PROCEEDINGS OF SCIENCE

Rare b decays - theory overview

Yue-Long Shen[*] [a,*]

[a] *College of Information Science and Technology, Ocean University of China, No. 238 Songling Road , Qingdao 266100, China*

E-mail: shenylmeteor@ouc.edu.cn

Highly suppressed in the Standard Model (SM), rare B-meson decays offer precise observables critical for new physics searches, yet pose significant theoretical challenges. Effective field theory (EFT) methods enable us to analyze amplitude factorization and explore power-suppressed contributions. For $B_q \to \gamma\gamma$ decays, we incorporate power corrections from diverse sources and find these corrections to be substantial. A physically motivated hard-collinear assignment for internal quark lines in long-distance quark-loop contributions introduces a novel soft function. Renormalization group (RG) evolution reveals its momentum-space argument is non-positive definite, and its asymptotic behavior ensures factorization of this contribution. Numerically, this effect significantly impacts mixing-induced CP violation. For exclusive $B \to \{K, \pi\}\ell^+\ell^-$ decays with an energetic light meson in the final state, we present the first next-to-leading-order (NLO) calculation of weak annihilation corrections. This fills a critical gap in the QCD corrections to hadronic operator matrix elements, and we demonstrate that one-loop corrections notably impact direct CP and isospin asymmetries in $B \to \pi\ell^+\ell^-$ decays. For pure leptonic decays, power-enhanced structure-dependent QED corrections are mandatory for $B \to \mu^+\mu^-$ calculations, yet negligible for $B \to \tau^+\tau^-$ due to the absence of logarithmic enhancement. In two-body nonleptonic decays, power-suppressed weak annihilation contributions play an important phenomenological role, but hard gluon exchange processes suffer from endpoint singularities. Previously neglected hard-collinear gluon exchange diagrams hold the potential to cancel these singularities, warranting further study. In pure annihilation decays, quark-loop diagrams contribute comparably to leading-order terms and significantly modify predicted CP-violating observables.

The 17th International Conference on Heavy Quarks and Leptons (HQL 2025)
15-19 September, 2025
Peking University, Beijing, China

[*]Email: shenylmeteor@ouc.edu.cn

[*]Speaker

PoS(HQL2025)081

1. Introduction

Among the diverse heavy-flavor decay processes, rare B-meson decays occupy a unique and critical role. Within the Standard Model (SM), they are typically suppressed by the GIM mechanism, which amplifies the potential impact of new physics (NP) effects—making them indispensable for indirect NP searches. Today, experimental collaborations such as LHCb, ATLAS, CMS, and Belle-II have conducted long-term precision measurements of rare B-meson decays. These high-luminosity facilities have propelled heavy-flavor physics into an era of high-precision measurements, heralding major breakthroughs in NP searches. With the accumulation of experimental data, physicists can now perform detailed final-state angular distribution analyses for processes like $B \to (P, V)\ell^+\ell^-$ [1–4]. Notable discrepancies persist between experimental measurements and SM predictions for the angular observable P_5'. Additionally, inconsistencies have been observed between the measured branching ratios, CP-violating asymmetries, and isospin asymmetries in $B \to K\ell^+\ell^-$ decays and their SM expectations.

Discrepancies between experiment and theory offer opportunities to search for NP, yet require sufficiently precise theoretical predictions to identify NP signals. Effective field theories (EFTs) provide a powerful framework for separating physics at different scales. Commonly used EFTs in heavy hadron decays include the weak effective Hamiltonian, Heavy Quark Effective Theory (HQET) [5, 6], and Soft-Collinear Effective Theory (SCET) [7, 8]. To enhance theoretical precision, one must compute QCD corrections to perturbative functions, reduce uncertainties in non-perturbative input parameters, and account for power corrections to decay amplitudes. In many processes, power-suppressed contributions can be as large as 10%–20%. Radiative leptonic B-meson decays [9–20] and double radiative B-meson decays [21] serve as ideal platforms to study power-suppressed contributions, where various next-to-leading power (NLP) effects have been considered. Extensive discussions have also been conducted on power-suppressed annihilation contributions in two-body non-leptonic decays [25]. Nevertheless, comprehensive and systematic research on power correction effects in heavy hadron decay amplitudes remains a significant challenge.

The remainder of this article is structured as follows: Section 2 discusses double radiative decays, focusing on power corrections from various sources. Section 3 introduces long-distance quark-loop corrections to these decays, which necessitate a novel B-meson soft function. Section 4 is dedicated to QCD corrections to annihilation diagrams in semi-leptonic $B \to (\pi, K)\ell^+\ell^-$ decays in the large-recoil region. QED corrections to B-meson decays are addressed in Section 5. Section 6 focuses on the weak-annihilation contribution in charmless nonleptonic B-decays. Finally, we present our conclusions in Section 7. A brief note: this article primarily centers on my own research contributions, and I apologize for any unintentional omissions of related work.

2. Double radiative B decays

The exclusive radiative decay amplitude $\bar{B}_q \to \gamma\gamma$ is:

$$\bar{\mathcal{A}}(\bar{B}_q \to \gamma\gamma) = -\frac{4G_F}{\sqrt{2}}\frac{\alpha_{\rm em}}{4\pi}\epsilon_1^{*\alpha}(p)\epsilon_2^{*\beta}(q)\sum_{p=u,c}V_{pb}V_{pq}^*\sum_{i=1}^{8}C_i T_{i,\alpha\beta}^{(p)}. \tag{1}$$

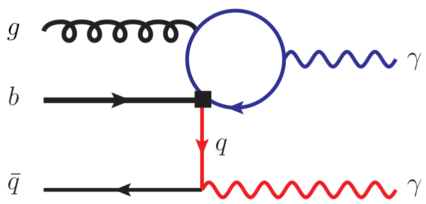

Figure 1: Diagrammatical representation of the soft-gluon emission from the factorizable quark loop generated by the effective four-quark operator and the electromagnetic current in the double radiative $\bar{B}_q \to \gamma\gamma$ decay process, where the symmetric diagram due to the exchange of two on-shell photons is not presented.

Using photon transversality and QED Ward-Takahashi identities, the hadronic tensors are parameterized via helicity form factors:

$$T^{(p)}_{i,\alpha\beta} = i m^3_{B_q} \left[\left(g^{\perp}_{\alpha\beta} - i\varepsilon^{\perp}_{\alpha\beta} \right) F^{(p)}_{i,L} - \left(g^{\perp}_{\alpha\beta} + i\varepsilon^{\perp}_{\alpha\beta} \right) F^{(p)}_{i,R} \right]. \tag{2}$$

Decay amplitudes have been computed at next-to-leading order (NLO) in α_s with some power corrections included [26, 27]. A comprehensive study [28] derived leading-power (LP) factorization formulas in SCET:

$$\sum_{i=1}^{8} C_i F^{(p),\mathrm{LP}}_{i,L} = -\frac{e_q f_{B_q} K^{-1}(\mu_{\mathrm{h}})}{m_{B_q}} \overline{m}_b(\nu) V^{(p)}_{7,\mathrm{eff}}(m_b,\mu,\nu) \mathcal{R}(m_b,\mu), \tag{3}$$

with $F^{(p),\mathrm{LP}}_{i,R} = 0$. Explicit forms of $V^{(p)}_{7,\mathrm{eff}}$ and $\mathcal{R}$ at next-to-leading logarithmic(NLL) level are given in [28]. In QCD factorization, tree-level formulas for subleading-power (NLP) corrections to $B_q \to \gamma\gamma$ were first derived in [28, 29], including: hard-collinear quark propagator subleading terms, non-vanishing light-quark masses, power-suppressed terms in the $\mathrm{SCET_I}$ heavy-to-light current, higher-twist B-meson distribution amplitudes (DAs), bottom-quark electromagnetic current, weak annihilation from four-quark operators, and resolved-photon soft contributions.

Numerically, NLL corrections to the LP predictions of $\mathrm{Re}\bar{\mathcal{A}}_L(\bar{B}_q \to \gamma\gamma)$ are estimated at 2-10%. Factorizable NLP contributions to $\mathrm{Re}\bar{\mathcal{A}}_L(\bar{B}_d \to \gamma\gamma)$ and $\mathrm{Im}\bar{\mathcal{A}}_L(\bar{B}_d \to \gamma\gamma)$ can be as large as 15% and 26%-35%, respectively. The resulting predictions of CP-averaged branching fraction are:

$$\mathcal{BR}(B_d \to \gamma\gamma) = \left(1.352^{+1.242}_{-0.745}\right) \times 10^{-8}, \qquad \mathcal{BR}(B_s \to \gamma\gamma) = \left(2.964^{+1.800}_{-1.614}\right) \times 10^{-7}. \tag{4}$$

This is consistent with the recent Belle measurement [30].

3. The long distance quark loop and B meson soft function

In a recent work [31], the authors presented the first calculation of the long-distance penguin contribution to the double radiative B-meson decay amplitudes via the QCD factorization approach, with the relevant Feynman diagrams shown in Fig. 1. The charm(up) quark in the loop propagates along a light-cone direction (denoted by $\bar{n}$), opposite to the momentum of the photon emitted from the soft quark within the B-meson. Consequently, a novel function Φ_G must be defined via the three-body HQET operator, where the partonic fields are localized on two distinct light-ray directions. The specific definition of this soft function is given as follows:

$$\begin{aligned}&\langle 0|(\bar{q}_s S_n)(\tau_1 n)\,(S_n^\dagger S_{\bar{n}})(0)\,(S_{\bar{n}}^\dagger g_s G_{\mu\nu} S_{\bar{n}})(\tau_2 \bar{n})\;\bar{n}^\nu\;\not{n}\gamma_\perp^\mu\gamma_5\,(S_{\bar{n}}^\dagger h_v)(0)|\bar{B}_v\rangle\\&= 2\,\tilde{f}_B(\mu)\,m_B\int_0^\infty d\omega_1\int_0^\infty d\omega_2 \exp\left[-i(\omega_1\tau_1+\omega_2\tau_2)\right]\;\Phi_G(\omega_1,\omega_2,\mu)\,.\end{aligned} \tag{5}$$

The arguments ω_1 and ω_2 are taken to cover the range $-\infty < \omega_{1,2} < +\infty$, which is dictated by the scale evolution of the soft function. The RG equation for Γ_G reads,

$$\frac{d}{d\ln\mu}\Phi_G(\omega_1,\omega_2,\mu) = -\int_{-\infty}^{+\infty} d\omega_1' \int_{-\infty}^{+\infty} d\omega_2' \Phi_G(\omega_1',\omega_2',\mu)\Gamma_G(\omega_1,\omega_2,\omega_1',\omega_2',\mu), \tag{6}$$

An exact solution to the RG evolution equation was derived with the Laplace transform technique [32–36]

$$\begin{aligned}\mathbf{\Phi}_G(\omega_1,\omega_2,\mu) &= \exp\left[V+2\gamma_E(a_1+a_2)\right]\int_0^\infty \frac{d\omega_1'}{\omega_1'}\int_0^\infty \frac{d\omega_2'}{\omega_2'}\left(\frac{\mu_0}{\omega_1'}\right)^{a_1}\left(\frac{\mu_0}{\omega_2'}\right)^{a_2}\\&\times\left[\mathcal{J}_1\otimes\mathcal{J}_2+\frac{1-e^{-i\pi a_2}}{4}\mathcal{K}_1\otimes\mathcal{K}_2\right]\mathbf{\Phi}_G(\omega_1',\omega_2',\mu_0).\end{aligned} \tag{7}$$

The evolution matrices $\mathcal{J}_i$ and $\mathcal{K}_i$ (with $i = 1,2$) are presented in the Supplemental Material of [37]. From the evolution equation, the asymptotic form of the soft function can be obtained. We are then led to conclude that all nonnegative moments of $\mathbf{\Phi}_G(\omega_1,\omega_2,\mu)$ are divergent, and the QCD factorization of the long-distance quark loop contribution is guaranteed.

Numerically, the power-suppressed soft-gluon radiative effects yield sizeable impacts on two specific CP-violating observables, $\mathcal{A}_{CP}^{\text{mix},\parallel}$ and $\mathcal{A}_{CP}^{\text{mix},\perp}$: numerical corrections of $O(30\%)$ arise for both $b\to d\gamma$ and $b\to s\gamma$ transitions with the default input parameters. Since the resulting numerical values of the CP-violating observables are insensitive to λ_{B_q}, we summarize the predictions here: $\left\{\mathcal{A}_{CP}^{\text{dir},\parallel},\mathcal{A}_{CP}^{\text{mix},\parallel},\mathcal{A}_{CP}^{\text{dir},\perp},\mathcal{A}_{CP}^{\text{mix},\perp}\right\} = \left\{13^{+4}_{-3}\%, -19^{+5}_{-8}\%, 34^{+7}_{-5}\%, 13^{+7}_{-6}\%\right\}$ for $\bar{B}_d\to\gamma\gamma$ at $\lambda_d =$ 275 MeV, and $\left\{\mathcal{A}_{CP}^{\text{dir},\parallel},\mathcal{A}_{CP}^{\text{mix},\parallel},\mathcal{A}_{CP}^{\text{dir},\perp},\mathcal{A}_{CP}^{\text{mix},\perp}\right\} = \left\{-0.67^{+0.15}_{-0.23}\%, 0.97^{+0.44}_{-0.27}\%, -1.8^{+0.3}_{-0.5}\%, -0.52^{+0.32}_{-0.41}\%\right\}$ for $\bar{B}_s\to\gamma\gamma$ at $\lambda_s = 325$ MeV.

4. Rare $B\to\{K,\pi\}\,\ell^+\ell^-$ Decays

In semileptonic $b\to s\ell^+\ell^-$ decays, the angular observable $P_5'(B^0\to K^{0*}\mu^+\mu^-)$ and integrated isospin asymmetry $A_I(B\to K\mu^+\mu^-)$[38–42], are both closely related to weak annihilation contributions. A field-theoretical framework for systematically addressing the hadronic matrix elements of electroweak penguin B-meson decays at large recoil has been established in the heavy quark mass limit based on the factorization formalism [43, 44]. The hadronic matrix elements of the two semileptonic operators $Q_{9,10}$ can be readily expressed in terms of the heavy-to-light B-meson decay form factors [45–47], which have been evaluated by Lattice simulation [48–64] or other QCD-inspired method such as light-cone sum rules[65–86]. The remaining contribution to

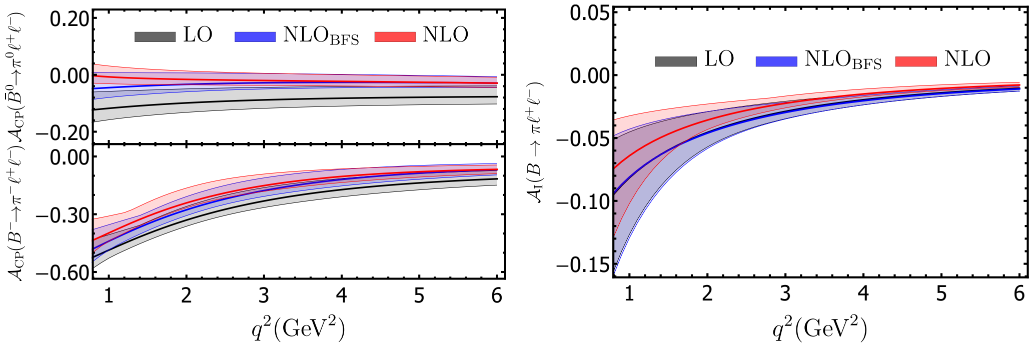

Figure 2: Theory predictions for the q^2 dependence of the direct CP asymmetries [top] and the isospin asymmetry [down] for the exclusive $B \to \pi \ell^+\ell^-$ decays at LO (grey bands), at NLO$_{\rm BFS}$ (blue bands) and at NLO (pink bands), where the uncertainties are obtained by adding the separate errors due to the variations of all input parameters in quadrature.

the decay amplitude can be determined by the contraction of the exclusive $B \to \{K,\pi\}\, \gamma^*$ matrix element with the vector lepton current $\gamma^* \to \ell^+\ell^-$. The $B \to \{K,\pi\}\, \gamma^*$ matrix element reads

$$\langle P(p')\, \gamma^*(q,\mu) | \mathcal{H}_{\rm eff}^{(t,u)} | \bar{B}(p)\rangle = -\frac{g_{\rm em}\, m_b}{4\,\pi^2}\, \frac{\mathcal{T}_P^{(t,u)}(q^2)}{m_B} \left[q^2\,(p_\mu + p'_\mu) - (m_B^2 - m_P^2)\, q_\mu \right], \tag{8}$$

The resulting factorization formulae for the hadronic quantities $\mathcal{T}_P^{(t,u)}$ at leading power can be cast in the form

$$\begin{aligned} \mathcal{T}_P^{(t,u)} &= C_P^{(t,u)}(q^2)\, f_{BP}^+(q^2) \\ &- \frac{\pi^2}{N_c}\, \frac{\mathcal{F}_B\, f_P}{m_B} \sum_{m=\pm} \int_0^\infty \frac{d\omega}{\omega} \int_0^1 du\, T_{P,m}^{(t,u)}(\omega,u)\, \phi_{B,m}(\omega)\, \phi_P(u)\,. \end{aligned} \tag{9}$$

The short-distance coefficient functions $C_P^{(i)}(q^2)$ and $T_{P,m}^{(i)}(\omega,u,\mu)$ $(i = u,\, t)$ are given in [43, 44]. In Ref. [87], the first NLO calculation of the weak annihilation contribution to rare $B \to \{K,\pi\}\,\ell^+\ell^-$ decays was presented at LP in the heavy quark expansion. This work provided the final missing piece for evaluating the QCD corrections to the amplitude. It was demonstrated that both twist-two and twist-three HQET DAs of the bottom meson are required to capture the soft QCD dynamics in the established factorization formula for the weak annihilation effect at $O(\alpha_s)$. Incorporating these higher-order corrections into the matrix elements of four-quark operators was shown to exert significant impacts on the theoretical predictions for both the direct CP asymmetry $\mathcal{A}_{\rm CP}(B^- \to \pi^-\ell^+\ell^-)$ and the CP-averaged isospin asymmetry $\mathcal{A}_{\rm I}(B \to \pi\ell^+\ell^-)$, as illustrated in Fig. 2.

5. Leptonic B decays and QED correction

At the leading order of $\alpha_{\rm em}$, the only relevant non-perturbative input in the leptonic decays $B_q \to \ell^+\ell^-$ is the B-meson decay constant f_B. To date, lattice QCD calculations have achieved most precise determinations of $f_{B_{u,d}}$ and f_{B_s}, with uncertainty less than 1%[88]. Therefore, a reliable assessment of QED corrections is highly desirable. Beneke *et al.* showed that virtual QED photon exchanges between one of the final-state leptons and the light spectator antiquark $\bar{q}$ inside the $\bar{B}_q$ meson gives rise to a power-enhanced ''non-local annihilation'' effect in the muonic decays of $\bar{B}_q$ mesons [90–92]. This power-enhanced QED effect also contains large double logarithms, and its magnitude is comparable to the non-parametric theoretical uncertainty (about 1.5% [93]). The structure-dependent QED effects were also considered in other processes[94–97].

For the QED correction to $B_q \to \tau^+\tau^-$ decays, a two-step matching procedure is required: starting from QCD $\times$ QED down to SCET$_{\rm I}$, and subsequently onto HQET $\times$ bHLET, in contrast to the direct matching onto SCET$_{\rm II}$ adopted for muonic decays. The resulting power enhanced amplitude $i\mathcal{A}_9$ reads

$$i\mathcal{A}_9 \propto \int_0^1 du\bar{u} \int_0^\infty \frac{d\omega}{\omega} \left[C_9^{\rm eff}(u,\mu) - \frac{2Q_\ell}{u} C_7^{\rm eff}(u,\mu)\right] \phi_+(\omega,\mu) \ln\left(1 + \frac{u}{\bar{u}} \frac{n_+ p_\ell \omega}{m_\ell^2}\right), \qquad (10)$$

The term proportional to $C_7^{\rm eff}$ in hard function will not cause endpoint-singularity as $u \to 0$ when convoluted by jet function, which is different from the case appeared in $B_{d,s} \to \mu^+\mu^-$. As a result, this channel does not exhibit the large enhanced QED effects observed in the muonic case. Numerically, after resumming the leading logarithmic contributions of the QCD and QED couplings, the combined QCD and QED corrections yield an overall enhancement of approximately 0.04% to the branching fraction [98], in stark contrast to the roughly 0.5% overall reduction found for $B_q \to \mu^+\mu^-$ decays.

6. Annhilation diagram in nonleptonic B decays

Several factorization approaches have been proposed and widely applied to various exclusive two-body charmless B-decay processes, including naive factorization [99, 100], generalized factorization [101, 102], QCD factorization [103–106], and the perturbative QCD (PQCD) approach [107, 108]. The QCD factorization (QCDF) approach works effectively at LP [109–112]. However, it cannot yield precise calculations of power-suppressed annihilation diagrams due to endpoint divergences. The annihilation contribution are usually estimated via process-(in)dependent parametrizations of logarithmically and linearly divergent integrals [103–106] (see also [113–122]). In [25], the contribution from hard-collinear gluon exchange which has been unfortunately neglected in all previous calculations was found to be at the same power with that from hard gluon exchange. The theory description of such hard-collinear gluon exchange mechanism will necessitate the introduction of a non-local four-body soft-collinear matrix element

$$\Theta_{B\to M}(\omega_1,\omega_2) = \int dt \int ds e^{i(\omega_1 t - \omega_2 s)} \langle M(p)| [\bar{q}_s(tn)\Gamma_1 h_v(0)] [\bar{\xi}_{\bar{c}}(0)\Gamma_2 q_{\overline{sc}}(sn)] \bar{Q}(p_B)\rangle, \quad (11)$$

the appearance of the soft-collinear field is due to the fact that one of the quark fields(in the final state mesons) attched by the hard collinear gluon can no longer be collinear, on the contrary, its momentum fraction must be at endpoint region. Motivated by the substantial progress on the soft-collinear factorization formalism of the bottom-quark induced contribution to the $h \to \gamma\gamma$

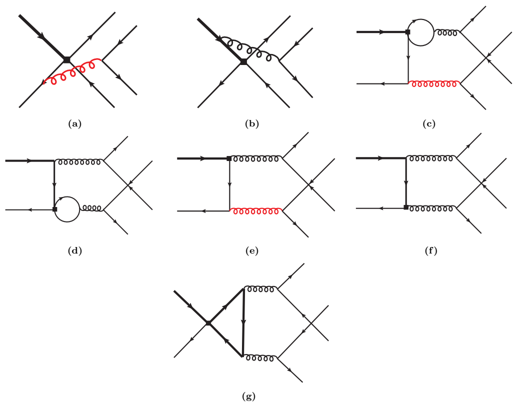

Figure 3: Sample Feynman diagrams for the weak annihilation $\bar{B}_q \to M_1 M_2$ decays at LO and NLO. The gluons marked with red colour can carry either the hard or hard-collinear momentum, while the remaining gluons can only possess the hard momentum. Symmetric diagrams that follow from (c)-(g) by exchanging the two off-shell gluons are not displayed.

amplitude [123–125], one can suppose that the rapidity divergence of the hard gluon exchange contribution can be canceled by

$$\int dy \mathcal{H}(y) \int d\omega_1 \int d\omega_2 \mathcal{J}(y, \omega_1, \omega_2) \Theta_{B\to M_2}(\omega_1, \omega_2) \phi_{M_1}(y). \tag{12}$$

An inspiring illustration is that if adopting ground-state factorization approximation for the soft-collinear matrix element

$$\Theta_{B\to M_2}(\omega_1, \omega_2)|_{\text{fac}} = \phi_B^+(\omega_1)\phi_{M_2}(1 - \omega_2/m_B)\theta(1 - \omega_2/m_B), \tag{13}$$

one can arrive at a compact form of the decay amplitude which is consistent with the original parameterization in QCD factorization.

In Ref. [25], the authors also uncovered a new perturbative mechanism induced by penguin contractions of the current-current operators (see Fig. 3), which an generate import impact on the decay amplitudes of pure annihilation processes albeit with an α_s suppression. This enhancement is more pronounced for penguin dominated processes since the penguin contraction diagrams(quark-loop diagrams) have large Wilson coefficient. The enhanced quark loop effects give rise to sizable numerical effects on the strong phases of pure annihilation amplitudes, consequently, the full spectrum of CP-violating observables in exclusive heavy-hadron decays is significantly modified. In Ref. [126], an improved PQCD analysis of pure annihilation-type charmless nonleptonic B-decays included enhanced quark-loop contributions, where both twist-2 and two-particle twist-3 DAs were included, confirmed the conclusion in [25].

7. Summary

In this presentation, I briefly review recent theoretical advances in the study of rare B-meson decay processes. Rare decays are highly suppressed within the SM, and precise predictions of their observables—an essential prerequisite for searching for new physics signals—pose a significant theoretical challenge. EFT techniques have been employed to analyze the factorization properties of decay amplitudes and to explore power-suppressed contributions. We focused on several representative B meson decay processes to discuss the precision calculation of B-decay amplitudes: The power-suppressed contributions in the double radiative decays $B_q \to \gamma\gamma$, especially the long-distance quark-loop effects; the NLO corrections to the annihilation contribution to $B \to \{K, \pi\}\ell^+\ell^-$ decays; the QED correction to pure leptonic B decays, and the annhilation diagrams in charmless nonleptonic B decays.

At both LP and NLP order, B-meson DAs represent one of the most critical input quantities and constitute the dominant source of theoretical uncertainties. Moving forward, further efforts are required to deepen our understanding of B-meson DAs in the following directions: investigating their renormalization properties beyond one-loop approximation and deriving perturbative constraints in the large-ω_i region; clarifying the rapidity/ultraviolet subtraction scheme to eliminate radiative tails; and performing nonperturbative determinations via lattice QCD simulations[127–132]. We also aim to further refine the QCD factorization formalism for subleading-power corrections, which includes developing a universal treatment of rapidity divergences in factorization formulae and establishing rigorous factorization proofs that account for Glauber gluon effects.

Acknowledgements

We acknowledges support from the National Natural Science Foundation of China with Grant No. 12575103, 12345004, 12175218, and the Natural Science Foundation of Shandong Province under the Grant No. ZR2024MA076.

References

[1] B. Capdevila, A. Crivellin and J. Matias, Eur. Phys. J. ST **1**, 20 (2023) doi:10.1140/epjs/s11734-023-01012-2 [arXiv:2309.01311 [hep-ph]].

[2] R. Aaij *et al.* [LHCb], JHEP **07**, 133 (2012) doi:10.1007/JHEP07(2012)133 [arXiv:1205.3422 [hep-ex]].

[3] R. Aaij *et al.* [LHCb], Phys. Rev. Lett. **111**, 191801 (2013) doi:10.1103/PhysRevLett.111.191801 [arXiv:1308.1707 [hep-ex]].

[4] R. Aaij *et al.* [LHCb], JHEP **06**, 133 (2014) doi:10.1007/JHEP06(2014)133 [arXiv:1403.8044 [hep-ex]].

[5] E. Eichten and B. R. Hill, Phys. Lett. B **234**, 511-516 (1990) doi:10.1016/0370-2693(90)92049-O

[6] H. Georgi, Phys. Lett. B **240**, 447-450 (1990) doi:10.1016/0370-2693(90)91128-X

[7] C. W. Bauer, S. Fleming, D. Pirjol and I. W. Stewart, ''An Effective field theory for collinear and soft gluons: Heavy to light decays,'' Phys. Rev. D **63**, 114020 (2001) [arXiv:hep-ph/0011336 [hep-ph]]

[8] M. Beneke, A. P. Chapovsky, M. Diehl and T. Feldmann, ''Soft collinear effective theory and heavy to light currents beyond leading power,'' Nucl. Phys. B **643**, 431-476 (2002) [arXiv:hep-ph/0206152 [hep-ph]].

[9] G. P. Korchemsky, D. Pirjol and T. M. Yan, Phys. Rev. D **61** (2000) 114510 [hep-ph/9911427].

[10] S. Descotes-Genon and C. T. Sachrajda, Nucl. Phys. B **650** (2003) 356 [hep-ph/0209216].

[11] E. Lunghi, D. Pirjol and D. Wyler, Nucl. Phys. B **649** (2003) 349 [hep-ph/0210091].

[12] S. W. Bosch, R. J. Hill, B. O. Lange and M. Neubert, Phys. Rev. D **67** (2003) 094014 [hep-ph/0301123].

[13] Z. L. Liu and M. Neubert, JHEP **06**, 060 (2020) doi:10.1007/JHEP06(2020)060 [arXiv:2003.03393 [hep-ph]].

[14] B. O. Lange and M. Neubert, Phys. Rev. Lett. **91** (2003) 102001 [hep-ph/0303082].

[15] V. M. Braun, Y. Ji and A. N. Manashov, Phys. Rev. D **100**, no.1, 014023 (2019) doi:10.3204/PUBDB-2019-02451 [arXiv:1905.04498 [hep-ph]].

[16] A. M. Galda, M. Neubert and X. Wang, JHEP **07**, 148 (2022) doi:10.1007/JHEP07(2022)148 [arXiv:2203.08202 [hep-ph]].

[17] A. M. Galda and M. Neubert, Phys. Rev. D **102**, 071501 (2020) doi:10.1103/PhysRevD.102.071501 [arXiv:2006.05428 [hep-ph]].

[18] M. Beneke and J. Rohrwild, *B meson distribution amplitude from $B \to \gamma \ell \bar{\nu}_\ell$,* Eur. Phys. J. C **71** (2011) 1818 [arXiv:1110.3228 [hep-ph]].

[19] M. Beneke, V. M. Braun, Y. Ji and Y. B. Wei, *Radiative leptonic decay $B \to \gamma \ell \nu_\ell$ with subleading power corrections,* JHEP **1807** (2018) 154 [arXiv:1804.04962 [hep-ph]].

[20] B. Y. Cui, Y. L. Shen, C. Wang and Y. B. Wei, Eur. Phys. J. C **85**, no.9, 1052 (2025) doi:10.1140/epjc/s10052-025-14772-z [arXiv:2308.16436 [hep-ph]].

[21] Y. L. Shen, Y. B. Wei, X. C. Zhao and S. H. Zhou, Chin. Phys. C **44**, no.12, 123106 (2020) doi:10.1088/1674-1137/abb6df [arXiv:2009.03480 [hep-ph]].

[22] V. M. Braun and A. Khodjamirian, Phys. Lett. B **718** (2013) 1014 [arXiv:1210.4453 [hep-ph]].

[23] Y. M. Wang, *Factorization and dispersion relations for radiative leptonic B decay,* JHEP **09** (2016) 159 [arXiv:1606.03080 [hep-ph]].

[24] Y. M. Wang and Y. L. Shen, *Subleading-power corrections to the radiative leptonic $B \to \gamma \ell \nu$ decay in QCD,* JHEP **05** (2018) 184 [arXiv:1803.06667 [hep-ph]].

[25] C. D. Lü, Y. L. Shen, C. Wang and Y. M. Wang, Nucl. Phys. B **990**, 116175 (2023) doi:10.1016/j.nuclphysb.2023.116175 [arXiv:2202.08073 [hep-ph]].

[26] S. Descotes-Genon and C. T. Sachrajda, *Universality of nonperturbative QCD effects in radiative B decays,* Phys. Lett. B **557** (2003) 213 [hep-ph/0212162].

[27] S. W. Bosch and G. Buchalla, *The Double radiative decays $B \to \gamma\gamma$ in the heavy quark limit,* JHEP **0208** (2002) 054 [hep-ph/0208202].

[28] Y. L. Shen, Y. M. Wang and Y. B. Wei, JHEP **12**, 169 (2020) doi:10.1007/JHEP12(2020)169 [arXiv:2009.02723 [hep-ph]].

[29] D. H. Li, L. Y. Li, C. D. Lü and Y. L. Shen, Phys. Rev. D **106**, no.9, 094038 (2022) doi:10.1103/PhysRevD.106.094038 [arXiv:2205.05528 [hep-ph]].

[30] I. Adachi *et al.* [Belle and Belle-II], Phys. Rev. D **110**, no.3, L031106 (2024) doi:10.1103/PhysRevD.110.L031106 [arXiv:2405.19734 [hep-ex]].

[31] Q. Qin, Y. L. Shen, C. Wang and Y. M. Wang, Phys. Rev. Lett. **131**, no.9, 091902 (2023) doi:10.1103/PhysRevLett.131.091902 [arXiv:2207.02691 [hep-ph]].

[32] G. Bell, T. Feldmann, Y. M. Wang and M. W. Y. Yip, JHEP **11**, 191 (2013) doi:10.1007/JHEP11(2013)191 [arXiv:1308.6114 [hep-ph]].

[33] S. J. Lee and M. Neubert, Phys. Rev. D **72**, 094028 (2005) doi:10.1103/PhysRevD.72.094028 [arXiv:hep-ph/0509350 [hep-ph]].

[34] P. Ball, V. M. Braun and E. Gardi, Phys. Lett. B **665**, 197-204 (2008) doi:10.1016/j.physletb.2008.06.004 [arXiv:0804.2424 [hep-ph]].

[35] H. N. Li, Y. L. Shen and Y. M. Wang, JHEP **02**, 008 (2013) doi:10.1007/JHEP02(2013)008 [arXiv:1210.2978 [hep-ph]].

[36] H. N. Li, Y. L. Shen and Y. M. Wang, JHEP **01**, 004 (2014) doi:10.1007/JHEP01(2014)004 [arXiv:1310.3672 [hep-ph]].

[37] Y. K. Huang, Y. Ji, Y. L. Shen, C. Wang, Y. M. Wang and X. C. Zhao, Phys. Rev. Lett. **133**, no.17, 171901 (2024) doi:10.1103/PhysRevLett.133.171901 [arXiv:2312.15439 [hep-ph]].

[38] T. Feldmann and J. Matias, JHEP **01**, 074 (2003) doi:10.1088/1126-6708/2003/01/074 [arXiv:hep-ph/0212158 [hep-ph]].

[39] A. Khodjamirian, T. Mannel and Y. M. Wang, JHEP **02**, 010 (2013) doi:10.1007/JHEP02(2013)010 [arXiv:1211.0234 [hep-ph]].

[40] J. Lyon and R. Zwicky, Phys. Rev. D **88**, no.9, 094004 (2013) doi:10.1103/PhysRevD.88.094004 [arXiv:1305.4797 [hep-ph]].

[41] W. S. Hou, M. Kohda and F. Xu, Phys. Rev. D **90**, no.1, 013002 (2014) doi:10.1103/PhysRevD.90.013002 [arXiv:1403.7410 [hep-ph]].

[42] C. Hambrock, A. Khodjamirian and A. Rusov, Phys. Rev. D **92**, no.7, 074020 (2015) doi:10.1103/PhysRevD.92.074020 [arXiv:1506.07760 [hep-ph]].

[43] M. Beneke, T. Feldmann and D. Seidel, *Systematic approach to exclusive $B \to V\ell\ell$, $V\gamma$ decays,* Nucl. Phys. B **612** (2001) 25 [hep-ph/0106067].

[44] M. Beneke, T. Feldmann and D. Seidel, *Exclusive radiative and electroweak $b \to d$ and $b \to s$ penguin decays at NLO,* Eur. Phys. J. C **41** (2005) 173 [hep-ph/0412400].

[45] M. Beneke and T. Feldmann, *Symmetry breaking corrections to heavy to light B meson form-factors at large recoil,* Nucl. Phys. B **592** (2001) 3 [hep-ph/0008255].

[46] C. W. Bauer, D. Pirjol and I. W. Stewart, Phys. Rev. D **67**, 071502 (2003) doi:10.1103/PhysRevD.67.071502 [arXiv:hep-ph/0211069 [hep-ph]].

[47] M. Beneke and T. Feldmann, Nucl. Phys. B **685**, 249-296 (2004) doi:10.1016/j.nuclphysb.2004.02.033 [arXiv:hep-ph/0311335 [hep-ph]].

[48] E. Dalgic, A. Gray, M. Wingate, C. T. H. Davies, G. P. Lepage and J. Shigemitsu, *''B meson semileptonic form-factors from unquenched lattice QCD,''* Phys. Rev. D **73** (2006), 074502; [Erratum: Phys. Rev. D **75** (2007), 119906] [arXiv:hep-lat/0601021 [hep-lat]].

[49] J. A. Bailey *et al.* [Fermilab Lattice and MILC], *''$|V_{ub}|$ from $B \to \pi\ell\nu$ decays and (2+1)-flavor lattice QCD,''* Phys. Rev. D **92** (2015) 014024 [arXiv:1503.07839 [hep-lat]].

[50] J. M. Flynn, T. Izubuchi, T. Kawanai, C. Lehner, A. Soni, R. S. Van de Water and O. Witzel, *'' $B \to \pi\ell\nu$ and $B_s \to K\ell\nu$ form factors and $|V_{ub}|$ from 2+1-flavor lattice QCD with domain-wall light quarks and relativistic heavy quarks,''* Phys. Rev. D **91** (2015) 074510 [arXiv:1501.05373 [hep-lat]].

[51] B. Colquhoun, R. J. Dowdall, J. Koponen, C. T. H. Davies and G. P. Lepage, *''$B \to \pi\ell\nu$ at zero recoil from lattice QCD with physical u/d quarks,''* Phys. Rev. D **93** (2016) 034502 [arXiv:1510.07446 [hep-lat]].

[52] B. Colquhoun *et al.* [JLQCD], *''Form factors of $B \to \pi\ell\nu$ and a determination of $|V_{ub}|$ with Möbius domain-wall fermions,''* Phys. Rev. D **106** (2022) 054502 doi:10.1103/PhysRevD.106.054502 [arXiv:2203.04938 [hep-lat]].

[53] J. A. Bailey *et al.* [Fermilab Lattice and MILC], *''$B \to \pi\ell\ell$ form factors for new-physics searches from lattice QCD,''* Phys. Rev. Lett. **115** (2015) 152002 [arXiv:1507.01618 [hep-ph]].

[54] C. M. Bouchard, G. P. Lepage, C. Monahan, H. Na and J. Shigemitsu, *''$B_s \to K\ell\nu$ form factors from lattice QCD,''* Phys. Rev. D **90** (2014) 054506 [arXiv:1406.2279 [hep-lat]].

[55] F. Bahr *et al.* [ALPHA], *''Continuum limit of the leading-order HQET form factor in $B_s \to K\ell\nu$ decays,''* Phys. Lett. B **757** (2016) 473 [arXiv:1601.04277 [hep-lat]].

[56] C. J. Monahan, C. M. Bouchard, G. P. Lepage, H. Na and J. Shigemitsu, *''Form factor ratios for $B_s \to K\,\ell\,\nu$ and $B_s \to D_s\,\ell\,\nu$ semileptonic decays and $|V_{ub}/V_{cb}|$,''* Phys. Rev. D **98** (2018) 114509 [arXiv:1808.09285 [hep-lat]].

[57] A. Bazavov *et al.* [Fermilab Lattice and MILC], *''$B_s \to K\ell\nu$ decay from lattice QCD,''* Phys. Rev. D **100** (2019) 034501 [arXiv:1901.02561 [hep-lat]].

[58] F. Bahr, D. Banerjee, F. Bernardoni, M. Koren, H. Simma and R. Sommer, *''Extraction of bare form factors for $B_s \to K\ell\nu$ decays in nonperturbative HQET,''* Int. J. Mod. Phys. A **34** (2019) 1950166 [arXiv:1903.05870 [hep-lat]].

[59] C. Bouchard *et al.* [HPQCD], *''Rare decay $B \to K\ell^+\ell^-$ form factors from lattice QCD,''* Phys. Rev. D **88** (2013) 054509; [Erratum: Phys. Rev. D **88** (2013) 079901] [arXiv:1306.2384 [hep-lat]].

[60] J. A. Bailey, A. Bazavov, C. Bernard, C. M. Bouchard, C. DeTar, D. Du, A. X. El-Khadra, J. Foley, E. D. Freeland and E. Gámiz, *et al.* *''$B \to Kl^+l^-$ Decay Form Factors from Three-Flavor Lattice QCD,''* Phys. Rev. D **93** (2016) 025026 [arXiv:1509.06235 [hep-lat]].

[61] W. G. Parrott *et al.* [HPQCD], *''$B \to K$ and $D \to K$ form factors from fully relativistic lattice QCD,''* [arXiv:2207.12468 [hep-lat]].

[62] Y. Aoki *et al.* [Flavour Lattice Averaging Group (FLAG)], *''FLAG Review 2021,''* Eur. Phys. J. C **82** (2022) 869 [arXiv:2111.09849 [hep-lat]].

[63] P. A. Boyle, B. Chakraborty, C. T. H. Davies, T. DeGrand, C. DeTar, L. Del Debbio, A. X. El-Khadra, F. Erben, J. M. Flynn and E. Gámiz, *et al.* *''A lattice QCD perspective on weak decays of b and c quarks Snowmass 2022 White Paper,''* [arXiv:2205.15373 [hep-lat]].

[64] A. Di Canto and S. Meinel, *''Weak Decays of b and c Quarks,''* [arXiv:2208.05403 [hep-ex]].

[65] V. M. Belyaev, A. Khodjamirian and R. Ruckl, *''QCD calculation of the $B \to \pi, K$ form-factors,''* Z. Phys. C **60** (1993) 349 [arXiv:hep-ph/9305348 [hep-ph]].

[66] A. V. Rusov, *''Higher-twist effects in light-cone sum rule for the $B \to \pi$ form factor,''* Eur. Phys. J. C **77** (2017) 442 [arXiv:1705.01929 [hep-ph]].

[67] A. Khodjamirian, R. Ruckl, S. Weinzierl and O. I. Yakovlev, *''Perturbative QCD correction to the $B \to \pi$ transition form-factor,''* Phys. Lett. B **410** (1997) 275 [arXiv:hep-ph/9706303 [hep-ph]].

[68] E. Bagan, P. Ball and V. M. Braun, *''Radiative corrections to the decay $B \to \pi e\nu$ and the heavy quark limit,''* Phys. Lett. B **417** (1998) 154 [arXiv:hep-ph/9709243 [hep-ph]].

[69] P. Ball and R. Zwicky, *''New results on $B \to \pi, K, \eta$ decay formfactors from light-cone sum rules,''* Phys. Rev. D **71** (2005) 014015 [arXiv:hep-ph/0406232 [hep-ph]].

[70] G. Duplancic, A. Khodjamirian, T. Mannel, B. Melic and N. Offen, *''Light-cone sum rules for $B \to \pi$ form factors revisited,''* JHEP **04** (2008) 014 [arXiv:0801.1796 [hep-ph]].

[71] A. Khodjamirian, T. Mannel, N. Offen and Y. M. Wang, *''$B \to \pi\ell\nu_l$ Width and $|V_{ub}|$ from QCD Light-Cone Sum Rules,''* Phys. Rev. D **83** (2011) 094031 [arXiv:1103.2655 [hep-ph]].

[72] A. Bharucha, *''Two-loop Corrections to the $B \to \pi$ Form Factor from QCD Sum Rules on the Light-Cone and $|V_{ub}|$,''* JHEP **05** (2012) 092 [arXiv:1203.1359 [hep-ph]].

[73] A. Khodjamirian, T. Mannel and N. Offen, ''B-meson distribution amplitude from the $B \to \pi$ form-factor,'' Phys. Lett. B **620** (2005) 52 [arXiv:hep-ph/0504091 [hep-ph]].

[74] A. Khodjamirian, T. Mannel and N. Offen, *''Form-factors from light-cone sum rules with B-meson distribution amplitudes,''* Phys. Rev. D **75** (2007) 054013 [arXiv:hep-ph/0611193 [hep-ph]].

[75] F. De Fazio, T. Feldmann and T. Hurth, ''Light-cone sum rules in soft-collinear effective theory,'' Nucl. Phys. B **733** (2006) 1; [Erratum: Nucl. Phys. B **800** (2008), 405] [arXiv:hep-ph/0504088 [hep-ph]].

[76] F. De Fazio, T. Feldmann and T. Hurth, *''SCET sum rules for $B \to P$ and $B \to V$ transition form factors,''* JHEP **02** (2008) 031 [arXiv:0711.3999 [hep-ph]].

[77] Y. M. Wang and Y. L. Shen, *''QCD corrections to $B \to \pi$ form factors from light-cone sum rules,''* Nucl. Phys. B **898** (2015), 563-604 [arXiv:1506.00667 [hep-ph]].

[78] Y. L. Shen, Y. B. Wei and C. D. Lü, *''Renormalization group analysis of $B \to \pi$ form factors with B-meson light-cone sum rules,''* Phys. Rev. D **97** (2018) no.5, 054004 [arXiv:1607.08727 [hep-ph]].

[79] C. D. Lü, Y. L. Shen, Y. M. Wang and Y. B. Wei, *''QCD calculations of $B \to \pi, K$ form factors with higher-twist corrections,''* JHEP **01** (2019), 024 [arXiv:1810.00819 [hep-ph]].

[80] N. Gubernari, A. Kokulu and D. van Dyk, *''$B \to P$ and $B \to V$ Form Factors from B-Meson Light-Cone Sum Rules beyond Leading Twist,''* JHEP **01** (2019) 150 [arXiv:1811.00983 [hep-ph]].

[81] J. Gao, C. D. Lü, Y. L. Shen, Y. M. Wang and Y. B. Wei, *''Precision calculations of $B \to V$ form factors from soft-collinear effective theory sum rules on the light-cone,''* Phys. Rev. D **101** (2020) 074035 [arXiv:1907.11092 [hep-ph]].

[82] Y. L. Shen and Y. B. Wei, *''$B \to P, V$ Form Factors with the B-Meson Light-Cone Sum Rules,''* Adv. High Energy Phys. **2022** (2022) 2755821 [arXiv:2112.01500 [hep-ph]].

[83] Y. M. Wang, Y. B. Wei, Y. L. Shen and C. D. Lü, *''Perturbative corrections to $B \to D$ form factors in QCD,''* JHEP **06** (2017), 062 [arXiv:1701.06810 [hep-ph]].

[84] J. Gao, T. Huber, Y. Ji, C. Wang, Y. M. Wang and Y. B. Wei, *''$B \to D$ form factors beyond leading power and extraction of $|V_{cb}|$ and $R(D)$,''* JHEP **05** (2022) 024 [arXiv:2112.12674 [hep-ph]].

[85] B. Y. Cui, Y. K. Huang, Y. M. Wang and X. C. Zhao, Phys. Rev. D **108**, no.7, L071504 (2023) doi:10.1103/PhysRevD.108.L071504 [arXiv:2301.12391 [hep-ph]].

[86] J. Gao, U. G. Meißner, Y. L. Shen and D. H. Li, Phys. Rev. D **112**, no.1, 1 (2025) doi:10.1103/yvjd-2ymn [arXiv:2412.13084 [hep-ph]].

[87] Y. K. Huang, Y. L. Shen, C. Wang and Y. M. Wang, Phys. Rev. Lett. **134**, no.9, 091901 (2025) doi:10.1103/PhysRevLett.134.091901 [arXiv:2403.11258 [hep-ph]].

[88] A. Bazavov, C. Bernard, N. Brown, C. Detar, A. X. El-Khadra, E. Gámiz, S. Gottlieb, U. M. Heller, J. Komijani and A. S. Kronfeld, *et al.* Phys. Rev. D **98**, no.7, 074512 (2018) doi:10.1103/PhysRevD.98.074512 [arXiv:1712.09262 [hep-lat]].

[89] N. Davidson, T. Przedzinski and Z. Was, Comput. Phys. Commun. **199**, 86-101 (2016) doi:10.1016/j.cpc.2015.09.013 [arXiv:1011.0937 [hep-ph]].

[90] M. Beneke, C. Bobeth and R. Szafron, Phys. Rev. Lett. **120**, no.1, 011801 (2018) doi:10.1103/PhysRevLett.120.011801 [arXiv:1708.09152 [hep-ph]].

[91] M. Beneke, C. Bobeth and R. Szafron, JHEP **10**, 232 (2019) [erratum: JHEP **11**, 099 (2022)] doi:10.1007/JHEP10(2019)232 [arXiv:1908.07011 [hep-ph]].

[92] T. Feldmann, N. Gubernari, T. Huber and N. Seitz, Phys. Rev. D **107**, no.1, 013007 (2023) doi:10.1103/PhysRevD.107.013007 [arXiv:2211.04209 [hep-ph]].

[93] C. Bobeth, M. Gorbahn, T. Hermann, M. Misiak, E. Stamou and M. Steinhauser, Phys. Rev. Lett. **112**, 101801 (2014) doi:10.1103/PhysRevLett.112.101801 [arXiv:1311.0903 [hep-ph]].

[94] M. Beneke, P. Böer, J. N. Toelstede and K. K. Vos, JHEP **11**, 081 (2020) doi:10.1007/JHEP11(2020)081 [arXiv:2008.10615 [hep-ph]].

[95] M. Beneke, P. Böer, G. Finauri and K. K. Vos, JHEP **10**, 223 (2021) doi:10.1007/JHEP10(2021)223 [arXiv:2107.03819 [hep-ph]].

[96] M. Beneke, P. Böer, J. N. Toelstede and K. K. Vos, JHEP **11**, 059 (2021) doi:10.1007/JHEP11(2021)059 [arXiv:2108.05589 [hep-ph]].

[97] C. Cornella, M. König and M. Neubert, Phys. Rev. D **108**, no.3, L031502 (2023) doi:10.1103/PhysRevD.108.L031502 [arXiv:2212.14430 [hep-ph]].

[98] Y. K. Huang, Y. L. Shen, X. C. Zhao and S. H. Zhou, JHEP **10**, 073 (2023) doi:10.1007/JHEP10(2023)073 [arXiv:2301.00697 [hep-ph]].

[99] M. Wirbel, B. Stech and M. Bauer, Z. Phys. C **29**, 637 (1985)

[100] M. Bauer, B. Stech and M. Wirbel, Z. Phys. C **34**, 103 (1987) doi:10.1007/BF01561122

[101] A. Ali, G. Kramer and C. D. Lu, Phys. Rev. D **58**, 094009 (1998) doi:10.1103/PhysRevD.58.094009 [arXiv:hep-ph/9804363 [hep-ph]].

[102] Y. H. Chen, H. Y. Cheng, B. Tseng and K. C. Yang, Phys. Rev. D **60**, 094014 (1999) doi:10.1103/PhysRevD.60.094014 [arXiv:hep-ph/9903453 [hep-ph]].

[103] M. Beneke, G. Buchalla, M. Neubert and C. T. Sachrajda, Phys. Rev. Lett. **83**, 1914-1917 (1999) doi:10.1103/PhysRevLett.83.1914 [arXiv:hep-ph/9905312 [hep-ph]].

[104] M. Beneke, G. Buchalla, M. Neubert and C. T. Sachrajda, Nucl. Phys. B **591**, 313-418 (2000) doi:10.1016/S0550-3213(00)00559-9 [arXiv:hep-ph/0006124 [hep-ph]].

[105] M. Beneke, G. Buchalla, M. Neubert and C. T. Sachrajda, Nucl. Phys. B **606**, 245-321 (2001) doi:10.1016/S0550-3213(01)00251-6 [arXiv:hep-ph/0104110 [hep-ph]].

[106] M. Beneke and M. Neubert, Nucl. Phys. B **675**, 333-415 (2003) doi:10.1016/j.nuclphysb.2003.09.026 [arXiv:hep-ph/0308039 [hep-ph]].

[107] Y. Y. Keum, H. N. Li and A. I. Sanda, Phys. Rev. D **63**, 054008 (2001) doi:10.1103/PhysRevD.63.054008 [arXiv:hep-ph/0004173 [hep-ph]].

[108] C. D. Lu, K. Ukai and M. Z. Yang, Phys. Rev. D **63**, 074009 (2001) doi:10.1103/PhysRevD.63.074009 [arXiv:hep-ph/0004213 [hep-ph]].

[109] M. Beneke, T. Huber and X. Q. Li, Nucl. Phys. B **832**, 109-151 (2010) doi:10.1016/j.nuclphysb.2010.02.002 [arXiv:0911.3655 [hep-ph]].

[110] G. Bell, M. Beneke, T. Huber and X. Q. Li, JHEP **04**, 055 (2020) doi:10.1007/JHEP04(2020)055 [arXiv:2002.03262 [hep-ph]].

[111] M. Beneke and S. Jager, Nucl. Phys. B **768**, 51-84 (2007) doi:10.1016/j.nuclphysb.2007.01.016 [arXiv:hep-ph/0610322 [hep-ph]].

[112] M. Beneke and S. Jager, Nucl. Phys. B **751**, 160-185 (2006) doi:10.1016/j.nuclphysb.2006.06.010 [arXiv:hep-ph/0512351 [hep-ph]].

[113] H. Y. Cheng and K. C. Yang, Phys. Rev. D **78**, 094001 (2008) [erratum: Phys. Rev. D **79**, 039903 (2009)] doi:10.1103/PhysRevD.79.039903 [arXiv:0805.0329 [hep-ph]].

[114] H. Y. Cheng and C. K. Chua, Phys. Rev. D **80**, 114008 (2009) doi:10.1103/PhysRevD.80.114008 [arXiv:0909.5229 [hep-ph]].

[115] H. Y. Cheng and C. K. Chua, Phys. Rev. D **80**, 114026 (2009) doi:10.1103/PhysRevD.80.114026 [arXiv:0910.5237 [hep-ph]].

[116] G. Zhu, Phys. Lett. B **702**, 408-412 (2011) doi:10.1016/j.physletb.2011.07.045 [arXiv:1106.4709 [hep-ph]].

[117] K. Wang and G. Zhu, Phys. Rev. D **88**, 014043 (2013) doi:10.1103/PhysRevD.88.014043 [arXiv:1304.7438 [hep-ph]].

[118] C. Bobeth, M. Gorbahn and S. Vickers, Eur. Phys. J. C **75**, no.7, 340 (2015) doi:10.1140/epjc/s10052-015-3535-1 [arXiv:1409.3252 [hep-ph]].

[119] Q. Chang, J. Sun, Y. Yang and X. Li, Phys. Lett. B **740**, 56-60 (2015) doi:10.1016/j.physletb.2014.11.027 [arXiv:1409.2995 [hep-ph]].

[120] Q. Chang, X. N. Li, J. F. Sun and Y. L. Yang, J. Phys. G **43**, no.10, 105004 (2016) doi:10.1088/0954-3899/43/10/105004 [arXiv:1610.02747 [hep-ph]].

[121] Q. Chang, X. Li, X. Q. Li and J. Sun, Eur. Phys. J. C **77**, no.6, 415 (2017) doi:10.1140/epjc/s10052-017-4980-9 [arXiv:1706.06138 [hep-ph]].

[122] T. Huber and G. Tetlalmatzi-Xolocotzi, Eur. Phys. J. C **82**, no.3, 210 (2022) doi:10.1140/epjc/s10052-022-10068-8 [arXiv:2111.06418 [hep-ph]].

[123] Z. L. Liu and M. Neubert, JHEP **04**, 033 (2020) doi:10.1007/JHEP04(2020)033 [arXiv:1912.08818 [hep-ph]].

[124] Z. L. Liu, B. Mecaj, M. Neubert and X. Wang, JHEP **01**, 077 (2021) doi:10.1007/JHEP01(2021)077 [arXiv:2009.06779 [hep-ph]].

[125] Z. L. Liu, B. Mecaj, M. Neubert and X. Wang, Phys. Rev. D **104**, no.1, 014004 (2021) doi:10.1103/PhysRevD.104.014004 [arXiv:2009.04456 [hep-ph]].

[126] H. Y. Sheng, F. C. Chu, Y. L. Shen and Z. T. Zou, Phys. Rev. D **111**, no.11, 114029 (2025) doi:10.1103/rzn1-2d27 [arXiv:2504.15002 [hep-ph]].

[127] W. Wang, Y. M. Wang, J. Xu and S. Zhao, *B-meson light-cone distribution amplitude from Euclidean quantities,* Phys. Rev. D **102** (2020) 011502 [arXiv:1908.09933 [hep-ph]].

[128] C. Han, W. Wang, J. L. Zhang and J. H. Zhang, Phys. Rev. D **110**, no.9, 094038 (2024) doi:10.1103/PhysRevD.110.094038 [arXiv:2408.13486 [hep-ph]].

[129] Z. F. Deng, W. Wang, Y. B. Wei and J. Zeng, Phys. Rev. D **110**, no.11, 114006 (2024) doi:10.1103/PhysRevD.110.114006 [arXiv:2409.00632 [hep-ph]].

[130] X. Y. Han *et al.* [Lattice Parton], Phys. Rev. D **111**, no.3, 034503 (2025) doi:10.1103/PhysRevD.111.034503 [arXiv:2410.18654 [hep-lat]].

[131] W. Wang, J. Xu, Q. A. Zhang and S. Zhao, [arXiv:2411.07101 [hep-ph]].

[132] T. Guo, C. Han, W. Wang and J. L. Zhang, Phys. Rev. D **112**, no.1, 016013 (2025) doi:10.1103/c97x-z7j9 [arXiv:2505.08611 [hep-ph]].