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A Relationship Between Spin and Geometry

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Abstract. In physics, spin is often seen exclusively through the lens of its phenomenological character: as an intrinsic form of angular momentum. However, there is mounting evidence that spin fundamentally originates as a quality of geometry, not of dynamics, and recent work further suggests that the structure of non-relativistic Euclidean three-space is sufficient to define it. In this paper, we directly explicate this fundamentally non-relativistic, geometric nature of spin by constructing non-commutative algebras of position operators which subsume the structure of an arbitrary spin system. These "Spin-s Position Algebras" are defined by elementary means and from the properties of Euclidean three-space alone, and constitute a fundamentally new model for quantum mechanical systems with non-zero spin, within which neither position and spin degrees of freedom, nor position degrees of freedom within themselves, commute. This reveals that the observables of a system with spin can be described completely geometrically as tensors of oriented planar elements, and that the presence of non-zero spin in a system naturally generates a non-commutative geometry within it. We will also discuss the potential for the Spin-s Position Algebras to form the foundation for a generalisation to arbitrary spin of the Clifford and Duffin–Kemmer–Petiau algebras.

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1. Introduction

1.1. Spin and Geometry

In non-relativistic physics, spin presents phenomenologically as a form of angular momentum which is intrinsic to some quantum mechanical systems.

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Typically, we model such systems using a tensor product between the position-momentum state space and the "internal" spin state space. The latter is taken to be an irreducible representation of the real Lie algebra $\mathfrak{su}(2,\mathbb{C})$. In this model, a clear line is drawn between the geometry of the space the system occupies, which is captured in the position-momentum degrees of freedom, and the spin degrees of freedom. However, the two are more connected than they might first appear.

To see how, let us first define,

Definition 1.1. ($\mathfrak{so}(3,\mathbb{R})$) The real Lie algebra $\mathfrak{so}(3,\mathbb{R})$,

$$\mathfrak{so}(3,\mathbb{R}) := \mathrm{span}_{\mathbb{R}}(\{S_1, S_2, S_3\}), \tag{1.1}$$

with Lie product,

$$S_a \times S_b = \sum_{c=1}^{3} \varepsilon_{abc} S_c. \tag{1.2}$$

It is well-known that $\mathfrak{so}(3,\mathbb{R})$ generates the Lie group $\mathrm{SO}(3,\mathbb{R})$, which is the group of rotations in Euclidean three-space [15]. These rotations preserve the Euclidean geometry of the space on which they act, directly connecting the structure of $\mathfrak{so}(3,\mathbb{R})$ to this geometry. On the other hand, $\mathfrak{su}(2,\mathbb{C})$ is isomorphic as a Lie algebra to $\mathfrak{so}(3,\mathbb{R})$ [15]. Thus, in principle, it should be possible to relate the spin degrees of freedom of a system, modelled by representations of $\mathfrak{su}(2,\mathbb{C})$ (equiv. $\mathfrak{so}(3,\mathbb{R})$), to the geometry of Euclidean three-space.

A connection between spin and geometry has been investigated in myriad ways by many authors. In the relativistic domain: Savasta et al. [25,26] associate spin with the $\mathrm{SO}(4,\mathbb{R})$ symmetry present between three-velocities and rates of change of proper time; whereas Kaparulin et al. [20] relate spin to geometric qualities of a particle worldline. In non-relativistic physics, Bühler [7] instead connects spin to polarisations of a wave in Euclidean three-space via the Lie group $\mathrm{SL}(2,\mathbb{R})$.

Of principle interest to this paper is the approach taken by Colatto et al. [8], who introduce a connection between spin generators and non-commutative position operator algebras in Euclidean three-space,

Definition 1.2. (*Position Operator Algebras*) Consider the set $\{\hat{x}_a\}$ of position operators in a physical model indexed over a set J. The $\{\hat{x}_a\}$ form a non-commutative position operator algebra iff $\exists a, b \in J$ such that,

$$\left[\hat{x}_a, \hat{x}_b\right] \neq 0. \tag{1.3}$$

Otherwise, the $\{\hat{x}_a\}$ form a commutative position operator algebra.

Non-commutative position operator algebras constitute non-commutative geometries in the sense of [3,13,29], which are employed by many as a way to incorporate gravity into quantum mechanics. In the case of Colatto et al., they consider a non-commutative position operator algebra satisfying, $\forall p, q \in \{1,2,3\}$,

$$\left[\hat{x}_p, \hat{x}_q\right] = -\frac{i\hbar}{m^2c^2} \sum_{r=1}^{3} \varepsilon_{pqr} \hat{S}_r, \tag{1.4}$$

with $\hat{S}_r$ the usual spin generators for a spin-$\frac{1}{2}$ system. Thus, encoding spin into quantum mechanical systems in this way invites the possibility of novel interactions between spin and gravity.

We will generalise this model of Colatto et al. in a number of ways. Firstly, we will construct real non-commutative position operator algebras. Secondly, our algebras will be independent of any notion of "internal" space relied on in most descriptions of spin. Finally, our non-commutative position operator algebras will model the position and spin degrees of freedom of a system with arbitrary spin. With these generalisations, we will establish elementary connections between arbitrary spin and the geometry of Euclidean three-space through non-commutative geometry. Exploring this connection will contribute to ongoing efforts to unify quantum mechanics and gravity, and is paramount to the efficacy of such models in the future.

1.2. Scope of this Paper

This paper is primarily concerned with developing non-commutative position operator algebras for systems of arbitrary spin in Euclidean three-space; this is because real algebraic descriptions for the structure of an arbitrary spin system (by which we mean their algebras of spin generators) are known in this case [6]. Despite this, many of the constructions given here are valid for space-times with non-degenerate signatures. Therefore, to lay the groundwork for future studies, we will develop our ideas using non-degenerate space-times as much as possible, and specialise to Euclidean three-space only when necessary to connect to these algebras of generators.

1.3. Desired Properties for a Non-Commutative Position Operator Algebra

Within this paper's approach to realising spin within a non-commutative geometry, there are a number of properties that the author wishes a non-commutative position operator algebra to have:

1. An action of the connected symmetry group for the space(-time) should be encoded within the algebra, enabling us to algebraically transform its elements;
2. The generators of the connected symmetry group for the space(-time) should exist as elements of the algebra and;
3. The structure of a system with arbitrary spin should be subsumed within the algebra of the generators.

While this may appear to be a challenging list of requirements, the Clifford and Duffin–Kemmer–Petiau [12, 16, 23] algebras famously possess all of these properties. Therefore, to gain a general understanding of the kind of algebras we wish to construct, let us first study the Clifford algebra.

1.4. The Clifford Algebra

1.4.1. Preliminary Definitions. In order to define and describe the Clifford algebra, we require some preliminary definitions. First, let us precisely define our space(-time) and the available geometric information about it,

Definition 1.3. ((V, g)) Minkowski space-time (V, g) is a pair consisting of a real $(p+q)$-dimensional vector space V and a non-degenerate symmetric bilinear map $g : V \times V \to \mathbb{R}$ with signature (p, q).

Definition 1.4. (*Metric*) In this paper, a "metric" is the map g of the Minkowski space-time (V, g), following the use of this term in relativity.

Remark. The metric g of (V, g) encodes the geometric information between elements of V. Replacing the metric g with another metric g' defined over the same vector space V may result in a Minkowski space-time with a different geometry. For the purposes of this paper, we shall interpret V as the real vector space of all position operators in some quantum mechanical system.

Of principal importance to this paper is Euclidean three-space,

Definition 1.5. ((E, δ)) Euclidean three-space (E, δ) is a three-dimensional Minkowski space-time whose metric has signature $(3, 0)$.

Remark. Since the metric of Euclidean three-space is a special case of a Minkowski space-time, we will discuss general Minkowski space-times where possible as specialise to Euclidean three-space only when necessary.

Next, there are some related algebras which will become important in the definition or to the structure of the Clifford algebra.

Definition 1.6. (*Tensor Algebra*) The tensor algebra[5] $T(V)$ of any real vector space V,

$$T(V) \cong \mathbb{R} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \ldots, \tag{1.5}$$

is a unital, associative algebra with product $\otimes$ and multiplicative identity 1, where,

$$V^{\otimes k} := \bigotimes_{j=1}^{k} V. \tag{1.6}$$

Elements of the tensor algebra are called "tensors".

Definition 1.7. (*Exterior Algebra*) The exterior algebra $\Lambda(V)$ of a real vector space V is a quotient algebra of the tensor algebra,

$$\Lambda(V) \cong \frac{T(V)}{I\,(v \otimes w + w \otimes v)}, \tag{1.7}$$

by the two-sided ideal $I\,(v \otimes w + w \otimes v)$ generated by all tensors of the given form, $\forall v, w \in V$. The product of $\Lambda(V)$, which we denote by $\wedge$, is antisymmetric and called the "wedge" product.

Of particular importance to the Clifford algebra are the objects of the Exterior algebra.

Definition 1.8. (*k-blade*) A k-blade is the wedge product of $k \in \mathbb{Z}^+$ vectors $\{v_j \in V\}$,

$$v_1 \wedge v_2 \wedge \ldots \wedge v_k = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \bigotimes_{j=1}^{k} v_{\sigma(j)}, \tag{1.8}$$

where S_k is the set of all permutations of k objects, and $\text{sgn}(\sigma)$ is the sign of the permutation σ. We may extend this definition to $k \in \mathbb{N}$ by defining 0-vectors to be scalars $\mathbb{R} \subset \Lambda(V)$.

Remark. k-blades are often interpreted as hypervolume elements with dimension k [10], and as such have a simple and definite geometrical character.

Definition 1.9. (k*-vector*) For a given $k \in \mathbb{N}$, a k-vector is a real linear combination of k-blades. We may also refer to k-vectors using prefixes, e.g. bivector for 2-vector. We denote the space of all k-vectors as $\Lambda^k(V)$.

1.4.2. Definition and Structure of the Clifford Algebra.

Definition 1.10. (*Clifford Algebra* $\text{Cl}(V, g)$) Given a Minkowski space-time (V, g), its Clifford algebra [9] $\text{Cl}(V, g)$ is the quotient algebra, $\forall v, w \in V$,

$$\text{Cl}(V, g) \cong \frac{T(V)}{I\,(v \otimes w + w \otimes v - 2g(v,w))}, \tag{1.9}$$

by the two-sided ideal $I\,(v \otimes w + w \otimes v - 2g(v,w))$ generated by all tensors of the given form, $\forall v, w \in V$. We will follow the community by leaving the product of $\text{Cl}(V, g)$ implicit.

We define the Clifford algebra in the above way to explicate its defining algebraic identity, $\forall v, w \in V$,

$$vw + wv = 2g(v,w). \tag{1.10}$$

This construction is identical to the usual one which makes use of a quadratic form,

Lemma 1.11. *As V is real, definition 1.10 is equivalent to the construction of* $\text{Cl}(V, g)$ *using a quadratic form.*

Proof. See [9]. □

We may understand the structure of $\text{Cl}(V, g)$ by relating its elements to those of the Exterior algebra $\Lambda(V)$.

Definition 1.12. (k-blade in $\text{Cl}(V, g)$) A k-blade in $\text{Cl}(V, g)$ is defined as in 1.8, with all tensor products $\otimes$ replaced with Clifford products. We abuse notation slightly by also denoting it by $v_1 \wedge v_2 \wedge \ldots \wedge v_k$, $\forall k \in \mathbb{Z}^+$.

Lemma 1.13. *As a vector space,* $\text{Cl}(V, g) \cong \Lambda(V)$*, and is spanned by the k-blades.*

Proof. See [11]. □

Remark. From Lemma 1.13, we see the Clifford algebra is constructed from objects which each have a definite geometric character. Furthermore, the defining algebraic structure of the Clifford algebra (1.10) is controlled entirely by the properties of the metric g. As such, it is an algebra with a strong and natural geometric character.

1.5. The Desired Properties in the Clifford Algebra

With the structure of the Clifford algebra understood, let us see how each of the properties defined in section 1.3 emerge within it.

1.5.1. Generators and the Action of $\mathbf{SO}^+(p,q,\mathbb{R})$ in $\mathbf{Cl}(V,g)$. The algebraic structure of $\mathrm{Cl}(V,g)$ admits a natural action of bivectors on vectors,

Definition 1.14. (u_{cl}[1]) For all $a,b,c \in V$,

$$\mathrm{u}_{cl}(a\wedge b)(c) := (a\wedge b)c - c(a\wedge b). \tag{1.11}$$

This action on vectors also outputs vectors.

Lemma 1.15. *For all* $a,b,c \in V$*,*

$$\mathrm{u}_{cl}(a\wedge b)(c) = -2\big(g(a,c)b - g(b,c)a\big). \tag{1.12}$$

Proof. Direct calculation using (1.10). □

Importantly, $\mathrm{u}_{cl}(a\wedge b)$ admits a natural extension of its action to the whole of $\mathrm{Cl}(V,g)$,

Lemma 1.16. *We may naturally extend* $\mathrm{u}_{cl}(a\wedge b)$ *to the whole of* $\mathrm{Cl}(V,g)$ *as a derivation* [5]*.*

Proof. For all $a,b \in V$, $A,B \in \mathrm{Cl}(V,g)$, we notice that,

$$(a\wedge b)(AB) - (AB)(a\wedge b) = \big((a\wedge b)A - A(a\wedge b)\big)B + A\big((a\wedge b)B - B(a\wedge b)\big).$$

This means the algebraic form of $\mathrm{u}_{cl}(a\wedge b)(c)$ within $\mathrm{Cl}(V,g)$ is naturally a derivation. This entails our extension. □

The action of u_{cl} on $\mathrm{Cl}(V,g)$ imparts the structure of a Lie algebra on the subspace of bivectors $\Lambda^2(V) \subset \mathrm{Cl}(V,g)$ with the commutator as its Lie product,

Lemma 1.17. *For all* $a,b,c,d \in V \subset \mathrm{Cl}(V,g)$*,*

$$(a\wedge b)(c\wedge d) - (c\wedge d)(a\wedge b) = \mathrm{u}_{cl}(a\wedge b)(c\wedge d) \in \Lambda^2(V). \tag{1.13}$$

Proof. Direct computation from the left-hand side utilising Lemma 1.15. The inclusion follows from the derivation property of $\mathrm{u}_{cl}(a \wedge b)$ by Lemma 1.16. □

Remark. Lemma 1.17 reveals the bivectors to be the objects of a Lie algebra within $\mathrm{Cl}(V,g)$, and $\mathrm{u}_{cl}(a\wedge b)$ to be the action of a Lie algebra on $\mathrm{Cl}(V,g)$.

To identify this Lie algebra, we must first define,

Definition 1.18. ($\mathfrak{so}(p,q,\mathbb{R})$) Given a Minkowski space-time (V,g) whose metric has signature (p,q), the Lie algebra $\mathfrak{so}(p,q,\mathbb{R})$ is the Lie algebra of all $X \in \mathrm{End}(V)$ such that $\forall v,w \in V$,

$$g(X(v),w) + g(v,X(w)) = 0, \tag{1.14}$$

with the commutator as the Lie product.

[1] We have chosen the symbol u for this and related actions for no other reason than its relationship to the maps t and μ defined in Definitions 2.16 and 1.19 respectively.

Lemma 1.19. *Given a Minkowski space-time* (V, g) *with metric signature* (p, q)*, and* $k \in \mathbb{R}$*, the map* μ*,* $\forall v, w, x \in V$*,*

$$\begin{aligned} \mu &: \Lambda^2(V) \to \mathfrak{so}(p, q, \mathbb{R}) \\ \mu(v \wedge w) &:= x \mapsto k\big(g(v,x)w - g(w,x)v\big), \end{aligned} \tag{1.15}$$

is a Lie algebra isomorphism if we endow $\Lambda^2(V)$ *with the Lie product* L*,* $\forall a, b, c, d \in V$*,*

$$L(a\wedge b, c\wedge d) := \mu(a\wedge b)(c)\wedge d + c\wedge\mu(a\wedge b)(d).$$

Proof. That μ is a vector space isomorphism is given in [14]. Showing,

$$\mu(L(a\wedge b, c\wedge d)) = [\mu(a\wedge b), \mu(c\wedge d)],$$

may be achieved by direct calculation. □

Corollary 1.20. *The Lie algebra of bivectors in* $\mathrm{Cl}(V, g)$ *is Lie algebra isomorphic to* $\mathfrak{so}(p, q, \mathbb{R})$*,*

Proof. This follows immediately from Lemma 1.19. □

Remark. Corollary 1.20 grants us an immediate geometric interpretation for the objects of $\mathfrak{so}(p, q, \mathbb{R})$: they are linear combinations of oriented planar elements. For $\mathfrak{so}(3, \mathbb{R})$, a 2-blade generates a rotation in $\mathrm{SO}(3, \mathbb{R})$, and encodes both the plane and angle of rotation.

To see the significance of this isomorphism, let us consider the symmetries of the geometry encoded by g.

Definition 1.21. ($\mathrm{O}(p, q, \mathbb{R})$) Given a Minkowski space-time (V, g) whose metric has signature (p, q), the Lie group $\mathrm{O}(p, q, \mathbb{R})$ is the group of all $A \in \mathrm{Aut}(V)$ such that $\forall v, w \in V$,

$$g(A(v), A(w)) = g(v, w), \tag{1.16}$$

with composition as the group product.

Remark. $\mathrm{O}(p, q, \mathbb{R})$ is the full group of linear transformations of the Minkowski space-time (V, g) which preserve the geometry imparted by g on V.

Definition 1.22. ($\mathrm{SO}^+(p, q, \mathbb{R})$) $\mathrm{SO}^+(p, q, \mathbb{R})$ is the maximal connected Lie subgroup of $\mathrm{O}(p, q, \mathbb{R})$.

Remark. $\mathrm{SO}^+(p, q, \mathbb{R})$ is the generalisation of $\mathrm{SO}^+(3, 1, \mathbb{R})$ and $\mathrm{SO}(3, \mathbb{R})$ to arbitrary non-degenerate signatures, preserving both spatial and temporal orientations.

Lemma 1.23. *The Lie algebra of bivectors generates* $\mathrm{SO}^+(p, q, \mathbb{R})$ [15].

Proof. By [14, 15, 30], the Lie group $\mathrm{SO}^+(p, q, \mathbb{R})$ is generated by finite products of elements of the form,

$$\exp(X) = \sum_{j=0}^{\infty} \frac{X^j}{j!}, \tag{1.17}$$

where $X \in \mathfrak{so}(p, q, \mathbb{R})$. Using Corollary 1.20, we are done. □

Lemma 1.24. *With* $\mathrm{u}_{cl}(a\wedge b)$ *extended as a derivation to all of* $\mathrm{Cl}(V,g)$*, there exists a natural action of* $\mathrm{SO}^+(p,q,\mathbb{R})$ *on* $\mathrm{Cl}(V,g)$ *which distributes over tensor products.*

Proof. See [21]. □

Remark. The algebraic form of $\mathrm{u}_{cl}(a\wedge b)$ is a result of the algebraic structure of $\mathrm{Cl}(V,g)$. The Duffin-Kemmer-Petiau algebra on V admits a similar $\mathfrak{so}(p,q,\mathbb{R})$-action, $\forall a,b,c\in V$,

$$\mathrm{u}_{dkp}(a\wedge b)(c) := (a\wedge b)c - c(a\wedge b) = -\frac{1}{2}\big(g(a,c)b - g(b,c)a\big),$$

which again is derived from the structure of its algebra. The properties of these actions, such as being a derivation, are determined by their algebraic forms, which by Lemma 1.24 necessarily affects the character of their induced $\mathrm{SO}^+(p,q,\mathbb{R})$-actions. As such, to make general statements about the relationship between spin, symmetries, and geometry, we must determine if $(a\wedge b)c - c(a\wedge b)$ is the general form for the $\mathfrak{so}(p,q,\mathbb{R})$-action in an arbitrary unital associative algebra, independent of any additional structure in the algebra. This shall be confirmed in Sect. 2.1.4.

1.5.2. Spin-$\frac{1}{2}$ Structure in $\mathrm{Cl}(E,\delta)$. Restricting our attention now to the Clifford algebra $\mathrm{Cl}(E,\delta)$ of Euclidean three-space, we may discover the structure of a spin-$\frac{1}{2}$ system embedded within the algebraic structure of its bivectors. To explicate this, let us introduce some notation.

Definition 1.25. (S'_p) Consider a Euclidean three-space (E,δ) and a basis $\{e_a\}$ which is orthonormal with respect to δ. Then, in its Clifford algebra $\mathrm{Cl}(E,\delta)$, $\forall p\in\{1,2,3\}$,

$$S'_p := -\frac{1}{4}\sum_{a,b=1}^{3}\varepsilon_{abp}\, e_a\wedge e_b. \tag{1.18}$$

Lemma 1.26. *In* $\mathrm{Cl}(E,\delta)$*,* $\forall p,q\in\{1,2,3\}$*,*

$$S'_pS'_q - S'_qS'_p = \sum_{r=1}^{3}\varepsilon_{pqr}S'_r. \tag{1.19}$$

Proof. Consider equation (1.13) on basis vectors. Then transform both bivectors on the left-hand side using (1.18) and simplify the right-hand side. □

Having recovered the canonical form of $\mathfrak{so}(3,\mathbb{R})$ from the bivectors of $\mathrm{Cl}(E,\delta)$, we quickly discover that $\mathbb{R}\oplus\Lambda^2(E)\subset\mathrm{Cl}(E,\delta)$ is an associative subalgebra.

Lemma 1.27. *In* $\mathrm{Cl}(E,\delta)$*,* $\forall p,q\in\{1,2,3\}$*,*

$$\frac{1}{2}(S'_pS'_q + S'_qS'_p) + \frac{1}{4}\delta_{pq} = 0. \tag{1.20}$$

Proof. See Appendix A. □

Corollary 1.28. *In* $\mathrm{Cl}(E,\delta)$, $\forall p,q \in \{1,2,3\}$,

$$S'_p S'_q = -\frac{1}{4}\delta_{pq} + \frac{1}{2}\sum_{r=1}^{3}\varepsilon_{pqr}S'_r. \tag{1.21}$$

Proof. Follows directly from Lemmas 1.26 and 1.27. □

And so we find,

Theorem 1.29. *The subalgebra* $\mathbb{R}\oplus\Lambda^2(E)\subset \mathrm{Cl}(E,\delta)$ *under the Clifford product is isomorphic as an algebra to the real Pauli algebra,*

$$1\mapsto I_2,\quad S'_x\mapsto \frac{1}{2}i\sigma_x,\quad S'_y\mapsto -\frac{1}{2}i\sigma_y,\quad S'_z\mapsto \frac{1}{2}i\sigma_z,$$

where I_2 *is the* 2×2 *identity matrix and* $\{\sigma_j\}$ *are the* 2×2 *Pauli matrices.*

Proof. This can be verified directly using Corollary 1.28. □

Remark. Theorem 1.29 shows that $\mathrm{Cl}(E,\delta)$ contains the structure of a spin-$\frac{1}{2}$ system algebraically within $\mathbb{R}\oplus\Lambda^2(E)$. This is widely known, for example in [1,2,10].

1.5.3. The Origin of these Properties in $\mathrm{Cl}(E,\delta)$. It is clear from the outset that, in the case of the Clifford algebra, these properties are connected to its defining identity (1.10). However, if we are to generalise these properties to arbitrary spin systems, we must understand the extent of this connection.

Lemma 1.30. *There is a hierarchy of identities within* $\mathrm{Cl}(V,g)$, $\forall a,b,c,d\in V$,

$$\begin{aligned} ab+ba &= 2g(a,b) \\ &\Downarrow \\ (a\wedge b)c - c(a\wedge b) &= -2\big(g(a,c)b - g(b,c)a\big) \\ &\Downarrow \\ (a\wedge b)(c\wedge d)-(c\wedge d)(a\wedge b) &= \mathrm{u}_{cl}(a\wedge b)(c\wedge d), \end{aligned} \tag{1.22}$$

where the implication $A\Rightarrow B$ *indicates that identity* B *may be derived within* $\mathrm{Cl}(V,g)$ *assuming identity* A *alone.*

Proof. The second implication follows from the proof of Lemma 1.17, and the first from the proof of Lemma 1.15. □

Theorem 1.31. *The spin-*$\frac{1}{2}$ *structure of* $\mathrm{Cl}(E,\delta)$ *is a direct result of its defining relation,* $\forall a,b\in E$,

$$ab+ba = 2\delta(a,b). \tag{1.23}$$

Proof. The proof of Lemma 1.27 in Appendix A derives Equation (1.20) directly from (1.23) on basis vectors, utilising elements of $\mathrm{Cl}(E,\delta)$ whose algebraic properties are also the result of (1.23). Since (1.19) is also a consequence of (1.23) by Lemma 1.30 we see Corollary 1.28 is as well. By Theorem 1.29, we are done. □

1.6. Limitations of Clifford Algebra-Based Approach

As an algebra of position operators satisfying the requirements of Sect. 1.3, the Clifford algebra is fundamentally limited to describing a spin-$\frac{1}{2}$ system by Theorem 1.31. As such, we cannot hope to retain its algebraic structure and describe a system with arbitrary spin. A standard approach to overcome this problem is to construct a tensor power algebra $\mathrm{Cl}(E,\delta)^{\otimes k}$; this algebra contains a Clifford substructure yet contains subalgebras with the structure of arbitrary spin systems [28] for large enough k. Indeed, subspaces of tensor products of this algebra also underpin the definition of many classic higher spin models [4,18,19].

However, considered as an algebra of position operators, this method also does not meet our requirements. To see why, first note that we are ultimately looking for algebras which may be compatible with quantum mechanical theories of systems with arbitrary spin. In principle, such theories should be able to accommodate Hamiltonians with arbitrary position-dependent potentials, including, for example, all polynomials of position operators. We can see from this consideration that the space of all position-dependent potentials is infinite-dimensional. However, regardless of the size of the tensor power, $\mathrm{Cl}(E,\delta)^{\otimes k}$ is always finite-dimensional; this severely limits the position-dependent potentials which may be algebraically realised within the Hamiltonians of the model.

As such, we cannot use the Clifford algebra as a base for the algebras of position operators that we seek. This immediately raises a number of challenges. Firstly, without the structure of the Clifford algebra, we cannot guarantee that our algebras will naturally entail an action of $\mathfrak{so}(p,q,\mathbb{R})$ within their structure. Secondly, we have no guarantee that the algebraic form of this action in our algebras will follow the commutator form of Definition 1.14; this forces us to consider whether the generators of spin are bivectors for arbitrary spin systems. Finally, it is unclear at this stage if algebras containing the structure of arbitrary spin systems can be constructed in a consistent and compatible way with the $\mathfrak{so}(p,q,\mathbb{R})$-action we are seeking to implement. To address these challenges in a way independent of the Clifford algebra, we must adopt a more synthetic approach.

1.7. Outline

The overall aim of this paper is to demonstrate that the geometry of Euclidean three-space (E,δ) is all that is required to construct algebras of position operators which naturally model and incorporate the structure of arbitrary spin systems. Furthermore, we will show that spin is a necessary consequence of geometry and that its presence naturally generates noncommutative geometries.

To do this, in Sect. 2.1 we will first derive the Lie algebra action for a general Minkowski space-time (V,g) through elementary arguments using Householder reflections. Once we have established this, we shall prove that the $\mathfrak{so}(p,q,\mathbb{R})$-action enjoys a unique algebraic form within any unital associative algebra. This will ensure that the algebraic form of this action is completely general and that our knowledge of the Clifford algebra form does not bias our construction.

Following this, in Sect. 2.2 we shall define the "Indefinite-Spin Position Algebra" $\mathrm{P}_\kappa(V, g)$ which incorporates this action, and show that this established both an $\mathrm{SO}^+(p, q, \mathbb{R})$-action, and the geometric character of it generators as oriented planar elements in general. Then, in Sect. 2.3, we shall outline the structure of arbitrary spin systems in terms of the spin algebras $A^{(s)}$, and, in Sect. 2.4 prove that the structure of $\mathrm{P}_\kappa(V, g)$ is compatible with an arbitrary spin structure.

We will complete our construction in Sect. 3, where we will derive from $\mathrm{P}_\kappa(V, g)$ the "Spin-s Position Algebra" $\mathrm{P}_\kappa^{(s)}(E, \delta)$. In so doing, we will reveal that the spin of a system naturally generates a non-commutative geometry. Finally, in Sect. 4, we will contrast the $\mathrm{P}_\kappa^{(s)}(E, \delta)$ with other higher spin models, and consider the implications of these algebras for quantum mechanics.

2. Method

2.1. The General Algebraic $\mathfrak{so}(p, q, \mathbb{R})$-action

In order to have a natural $\mathrm{SO}^+(p, q, \mathbb{R})$ action in our position operator algebras, we must first establish the general form of an $\mathfrak{so}(p, q, \mathbb{R})$-action within it. So far, we have only seen the $\mathfrak{so}(p, q, \mathbb{R})$-action in the context of the Clifford algebra, wherein it has a particular form and algebraic implementation. A priori, it is unclear which aspects of this are particular to the Clifford algebra. Therefore, to ensure the generality of the non-commutative position operator algebra we are constructing, it is prudent to derive both the form and algebraic implementation of the $\mathfrak{so}(p, q, \mathbb{R})$-action by elementary means and using only the properties of g to guide us.

2.1.1. *g*-adjoints. To begin our investigation, let us consider how the metric g structures the endomorphism algebra $\mathrm{End}(V)$, of which $\mathrm{O}(p, q, \mathbb{R})$ is a part. To this end we define,

Definition 2.1. (g-adjoint) A g-adjoint of an endomorphism $A \in \mathrm{End}(V)$ as an endomorphism $B \in \mathrm{End}(V)$ such that $\forall v, w \in V$,

$$g(A(v),w) = g(v,B(w)). \tag{2.1}$$

Lemma 2.2. *Suppose $B \in \mathrm{End}(V)$ is a g-adjoint of $A \in \mathrm{End}(V)$. Then B is unique, and A is the g-adjoint of B.*

Proof. Suppose $C \in \mathrm{End}(V)$ is also a g-adjoint of A. Then, $\forall v, w \in V$,

$$g(A(v),w) = g(v,B(w)) = g(v,C(w)),$$

thus,

$$g(v,(B - C)(w)) = 0.$$

By the non-degeneracy of g, we have, $\forall w \in V$,

$$(B - C)(w) = 0.$$

Therefore, $B = C$. Furthermore, let $D \in \mathrm{End}(V)$ be the g-adjoint of B. Then, $\forall v, w \in V$,

$$g(v,D(w)) = g(B(v),w) = g(w,B(v)) = g(A(w),v) = g(v,A(w)),$$

and so from the previous argument $D = A$. □

Definition 2.3. (A^g) Let $A^g \in \mathrm{End}(V)$ denote the unique g-adjoint of $A \in \mathrm{End}(V)$.

Remark. The notation A^g is intended to mirror the notations for other important involutions, such as the transpose M^T and Hermitian adjoint $M^\dagger$.

Lemma 2.4. *The g-adjoint of the composition $A \circ B$ is $B^g \circ A^g$.*

Proof. For all $v, w \in V$,

$$g(A \circ B(v),w) = g(B(v),A^g(w)) = g(v,B^g \circ A^g(w)),$$

which is unique by Lemma 2.2. □

To summarise the developments above, the metric g assigns to each endomorphism A a unique partner A^g whose action on V is in some sense compatible with that of A when considered through the structure of g. The g-adjoint is entirely dependent on the structure of g, and provides us with another tool to probe it.

The g-adjoints for some endomorphisms are particularly simple,

Definition 2.5. An endomorphism $A \in \mathrm{End}(V)$ is self-g-adjoint when $A^g = A$, and anti-self-g-adjoint when $A^g = -A$.

To capture this more precisely, let us define,

Definition 2.6. For all $A \in \mathrm{End}(V)$,

$$a_\pm(A) := \frac{1}{2}\big(A \pm A^g\big). \tag{2.2}$$

Lemma 2.7. *For all $A \in \mathrm{End}(V)$, $a_+(A)$ is self-g-adjoint, and $a_-(A)$ is anti-self-g-adjoint.*

Proof. Direct computation. □

Corollary 2.8. *Each $A \in \mathrm{End}(V)$ may be decomposed into self-g-adjoint and anti-self-g-adjoint parts,*

$$A = a_+(A) + a_-(A). \tag{2.3}$$

Proof. This follows immediately from Definition 2.6 and Lemma 2.7. □

The ability to isolate self- and anti-self-g-adjoint parts of an endomorphism will facilitate a rapid identification of the $\mathfrak{so}(p, q, \mathbb{R})$-action on vectors in section 2.1.3.

2.1.2. Householder Reflections and $\mathrm{O}(p,q,\mathbb{R})$. Now, let us limit our efforts to understanding the structure of $\mathrm{O}(p,q,\mathbb{R})$. To begin, let us define a Householder reflection,

Definition 2.9. (*Householder Reflection*$R(a)$) Given a Minkowski space-time (V,g), we define the Householder reflection $R(a)$ for non-null $a \in V$,

$$R(a) := v \mapsto v - 2\frac{g(a,v)}{g(a,a)}a.$$

The Householder reflections, which will henceforth be referred to simply as "reflections", are important to the study of $\mathrm{O}(p,q,\mathbb{R})$ for a number of reasons. First,

Lemma 2.10. *For all non-null $a \in V$, $R(a) \in \mathrm{O}(p,q,\mathbb{R})$.*

Proof. We may direct compute that, $\forall v, w \in V$,

$$g\big(R(a)(v), R(a)(w)\big) = g(v,w). \tag{2.4}$$

□

Next,

Lemma 2.11. *Given a finite set of non-null vectors $\{a_j \in V\}$ indexed over the set $\{1,\dots,N\}$, their composition is an orthogonal transformation,*

$$R(a_1)\circ\ldots\circ R(a_N) \in \mathrm{O}(p,q,\mathbb{R}). \tag{2.5}$$

Proof. Direct computation. □

Finally, we may establish the famous Cartan-Dieudonné Theorem,

Theorem 2.12. (Cartan-Dieudonné) *All transformations from $\mathrm{O}(p,q,\mathbb{R})$ are compositions of at most $(p+q)$ reflections.*

Proof. See [22,24]. □

Thus, studying the reflections grants us access the entirety of $\mathrm{O}(p,q,\mathbb{R})$, which includes $\mathrm{SO}^+(p,q,\mathbb{R})$. As such, we should be able to identify $\mathfrak{so}(p,q,\mathbb{R})$ using them.

2.1.3. The $\mathfrak{so}(p,q,\mathbb{R})$-Action from the Algebraic Structure of Reflections. Now, let us combine the results of the previous two sections by studying the algebraic structure of the reflections.

Lemma 2.13. *All reflections $R(a)$ are self-g-adjoint.*

Proof. For all $v, w \in V$,

$$\begin{aligned} g\big(R(a)(v), w\big) &= g\Big(v - 2\frac{g(a,v)}{g(a,a)}a, w\Big) \\ &= g\Big(v, w - 2\frac{g(a,v)}{g(a,a)}a\Big) \\ &= g\big(v, R(a)(w)\big). \end{aligned}$$

□

With this in mind, let us consider the properties of a composition of two reflections $R(a)\circ R(b)$.

Lemma 2.14. *For all $v\in V$,*

$$a_-\big(R(a)\circ R(b)\big)(v) = -\frac{2g(a,b)}{g(a,a)g(b,b)}\big(g(a,v)b - g(b,v)a\big) \tag{2.6a}$$

$$a_+\big(R(a)\circ R(b)\big)(v) = v + \frac{2g(a,v)}{g(a,a)g(b,b)}\big(g(a,b)b - g(b,b)a\big) \tag{2.6b}$$

$$-\frac{2g(b,v)}{g(a,a)g(b,b)}\big(g(a,a)b - g(b,a)a\big). \tag{2.6c}$$

Proof. Direct computation. □

This immediately reveals that,

Corollary 2.15. *The set of all reflections is not closed under composition.*

Proof. We note that for two non-null vectors a and b,

$$a_-\big(R(a)\circ R(b)\big)\neq 0,$$

when $g(a,b)\neq 0$, and thus $R(a)\circ R(b)$ cannot be a reflection. □

Corollary 2.15 is well-known, but our method of proving it has revealed the internal structure of a product of reflections. In particular, Lemma 2.14 reveals the $\mathfrak{so}(p,q,\mathbb{R})$-action is present in both expressions. Let us capture this through a bilinear map,

Definition 2.16. For all $a,b\in V$,

$$\begin{aligned} t &: V\times V \to \mathrm{End}(V)\\ t(a,b) &:= v \mapsto g(a,v)b - g(b,v)a. \end{aligned} \tag{2.7}$$

Remark. We are not ready to write $t(a,b)$ as the action of a bivector $\mathrm{u}(a\wedge b)$; first, we would like to be certain that the bivector $a\wedge b$ itself enters the algebraic form of $t(a,b)$.

Corollary 2.17. *The product of any two reflections is,*

$$R(a)\circ R(b) = \mathrm{id} - \frac{2g(a,b)}{g(a,a)g(b,b)}t(a,b) + \frac{2}{g(a,a)g(b,b)}t(a,b)\circ t(a,b). \tag{2.8}$$

Proof. This is the sum of the results in Lemma 2.14 with a notational change to $t(a,b)$. □

Remark. Corollary 2.17 shows that the product in the algebra of reflections is controlled by the $\mathfrak{so}(p,q,\mathbb{R})$-action, hinting at the relationship between compositions of reflections and the elements of $\mathrm{SO}^+(p,q,\mathbb{R})$.

2.1.4. The Algebraic Form of the $\mathfrak{so}(p,q,\mathbb{R})$-Action on Vectors. Having motivated the $\mathfrak{so}(p,q,\mathbb{R})$-action on V in the form of the map t from reflections, we must consider how to implement it as an algebraic identity within an associative algebra. To achieve this aim we seek a third-order tensor $f(a,b,c)$

Definition 2.18. For all $a,b,c \in V$, and some $\alpha,\beta,\gamma,\delta,\epsilon,\zeta \in \mathbb{R}$,

$$\begin{aligned} f : V \times V \times V &\to T(V) \\ f(a,b,c) =& \alpha(a \otimes b \otimes c) + \beta(a \otimes c \otimes b) + \gamma(b \otimes c \otimes a) \\ &+ \delta(b \otimes a \otimes c) + \epsilon(c \otimes a \otimes b) + \zeta(c \otimes b \otimes a), \end{aligned} \tag{2.9}$$

whose properties match those of $t(a,b)(c)$; this will enable us to construct a quotient of $T(V)$ by the two-sided ideal $\forall a,b,c \in V$,

$$I\left(f(a,b,c) - t(a,b)(c)\right) \tag{2.10}$$

yielding the most general non-trivial algebra possible which contains the action of $\mathfrak{so}(p,q,\mathbb{R})$ within its structure.

Theorem 2.19. *For arbitrary Minkowski space-times (V,g), the only $f(a,b,c)$ which shares all properties of $t(a,b)(c)$ is,*

$$f(a,b,c) = k\big((a \otimes b - b \otimes a) \otimes c - c \otimes (a \otimes b - b \otimes a)\big), \tag{2.11}$$

where $k \in \mathbb{R}$.

Proof. See Appendix B. □

Remark. Theorem 2.19 shows that $f(a,b,c)$ is defined only up to an arbitrary scaling. This is expected and unproblematic. Understanding the equivalence of quotient algebras defined using different values of k is beyond the scope of this paper.

2.2. The Indefinite-Spin Position Algebra

Using Theorem 2.19 we may now define the most general algebra of position operators which contains a natural $\mathfrak{so}(p,q,\mathbb{R})$-action on vectors, or more precisely a family of such algebras,

Definition 2.20. Given a Minkowski space-time (V,g), and choosing a value of $k = \frac{1}{2}$ in (2.11), we define the "Indefinite-Spin Position Algebra" $\mathrm{P}_\kappa(V,g)$ as the quotient, $\forall a,b,c \in V$,

$$\mathrm{P}_\kappa(V,g) \cong \frac{T(V)}{I\left((a \wedge b) \otimes c - c \otimes (a \wedge b) - \kappa(g(a,c)b - g(b,c)a)\right)}, \tag{2.12}$$

by the two-sided ideal (2.10), for some $\kappa \in \mathbb{R}$. In this paper, we shall abuse notation slightly and denote the product of $\mathrm{P}_\kappa(V,g)$ by $\otimes$, as this is consistent with the developments of [6].

Remark. We include the κ in this definition to ensure these algebras are eventually consistent with the Clifford and Duffin-Kemmer-Petiau algebras defined on (V,g), for which $\kappa = -2$ and $\kappa = -\frac{1}{2}$ respectively. More precisely, for a family of Clifford algebras defined by, $\forall a,b,c \in V$,

$$ab + ba = 2\kappa_{cl}\, g(a,b),$$

and a family of Duffin-Kemmer-Petiau algebras defined by, $\forall a, b, c \in V$,

$$abc + cba = \kappa_{dkp}(g(a,b)c + g(c,b)a),$$

for some $\kappa_{cl}, \kappa_{dkp} \in \mathbb{R}$, then their $\mathfrak{so}(p, q, \mathbb{R})$-actions are consistent with the $\mathrm{P}_\kappa(V, g)$ when $\kappa = -2\kappa_{cl}$ and $\kappa = -\frac{1}{2}\kappa_{dkp}$ respectively.

Remark. The term "indefinite-spin" here foreshadows Theorem 2.44, where we will show that $\mathrm{P}_\kappa(E, \delta)$ does not already contain the structure of any particular spin. As such, in Definition 3.1 we will quotient it further to define a new family of algebras each subsuming a particular spin structure.

We must now ensure that our algebraic implementation of the $\mathfrak{so}(p, q, \mathbb{R})$-action in $\mathrm{P}_\kappa(V, g)$ extends to an $\mathrm{SO}^+(p, q, \mathbb{R})$-action. To facilitate this discussion, let us capture the $\mathfrak{so}(p, q, \mathbb{R})$-action on vectors in $\mathrm{P}_\kappa(V, g)$ through the multilinear map,

Definition 2.21. (u)

$$\mathrm{u}(a\wedge b) := c \mapsto (a\wedge b)\otimes c - c\otimes(a\wedge b) = \kappa t(a,b)(c). \tag{2.13}$$

Now, let us consider its properties,

Lemma 2.22. *For all $a, b \in V$, $\mathrm{u}(a\wedge b)$ may be naturally extended to the whole of $\mathrm{P}_\kappa(V, g)$ as a derivation. Furthermore,*

$$(a\wedge b)\otimes(c\wedge d) - (c\wedge d)\otimes(a\wedge b) = \mathrm{u}(a\wedge b)(c\wedge d). \tag{2.14}$$

Proof. Noting that $\mathrm{u}(a\wedge b)$ and $\mathrm{u}_{cl}(a\wedge b)$ (recall Definition 1.14) have the same implementation within their respective algebras (up to scaling), the derivation property and (2.14) follow the same proofs as for Lemmas 1.16 and 1.17 respectively. □

Remark. Since $\mathrm{u}(a\wedge b)$ is the unique tensorial implementation of $t(a,b)$ in $\mathrm{P}_\kappa(V, g)$ (up to scaling), we may consider its properties to be the natural extension of $t(a,b)$ to $\mathrm{P}_\kappa(V, g)$.

Lemma 2.23. *With $\mathrm{u}(a\wedge b)$ extended as a derivation to all of $\mathrm{P}_\kappa(V, g)$, there exists a natural action of $\mathrm{SO}^+(p, q, \mathbb{R})$ on $\mathrm{P}_\kappa(V, g)$ which distributes over tensor products.*

Proof. See [21]. □

With Lemma 2.23, we have succeeded in constructing a general non-commutative algebra of position operators which encodes an action of $\mathrm{SO}^+(p, q, \mathbb{R})$ and contains its generators as algebraic elements, in this case the bivectors $\Lambda^2(V)$. This construction holds for an arbitrary Minkowski space-time.

2.3. The Algebraic Structure of Arbitrary Spin Systems

To achieve our final aim from Sect. 1.3, to construct non-commutative position operators algebras which subsume the structure of an arbitrary spin system, we will specialise our arguments to Euclidean three-space (E, δ). Doing so will enable us to complete this task by utilising the real arbitrary spin algebras developed in [6]. For clarity, we will discuss the construction and composition of these arbitrary spin algebras before proceeding further.

2.3.1. The Universal Enveloping Algebra $U(\mathfrak{so}(3,\mathbb{R}))$ of $\mathfrak{so}(3,\mathbb{R})$. By definition, all spin systems implement the Lie product of $\mathfrak{so}(3,\mathbb{R})$ within their algebra of spin operators, more specifically within the commutator between two spin generators. The most general algebra which implements such a structure is the universal enveloping algebra $U(\mathfrak{so}(3,\mathbb{R}))$ of $\mathfrak{so}(3,\mathbb{R})$,

Definition 2.24. ($U(\mathfrak{so}(3,\mathbb{R}))$) Given the Lie algebra $\mathfrak{so}(3,\mathbb{R})$, its universal enveloping algebra[17] $U(\mathfrak{so}(3,\mathbb{R}))$ is the quotient algebra,

$$U(\mathfrak{so}(3,\mathbb{R})) \cong \frac{T(\mathfrak{so}(3,\mathbb{R}))}{I\left(S_a \otimes S_b - S_b \otimes S_a - S_a \times S_b\right)}, \tag{2.15}$$

by the two-sided ideal $I\left(S_a \otimes S_b - S_b \otimes S_a - S_a \times S_b\right)$ generated by all tensors of the given form, $\forall S_a, S_b \in \mathfrak{so}(3,\mathbb{R})$. We will abuse notation for a third time by also denoting its product by $\otimes$. This is to ensure consistency with [6].

We may probe the structure of $U(\mathfrak{so}(3,\mathbb{R}))$ using an action of generators which captures the structure of the Lie product, namely the adjoint action [14] ad of $U(\mathfrak{so}(3,\mathbb{R}))$ on itself,

Definition 2.25. (*Adjoint action*)

$$\begin{aligned} \mathrm{ad} : U(\mathfrak{so}(3,\mathbb{R})) &\to \mathrm{End}(U(\mathfrak{so}(3,\mathbb{R}))) \\ \mathrm{ad}(u) \coloneqq v &\mapsto \begin{cases} uv & u \in \mathbb{R} \\ u \otimes v - v \otimes u & u \in \mathfrak{so}(3,\mathbb{R}) \\ \mathrm{ad}(a) \circ \mathrm{ad}(b)(v) & u = a \otimes b. \end{cases} \end{aligned} \tag{2.16}$$

In particular, we may decompose $U(\mathfrak{so}(3,\mathbb{R}))$ using the adjoint action of the Casimir element $S^2 \in U(\mathfrak{so}(3,\mathbb{R}))$,

Definition 2.26. (S^2)

$$S^2 \coloneqq \sum_{a=1}^{3} S_a \otimes S_a. \tag{2.17}$$

Doing so naturally identifies the "multipoles",

Definition 2.27. (*Multipoles*) For all $k \in \mathbb{N}$, the multipole tensors[6] are defined recursively, $\alpha \in \mathbb{R}$, $v \in \mathfrak{so}(3,\mathbb{R})$, $B_k \in \mathfrak{so}(3,\mathbb{R})^{\otimes k}$,

$$\begin{aligned} M^{(k)} : \mathfrak{so}(3,\mathbb{R})^{\otimes k} &\to U(\mathfrak{so}(3,\mathbb{R})) \\ M^{(0)}(\alpha) &= \alpha \\ M^{(k+1)}(v \otimes B_k) &= \frac{\mathrm{ad}(S^2 + k(k-1)) \circ \mathrm{ad}(S^2 + k(k+1))}{4(k+1)(2k+1)} \big(v \otimes M^{(k)}(B_k)\big). \end{aligned} \tag{2.18}$$

The images of the first few multipoles can be seen Table 1,

The multipoles $\{M^{(k)}\}$ enjoy a simple relationship with the action of ad, and therefore with the algebraic structure of $U(\mathfrak{so}(3,\mathbb{R}))$,

Lemma 2.28. *For all $k \in \mathbb{N}$, $\forall v \in \mathfrak{so}(3,\mathbb{R})$,*

$$\mathrm{Im}(\mathrm{ad}(v) \circ M^{(k)}) \subseteq \mathrm{Im}(M^{(k)}) \tag{2.19a}$$

$$\mathrm{ad}(S^2) \circ M^{(k)} = -k(k+1) M^{(k)}. \tag{2.19b}$$

Table 1. Images of the multipoles $k = 0, \ldots, 3$ on k-adic tensors. Adapted from [6]

Multipole	Explicit form
$M^{(0)}(1)$	1
$M^{(1)}(S_a)$	S_a
$M^{(2)}(S_a \otimes S_b)$	$\sum_{\sigma \in \mathrm{Perm}(\{a,b\})} \frac{1}{2} S_{\sigma(a)} \otimes S_{\sigma(b)} - \frac{1}{6} \delta_{\sigma(a)\sigma(b)} S^2$
$M^{(3)}(S_a \otimes S_b \otimes S_c)$	$\sum_{\sigma \in \mathrm{Perm}(\{a,b,c\})} \frac{1}{6} S_{\sigma(a)} \otimes S_{\sigma(b)} \otimes S_{\sigma(c)} - \frac{1}{30} \delta_{\sigma(a)\sigma(b)} (3S^2 + 1) \otimes S_{\sigma(c)}$

Proof. See [6]. □

Furthermore, the multipoles are multiplicatively closed in the following sense,

Lemma 2.29. *For any $p, q \in \mathbb{N}$, consider $A \in \mathrm{Im}(M^{(p)})$ and $B \in \mathrm{Im}(M^{(q)})$. Then, the product $A \otimes B$ can be written as an $\mathbb{R}[S^2]$-linear combination of elements $C \in \bigcup_{j=|p-q|}^{p+q} \mathrm{Im}(M^{(j)})$.*

Proof. See [6]. □

Together, these properties reveal the significance of the multipoles $\{M^{(k)}\}$ to $U(\mathfrak{so}(3, \mathbb{R}))$: they form a natural "basis" for it, in the sense that,

Lemma 2.30. *All elements of $U(\mathfrak{so}(3, \mathbb{R}))$ can be written as an $\mathbb{R}[S^2]$-linear combination of objects from $\mathrm{Im}(\{M^{(k)}\})$, where $\mathbb{R}[S^2]$ is the ring of real polynomials of S^2.*

Proof. See [6]. □

2.3.2. The Spin Algebras $A^{(s)}$. In light of Lemma 2.30, we can see that the multipoles $\{M^{(k)}\}$ are closely related to the irreducible representations of $\mathfrak{so}(3, \mathbb{R})$, which all derive their structure from $U(\mathfrak{so}(3, \mathbb{R}))$. We may make this connection explicit by quotienting $U(\mathfrak{so}(3, \mathbb{R}))$ by a given multipole to define the "spin algebras",

Definition 2.31. ($A^{(s)}$) The spin algebra for spin s is the quotient algebra,

$$A^{(s)} := \frac{U(\mathfrak{so}(3, \mathbb{R}))}{I\left(\mathrm{Im}(M^{(2s+1)})\right)}. \tag{2.20}$$

This quotient by a given multipole entails that,

Lemma 2.32. *For all $k \in \mathbb{Z}^+$ such that $k \geq 2s + 1$, $M^{(k)} = 0$ in $A^{(s)}$.*

Proof. The case $k = 2s + 1$ follows from the definition of $A^{(s)}$, and the case $k > 2s + 1$ follows from the recursive relationship between multipoles of Definition 2.27. □

Furthermore,

Lemma 2.33. *In* $A^{(s)}$, $\forall s \in \{0, \frac{1}{2}, 1, \frac{3}{2} \dots\}$,

$$S^2 = -s(s+1). \tag{2.21}$$

Proof. See [6]. □

Corollary 2.34. *For all* $s \in \{0, \frac{1}{2}, 1, \frac{3}{2} \dots\}$, $A^{(s)}$ *is a finite-dimensional real algebra spanned by* $\{\mathrm{Im}(M^{(k)}) \mid 0 \le k \le 2s\}$ *with dimension* $\dim(A^{(s)}) = (2s+1)^2$.

Proof. This follows from Lemmas 2.30, 2.32, and 2.33. The proof of the dimension is given in [6]. □

Remark. The algebraic structure of the spin algebras is exactly that of the unital associative algebra of generators for a spin-s system, and as such all irreducible representations of $\mathfrak{so}(3,\mathbb{R})$ derive their structure from them. In a full quantum mechanical theory, the multipoles $M^{(k)}$ are the physically distinct observables for an arbitrary spin system. See [6] for further discussion.

2.4. Suitability of $\mathrm{P}_\kappa(E,\delta)$ to Subsume Arbitrary Spin Algebras

In light of the previous section, if we hope to use $\mathrm{P}_\kappa(E,\delta)$ to define new algebras containing arbitrary spin structure, then we must ensure that it contains no existing spin structure. More precisely, we must ensure that it contains a subalgebra isomorphic as unital associative algebras to the whole of $U(\mathfrak{so}(3,\mathbb{R}))$.

2.4.1. Bivectors and Spin Generators. To begin, let us abuse notation slightly and define,

Definition 2.35. (S_p) Given an orthonormal basis $\{e_a\}$ of (E,δ), $a \in \{1,2,3\}$, we define, $\forall p \in \{1,2,3\}$,

$$S_p := \frac{1}{2\kappa}\sum_{a,b=1}^{3} \varepsilon_{abp}\, e_a \wedge e_b, \tag{2.22}$$

which has inverse transformation,

$$e_a \wedge e_b = \kappa \sum_{p=1}^{3} \varepsilon_{abp} S_p. \tag{2.23}$$

Lemma 2.36. *In* $\mathrm{P}_\kappa(E,\delta)$, $\forall p,q \in \{1,2,3\}$

$$S_p \otimes S_q - S_q \otimes S_p = \sum_{r=1}^{3} \varepsilon_{pqr} S_r. \tag{2.24}$$

Proof. Apply (2.22) to (2.14) evaluated on basis vectors $\{e_a\}$. □

Lemma 2.37. *In* $\mathrm{P}_\kappa(E,\delta)$,

$$S^2 = \frac{1}{2\kappa^2}\sum_{a,b=1}^{3} (e_a \wedge e_b) \otimes (e_a \wedge e_b), \tag{2.25}$$

and commutes with all bivectors.

Proof. The form of S^2 can be seen by applying (2.22) to Definition 2.26. That it commutes with all bivectors can be directly computed. □

2.4.2. $\mathrm{P}_\kappa(E,\delta)$ and $U(\mathfrak{so}(3,\mathbb{R}))$. Now that we have established an explicit Lie algebra isomorphism between the bivectors and spin generators, we must determine if the algebra of bivectors contained in $\mathrm{P}_\kappa(E,\delta)$ is isomorphic to $U(\mathfrak{so}(3,\mathbb{R}))$. This is important, as our algebraic implementation of the $\mathfrak{so}(p,q,\mathbb{R})$-action may have unintentionally restricted this algebra à la Theorem 1.31.

First, let us establish that the Casimir element S^2 is not a scalar, as this would certainly indicate that the structure of $U(\mathfrak{so}(3,\mathbb{R}))$ has been disturbed in some way.

Lemma 2.38. *In $\mathrm{P}_\kappa(E,\delta)$, S^2 is not equal to a scalar.*

Proof. Considering the commutator of S^2 with an arbitrary basis vector e_d we find, $\forall d \in \{1,2,3\}$,

$$S^2 \otimes e_d - e_d \otimes S^2 = -\frac{1}{2\kappa}\sum_{a=1}^{3}\big((e_a \odot e_a)\otimes e_d - e_d \otimes (e_a \odot e_a)\big), \tag{2.26}$$

where we have introduced the symmetric product,

$$a \odot b \coloneqq \frac{1}{2}(a\otimes b + b\otimes a).$$

The right-hand side is clearly non-zero in $T(E)$ and not an element of the ideal, $\forall a,b,c \in E$,

$$I\left((a\wedge b)\otimes c - c\otimes(a\wedge b) - \kappa(g(a,c)b - g(b,c)a)\right),$$

used to construct $\mathrm{P}_\kappa(E,\delta)$. Thus, it is non-zero in $\mathrm{P}_\kappa(E,\delta)$, and so S^2 does not generally commute with vectors $v \in E \subset \mathrm{P}_\kappa(E,\delta)$. Therefore, it cannot be a scalar. □

Remark. Notice that in $\mathrm{Cl}(E,\delta)$ the right-hand side of (2.26) is zero, so S^2 commutes with the whole algebra. This is a clear example of how the hierarchy of Lemma 1.30 induces differences in the algebraic structures of $\mathrm{Cl}(E,\delta)$ and $\mathrm{P}_\kappa(E,\delta)$.

With that in hand, let us abuse notation to define multipoles of bivectors,

Definition 2.39. The multipoles $M^{(k)}$ written in terms of bivectors are, $\forall n \in \mathbb{Z}^+$, $\forall j \in \{1,\ldots,n\}$, $\forall a_j, b_j \in \{1,2,3\}$,

$$M^{(n)}\Big(\bigotimes_{j=1}^{n} e_{a_j}\wedge e_{b_j}\Big) = \kappa^n \sum_{p_1,\ldots,p_n=1}^{3}\prod_{j=1}^{n}\varepsilon_{a_j b_j p_j} M^{(n)}\Big(\bigotimes_{m=1}^{n} S_{p_m}\Big), \tag{2.27}$$

and $M^{(0)}(\alpha) = \alpha$, $\forall \alpha \in \mathbb{R}$.

For example, $\forall a,b,c,d \in E$,

$$\begin{aligned} M^{(2)}\big((a\wedge b)\otimes(c\wedge d)\big) &= \frac{1}{2}\big((a\wedge b)\otimes(c\wedge d) + (c\wedge d)\otimes(a\wedge b)\big) \\ &\quad - \frac{\kappa^2}{3} S^2\big(\delta(a,c)\delta(b,d) - \delta(a,d)\delta(b,c)\big). \end{aligned} \tag{2.28}$$

Now, let us explore the possibility that some multipoles of bivectors may have become zero during the quotient, which would limit the kinds of spin structures that we could implement within $\mathrm{P}_\kappa(E,\delta)$.

Lemma 2.40. $\nexists n \in \mathbb{Z}^+ : M^{(n)} = 0$.

Proof. [6] shows that in $U(\mathfrak{so}(3,\mathbb{R}))$, $M^{(n)} = 0$ entails $S^2 = -\frac{(n-1)(n+1)}{4}$. By Lemma 2.38, S^2 is not a scalar in $\mathrm{P}_\kappa(E,\delta)$, so by contraposition no multipoles are zero in $\mathrm{P}_\kappa(E,\delta)$. □

With Lemma 2.40 established, we may now show that,

Lemma 2.41. *The unital associative algebra homomorphism, $\forall p \in \{1,2,3\}$,*

$$\phi : U(\mathfrak{so}(3,\mathbb{R})) \to \mathrm{P}_\kappa(E,\delta)$$
$$\phi \coloneqq S_p \mapsto \frac{1}{2\kappa}\sum_{a,b=1}^{3} \varepsilon_{abp}\, e_a \wedge e_b \tag{2.29}$$

is injective.

Proof. That ϕ as defined on generators can be extended to a unital associative algebra homomorphism follows from Lemma 2.36. Furthermore, Lemma 2.40 demonstrates that no multipoles in $\mathrm{P}_\kappa(E,\delta)$ are zero, and so by Lemma 2.30 we are done. □

We may draw this connection between $U(\mathfrak{so}(3,\mathbb{R}))$ and $\mathrm{P}_\kappa(E,\delta)$ closer by relating the actions of their generators. First,

Definition 2.42. Let us extend u to be a unital associative algebra action, $\forall \alpha \in \mathbb{R}$, $A, B \in T(\Lambda^2(V))$,

$$\mathrm{u} : T(\Lambda^2(V)) \to \mathrm{End}(\mathrm{P}_\kappa(V,g))$$
$$\mathrm{u}(A) \coloneqq \begin{cases} D \mapsto AD & A \in \mathbb{R} \\ \mathrm{u}(B)\circ\mathrm{u}(C) & A = B \otimes C. \end{cases} \tag{2.30}$$

Then,

Corollary 2.43. *For all, $A \in T(\Lambda^2(E))$,*

$$\mathrm{u}(A)|_{T(\Lambda^2(V))} = \mathrm{ad}(A), \tag{2.31}$$

where we translate bivectors into spin generators and back here appropriate using Definition 2.35.

Proof. We need only compare the definitions of u and ad in light of Lemma 2.41. □

Finally, we are in a position where we may conclude,

Theorem 2.44. $\mathrm{P}_\kappa(E,\delta)$ *can support any spin structure.*

Proof. Since by Lemma 2.41 $U(\mathfrak{so}(3,\mathbb{R})) \subset \mathrm{P}_\kappa(E,\delta)$, $\mathrm{P}_\kappa(E,\delta)$ lacks the additional structure of any spin algebra $A^{(s)}$. Therefore, we are not limited in the kinds of spin structures we can impose on $\mathrm{P}_\kappa(E,\delta)$. □

3. Results

3.1. The Indefinite-Spin Position Algebra

The aim of this paper has been to explore the relationship between spin and geometry by constructing algebras of position operators from a Minkowski space-time (V, g) which meet three requirements as outlined in Sect. 1.3. This first two of these requirements are: algebraically containing an action of the connected symmetry group for the metric g; and containing the generators of this symmetry group as elements of the algebra. These goals are achieved for general Minkowski space-times (V, g) within the Indefinite-Spin Position Algebra, for $\kappa \in \mathbb{R}$, $\forall a, b, c \in V$,

$$\mathrm{P}_\kappa(V, g) \cong \frac{T(V)}{I\left((a\wedge b)\otimes c - c\otimes(a\wedge b) - \kappa(g(a,c)b - g(b,c)a)\right)}. \tag{3.1}$$

Our third requirement is to subsume within such an algebra the structure of a system with arbitrary spin. To do this, we restrict our attention to Euclidean three-space (E, δ), where we may utilise the spin algebras $A^{(s)}$ described in Sect. 2.3.2. In Theorem 2.44, we showed that $\mathrm{P}_\kappa(E, \delta)$ has no pre-existing spin structure, and so is the ideal foundational algebra within which to achieve this aim for arbitrary spin (except for $s = 0$, which will be discussed shortly).

3.2. The Spin-s Position Algebras

3.2.1. General Definition. We may now implement the final property from section 1.3 and complete our construction of non-commutative position operator algebras which subsume the structure of an arbitrary spin system,

Definition 3.1. ($\mathrm{P}^{(s)}_\kappa(E, \delta)$) The Spin-$s$ Position Algebra $\mathrm{P}^{(s)}_\kappa(E, \delta)$ for $s \in \{0, \frac{1}{2}, 1, \dots\}$ is the quotient algebra,

$$\mathrm{P}^{(s)}_\kappa(E, \delta) \cong \begin{cases} \dfrac{\mathrm{P}_\kappa(E, \delta)}{I\left(\mathrm{Im}(M^{(2s+1)})\right)} & s \neq 0 \\ \dfrac{T(E)}{\mathrm{Im}(M^{(1)})} & s = 0, \end{cases} \tag{3.2}$$

with the multipole $M^{(2s+1)}$ understood to be tensors of bivectors as in Lemma 2.40.

Remark. When $s \neq 0$, by Lemma 2.41 we are effectively embedding the structure of $A^{(s)}$ within the algebra of bivectors in $\mathrm{P}_\kappa(E, \delta)$. When $s = 0$, we embed the structure of $A^{(0)}$ into $T(E)$ instead (this difference shall be explained shortly). Therefore, we find the geometric realisation of spin generators as bivectors is robust in the presence of arbitrary spin.

Let us explore some immediate consequences of this definition,

Lemma 3.2. *For all $s \in \{0, \frac{1}{2}, 1, \dots\}$, in $\mathrm{P}^{(s)}_\kappa(E, \delta)$, $S^2 = -s(s + 1)$.*

Proof. When $s = 0$, this is clear from the definition, and the case $s \neq 0$ follows from Lemmas 2.33 and 2.41. □

Lemma 3.3.

$$\mathrm{P}_\kappa^{(0)}(E,\delta) \cong \mathrm{Sym}(E). \tag{3.3}$$

Proof. The quotient by the ideal $I\left(\mathrm{Im}(M^{(1)})\right)$ in Definition 3.1 entails that in $\mathrm{P}_\kappa^{(0)}(E,\delta)$,

$$[a \wedge b = 0] \Rightarrow [a \otimes b = b \otimes a].$$

Thus, the definition of $\mathrm{P}_\kappa^{(0)}(E,\delta)$ is identical to that of the symmetric algebra $\mathrm{Sym}(E)$. □

Remark. For $\mathrm{P}_\kappa^{(0)}(E,\delta)$, the value of κ changes nothing since it is only defined in the $\mathfrak{so}(3,\mathbb{R})$-action on vectors.

Lemma 3.4. *For all $s \in \{0, \frac{1}{2}, 1, \dots\}$, $\mathrm{P}_\kappa^{(s)}(E,\delta)$ is infinite-dimensional.*

Proof. It is easy to see that, $\forall s \in \{0, \frac{1}{2}, 1, \dots\}$, $\forall n \in \mathbb{Z}^+$, $\forall j \in \{1, \dots, n\}$, $\forall v_j \in E$, the vector space homomorphism,

$$\begin{aligned} f : \mathrm{Sym}(E) &\to \mathrm{P}_\kappa^{(s)}(E,\delta) \\ f := 1 &\mapsto 1 \\ f := v_1 &\mapsto v_1 \\ f := v_1 \odot \dots \odot v_n &\mapsto \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}, \end{aligned}$$

is injective. □

3.2.2. $\mathbf{P}_\kappa^{(0)}(E,\delta)$ and Commutative Geometry.

To begin our analysis of the $\mathrm{P}_\kappa^{(s)}(E,\delta)$, let us consider the case when $s = 0$. In Definition 3.1, the $s = 0$ case derived from $T(E)$ rather than $\mathrm{P}_\kappa(E,\delta)$; this is to ensure that $\mathrm{P}_\kappa^{(0)}(E,\delta)$ is identical to the standard commutative position operator algebra of quantum mechanics. In terms of geometry,

Corollary 3.5. *In the sense of commuting position operators, the structure of a spin $s = 0$ system subsumed within $\mathrm{P}_\kappa^{(0)}(E,\delta)$ generates a commutative geometry.*

Proof. The commutative structure in Lemma 3.3 is a direct consequence of the defining property of $A^{(0)}$, i.e. that all spin generators are zero. □

The zeroing of spin generators in an $s = 0$ system requires us to treat this case separately, to ensure consistency with the standard position operator algebra of quantum mechanics. Otherwise,

Lemma 3.6. *Suppose we define,*

$$\widetilde{\mathrm{P}}_\kappa^{(0)}(E,\delta) \cong \frac{\mathrm{P}_\kappa(E,\delta)}{I\left(\mathrm{Im}(M^{(1)})\right)},$$

consistently with the $\mathrm{P}_\kappa^{(s\neq 0)}(E,\delta)$. Then, for $\kappa \neq 0$, $\forall a, b \in E$,

$$\widetilde{\mathrm{P}}_\kappa^{(0)}(E,\delta) \cong \mathbb{R}[x]. \tag{3.4}$$

Proof. In $\widetilde{\mathrm{P}}_\kappa^{(0)}(E,\delta)$, the quotient entails, $\forall a,b \in E$,

$$\kappa(a\wedge b) = 0,$$

and thus, $\forall a,b,c \in E$,

$$(a\wedge b)\otimes c - c\otimes(a\wedge b) = 0 = \kappa\big(\delta(a,c)b - \delta(b,c)a\big).$$

Let us now choose $\{a,b\}$ linearly independent in the above. Then, since $\kappa \neq 0$, $\forall c \in E$,

$$[\delta(a,c) = 0] \wedge [\delta(b,c) = 0] \Rightarrow [a = 0] \wedge [b = 0],$$

by the non-degeneracy of δ, contradicting our assumption. Therefore, no such linearly independent sets of $a,b \in E$ exist in $\widetilde{\mathrm{P}}_\kappa^{(0)}(E,\delta)$, so all $a \in E \subset \widetilde{\mathrm{P}}_\kappa^{(0)}(E,\delta)$ are linearly dependent. The isomorphism with $\mathbb{R}[x]$ then follows. □

Remark. Basing $\mathrm{P}_\kappa^{(0)}(E,\delta)$ on $T(E)$ instead of $\mathrm{P}_\kappa(E,\delta)$ means that it does not contain an $\mathfrak{so}(3,\mathbb{R})$-action on its elements. Therefore, no rotations of its elements are possible within $\mathrm{P}_\kappa^{(0)}(E,\delta)$. While this might seem alarming at first, recall that in quantum mechanics we may only rotate position operators using generators of *orbital* angular momentum, i.e. tensors of both position and momentum operators, with the action determined by the Heisenberg algebra. This structure is not present in $\mathrm{P}_\kappa^{(0)}(E,\delta)$, and so there is no contradiction.

3.2.3. $\mathrm{P}_\kappa^{(s\neq 0)}(E,\delta)$ and Non-Commutative Geometry. We established in Corollary 3.5 that the structure of a spin-0 system entails a commutative geometry within our algebra of position operators. We may generalise this statement to all spins,

Theorem 3.7. *In the sense of non-commuting position operators [3, 13, 29], the structure of a spin-s system subsumed within $\mathrm{P}_\kappa^{(s)}(E,\delta)$ generates a non-commutative geometry within $\mathrm{P}_\kappa^{(s)}(E,\delta)$ iff $s \neq 0$.*

Proof. We may prove the above statement by proving its contraposition. One direction of this is proven in Corollary 3.5, so let us prove its converse. Assume a commutative geometry within $T(E)$, so, $\forall a,b \in E$

$$\Lambda^2(E) = \{0\}.$$

Since $\mathrm{Im}(M^{(1)}) \cong \Lambda^2(E)$, our algebra must subsume the structure of a spin-0 system. □

This is the central result of this paper: the presence of spin in a system entails a natural, spin-dependent non-commutative geometry for that system. Such non-commutative geometries are much weaker than those common to the literature [3, 13, 29], which typically place the position operators into a Heisenberg-like [27, 31] algebra.

4. Discussion

As a non-commutative algebra of position operators, the $\mathrm{P}_\kappa^{(s)}(E, \delta)$ form the foundation for a fundamentally new approach to incorporating spin into quantum mechanical theories. Of course, each $\mathrm{P}_\kappa^{(s)}(E, \delta)$ is not a complete model for a quantum mechanical point particle. However, it can form part of such a model which includes additional observables, such as momentum, and additional algebraic structure such as dynamical (symplectic) transformations [9, 14]. Despite this difference in scope, it is important that we contrast the $\mathrm{P}_\kappa^{(s)}(E, \delta)$ with standard quantum mechanical models of arbitrary spin systems.

In such a model, position and spin degrees of freedom always commute [32], as do the position degrees of freedom within themselves. This shows that these observables are not independent in any $\mathrm{P}_\kappa^{(s\neq 0)}(E, \delta)$, and necessarily entails higher-order correlations between position and spin, and between position observables themselves. Clearly such corrections to the commuting theory must be small to have not yet been observed, but through our approach they are naturally motivated. The only exception to this statement is the spin-0 model, which is completely equivalent to the standard quantum mechanical model. Beyond non-relativistic quantum mechanics, the non-commutative geometric aspects of relativistic generalisations of the $\mathrm{P}_\kappa^{(s\neq 0)}(E, \delta)$ may prove useful in the construction of theories of quantum gravity which incorporate both non-commutative geometry and spin.

It is also instructive to contrast the $\mathrm{P}_\kappa^{(s\neq 0)}(E, \delta)$ with other known descriptions of spin-s systems with $s \neq 0$, principally between $\mathrm{P}_\kappa^{(1/2)}(E, \delta)$ and the Clifford, and $\mathrm{P}_\kappa^{(1)}(E, \delta)$ and the Duffin–Kemmer–Petiau algebras for Euclidean three-space respectively. The Clifford and Duffin–Kemmer–Petiau algebras are both finite-dimensional whereas both $\mathrm{P}_\kappa^{(1/2)}(E, \delta)$ and $\mathrm{P}_\kappa^{(1)}(E, \delta)$ are infinite-dimensional by Lemma 3.4. This makes the $\mathrm{P}_\kappa^{(s)}(E, \delta)$ more appropriate for constructing models with position-dependent potentials. Furthermore, we may always find a value of κ such that $\mathfrak{so}(3, \mathbb{R})$-actions are identical between $\mathrm{P}_\kappa^{(1/2)}(E, \delta)$ and the Clifford algebra, and between $\mathrm{P}_\kappa^{(1)}(E, \delta)$ and the Duffin–Kemmer–Petiau algebra. This suggests the Clifford and Duffin–Kemmer–Petiau algebras for Euclidean three-space may be recovered from $\mathrm{P}_\kappa^{(1/2)}(E, \delta)$ and $\mathrm{P}_\kappa^{(1)}(E, \delta)$ respectively (for particular values of κ). This may be achieved by further quotienting $\mathrm{P}_\kappa^{(1/2)}(E, \delta)$ and $\mathrm{P}_\kappa^{(1)}(E, \delta)$ by ideals generated by their defining algebraic relations. Furthermore, this raises the possibility that $\mathrm{P}_\kappa^{(s)}(E, \delta)$ may be used to derive finite-dimensional, higher-spin generalisations of both the Clifford and Duffin–Kemmer–Petiau algebras.

5. Conclusion

In this paper, we revealed the relationship between spin and geometry in the non-relativistic setting through the construction of non-commutative position operator algebras. We first derived the “Indefinite-Spin Position Algebra” $\mathrm{P}_\kappa(V, g)$: the most general such algebra incorporating within its structure a natural action of the connected symmetry group for the Minkowski spacetime (V, g). Showing that this algebra contained no spin structure itself, we constructed from it the "Spin-s Position Algebras" a family of position operator algebras which subsume the structure of a system with arbitrary spin. When $s \neq 0$, these algebras constitute a fundamentally new model for quantum mechanical systems with non-zero spin in which position and spin operators naturally do not commute, nor do position operators themselves. These developments demonstrate that: spin naturally arises from Euclidean geometry; spin can be described completely in terms of oriented planar elements; and the presence of spin in a system naturally generates a spin-dependent non-commutative geometry in that system. The possibility of using these algebras to generalise the Clifford and Duffin–Kemmer–Petiau algebras was also discussed.

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Declarations

Conflict of interest The author has no conflict of interest to declare that are relevant to the content of this article. There is no data associated with this article.

Appendix A. Proof that $M^{(2)}(S'_p \otimes S'_q) = 0$ in $\mathrm{Cl}(E, \delta)$

Definition A.1. (I) Consider a Euclidean three-space (E, δ) and a basis $\{e_a\}$ which is orthonormal with respect to δ. Then, we define $I \in \mathrm{Cl}(E, \delta)$,

$$I := e_1 \wedge e_2 \wedge e_3. \tag{A.1}$$

This element I is called the "pseudoscalar" element[10] of $\mathrm{Cl}(E, \delta)$.

Lemma A.2. *In* $\mathrm{Cl}(E, \delta)$, *the orthonormal set of vectors* $\{e_a\}$ *satisfy,* $\forall a, b \in \{1, 2, 3\}$,

$$\frac{1}{2}\big(e_a e_b + e_b e_a\big) = \delta_{ab}. \tag{A.2}$$

Proof. By the definition of $\mathrm{Cl}(E, \delta)$. □

Corollary A.3.

$$I = e_1 e_2 e_3. \tag{A.3}$$

Proof. By Lemma A.2 $\forall \sigma \in S_k$,

$$\mathrm{sgn}(\sigma) e_{\sigma(1)} e_{\sigma(2)} e_{\sigma(3)} = e_1 e_2 e_3.$$

□

Corollary A.4.

$$II = -1. \tag{A.4}$$

Proof. Direct computation. □

Lemma A.5. *For all* $a \in \{1, 2, 3\}$,

$$e_a I = I e_a. \tag{A.5}$$

Proof. Fix $k \in \{1, 2, 3\}$, then,

$$e_k I = e_k e_1 e_2 e_3 = (-1)^2 e_1 e_2 e_3 e_k = I e_k.$$

□

Corollary A.6. *For all* $A \in \mathrm{Cl}(E, \delta)$,

$$AI = IA. \tag{A.6}$$

Proof. This follows directly from A.5 as all elements of $\mathrm{Cl}(E, \delta)$ are algebraic combinations of $\mathbb{R}$ and the $\{e_a\}$. □

Lemma A.7. *For all* $a \in \{1, 2, 3\}$,

$$I e_a = \frac{1}{2} \sum_{b,c=1}^{3} \varepsilon_{abc}\, e_b \wedge e_c. \tag{A.7}$$

Proof. Direct computation. □

Corollary A.8. *For all* $b, c \in \{1, 2, 3\}$,

$$e_b \wedge e_c = \sum_{a=1}^{3} \varepsilon_{abc}\, I e_a. \tag{A.8}$$

Proof. Immediate consequence of Lemma A.7. □

Lemma A.9. *For all* $p, q \in \{1, 2, 3\}$,

$$\frac{1}{2}(S'_p S'_q + S'_q S'_p) + \frac{1}{4}\delta_{pq} = 0. \tag{A.9}$$

Proof. From Lemma A.2 we apply,

$$\frac{1}{2}\sum_{a=1}^{3}\sum_{b=1}^{3} \varepsilon_{acd}\varepsilon_{bfg}\, II\big(e_a e_b + e_b e_a\big) = \sum_{a=1}^{3}\sum_{b=1}^{3} \varepsilon_{acd}\varepsilon_{bfg}\, II\delta_{ab}.$$

Applying Corollaries A.6 and A.8 to the left-hand side, and Corollary A.4 to the right, we find,

$$\frac{1}{2}\big((e_c \wedge e_d)(e_f \wedge e_g) + (e_f \wedge e_g)(e_c \wedge e_d)\big) = -(\delta_{cf}\delta_{dg} - \delta_{cg}\delta_{df}).$$

Applying the transformation (1.18) to both sides yields,

$$\frac{1}{2}(S'_p S'_q + S'_q S'_p) = -\frac{1}{4}\delta_{pq}.$$

□

Appendix B. Proof of the Algebraic Form of $t(a,b)(c)$

Let us begin by showing some important properties of $t(a,b)(c)$.

Lemma B.1. *For all* $a, b, e \in V$,

$$t(b,a)(e) = -t(a,b)(e). \tag{B.1}$$

Proof. Direct computation. □

Lemma B.2. *For all* $a, b, e \in V$,

$$t(a,b)(e) + t(b,e)(a) + t(e,a)(b) = 0. \tag{B.2}$$

Proof. Direct computation. □

Lemma B.3. *For all* $a, b, c, d, e \in V$,

$$\begin{aligned} &t(a,b)\circ t(c,d)(e) - t(c,d)\circ t(a,b)(e) = \\ &\qquad t\big(t(a,b)(c), d\big)(e) + t\big(c, t(a,b)(d)\big)(e). \end{aligned} \tag{B.3}$$

Proof. Direct computation. □

Remark. Lemma B.3 captures the fact that the vector space of $\{t(a,b)\}$ is closed under commutator; this is an explicit realisation of the same fact for $\mathfrak{so}(p, q, \mathbb{R})$, which the $\{t(a,b)\}$ is isomorphic to as a Lie algebra. As such, we are searching for a third-order tensor whose properties support those required to encode the Lie algebra structure of $\mathfrak{so}(p, q, \mathbb{R})$.

Lemma B.4. *The properties of Lemmas B.1, B.2, and B.3 entail the constraints,* $\forall a, b, c, d, e \in V$*,*

$$f(b, a, e) + f(a, b, e) = 0 \tag{B.4a}$$

$$f(a, b, e) + f(b, e, a) + f(e, a, b) = 0 \tag{B.4b}$$

$$\begin{aligned} &f(a, b, f(c, d, e)) - f(c, d, f(a, b, e)) \\ &\quad - f(f(a, b, c), d, e) - f(c, f(a, b, d), e) = 0 \end{aligned} \tag{B.4c}$$

respectively.

Proof. In any algebra where $f(a, b, c) = t(a,b)(c)$ (replacing the tensor products in f with the product of the algebra), we may translate the properties of Lemmas B.1, B.2, and B.3 into constraints on f by direct substitution. □

Theorem B.5. *For arbitrary Minkowski space-times, the system of constraints in Lemma B.4 is satisfied only when,*

$$f(a, b, c) = k\big((a\otimes b - b\otimes a)\otimes c - c\otimes(a\otimes b - b\otimes a)\big),$$

where $k \in \mathbb{R}$.

Proof. In general, the form of f which satisfies the constraints (B.4a), (B.4b), and (B.4c) will depend on the dimension of V; this is because in low dimension we cannot assume that the tensors in our expressions are all linearly independent.

When $\dim(V) = 1$, all non-zero vectors are linearly dependent, so writing $a = k_a x$ etc. for some non-zero $x \in V$ we have,

$$f(a, b, c) = (\alpha + \beta + \gamma + \delta + \epsilon + \zeta)k_a k_b k_c \, x\otimes x\otimes x.$$

Applying the constraint (B.4a), we find,

$$\alpha + \beta + \gamma + \delta + \epsilon + \zeta = 0,$$

and so $f(a, b, c) = 0$. This trivially satisfies (B.4b) and (B.4c), and is of the form (B.5).

When imposing the constraints (B.4a)-(B.4c) for $\dim(V) \geq 2$, we will assume where possible that our set of input vectors are linearly independent. However, linear independence of this set may fail when the dimension of the space considered is too low. To overcome this difficulty, we will employ the following combinatorial argument. Consider a basis $I = \{b_j\}$ for V indexed over a set J, and a set of parameterised linear combinations $D = \{\sum_{j\in J} \lambda_j^{(k)} b_j\}$ indexed over a set K. Then, to impose a constraint defined with n vectors for $n > \dim(V)$, let us fix $|D| = n - \dim(V)$ and evaluate it using all n-tuples of vectors $(\sigma(1), \ldots, \sigma(n))$ for all bijective $\sigma : \{1, \ldots, n\} \to I \cup D$. We consider valid only those solutions which are independent of the parameterisation of the elements of D, and independent of the choice of σ.

Now let us continue with the proof, and implicitly utilise either linear independence or this combinatorial argument when necessary. When $\dim(V) \geq 3$, imposing the constraint (B.4a) entails,

$$[\alpha = -\delta] \wedge [\beta = -\gamma] \wedge [\epsilon = -\zeta], \tag{B.5}$$

and so,

$$f(a,b,c) = \alpha(a\otimes b - b\otimes a)\otimes c + \epsilon\, c\otimes(a\otimes b - b\otimes a) + \beta\big(a\otimes c\otimes b - b\otimes c\otimes a\big). \tag{B.6}$$

Next, imposing (B.4b) on (B.6) entails,

$$[\alpha + \epsilon = \beta], \tag{B.7}$$

thus,

$$\begin{aligned} f(a,b,c) = \alpha(a\otimes b - b\otimes a)\otimes c + \epsilon\, c\otimes(a\otimes b - b\otimes a) \\ + (\alpha+\epsilon)\big(a\otimes c\otimes b - b\otimes c\otimes a\big). \end{aligned} \tag{B.8}$$

Finally, imposing (B.4c) on (B.8) entails,

$$[\alpha^2 + \alpha\epsilon = 0] \wedge [\epsilon^2 + \alpha\epsilon = 0] \wedge [(\alpha+\epsilon)^2 = 0],$$

which reduces to,

$$[\alpha = -\epsilon], \tag{B.9}$$

yielding (B.5).

There is an edge case when $\dim(V) = 2$ which we shall now address. Imposing (B.4a) yields (B.5) as before, however, in two-dimensions, all tensors of the form (B.6) satisfy the constraint (B.4b). Thus, we must impose (B.4c) directly on (B.6), yielding,

$$\big([\alpha = -\epsilon] \wedge [\beta = 0]\big) \vee \big([\alpha = \epsilon] \wedge [\beta = -\epsilon]\big).$$

The first solution in the above gives the general case, and the second solution yields only a trivial solution,

$$f(a,b,c) = \alpha\big(a\wedge b\wedge c\big) = 0,$$

with the final equality following from the necessary linear dependence of $\{a,b,c\}$. □

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