Burge multipartitions and the AGT correspondence

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Abstract. Burge multipartitions are tuples of partitions that satisfy a cyclic embedding condition. When uncoloured, Burge m-multipartitions give a combinatorial model for characters of the W_m algebras (W_2 is the Virasoro algebra). When n-coloured, they generalise the "cylindric partitions" that provide a combinatorial model (and a crystal graph) for integrable characters of the affine Lie algebra $\widehat{\mathfrak{sl}}(n)$.

Here, we show that the n-coloured Burge m-multipartitions yield the characters of the CFT cosets $\widehat{\mathfrak{gl}}(d)_m/\widehat{\mathfrak{gl}}(d-n)_m$. Having previously shown that the same combinatorial objects give the $SU(m)$ instanton partition functions in $\mathcal{N}=2$ supersymmetric gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$, we have thus established a wide-ranging extension of the AGT correspondence.

This talk is based on a collaboration with N.Macleod (Melbourne).

1. Introduction

In this work, we show how tuples of partitions, subject to certain restrictions, can be used to enumerate the characters of the affine Lie algebra $\widehat{\mathfrak{sl}}(n)$, WZW models of CFT, and various cosets that arise from the WZW models [1]. Based on [2, 3], we explain first how the characters of $\widehat{\mathfrak{sl}}(n)$ and their string functions are enumerated by coloured cylindric partitions. Beyond this, we are particularly interested in tuples of partitions known as Burge multipartitions, these having arisen previously (uncoloured) in the calculation of characters of the Virasoro algebra and the more general W_m algebras [4, 5]. These multipartitions were first defined in [6], generalising the partition pairs defined by [7]. Coloured versions of these multipartitions then arose in the calculation of the $SU(m)$ instanton partition functions in 4D $\mathcal{N}=2$ supersymmetric gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ [8, 9]. There, it was also pointed out that in the subclass for which the multiparitions are cylindric, these functions are WZW characters, thereby generalising the original AGT correspondence [10] which related 4D quiver $\mathcal{N}=2$ supersymmetric gauge theories to Virasoro conformal blocks. Here, we further extend this correspondence by showing how the full class of coloured Burge multipartitions enumerate certain repeated tensor products of $\widehat{\mathfrak{sl}}(n)$ representations, leading then to characters of the coset $\widehat{\mathfrak{gl}}(d)_m/\widehat{\mathfrak{gl}}(d-n)_m$.

In fact, a special case of our result shows that the string functions of level m integrable characters of $\widehat{\mathfrak{sl}}(n)$ are equal to certain n-fold tensor products of level 1 representations of $\widehat{\mathfrak{sl}}(m)$. We show how this level-rank type duality arises from our construction, using it to motivate our more general result. A complete proof of our construction appears in [11].

2. Partitions, multipartitions and characters

A *partition* λ is a non-decreasing sequence of non-negative integers $(\lambda_1, \lambda_2, \ldots)$ such that there exists $\ell \geq 0$ for which $\lambda_i = 0$ for all $i > \ell$. Let $\mathrm{len}(\lambda)$ denote the smallest such ℓ, and call it the *length* of λ. The set of all partitions is denoted Par.

For a *rank* n and a *charge* κ, colour the box at position (i, j) of the diagram of λ with the entry

$$(\kappa - i + j) \bmod n\,. \tag{1}$$

For example, with $n = 5$ and $\kappa = 1$, the partition $\lambda = (97331)$ is coloured as in Fig. 1.

1	2	3	4	0	1	2	3	4
0	1	2	3	4	0	1		
4	0	1						
3	4	0						
2								

Figure 1. Partition (97331) coloured for rank 5 and charge 1

Then define

$$N_\kappa(\lambda) = \left(N_\kappa^0(\lambda), N_\kappa^1(\lambda), \ldots, N_\kappa^{n-1}(\lambda)\right) \tag{2}$$

where $N_\kappa^i(\lambda)$ denotes the number of entries i in the coloured diagram λ with charge κ. For example, $N_1(97331) = (5, 5, 4, 4, 5)$. It will also be useful to define

$$N_\kappa^\Delta(\lambda) = \left(N_\kappa^1(\lambda) - N_\kappa^0(\lambda), N_\kappa^2(\lambda) - N_\kappa^0(\lambda), \ldots, N_\kappa^{n-1}(\lambda) - N_\kappa^0(\lambda)\right). \tag{3}$$

Then $N_1^\Delta(97331) = (0, -1, -1, 0)$.

In what follows, we use a set of parameters $\mathbf{z} = (z_1, z_2, \ldots, z_{n-1})$, and for any vector $N = (N^1, N^2, \ldots, N^{n-1})$, define

$$\mathbf{z}^N = z_1^{N^1} z_2^{N^2} z_2^{N^2} \cdots z_{n-1}^{N^{n-1}}\,. \tag{4}$$

We can then write the characters of the fundamental representations (level 1) of $\widehat{\mathfrak{sl}}(n)$ as generating functions of coloured partitions. Let $\mathfrak{h}$ denote the Cartan subalgebra of $\widehat{\mathfrak{sl}}(n)$, with $\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$ its simple roots, and $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}$ the null root.[1] We set $z_i = e^{-\alpha_i}$ for $0 \le i < n$, and $q = e^{-\delta}$, so that $q = z_0 z_1 \cdots z_{n-1}$. For $\Lambda \in \mathfrak{h}^*$, let $L(\Lambda)$ denote the irreducible representation of $\widehat{\mathfrak{sl}}(n)$ having highest weight Λ, and let $\chi_\Lambda^{\widehat{\mathfrak{sl}}(n)}(q, \mathbf{z})$ denote its character.

Then, if $\{\Lambda_0, \Lambda_1, \ldots, \Lambda_{n-1}\}$ are the fundamental weights of $\widehat{\mathfrak{sl}}(n)$,

$$\chi_{\Lambda_\kappa}^{\widehat{\mathfrak{sl}}(n)}(q, \mathbf{z}) = (q; q)_\infty \sum_{\lambda \in Par} q^{N_\kappa^0(\lambda)}\, \mathbf{z}^{N_\kappa^\Delta(\lambda)} \tag{5}$$

where $(q; q)_\infty = \prod_{k>0}(1 - q^k)$ is the *q-Pochhammer* symbol [13]. In fact, we can eliminate the $(q; q)_\infty$ factor, by instead summing over just the so-called n-regular partitions,[2] but we won't need that expression here.

Below, we give an analogous expression for higher levels. This requires the use of multipartitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)})$.

In what follows, it will be useful to denote the $\widehat{\mathfrak{sl}}(n)$ Dynkin weight $\sum_{i=0}^{n-1} a_i \Lambda_i$ by the symbol $[a_0, a_1, \ldots, a_{n-1}]$. Then, for $n \in \mathbb{Z}_+$ and $m \in \mathbb{Z}_{0+}$, define

$$\begin{aligned} P_{n,m} &= \{[a_0, a_1, \ldots, a_{n-1}] \,:\, a_i \in \mathbb{Z}\,\forall i, a_0 + a_1 + \cdots + a_{n-1} = m\} \\ P_{n,m}^+ &= \{[a_0, a_1, \ldots, a_{n-1}] \in P_{n,m} \,:\, a_i \ge 0\,\forall i\}\,. \end{aligned} \tag{6}$$

[1] See [12] for a comprehensive account of the theory of affine Lie algebras and their representations.
[2] A partition λ is n-regular if $\lambda_i < \lambda_{i+n}$ whenever $0 < i \le \operatorname{len}(\lambda)$.

These are the set of *integral weights* of level m, and the set of *dominant integral weights* of level m. The *class* of a weight $\Lambda = [a_0, a_1, \ldots, a_{n-1}]$ is defined by

$$\operatorname{cls}(\Lambda) = \left(\sum_{i=1}^{n-1} i a_i\right) \bmod n\,. \tag{7}$$

To express the character $\chi_\Lambda^{\widehat{\mathfrak{sl}}(n)}(q, \mathbf{z})$ of highest weight $\Lambda \in P^+_{n,m}$, set $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \ldots, \kappa_m)$ to be a non-decreasing sequence such that

$$\Lambda_{\kappa_1} + \Lambda_{\kappa_2} + \cdots + \Lambda_{\kappa_m} = \Lambda\,. \tag{8}$$

Then for each multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)})$, define the vector

$$N^\Delta_{\boldsymbol{\kappa}}(\boldsymbol{\lambda}) = N^\Delta_{\kappa_1}(\lambda^{(1)}) + N^\Delta_{\kappa_2}(\lambda^{(2)}) + \cdots + N^\Delta_{\kappa_m}(\lambda^{(m)})\,. \tag{9}$$

Then if $N^i_{\boldsymbol{\kappa}}(\boldsymbol{\lambda})$ denotes the number of boxes coloured i throughout the m components of $\boldsymbol{\lambda}$, with $\lambda^{(j)}$ coloured for charge κ_j, the ith component of $N^\Delta_{\boldsymbol{\kappa}}(\boldsymbol{\lambda})$ is $N^i_{\boldsymbol{\kappa}}(\boldsymbol{\lambda}) - N^0_{\boldsymbol{\kappa}}(\boldsymbol{\lambda})$.

Let $\mathcal{C}^\Lambda$ denote the set of all multipartitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)})$ for which

$$\lambda^{(j)}_i \geq \lambda^{(j+1)}_{i+\kappa_{j+1}-\kappa_j} \qquad (i \geq 1, 1 \leq j < m) \tag{10a}$$

$$\lambda^{(m)}_i \geq \lambda^{(1)}_{i+\kappa_1-\kappa_m+n} \qquad (i \geq 1). \tag{10b}$$

Such multipartitions are known as *cylindric multipartitions*. Instead of using these to enumerate the characters of the representations $L(\Lambda)$ of $\widehat{\mathfrak{sl}}(n)$, it is simpler to use the corresponding representations of $\widehat{\mathfrak{gl}}(n)$ which we denote $\hat{L}(\Lambda)$, with their characters denoted by $\chi_\Lambda^{\widehat{\mathfrak{gl}}(n)}(q, \mathbf{z})$. The two are related by

$$\chi_\Lambda^{\widehat{\mathfrak{sl}}(n)}(q, \mathbf{z}) = (q; q)_\infty\, \chi_\Lambda^{\widehat{\mathfrak{gl}}(n)}(q, \mathbf{z})\,. \tag{11}$$

This expression reflects the fact that $\widehat{\mathfrak{gl}}(n) = \mathcal{H} \times \widehat{\mathfrak{sl}}(n)$ where $\mathcal{H}$ is the *Heisenberg algebra*.

We now have [2, 3]

$$\chi_\Lambda^{\widehat{\mathfrak{gl}}(n)} = \sum_{\boldsymbol{\lambda} \in \mathcal{C}^\Lambda} q^{N^0_{\boldsymbol{\kappa}}(\boldsymbol{\lambda})}\, \mathbf{z}^{N^\Delta_{\boldsymbol{\kappa}}(\boldsymbol{\lambda})}\,. \tag{12}$$

In fact, there is defined a set of multipartitions such that $\chi_\Lambda^{\widehat{\mathfrak{sl}}(n)}$ is precisely their generating function. They are known as FLOTW multipartitions (see [3, Prop. 2.11]).

The characters of the WZW models of CFT (see [1, Chapter 15]) are now obtained from the *string functions* $\sigma^\Lambda_M(q)$ of $\chi_\Lambda^{\widehat{\mathfrak{sl}}(n)}$, where also $M \in P^+_{n,m}$. Specifically,

$$\sigma^\Lambda_M(q) = [z_1^{k_1} z_2^{k_2} \cdots z_{n-1}^{k_{n-1}}]\, \chi_N^{\widehat{\mathfrak{sl}}(n)}(q, \mathbf{z}) \tag{13}$$

where, on setting $K = \Lambda - M$, the vector $\mathbf{k} = (k_1, k_2, \ldots, k_{n-1})$ is determined by

$$K = \sum_{i=1}^{n-1} k_i \alpha_i\,. \tag{14}$$

Again, it is simpler to focus instead on the $\widehat{\mathfrak{gl}}(n)$ analogue, which we denote $\hat{\sigma}^\Lambda_M$. From (11),

$$\hat{\sigma}^\Lambda_M(q) = \frac{1}{(q; q)_\infty}\, \sigma^\Lambda_M(q)\,. \tag{15}$$

3. Burge multipartitions

In this section, we introduce a more general class of multipartitions, that enables the evaluation of characters of CFTs that arise from taking tensor products of, and applying the coset construction (see [12, Chap. 12], for example) to, the WZW models.

For $m, n, d \in \mathbb{Z}_+$ with $n \leq d$, let $\Xi \in P^+_{m,d-n}$ and $Z \in P^+_{m,d}$. A *Burge multipartition* $\boldsymbol{\lambda}$ of shape (Ξ, Z) is then a multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)})$ for which

$$\lambda^{(j)}_i \geq \lambda^{(j+1)}_{i+Z_j} - \Xi_j \qquad (0 \leq j < m, i \geq 1) \tag{16}$$

where we identify $\lambda^{(0)} \equiv \lambda^{(m)}$.

The vector $N^{\Delta}_{\boldsymbol{\kappa}}(\boldsymbol{\lambda})$ is then defined by (9) after setting the charges $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \ldots, \kappa_m)$ using

$$\kappa_j = \big(\kappa + \sum_{i=0}^{j-1} (Z_i - \Xi_i)\big) \bmod n \tag{17}$$

where κ is an overall charge (as yet unspecified). Fig. 2 depicts a Burge multipartition of shape (Ξ, Z), along with its charges.

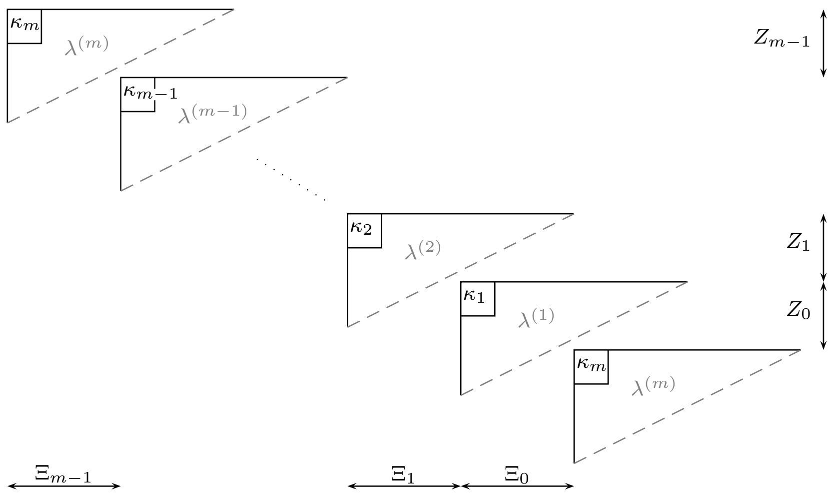

Figure 2. Schematic depiction of Burge multipartition of shape (Ξ, Z)

We then define $\mathcal{C}^{\Xi,Z;n,\kappa}$ to be the set of all such Burge multipartitions, and define its generating function by

$$\Psi^*_{\Xi,Z;n,\kappa}(q; \mathbf{z}) = \sum_{\lambda \in \mathcal{C}^{\Xi,Z;n,\kappa}} q^{N^0_{\boldsymbol{\kappa}}(\boldsymbol{\lambda})} \, \mathbf{z}^{N^{\Delta}_{\kappa}(\boldsymbol{\lambda})} . \tag{18}$$

Moreover, for a vector $\mathbf{k} \in \mathbb{Z}^{n-1}$, we define an analogue of the string function (13) by

$$\psi^{\Xi,Z;n,\kappa}_{\mathbf{k}}(q) = [z_1^{k_1} z_2^{k_2} \cdots z_{n-1}^{k_{n-1}}] \, \Psi^*_{\Xi,Z;n,\kappa}(q; \mathbf{z}) \tag{19}$$

for $\mathbf{k} = (k_1, k_2, \ldots, k_{n-1})$.

From the AGT perspective, the Burge multipartitions are significant because they enumerate the generating function for $SU(m)$ instantons in 4D $\mathcal{N} = 2$ supersymmetric gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$. See [8] for details.

Note that for $\Xi \in P^+_{m,p-m}$ and $Z \in P^+_{m,p'-m}$, the uncoloured ($n = 1$) Burge multipartitions of shape (Ξ, Z) enumerate (p, p')-minimal model characters of the W_m conformal algebra (W_2 being the Virasoro algebra) [5].

Also note that the cylindric multipartitions $\mathcal{C}^\Lambda$ that appear in (12) are special cases of the Burge multipartitions $\mathcal{C}^{\Xi,Z;n,\kappa}$ with $\Xi = 0$ and $Z \in P^+_{m,n}$. Specifically, if Λ is expressed in the form (8), then $Z = (Z_0, \dots, Z_{m-1})$ is determined by $Z_i = \kappa_{i+1} - \kappa_i$ for $0 \le i < m$, having set $\kappa = \kappa_0 = \kappa_m$. The $0 < j < m$ cases of (16) are then equivalent to (10a), while the $j = 0$ case is equivalent to (10b). Moreover, (19) then implies, via (12), (13) and (15), that

$$\psi^{\Xi,Z;n,\kappa}_{\mathbf{k}}(q) = \hat{\sigma}^\Lambda_M(q) \tag{20}$$

where $M = \Lambda - K$ with K given by (14).

4. Tensor products

For $m, n, d \in \mathbb{Z}_+$ with $n \le d$, $H \in P^+_{m,n}$, $\Xi \in P^+_{m,d-n}$, define, for each $Z \in P^+_{m,d}$, the q-series $\hat{b}^Z_{H,\Xi}(q)$ such that

$$\chi^{\widehat{\mathfrak{gl}}(m)}_{\Xi}(q,\mathbf{z})\,\chi^{\widehat{\mathfrak{gl}}(m)}_{H^{(1)}}(q,\mathbf{z})\,\chi^{\widehat{\mathfrak{gl}}(m)}_{H^{(2)}}(q,\mathbf{z})\cdots\chi^{\widehat{\mathfrak{gl}}(m)}_{H^{(n)}}(q,\mathbf{z}) = \sum_{Z\in P^+_{m,d}} \hat{b}^Z_{H,\Xi}(q)\,\chi^{\widehat{\mathfrak{gl}}(m)}_{Z}(q,\mathbf{z}) \tag{21}$$

where $H^{(1)}, H^{(2)}, \dots, H^{(n)} \in P^+_{m,1}$ are such that $H = \sum_{i=1}^n H^{(i)}$. The coefficients $\hat{b}^Z_{H,\Xi}(q)$ here are known as *branching coefficients*. The coefficient of q^k therein gives the multiplicity of the $\widehat{\mathfrak{gl}}(m)$ representation $\hat{L}(Z - k\delta)$ in the $(n+1)$-fold tensor product $\hat{L}(\Xi) \otimes \hat{L}(H^{(1)}) \otimes \hat{L}(H^{(2)}) \otimes \cdots \otimes \hat{L}(H^{(n)})$.

The following result, which uses notation defined subsequently, shows that the branching coefficients $\hat{b}^Z_{H,\Xi}(q)$ are given by enumerating certain Burge multipartitions:

Theorem 4.1 *For $m, n, d \in \mathbb{Z}_+$ with $n \le d$, let $H \in P^+_{m,n}$, $\Xi \in P^+_{m,d-n}$ and $Z \in P^+_{m,d}$. Then,*

$$\hat{b}^Z_{H,\Xi}(q) = \begin{cases} q^{|\tilde{\theta}|_n}\,\psi^{\Xi,Z;n,\kappa}_{\mathbf{k}}(q) & \textit{if } \operatorname{cls}(Z) = \operatorname{cls}(\Xi + H), \\ 0 & \textit{otherwise}, \end{cases} \tag{22}$$

where $\eta = \operatorname{par}(H)$, $\theta = \operatorname{par}(Z - \Xi)$, $\kappa = (|\theta| - |\eta|)/m$ and, having written $\theta = (\theta_1, \theta_2, \dots, \theta_m)$, we set $\tilde{\theta} = (\theta_1 - \kappa, \theta_2 - \kappa, \dots, \theta_m - \kappa)$. The vector $\mathbf{k} = (k_1, k_2, \dots, k_{n-1})$ is then determined by (14) on setting $K = \operatorname{aff}_n(\tilde{\theta} - \eta)$.

Given an element $\Pi \in P_k$ with $\Pi = [\Pi_0, \Pi_1, \dots, \Pi_{k-1}]$, we define $\operatorname{par}(\Pi)$ to be the k-element sequence $\operatorname{par}(\Pi) = (\pi_1, \pi_2, \dots, \pi_k)$ by

$$\pi_j = \sum_{i=j}^{k-1} \Pi_i \tag{23}$$

for $1 \le j \le k$. Note that if $\pi = \operatorname{par}(\Pi)$ then $\pi_k = 0$ always. Moreover, if $\Pi \in P^+_{k,\ell}$, then π is non-increasing and thus defines a partition with at most $k - 1$ parts. Moreover, $\pi_1 \le \ell$ in this case.

Now, given a sequence $\pi = (\pi_1, \pi_2, \dots, \pi_k)$ and $\ell \in \mathbb{Z}_+$, we define $\operatorname{aff}_\ell(\pi) \in P^+_{\ell,m}$ by

$$\operatorname{aff}_\ell(\pi) = \sum_{i=1}^m \Lambda_{-\pi_i} \tag{24}$$

where the indices are taken modulo ℓ.

Typically, we combine the above two maps to obtain a map from $P_{k,\ell}$ to $P^+_{\ell,k}$. Specifically, if $\Pi \in P_{k,\ell}$ and $\pi = \mathrm{par}(\Pi)$ then $\mathrm{aff}_\ell(\pi) \in P^+_{\ell,k}$. This map then provides a rank-level dual. However, it is not a bijection. This is also the case if we restrict the domain to $P^+_{k,\ell}$.

Given a sequence $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$, its norm $|\pi|$ is defined by $|\pi| = \sum_{i=1}^k \pi_i$. In addition, for $\ell \in \mathbb{Z}_+$, we also define a generalised norm $|\pi|_\ell$ by

$$|\pi|_\ell = \sum_{i=1}^{k} |\pi_i|_\ell \tag{25}$$

where for each $p \in \mathbb{Z}$, the value $|p|_\ell$ is defined by first setting $t, r \in \mathbb{Z}$ such that $p = \ell t + r$ with $0 \le r < \ell$, and then setting

$$|p|_\ell = \frac{1}{2}\ell t(t-1) + rt\,. \tag{26}$$

Note that if $0 \le p \le \ell$ then $|p|_\ell = 0$.

Note that Theorem 4.1 can be made more general by performing automorphisms on the $\widehat{\mathfrak{sl}}(n)$ weights. This more general result will be presented explicitly elsewhere [11].

Combining the special case of Theorem 4.1 for which $\Xi = 0$ with (20) leads to the following result which shows that the branching functions for n-fold tensor products of level 1 representations of $\widehat{\mathfrak{gl}}(m)$ are given by the string functions of level m representations of $\widehat{\mathfrak{gl}}(n)$.

Corollary 4.2 *Let $m, n \in \mathbb{Z}_+$.*
If $H, Z \in P^+_{m,n}$ with $\mathrm{cls}(Z) = \mathrm{cls}(H)$. Then

$$\hat{b}^Z_{H,0}(q) = q^{|\tilde{\theta}|_n}\, \hat{\sigma}^N_M(q) \tag{27}$$

where after setting $\eta = \mathrm{par}(H)$ and $\theta = \mathrm{par}(Z)$ with $\theta = (\theta_1, \theta_2, \ldots, \theta_m)$, we define $\kappa = (|\theta| - |\eta|)/m$ and $\tilde{\theta} = (\theta_1 - \kappa, \theta_2 - \kappa, \ldots, \theta_m - \kappa)$. Then $N, M \in P^+_{n,m}$ are defined by $N = \mathrm{aff}_n(\tilde{\theta})$ and $M = \mathrm{aff}_n(\eta)$.

On the other hand, if $N, M \in P^+_{n,m}$ with $\mathrm{cls}(N) = \mathrm{cls}(M)$ then

$$\hat{\sigma}^N_M(q) = q^{|\tilde{\mu}|_m}\, \hat{b}^Z_{H,0}(q) \tag{28}$$

where, after setting $\nu = \mathrm{par}(N)$ and $\mu = \mathrm{par}(M)$, then with $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and $\delta = (|\mu| - |\nu|)/n$, defining $\tilde{\mu} = (\mu_1 - \delta, \mu_2 - \delta, \ldots, \mu_n - \delta)$, $Z = \mathrm{aff}_m(\nu)$ and $H = \mathrm{aff}_m(\tilde{\mu})$.

Note that this result is easily converted to an equivalent result for $\widehat{\mathfrak{sl}}(m)$ and $\widehat{\mathfrak{sl}}(n)$. In analogy with (21), define, for each $Z \in P^+_{m,d}$, the branching coefficients $b^Z_{H,\Xi}(q)$ such that

$$\chi^{\widehat{\mathfrak{sl}}(m)}_{\Xi}(q,\mathbf{z})\, \chi^{\widehat{\mathfrak{sl}}(m)}_{H^{(1)}}(q,\mathbf{z})\, \chi^{\widehat{\mathfrak{sl}}(m)}_{H^{(2)}}(q,\mathbf{z}) \cdots \chi^{\widehat{\mathfrak{sl}}(m)}_{H^{(n)}}(q,\mathbf{z}) = \sum_{Z \in P^+_{m,d}} b^Z_{H,\Xi}(q)\, \chi^{\widehat{\mathfrak{sl}}(m)}_{Z}(q,\mathbf{z})\,. \tag{29}$$

Using (12) then implies that

$$\hat{b}^Z_{H,\Xi}(q) = \frac{1}{(q;q)^n_\infty}\, b^Z_{H,\Xi}(q)\,. \tag{30}$$

This along with (15) enables the conversion of (27) and (28).

5. Explicit coset decomposition

Combining Theorem 4.1 with Corollary 4.2 now enables us to give explicit decompositions of $\widehat{\mathfrak{gl}}(d)$ string functions that respect the coset factorisation $\widehat{\mathfrak{gl}}(d)_m/\widehat{\mathfrak{gl}}(d-n)_m$.

First notice that if $Z, A \in P^+_{m,d}$, and $B \in P^+_{m,d-n}$ and $H \in P^+_{m,n}$ are such that $A = B + H$ then (21) implies that

$$\hat{b}^Z_{A,0}(q) = \sum_{\Xi \in P^+_{m,d-n}} \hat{b}^\Xi_{B,0}(q)\, \hat{b}^Z_{H,\Xi}(q)\,. \tag{31}$$

Then, on using (27) to convert each of $\hat{b}^Z_{A,0}(q)$ and $\hat{b}^\Xi_{B,0}(q)$ to a string function leads to an expression of the desired form. The coefficients $\hat{b}^Z_{H,\Xi}(q)$ in the expansion are then generating functions of Burge multipartitions by (22).

As an example, consider the string function $\hat{\sigma}^{[52000]}_{[11203]}(q)$ in the case $m = 7$, $d = 5$. Using (28) followed by (31) in the $n = 3$ case results in

$$\begin{aligned}\hat{\sigma}^{[52000]}_{[11203]}(q) &= q^3\, \hat{b}^{[4000010]}_{[2001110],[0000000]}(q)\\ &= q^3 \sum_{\Xi \in P^+_{7,2}} \hat{b}^\Xi_{[1001000],[0000000]}(q) \cdot \hat{b}^{[4000010]}_{[1000110],\Xi}(q)\,.\end{aligned} \tag{32}$$

In view of (22), the summation here can be restricted further to those Ξ for which $\mathrm{cls}(\Xi) = 3$ because other terms are zero. This then restricts the summation to the four terms

$$\begin{aligned}\hat{\sigma}^{[52000]}_{[11203]}(q) &= q^3\, \hat{b}^{[0110000]}_{[1001000],[0000000]}(q) \cdot \hat{b}^{[4000010]}_{[1000110],[0110000]}(q)\\ &\quad + q^3\, \hat{b}^{[1001000]}_{[1001000],[0000000]}(q) \cdot \hat{b}^{[4000010]}_{[1000110],[1001000]}(q)\\ &\quad + q^3\, \hat{b}^{[0000101]}_{[1001000],[0000000]}(q) \cdot \hat{b}^{[4000010]}_{[1000110],[0000101]}(q)\\ &\quad + q^3\, \hat{b}^{[0000020]}_{[1001000],[0000000]}(q) \cdot \hat{b}^{[4000010]}_{[1000110],[0000020]}(q)\,.\end{aligned} \tag{33}$$

Finally, applying (27) to the first factor in each of these summands, and (22) to the second factor then yields

$$\begin{aligned}\hat{\sigma}^{[52000]}_{[11203]}(q) &= q^3\, \hat{\sigma}^{[61]}_{[43]}(q) \cdot \psi^{[0110000],[4000010];3,2}_{(0,1)}(q)\\ &\quad + q^3\, \hat{\sigma}^{[43]}_{[43]}(q) \cdot \psi^{[1001000],[4000010];3,2}_{(0,2)}(q)\\ &\quad + q^4\, \hat{\sigma}^{[25]}_{[43]}(q) \cdot \psi^{[0000101],[4000010];3,1}_{(0,2)}(q)\\ &\quad + q^5\, \hat{\sigma}^{[07]}_{[43]}(q) \cdot \psi^{[0000020],[4000010];3,1}_{(0,2)}(q)\end{aligned} \tag{34}$$

corresponding to the formal coset decomposition

$$\widehat{\mathfrak{gl}}(5)_7 \sim \widehat{\mathfrak{gl}}(2)_7 \times \frac{\widehat{\mathfrak{gl}}(5)_7}{\widehat{\mathfrak{gl}}(2)_7}\,. \tag{35}$$

6. Conclusion

In this note, we have shown that the n-coloured Burge multipartition generating functions $\psi^{\Xi,Z;n,\kappa}_{\mathbf{k}}(q)$ are the tensor product branching functions $\hat{b}^Z_{H,\Xi}(q)$, and have indicated how they lead to characters of the coset $\widehat{\mathfrak{gl}}(d)_m/\widehat{\mathfrak{gl}}(d-n)_m$. Having previously shown [8, 9] that these same generating functions enumerate $SU(m)$ instantons in 4D $\mathcal{N} = 2$ supersymmetric gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$, we have found thus an extensive generalisation of the AGT correspondence. A more detailed exposition along with complete proofs appear in [11].

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