Journal of Physics: Conference Series **3177** (2026) 012134 doi:10.1088/1742-6596/3177/1/012134

Tracking Topology in Casual Dynamical Triangulations: A Toy Model in Dimension 1+1

Christopher L Duston[1]

[1]Mathematics and Physics Department, Merrimack College, N Andover, USA

E-mail: dustonc@merrimack.edu

Abstract. This work presents an approach to track the spacetime topology in casual dynamical triangulations. The focus will be on a basic demonstration of the validity of the model through simulations in dimension 1+1. We will evaluate this model using standard tools, such as the spectral dimension, and will also present a calculation of the expectation value of a topological invariant, the Euler Characteristic. We will also briefly mention the recently completed dimension 3 case, in which we implemented an algorithm to find the fundamental group in this framework.

1 Background

The essential feature of our proposed construction is the use of a *branched covering space* to represent the spacetime. The prototypes for these are the Riemann surfaces, shown in Figure 1. Although this embedding $S \to \mathbb{R}^3$ is a common representation, it gives the impression that the surface S is somehow not a smooth manifold. This is just an artifact of the embedding; these are smooth manifolds, which we illustrate in Figure 2.

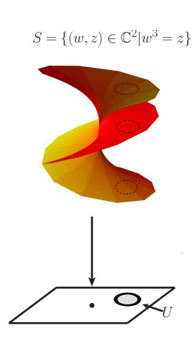

Figure 1: A Riemann surface, the prototype for our branched covers. For every open set U in the base, there are three inverse images $F^{-1}(U)_i$ in the cover, with the exception of a single point, in which there is only one inverse image under the covering map F.

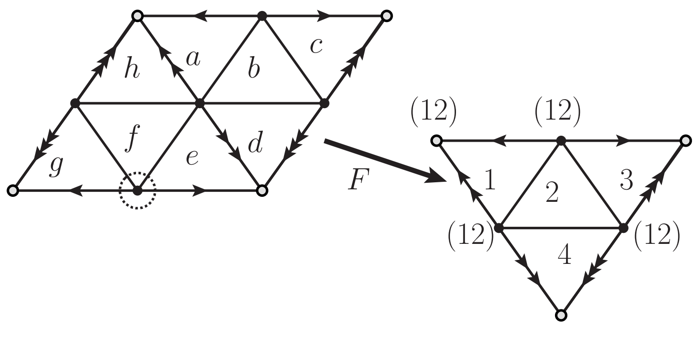

Figure 2: An explicit construction of the two-fold branched covering $F : \mathbb{T}^2 \to \mathbb{S}^2$. The map F is two-to-one on each triangle $((a,e) \to 1,\ etc)$ and edge, but the four points in the base $\mathbb{S}^2$ each have only one image in $\mathbb{T}^2$. Each vertex is labeled $(12) \in \mathcal{S}_2$, which corresponds to a transition between sheets $abdc$ and $efgh$. Any smooth surface should have such a representation.

The surface in Figure 1 can be constructed using a labeling $(123) \in \mathcal{S}_3$. Other elements of the permutation group will give us other topological structures, and by adding further points to the branch locus we can create any Euler Characteristic we want via the Riemann-Hurwitz formula $\chi(S) = deg(f)\chi(B) - b$ for branch index b.

These structures have been previously used to study topology change in Loop Quantum Gravity [1, 2, 3, 4]. Utilizing these structures in CDT[5] will be pretty straightforward - we just need to add a permutation label to each vertex of the triangulation. Thus the branched covering space will be the model for our physical spacetime and a cylinder in the base (the space upon which we perform the simulations) will be an auxiliary space. The action is the standard 2d one, written using $\chi = N_0 - N_1 + N_2$:

$$Z = \sum_{\mathcal{T}} \frac{1}{C_T} \exp(-k_1 N_2 + k_2(N_0 - N_1)) \equiv \exp(-k_1 N_2) Z(N_2, k_2),$$

where Z is the grand partition function (where the volume can in principle vary), and the canonical one $Z(N_2, k_2)$.

We need to make an important point here about the spatial sections in this manifold - as they are necessarily described by 3 fold covers over a 1-sphere, branched over a codimension 1 locus, *they will not generally be smooth manifolds* (see Figure 4). This pathological behavior is simply due to our choice in dimension, and need not be true in higher dimensions.

We fix the initial size of the simulations to be 10 x 10 (10 vertices along the spatial circle, and 10 sections), and allow them to evolve upwards to some thermalization volume (typically a few thousand triangles). Once this volume has been reached, we run the simulations to perform the measurements (typically 200 steps), and measure the spectral dimensions and topological parameters.

2 Simulation Results

These simulations were run at the coupling constant values listed above Figure 4. Figure 3 illustrates a typical run. The Hausdorff dimensions (for details, see [5]) are shown in Figures 5 and 6. We note that the significant spread in the average neighbor distance $\langle r \rangle$ is to be expected; in high-genus spaces, there can be a wide variety of paths between any two triangles.

The expectation value of the Euler Characteristic $\chi = 2 - 2g$, is shown in Figure 7. The direct relationship to volume is fully expected given the conditions of our simulations. In a thermalized system

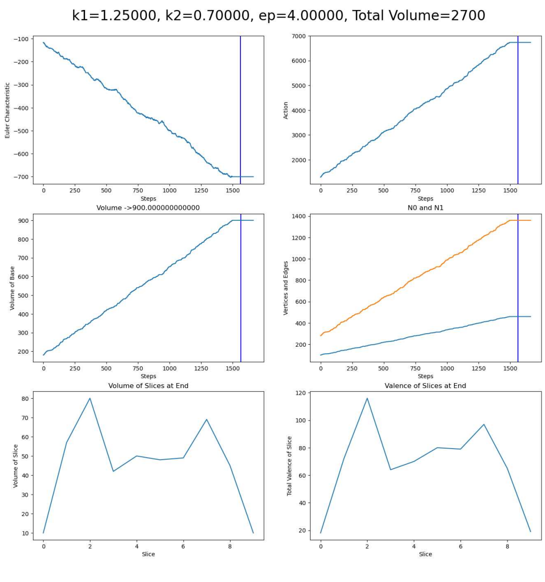

Figure 3: Details of one particular simulation (parameters at the top). The thermalization time is the vertical line, after which measurements are performed, and the fixed-volume condition can be seen in the middle row.

k_1	k_2
$(0.05, 0.06, 0.07, 0.08, 0.09)$	$(0.3, 0.4, 0.5, 0.6, 0.7)$
$(0.06, 0.07, 0.08, 0.09)$	$(0.10, 0.15, 0.20, 0.25)$
$(1.0, 1.25, 1.5, 1.75, 2.0)$	$(0.3, 0.4, 0.5, 0.6, 0.7)$

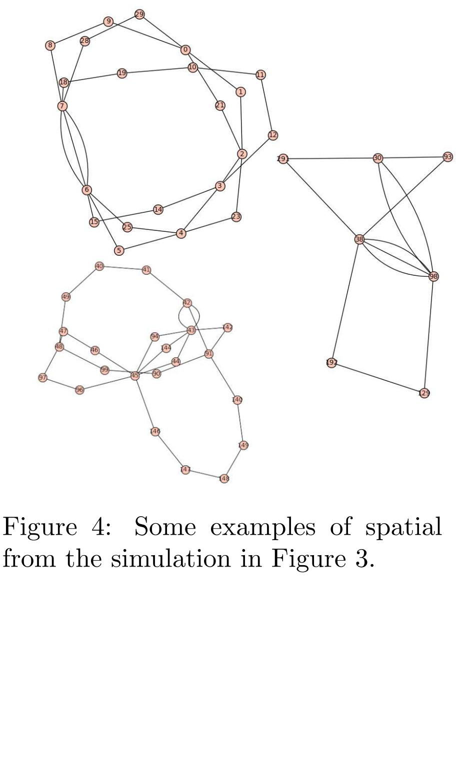

Figure 4: Some examples of spatial sections from the simulation in Figure 3.

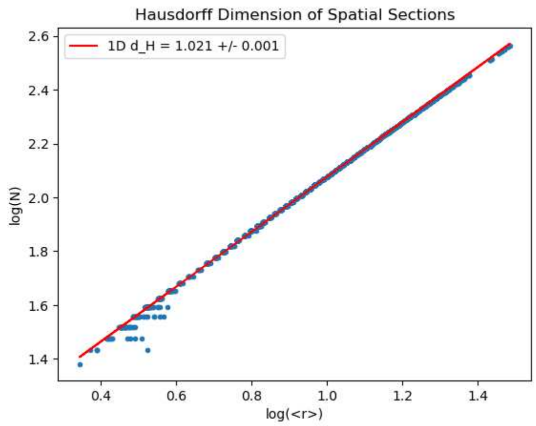

Figure 5: The Hausdorff dimension of spatial sections. This plot shows all the simulations run for the initially random topological state, at all parameter values. The slope is consistent with a Hausdorff dimension of 1.

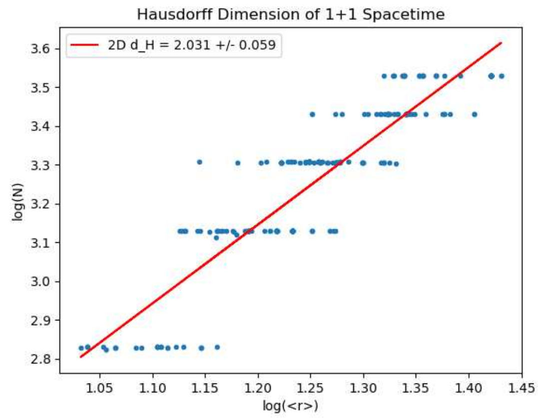

Figure 6: The Hausdorff dimension of the 1+1 spacetime. This plot shows all the simulations run for the initially random topological state, at all parameter values. The slope is consistent with a Hausdorff dimension of 2.

like the ones we are creating, we would expect global parameters like this to achieve generic values with some distribution, rather than "wander off" into some special location in parameter space.

Finally, we present an initial study of the numerical behavior of the canonical ensemble as Figure 8. This negative relationship is unexpected, particularly since traditional approaches consistently demonstrate that this entropy factor should be growing as $N_2!$ [6]. The behavior we observe is due the form of the partition function - the finite $N_0 - N_1$ is being suppressed by the factor $C_T = N_0!$ - but we have not explored this further. At this level, there is no evidence for critical behavior in the partition function.

As we think about extending these results to higher dimensions, we note that the Euler Characteristic is not sufficient for topological classification. However, finding the fundamental group π_1 gives one access to the complete set of homology groups, and there is an algorithm in this setting for finding it. In a recent study, we successfully implemented this algorithm and determined the topologies that are created in this dimension for several special cases [7]. Since we are primarily interested in 2+1 and 3+1, this algorithm will be (nearly) sufficient for us to track the topology of the spatial sections, with the full spacetimes being simple foliations created by such topological spaces.

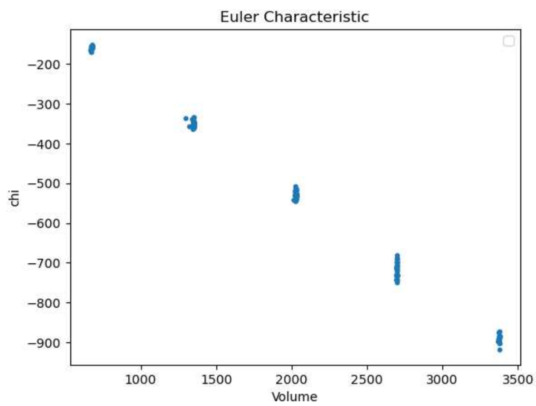

Figure 7: The Euler characteristic of the 1+1 spacetime. This plot shows all the simulations run for the initially random topological state, at all parameter values.

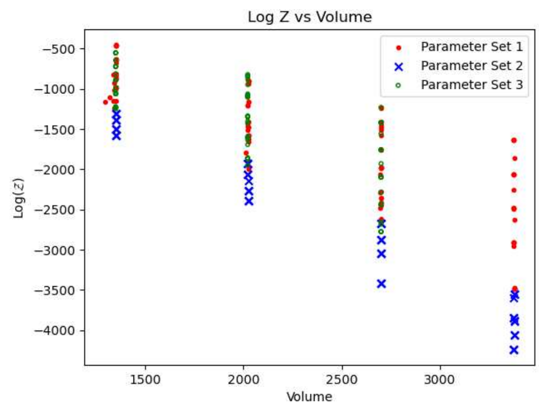

Figure 8: The partition function of the canonical ensemble of the 1+1 spacetime. This plot shows all the simulations run for the initially random topological state, with the various collections of parameter values indicated.

Journal of Physics: Conference Series **3177** (2026) 012134 doi:10.1088/1742-6596/3177/1/012134

References

[1] Duston C L 2012 *Classical and Quantum Gravity* **29** 205015 URL https://doi.org/10.1088%2F0264-9381%2F29%2F20%2F205015

[2] Duston C L 2013 The fundamental group of a spatial section represented by a topspin network (*Preprint* 1308.2934)

[3] Duston C L 2020 *IOP SciNotes* **1** 025205 URL https://dx.doi.org/10.1088/2633-1357/aba5a2

[4] Villani M 2021 *Classical and Quantum Gravity* **38**

[5] Ambjørn J, Görlich A, Jurkiewicz J and Loll R 2012 *Physics Reports* **519** 127–210 URL https://doi.org/10.1016%2Fj.physrep.2012.03.007

[6] Ambjørn J, Durhuus B and Jónsson T 1997 *Quantum geometry: a statistical field theory approach* (Cambridge University Press)

[7] Duston C L 2025 The realization of 3d topological spaces branched over graphs (*Preprint* 2508.09026) URL https://arxiv.org/abs/2508.09026