess this metric from very heuristic hypothesis and then check it possesses the proper symmetries. Considering that in the weak-field approximation the metric in presence of a plane wave has the form

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \kappa h_{ij}(\phi) dx^i dx^j = 2dx^+ d\phi + (\kappa h_{ij}(\phi) - \delta_{ij}) dx^i dx^j , \tag{5.7}$$

one of the most natural assumptions we can come up with is to assume the general form

$$ds^2 = 2dx^+ d\phi + \gamma_{ij}(\phi) dx^i dx^j \tag{5.8}$$

for arbitrarily strong plane waves. This metric is known as Rosen (or Einstein-Rosen) metric [78] and it is actually the right one to describe our spacetime. However, it has a problem, it shows spurious coordinate singularities. In other words this chart will not cover all the spacetime, we will have to glue a certain amount of its copies in order to describe a plane wave globally. Despite this problem the Rosen map will turn out to be extremely useful both from the mathematical and the conceptual point of view, especially because it posses many explicit symmetries. The metric depends on ϕ only, thus the three translations discussed for the electromagnetic case are clearly present. The other two Killing vectors are natural extensions of the χ_i previously discussed

$$\chi^j = 2i\int^{\phi}\gamma^{ji}\frac{\partial}{\partial x^i} - x^j\frac{\partial}{\partial x^+} \quad \text{to be compared to} \quad \chi^j_{\text{flat}} = 2i\int^{\phi}\eta^{ji}\frac{\partial}{\partial x^i} - x^j\frac{\partial}{\partial x^+}. \tag{5.9}$$

Later in this chapter we will see how to connect this chart to a globally defined one covering all the spacetime which goes under the name of Brinkmann chart. This will complete the toolkit we need in order to properly work in this spacetime and will allow us to calculate S-matrix elements.

5.2.2 An interesting connection between symmetries and dynamics

For the sake of clarity, it is useful to discuss here in more detail the familiar case of an electromagnetic plane wave in flat spacetime $A^\alpha(\phi)$. As previously observed, this field is gauge transformed by the Lorentz-like generators $f_i^{\alpha\beta}$. This being said, there is a particular combination of these generators which is the key to solve the dynamics in this background, namely

$$\omega^{\alpha\beta}(\phi, A) = eA^i(\phi)f_i^{\alpha\beta} \quad \text{or equivalently} \quad \omega^{\alpha\beta}(\phi, A) = e\int^{\phi} d\tilde{\phi}F^{\alpha\beta}(\tilde{\phi}). \tag{5.10}$$

As already mentioned, for generic functions $A^i(\phi)$ these define the local little group $E_2(n)$ associated to the null vector n^α [50, 121, 144]. In addition, when these functions are the plane wave components it turns out that these Lorentz-like generators completely solve the motion. Namely, one can easily show that the momentum of a charged particle in a plane wave is described by [121, 160]

$$\pi_p^\alpha(\phi) = \Lambda^\alpha_{p,\beta}(\phi, A)p^\beta \quad , \qquad \Lambda_p(\phi, A) = \exp\left(e\int^{\phi}\frac{d\tilde{\phi}}{p^-}F(\tilde{\phi})\right) \equiv e^{\frac{\omega(\phi,A)}{p^-}}. \tag{5.11}$$

This is what we obtained solving the Lorentz equation by exponentiation in (1.8). However now he want to spend few more words about the matrix $\Lambda^\alpha_{p,\beta}$. The transformation $\Lambda^\alpha_{p,\beta}$ has the following properties [144, 50, 121]:

$$\Lambda^\alpha_{p,\beta}\Lambda^{\gamma\beta}_p = \eta^{\alpha\gamma} \quad , \quad \Lambda^\alpha_{p,\beta}n^\beta = n^\alpha \quad \Rightarrow \quad \partial^\beta\Lambda^\alpha_{p,\beta} = 0 \quad , \quad \Lambda^\alpha_{p,\beta}A^\beta = A^\alpha + \partial^\alpha g, \tag{5.12}$$

with $g(\phi) = \frac{e}{p^-}\int^{\phi} d\tilde{\phi}A^\alpha(\tilde{\phi})A_\alpha(\tilde{\phi})$. The physical meaning of the above equations is that $\Lambda^\alpha_{p,\beta}$ is a Lorentz-like transformation, it describes the Wigner little group of the null vector n^α and finally it defines a U(1) gauge transformation. The momentum evolution tells us something interesting. The motion of a particle in a plane wave can be described by local Lorentz-like transformations of the initial momentum on the constant-ϕ hypersurfaces. These transformations depend on the plane wave components. They do not alter the background because they act as gauge transformations and they do not change the phase direction due to the fact that they belong to its little group. These features will turn out to be very general and surprisingly useful also in the gravitational generalization. In fact, we will show that they are manifestly present in the Rosen metric. The underlying connection between gauge and spacetime symmetries basically turns the problem of finding quantum field states in a plane wave background into a simple procedure. This construction will be discussed in detail later.

5.3 Two complementary charts: Brinkmann and Rosen coordinates

5.3.1 Prelude

As we have already anticipated, there are two charts particularly useful to describe a plane wave spacetime: the Brinkmann [48] and the Rosen [78] coordinates. These metrics have the form

$$\mathcal{G}_{\mu\nu} = \eta_{\mu\nu} + H_{ij}(\phi)X^iX^jn_\mu n_\nu \quad \text{and} \quad g_{\mu\nu} = 2n_{(\mu}\tilde{n}_{\nu)} + \gamma_{ij}(\phi)\delta^i_\mu\delta^j_\nu, \tag{5.13}$$

respectively. While the Rosen metric shows manifestly three of the five symmetries of plane waves, it has the drawback of being not global: in general, at least two Rosen charts are needed in order to cover the whole spacetime. On the other hand, the Brinkmann chart has only one manifest symmetry but it is global and the Einstein equations have a trivial form when described in these coordinates. In the following, we will describe the main properties and the geodesic motion in both these charts, underlying the natural interplay between them.

The key element connecting these metrics will be the Rosen vierbein $e_{\alpha\mu}$ defined by

$$e_{\alpha\mu}e_{\beta\nu}\eta^{\alpha\beta} = g_{\mu\nu}. \tag{5.14}$$

The matrix $e_{\beta\nu}$ will appear in the discussions of both the Rosen and the Brinkmann charts, the link between the two metrics being encoded in the evolution of its transverse part $\ddot{e}_{ij} = H_{ik}e^k{}_j$. This equation will be properly derived in the following.

It is worth emphasizing again that the symbol $e_{\beta\nu}$ will only refer to the Rosen vierbein from now on, while $E_{\beta\nu}$ will refer to the Brinkmann vierbein. Moreover, it is important to note that the Rosen vierbein's first index is be assumed to be raised and lowered by the Minkowski metric while the second one by the Rosen metric. Thus, $e_{\beta\nu}$ is not a tensor in Brinkmann coordinates but just a matrix. Furthermore, we will assume without losing generality the symmetry condition [41]

$$\dot{e}_\alpha{}^\mu e_{\beta\mu} = \dot{e}_\beta{}^\mu e_{\alpha\mu}. \tag{5.15}$$

5.3.2 The Brinkmann chart

A plane wave is described throughout all the spacetime by the Brinkmann metric $\mathcal{G}_{\mu\nu}(X)$ [48]

$$\mathcal{G}_{\mu\nu} = \eta_{\mu\nu} + H_{ij}(\phi)X^iX^jn_\mu n_\nu, \tag{5.16}$$

where $\phi = n \cdot X = T - Z$ is the same light-cone coordinate introduced in the previous sections. The independence of $\mathcal{G}_{\mu\nu}$ on $X^+ = (T + Z)/2$ makes manifest the symmetry associated to the Killing vector field ∂_+. It is worth noting that only one of the five symmetries of plane waves is explicit in this metric, for this reason the equations of motion in this chart are not as trivial as in the Rosen one, where three symmetries out of five are manifest. The fact that this metric is in the Kerr-Schild form assures three easily verified facts [117]: the inverse metric is simply $\mathcal{G}^{\mu\nu} = \eta^{\mu\nu} - H_{ij}(\phi)X^i X^j n^\mu n^\nu$, the vector n^μ is null with respects to both $\mathcal{G}_{\mu\nu}$ and $\eta_{\mu\nu}$, and, finally, the metric determinant is constant and equal to the Minkowskian one $\det \mathcal{G}_{\mu\nu} = \det \eta_{\mu\nu} = -1$.

After a straightforward calculation one finds the following expression for the Christoffel symbols in this chart

$$\Gamma^\lambda_{\mu\nu} = n^\lambda \left(\frac{1}{2} \dot{H} n_\mu n_\nu + 2H_{(\mu} n_{\nu)} \right) - H^\lambda n_\mu n_\nu, \tag{5.17}$$

where we introduced the notation $H = H_{ij}X^i X^j$ and $H_\mu = H_{\mu i}X^i$. From these we can calculate the Ricci tensor, which has the simple form [167]

$$R_{\mu\rho} = \partial_\rho \Gamma^\nu_{\mu\nu} - \partial_\nu \Gamma^\nu_{\mu\rho} + \Gamma^\lambda_{\mu\nu}\Gamma^\nu_{\lambda\rho} - \Gamma^\lambda_{\mu\rho}\Gamma^\nu_{\nu\lambda} = H_i{}^i n_\mu n_\rho. \tag{5.18}$$

From this expression we see that the matrix H_{ij} encodes the physical information about the spacetime curvature. Thus, the Einstein's equations in this spacetime read

$$H_i{}^i n_\mu n_\nu = -8\pi G T_{\mu\nu} \tag{5.19}$$

and therefore the trace of H_{ij} represents the energy density of the source which induces the gravitational perturbation. This is clearly zero for vacuum solutions, the Brinkmann profile H_{ij} in this case satisfies the same traceless condition as the weak-field perturbation h_{ij} in the physical transverse traceless (TT)-gauge. The pure gravitational wave is thus represented by a 2×2 traceless matrix H_{ij} and we can identify the two polarizations usually introduced under the weak-field approximation [41, 100] (see Sec. 3.3.2)

$$H_{ij} = \begin{pmatrix} H_+ & H_\times \\ H_\times & -H_+ \end{pmatrix}. \tag{5.20}$$

Let us go back to the general case, where $T_{\mu\nu} \neq 0$. The weak energy condition $T_{00} \geq 0$ implies $H_i{}^i \leq 0$ [91]. Exploiting the simple form of the Riemann tensor (see App. A.10) it is possible to verify that its traceless part, the Weyl tensor, depends on the traceless part of the matrix H_{ij}. It thus follows that the spacetime is conformally flat if and only if the Brinkmann profile is a pure trace $H_{ij} \propto \delta_{ij}$. This is the case for an electromagnetic wave perturbating the spacetime [145, 41, 76], while pure gravitational waves are represented by the traceless components and are Ricci-flat so their Weyl tensor is equal to the Riemann one. This reflects the nature of the tidal forces acting on geodesics: in the electromagnetic case these are acting on the volume of a body while in the gravitational case they deform its shape along the two polarizations $+, \times$. Recalling that the energy momentum tensor of an electromagnetic wave has the form

$$T^{\mu\nu}(\phi) = -\dot{A}^2(\phi) n^\mu n^\nu \equiv \rho(\phi) n^\mu n^\nu, \tag{5.21}$$

we clearly see that this can be a legitimate source for a gravitational wave. Indeed this expression has the correct tensor structure required by (5.19)

$$H_i{}^i(\phi) n_\mu n_\nu = -8\pi G \rho(\phi) n_\mu n_\nu. \tag{5.22}$$

This is a peculiar feature of nonlinear gravitational waves. In fact, in Sec. 4.2.1 we proved that an electromagnetic wave cannot produce a linear gravitational wave. Here, we observe that working within the full theory of general relativity uncovers interesting scenarios that are otherwise hidden in first-order approximations.

Studying the geodesics we will see that the actual connection between the Brinkmann profile H_{ij} and the usual weak-field perturbation is given by $H_{ij} = \frac{\kappa}{2}\ddot{h}_{ij} + \mathcal{O}(h^2)$. Despite the similarity between H_{ij} and h_{ij}, the Brinkmann chart is not the best candidate to generalize the TT-gauge because of its dependence on the transverse coordinates, indeed the Rosen chart will be more suitable for this purpose.

5.3.3 The Rosen chart

The other chart commonly employed to describe plane wave spacetimes is the Rosen one $g_{\mu\nu}(\phi)$ [78]

$$g_{\mu\nu}(\phi) = 2n_{(\mu}\tilde{n}_{\nu)} + \gamma_{ij}(\phi)\delta^i_\mu \delta^j_\nu, \tag{5.23}$$

where ϕ is the same as in Brinkmann coordinates $\phi = n \cdot x$ and γ_{ij} is a 2×2 matrix. In this chart three of the five Killing vector fields are clearly manifest: ∂_+, ∂_i. This will be reflected in the simplicity of the equations of motion, as we will see in the following. An important feature of this chart is that, unlike the Brinkmann one, it does not cover the whole spacetime. Indeed, it exhibits spurious coordinates singularities, as anticipated in the previous paragraph. This will become clear studying the geodesics. For this reason it is generally necessary to work in Brinkmann coordinates whenever global problems such as scattering processes are studied. Nonetheless this chart is very important in order to exploit the symmetries of plane waves and it will be essential in solving field equations. Moreover, the Rosen chart can be chosen to generalize the concept of TT-gauge usually introduced in the weak-field approximation. In fact, the Rosen profile γ_{ij} is transverse and depends on ϕ only, as the perturbation h_{ij}. The traceless property, which the Brinkmann profile shares with h_{ij}, is not satisfied by γ_{ij}. However, a perturbative expansion in vacuum shows that all the odd orders of γ_{ij} are traceless while all the even ones are pure traces [101]. Thus, in the linear limit $\gamma_{ij} = \eta_{ij} + \kappa h_{ij}$, with the perturbation in the TT-gauge.

The Christoffel symbols in this metric are simply

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\left(2g^{\lambda\rho}\dot{\gamma}_{\rho(\mu}n_{\nu)} - \dot{\gamma}_{\mu\nu}n^\lambda\right) \tag{5.24}$$

and from these one can find the Ricci tensor to be

$$R_{\mu\delta} = \left(\frac{1}{4}\dot{\gamma}^{kl}\dot{\gamma}_{lm} + \frac{1}{2}\gamma^{kl}\ddot{\gamma}_{lm}\right) n_\mu n_\delta. \tag{5.25}$$

In order to connect the profiles γ_{ij} and H_{ij} one can for example compare the Rosen and Brinkmann Riemann tensors in the Rosen chart. With the usual symmetries understood, one finds (see App. A.10)

$$\frac{1}{4}\dot{\gamma}^{kl}\dot{\gamma}_{lm} + \frac{1}{2}\gamma^{kl}\ddot{\gamma}_{lm} = e^{ik}e^j{}_m H_{ij}. \tag{5.26}$$

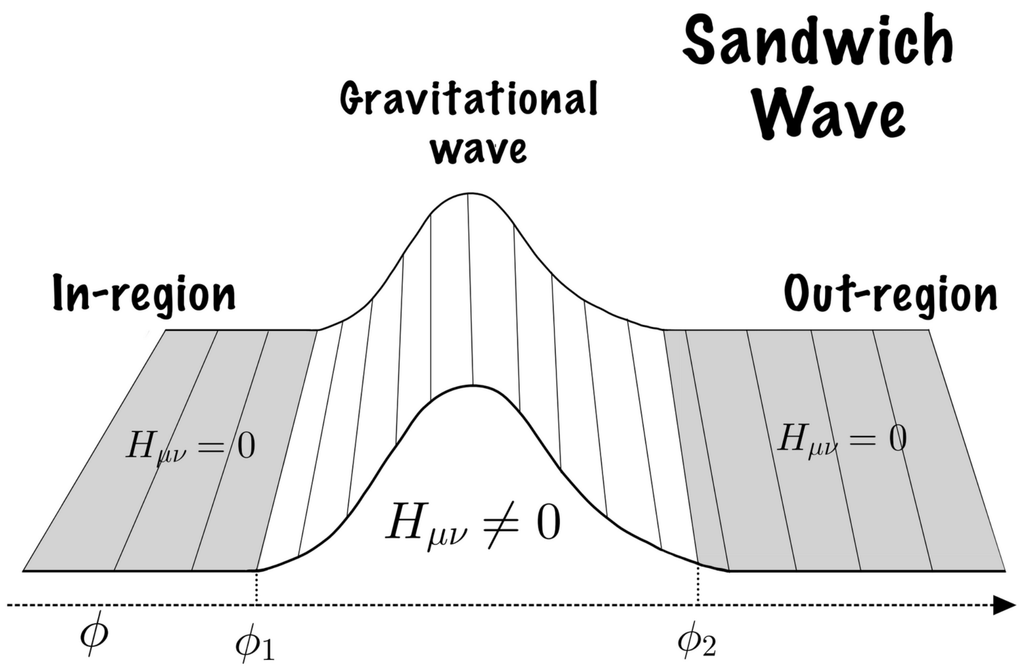


Figure 5.1: Gravitational sandwich plane wave.

We can rewrite this relation in terms of the vierbein only as

$$\ddot{e}_{ij} = H_{ik} e^k{}_j, \tag{5.27}$$

recalling that we assumed without losing generality that the symmetry condition $\dot{e}_\alpha{}^\mu e_{\beta\mu} = \dot{e}_\beta{}^\mu e_{\alpha\mu}$ is fulfilled [41]. The link between the Rosen and the Brinkmann charts can be completed observing that the Brinkmann transverse coordinates, being flat, are just the Rosen vierbein projection of the Rosen ones $X^i = e^i{}_j x^j$, with $e_{\alpha\mu} e^\alpha{}_\nu = g_{\mu\nu}$.

This finally exhibits the connection between the Rosen and the Brinkmann charts in a clear way. For completeness we report here the full coordinate transformation that connects the two systems

$$x^\mu \Rightarrow X^\mu \qquad \begin{cases} x^- = X^- \\ x^i = e_j{}^i X^j \\ x^+ = X^+ + \frac{1}{2}\sigma_{ij} X^i X^j \end{cases} , \quad \ddot{e}_{ij} = H_{ik} e^k{}_j, \tag{5.28}$$

where we introduced the symmetric tensor $\sigma_{ij} = \dot{e}_{ik} e_j{}^k$. Another way to look at this connection comes from the observation that Brinkmann coordinates are Fermi normal coordinates constructed on null geodesics [42] (see App. A.9 for further details).

5.3.4 Focusing of null geodesics and singularities of the Rosen metric

In 1965 [145] Roger Penrose showed that it is impossible to define a global spacelike Cauchy hypersurface in plane waves spacetime because of the focusing effect that plane waves exert on null geodesics (cones). Let us now briefly discuss the concepts of this paper, these will allow us to introduce some essential elements useful in the following.

We consider a sandwich plane wave as in [145] (see Fig. 5.1) that is bounded between two values of the phase $\phi_1 < \phi_2$. We then have our spacetime divided into three regions: in-region, wave region, out-region. The in- and out-regions are flat while the wave region is clearly not. We now pick a generic point Q in the flat in-region and study the null cone based on it, that is the family of null geodesics that pass through Q. We can choose Q to have the coordinates $(\phi_Q, 0, 0, X^+_Q)$ without loosing generality. Let us work in Brinkmann coordinates, which we know being well defined everywhere

$$s^2 = \eta_{\alpha\beta} X^\alpha X^\beta + H_{ij} X^i X^j \phi^2. \tag{5.29}$$

Now, from the form of the metric one finds that the null cone originating from Q has the general form

$$X^+ = X^+_Q - \frac{1}{2}\sigma_{ij}(\phi) X^i X^j. \tag{5.30}$$

In the flat in-region

$$2(X^+ - X^+_Q)(\phi - \phi_Q) + X^i X_i = 0, \tag{5.31}$$

such that in this region we have

$$\sigma_{ij} = \frac{\eta_{ij}}{(\phi - \phi_Q)}. \tag{5.32}$$

Now, asking this surface to be a light-cone through all the spacetime corresponds to asking its normal vector to be null. This vector is easily found to be

$$N_\delta = \partial_\delta \Sigma = \partial_\delta (X^+ - X^+_Q + \frac{1}{2}\sigma_{ij} X^i X^j) = \tilde{n}_\delta + \frac{1}{2}\dot{\sigma}_{ij} X^i X^j n_\delta + \sigma_{\delta j} X^j \tag{5.33}$$

such that we have

$$N_\mu N_\nu \mathcal{G}^{\mu\nu} = 0 = (\dot{\sigma}_{ij} + \sigma_{ik}\sigma^k_j - H_{ij}) X^i X^j \tag{5.34}$$

which considering the arbitrariness of the transverse coordinates corresponds to

$$\dot{\sigma}_{ij} = H_{ij} - \sigma_{ik}{\sigma_j}^k. \tag{5.35}$$

As we observed in Sec. 5.3.2, the weak energy condition imposes ${H_i}^i \leq 0$. We can thus trace our equation and observe that by Schwartz inequality

$${\dot{\sigma}_i}^{\ i} + ({\sigma_i}^i)^2 \leq 0, \tag{5.36}$$

which is equivalent to [145]

$$\partial^2_\phi \exp \int^\phi d\tilde{\phi} {\sigma_i}^i \leq 0. \tag{5.37}$$

Now, before we continue let us think for a moment. We found how the coordinate X^+ has to change in order to keep a surface null in the wave region. However we know that a chart exists in which the x^+ coordinate is flat: the Rosen one. From (5.28) we know that

$$x^+ - x^+_Q = X^+ - X^+_Q + \frac{1}{2}\sigma_{ij} X^i X^j. \tag{5.38}$$

If we fix $x^+ = x^+_Q$ we see that the equation that describes a fixed value of x^+ reproduces (5.30), where now we can identify $\sigma_{ij} = \dot{e}_{ik} e_j^{\ k}$. Let us consider a point Q such that $\phi_Q \to -\infty$ and define $\sigma^{\uparrow}_{ij}$ to be the matrix associated to this choice. In this case we see from (5.30) that

$$\sigma^{\uparrow}_{ij}(\phi) = 0 \qquad \text{for} \quad \phi \in \text{in-region}. \tag{5.39}$$

From this condition it follows that for $\phi \in$ in-region

$$\partial_\phi \exp \int^{\phi} d\tilde{\phi} \sigma^{\uparrow,i}_{i} = 0. \tag{5.40}$$

This expression, plus (5.37), implies that for some finite ϕ^* outside of the in-region

$$\exp \int^{\phi^*} d\tilde{\phi} \sigma^{\uparrow,i}_{i} = 0, \tag{5.41}$$

which means that some components of $\sigma^{\uparrow}_{ij}$ must become infinite. Now, let us assume for simplicity that this occurs in the out-region. The evolution equation (5.35) then implies that in the out-region, where $H_{ij} = 0$

$$\partial_\phi \left(\sigma^{\uparrow}_{ij}\right)^{-1} = \eta_{ij}, \tag{5.42}$$

from which it follows that in this region

$$\left(\sigma^{\uparrow}_{ij}\right)^{-1} = \phi\eta_{ij} - c_{ij}. \tag{5.43}$$

Here we see that whenever ϕ is an eigenvalue of the matrix c_{ij} we get a singularity for $\sigma^{\uparrow}_{ij}$. In the case in which the two eigenvalues of c_{ij} are the same (anastygmatic case), such that $c_{ij} = \phi^* \eta_{ij}$, the matrix $\sigma^{\uparrow}_{ij}$ gets the form

$$\sigma^{\uparrow}_{ij} = \frac{\eta_{ij}}{\phi - \phi^*}. \tag{5.44}$$

Here we see that the null cone originating from the point Q is here refocused in another point R with coordinates $R = (\phi^*, 0, 0, X^+_Q)$. This is the case of a gravitational wave produced by an electromagnetic wave, the anastygmatic case discussed in Sec 5.3.2. if pure gravitational waves are present one gets also an astygmatic focusing: the cone is focused in a spacelike line passing through R. From this property it follows that it is impossible to define a global Cauchy surface, indeed this surface should intersect every null geodesic only once, but it is impossible to construct such a surface here.

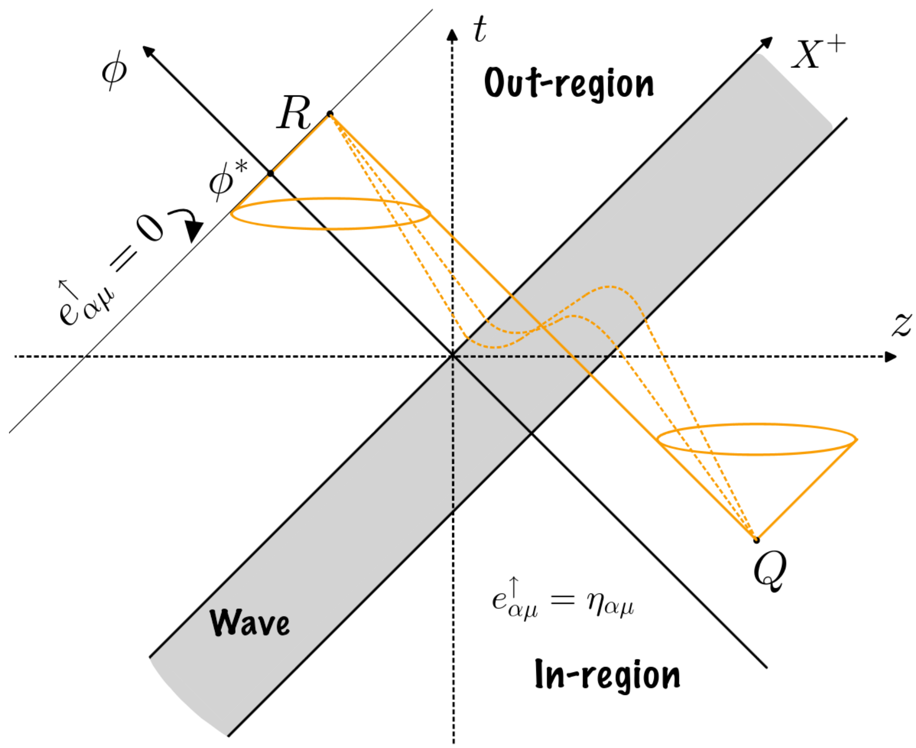


Figure 5.2: The null cone originating from Q is refocused by the gravitational wave in the point R at some phase ϕ^*. Moreover, at the same phase the Rosen chart initially well defined in the in-region becomes singular. The focusing, and with it the Rosen singularity, does not have to be in the out-region but it can also happen to be in the wave. Here, as in the text, we assumed a "weak condition" as in [145], such that ϕ^* belongs to the out-region.

To sum up, we found that in a gravitational plane wave null cones are inevitably refocused at some phase ϕ^*. This is not all, in fact we also proved that a Rosen metric initially well defined in the in-region will have a singularity at the same phase ϕ^*. Indeed, we can introduce the Rosen metric

$$g^{\uparrow}_{\mu\nu} = e^{\uparrow\alpha}{}_{\mu} e^{\uparrow}_{\alpha\nu} \tag{5.45}$$

with the initial condition

$$g^{\uparrow}_{\mu\nu} = \eta_{\mu\nu} \qquad \text{for} \quad \phi \in \text{in-region}. \tag{5.46}$$

The vierbein defined by this metric is clearly the one generating $\sigma^{\uparrow}_{ij} = \dot{e}^{\uparrow}_{ik} e^{\uparrow,k}_{j}$ and from the previous discussion we see that at the focusing phase $e^{\uparrow,k}_{j}(\phi^*) \to \infty$. Thus, at this phase the Rosen metric $g^{\uparrow,\mu\nu}$ becomes singular and it cannot be employed to describe the spacetime.

5.3.5 Classical dynamics in the Brinkmann chart

While the Einstein's equations in the Brinkmann chart are algebraic, the equations of motion are not completely trivial. The trajectories can be extracted from the Lagrangian

$$\mathcal{L} = m^{-1} \Pi^{\mu} \Pi^{\nu} \mathcal{G}_{\mu\nu}. \tag{5.47}$$

The fact that $n^\mu\partial_\mu$ is a Killing vector implies that Π^- is conserved such that $\Pi^- = p^-$, where p is the initial momentum. Exploiting the Euler-Lagrange equations, the transverse components are easily found to follow the harmonic equation

$$\ddot{X}^i = H^i{}_j X^j. \tag{5.48}$$

The last component Π^+ can be found algebraically from the on-shell condition $\Pi^\mu\Pi^\nu\mathcal{G}_{\mu\nu} = m^2$. A key point to study the geodesics in this chart is to find its Killing vectors. Studying the Killing equation $\mathcal{K}_{(\mu;\nu)} = 0$, it is possible to identify two important symmetries. By introducing a Rosen vierbein (here just a matrix) e_{ij} satisfying $\ddot{e}_{ij} = H_{ik}e^k{}_j$ and $\dot{e}_{[i}{}^j e_{k]j} = 0$, the Killing vectors take the form [8, 43]

$$\mathcal{K}^\mu_j = \gamma_{jk}\partial^\mu e_i{}^k X^i, \tag{5.49}$$

where $\gamma_{ij} = e_{li}e^l{}_j$. Having in mind the details of Rosen metrics and their connection to the Brinkmann chart, it is trivial to obtain these Killing vectors. Indeed, we know that the coordinates x^i are cyclic in the Rosen chart, from which it follow the two Killing vectors $\mathcal{K}^\mu_i = \gamma_{ij}\partial^\mu x^j$. Transforming these vectors to the Brinkmann chart and employing the transformations (5.28) one immediately gets Eq. (5.49).

From these two Killing vectors follow two conservation laws: let $\dot{X}^\mu$ be a geodesics, then the quantities $\mathcal{K}^\mu_j\dot{X}_\mu$ are conserved. It is immediate to see that these are equivalent to $\partial_\phi(e_i{}^j X^i) = c_k\gamma^{kj}$ with c_k being constant. Let us now suppose that the gravitational wave belongs to the sandwich-type, such that $H_{ij}(\phi) \neq 0$ only in the interval $\phi_1 < \phi < \phi_2$ for arbitrary fixed $\phi_{1,2}$. If we assume boundary conditions in the past flat region $\phi < \phi_1$, then we can choose $e_{ij} = \eta_{ij}$ initially and write these conservation laws as $p^-\partial_\phi(e_i{}^j X^i) = p_k\gamma^{kj}$, where p_k is the particle initial momentum before the interaction with the wave. Expanding the derivative we get the following expression for the transverse momentum

$$\Pi^i_p = e^{ij}p_j + p^-\sigma^i{}_j X^j \equiv p^i + \Delta^i_p, \tag{5.50}$$

where for future convenience we have introduced the symbol $\Delta^i_p = \Pi^i_p - p^i = \Delta e^{ij}p_j + p^-\sigma^i{}_j X^j$ to represent the correction to the transverse constant momentum one would have in flat spacetime. It is worth noting that the knowledge of Π^i still presumes the ability to solve the second-order differential equation $\ddot{e}_{ij} = H_{ik}e^k{}_j$ and the number of wave profiles H_{ij} allowing to find analytical solutions is extremely limited. However, this expression can be useful to study the geodesic congruences behavior. Exploiting the on-shell condition we can write the full momentum as [10, 9]

$$\Pi^\mu_p = \eta^{\mu\alpha}p_\alpha + \Delta^\mu_p - \frac{1}{2p^-}\Big[2p_i\Delta^i_p + \Delta_{p,i}\Delta^i_p + (p^-)^2 H\Big]n^\mu, \tag{5.51}$$

where $\Delta^\mu_p = \delta^\mu_i\Delta^i_p$ and $H = H_{ij}X^iX^j$. While by definition of plane waves $n_{\mu;\nu} = 0$, the geodesic congruence generated by ∂_- has the non trivial deformation tensor σ_{ij} [163, 8] as we have seen studying the null cones in Sec. 5.3.4.

Another important property of plane wave spacetimes concerns the imprint the wave leaves on particles after its passage. These features are known as memory effects [47, 101, 97, 186]. Let us consider a pair of particles in a sandwich-wave, which we know being Minkowskian in the in-region $\phi < \phi_1$ and out-region $\phi > \phi_2$, both initially at rest $p^\nu = (m, 0, 0, 0)$ for simplicity. If their initial perpendicular displacement is ΔX_0^i then, exploiting the fact that $e_i^{\ j} \Delta X^i = \Delta X_0^j$ due to the symmetries discussed above Eq. (5.50), we get for the momentum difference in the out-region the following expression [186]

$$\Delta\Pi_\nu\big|_{\text{out-region}} = m\dot{e}_{\nu i}(\phi_2)\Delta X_0^i - \frac{m}{2}\left[\Delta X_0^i \dot{e}_{ji}(\phi_2)\dot{e}^j_{\ k}(\phi_2)\Delta X_0^k\right] n_\nu. \tag{5.52}$$

This is an example of velocity memory effect: the vierbein is trivial in the in-region $e_{ij} = \eta_{ij}$ but it is constrained by the differential equation $\ddot{e}_{\nu i}(\phi_2) = 0$ in the out-region, such that its first derivative will generally be different from zero. This means that two particles initially at rest but displaced acquire a relative momentum after the passage of the wave. As we will see this property is also connected to the classical and quantum scattering of a particle by the gravitational wave itself.

5.3.6 Classical dynamics in the Rosen chart

In the Rosen metric we can exploit the large number manifest symmetries to find the geodesics. From the three Killing vector fields ∂_+, ∂_i we deduce the corresponding conserved momenta $\pi_p^- = p^-$, $\pi_{p,i} = p_i$. The last component π_p^+ can be derived from the on-shell condition $\pi_p^\mu \pi_p^\nu g_{\mu\nu} = m^2$. The complete momentum with initial conditions p_μ can be conveniently written as

$$\pi_{p,\mu} = p_\mu - \frac{1}{2p^-}\left(g^{\rho\nu}p_\rho p_\nu - m^2\right) n_\mu = p_\mu - \frac{p_i p_j}{2p^-}\int_{-\infty}^{\phi} d\tilde{\phi}\dot{\gamma}^{ij}(\tilde{\phi}) n_\mu. \tag{5.53}$$

It is interesting to observe that the vierbein projection of geodesics in Rosen coordinates is in a one-to-one correspondence with the motion of a charged particle in an electromagnetic plane wave [21]. Recalling the definition $e_\alpha^{\ \mu} = \delta_\alpha^{\ \mu} + \Delta e_\alpha^{\ \mu}$, where $\Delta e_\alpha^{\ \mu}$ is a 2×2 matrix with transverse indices, we introduce the notation

$$\Delta e_\alpha^{\ \mu} p_\mu = -\kappa P_\alpha. \tag{5.54}$$

The vierbein projected momentum $\bar{\pi}_p^\alpha = e^\alpha_{\ \mu}\pi_p^\mu$ is easily found to have the form

$$\bar{\pi}_p^\alpha = p^\alpha - \kappa P^\alpha + \frac{\kappa}{p^-}\left(p_\beta P^\beta - \frac{\kappa}{2}P_\beta P^\beta\right) n^\alpha, \tag{5.55}$$

with $p_\alpha = \delta_\alpha^\mu p_\mu$. This is exactly the momentum of a charged particle with charge e moving in an electromagnetic plane wave $A^\alpha(\phi)$ in flat spacetime, with the substitution $\kappa P^\alpha \leftrightarrow eA^\alpha$. It is worth noting that the vector field P^α has only transverse degrees of freedom by definition, as the electromagnetic wave. This formal equivalence is a consequence of the symmetries shared between the Rosen metric and a plane wave in flat spacetime. In particular, one can consider the vierbein projected geodesic equation $p^-\dot{\bar{\pi}}_\beta + \bar{\pi}^\alpha\omega_{\alpha\beta\delta}\bar{\pi}^\delta = 0$, where $\omega_{\alpha\beta\delta}$ are the spin connection coefficients $\omega_{\alpha\beta\delta} = e_\alpha^{\ \mu} e_\beta^{\ \nu} e_{\delta\nu;\mu}$. The coefficients depend on ϕ only, moreover the contraction $\bar{\pi}^\alpha\omega_{\alpha\beta\delta}\bar{\pi}^\delta$ is actually linear in the geodesic momentum because of the conservation of p_i and p^-. It is easy to show that $\bar{\pi}^\alpha\omega_{\alpha\beta\delta} = \bar{p}^\alpha\omega_{\alpha\beta\delta} = -\kappa\mathcal{P}_{\beta\delta}$, where we introduced the analog of the plane wave Maxwell tensor $\mathcal{P}_{\beta\delta} = 2\bar{\partial}_{[\beta}P_{\delta]}$. Exploiting this result we obtain a Lorentz equation for the vierbein projected geodesics

$$\dot{\bar{\pi}}_p^\alpha = \frac{\kappa}{p^-}\mathcal{P}^{\alpha\beta}\bar{\pi}_{p,\beta}. \tag{5.56}$$

As before this is formally equivalent to the Lorentz equation of a particle with charge κ in flat spacetime in presence of an electromagnetic plane wave $P^\alpha(\phi)$. It follows from the analysis developed in Sec. 5.2 that we can write the momentum as a local Lorentz transformation of the initial one, now depending on the vierbein components

$$\bar{\pi}^\alpha_p(\phi) = \Lambda^\alpha_{p,\beta}(\phi, P)p^\beta \quad , \qquad \Lambda_p(\phi, P) = \exp\left(\kappa \int_{-\infty}^{\phi} \frac{d\tilde{\phi}}{p^-} \boldsymbol{\mathcal{P}}(\tilde{\phi})\right) . \tag{5.57}$$

The transformation $\Lambda^\alpha_{p,\beta}(\phi, P)$ is a gauge transformation for the vierbein, therefore $\tilde{e}_\alpha{}^\mu = e_\beta{}^\mu \Lambda^\beta_{p,\alpha}$ defines a local reference system in which the particle has a constant momentum p^α throughout all the trajectory. This map is not only a formal interesting feature, it is very useful for translating some known results in electromagnetism and QED to systems in plane wave spacetimes and offers a very direct way to compare the linear results in these different contexts. One example is the motion of a charged particle moving in an electromagnetic plane wave keeping into account the spacetime curvature produced by the latter. The vierbein projected motion in this case is described by $p^- \dot{\bar{\pi}}^\alpha = \left(\kappa \mathcal{P}^{\alpha\beta} + e\bar{F}^{\alpha\beta}\right) \bar{\pi}_\beta$, therefore the momentum is formally identical to the one of a free-falling particle with the substitution $\kappa P^\alpha \to \kappa P^\alpha + e\bar{A}^\alpha$. While the formal analogy is as simple as clear, in the physical interpretation one has to be more careful. In fact, the connection between P_α and the physical complete gravitational wave profile H_{ij} is $\ddot{P}_i = H_i{}^j(p_j + P_j)$. Nonetheless, in the linear limit it is $P_\alpha = \frac{1}{2}p^\beta h_{\beta\alpha}$ and this is the most natural representative of the gravitational perturbation in the weak-field limit.

5.3.7 About the analytically solvable motions

As we have illustrated, given a metric in the Rosen form it is immediate to solve the geodesic equations and express the motion in terms of integrals of the metric itself. Now, once we have a specific Rosen form it is rather not trivial, and many times impossible, to find the Brinkmann waveform which corresponds to it. In fact, this is equivalent to solve the equation

$$\ddot{e}_{\alpha\mu} = H_{\alpha\beta} e^\beta{}_\mu \tag{5.58}$$

which is of the same order of complexity of the Einstein's equation in the Rosen form. Here we observe a sort of "conservation of complexity": in Brinkmann coordinates the vacuum Einstein equation becomes a trivial algebraic relation $H_\alpha{}^\alpha = 0$, while the equations of motion are second order ODE. On the other hand, in Rosen coordinates the Einstein's equation is a second order ODE, while the equations of motion are almost algebraic relations. In other words, we can choose where we want the problems to arise, but we cannot avoid them.

This being said, the object that reflects the gravitational properties in vacuum is $H_{\alpha\beta}$. We thus have to face the following problem: in which cases can we analytically solve the vierbein oscillator equation? Few solutions have already been considered in the literature [17],[18], here we report two simple but instructive cases.

Constant $H_{\alpha\beta}$

Let us warn the reader before we start: this spacetime is not flat at $\pm\infty$, therefore it is not a suitable choice for describing a sandwich wave. Nonetheless, it can be useful to study general properties of the motion. If the two polarizations $+, \times$ are linearly independent than the wave is in linear polarization. Here we will assume

$$H_{\alpha\beta} = H_+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.59}$$

In this case, assuming a diagonal vierbein we get the two differential equations

$$\ddot{e}_{11} = -H_+ e_{11} \qquad , \qquad \ddot{e}_{22} = H_+ e_{22} \tag{5.60}$$

and therefore, if we define $\omega_+ = \sqrt{H_+}$ and we impose the boundary conditions $e_{ij}(\phi_0) = \eta_{ij}$ and $\dot{e}_{ij}(\phi_0) = 0$ we get

$$e_{ij} = \begin{pmatrix} -\cos\omega_+(\phi - \phi_0) & 0 \\ 0 & -\cosh\omega_+(\phi - \phi_0) \end{pmatrix}. \tag{5.61}$$

Note that if H_+ is small we find to the first order

$$e_{ij} \simeq \eta_{ij} + \frac{1}{2} \begin{pmatrix} H_+(\phi - \phi_0)^2 & 0 \\ 0 & -H_+(\phi - \phi_0)^2 \end{pmatrix}. \tag{5.62}$$

And this is what we expected, a traceless perturbation quadratic in the phase. It is however interesting to note that starting with a constant traceless matrix one gets a vierbein which is oscillatory in one transverse direction and exponential in the other, two completely different behaviors. It is also interesting to note that the orders H_+^n with odd n in the expansion are traceless, while the even ones are pure traces. This behavior will be discussed again later when we will deal with the perturbative approach in Sec. 5.3.8 [101].

The conformal linear polarization

There are only three families of non-conformally flat vacuum solutions with conformal symmetry and all of them are plane waves [17], [74]. In particular the following waveform linearly polarized and its circular generalization belong to this class

$$H_{\alpha\beta} = \frac{a}{(\phi^2 + \epsilon^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.63}$$

Choosing $a = \epsilon^3$ and take the limit $\epsilon \to 0$, this waveform reduces to the impulsive case, in which $H_{\alpha\beta} \propto \delta(\phi)$ [8, 14]. Because of the many symmetries of this specific spacetime (the conformal group is 7-dimensional), we can actually solve analytically the equations

$$\ddot{e}_{\alpha\mu} = H_{\alpha\beta} e^{\beta}{}_{\mu}. \tag{5.64}$$

Being $H_{\alpha\beta}$ diagonal, we are free to choose the vierbein to be diagonal as well, we then get the two equations

$$\ddot{e}_{ij} = -(1)^i \frac{a}{(\phi^2+\epsilon^2)^2} e_{ij} \tag{5.65}$$

where the indices are not contracted. If $a < \epsilon^2$ the solutions can be written in the following form [17]

$$e^{\uparrow}_{ij} = \eta_{ij} \frac{\sqrt{\phi^2+\epsilon^2}}{\epsilon c_i} \sin\left[c_i\left(\arctan\left(\frac{\phi}{\epsilon}\right)+\frac{\pi}{2}\right)\right] \equiv \eta_{ij}\frac{\sqrt{\phi^2+\epsilon^2}}{\epsilon c_i}\sin\left(c_i\Lambda(\phi)\right), \tag{5.66}$$

where $c_i = \sqrt{1-(-1)^i \frac{a}{\epsilon^2}}$ and we imposed the initial conditions $e_{ij}(-\infty) = \eta_{ij}$ and $\dot{e}_{ij}(-\infty) = 0$. The out solutions is easily found to be

$$e^{\downarrow}_{ij}(\phi) = e^{\uparrow}_{ij}(-\phi). \tag{5.67}$$

Let us note that $0 < \Lambda(\phi) < \pi$, this means that the vierbein components get singular when $c_i\Lambda = \pi$ and this can happen only if $c_i > 1$ and this is true only for c_1. We conclude that only ${e_1}^1$ will be singular at $c_1\Lambda = \pi$, which corresponds to

$$\phi^* = -\varepsilon \cot\left(\frac{\pi}{c_1}\right). \tag{5.68}$$

This point is a geodesic focusing point [17], as discussed in Sec. 5.3.4. Moreover, the first derivative of the vierbein has the form

$$\dot{e}^{\uparrow}_{ij} = \eta_{ij}\frac{1}{\epsilon c_i\sqrt{\phi^2+\epsilon^2}}\left[\phi\sin\left(c_i\Lambda(\phi)\right)+\epsilon c_i\cos\left(c_i\Lambda(\phi)\right)\right] \tag{5.69}$$

and from this it follows that the deformation tensor

$$\sigma^{\uparrow}_{ij} = \delta_{ij}\frac{\phi+\epsilon c_i\cot\left(c_i\Lambda(\phi)\right)}{\phi^2+\epsilon^2}. \tag{5.70}$$

Clearly this tensor inherits the singular behavior from the vierbein, therefore $\sigma^{\uparrow}_{11}$ is singular in ϕ^*. From the previous arguments we know that $\sigma^{\downarrow}_{ij}(\phi) = -\sigma^{\uparrow}_{ij}(-\phi)$ and consequently that $\sigma^{\downarrow}_{11}$ has a singularity in $\phi = -\phi^*$.

5.3.8 Perturbative considerations

Let us introduce a Brinkmann waveform [101]

$$H_{\alpha\beta}(\epsilon,\phi) = \epsilon\mathcal{H}_{\alpha\beta}(\phi) \tag{5.71}$$

such that $\epsilon = \kappa\beta$ regulates the amplitude of the wave, β being a real number. In the same way we introduce an expansion for the vierbein and the related Rosen metric

$$e_{\alpha\mu}(\epsilon,\phi) = \sum_{a=0}^{\infty}\epsilon^a e^{(a)}_{\alpha\mu} \qquad , \qquad g_{\mu\nu}(\epsilon,\phi) = \sum_{a=0}^{\infty}\epsilon^a g^{(a)}_{\mu\nu}(\phi) \tag{5.72}$$

with

$$g^{(n)}_{\mu\nu} = \sum_{a+b=n} e^{(a)}_{\alpha\mu} e^{(b)\alpha}{}_{\nu}. \tag{5.73}$$

The oscillator equation now links different orders in ϵ

$$\ddot{e}^{(n)}_{\alpha\mu} = \mathcal{H}_{\alpha}{}^{\beta} e^{(n-1)}_{\beta\mu} \qquad , \qquad \ddot{e}^{(0)}_{\alpha\mu} = 0. \tag{5.74}$$

Let us impose the initial conditions such that the vierbein and the associated Rosen metric reduces to the Minkowskian form in the in-region

$$\lim_{\phi\to-\infty} e^{\uparrow}_{\alpha\mu}(\epsilon,\phi) = \eta^{\uparrow}_{\alpha\mu} \qquad , \qquad \lim_{\phi\to-\infty} \dot{e}^{\uparrow}_{\alpha\mu}(\epsilon,\phi) = 0, \tag{5.75}$$

from which it follow $e^{\uparrow(0)}_{\alpha\mu} = \eta_{\alpha\mu}$ and $\ddot{e}^{\uparrow(1)}_{\alpha\mu} = \mathcal{H}_{\alpha\mu}$. In general

$$e^{(n)}_{\alpha\mu} = \int_{-\infty}^{\phi} d\phi_1 \int_{-\infty}^{\phi_1} d\phi_2 \mathcal{H}_{\alpha}{}^{\beta}(\phi_2) e^{(n-1)}_{\beta\mu}(\phi_2). \tag{5.76}$$

It is worth observing that if we are in vacuum then $\mathcal{H}$ is traceless and therefore all the odd terms in the expansion are traceless, while the even ones are pure traces. Here we can connect the picture with the weak field limit identifying $\mathcal{H}_{\alpha\beta} = \frac{1}{2}\ddot{h}_{\alpha\beta}$ such that

$$e^{\uparrow}_{\alpha\mu}(\epsilon,\phi) = \eta_{\alpha\mu} + \frac{\epsilon}{2} h_{\alpha\mu}(\phi) + \frac{\epsilon^2}{4} \int_{-\infty}^{\phi} d\phi_1 \int_{-\infty}^{\phi_1} d\phi_2 \ddot{h}_{\alpha}{}^{\beta}(\phi_2) h_{\beta\mu}(\phi_2) + \mathcal{O}(\epsilon^3) \tag{5.77}$$

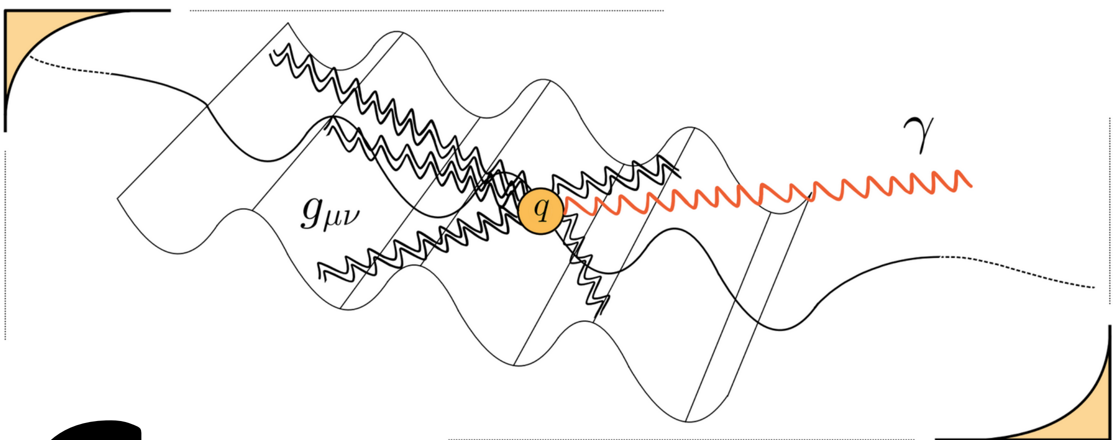
and the corresponding Rosen metric, integrating by parts reads

$$\begin{aligned} g^{\uparrow}_{\mu\nu}(\epsilon,\phi) = \eta_{\mu\nu} &+ \epsilon h_{\mu\nu}(\phi) \\ &+ \frac{\epsilon^2}{4}\left[h_{\gamma\beta}(\phi)h^{\beta\gamma}(\phi) - \int_{-\infty}^{\phi} d\phi_1 \int_{-\infty}^{\phi_1} d\phi_2 \dot{h}_{\gamma\beta}(\phi_2)\dot{h}^{\beta\gamma}(\phi_2)\right]\eta_{\mu\nu} + \mathcal{O}(\epsilon^3). \end{aligned} \tag{5.78}$$

Moreover, it is worth noting that the out-vierbein reducing to the Minkowski form in the out-region has the form

$$e^{\downarrow}_{\alpha\mu}(\epsilon,\phi) = \eta_{\alpha\mu} + \frac{\epsilon}{2} h_{\alpha\mu}(\phi) + \frac{\epsilon^2}{4}\left[\int_{\infty}^{\phi} d\phi_1 \int_{\infty}^{\phi_1} d\phi_2 \ddot{h}_{\gamma\beta}(\phi_2) h^{\beta\gamma}(\phi_2)\right]\eta_{\alpha\mu} + \mathcal{O}(\epsilon^3) \tag{5.79}$$

so clearly $e^{\downarrow}_{\alpha\mu} - e^{\uparrow}_{\alpha\mu} = \mathcal{O}(\epsilon^2)$. Here we see that in the weak-field approximation the in- and out-verbein can be identified. Nonetheless, the second order contributions inevitably distinguish them. The interesting fact we want to highlight here is that the second order contributions involve the integral of $\dot{h}_{\gamma\beta}\dot{h}^{\beta\gamma}$, which is a positive quantity. Thus there is a length scale at which the second order contributions exceed the first order. This is a very peculiar fact and it make it worth studying the connection between the description of waves in the full general relativity and with the weak-field approximation usually employed. Indeed, it could be possible that non-leading order correction are not negligible even if the wave amplitude is small, when long-scale interactions are considered [101].



6

Quantum states and photon emission in nonlinear gravitational waves

6.1 Fields with arbitrary spin in curved spacetime

6.1.1 Covariant differentiation

It is worth introducing the general formalism suited for dealing with fields in curved spacetime. The key element is the vierbein briefly introduced in Sec. 3.3.1. We recall that this matrix is defined by the following two equations

$$e^{\alpha}{}_{\mu} e_{\alpha\nu} = g_{\mu\nu} \quad , \quad e_{\alpha}{}^{\mu} e_{\beta\mu} = \eta_{\alpha\beta}, \tag{6.1}$$

where $g_{\mu\nu}$ is the metric of the curved spacetime considered and $\eta_{\alpha\beta}$ is the usual Minkowski metric. With the vierbein we can introduce the spin connection Γ_{μ}, which generalizes the covariant differentiation to arbitrary spin fields (see, e.g., [178, 33]). Namely, one can define for any spin representation of the Lorentz group the derivative

$$\nabla_\mu = \partial_\mu + \Gamma_\mu \tag{6.2}$$

and define the spin connection requiring this to properly transform under general coordinate and local Lorentz transformations. Let Ψ^a be a generic field, under the arbitrary local Lorentz transformation

$$e^\alpha_{\ \mu}(x) \to \Lambda^\alpha_{\ \beta}(x) e^\beta_{\ \mu}(x), \tag{6.3}$$

it will transform as $\Psi^a \to R^a_{\ b}(\Lambda)\Psi^b$, where $R^a_{\ b}(\Lambda)$ is a proper representation of the Lorentz transformation $\Lambda^\alpha_{\ \beta}$ [see Eq. (3.40) for the Dirac spinor case]. Now we ask ∇_μ to be a scalar with respect to the local Lorentz transformation, namely

$$\nabla_\mu \Psi^a \to R^a_{\ b}(\Lambda) \nabla_\mu \Psi^b. \tag{6.4}$$

This fixes the spin connection to be [33, 178]

$$\Gamma_\mu = \frac{i}{2} \omega_{\mu\alpha\beta} \Sigma^{\alpha\beta}, \tag{6.5}$$

where $\Sigma^{\alpha\beta}$ are the generators of the Lorentz algebra in the Ψ representation

$$[\Sigma^{\alpha\delta}, \Sigma^{\beta\gamma}] = -i \left(\eta^{\alpha\gamma} \Sigma^{\delta\beta} + \eta^{\delta\beta} \Sigma^{\alpha\gamma} - \eta^{\alpha\beta} \Sigma^{\delta\gamma} - \eta^{\delta\gamma} \Sigma^{\alpha\beta} \right) \tag{6.6}$$

and $\omega_{\mu\alpha\beta}$ are the spin coefficients defined by

$$\omega_{\mu\alpha\beta} = e_\alpha^{\ \nu} e_{\beta\nu;\mu}. \tag{6.7}$$

Note that here we define the derivative $;\mu$ to act only on curved spacetime indices while ∇_μ is the full covariant derivative extended to both Lorentz flat and curved spacetime indices. To illustrate this more clearly, one can check that

$$\nabla_\mu e^\alpha_{\ \nu} = 0, \tag{6.8}$$

which is a fundamental property of ∇_μ, while in general $e^\alpha_{\ \nu;\mu} \neq 0$ unless the spin coefficients are null.

As an example we can consider a very familiar object, a vector. If we want a vector with respect to local Lorentz transformations behaving as a scalar under general coordinate transformations we can simply project a vector field A^μ onto the vierbein

$$\bar{A}^\alpha = e^\alpha_{\ \mu} A^\mu. \tag{6.9}$$

From now on, unless otherwise stated, an overbar stands for a vierbein projection. For the vector representation $\left(\frac{1}{2}, \frac{1}{2}\right)$ of the Lorentz algebra we have

$$(\Sigma^{\alpha\beta})^\gamma_{\ \delta} = -i(\eta^{\alpha\gamma}\eta^\beta_{\ \delta} - \eta^{\beta\gamma}\eta^\alpha_{\ \delta}) \tag{6.10}$$

and therefore the spin connection in this representation is simply given by the spin coefficients

$$(\Gamma_\mu^{(\frac{1}{2},\frac{1}{2})})^\alpha{}_\beta = \omega_\mu{}^\alpha{}_\beta. \tag{6.11}$$

Thus we finally find that

$$\nabla_\mu \bar{A}^\alpha = \partial_\mu \bar{A}^\alpha + \omega_\mu{}^\alpha{}_\beta \bar{A}^\beta \tag{6.12}$$

and this could actually be taken as a definition of the spin connection: it generalizes the usual vector covariant derivative to Lorentz flat indices. Recalling the definition of covariant derivative of a vector with respect to curved spacetime indices $\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda} A^\lambda$ and exploiting the property (6.8) one can easily derive the form of the spin coefficients of Eq. (6.7).

Now we have a proper extended version of covariant differentiation we a re ready to deal with the generalizations of field equations in curved spacetime, this will be the topic of the next section.

6.1.2 Field equations

In this section we will discuss the equations of motion for scalar, Dirac and vector fields in a general curved spacetime. A scalar field in flat spacetime satisfies the Klein-Gordon equation

$$(\partial_\alpha \partial^\alpha + m^2)\mathit{\Phi} = 0. \tag{6.13}$$

It is worth observing that there is not only one way to extend this equation to general relativity. Indeed, any equation which respects general covariance and reduces to (6.13) in the flat spacetime limit could be an acceptable choice. The most general equation involving scalar, local couplings with the right dimensions in this case is

$$(\nabla_\mu \nabla^\mu + m^2 + \xi R)\Phi = 0, \tag{6.14}$$

where $\nabla_\mu \nabla^\mu \phi = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\partial^\mu \Phi)$, ξ is a number and R is the Ricci scalar. Considering specific values of the coefficients ξ can be very interesting (see,e.g., [33]), however here we are interested in the simple minimal coupling between the field and the background, such that $\xi = 0$. From this point forward, we will assume that the following equation holds

$$(\nabla_\mu \nabla^\mu + m^2)\Phi = 0. \tag{6.15}$$

Let us move on to the spinor case. The Dirac equation reads

$$(i\bar{\gamma}^\alpha \partial_\alpha - m)\mathit{\Psi} = 0. \tag{6.16}$$

Now, the Dirac matrices $\bar{\gamma}^\alpha$ are just matrices and not a vector. Thus, we cannot just substitute the flat indices with curved ones and write $\bar{\gamma}^\alpha \partial_\alpha \to \bar{\gamma}^\mu \nabla_\mu$, this would not transform properly. However, we know form the discussion of Sec. 3.3.1 that spinors behave like scalars under general coordinate transformations. Moreover the Dirac equation is Lorentz-covariant. Thus, we need a covariant derivative transforming as a vector under local Lorentz transformations and this is just $\bar{\nabla}^\alpha = e^{\alpha\mu}\nabla_\mu$. The proper generalization of the Dirac equation to curved spacetime is then

$$(i\bar{\gamma}^\alpha\bar{\nabla}_\alpha - m)\Psi = 0. \tag{6.17}$$

This equation can be alternatively written in terms curved spacetime indices if we define the generalized Dirac matrices

$$\gamma^\mu = e_\alpha{}^\mu\bar{\gamma}^\alpha, \tag{6.18}$$

such that

$$(i\gamma^\mu\nabla_\mu - m)\Psi = 0. \tag{6.19}$$

In Sec. 3.4 we described the interactions between spinors and gravitons at the first order in κ. This has to correspond to the linearization of the spinor Lagrangian obtained with the fully covariant equation (6.19). Let us check this is actually true in order to test the consistency of our picture. The spinor Lagrangian in curved spacetime is clearly

$$\mathcal{L}_f = \sqrt{g}\left(\frac{i}{2}\,\bar{\Psi}\gamma^\mu\overleftrightarrow{\nabla}_\mu\Psi - m\bar{\Psi}\Psi\right), \tag{6.20}$$

where the metric determinant is needed in order to have an invariant action $S_f = \int d^4x\mathcal{L}_f$. Now, from the vierbein definition $e^\alpha{}_\mu e_{\alpha\nu} = g_{\mu\nu}$ and assuming to work in the weak-field approximation $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, it is straightforward to find that

$$e^\alpha{}_\mu = \delta^\alpha_\mu + \frac{\kappa}{2}h^\alpha{}_\mu + \mathcal{O}(\kappa^2). \tag{6.21}$$

The spin connection term vanishes at the first order, thus we are left with

$$\mathcal{L}_f = \mathcal{L}_f^{(0)} - \frac{\kappa}{2}\,h_{\alpha\beta}\left[\frac{i}{2}\,\bar{\Psi}\bar{\gamma}^\alpha\overleftrightarrow{\partial}^\beta\,\Psi - \eta^{\alpha\beta}\bar{\Psi}(\frac{i}{2}\overleftrightarrow{\not{\partial}} - m)\Psi\right] + \mathcal{O}(\kappa^2). \tag{6.22}$$

The order $\mathcal{O}(\kappa)$ reproduces exactly the linear interaction between fermions and gravitons we found in Eq. (3.71).

We can now proceed and deal with spin-1 abelian fields. Let us start from the massless case. The familiar inhomogeneous Maxwell equations in vacuum can be covariantly extended to curved spacetime as

$$\nabla_\mu F^{\mu\nu} = 0, \tag{6.23}$$

where $F_{\mu\nu} = \nabla_\mu\mathcal{A}_\nu - \nabla_\nu\mathcal{A}_\mu = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu$ due to the torsionless nature of gravity. Using the identity $[\nabla_\mu, \nabla_\nu]\mathcal{A}^\mu = -R_\nu{}^\mu\mathcal{A}_\mu$ we can expand these equations in the form

$$\nabla_\mu F^{\mu\nu} = \nabla_\mu\nabla^\mu\mathcal{A}^\nu - \nabla_\mu\nabla^\nu\mathcal{A}^\mu = \nabla_\mu\nabla^\mu\mathcal{A}^\nu + R_\mu{}^\nu\mathcal{A}^\mu - \nabla^\nu\nabla_\mu\mathcal{A}^\mu. \tag{6.24}$$

If we choose to work in the covariant Lorenz gauge $\nabla_\mu\mathcal{A}^\mu = 0$ we can simplify this expression to

$$\nabla_\mu\nabla^\mu\mathcal{A}^\nu + R^\nu{}_\mu\mathcal{A}^\mu = 0. \tag{6.25}$$

We are now free to include a mass term to generalize the equation to massive bosons

$$(\nabla_\mu \nabla^\mu + m^2)\mathcal{A}^\nu + R^\nu{}_\mu \mathcal{A}^\mu = 0. \tag{6.26}$$

The spin-2 wave equation for a massless perturbation in vacuum can be found computing the Einstein's equations in a metric $g_{\mu\nu} + \kappa h_{\mu\nu}$ to the first order in κ, for arbitrary $g_{\mu\nu}$. This is a very straightforward procedure and we refer to [176] for the details, the resulting equation in the TT-gauge reads

$$\nabla_\rho \nabla^\rho h_{\mu\nu} + 2R^\rho{}_{\mu\nu\delta} h_\rho{}^\delta = 0. \tag{6.27}$$

6.2 Quantum fields in a plane wave

6.2.1 In flat spacetime

In Sec. 5.2 we noticed that in an electromagnetic background the Lorentz-like transformation generated by $\omega^{\alpha\beta}(\phi, A) = e\int^\phi d\tilde{\phi} F^{\alpha\beta}(\tilde{\phi})$ plays a very important role. Indeed, $\Lambda_p = \exp(\omega/p^-)$ completely solves the dynamics acting as an evolution operator on the initial four-momentum: $\pi_p^\alpha = \Lambda^\alpha_{p,\beta} p^\beta$. The importance of this transformation is not limited to particle dynamics but it is essential in solving fields equations. Let us recall the form of the Hamilton-Jacobi action of Eq. (3.6)

$$S_p(x) = -p\cdot x - \frac{e}{p^-}\int^\phi d\tilde{\phi}\left[p\cdot A(\tilde{\phi}) - \frac{e}{2}A^2(\tilde{\phi})\right], \tag{6.28}$$

which satisfies

$$e^{-iS_p} iD_\alpha e^{iS_p} = e^{-iS_p}(i\partial_\alpha - eA_\alpha)e^{iS_p} = \pi_{p,\alpha}. \tag{6.29}$$

From the momentum evolution Eq. (5.57) we find that

$$e^{-iS_p} iD_\alpha e^{iS_p} = \Lambda^\alpha_{p,\beta} p^\beta. \tag{6.30}$$

This means that the quantum operator $\mathcal{Y} = \exp i\,(S_p + p\cdot x)\,|_{p\to i\partial}$ constructed from the difference between the dressed and the free action is a unitary transformation that changes the covariant derivative into the locally Lorentz transformed free derivative [50, 121]. If we define the operator $\Lambda = \Lambda_p|_{p\to i\partial}$ we find the useful relation

$$\mathcal{Y}^{-1} D^\alpha \mathcal{Y} = \Lambda^\alpha{}_\beta \partial^\beta. \tag{6.31}$$

This can be seen as an evolution operator for the free field $\varphi_p = e^{-ip\cdot x}$ such that

$$\varPhi_p = \mathcal{Y}\varphi_p \tag{6.32}$$

solves the Klein-Gordon equation $\left(D^\alpha D_\alpha + m^2\right)\varPhi_p = 0$. It is indeed immediate to check this exploiting the identity

$$\mathcal{Y}^{-1} D^2 \mathcal{Y}\varphi_p = \partial^2 \varphi_p, \tag{6.33}$$

which in turn follows from the properties $\Lambda^{\alpha}{}_{\beta}\Lambda^{\gamma\beta} = \eta^{\alpha\gamma}$ and $\Lambda^{\alpha}{}_{\beta}n^{\beta} = n^{\alpha}$, recalling that $\Lambda^{\alpha}{}_{\beta}$ depends only on ϕ.

We now proceed considering Dirac spinors and massive charged vector fields [50, 138], as we will see there is a very natural way to exploit Λ in order to find these quantum states. Let us introduce the generalized quantum operator

$$\mathbf{\Lambda}(\phi, A) = e^{\frac{e}{2\partial_+}\int^{\phi} d\tilde{\phi} F^{\alpha\beta}(\tilde{\phi})\mathbf{\Sigma}_{\alpha\beta}}, \tag{6.34}$$

where $\mathbf{\Sigma}_{\alpha\beta}$ are the generators of the Lorentz algebra in a general representation. Above, the four-vector representation was considered, where

$$(\mathbf{\Sigma}_{\alpha\beta})^{\gamma\delta} = -i(\delta_{\alpha}{}^{\gamma}\delta_{\beta}{}^{\delta} - \delta_{\alpha}{}^{\delta}\delta_{\beta}{}^{\gamma}). \tag{6.35}$$

The wave equations for fields $\boldsymbol{\Phi}_p$ of spin $\left\{0, \frac{1}{2}, 1\right\}$ in an electromagnetic plane wave can be written as

$$(\tilde{\nabla}^{\alpha}\tilde{\nabla}_{\alpha} + m^2)\boldsymbol{\Phi}_p = 0 \qquad , \quad \text{with} \qquad \tilde{\nabla}_{\alpha} = D_{\alpha}\mathbf{1} - \frac{e}{2\partial_+}n_{\alpha}F^{\gamma\beta}\mathbf{\Sigma}_{\gamma\beta} = \mathbf{\Lambda}D_{\alpha}\mathbf{\Lambda}^{-1}. \tag{6.36}$$

It is worth observing that in order to prove the last equality one has to exploit the identity

$$[F^{\alpha\beta}(\phi)\mathbf{\Sigma}_{\alpha\beta}, F^{\gamma\delta}(\phi')\mathbf{\Sigma}_{\gamma\delta}] = 0 \tag{6.37}$$

for arbitrary values of ϕ and ϕ', which derives from the fact that $\mathbf{\Sigma}_{\alpha\beta}$ are the generators of the Lorentz algebra. One can expand $\tilde{\nabla}^{\alpha}\tilde{\nabla}_{\alpha} = D_{\alpha}D^{\alpha} - eF^{\gamma\beta}\mathbf{\Sigma}_{\gamma\beta}$ and check that this reproduces the equations studied for example in [50, 138] for the scalar, Dirac and vector fields

$$\begin{aligned}
&(D_{\alpha}D^{\alpha} + m^2)\Phi_p = 0, \\
&(D_{\alpha}D^{\alpha} + ie\not{n}\not{A} + m^2)\Psi_p = 0, \\
&[(D_{\alpha}D^{\alpha} + m^2)\delta^{\beta}_{\gamma} + 2ieF^{\beta}{}_{\gamma}]\mathcal{A}^{\gamma}_p = 0 \quad , \quad D_{\alpha}\mathcal{A}^{\alpha}_p = 0.
\end{aligned} \tag{6.38}$$

Now, looking at the wave equations in the form of Eq. (6.36) we see that the property $\tilde{\nabla}_{\alpha} = \mathbf{\Lambda}D_{\alpha}\mathbf{\Lambda}^{-1}$ allows us to get rid of the spin structure, reducing the equation to the scalar case. We can indeed write the equations $(\tilde{\nabla}^{\alpha}\tilde{\nabla}_{\alpha} + m^2)\boldsymbol{\Phi}_p = 0$ as $(D^{\alpha}D_{\alpha} + m^2)\mathbf{\Lambda}^{-1}\boldsymbol{\Phi}_p = 0$, such that the problem is reduced to finding the scalar field solution. We already know the solution this problem which is $\Phi_p = \mathcal{Y}\varphi_p$ and we thus conclude that $\mathcal{Y}\mathbf{\Lambda}$ is the evolution operator for fields up to spin one

$$\boldsymbol{\Phi}_p = \mathbf{\Lambda}\mathcal{Y}\boldsymbol{\varphi}_p, \tag{6.39}$$

where $\boldsymbol{\varphi}_p$ is the initial condition in absence of the background wave. This clearly corresponds to the fact that $\mathbf{\Lambda}$ satisfies the equation $\mathbf{\Lambda}^{-1}\mathcal{Y}^{-1}\tilde{\nabla}^{\alpha}\mathcal{Y}\mathbf{\Lambda} = \Lambda^{\alpha}{}_{\beta}\partial^{\beta}$ and from this it follows $\mathbf{\Lambda}^{-1}\mathcal{Y}^{-1}\tilde{\nabla}^2\mathcal{Y}\mathbf{\Lambda} = \partial^2$. For the sake of clarity, let us introduce the ϕ-dependent polarization $\mathbf{S}_p(\phi)$ such that

$$\boldsymbol{\Phi}_p = \mathbf{S}_p e^{iS_p}. \tag{6.40}$$

Defining $\mathbf{\Lambda}_p = \mathbf{\Lambda}|_{\partial \to -ip}$ it is easy to check that this can be explicitly written as

$$\mathbf{S}_p = \mathbf{\Lambda}_p \boldsymbol{s}_p, \tag{6.41}$$

where $\boldsymbol{s}_p = \{1, u_p, \varepsilon_p^\alpha\}$ are the free polarizations for scalar, Dirac, and vector fields, respectively. For example, in the well known positive-energy Volkov solution [173] we studied in Sec. 3.1 $\mathcal{U}_p = U_p e^{iS_p}$ the spinorial term can be found as $U_p^a = \mathbf{\Lambda}^a_{p,b} u_p^b$, such that

$$\mathcal{U}_p = e^{iS_p}\left(1 + e\frac{\not{n}\not{A}}{2p^-}\right)u_p. \tag{6.42}$$

In the same way, the spin-one solution $\mathit{\Phi}_p^\alpha = \mathcal{E}_p^\alpha e^{iS_p}$ is found to be [50]

$$\mathit{\Phi}_p^\alpha = e^{iS_p}\left[\varepsilon_p^\alpha - \frac{\varepsilon_p^-}{p^-}eA^\alpha + \frac{e}{p^-}\left(\varepsilon_p^\beta A_\beta - \frac{\varepsilon_p^-}{2p^-}eA_\beta A^\beta\right)n^\alpha\right]. \tag{6.43}$$

Being in the vector representation, $\mathcal{E}_p^\alpha = \Lambda^\alpha_{p,\beta}\varepsilon_p^\beta$ reduces to the particle classical momentum if ε_p^α is replaced by p^α. The Lorenz condition corresponds to the natural requirement $\mathcal{E}_p^\alpha \pi_{p,\alpha} = \varepsilon_p^\alpha p_\alpha = 0$.

As a side note, let us mention that the symbol $\tilde{\nabla}_\alpha$ was not randomly chosen. In fact, it defines a sort of parallel transport rule for the field. From the properties of the operator $\tilde{\nabla}_\alpha$ it follows that $\tilde{\nabla}^\alpha \boldsymbol{\mathit{\Phi}}_p = -i\Lambda^\alpha_{p,\beta} p^\beta \boldsymbol{\mathit{\Phi}}_p$. Thus, a contraction with the momentum gives $\pi_p^\alpha \tilde{\nabla}_\alpha \boldsymbol{\mathit{\Phi}}_p = -im^2 \boldsymbol{\mathit{\Phi}}_p$. If we insert the expression $\boldsymbol{\mathit{\Phi}}_p = \mathbf{S}_p e^{iS_p}$ this equation reduces to a condition on the polarization function, which is formally equivalent to a parallel transport

$$\pi_p^\alpha \tilde{\nabla}'_{p,\alpha} \mathbf{S}_p = 0, \tag{6.44}$$

where $\tilde{\nabla}'_{p,\alpha} = \partial_\alpha \mathbf{1} - i\frac{e}{2p^-} n_\alpha F^{\gamma\beta} \mathbf{\Sigma}_{\gamma\beta} = \mathbf{\Lambda}_p \partial_\alpha \mathbf{\Lambda}_p^{-1}$.

It is also worth noting that if we choose the vector representation then

$$\pi_p^\alpha \tilde{\nabla}'_{p,\alpha} \pi_p^\beta = 0 \tag{6.45}$$

is actually equivalent to the Lorentz equation. We could rephrase the preceding discussion as follows: to construct the fields previously considered one can first operate a unitary transformation $\mathcal{Y}$ that turns the gauge derivative D_α into $\Lambda^\alpha{}_\beta \partial^\beta$, then transport the polarization function through the operator $\tilde{\nabla}'_{p,\alpha}$. We will now see how these considerations can be translated in a plane wave spacetime.

6.2.2 In the Rosen metric

The Rosen chart is a particularly appropriated choice to solve fields equations. Let us start with the scalar field satisfying

$$(\nabla_\mu \nabla^\mu + m^2)\Phi = 0. \tag{6.46}$$

In vacuum flat spacetime we are used to expand fields in terms of momentum eigenstates, but in a general curved spacetime this is not possible because there is no clear global notion of Fourier components. Nonetheless, the plane wave spacetime is quite peculiar because in this space the scalar wave satisfies the Huygens' principle [87, 177]. By exploiting the identity $\nabla_\mu \nabla^\mu \Phi = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} \partial^\mu \Phi)$ we can rewrite the wave equation as

$$(\dot{\log}(\sqrt{g})\partial_+ + \partial^\mu \partial_\mu + m^2)\Phi = 0. \tag{6.47}$$

Now, being ∂_i, ∂_+ symmetries of the spacetime under consideration, we will assume the solution to be an eigenstate of these operators. It is then easy to reduce the equation to

$$(\partial^\mu \partial_\mu + m^2) g^{\frac{1}{4}} \Phi = 0. \tag{6.48}$$

From this equation one immediately sees that $g^{\frac{1}{4}} \Phi_p = e^{iS_p}$ is a solution with initial momentum p if S_p is the classical action satisfying

$$g^{\mu\nu} \partial_\mu S_p \partial_\nu S_p = m^2 \quad \text{and} \quad \partial^\mu \partial_\mu S_p = 0. \tag{6.49}$$

Introducing the notation $\Omega = g^{-\frac{1}{4}}$ we can write the scalar wave as

$$\Phi_p = \Omega e^{iS_p}. \tag{6.50}$$

The action can be easily computed exploiting the symmetries of the metric, but it can also be borrowed from electromagnetism. If we project the derivatives in Eq. (6.48) onto the vierbein and we split the latter as in Sec. 5.3.6 $e_\alpha{}^\mu = \delta_\alpha{}^\mu + \Delta e_\alpha{}^\mu$, we obtain

$$[\eta^{\alpha\beta}(\partial_\alpha + \Delta e_\alpha{}^i \partial_i)(\partial_\beta + \Delta e_\beta{}^j \partial_j) + m^2]\Omega^{-1}\Phi_p = 0. \tag{6.51}$$

By observing that $\eta^{\alpha\beta}[\Delta e_\alpha{}^\mu \partial_\mu, \partial_\beta + \Delta e_\beta{}^\nu \partial_\nu] = 0$ and by recalling that Φ_p is eigenstate of ∂_i, this equation can be written as

$$[\eta^{\alpha\beta}(\partial_\alpha + i\kappa P_\alpha)(\partial_\beta + i\kappa P_\beta) + m^2]\Omega^{-1}\Phi_p = 0. \tag{6.52}$$

Now, this is formally the scalar wave equation for the field $\Omega^{-1}\Phi_p$ in an electromagnetic plane wave with the potential and charge product replaced by κP_β. It then follows that the action $S_p(x)$ is just the one of a charged particle moving in an electromagnetic plane wave under the same substitution:

$$S_p(x) = -p_\mu x^\mu - \frac{\kappa}{p^-} \int^\phi d\tilde{\phi} \left(p_\alpha P^\alpha - \frac{\kappa}{2} P_\alpha P^\alpha \right). \tag{6.53}$$

One can easily verify that

$$-\partial_\alpha S_p(x) = p_\alpha + \frac{\kappa}{p^-} \left(p_\beta P^\beta - \frac{\kappa}{2} P_\beta P^\beta \right) n_\alpha = \bar{\pi}_{p,\alpha} + \kappa P_\alpha, \tag{6.54}$$

in complete analogy with the electromagnetic case [see Eq. (3.7) and Eq. (5.55)]. To summarize, we found that the solution of the Klein-Gordon equation in the Rosen metric is $\Phi_p = \Omega \mathit{\Phi}_p$, where $\mathit{\Phi}_p$ is formally the field with charge κ satisfying the Klein-Gordon equation in flat spacetime with an electromagnetic plane wave potential P_β.

Interestingly, the same is true for Dirac fields, such that the spinor in a plane wave spacetime is $\Psi_p = \Omega \mathit{\Psi}_p$, where $\mathit{\Psi}_p$ is the Volkov state [173] of charge κ in a wave potential P_β we discussed in Sec. 3.1

$$\Psi_p = \Omega e^{iS_p}\left(1 + \kappa\frac{\not{n}\not{P}}{2p^-}\right)u_p. \tag{6.55}$$

This can be verified observing that the Dirac equation in this spacetime (see Sec. 6.1.2)

$$(i\gamma^\mu \nabla_\mu - m)\Psi_p = 0 \tag{6.56}$$

can be reduced to

$$[i(\not{\partial} + i\kappa\not{P}) - m]\Omega^{-1}\Psi_p \tag{6.57}$$

and this is once again formally the Dirac equation in flat spacetime in presence of a plane wave field κP^α. In this case, the spin connection term $\gamma^\mu\Gamma_\mu$ is exactly compensated by the Ω prefactor. Indeed, in this metric the spin coefficients read (see App. A.10) $\omega_{\mu\alpha\beta} = -2n_{[\alpha}\dot{e}_{\beta]\mu}$, such that the spin connection when contracted with a Dirac matrix gets the simple form

$$\gamma^\mu\Gamma_\mu = \frac{1}{4}\dot{\log}(g)\not{n} = -\dot{\log}(\Omega)\not{n}. \tag{6.58}$$

Inserting this expression in the Dirac equation one finds

$$(i\gamma^\mu\partial_\mu - i\dot{\log}(\Omega)\not{n} - m)\Psi_p = \Omega(i\gamma^\mu\partial_\mu - m)\Omega^{-1}\Psi_p = 0, \tag{6.59}$$

from which it directly follows Eq. (6.57). We will now proceed and consider vector fields.

The spin-one massive case is slightly more involved. Let us introduce the notation

$$\bar{\mathcal{A}}^\alpha = \Omega\bar{\mathit{\Phi}}^\alpha, \tag{6.60}$$

where $\bar{\mathcal{A}}^\alpha = e^\alpha{}_\mu\mathcal{A}^\mu$ is the vierbein-projected vector field. Keeping into account the condition $\nabla_\mu\mathcal{A}^\mu = 0$ it is possible to reduce the field equation (6.26)

$$(\nabla_\mu\nabla^\mu + m^2)\mathcal{A}^\nu + R^\nu{}_\mu\mathcal{A}^\mu = 0 \tag{6.61}$$

to the form (see S.m. 6.5.1)

$$(\bar{\partial}^\beta\bar{\partial}_\beta + m^2)\bar{\mathit{\Phi}}^\alpha + 2i\kappa\hat{\mathcal{P}}^\alpha{}_\gamma\bar{\mathit{\Phi}}^\gamma + \frac{1}{2}\ddot{\log}(g)\bar{\mathit{\Phi}}^- n^\alpha = 0, \tag{6.62}$$

where we introduced the quantum analog of the Maxwell-like tensor $\mathcal{P}_{\alpha\beta}$ substituting $p_i \to i\partial_i$, namely $\kappa\hat{\mathcal{P}}^{\alpha\beta} = -f^{\alpha\beta}_j\dot{e}^{jk}i\partial_k$. Here, we see that the last term in the l.h.s., which comes from the contraction of two spin connections, is a nonlinear term absent in flat spacetime [see the last of Eqs. (6.38)].

However, contracting with n^α one finds that

$$(\bar{\partial}^\beta\bar{\partial}_\beta + m^2)\bar{\mathit{\Phi}}^- = 0 \tag{6.63}$$

and therefore, assuming an initial momentum p it follows that

$$\bar{\Phi}^-_p = \varepsilon^-_p e^{iS_p}, \tag{6.64}$$

with ε^-_p constant. Exploiting the property $\bar{\partial}_\beta \bar{\Phi}^-_p = -i\bar{\pi}_{p,\beta}\bar{\Phi}^-_p$, one can check that the shift $\bar{\Phi}'^\alpha_p = \bar{\Phi}^\alpha_p + \frac{i}{4p^-}\dot{\log}(g)\bar{\Phi}^-_p n^\alpha$ restores the form of a flat spacetime field equation in an electromagnetic wave background Eq. (6.38)

$$(\bar{\partial}^\beta \bar{\partial}_\beta + m^2)\bar{\Phi}'^\alpha_p + 2i\kappa \mathcal{P}^\alpha{}_\gamma \bar{\Phi}'^\gamma_p = 0. \tag{6.65}$$

Indeed, this field satisfies the condition $\bar{\partial}_\alpha \bar{\Phi}'^\alpha_p = (\partial_\alpha + i\kappa P_\alpha)\bar{\Phi}'^\alpha_p = 0$, which is the analog of $D_\alpha \Phi^\alpha_p = 0$ found in flat spacetime in the previous section.

Finally, we can report the solution for the massive vector field in the form

$$\bar{\mathcal{A}}^\alpha_p = \Omega e^{iS_p}\bar{\mathcal{E}}^\alpha_p, \tag{6.66}$$

namely [21]

$$\bar{\mathcal{A}}^\alpha_p = \Omega e^{iS_p}\left[\varepsilon^\alpha_p - \frac{\varepsilon^-_p}{p^-}\kappa P^\alpha + \frac{\kappa}{p^-}\left(\varepsilon^\beta_p P_\beta - \frac{\varepsilon^-_p}{2p^-}\kappa P_\beta P^\beta\right)n^\alpha + i\dot{\log}(\Omega)\frac{\varepsilon^-_p}{p^-}n^\alpha\right]. \tag{6.67}$$

Here one can explicitly observe once again the formal analogy with the case of an electromagnetic background in flat spacetime. Once the shift previously described is operated the polarization $\bar{\mathcal{E}}^\alpha_p$ can be found substituting $eA^\alpha \to \kappa P^\alpha$ in the flat spacetime solution of Eq. (6.43).

If the massless case is considered one can exploit the gauge freedom to fix

$$\bar{\partial}_\alpha \bar{\Phi}^\alpha_l = 0 = \bar{\Phi}^-_l, \tag{6.68}$$

which implies $\nabla_\mu \mathcal{A}^\mu_l = 0$ as well. In this case the additional term proportional to $\ddot{\log}(g)$ in Eq. (6.62) vanishes and the solution is once again in direct correspondence with the one in flat spacetime, without any shift needed

$$\bar{\mathcal{A}}^\alpha_l = \Omega e^{iS_l}\left[\varepsilon^\alpha_l + \frac{\kappa}{l^-}\varepsilon^\beta_l L_\beta n^\alpha\right], \tag{6.69}$$

where here L^α is the equivalent of P^α but with initial momentum l^α. We will deal with this choice studying the field in Brinkmann coordinates.

The massive spin-two field is not of our interest in the present work and we will confine ourselves to the massless case, which is easily treated in Brinkmann coordinates once the proper gauge is chosen [8]. For completeness we provide a brief treatment of this topic in Rosen coordinates in S.m. 6.5.2 and we direct the reader to it for further details.

We can now retrace the discussion carried out in flat spacetime, i.e., that the operator constructed from the difference between the free and the curved spacetime actions $\mathcal{U} = \exp i\,(S_p + p_\mu x^\mu)\,|_{p\to i\partial}$ is a unitary transformation that changes the vierbein projected derivative into the Lorentz transformed curved one $\mathcal{Y}^{-1}\bar{\partial}_\alpha \mathcal{Y} = \Lambda_\alpha{}^\beta \partial_\beta$, with $\partial_\alpha = \delta^\mu_\alpha \partial_\mu$. We can introduce the gravitational analog of the Lorentz-like transformation exploited in the electromagnetic wave background treatment, which can be obtained through the simple substitution emerged in the study of the dynamics $e\dot{A}^j \to \kappa \hat{P}^j = -i\kappa\Delta\dot{e}^{jk}\partial_k$

$$\mathbf{\Lambda} = e^{\frac{\kappa}{2\partial_+}\int_{-\infty}^{\phi} d\tilde{\phi}\hat{\mathcal{P}}^{\alpha\beta}(\tilde{\phi})\mathbf{\Sigma}_{\alpha\beta}}. \tag{6.70}$$

The wave equation for the fields $\{\Phi, \Psi, \bar{A}'^{\alpha}\} \equiv \bar{\mathbf{\Phi}}_p$ can be written as

$$(\tilde{\nabla}^{\alpha}\tilde{\nabla}_{\alpha} + m^2)\Omega^{-1}\bar{\mathbf{\Phi}}_p = 0 \qquad , \quad \text{with} \qquad \tilde{\nabla}_{\alpha} = \bar{\partial}_{\alpha}\mathbf{1} - \frac{\kappa}{2\partial_+}n_{\alpha}\hat{\mathcal{P}}^{\gamma\beta}\mathbf{\Sigma}_{\gamma\beta} = \mathbf{\Lambda}\bar{\partial}_{\alpha}\mathbf{\Lambda}^{-1}, \tag{6.71}$$

where we recall that the spin-one field now involves the shift $\bar{\mathcal{A}}'^{\alpha} = \bar{\mathcal{A}}^{\alpha} - \frac{i}{p^-}\dot{\log}(\Omega)\bar{\mathcal{A}}^{-}n^{\alpha}$. Let us point out that $\tilde{\nabla}_{\alpha}$ is not the vierbein projected covariant derivative. This operator is defined by the representation of the considered field and not by the object it is acting on. This is where its relevance comes from as it allows one to write the wave equation as the actual square of an operator. However, when taken along a geodesic this operator is actually equivalent to the vierbein-projected covariant derivative for the representations here discussed, namely

$$\bar{\pi}_p^{\alpha}\tilde{\nabla}_{p,\alpha} = \bar{\pi}_p^{\alpha}\bar{\nabla}_{\alpha}, \tag{6.72}$$

with $\tilde{\nabla}_{p,\alpha} = \bar{\partial}_{\alpha}\mathbf{1} - i\frac{\kappa}{2p^-}n_{\alpha}\mathcal{P}^{\gamma\beta}\mathbf{\Sigma}_{\gamma\beta}$. This is quickly verified exploiting the spin coefficients property $\bar{\pi}^{\alpha}\omega_{\alpha\beta\delta} = -\kappa\mathcal{P}_{\beta\delta}$. To complete the correspondence we have the properties

$$\mathbf{\Lambda}^{-1}\mathcal{Y}^{-1}\tilde{\nabla}_{\alpha}\mathcal{Y}\mathbf{\Lambda} = \Lambda_{\alpha}^{\ \beta}\partial_{\beta} \qquad , \qquad \mathbf{\Lambda}^{-1}\mathcal{Y}^{-1}\tilde{\nabla}^2\mathcal{Y}\mathbf{\Lambda} = \eta^{\alpha\beta}\partial_{\alpha}\partial_{\beta}. \tag{6.73}$$

This allows us to easily find the aforementioned fields as

$$\bar{\mathbf{\Phi}}_p = \Omega\mathbf{\Lambda}\mathcal{U}\boldsymbol{\varphi}_p, \tag{6.74}$$

where $\boldsymbol{\varphi}_p$ is the flat initial condition. Once again we can introduce the polarization $\mathbf{S}_p(\phi)$ such that

$$\bar{\mathbf{\Phi}}_p = \Omega\bar{\mathbf{S}}_p(\phi)e^{iS_p}. \tag{6.75}$$

As in Sec. 6.2.1 one can check that this can be written as

$$\bar{\mathbf{S}}_p = \mathbf{\Lambda}_p\boldsymbol{s}_p, \tag{6.76}$$

where $\boldsymbol{s}_p = \{1, u_p, \varepsilon_p^{\alpha}\}$ are the flat-spacetime free polarizations for scalar, Dirac and vector fields respectively.

By adapting the discussion of Sec. 6.2.1 we find that $\tilde{\nabla}_{\alpha}\Omega^{-1}\bar{\mathbf{\Phi}}_p = -i\Lambda_{\alpha}^{\ \beta}p_{\beta}\Omega^{-1}\bar{\mathbf{\Phi}}_p$ and from this $\bar{\pi}^{\alpha}\tilde{\nabla}_{\alpha}\Omega^{-1}\bar{\mathbf{\Phi}}_p = -im^2\Omega^{-1}\bar{\mathbf{\Phi}}_p$. Inserting into this equation the form $\bar{\mathbf{\Phi}}_p = \Omega\bar{\mathbf{S}}_p(\phi)e^{iS_p}$ we obtain a condition for the polarization which reads $\bar{\pi}_p^{\alpha}\tilde{\nabla}'_{p,\alpha}\bar{\mathbf{S}}_p = 0$, where $\tilde{\nabla}'_{p,\alpha} = \partial_{\alpha}\mathbf{1} - i\frac{\kappa}{2p^-}n_{\alpha}\mathcal{P}^{\gamma\beta}\mathbf{\Sigma}_{\gamma\beta}$. Now, observing that

$$\bar{\pi}_p^{\alpha}\tilde{\nabla}'_{p,\alpha}\bar{\mathbf{S}}_p = \bar{\pi}_p^{\alpha}\tilde{\nabla}_{p,\alpha}\bar{\mathbf{S}}_p = \bar{\pi}_p^{\alpha}\bar{\nabla}_{p,\alpha}\bar{\mathbf{S}}_p, \tag{6.77}$$

we finally find that the polarization function is defined by a proper parallel transport $\pi_p^\mu \nabla_\mu \mathbf{S}_p = 0$, or [21]

$$\frac{D\mathbf{S}_p}{D\phi} = 0. \tag{6.78}$$

This is a covariant equation and it holds in other charts as well. In particular, it can be used as an alternative way to find the states in Brinkmann coordinates. Moreover, this equation underlines once again the semiclassical behaviour of quantum states in a plane wave background and offers an intuitive geometric picture. For example, the polarization of the shifted spin-one field evolves as a free falling gyroscope [178] being just parallel transported.

6.2.3 In the Brinkmann metric

It is needless to say that we could just transform the fields obtained in the Rosen chart and find the states in Brinkmann coordinates. However here we want to underline the role of the operator $\mathbf{\Lambda}$ in this chart and also the form this takes in a particular gauge reproducing the spin-raising operator studied in [8] for massless fields. The scalar field can be clearly expressed as $\Phi_p = \Omega e^{iS_p}$ where S_p is the classical action for a particle in Brinkmann coordinates [8]

$$S_p(X) = -p^- X^+ - X^i e_i{}^j p_j - \frac{p^-}{2}\sigma_{ij} X^i X^j + \frac{1}{2p^-}\int^\phi d\tilde{\phi}(p_i p_j \gamma^{ij} - m^2). \tag{6.79}$$

The operator $\mathbf{\Lambda}$ can be written in this chart as

$$\mathbf{\Lambda} = e^{-\frac{i}{2\partial_+}\mathbf{\Sigma}_{\alpha\beta} f_i^{\alpha\beta}\eta^{ij}(\partial_{X^j} - e^k{}_j \partial_{X^k})}. \tag{6.80}$$

If we choose the natural vierbein associated to this chart

$$E_{\alpha\mu} = \eta_{\alpha\mu} + \frac{H}{2} n_\alpha n_\mu \tag{6.81}$$

we can immediately find the spinor solutions, in fact $E_{ij} = e'_{ij}$ where $e'_{ij} = e_{ik}\frac{\partial x^k}{\partial X^j}$ is the Rosen transverse vierbein in Brinkmann coordinates. We can thus expand the operator to obtain

$$\Psi_p = \Omega e^{iS_p}\left(1 + \frac{\not{\Delta}_p \not{n}}{2p^-},\right) u_p, \tag{6.82}$$

where we recall that $\Delta_p^i = \Pi_p^i - p^i$ as in Sec. 5.3.5. Once again the solution is similar to the Volkov one, with the substantial difference that now Δ_p^i depends on the three coordinates ϕ, X^i and not just on ϕ. This will be a major difference as compared to the electromagnetic case when we will calculate S-matrix elements, along with the definition of in- and out-states discussed in the next section. This solution can be verified to match the one found in the Rosen chart, in fact the transformation of the spinor field is generated by the Lorentz transformation connecting the Brinkmann chosen vierbein and the Rosen one expressed in Brinkmann coordinates $E^\alpha{}_\mu = \tilde{\Lambda}^\alpha{}_\beta e^\beta{}_\nu \frac{\partial x^\nu}{\partial X^\mu} = \tilde{\Lambda}^\alpha{}_\beta e'^\beta{}_\mu$. Expanding $\tilde{\Lambda}_{\alpha\beta} = \eta_{\alpha\beta} - \omega_{\alpha\beta}$ the infinitesimal Lorentz transformation is found to be $\omega_{\alpha\beta} = 2n_{[\alpha}\sigma_{\beta]i}X^i$ and accordingly the transformed spinor follows as [see Eq. (3.40)]

$$\begin{aligned}\Psi_p &= e^{-\frac{i}{2}\omega_{\alpha\beta}\boldsymbol{\Sigma}^{\alpha\beta}}\Omega e^{iS_p}\left(1+\kappa\frac{\not{n}\not{P}}{2p^-}\right)u_p \\ &= \Omega e^{iS_p}e^{\frac{1}{2}\sigma_{\alpha i}X^i\bar{\gamma}^\alpha\not{n}}\left(1+\kappa\frac{\not{n}\not{P}}{2p^-}\right)u_p \\ &= \Omega e^{iS_p}\left(1+\frac{\not{\Delta}_p\not{n}}{2p^-}\right)u_p.\end{aligned} \tag{6.83}$$

One can also verify that the spinorial matrix satisfies the parallel transport equation (6.78), as discussed in the context of the Rosen metric.

Let us now consider the photon field. If we fix the gauge such that only the transverse components of the flat polarization are left $\varepsilon^\mu = \delta^\mu_i\varepsilon^i$ and they obey the Lorentz condition $l_i\varepsilon^i = 0$, we can reduce the operator action to

$$\mathcal{A}^\mu_l = \Omega\boldsymbol{\Lambda}^\mu{}_i\varepsilon^i e^{iS_l} \equiv \Omega R^\mu e^{iS_l}, \tag{6.84}$$

where we introduced the spin raising operator studied in [8, 130]

$$R^\mu = \boldsymbol{\Lambda}^\mu{}_i\varepsilon^i = \frac{1}{\partial_+}\left[\delta^\mu_i\partial_+ - \partial_i n^\mu\right]\varepsilon^i = -\frac{1}{\partial_+}\varepsilon^i f^{\mu\nu}_i\partial_\nu. \tag{6.85}$$

In this gauge the massless vector field is thus given by

$$\mathcal{A}^\mu_l = \Omega e^{iS_l}\left(\delta^\mu_i - \frac{1}{l^-}\Delta_{l,i}n^\mu\right)\varepsilon^i. \tag{6.86}$$

This is clearly in agreement with the field found in the Rosen chart once the same gauge is imposed, as in Eq. (6.69).

As noticed in [8], in vacuum we can choose for a massless spin-2 field the covariant TT-gauge and fix $n^\mu h_{\mu\nu} = 0$. As a result the field can be found applying two times the spin raising operator, such that $h^{\mu\nu}_l = \Omega R^\mu R^\nu \Phi_l$. The polarization tensor is easily found to be $\mathcal{E}^{\mu\nu}_l = \mathcal{E}^\mu_l\mathcal{E}^\nu_l - \frac{i}{l^-}\sigma_{ij}\varepsilon^i\varepsilon^j n^\mu n^\nu$ such that

$$h^{\mu\nu}_l = \Omega e^{iS_l}\left[\left(\delta^\mu_i - \frac{\Delta_{l,i}n^\mu}{l^-}\right)\left(\delta^\nu_j - \frac{\Delta_{l,j}n^\nu}{l^-}\right) - \frac{i}{l^-}\sigma_{ij}n^\mu n^\nu\right]\varepsilon^i\varepsilon^j. \tag{6.87}$$

6.3 Photon emission

6.3.1 Definition of scattering states

When dealing with scattering problems we have to work in Brinkmann coordinates, because the regularity of fields over the whole spacetime is essential. Due to the presence of a gravitational wave in- and out-states are generally different. Let us consider for simplicity the scalar case $\Phi(X) = \Omega(\phi)e^{iS(X)}$. Moreover we will assume the spacetime to be flat at

$\phi = \pm\infty$. We can impose the free plane wave boundary condition in the in- or out-region and define the positive energy in- and out-states, respectively $\Phi^{\uparrow}, \Phi^{\downarrow}$ as [91, 8]

$$\Phi_p^{\updownarrow} = \Omega^{\updownarrow} e^{iS_p^{\updownarrow}} \qquad , \qquad \lim_{\phi \to \mp\infty} \Phi_p^{\uparrow,\downarrow} = e^{-ip\cdot X}. \tag{6.88}$$

As observed in the previous sections, these fields depend on a matrix $e_{\alpha\mu}$, which is also the vierbein of a Rosen metric. This is the reason why in- and out-states are different, they depend on vierebeins which do not represent the physical wave profile H_{ij} but they are connected to it through the differential equation $\ddot{e}_{ik} = H_{ij} e^j{}_k$. A vierbein does not have to be trivial $e_{\alpha\mu} = \eta_{\alpha\mu}$ in order to describe a flat region, but it can be at most linear in ϕ such that $\ddot{e}_{ik} = 0$. In other words a Rosen metric does not have to be Minkowskian in a flat region and if we require it to be so, for example in the in-region, it will not be Minkowskian in the out-region. The boundary conditions are thus transferred to the vierbeins, such that

$$\lim_{\phi\to\mp\infty} (e^{\uparrow,\downarrow})^{\alpha}{}_{\mu} = \delta^{\alpha}_{\mu} \qquad , \qquad (e^{\uparrow,\downarrow})^{\alpha}{}_{\mu} = (b^{\uparrow,\downarrow})^{\alpha}{}_{\mu}\phi + (c^{\uparrow,\downarrow})^{\alpha}{}_{\mu} \quad \text{for} \quad \phi \to \pm\infty. \tag{6.89}$$

There are only two constrains on these coefficients, one comes from the symmetry condition $\dot{e}_{[\beta}{}^{\mu} e_{\alpha]\mu} = 0$ and reads [8] $(b^{\uparrow,\downarrow})^{\alpha}{}_{[\mu}(c^{\uparrow,\downarrow})_{\alpha\nu]} = 0$, the other one comes from the conservation of the Wronskian related to the harmonic oscillator equation and links in- and out-states [91] $b^{\downarrow}_{\alpha\mu}\delta^{\alpha}_{\nu} = -b^{\uparrow}_{\alpha\nu}\delta^{\alpha}_{\mu}$. It is worth recalling that the coefficients $b^{\updownarrow}$ represent the velocity memory effect induced by the passage of the wave. Now that the states are well defined we should introduce a scalar product, however the standard definition [33] needs a well-defined Cauchy surface Σ

$$\langle \Psi_2 | \Psi_1 \rangle = -i \int_{\Sigma} \sqrt{\mathcal{G}_{\Sigma}} d\Sigma^{\mu} \Psi_2 \overleftrightarrow{\partial}_{\mu} \Psi_1^*. \tag{6.90}$$

As we have already discussed, it is impossible to define a Cauchy surface Σ in this spacetime [145], nevertheless the surfaces of constant ϕ are intersected by almost all the geodesics except the ones with constant ϕ [92], therefore they can be chosen to construct a foliation of the spacetime. We can thus choose an arbitrary ϕ and rewrite the product definition as

$$\langle \Psi_2 | \Psi_1 \rangle = -i \int_{\phi=const.} dX^+ d^2 X_{\perp} \Psi_2 \overleftrightarrow{\partial}_{+} \Psi_1^*, \tag{6.91}$$

where we recall that the absolute value of the Brinkmann determinant is always equal to one due to its Kerr-Schild form. A straightforward calculation shows that the Bogoliubov coefficient encoding particle creation is zero. This corresponds to the amplitude for a positive frequency in-state to develop a negative frequency out-state component: $\langle \Phi_1^{\downarrow} | \Phi_2^{\uparrow *} \rangle = 0$ [92, 91, 8]. It follows that the in- and out-vacua can be identified and no particle is created by the background, as in the electromagnetic case. On the other hand, one can show that the in-to-out scattering probability depends on the determinant of $\dot{e}^{\uparrow}_{ij}$ in the out-region [91]

$$\left| \langle \Phi_1^{\downarrow} | \Phi_2^{\uparrow} \rangle \right|^2 \propto \left. \frac{\delta(p_1^- - p_2^-)}{|\det \dot{e}^{\uparrow}_{ij}|} \right|_{\text{out-region}}. \tag{6.92}$$

Thus the probability for a scalar field to be scattered from the gravitational wave background is directly correlated to the velocity memory effect. This is what one would expect, because the difference between the initial and final momenta induced by the background corresponds to this effect. The same is true for the classical cross section describing the interaction between a particle and the gravitational wave [91].

6.3.2 S-matrix element

In the previous sections we prepared all the tools needed in order investigate the emission of a single photon by an electron (Compton scattering) in a sandwich plane wave spacetime. The amplitude for the process $e(p) \to e(p') + \gamma(k')$ in this background is

$$S_G^\gamma = -ie \int d^4X \bar{\Psi}_{p'}^{\downarrow} \gamma^\mu \Psi_p^{\uparrow} \mathcal{A}_{k',\mu}^{\downarrow *}. \tag{6.93}$$

As a first consistency check it is interesting to consider its linear order in κ in the monochromatic assumption. This has to coincide with the known results for the inverse graviton photoproduction $g(k) + e(p) \to e(p') + \gamma(k')$ in vacuum quantum field theory, where $k^\mu = \omega n^\mu$ is the momentum of the gravitational wave. In this approximation we are free to use the quantum states in Rosen coordinates, in fact the difference between the in- and the out-vacua emerges at the second order. We will thus drop the in and out labels and consider $S_G^\gamma = -ie \int d^4x \bar{\Psi}_{p'} \bar{\gamma}^\alpha \Psi_p \bar{\mathcal{A}}_{k',\alpha}^* + \mathcal{O}(\kappa^2)$, where the states will be defined below and where the metric determinant does not contribute at this order. Everything we need for this approximation is encoded in the vierbein, which is now expanded as

$$e_{\alpha\mu} = \eta_{\alpha\mu} + \frac{\kappa}{2} h_{\alpha\mu} \quad , \quad e_\alpha{}^\mu = \delta_\alpha^\mu - \frac{\kappa}{2} h_\alpha{}^\mu, \tag{6.94}$$

where $h_{\alpha\mu}$ is the graviton field. We can now introduce the monochromatic assumption defining $h_{\mu\nu} = \varepsilon_\mu \varepsilon_\nu e^{-ik\cdot x}$, where the graviton polarization tensor has been written as a product of two photon polarization vectors $\varepsilon_{\mu\nu}(k) = \varepsilon_\mu(k)\varepsilon_\nu(k)$ satisfying $\varepsilon \cdot \varepsilon = 0$, $k \cdot \varepsilon = 0$ and $\varepsilon \cdot \varepsilon^* = -1$ [98]. This, together with the energy-momentum conservation leads to the simple substitution rule

$$P_\alpha \to \frac{1}{2} e^{-ik\cdot x} \varepsilon \cdot p \varepsilon_\alpha \tag{6.95}$$

in Ψ_p, in $\bar{\Psi}_{p'}$ (with $p \to p'$), and in $\bar{\mathcal{A}}_{k',\alpha}^*$ (with $p \to k'$). With this in mind we can easily find the states under these assumptions in the form

$$\begin{aligned} \Psi_p &\simeq e^{-ip\cdot x} \left[1 + e^{-ik\cdot x} \kappa \frac{p\cdot\varepsilon}{4p\cdot k} (2p\cdot\varepsilon + \not{k}\not{\varepsilon}) \right] u_p + \mathcal{O}(\kappa^2), \\ \bar{\mathcal{A}}_{k',\alpha}^* &\simeq e^{ik'\cdot x} \left[\varepsilon_\alpha'^* + e^{-ik\cdot x} \kappa \frac{k'\cdot\varepsilon}{2k\cdot k'} \left(\varepsilon'^* \cdot \varepsilon k_\alpha - k' \cdot \varepsilon \varepsilon_\alpha'^* - \varepsilon'^* \cdot k \varepsilon_\alpha \right) \right] + \mathcal{O}(\kappa^2). \end{aligned} \tag{6.96}$$

Now, one can plug these expressions into the amplitude previously defined. Introducing the notation $S_G^\gamma = (2\pi)^4 \delta^{(4)}(p + k - p' - k') M + \mathcal{O}(\kappa^2)$, the following result is found after a proper rearrangement of the various terms

$$\begin{aligned} M = -ie\frac{\kappa}{2} \bar{u}_{p'} \Bigg[& \frac{p'\cdot\varepsilon}{2p'\cdot k} (-2p'\cdot\varepsilon + \not{\varepsilon}\not{k}) \not{\varepsilon}'^* + \frac{p\cdot\varepsilon}{2p\cdot k} \not{\varepsilon}'^* (2p\cdot\varepsilon + \not{k}\not{\varepsilon}) \\ & + \frac{\varepsilon\cdot k'}{k\cdot k'} \left(\varepsilon'^* \cdot \varepsilon \not{k} - \varepsilon \cdot k' \not{\varepsilon}'^* - \varepsilon'^* \cdot k \not{\varepsilon} \right) \Bigg] u_p, \end{aligned} \tag{6.97}$$

which agrees with the vacuum QFT result as expected (see e.g. [56, 111] and Sec. 3.4.1). The first-order expansion of the dressed fermions naturally generates the s and u channels contributions, while the expansion of the photon state reduces to the sum of the γ-pole and the seagull diagrams [56]. The contact term never shows up explicitly in our calculation as a consequence of graviton gauge invariance [54]: the seagull diagram only provides a momentum-independent contribution which cancels out a term of the same kind in the γ-pole diagram.

Let us now go back to the original amplitude. In the fully nonlinear case we have to work in the Brinkmann chart in order to have a well defined matrix element. Choosing the gauge $\nabla_\mu \mathcal{A}^\mu = 0 = \mathcal{A}^-$ for the photon field, the amplitude gets the following form

$$S^\gamma_G = -ie\int d^4X(\Omega^\downarrow)^2\Omega^\uparrow e^{i(S^\uparrow_p - S^\downarrow_{p'} - S^\downarrow_{k'})}\bar{u}_{p'}\left(1 - \frac{\not{\Delta}^\downarrow_{p'}\not{k}}{2k\cdot p'}\right)\left(\not{\varepsilon}^* - \frac{\varepsilon^{i*}\Delta^\downarrow_{k',i}\not{k}}{k\cdot k'}\right)\left(1 - \frac{\not{k}\not{\Delta}^\uparrow_p}{2k\cdot p}\right)u_p, \tag{6.98}$$

The unpolarized squared amplitude averaged over the initial spin reads

$$\begin{aligned}\frac{1}{2}\sum_{s,s',\varepsilon'} S^{\gamma\dagger}_G S^\gamma_G = \frac{e^2}{2}\int d^4X\mathfrak{I}(X)\int d^4X'\mathfrak{I}^*(X')\times \\ \times \operatorname{Tr}\Bigg[(\not{p}'+m)\left(1 - \frac{\not{\Delta}^\downarrow_{p'}(X)\not{k}}{2k\cdot p'}\right)\left(\bar{\gamma}^\alpha - \frac{\Delta^{\downarrow,\alpha}_{k'}(X)\not{k}}{k\cdot k'}\right)\left(1 - \frac{\not{k}\not{\Delta}^\uparrow_p(X)}{2k\cdot p}\right) \\ \times(\not{p}+m)\left(1 - \frac{\not{\Delta}^\uparrow_p(X')\not{k}}{2k\cdot p}\right)\left(\bar{\gamma}^\beta - \frac{\Delta^{\downarrow,\beta}_{k'}(X')\not{k}}{k\cdot k'}\right)\left(1 - \frac{\not{k}\not{\Delta}^\downarrow_{p'}(X')}{2k\cdot p'}\right)\Bigg] \\ \times\left(-\eta_{\alpha\beta} + \frac{k_\alpha k'_\beta + k'_\alpha k_\beta}{k\cdot k'}\right),\end{aligned} \tag{6.99}$$

where the prefactor is defined as $\mathfrak{I} = (\Omega^\downarrow)^2\Omega^\uparrow e^{i(S^\uparrow_p - S^\downarrow_{p'} - S^\downarrow_{k'})}$. Once the trace is computed this expression reduces to the following form

$$\begin{aligned}\frac{1}{2}\sum_{s,s',\varepsilon'} S^{\gamma\dagger}_G S^\gamma_G = \frac{e^2}{2}\int d^4X\mathfrak{I}(X)\int d^4X'\mathfrak{I}^*(X') \\ \times\Bigg\{\frac{k\cdot p'}{k\cdot k'}\left(\Pi^{\uparrow i}_p(X)\Pi^\downarrow_{k',i}(X') + \Pi^{\downarrow i}_{k'}(X)\Pi^\uparrow_{p,i}(X') + 2p^+k'^- + 2p^-k'^+\right) \\ + \frac{k\cdot p}{k\cdot k'}\left(\Pi^{\downarrow i}_{p'}(X)\Pi^\downarrow_{k',i}(X') + \Pi^{\downarrow i}_{k'}(X)\Pi^\uparrow_{p',i}(X') + 2p'^+k'^- + 2p'^-k'^+\right) \\ - \frac{k\cdot p'}{k\cdot p}\left(\Pi^{\uparrow i}_p(X)\Pi^\uparrow_{p,i}(X') + 2p^+p^-\right) - \frac{k\cdot p}{k\cdot p'}\left(\Pi^{\downarrow i}_{p'}(X)\Pi^\downarrow_{p',i}(X') + 2p'^+p'^-\right) \\ - 2\frac{k\cdot pk\cdot p'}{(k\cdot k')^2}\left(\Pi^{\downarrow i}_{k'}(X)\Pi^\downarrow_{k',i}(X') + 2k'^+k'^-\right) + \frac{m^2(k\cdot k')^2}{k\cdot pk\cdot p'}\Bigg\}.\end{aligned} \tag{6.100}$$

This expression can be proved to give the correct linear limit of (inverse) graviton photo-production employing the FORM code reported in App. A.11.2. With the same code it is possible to check the photon-gauge invariance of the result. It is interesting to observe that the terms in the amplitude quadratic in the gravitational field disappear in the unpolarized sum. For this reason, the integrals in the three coordinates X^+, X^i are easily carried out by observing that the prefactor is Gaussian in X^i. Namely

$$\int d^4X \mathfrak{I}(X) = -i(2\pi)^2\delta(p^- - p'^- - k'^-) \int d\phi \frac{(\Omega^{\downarrow})^2\Omega^{\uparrow}}{p^-\sqrt{C}} e^{-i(p^+ - p'^+ - k'^+)\phi}$$
$$\times\, e^{\frac{i}{2p^-}B^i(C^{-1})_{ij}B^j - i\frac{\kappa}{p^-}\int_{-\infty}^{\phi} d\tilde{\phi}\left(p_\alpha P^\alpha - \frac{\kappa}{2}P_\alpha P^\alpha\right) + i\frac{\kappa}{p'^-}\int_{\infty}^{\phi} d\tilde{\phi}\left(p'_\alpha P'^\alpha - \frac{\kappa}{2}P'_\alpha P'^\alpha\right) + i\frac{\kappa}{k'^-}\int_{\infty}^{\phi} d\tilde{\phi}\left(k'_\alpha K'^\alpha - \frac{\kappa}{2}K'_\alpha K'^\alpha\right)}, \tag{6.101}$$

where $C_{ij} = \sigma^{\uparrow}_{ij} - \sigma^{\downarrow}_{ij}$, $C = \det C_{ij}$ and $B_i = e^{\uparrow j}_i p_j - e^{\downarrow j}_i(p'_j + k'_j)$. By setting $\int d^4X\mathfrak{I}(X) = i(2\pi)^2\delta(p^- - p'^- - k'^-)\int d\phi\tilde{\mathfrak{I}}(\phi)$, the only other integral needed is the one with a prefactor linear in the transverse coordinate X^i, namely

$$\int d^4X\mathfrak{I}(X)X^i = i(2\pi)^2\delta(p^- - p'^- - k'^-)\int d\phi\tilde{\mathfrak{I}}(\phi)\frac{(C^{-1})^i{}_jB^j}{p^-}. \tag{6.102}$$

We can finally write down the partially integrated transverse momentum as

$$\int d^4X\mathfrak{I}(X)\Pi^{\uparrow i}_p(X) = -i(2\pi)^2\delta(p^- - p'^- - k'^-)\int d\phi\tilde{\mathfrak{I}}(\phi)\left(e^{\uparrow ij}p_j - \sigma^{\uparrow i}_j(C^{-1})^j{}_kB^k\right), \tag{6.103}$$

with this substitution rule the squared amplitude can be written as an integral in ϕ, ϕ' only. A further simplification can be achieved exploiting the identity $C^{-1}_{ij} = \frac{1}{C}(C_k{}^k\eta_{ij} - C_{ij})$ for rank-two matrices. Having derived the full nonlinear squared amplitude it would be natural to consider some physically interesting examples, however one runs very soon into the mathematical complexity of the problem. In particular, we recall that once the Brinkmann profile is chosen one has to solve the differential equation $\ddot{e}_{ij} = H_{ik}e^k{}_j$, in fact the knowledge of the in and out vierbein is a key element. There are very few profiles H_{ij} for which an analytical solution of this equation is available (see Sec. 5.3.7). These include for example impulsive waves $H_{ij} = \delta(\phi)d_{ij}$ where d_{ij} is a constant matrix, these solutions are relevant as they represent the gravitational field produced by a massless particle moving at the speed of light [14, 146, 72]. Other possibilities are profiles with conformal symmetry of the form $H_{ij} = a(\phi^2 + b^2)^{-1}\epsilon_{ij}$ studied in [17]. However, physically interesting scenarios for the process considered here would require proper approximations and numerical evaluations. Also the concept of formation length, of great importance in electromagnetism, could be worth to be studied in this context. In fact, despite the non-locality of the interaction and the clear implications of a curved spacetime background, the high amount of symmetries of the problem could lead to some interesting developments. A detailed study of this and similar processes in the aforementioned directions will be left for future works.

6.4 Discussion

We examined the one-to-one map connecting the dynamics of a charged particle in an electromagnetic plane wave and a particle in a nonlinear gravitational plane wave. This map enables to translate interesting results known in electromagnetism to the context of plane wave spacetimes. The dynamics in an electromagnetic plane wave is completely determined by a specific Lorentz-like transformation $\Lambda^{\alpha}{}_{\beta}$, which provides a way to transport the initial momentum of a free particle through the wave such that $\pi^{\alpha} = \Lambda^{\alpha}{}_{\beta} p^{\beta}$. This transformation has a number of interesting properties. Besides belonging to the Lorentz group, it operates on constant-ϕ hypersurfaces and it acts as a gauge transformation on the background wave. For these reasons its proper generalization as a quantum operator $\mathbf{\Lambda}$ turns out to be a key element to construct quantum states in an electromagnetic plane wave background. Exploiting the aforementioned analogy between electromagnetic waves in flat spacetime and gravitational nonlinear waves we showed that the curved spacetime generalization of $\mathbf{\Lambda}$ can be used to find quantum states in gravitational plane waves as well. In particular we showed how scalar, spinor, and vector fields can be naturally mapped from one context to the other. Finally, in the last part of the paper we applied the found states for spin 1/2 and spin 1 particles to consider Compton scattering in a nonlinear sandwich gravitational wave. Here we provided the spin- and polarization-summed squared amplitude for the process, exact in the gravitational plane wave. A detailed study of the Compton cross section for physically interesting plane wave gravitational backgrounds is left for a future work. Other processes could be naturally investigated as well in the future, an interesting example being the graviton emission by a moving particle. This could provide insights about its relation with photon emission, a relation which has been studied in vacuum QFT [56, 110] and in the presence of a background electromagnetic plane wave within strong-field QED [20, 90].

6.5 Support material

6.5.1 Vierbein projection of the vector wave equation

The spin-1 wave equation reads

$$(\bar{\nabla}_\beta \bar{\nabla}^\beta + m^2)\bar{\mathcal{A}}^\alpha + \bar{R}^\alpha{}_\beta \bar{\mathcal{A}}^\beta = 0. \tag{6.104}$$

Observing that $\bar{\nabla}_\beta \bar{\mathcal{A}}^\alpha = \bar{\partial}_\beta \bar{\mathcal{A}}^\alpha + \omega_\beta{}^\alpha{}_\gamma \bar{\mathcal{A}}^\gamma$, we can expand the double derivative as

$$\begin{aligned} \bar{\nabla}_\beta \bar{\nabla}^\beta \bar{\mathcal{A}}^\alpha &= \frac{1}{2}\dot{\log}(g)\partial_\phi \bar{\mathcal{A}}^\alpha + \bar{\partial}^\beta(\bar{\partial}_\beta \bar{\mathcal{A}}^\alpha + \omega_\beta{}^\alpha{}_\gamma \bar{\mathcal{A}}^\gamma) + \omega_\beta{}^\alpha{}_\gamma(\bar{\partial}^\beta \bar{\mathcal{A}}^\gamma + \omega^{\beta\gamma}{}_\delta \bar{\mathcal{A}}^\delta) \\ &= \frac{1}{2}\dot{\log}(g)\partial_\phi \bar{\mathcal{A}}^\alpha + \bar{\partial}^\beta \bar{\partial}_\beta \bar{\mathcal{A}}^\alpha + 2\omega_\beta{}^\alpha{}_\gamma \bar{\partial}^\beta \bar{\mathcal{A}}^\gamma + \omega_\beta{}^\alpha{}_\gamma \omega^{\beta\gamma}{}_\delta \bar{\mathcal{A}}^\delta \\ &= \frac{1}{2}\dot{\log}(g)\partial_\phi \bar{\mathcal{A}}^\alpha + \bar{\partial}^\beta \bar{\partial}_\beta \bar{\mathcal{A}}^\alpha + 2\omega_\beta{}^\alpha{}_\gamma \bar{\partial}^\beta \bar{\mathcal{A}}^\gamma + \frac{1}{4}\dot{\gamma}^{\lambda\rho}\dot{\gamma}_{\lambda\rho}\bar{\mathcal{A}}^- n^\alpha. \end{aligned} \tag{6.105}$$

Now, recalling that the Ricci tensor in Rosen coordinates has the form (see App. A.10)

$$R_{\mu\delta} = \left(\frac{1}{4}\dot{\gamma}^{\rho\lambda}\dot{\gamma}_{\lambda\rho} + \frac{1}{2}\gamma^{\rho\lambda}\ddot{\gamma}_{\lambda\rho}\right) n_\mu n_\delta \tag{6.106}$$

we can write the full wave equation as

$$\frac{1}{2}\dot{\log}(g)\partial_\phi \bar{\mathcal{A}}^\alpha + \bar{\partial}^\beta \bar{\partial}_\beta \bar{\mathcal{A}}^\alpha + 2\omega_\beta{}^\alpha{}_\gamma \bar{\partial}^\beta \bar{\mathcal{A}}^\gamma + \frac{1}{2}\ddot{\log}(g)\bar{\mathcal{A}}^- n^\alpha = 0. \tag{6.107}$$

If we now introduce the notation $\bar{\mathcal{A}}^\alpha = \Omega \bar{\Phi}^\alpha$ and observe that $\omega_\beta{}^\alpha{}_\gamma \bar{\partial}^\beta \bar{\mathcal{A}}^\gamma = i\kappa \hat{\mathcal{P}}^\alpha{}_\gamma \bar{\mathcal{A}}^\gamma$, we can finally write the equation for the rescaled field as in Eq. (6.62)

$$(\bar{\partial}^\beta \bar{\partial}_\beta + m^2)\bar{\Phi}^\alpha + 2i\kappa \hat{\mathcal{P}}^\alpha{}_\gamma \bar{\Phi}^\gamma + \frac{1}{2}\ddot{\log}(g)\bar{\Phi}^- n^\alpha = 0, \tag{6.108}$$

6.5.2 Massless spin-2 fields in Rosen coordinates

The linearized Einstein equations on a curved background in the TT-gauge $\nabla_\mu h^\mu{}_\nu = 0$, $h^\mu{}_\mu = 0$ read [176]

$$\nabla_\rho \nabla^\rho h_{\mu\nu} + 2R^\rho{}_{\mu\nu\delta} h_\rho{}^\delta = 0. \tag{6.109}$$

In a plane wave we are allowed to fix the gauge condition $n^\mu h_{\mu\nu} = 0$, due to the invariance of n^μ. Let us now deal with the curvature interaction $2R^\rho{}_{\mu\nu\delta} h_\rho{}^\delta$. Due to the condition $n^\mu h_{\mu\nu} = 0$ this product will select all the terms of the Riemann tensor (see App. A.10) that do not involve neither n^ρ nor n_δ. It is easy to check that one is left with

$$2R^\rho{}_{\mu\nu\delta} h_\rho{}^\delta = -\left(g^{\rho\lambda}\ddot{g}_{\delta\lambda} + \frac{1}{2}\dot{g}^{\rho\lambda}\dot{g}_{\delta\lambda}\right) h_\rho{}^\delta n_\mu n_\nu, \tag{6.110}$$

such that the full equation reads

$$\nabla_\rho \nabla^\rho h_{\mu\nu} - \left(g^{\rho\lambda}\ddot{g}_{\delta\lambda} + \frac{1}{2}\dot{g}^{\rho\lambda}\dot{g}_{\delta\lambda}\right) h_\rho{}^\delta n_\mu n_\nu = 0. \tag{6.111}$$

We can now assume the ansatz $h_{l,\mu\nu} = \Phi_l \mathcal{E}_{l,\mu\nu}(\phi)$ with the scalar field satisfying the massless Klein-Gordon equation $\nabla_\rho \nabla^\rho \Phi_l = 0$. The first term under these hypothesis becomes

$$\begin{aligned}\nabla_\rho \nabla^\rho h_{l,\mu\nu} &= 2\nabla^\rho \Phi_l \nabla_\rho \mathcal{E}_{l,\mu\nu} + \Phi_l \nabla^\rho \nabla_\rho \mathcal{E}_{l,\mu\nu} \\ &= -2il^\rho \Phi_l \nabla_\rho \mathcal{E}_{l,\mu\nu} + 2\Phi_l g^{\rho\lambda} \Gamma^\eta_{\lambda\nu} \Gamma^\sigma_{\rho\mu} \mathcal{E}_{l,\eta\sigma}.\end{aligned} \tag{6.112}$$

It easy now to show that

$$g^{\rho\lambda} \Gamma^\eta_{\lambda\nu} \Gamma^\sigma_{\rho\mu} \mathcal{E}_{l,\eta\sigma} = -\frac{1}{4} \dot{g}^{\rho\lambda} \dot{g}_{\delta\lambda} \mathcal{E}_{l,\rho}^{\ \ \delta} n_\mu n_\nu. \tag{6.113}$$

We can now collect all the terms and find a rather compact equation for the polarization tensor

$$-2il^\rho \nabla_\rho \mathcal{E}_{l,\mu\nu} - \left(g^{\rho\lambda} \ddot{g}_{\delta\lambda} + \dot{g}^{\rho\lambda} \dot{g}_{\delta\lambda} \right) \mathcal{E}_{l,\rho}^{\ \ \delta} n_\mu n_\nu = 0. \tag{6.114}$$

Here one immediately notices that $\mathcal{E}_{l,\mu\nu}$ cannot be just the product of two spin-1 polarization vectors as in the flat case [8] $\mathcal{E}_{l,\mu\nu} \neq \mathcal{E}_{l,\mu} \mathcal{E}_{l,\nu}$. Indeed, in this case the vector would satisfy the parallel transport equation $l^\rho \nabla_\rho \mathcal{E}_{l,\mu} = 0$ and this assumption would directly cancel the first term in this equation, leaving us with $\ddot{g}_{\delta\lambda} \mathcal{E}_l^\lambda \mathcal{E}_l^\delta = 0$. Let us now make the gauge condition $\nabla^\mu h_{l,\mu\nu} = 0$ explicit, this reads

$$-il^\mu \mathcal{E}_{l,\mu\nu} - \Gamma^\lambda_{\mu\nu} \mathcal{E}^\mu_{l,\lambda} = 0 \qquad \text{or} \qquad -il^\mu \mathcal{E}_{l,\mu\nu} - \frac{1}{2} \dot{g}_{\mu\rho} \mathcal{E}_l^{\mu\rho} n_\nu = 0. \tag{6.115}$$

Given the form of the polarization equation, it is sensible to assume

$$\mathcal{E}_{l,\mu\nu} = \mathcal{E}_{l,\mu} \mathcal{E}_{l,\nu} + \mathcal{E}_l^{++} n_\mu n_\nu, \tag{6.116}$$

where $\mathcal{E}_{l,\mu}$ is the spin-1 polarization vector. With this assumption and recalling that $l^\mu \mathcal{E}_{l,\mu} = 0$, the gauge condition reduces to

$$\mathcal{E}_l^{++} = \frac{i}{2l^-} \dot{g}_{\lambda\rho} \mathcal{E}_l^\lambda \mathcal{E}_l^\rho \tag{6.117}$$

and this is enough to determine the full solution

$$\mathcal{E}_{l,\mu\nu} = \mathcal{E}_{l,\mu} \mathcal{E}_{l,\nu} - \frac{i}{2l^-} \dot{g}^{\lambda\rho} \mathcal{E}_{l,\lambda} \mathcal{E}_{l,\rho} n_\mu n_\nu. \tag{6.118}$$

Appendix

A.1 Light-cone coordinates

A plane wave depends only on the phase $\varphi = k \cdot x$, where k^μ is the associated momentum. It is useful to define a reference system in which this dependence is explicit, a system in which this scalar involves only one coordinate. Such a reference goes under the name of light-cone basis. Let us assume that the momentum is

$$k^\mu = \omega(1, \boldsymbol{n}) \tag{119}$$

where $\boldsymbol{n}$ is the direction of propagation of the wave. The goal is to define coordinates in which this four-vector is defined by a single number, i.e., has only one non-zero component. Here, as in the main text, we assume the wave is propagating along $\boldsymbol{z}$, such that $\boldsymbol{n} = \boldsymbol{z}$. A natural basis can thus be chosen as $\{n^\alpha, \delta_i^\alpha, \tilde{n}^\alpha\}$

$$n^\alpha = (1,0,0,1), \quad \tilde{n}^\alpha = \frac{1}{2}(1,0,0,-1) \tag{120}$$

and choose the following notation for the coordinates in this basis

$$\{x^-, \boldsymbol{x}^\perp, x^+\}. \tag{121}$$

The Minkowski metric in this basis has the form

$$\eta'_{\mu\nu} = \eta_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{122}$$

such that $x^+ = x_-$ and $x^- = x_+$. Clearly in terms of this coordinates the phase is simply $\varphi = \omega x^-$. It is worth noting that the Jacobian for this change of basis is equal to one, thus the measure has the natural form $d^4x = dx^- dx^+ d^2\boldsymbol{x}_\perp$.

A.2 The Landau-Lifshitz equation in a constant magnetic field

The Landau-Lifshitz equation in a constant magnetic field is given by

$$\frac{d\pi^\alpha}{d\tau} = \frac{e}{m} B^\alpha{}_\beta \pi^\beta + \frac{e^2}{\omega_e m^2}\left[B^\alpha{}_\gamma B^\gamma{}_\beta \pi^\beta + \frac{1}{m^2}(B_{\beta\gamma}\pi^\gamma)^2\pi^\alpha\right], \tag{123}$$

where the electron fundamental frequency is defined as $\omega_e = \frac{3m}{2\alpha} \simeq 1.6 \times 10^{23}$ Hz [161] and $B_{\alpha\beta}$ is the magnetic field Maxwell tensor. Observing that $(B)^2_{ij} = -H^2\eta_{ij}$, we can collect the constant interaction terms and define the following transformation

$$\pi^\alpha = \exp\left[\left(\frac{e}{m}B^\alpha{}_\beta - \frac{\omega_H^2}{\omega_e}\eta^\alpha_{\perp\beta}\right)\tau\right]\Pi^\beta, \tag{124}$$

where $\eta_\perp^{\alpha\beta} = \eta^{ij}\delta_i^\alpha\delta_j^\beta$. such that the equation reduces to:

$$\frac{d\Pi^\alpha}{d\tau} = \frac{\omega_H^2}{\omega_e m^2}\exp\left(-2\frac{\omega_H^2}{\omega_e}\tau\eta^\alpha_{\perp\beta}\right)\Pi_\perp^2\Pi^\alpha. \tag{125}$$

Multiplying by $\Pi_\perp^\alpha$ yields:

$$\frac{d\Pi_\perp^2}{d\tau} = 2\frac{\omega_H^2}{\omega_e m^2}\exp\left(-2\frac{\omega_H^2}{\omega_e}\tau\right)\Pi_\perp^4 \quad \Rightarrow \quad \frac{d\Pi_\perp^{-2}}{d\tau} = -2\frac{\omega_H^2}{\omega_e m^2}\exp\left(-2\frac{\omega_H^2}{\omega_e}\tau\right). \tag{126}$$

Solving this equation, we find:

$$\Pi_\perp^2 = \frac{m^2 p_\perp^2}{p_\perp^2\exp\left(-2\frac{\omega_H^2}{\omega_e}\Delta\tau\right) + 2p^+p^-}, \tag{127}$$

where $\Delta\tau = \tau - \tau_0$ and τ_0 is the time at which the interaction starts. The next step is to solve:

$$\begin{aligned}\frac{d\Pi^\alpha}{d\tau} =& \frac{\omega_H^2}{\omega_e}\frac{p_\perp^2}{p_\perp^2 + 2p^+p^-\exp\left(2\frac{\omega_H^2}{\omega_e}\Delta\tau\right)}(\Pi^+n^\alpha + \Pi^-\tilde{n}^\alpha) \\ &+ \frac{\omega_H^2}{\omega_e}\frac{2p^+p^-}{2p^+p^- + p_\perp^2\exp\left(-2\frac{\omega_H^2}{\omega_e}\Delta\tau\right)}\Pi_\perp^\alpha,\end{aligned} \tag{128}$$

for each component individually. This is easily done and the final result reads

$$\pi^\alpha = \frac{m}{\sqrt{p_\perp^2\exp\left(-2\frac{\omega_H^2}{\omega_e}\Delta\tau\right) + 2p^+p^-}}\left[p^+n^\alpha + p^-\tilde{n}^\alpha + \exp\left(-\frac{\omega_H^2}{\omega_e}\Delta\tau\right)\pi^\alpha_{L,\perp}\right], \tag{129}$$

where the transverse solution of the Lorentz equation $\pi^\alpha_{L,\perp}$ is given by

$$\pi^\alpha_{L,\perp} = \exp\left(\frac{e}{m}B^\alpha{}_\beta\tau\right)p^\beta. \tag{130}$$

Let us now consider the motion to be initially (and thus during all the interaction) in the transverse plane, with $p_\parallel = 0$. In this case, the Lorentz factor γ extracted from $\pi^0 = m\gamma$ reads

$$\gamma(\tau) = \frac{1}{\sqrt{1 - |\boldsymbol{v}|^2 \exp\left(-2\frac{\omega_H^2}{\omega_e}\Delta\tau\right)}}, \tag{131}$$

where we observe the expected exponential damping of the transverse velocity. As a function of t, the Lorentz factor can be found by using (see e.g. [166] eq. 7.19)

$$\frac{d\gamma}{dt} = \frac{d}{d\tau}\log(\gamma) = -\frac{\omega_H^2}{\omega_e}\frac{|v|^2 \exp\left(-2\frac{\omega_H^2}{\omega_e}\Delta\tau\right)}{1 - |v|^2 \exp\left(-2\frac{\omega_H^2}{\omega_e}\Delta\tau\right)} = -\frac{\omega_H^2}{\omega_e}(\gamma^2 - 1). \tag{132}$$

This equation can be integrated to find [108]

$$\gamma(t) = \frac{\gamma_0 + 1 + (\gamma_0 - 1)\exp\left(-2\frac{\omega_H^2}{\omega_e}\Delta t\right)}{\gamma_0 + 1 - (\gamma_0 - 1)\exp\left(-2\frac{\omega_H^2}{\omega_e}\Delta t\right)}. \tag{133}$$

This result can also be derived from the LL equation for the gamma factor in a constant magnetic field [see Eq. (137)], assuming an initially transverse velocity

$$\frac{d\gamma}{dt} = -\frac{e^2\tau_e\gamma^2}{m^2c^2}|\boldsymbol{v}\times\boldsymbol{B}|^2 \quad \text{with} \quad \boldsymbol{v} = \boldsymbol{v}_\perp \quad \Rightarrow \quad \frac{d\gamma}{dt} = -\tau_e\omega_H^2\gamma^2\boldsymbol{\beta}^2 = -\tau_e\omega_H^2(\gamma^2 - 1), \tag{134}$$

which matches the earlier result. The scale that governs the energy loss due to radiation reaction effects is set by

$$2\frac{\omega_H^2}{\omega_e}\Delta t \sim 1 \quad \Rightarrow \quad \omega_H^2\Delta t \sim 10^{23}\ \text{Hz}. \tag{135}$$

If we measure time in units of the cyclotron frequency, we have the condition $\omega_H\Delta t' \sim \omega_e$, where $\Delta t' = \omega\Delta t$. To observe RR effects within $\Delta t'/2\pi \sim 100$ cycles for example, an incredibly large frequency $\omega_H \sim 10^{20}$ Hz is required.

A.3 The average rest frame

The average rest frame is a useful choice of reference frame when we want to visualize the motion in a plane wave. Below, the subscript L (R) indicates quantities in the laboratory and the average rest frames, respectively. The latter has a relative (drift) velocity as compared to the former given by $\boldsymbol{v}_d = v_d\boldsymbol{n}$, with v_d defined via the relation

$$\langle\boldsymbol{\pi}_R(\phi)\cdot\boldsymbol{n}\rangle = \gamma(v_d)\big[\langle\boldsymbol{\pi}_L(\phi)\cdot\boldsymbol{n}\rangle - v_d\langle\varepsilon_L(\phi)\rangle\big] = 0. \tag{136}$$

Here the averages are taken over a plane wave period, thus this definition defines the frame in which there is no drift velocity along the wave direction but just an oscillating component. In the average rest frame the electron energy corresponds to the so-called effective mass $m_* = m\gamma_*$ [141], such that $\gamma_* = \sqrt{(1+v_d)/(1-v_d)}$ describes the relativistic Doppler effect: $\omega_{0,L} = \gamma_*\omega_{0,R}$ (recall that $\boldsymbol{v}_d$ is along the propagation direction of the laser).

A.4 Explicit form of the LL equation

For the sake of completeness let us report the LL equation in terms of the physical electromagnetic fields (see, e.g., [122] or [183] Eq. 14a, 14b)

$$\begin{aligned}\frac{d(\gamma \boldsymbol{v})}{dt} &= \frac{e}{m}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}) + \frac{e\tau_e\gamma}{m}\left(\frac{d\boldsymbol{E}}{dt}+\boldsymbol{v}\times\frac{d\boldsymbol{B}}{dt}\right)\\ &+ \frac{e^2\tau_e}{m^2c}\left[\frac{1}{c}(\boldsymbol{v}\cdot\boldsymbol{E})\boldsymbol{E}+c(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\times\boldsymbol{B}\right]\\ &+ \frac{e^2\tau_e\gamma^2}{m^2c^2}\left[\frac{1}{c^2}(\boldsymbol{v}\cdot\boldsymbol{E})^2-|\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}|^2\right]\boldsymbol{v},\end{aligned} \tag{137}$$

$$\begin{aligned}\frac{d\gamma}{dt} &= \frac{e}{mc^2}\boldsymbol{v}\cdot\boldsymbol{E} + \frac{e\tau_e\gamma}{mc^2}\boldsymbol{v}\cdot\frac{d\boldsymbol{E}}{dt}\\ &+ \frac{e^2\tau_e}{m^2c^2}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{E}\\ &+ \frac{e^2\tau_e\gamma^2}{m^2c^2}\left[\frac{1}{c^2}(\boldsymbol{v}\cdot\boldsymbol{E})^2-|\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}|^2\right],\end{aligned} \tag{138}$$

with $\tau_e = \frac{2\alpha\hbar}{3mc^2} = 6.27\times10^{-24}s$ in physical units and $\frac{d\boldsymbol{E}}{dt} = \frac{\partial\boldsymbol{E}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{E}$. Combining the two equations we can explicitly write the evolution of the velocity in the laboratory frame

$$\begin{aligned}\frac{d\boldsymbol{v}}{dt} &= \frac{e}{m\gamma}\left[\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}-\frac{1}{c^2}(\boldsymbol{v}\cdot\boldsymbol{E})\boldsymbol{v}\right] + \frac{e\tau_e}{m}\left[\frac{d\boldsymbol{E}}{dt}+\boldsymbol{v}\times\frac{d\boldsymbol{B}}{dt}-\left(\boldsymbol{v}\cdot\frac{d\boldsymbol{E}}{dt}\right)\boldsymbol{v}\right]\\ &+ \frac{e^2\tau_e}{m^2c\gamma}\left[\frac{1}{c}(\boldsymbol{v}\cdot\boldsymbol{E})\boldsymbol{E}+c(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\times\boldsymbol{B}-\frac{1}{c}[(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{E}]\boldsymbol{v}\right].\end{aligned} \tag{139}$$

A.5 The graviton propagator

The free graviton action reads

$$S_h^{\text{g.f.}} = \int d^4x\left\{\frac{1}{2}\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} - \frac{1}{4}\partial^\mu h\partial_\mu h\right\}. \tag{140}$$

This is equivalent to the more symmetrical form

$$S_h^{\text{g.f.}} = -\int d^4x h^{\alpha\beta}\Box h^{\mu\nu}\mathcal{P}_{\alpha\beta\mu\nu}, \tag{141}$$

where $\mathcal{P}_{\alpha\beta\mu\nu}$ is the projector

$$\mathcal{P}_{\alpha\beta\mu\nu} = \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\eta_{\mu\nu}\eta_{\alpha\beta}\right). \tag{142}$$

We can now apply the general procedure to find a free propagator. Going to momentum space we have to solve

$$k^2\,\mathcal{P}^{\alpha\beta}_{\ \ \mu\nu}D^{\mu\nu\rho\delta} = i\eta^{\alpha\rho}\eta^{\beta\delta} \tag{143}$$

This is trivially done observing that $\mathcal{P}^{\alpha\beta}_{\ \ \mu\nu}\mathcal{P}^{\mu\nu\rho\delta} = \eta^{\alpha\rho}\eta^{\beta\delta}+\eta^{\alpha\delta}\eta^{\beta\rho}$ which is actually the unity for symmetric contractions. Therefore

$$D_{\alpha\beta\mu\nu}(k) = \frac{i}{2\,k^2}\mathcal{P}_{\alpha\beta\mu\nu}. \tag{144}$$

A.6 Graviton photoproduction in scalar-QED

We recall here the scalar QED Lagrangian

$$\mathcal{L}_{SQED} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D_\mu\phi)^\dagger D^\mu\phi - m^2\phi^\dagger\phi. \tag{145}$$

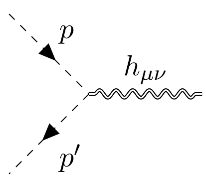
When we include a graviton in this theory the Lagrangian takes the form

$$\mathcal{L} = \mathcal{L}_{SQED} + \mathcal{L}_h + \mathcal{L}_{h\mathcal{A}} + \mathcal{L}_{hs} + \mathcal{L}_{h\mathcal{A}s} + \mathcal{L}_{h\mathcal{A}\mathcal{A}s}. \tag{146}$$

For the purpose of calculating the graviton photoproduction we can forget about the last interaction which involves 5 particles. The Feynman rules can be extracted from the quantity

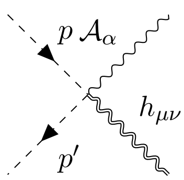
$$\eta^{\mu\nu}(D_\alpha\phi)^\dagger D^\alpha\phi - 2(D^\mu\phi)^\dagger D^\nu\phi, \tag{147}$$

from which one finds



$$= i\frac{\kappa}{2}\left[p^\mu p'^\nu + p^\nu p'^\mu - \eta^{\mu\nu}(p\cdot p' + m^2)\right] \tag{148}$$

and



$$= ie\frac{\kappa}{2}\left[\eta^{\alpha\mu}(p-p')^\nu + \eta^{\alpha\nu}(p-p')^\mu - \eta^{\mu\nu}(p-p')^\alpha\right]. \tag{149}$$

One can now verify by hand or with the following simple FORM code that the proportionality described in Sec. 3.4.1 occurs.

```
s e,K,m;
s s,t,u,a,A,B,C,D,E,F;
s wi,wf, cth, sth;
v ki,kf,pi,pf,p1,p2,p3,p4, epA,eph, epss;
i i,j, mu , nu , rho , al , be, si, de, th ;
f P;
auto f V;
***

l GravitonPhotoprodA  = (
VAss(al,pi, - pi - ki )*Vhss(mu,nu,pi+ki,-pf)*i_/(2*pi.ki) -
VAss(al, pi - kf , -pf)*Vhss(mu,nu,pi,-pi + kf)*i_/(2*pi.kf) -
VhAA(mu,nu,al,be,ki,-ki+kf)*VAss(be,pi,-pf)*i_/(2*ki.kf)+
VhAss(mu,nu, al, pi,-pf) )*epA(al)*eph(mu)*eph(nu);

l ScalarComptonA =  (
VAss(mu,pi, - pi - ki )*VAss(nu,pi+ki,-pf)*i_/(2*pi.ki) -
VAss(mu, pi - kf , -pf)*VAss(nu,pi,-pi + kf)*i_/(2*pi.kf) +
```

```
VAAss(mu,nu) )*epA(mu)*eph(nu);

l H = - K/(2*e)*(eph.pf*pi.kf - eph.pi*pi.ki)/ki.kf;

l proportionalitytest = ki.kf*(GravitonPhotoprodA - H*ScalarComptonA);


#include amplitude.h

id epA.ki = 0;
id eph.kf = 0;
id eph.eph = 0;
id kf.epA = pi.epA - pf.epA;
id ki.eph = pf.eph - pi.eph;
id ki.ki = 0;
id ki.kf = ki.pi- ki.pf;
id ki.pf = kf.pi;


bracket e,K,i_;

Print +s;
.end
```

The header amplitude.h is reported in A.11.3.

A.7 Massless states little group and the Pauli-Lubanski pseudovector

Here we want to briefly investigate the massless states little group [179]. We choose without losing generality a reference null vector $k^\mu = \omega(1,0,0,1)$ and we want to find the transformations satisfying

$$W^\mu_{\ \nu} k^\nu = k^\mu. \tag{150}$$

Now, remember that we can collect the Lorentz generators $M_{\mu\nu}$ introducing rotations and boosts

$$J^i = -\frac{1}{2}\varepsilon^{ijk} M_{jk} \quad , \quad K^i = M^{0i} \tag{151}$$

with the angles and rapidity

$$\theta^k = -\frac{1}{2}\varepsilon^{kij}\omega_{jk} \quad , \quad \eta^i = \omega^{0i} \tag{152}$$

such that

$$\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu} = i(\eta_i K^i - \theta_i J^i) \qquad \text{and} \qquad \Lambda^\mu_{\ \nu} = e^{-i(\eta\cdot K - \theta\cdot J)}. \tag{153}$$

Our little group condition is transferred to the algebra as $\omega_{\mu\nu} n^\nu = \omega_{\mu 0} + \omega_{\mu 3} = 0$ and therefore we get the 3 conditions

$$\eta^1 = -\theta^2 \quad , \quad \eta^2 = \theta^1 \quad , \quad \eta^3 = 0 \tag{154}$$

we see here that the rotation along the propagation direction can be whatever we want, there is no constrain on θ^3, this will be shown to be connected to the helicity of the states. We are than left with the generators (see e.g. [179] p. 71)

$$K_1 + J_2 \quad , \quad K_2 - J_1 \quad , \quad J_3. \tag{155}$$

The first two generators can be put in the form

$$\chi^l = \frac{1}{2} f_l^{\mu\nu} M_{\mu\nu}, \tag{156}$$

in fact

$$\chi^l = \frac{1}{2} f_l^{\mu\nu} M_{\mu\nu} = n^\mu \eta^{l\nu} M_{\mu\nu} = n^i \eta^{lj} M_{ij} + n^0 \eta^{lj} M_{0j} \tag{157}$$

and in terms of boosts and rotations we find

$$\chi^l = n^i \eta^{lj} M_{ij} + 2n^0 \eta^{lj} M_{0j} = \eta^{i3} \eta^{lj} \varepsilon_{ijk} J^k + \eta^{lj} K_j = \varepsilon_{3lk} J^k + K^l = \varepsilon^{lk} J^k + K^l \tag{158}$$

such that

$$\chi^1 = K^1 + J^2 \quad , \quad \chi^2 = K^2 - J^1. \tag{159}$$

The generators of the little group can be collected in the Pauli-Lubanski pseudovector [127], [25]

$$W_\mu = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} M^{\nu\alpha} P^\beta. \tag{160}$$

Indeed expanding this object we find

$$W^i = P^0 J^i - \varepsilon^{ijk} P^j K^k \quad \text{and} \quad W^0 = -J^i P_i. \tag{161}$$

Choosing $P^\mu = k^\mu = \omega(1, 0, 0, 1)$, then

$$W^1 = \omega(J^1 + K^2) \quad , \quad W^2 = \omega(J^2 - K^1) \quad , \quad W^0 = \omega J^3. \tag{162}$$

We can than express our three generators as

$$\chi_\mu = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} M^{\nu\alpha} n^\beta. \tag{163}$$

The take home message is that the Pauli-Lubanski operator represents the generators of the little group (see [25] pag. 215), defining the angular momentum of moving particles (spin included). Note that, being by definition $W^\mu P_\mu = 0$ only 3 components are independent, and in fact we have 3 generators. This was easy to guess considering that by definition $W_\mu k^\mu = k_\mu k^\mu = 0$

A.8 The Carroll group and its connection to plane wave symmetries

The plane wave symmetries discussed in Ch. 5 are connected to the so-called Carroll group (from the Alice in Wonderland author), introduced as a limit of the Poincarè group by Lèvy-Leblond in [123], therefore let us spend few words about this topic. We start recalling that a general Poincarè transformation can be written in the form

$$\text{Poincarè transformation} \begin{cases} x_0' = \gamma(x_0 + \beta^i R_{ij} x^j) + d_0 \\ x'^i = R^{ij} x_j + \frac{\gamma - 1}{\beta^2} \beta^j R_{jk} x^k \beta^i + \gamma x_0 \beta^i + a^i \end{cases}, \tag{164}$$

where R is a rotation matrix. Now, let us set [123]

$$s = Cx^0 \quad , \quad b^i = C\beta^i \quad , \quad a^0 = Cd^0 \tag{165}$$

and take the limit $C \to \infty$ keeping s, b^i, d^i fixed. In this case we get

$$\text{Carroll transformation} \begin{cases} s' = s + b^i R_{ij} x^j + a^0 \\ x'^i = R^{ij} x_j + a^i \end{cases} \tag{166}$$

Now, this is in four dimensions, but we can restrict our attention at the slices of constant phase of the Minkowski space (x^+, x^i). This restriction is a *Carroll manifold*, because it contains a constant isometry ∂_+ and the simple metric $dx^i dx_i$ [73]. On this manifold we have a Carroll symmetry group. The phase is not part of this space so we can just use it as a parameter of translations and redefine Carroll transformations on this manifold as

$$\begin{cases} x'^+ = x^+ + b^i R_{ij}(x^j + a^j - \frac{1}{2} b^j x^-) + a^+ \\ x'^i = R^{ij} x_j + b^i x^- + a^i \end{cases} \tag{167}$$

where we also introduces in a proper way a^+. Now, if we consider the case in which there is no rotation in the x^i plane we clearly get back our plane wave isometries. Therefore we conclude that

The plane wave isometry group in d dimensions is the Carroll group in $d-1$ dimensions with broken rotations.

A.9 Brinkmann coordinates as null Fermi coordinates

A very interesting property of Brinkmann coordinates has been pointed out in [42], namely that Brinkmann coordinates are Fermi normal coordinates constructed on null geodesics. Let us discuss the general construction of Fermi coordinates on null geodesics (see [42] Sec. 4). First of all we define a null geodesic $\gamma^\mu(\phi)$ and we introduce a tetrad e_α. As usual we identify one of the tetrad vectors with the velocity tangent to the geodesics. In this case

$$e^+ = \dot{\gamma} \partial_-, \tag{168}$$

with the orthonormality conditions along the curve being

$$ds^2|_\gamma = 2e^+e^- + e^i e_i. \tag{169}$$

We now consider a point on geodesic $\gamma(0) = x_0$ and we construct from it spacelike geodesics $x^\mu(\lambda)$ originating from it and perpendicular to $e^-_\mu(x_0)$ on the geodesic:

$$\dot{x}^\mu(0)e^-_\mu(x_0) = 0 \tag{170}$$

Usually the time vector of the tetrad is chosen as the tangent velocity and the spacelike geodesics are required to be orthogonal to it. In this case both $e^+_\mu(x_0)$ and $e^-_\mu(x_0)$ are involved [42]. We can introduce the Fermi coordinates y^α of the point $x^\mu(\lambda)$ as follows

$$y^\alpha(x(\lambda)) = (x^-, \lambda\dot{x}^\mu(0)e^\alpha{}_\mu(x_0)) \tag{171}$$

Now, by definition we have that on the geodesic the relation between the general and the Fermi coordinates is represented by the vierbein, in fact

$$\left.\frac{\partial y^\alpha}{\partial x^\mu}\right|_{\lambda=0} = \left.\frac{\partial \dot{y}^\alpha}{\partial \dot{x}^\mu}\right|_{\lambda=0} = e^\alpha{}_\mu(x_0). \tag{172}$$

With this considerations in mind we can now study the connection between Rosen and Brinkmann coordinates in these terms.

Let us start with the Rosen metric and consider a null geodesic congruence with tangent vector $e^+ \propto \partial_-$. We can use the usual vierbein for the Rosen metric $e_{\alpha\mu}e^\alpha{}_\nu = g_{\mu\nu}$. Now we require it to be parallel transported along the congruence

$$e_\alpha{}^\mu{}_{;-} = 0. \tag{173}$$

Recalling the form of Christoffel symbols in this metric we get

$$e_\alpha{}^\mu{}_{;-} = \tilde{n}^\mu(\partial_\mu e_\alpha{}^\lambda + \Gamma^\lambda_{\mu\nu}e_\alpha{}^\nu) = \partial_- e_\alpha{}^\lambda + \frac{1}{2}g^{\lambda\rho}\partial_- g_{\rho\mu}e_\alpha{}^\mu = 0 \tag{174}$$

and this is just the symmetry condition (5.15). Therefore, we learn here that this condition is enough to state that the canonical Rosen vierbein is parallel transported along null geodesics. As a next step we introduce the geodesics $x^\mu(\lambda)$ starting from the original one in x_0 and we require them to have no components tangent to it

$$\dot{x}^\mu(0)e_{-\mu}(x_0) = \dot{x}^-(0) = 0. \tag{175}$$

These geodesics are defined by

$$\dot{u}_\nu - \frac{1}{2}u^\mu u^\rho \dot{g}_{\rho\mu}n_\nu = 0, \tag{176}$$

the condition $u^-(0) = 0$ implies that $x^- = x_0^-$. The metric depends only on the phase, therefore we have

$$\dot{u}_\nu(\lambda) - \frac{1}{2}u^\mu(\lambda)u^\rho(\lambda)\dot{g}_{\rho\mu}(x_0)n_\nu = 0. \tag{177}$$

From this equation it is clear that the transverse components are linear in the affine parameter λ, thus

$$x^i = \lambda u^i_{0,\perp} \qquad , \qquad x^+ = u_0^+ \lambda + \frac{1}{4} u_0^i u_0^j \dot{\gamma}_{ij}(x_0)\lambda^2. \tag{178}$$

By definition the transverse Fermi coordinates are just the vierbein projection of these

$$X^i(\lambda) = e^i{}_j x^j \qquad , \qquad X^+(\lambda) = x^+ + \frac{1}{4} \dot{g}_{\rho\mu} x^\mu_\perp x^\rho_\perp \tag{179}$$

and this is exactly the connection between Brinkmann and Rosen coordinates (5.28). This finally proves that Brinkmann coordinates are null Fermi coordinates. Moreover, we can expand the metric around the geodesics in term of Fermi coordinates, a general expression for null geodesics was given in [42] eq. (1.5). The interesting fact is that in the case of GW this expansion stops at quadratic order reproducing the Brinkmann metric

$$ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu + H_{ij}(\phi) X^i X^j d\phi^2. \tag{180}$$

It follows that the Brinkmann metric is the metric seen by a free falling observer following the considered null geodesics.

A.10 Gravitational wave metrics handbook

In the following table, we collect a few known but useful formulas regarding the Brinkmann and the Rosen metrics. We refer to the main text for the notation.

Brinkmann	
The metric	$ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu + H d\phi^2$
Christoffel symbols	$\Gamma^\lambda_{\mu\nu} = n^\lambda \left(\frac{1}{2}\dot{H} n_\mu n_\nu + 2H_{(\mu} n_{\nu)}\right) - H^\lambda n_\mu n_\nu$
Riemann tensor	$R_{\rho\mu\nu\delta} = [H_{\rho\nu} n_\mu n_\delta - (\rho \leftrightarrow \mu)] - (\nu \leftrightarrow \delta)$
Ricci tensor	$R_{\mu\delta} = H^i{}_i n_\mu n_\delta$
Vierbein	$E_{\alpha\mu} = \eta_{\alpha\mu} + \frac{H}{2} n_\alpha n_\mu$
Spin connection	$\omega_{\mu\alpha\beta} = 2 n_\mu H_{[\beta} n_{\alpha]}$
Rosen	
The metric	$ds^2 = 2dx^+ d\phi + \gamma_{ij} dx^i dx^j$
Christoffel symbols	$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(n_\mu \dot{\gamma}_{\rho\nu} + n_\nu \dot{\gamma}_{\rho\mu} - n_\rho \dot{\gamma}_{\mu\nu}\right)$
Riemann tensor	$R_{\rho\mu\nu\delta} = \left[\left(\frac{1}{4}\gamma_{\rho\sigma}\dot{\gamma}^{\sigma\lambda}\dot{\gamma}_{\lambda\nu} + \frac{1}{2}\ddot{\gamma}_{\rho\nu}\right) n_\mu n_\delta - (\rho \leftrightarrow \mu)\right] - (\nu \leftrightarrow \delta)$
Ricci tensor	$R_{\mu\delta} = \left(\frac{1}{4}\dot{\gamma}^{\rho\lambda}\dot{\gamma}_{\lambda\rho} + \frac{1}{2}\gamma^{\rho\lambda}\ddot{\gamma}_{\lambda\rho}\right) n_\mu n_\delta$
Vierbein	$e_{\alpha\mu}$
Spin connection	$\omega_{\mu\alpha\beta} = -2 n_{[\alpha} \dot{e}_{\beta]\mu}$

A.11 Codes

A.11.1 Nonlinear Compton scattering (ComptonEWlinear.frm)

The following FORM code calculates the unpolarized squared amplitudes for the process of nonlinear Compton scattering in QED and in scalar QED. Moreover, this code calculates the linear limit of these squared amplitudes recovering the known results for Compton scattering in vacuum.

Here J stands for an integration times the exponential phases [see Eq. (3.20)]

$$J = \int d\phi e^{ig(\phi)} \tag{181}$$

and JA^a has to be interpreted as

$$JA^a = \int d\phi e^{ig(\phi)} A^a(\phi). \tag{182}$$

The incoming momenta are p_i, k_i while the outgoing ones p_f, k_f.

```
s e,m,Xi,G,Gc, w, wf,K,J,Jc,E,[1-cos],cos,sin;
v A,Ac,k,kf,ki,pi,pf,Pi,Kf,Pf, Pic, Kfc, Pfc,kfperp, nt, eps, epsc, p1,...,p10,q,p;
i i,j,mu,nu,al,be,rho,de,si,eta;
cf f0,f1,f2,g0,g1,g2,D,Q,Qc,B,C,epA,eph,PI, PIc;
s E,Ep,wp,cth,sth,cph,kfplus,f,fc;
***
#include amplitude.h
off statistics ;
************************************************************************
*@@ 1/4*SUM_(spin,pol) |M|^2  for NONLINEAR COMPTON SCATTERING in QED @@@
************************************************************************
*** J represents the integral times esponential phases, ***
*** Jc is its complex conjugate ***

l SquaredAmpSpinor = 1/4*J*Jc*
e^2*(g_(1,pf) + m)*
( 1 - e/(2*ki.pf)*g_(1,ki,A))*
( g_(1,mu)  )*
( 1 - e/(2*ki.pi)*g_(1,A,ki))*
(g_(1,pi) + m)*
( 1 - e/(2*ki.pi)*g_(1,ki,Ac))*
( g_(1,nu) )*
( 1 - e/(2*ki.pf)*g_(1,Ac,ki))*
-d_(mu,nu)   ;

**************************************************************************
*@@ 1/2*SUM_(pol) |M|^2  for NONLINEAR COMPTON SCATTERING in scalar QED @@@
**************************************************************************
l SquaredAmpScalar = 1/2*J*Jc*
( - e*( PI(mu,pi) + PI(mu,pf) )
+ 2*e^2*A(mu) )*
( - e*( PIc(nu,pi) + PIc(nu,pf) )
+ 2*e^2*Ac(nu) )*
-d_(mu,nu)   ;
***
trace4,1;
contract;
***
id PI(mu?,q?) = q(mu) + e*q.A/ki.q*ki(mu) - e^2*A.A/(2*ki.q)*ki(mu) ;
```

```
id PIc(mu?,q?) = q(mu) + e*q.Ac/ki.q*ki(mu) - e^2*Ac.Ac/(2*ki.q)*ki(mu) ;
***
id ki.A = 0;
id ki.Ac = 0;
id ki.ki = 0;
id kf.kf = 0;
id pi.pi = m^2;
id pf.pf = m^2;
*
*** Print the full nonlinear result  ***
*
bracket e,G,Gc,m,J,Jc ;
print +s   SquaredAmpSpinor, SquaredAmpScalar;
.sort
********************
*@@ LINEAR LIMIT @@@
********************
*** there is no contribution from the terms quadratic in A ***
id A.A = 0;
id Ac.Ac = 0;
*** monochromatic plane wave hypothesis, f = e^(ik.x) ***
id A.Ac = f*fc*epA(rho,ki)*epA(rho,ki);
id Ac.A = f*fc*epA(rho,ki)*epA(rho,ki);
id A.q? = f*epA(al,ki)*q(al) ;
id Ac.q? = fc*epA(be,ki)*q(be) ;
*** first order expansion of the phase factors ***
id J = 1 + f*e*C(si)*epA(si,ki) ;
id Jc = 1 + fc*e*C(eta)*epA(eta,ki) ;
id C(mu?) = pi(mu)/ki.pi - pf(mu)/ki.pf ;
*** keep only the order e^4, momentum conservation requires the product f*fc ***
id e^4 = E;
id e = 0;
id E = e^4;
*
id f^2 = 0;
id fc^2 = 0;
*** sum over the incoming wave polarizations ***
id epA(ki,ki) = 0;
id epA(q?,ki)*epA(p?,ki) = -p(mu)*q(nu)*(d_(mu,nu) - (ki(mu)*kf(nu) + kf(mu)*ki(nu))/ki.kf );
id epA(mu?,ki)*epA(mu?,ki) = -(d_(mu,nu) - (ki(mu)*kf(nu) + kf(mu)*ki(nu))/ki.kf )*d_(mu,nu);
*** momentum conservation ***
id pi.pf = pi.ki - pi.kf + m^2 ;
id pi.pi = m^2;
id pf.pf = m^2;
id ki.ki = 0;
id kf.kf = 0;
id kf.pf = pi.ki;
id kf.pf^-1 = pi.ki^-1;
id pf.ki = pi.kf;
id pf.ki^-1 = pi.kf^-1;
***
*
#procedure labframe
id pi.ki^-2 = (w*m)^-2    ;
id pi.ki^-1 = (w*m)^-1    ;
id pi.ki = w*m    ;
id pi.kf^-2 = (wp*m)^-2    ;
id pi.kf^-1 = (wp*m)^-1    ;
id pi.kf = wp*m ;
id ki.kf^-2 = (wp*w*[1-cos])^-2    ;
id ki.kf^-1 = (wp*w*[1-cos])^-1    ;
id ki.kf = wp*w*[1-cos] ;
```

```
**
id wp^-2 = (w^-1 + [1-cos]*m^-1)*wp^-1 ;
id w^-2 = (wp^-1 -  [1-cos]*m^-1)*w^-1  ;
id w^-1*wp = A ;
id wp^-1*w = B ;
id wp^-1 = w^-1 + [1-cos]*m^-1 ;
id w^-1 = wp^-1 -  [1-cos]*m^-1  ;
id A = w^-1*wp ;
id B = wp^-1*w ;
id [1-cos] = 1-cos;
id [1-cos]^-1 = (1-cos)^-1;
**
#endprocedure
*
*** factorize and print the linear order ***
.sort
#$fac = factorin_(SquaredAmpSpinor);
Multiply 1/('$fac');
#Optimize SquaredAmpSpinor
bracket e,G,Gc,m,f ;
Print +s SquaredAmpSpinor,SquaredAmpScalar;
.sort
Skip;
#write <> "
Squarelinear =  (%$)*Squarelinear ;",$fac
*
*** print the lab frame ***
.sort
*#call labframe
bracket e,G,Gc,m,f ;
*Print +s SquaredAmpSpinor,SquaredAmpScalar;
.sort
.end
```

A.11.2 Photon emission in a gravitational wave (ComptonGW3.frm)

The following FORM code calculates the unpolarized squared amplitudes for the process of photon emission in a nonlinear gravitational wave. Moreover, this code calculates the linear limit of these squared amplitudes recovering the known results for graviton photoproduction. As in A.11.1 the J stands for the integration involving the exponential terms. Moreover, this code checks the independence of the probability on the gauge choice. The incoming momenta are p_i, k_i while the outgoing ones p_f, k_f.

```
s e,m,Xi,G,Gc, w, wf,K,A,[1-cos],cos,sin;
v k,kf,ki,pi,pf,Pi,Kf,Pf, Pic, Kfc, Pfc,kfperp, nt, eps, epsc, PI, PIc, p1,...,p10,q1,...,q10,q,p;
v k5,k6,k7;
i i,j,mu,nu,al,be,rho,de,sig,eta;
i mu1,...,mu40,nu1,...,nu40,i1,...,i40,al1,...,al20;
cf f0,f1,f2,g0,g1,g2,D,Q,Qc,B, eph, epA,P, PG,L,ub,u,vb,v,g,gstring,fprop,Aprop,gprop,prop;
AutoDeclare f V;
s E,Ep,wp,cth,sth,cph,kfplus,f,fc,J,Jc;
***
off stat;
#include amplitude.h
**********************************************************************
*@@ 1/4*SUM_(spin,pol) |M|^2 for NONLINEAR COMPTON SCATTERING IN GW @@@
**********************************************************************
*** Calculation done assuming only transverse polarizations for the incoming graviton ***
*** To check the correctness of the C.S. polarization sum, we calculate the           ***
*** squared amplitude first substituting the flat polarization sum and than           ***
*** the C.S. polarization sum. The result has to be the same.                         ***
*** J represents the integral times esponential phases, Jc is its complex conjugate.  ***
*
l SquaredAmp1 = 1/4*e^2*J*Jc*
(g_(1,pf) + m)*
( gi_(1) - 1/(2*k.pf)*g_(1,Pf,k))*
( g_(1,mu)  -  Kf(mu)/(k.kf)*g_(1,k) )*
( gi_(1) - 1/(2*k.pi)*g_(1,k,Pi))*
(g_(1,pi) + m)*
( gi_(1) - 1/(2*k.pi)*g_(1,Pic,k))*
( g_(1,nu)  -  Kfc(nu)/(k.kf)*g_(1,k)  )*
( gi_(1) - 1/(2*k.pf)*g_(1,k,Pfc))*
(-d_(mu,nu) + ( k(mu)*kf(nu) + k(nu)*kf(mu))/k.kf )  ;
*
l SquaredAmp2 =         1/4*e^2*J*Jc*
(g_(1,pf) + m)*
( gi_(1) - 1/(2*k.pf)*g_(1,Pf,k))*
( g_(1,mu) )*
( gi_(1) - 1/(2*k.pi)*g_(1,k,Pi))*
(g_(1,pi) + m)*
( gi_(1) - 1/(2*k.pi)*g_(1,Pic,k))*
( g_(1,nu) )*
( gi_(1) - 1/(2*k.pf)*g_(1,k,Pfc))*
( - d_(mu,nu) + 1/2*( Kf.Kf + Kfc.Kfc - 2*Kf.Kfc )*k(mu)*k(nu)/(k.kf)^2 +
( k(mu)*(kf(nu) + Kf(nu) - ( kf.Kf + Kf.Kf/2 )*k(nu)/(k.kf) ) +
k(nu)*(kf(mu) + Kfc(mu) - ( kf.Kfc + Kfc.Kfc/2 )*k(mu)/(k.kf) ))/k.kf );
*
l PolarizationSumTest = SquaredAmp1 - SquaredAmp2 ;
*
***
trace4,1;
contract;
***
id k.k = 0;
```

```
id k.Pi = 0;
id k.Pf = 0;
id k.Kf = 0;
id k.Pic = 0;
id k.Pfc = 0;
id k.Kfc = 0;
id kf.kf = 0;
id pi.pi = m^2;
id pf.pf = m^2;
bracket e,G,Gc,m,J,Jc ;
print +s  SquaredAmp1;
.sort
*******************
*@@ LINEAR LIMIT @@@
*******************
Multiply replace_(k,ki) ;
*** monochromatic plane wave hypothesis, f = e^(ik.x) ***
id Pi.Pic =  f*fc*G^2/4*eph(pi,mu1)*eph(pi,mu1);
id Pf.Pfc =  f*fc*G^2/4*eph(pf,mu2)*eph(pf,mu2);
*
** obs: Kf.Kfc does not contribute in the linear limit **
id Kf.Kfc =  A*f*fc*G^2/4*eph(kf,mu3)*eph(kf,mu3);
*
id Kf.Pic =  Pi.Kfc ;
id Pi.Kfc =  f*fc*G^2/4*eph(pi,mu4)*eph(kf,mu4);
id Kf.Pfc =  Pf.Kfc ;
id Pf.Kfc =  f*fc*G^2/4*eph(pf,mu5)*eph(kf,mu5);
*
id Pi.q? = - f*G/2*eph(pi,q);
id Pf.q? = - f*G/2*eph(pf,q);
id Kf.q? = - f*G/2*eph(kf,q);
id Pic.q? = - fc*G/2*eph(pi,q);
id Pfc.q? = - fc*G/2*eph(pf,q);
id Kfc.q? = - fc*G/2*eph(kf,q);
*
*** first order expansion of the phase factors ***
id J = 1 + f*G/2*B(al,be)*eph(al,be);
id Jc = 1 + fc*G/2*B(sig,eta)*eph(sig,eta);
id B(mu?,nu?) = pi(mu)*pi(nu)/ki.pi - pf(mu)*pf(nu)/ki.pf - kf(mu)*kf(nu)/ki.kf   ;
*
*** Print the full nonlinear result ***
bracket e,G,Gc,m,J,Jc ;
print +s   SquaredAmp1  ;
.sort
*** keep only the order e^2*G^2, momentum conservation requires the product f*fc ***
id e^2*G^2 = E;
id e = 0;
id G = 0;
id E = e^2*G^2;
id fc^2 = 0;
id f^2 = 0;
id f*fc = 1;
*
*** sum over the incoming wave polarizations ***
id eph(p1?,p2?)*eph(p3?,p4?) =  PG(mu6,mu7,mu8,mu9)*p1(mu6)*p2(mu7)*p3(mu8)*p4(mu9)  ;
id eph(p1?,mu1?)*eph(p3?,mu1?) =  PG(mu10,mu11,mu12,mu13)*p1(mu10)*p3(mu12)*d_(mu11,mu13)  ;
id PG(al?,be?,mu?,nu?) = 1/2*( L(al,mu)*L(be,nu) +  L(al,nu)*L(be,mu) - L(mu,nu)*L(al,be) );
id L(mu?,nu?) = d_(mu,nu) - (ki(mu)*kf(nu) + ki(nu)*kf(mu))/ki.kf ;
***
.sort
****************************************************************************
* Calculation of the linear squared amplitude from usual linear Feynman rules *
```

```
******************************************************************************
l ComptonH =           -1/2*
G/(2*e)*(pf(mu1)*pi.kf - pi(mu1)*pi.ki)/ki.kf*
(
ub(i1,pf,m)*VAff(mu2,i1,i2)
*fprop(i2,i3,q1,m)
*VAff(al1,i3,i4)*u(i4,pi,m)
+
ub(i1,pf,m)*VAff(al1,i1,i2)
*fprop(i2,i3,q2,m)
*VAff(mu2,i3,i4)*u(i4,pi,m)
)*epA(al1,ki)*eph(mu1,mu2,kf);
*
*** We square the amplitude, summing over the spin, pol. ***
#call vertices
#call squareamplitude(ComptonH,value)
*
*
* We can replace the propagators
*
id prop(q1.q1-m^2)^2 = (2*pi.ki)^-2;
id prop(q2.q2-m^2)^2 = (-2*pi.kf)^-2;
id prop(q1.q1-m^2)^1 = (2*pi.ki)^-1;
id prop(q2.q2-m^2)^1 = (-2*pi.kf)^-1;
id prop(q3.q3)^2 = (-2*ki.kf)^-2;
id prop(q3.q3)^1 = (-2*ki.kf)^-1;
id q1 = pi + ki;
id q2 = pi - kf;
id q3 = pi - pf;
*
.sort
*
l LinearTest = SquaredAmp1 - value ;
*
Multiply ki.kf^2;
*** momentum conservation ***
id ki.kf = pi.ki -pi.kf ;
id pi.pf = pi.ki -pi.kf + m^2 ;
id pi.pi = m^2;
id pf.pf = m^2;
id ki.ki = 0;
id kf.kf = 0;
id kf.pf = pi.ki;
id kf.pf^-1 = pi.ki^-1;
id pf.ki = pi.kf;
id pf.ki^-1 = pi.kf^-1;
***
#procedure labframe
id pi.ki^-2 = (w*m)^-2;
id pi.ki^-1 = (w*m)^-1;
id pi.ki = w*m;
id pi.kf^-2 = (wp*m)^-2;
id pi.kf^-1 = (wp*m)^-1;
id pi.kf = wp*m ;
id ki.kf^-2 = (wp*w*[1-cos])^-2;
id ki.kf^-1 = (wp*w*[1-cos])^-1;
id ki.kf = wp*w*[1-cos] ;
**
id wp^-2 = (w^-1 + [1-cos]*m^-1)*wp^-1 ;
id w^-2 = (wp^-1 - [1-cos]*m^-1)*w^-1 ;
id w^-1*wp = A ;
id wp^-1*w = B ;
```

```
id wp^-1 = w^-1 + [1-cos]*m^-1 ;
id w^-1 = wp^-1 - [1-cos]*m^-1 ;
id A = w^-1*wp ;
id B = wp^-1*w ;
id [1-cos] = 1-cos;
*id [1-cos]^-1 = (1-cos)^-1;
**
#endprocedure
*
*** facatorization and print of the linear limit ***
#$fac = factorin_(SquaredAmp1);
Multiply 1/('$fac');
Bracket m,K,e,A;
Print +s SquaredAmp1,SquaredAmp2,PolarizationSumTest,value;
.sort
Skip;
#write <> "
PeheA =  (%$)*PeheA ;",$fac
.sort
*** In order to reduce as much as possible the denominators we go to the Lab frame ***
#call labframe
Multiply [1-cos];
#call labframe
id w*wp*cos = m*wp - m*w  + w*wp ;
*
Bracket m,K,e;

Print +s LinearTest, PolarizationSumTest ;
#write <> "
HERE THE TWO TESTS "

.end
```

A.11.3 (amplitude.h)

```
#procedure squareamplitude(Amp,Mat)
.sort
*
* We skip everything but Amp. In Amp we look for the highest
* i and al indices
*
Skip;
NSkip 'Amp';
#$imax = 0;
#do i = 1,40
if ( match(vb(i'i',?a)) || match(v(i'i',?a))
|| match(ub(i'i',?a)) || match(u(i'i',?a))
|| match(g(i'i',?a)) || match(g(i?,i'i',?a))
|| match(fprop(i'i',?a))
|| match(fprop(i?,i'i',?a)) );
$imax = 'i';
endif;
#enddo
#$almax = 0;
#do al = 1,20
if ( match(g(?a,al'al')) ||
match(Aprop(al'al',?a)) ||
match(Aprop(al?,al'al',?a)) ||
match(epA(al'al',?b))   ||
match(epA(?a,al'al')) );
$almax = 'al';
endif;
#enddo
#$mumax = 0;
#do mu = 1,20
if ( match(g(?a,mu'mu')) ||
match(eph(mu'mu',mu?,?b)) ||
match(eph(mu?,mu'mu',?b))    );
$mumax = 'mu';
endif;
#enddo
.sort
*
*just for a check we print the highest i and al indices
*
#message highest i is i'$imax', highest al is al'$almax', highest mu is mu'$mumax';
*
*Now construct the conjugate
*
Skip;
L 'Amp'C = 'Amp';
id i_ = -i_;
*
* Make a new set of dummy indices above $imax and $almax.
*
Multiply replace_(
<i1,i{'$imax'+1}>,...,<i'$imax',i{2*'$imax'}>);
Multiply replace_(
<al1,al{'$almax'+1}>,...,<al'$almax',al{2*'$almax'}>);
Multiply replace_(
<mu1,mu{'$mumax'+1}>,...,<mu'$mumax',mu{2*'$mumax'}>);
*
* Exchange rows and columns
*
id g(i1?,i2?,al?) = g(i2,i1,al);
```

```
id g(i1?,i2?) = g(i2,i1);
id fprop(i1?,i2?,?a) = fprop(i2,i1,?a);
id Aprop(al1?,al2?,p?) = Aprop(al2,al1,p);
*
* and exchange u and ub, v and vb
*
Multiply replace_(ub,u,u,ub,vb,v,v,vb);
*
* gamma5 gets a minus sign. Hence k6 <--> k7
*
Multiply replace_(k6,k7,k7,k6);
id g(?a,k5) = -g(?a,k5);
.sort
*
* Now multiply Amp and AmpC to get the matrix element squared.
*
Skip;
Drop,'Amp','Amp'C;
L 'Mat' = 'Amp'*'Amp'C;
*
* Spin sums
*
id u(i1?,p?,m?)*ub(i2?,p?,m?) = g(i1,i2,p)+g(i1,i2)*m;
id v(i1?,p?,m?)*vb(i2?,p?,m?) = -g(i1,i2,p)+g(i1,i2)*m;
id epA(al1?,p?)*epA(al2?,p?) = -d_(al1,al2);
id eph(al1?,al2?,p?)*eph(al3?,al4?,p?) = P(al1,al2,al3,al4);
id P(al?,be?,mu?,nu?) = 1/2*( d_(al,mu)*d_(be,nu) +  d_(al,nu)*d_(be,mu) - d_(mu,nu)*d_(al,be) );

*
* Propagators
*
id fprop(i1?,i2?,p?,m?) = (g(i1,i2,p)+g(i1,i2)*m)*prop(p.p-m^2);
id Aprop(al1?,al2?,q?) = d_(al1,al2)*prop(q.q);
*
* String the gamma matrices together in traces.
*
repeat id g(i1?,i2?,?a)*g(i2?,i3?,?b) = g(i1,i3,?a,?b);
.sort
Skip;
NSkip 'Mat';
*
* Now put the traces one by one in
* terms of the built in gammas
*
#do i = 1,10
id,once,g(i1?,i1?,?a) = g_('i',?a);
id g_('i',k7) = g7_('i');
id g_('i',k6) = g6_('i');
id g_('i',k5) = g5_('i');
#enddo
.sort
*
* Finally take the traces
*
#do i = 1,10
Trace4,'i';
#enddo
#endprocedure


#procedure vertices
******************
```

```
*** CONVENTIONS ***
**In the notation Vabc.. the fields are in decresing spin order, i.e. VhAff.
**All the momenta are incoming.
*******************

*** SCALAR QED ***

id VAss(mu?,p2?,p3?) = -i_*e*( p2(mu) - p3(mu) );
*
id VAAss(mu?,nu?) = 2*i_*e^2*d_(mu,nu);

*** QED ***

id VAff(mu?,i1?,i2?) = -i_*e*g(i1,i2,mu);

*** GRAVITON - QED ***

id VhAA(mu?,nu?,al?,be?,p2?,p3?) = i_*K/2*(
p2.p3*2*P(al,be,mu,nu) +
d_(mu,nu)*p2(al)*p3(be) +
d_(al,be)*( p2(mu)*p3(nu) + p3(mu)*p2(nu))-
(d_(al,mu)*p3(nu) + d_(al,nu)*p3(mu))*p2(be)-
(d_(be,mu)*p2(nu) + d_(be,nu)*p2(mu))*p3(al) );
**
id VhAff(mu?,nu?,al?,p3?,p4?,i3?,i4?) = -i_*e*K/4*(
2*d_(mu,nu)*g(i3,i4,al) -
d_(al,mu)*g(i3,i4,nu) -
d_(al,nu)*g(i3,i4,mu)  );
**
id Vhff(mu?,nu?, p1?,p2?,i1?,i2?) = i_*K/8*(
(p2(mu) - p1(mu))*g(i1,i2,nu) +
(p2(nu) - p1(nu))*g(i1,i2,mu) +
2*(g(i1,i2,p1) - g(i1,i2,p2) - 2*m*g(i1,i2) )*d_(mu,nu) );


*** GRAVITON - SQED ***

id Vhss(mu?,nu?,p2?,p3?) = i_*K/2*(
p2(mu)*p3(nu) + p3(mu)*p2(nu) -
d_(mu,nu)*( p2.p3 + m^2) );
*
id VhAss(mu?,nu?,al?,p3?,p4?) = i_*e*K/2*(
d_(al,mu)*(p3(nu)-p4(nu)) +
d_(al,nu)*(p3(mu)-p4(mu)) -
d_(mu,nu)*(p3(al)-p4(al)) );
***
id P(al?,be?,mu?,nu?) = 1/2*( d_(al,mu)*d_(be,nu) +  d_(al,nu)*d_(be,mu) - d_(mu,nu)*d_(al,be) );
***



#endprocedure


#procedure mandelstam(p1,p2,p3,p4,m1,m2,m3,m4)
id 'p1'.'p2'^-2 = (1/2)^-2*(S - 'm1'^2 - 'm2'^2)^-2  ;
id 'p3'.'p4'^-2  = (1/2)^-2*(S - 'm3'^2 - 'm4'^2)^-2  ;
id 'p1'.'p3'^-2  = (-1/2)^-2*(T - 'm1'^2 - 'm3'^2)^-2  ;
id 'p2'.'p4'^-2  = (-1/2)^-2*(T - 'm2'^2 - 'm4'^2)^-2  ;
id 'p1'.'p4'^-2  = (-1/2)^-2*(U - 'm1'^2 - 'm4'^2)^-2 ;
id 'p2'.'p3'^-2 = (-1/2)^-2*(U - 'm2'^2 - 'm3'^2)^-2 ;
*
```

```
id 'p1'.'p2'^-1 = (1/2)^-1*(S - 'm1'^2 - 'm2'^2)^-1 ;
id 'p3'.'p4'^-1  = (1/2)^-1*(S - 'm3'^2 - 'm4'^2)^-1 ;
id 'p1'.'p3'^-1  = (-1/2)^-1*(T - 'm1'^2 - 'm3'^2)^-1  ;
id 'p2'.'p4'^-1  = (-1/2)^-1*(T - 'm2'^2 - 'm4'^2)^-1  ;
id 'p1'.'p4'^-1  = (-1/2)^-1*(U - 'm1'^2 - 'm4'^2)^-1 ;
id 'p2'.'p3'^-1 = (-1/2)^-1*(U - 'm2'^2 - 'm3'^2)^-1 ;
*
id 'p1'.'p2' = 1/2*(S - 'm1'^2 - 'm2'^2) ;
id 'p3'.'p4' = 1/2*(S - 'm3'^2 - 'm4'^2) ;
id 'p1'.'p3' = -1/2*(T - 'm1'^2 - 'm3'^2) ;
id 'p2'.'p4' = -1/2*(T - 'm2'^2 - 'm4'^2) ;
id 'p1'.'p4' = -1/2*(U - 'm1'^2 - 'm4'^2);
id 'p2'.'p3' = -1/2*(U - 'm2'^2 - 'm3'^2) ;
#endprocedure
```

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