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Gauge-Invariant Lagrangian Formulations for Mixed-Symmetry Higher-Spin Bosonic Fields in AdS Spaces

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Abstract: We deduce a non-linear commutator higher-spin (HS) symmetry algebra which encodes unitary irreducible representations of the AdS group—subject to a Young tableaux $Y(s_1, \dots, s_k)$ with $k \geq 2$ rows—in a d-dimensional anti-de Sitter space. Auxiliary representations for a deformed non-linear HS symmetry algebra in terms of a generalized Verma module, as applied to additively convert a subsystem of second-class constraints in the HS symmetry algebra into one with first-class constraints, are found explicitly in the case of a $k = 2$ Young tableaux. An oscillator realization over the Heisenberg algebra for the Verma module is constructed. The results generalize the method of constructing auxiliary representations for the symplectic $sp(2k)$ algebra used for mixed-symmetry HS fields in flat spaces [Buchbinder, I.L.; et al. Nucl. Phys. B 2012, 862, 270–326]. Polynomial deformations of the $su(1,1)$ algebra related to the Bethe ansatz are studied as a byproduct. A nilpotent BRST operator for a non-linear HS symmetry algebra of the converted constraints for $Y(s_1, s_2)$ is found, with non-vanishing terms (resolving the Jacobi identities) of the third order in powers of ghost coordinates. A gauge-invariant unconstrained reducible Lagrangian formulation for a free bosonic HS field of generalized spin (s_1, s_2) is deduced. Following the results of [Buchbinder, I.L.; et al. Phys. Lett. B 2021, 820, 136470.; Buchbinder, I.L.; et al. arXiv 2022, arXiv:2212.07097], we develop a BRST approach to constructing general off-shell local cubic interaction vertices for irreducible massive higher-spin fields (being candidates for massive particles in the Dark Matter problem). A new reducible gauge-invariant Lagrangian formulation for an antisymmetric massive tensor field of spin $(1, 1)$ is obtained.

Keywords: higher spin field theory; anti-de-Sitter spaces; Lagrangian formulation; BRST operator; cubic interaction vertices; Verma nodules; Non-linear algebras

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1. Introduction

The great interest in higher-spin field theory is mainly explained by a hope to re-examine the problems of a unified description for the variety of elementary particles and all of the known interactions beyond the Standard Model, as well as by a possible insight into the origin of Dark Matter [1,2] beyond the scope of models with sterile neutrinos [3] or vector massive fields [4]. Massive higher-spin fields in constant-curvature spaces are legitimate candidates for Dark Matter, providing a contribution to the total energy density at the level of about 27% in a universe with the geometry of a 4D de Sitter space; see [5,6] for a review. Such a model calls for a study involving the compactification of extra dimensions after the Big Bang. In the case of a Big Bang singularity, this indicates that the physics is no longer described by classical gravity but rather by a quantum version of gravity involving higher-spin fields [7], or, perhaps, a superstring field theory model is required. The problems of Dark Matter and Dark Energy provide insight into the inflation theory, which consistently describes the post-Planck classical early universe and plays a crucial role in explaining the late-time cosmic acceleration; for a review, see, for example, ref. [8]

and the references therein. Note that the simplest candidate for Dark Energy is given by a positive-valued Einstein cosmological constant Λ, constructed along with Cold Dark Matter, which has proved to be quite successful in describing a large span of observational data related to polarization anisotropies in Cosmic Microwave Background Radiation; see, for example, refs. [9–11].

Should these expectations prove to be viable, it appears that the development of higher-spin field theory will be of high relevance in view of its close relation to superstring theory in constant-curvature spaces, which operates an infinite set of massive and massless bosonic and fermionic HS fields subject to a multi-row Young tableaux (YT) $Y(s_1, \ldots, s_k)$, $k \geq 1$; for a review, see, for example, refs. [12–18] and the references therein.

A description of such theories, aimed at the final purpose of Lagrangian formulation, demands some advanced and sophisticated group-theoretical techniques, related to constructing different representations of the (super)algebras underlying the theories in question. Whereas for a Lie (super)algebra—being the case relevant to HS fields in flat spaces—the deduction and the structure of such ingredients as Verma modules and generalized Verma modules [19] are rather well established, the similar case of non-linear algebraic and superalgebraic structures—which corresponds to HS fields in AdS spaces—has not been classified until now, with the exception of totally symmetric bosonic [20,21] and fermionic [22,23] HS fields.

There are two well-established approaches to constructing gauge-invariant Lagrangians for higher-spin fields by using the respective metric and frame-like formulations. In the first approach, the fields remain unconstrained, i.e., all the conditions (d'Alembert equation, Young symmetry, absence of trace and divergence) that select an irreducible Poincaré [24–26] or (A)dS [27–29] group representation of a given mass and spin are implemented on equal footing to obtain a Lagrangian. In the second approach, some of the conditions (normally, those related to the trace) for the fields and gauge parameters are implemented in a consistent manner which is imposed by hand, outside the construction of a Lagrangian. At the level of free fields, there exist two efficient approaches to the above objectives, known as BRST methods (initially developed to quantize gauge field theories [30–32], constrained dynamical systems using the BFV–BRST procedure [33–35]), with respective complete (e.g., [36–38]) and incomplete [39] BRST operators (implied by String Field Theory [40,41]), whose Lagrangian descriptions for one and the same higher-spin field in a d-dimensional Minkowski space-time are shown to be equivalent [42]. For HS fields in AdS spaces, descriptions of irreducible representations (in the metric-like formalism) for the (A)dS group with both integer and half-integer spins are rather different from those for the Poincaré group in a flat space-time, even for free theories. In all of the known cases, Lagrangian descriptions for one and the same higher-spin field deduced using the constrained (incomplete) BRST approach with additional holonomic constraints and the approach with a complete BRST operator do not coincide. Successful Lagrangian descriptions for integer and half-integer spins in AdS spaces using the approach of a complete BRST operator are suggested for massless and massive particles of integer spins in [21,37], and for massive particles of half-integer spins in [22]. Some problems of this approach have not been solved (in view of the yet unknown form of holonomic constraints consistent with an incomplete BRST operator), even for free fields of a given higher spin.[1]

The present article develops the research line initiated by [44], and its objectives are achieved using a regular method of constructing a Verma module and a Fock space for quadratic algebras, whose negative/positive-root vectors in a Cartan-like triangular decomposition are entangled due to the presence of a special parameter r, being the inverse square of the AdS radius. The resulting ingredients allow one to obtain Lagrangian formulations (LFs) for free mixed-symmetric integer HS fields in a d-dimensional AdS space with $Y(s_1, \ldots, s_k)$ by using the Fronsdal metric-like formalism [45–47] in the BRST approach. Such an LF is the starting point for an interacting HS field theory in the scope of conventional Quantum Field Theory. An application of the BRST construction with a complete BRST operator to a free HS field theory in AdS spaces consists of several stages and solves a *problem being inverse* to that of the method [33–35], thus reflecting the concept of BV–BFV duality [48–50] as a particular case of the AKSZ model[2] [52]. First of all, the conditions that determine representations of a given mass and spin are regarded as a topological (having no Hamiltonian) gauge system of mixed-class operator constraints, o_I, $I = 1, 2, \ldots$, in an auxiliary Fock space $\mathcal{H}$. Second, the entire system of o_I, which forms a quadratic commutator algebra, is additively converted (see [53,54] for the conversion methods) within a deformed (in powers of the parameter r) algebra of O_I, $O_I = o_I + o'_I$, defined in a larger Fock space, $\mathcal{H} \bigotimes \mathcal{H}'$, with first-class constraints O_α, $O_\alpha \subset O_I$. Third, one needs to find a Hermitian and nilpotent BRST operator Q' for a non-linear algebra of converted operators O_I, which contains a BRST operator Q for the subsystem O_α. Fourth, a Lagrangian action $\mathcal{S}$ is constructed for a given HS field through an inner product $\langle\ |K|\ \rangle$ in a total Hilbert space $\mathcal{H}_{tot}$, $\mathcal{S} \sim \langle\chi|KQ|\chi\rangle$ so that $\mathcal{S}$ obeys reducible gauge transformations $\delta|\chi\rangle = Q|\Lambda\rangle$, with $|\chi\rangle$ containing the initial and auxiliary fields. As a result, the corresponding equations of motion are to reproduce the initial AdS group conditions.

The above algorithm, as applied to bosonic [55,56] and fermionic [57,58] HS fields in flat spaces, does not encounter any problems in view of a linear Lie structure of the initial constraint algebra $[o_I, o_J\} = f^K_{IJ} o_K$ with structure constants f^K_{IJ}. Indeed, for the same algebra of additional parts o'_I, it is sufficient to use the construction of a Verma module (VM) for the integer HS symmetry algebra $sp(2k)$ [59], which is in one-to-one correspondence with the Lorentz $so(1, d-1)$ algebra of irreducible unitary representations subject to $Y(s_1, \ldots, s_k)$, $k \leq \left[\frac{d-2}{2}\right]$, for massless fields, due to the Howe duality [60]. Then, an oscillator realization of the symplectic algebra $sp(2k)$ in a Fock space $\mathcal{H}'$ [59] (orthosymplectic algebra $osp(1|2k)$ for half-integer massless field representations [61]) has a polynomial form as compared to totally symmetric HS fields in AdS spaces [22,62], where the (super)algebras of o_I are non-linear and different from those of o'_I [20,21,23]. A problem of equal complexity arises in constructing a BRST operator Q' for the algebra of converted constraints O_I, which, in the transition to AdS spaces, does not have a form, being quadratic in ghost coordinates $\mathcal{C}$, and requires a complete analysis of the relations, starting from a resolution of the Jacobi identities; see [63] for the details of finding a relevant operator Q' and [64] for classical quadratic algebras.

The details of a Lagrangian description for mixed-symmetry massless HS tensors on (A)dS backgrounds have been studied in a "frame-like" formulation [65–69], whereas such descriptions for mixed-symmetry massive bosonic fields with off-shell traceless constraints in the (A)dS case are known for a Young tableaux with two rows [70–72], the main result for Lagrangian formulation being [72]. In a metric-like formalism, (non-Lagrangian) equations of motion from the viewpoint of a BRST description with an incomplete BRST operator for arbitrary massless HS fields is discussed in [73,74], whereas in the BRST–BV approach with an incomplete BRST operator and also in the light-cone approach, mixed-symmetric HS fields are examined in [75–78]. Diverse aspects of mixed-symmetry HS field Lagrangian dynamics in Minkowski spaces are discussed in [79] and the references therein, and the case of interacting mixed-symmetry HS fields in AdS spaces is considered in [80,81]. In the case of cubic interaction (see [82–92] for the study of cubic vertices in different approaches), irreducible higher-spin fields in flat spaces are classified by Metsaev [93] using the light-cone formalism. We intend to follow the application of the BRST approach to Lagrangian cubic interaction vertices for irreducible totally symmetric bosonic fields in flat spaces [38,94–96], however, as applied to HS fields (including mixed-symmetric ones) in AdS spaces.

The article is devoted to solving the following problems. First, we deduce an HS symmetry algebra for bosonic HS fields in a d-dimensional AdS space subject to an arbitrary $Y(s_1, \ldots, s_k)$; second, we develop the Verma module construction for an HS symmetry algebra, with a two-row YT $Y(s_1, s_2)$ and an oscillator realization for a given non-linear algebra as a formal power series in the oscillators of the related Heisenberg algebra; third, we construct a BRST operator for a converted HS symmetry algebra and an unconstrained LF for free bosonic HS fields in AdS spaces subject to $Y(s_1, s_2)$; and fourth, we develop a procedure for constructing cubic interaction vertices by using three copies of massive HS fields in AdS spaces.

The article is organized as follows. In Section 2, we examine bosonic HS fields, which includes, in Section 2.1, the construction of an HS symmetry algebra $\mathcal{A}(Y(k), AdS_d)$ for HS fields subject to a Young tableaux with k rows; then an auxiliary theorem on a deformation of polynomial general commutator algebras under an additive conversion procedure is presented in Section 2.2. In Section 3, we deduce a manifest form of an HS symmetry algebra of the additional parts for arbitrary HS fields, formulate the problem of Verma module construction for the algebra $\mathcal{A}'(Y(k), AdS_d)$ in Section 3.1 and solve it explicitly for a quadratic algebra $\mathcal{A}'(Y(2), AdS_d)$ of the additional parts o'_I in Section 3.2. We find a Fock space realization for $\mathcal{A}'(Y(2), AdS_d)$ in Section 3.3. In Section 4, we manifestly deduce a non-linear algebra $\mathcal{A}_c(Y(k), AdS_d)$ of converted operators O_I in Section 4.1; we also present the form of a BRST operator for this algebra with $k = 2$ in Section 4.2 and develop an unconstrained Lagrangian formulation for massive bosonic HS fields with a two-row Young tableaux $Y(s_1, s_2)$ in Section 4.3. The general concept of finding cubic interaction vertices for massive HS fields in AdS spaces, using the approach with a complete BRST operator, is examined in Section 5. Some examples of fields with lower spins are presented in Section 6. In the Conclusion section, we summarize our results and discuss some open problems. In Appendix A, we prove a certain proposition, and then in Appendix B, we consider its application to a polynomial algebra related to the Bethe ansatz. In respective Appendices C.1 and D, we present calculations for the Verma module construction with the algebra $\mathcal{A}'(Y(2), AdS_d)$ in terms of a series of lemmas and also give its oscillator representation. Finally, in Appendix E, we prove that the resulting Lagrangian does indeed reproduce appropriate conditions for a field to determine an irreducible representation of the AdS group.

As a rule, we apply the notation of conventions of [38], and also the respective notation $\epsilon(F)$, $gh(F)$, $[H, G\}$, $[x]$, $\vec{s}_k$, $(\vec{s})^3$ for the Grassmann parity and ghost number of a homogeneous quantity F, as well as for a supercommutator, the integer part of a real-valued x, an integer-valued vector $(s_1, s_2, \dots, s_k)$, and a triple $(\vec{s}^1_{k_1}, \vec{s}^2_{k_2}, \vec{s}^3_{k_3})$.

2. HS Fields of Integer Spin in AdS Spaces

In this section, we obtain a number of special HS symmetry non-linear algebras which encode mixed-symmetry tensor fields as elements of irreducible AdS group representations with a generalized spin $\mathbf{s} = (s_1, \dots, s_k)$ and mass m in AdS_d space-times. We examine the problem of a Verma module construction for one of the algebras and solve it explicitly for a non-linear algebra with a two-row Young tableaux, as well as for a polynomial algebra of order $(n-1)$. The construction of a Fock space representation for a non-linear algebra with a resulting Verma module finalizes the solution of the problem under consideration.

2.1. HS Symmetry Algebra $\mathcal{A}(Y(k), AdS_d)$ for Mixed-Symmetry Tensor Fields with $Y(s_1, \dots, s_k)$

A massive generalized integer spin $\mathbf{s} = (s_1, \dots, s_k)$ $(s_1 \geq s_2 \geq \dots \geq s_k > 0, k \leq [d/2])$ AdS group irreducible representation in an AdS_d space is realized on mixed-symmetry tensors $\Phi_{\mu^1(s_1),\mu^2(s_2),\dots,\mu^k(s_k)} \equiv \Phi_{\mu^1_1\dots\mu^1_{s_1},\mu^2_1\dots\mu^2_{s_2},\dots,\mu^k_1\dots\mu^k_{s_k}}(x)$, which correspond to the Young tableaux

$$\Phi_{\mu^1(s_1),\mu^2(s_2),\dots,\mu^k(s_k)} \longleftrightarrow \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline \mu^1_1 & \mu^1_2 & \cdot\ \cdot\ \cdot & \cdot & \cdot & \cdot & \cdot & \mu^1_{s_1} \\ \hline \mu^2_1 & \mu^2_2 & \cdot\ \cdot\ \cdot & \cdot & \cdot & \mu^2_{s_2} & & \\ \hline \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot & \cdot & & \\ \hline \mu^k_1 & \mu^k_2 & \cdot\ \cdot\ \cdot & \cdot & \mu^k_{s_k} & & & \\ \hline \end{array} \qquad (1)$$

subject to the Klein–Gordon (2), divergence-free (3), tracelessness (4), and mixed-symmetry (5) equations for $\beta = (2; 3; \dots; k+1) \Longleftrightarrow (s_1 > s_2; s_1 = s_2 > s_3; \dots; s_1 = s_2 = \dots = s_k)$ [28], namely,

$$\{\nabla^2 + r[(s_1 - \beta - 1 + d)(s_1 - \beta) - \sum_{i=1}^{k} s_i] + m^2\}\Phi_{\mu^1(s_1),\mu^2(s_2),\ldots,\mu^k(s_k)} = 0, \tag{2}$$

$$\nabla^{\mu^i_{l_i}}\Phi_{\mu^1(s_1),\mu^2(s_2),\ldots,\mu^k(s_k)} = 0, \quad i,j = 1,\ldots,k; \quad l_i, m_i = 1,\ldots,s_i\,, \tag{3}$$

$$g^{\mu^i_{l_i}\mu^i_{m_i}}\Phi_{\mu^1(s_1),\mu^2(s_2),\ldots,\mu^k(s_k)} = g^{\mu^i_{l_i}\mu^j_{m_j}}\Phi_{\mu^1(s_1),\mu^2(s_2),\ldots,\mu^k(s_k)} = 0, \quad l_i < m_i,\,, \tag{4}$$

$$\Phi_{\mu^1(s_1),\ldots,\{\mu^i(s_i)\underbrace{,\ldots,\mu^j_1\ldots}\mu^j_{l_j}\}\ldots\mu^j_{s_j},\ldots\mu^k(s_k)} = 0, \quad i<j,\ 1 \le l_j \le s_j, \tag{5}$$

where the underlined brackets denote the fact that the indices inside do not take part in symmetrization, i.e., the present symmetrization involves only the indices $\mu^i(s_i), \mu^j_{l_j}$ in $\{\mu^i(s_i)\underbrace{,\ldots,\mu^j_1\ldots}\mu^j_{l_j}\}$.

To obtain the HS symmetry algebra of o_I for a description of all integer HS fields, we introduce, in a standard manner, a Fock space $\mathcal{H}$ generated by k pairs of bosonic creation $a^i_{\mu^i}(x)$ and annihilation $a^{j+}_{\nu^j}(x)$ operators,[3] $i,j = 1,\ldots,k, \mu^i, \nu^j = 0, 1\ldots, d-1$,

$$[a^i_{\mu^i}, a^{j+}_{\nu^j}] = -g_{\mu^i\nu^j}\delta^{ij}\,, \qquad \delta^{ij} = diag(1,1,\ldots 1)\,, \tag{6}$$

and a set of constraints for an arbitrary string-like vector $|\Phi\rangle \in \mathcal{H}$, which we call a basic vector,

$$|\Phi\rangle = \sum_{s_1=0}^{\infty}\sum_{s_2=0}^{s_1}\cdots\sum_{s_k=0}^{s_{k-1}} \frac{\imath^{s_1+\ldots+s_k}}{s_1!\ldots s_k!}\Phi_{\mu^1(s_1),\mu^2(s_2),\ldots,\mu^k(s_k)}(x)\prod_{i=1}^{k}\prod_{l_i=1}^{s_i} a_i^{+\mu^i_{l_i}}|0\rangle, \tag{7}$$

$$\tilde{l}_0|\Phi\rangle = \big(l_0 + \tilde{m}_b^2 + r\big((g^1_0 - 2\beta - 2)g^1_0 - \sum_{i=2}^{k} g^i_0\big)\big)|\Phi\rangle = 0, \quad l_0 = [D^2 - r\tfrac{d(d-2(k+1))}{4}], \tag{8}$$

$$\big(l^i, l^{ij}, t^{i_1j_1}\big)|\Phi\rangle = \big(-ia^i_\mu D^\mu, \tfrac{1}{2}a^i_\mu a^{j\mu}, a^{i_1+}_\mu a^{j_1\mu}\big)|\Phi\rangle = 0, \quad i \le j;\ i_1 < j_1, \tag{9}$$

with particle number operators, a central charge, and a covariant derivative in $\mathcal{H}$, respectively,[4]

$$g^i_0 = -\frac{1}{2}\{a^{i+}_\mu, a^{\mu i}\}, \qquad \tilde{m}_b^2 = m^2 + r\beta(\beta+1), \tag{10}$$

$$D_\mu = \partial_\mu - \omega^{ab}_\mu(x)\Big(\sum_i a^+_{ia}a_{ib}\Big), \qquad a_i^{\mu(+)}(x) = e^\mu_a(x)a_i^{a(+)}, \tag{11}$$

with a vielbein e^μ_a, a spin connection ω^{ab}_μ, and tangent indices a, b, $a = 0, 1\ldots, d-1$. The operator D_μ is equivalent (as applied in $\mathcal{H}$) to the covariant derivative ∇_μ with the d'Alembertian $D^2 = (D_a + \omega^b{}_{ba})D^a$. The set of $k(k+1)$ primary constraints (8) and (9) with $\{o_\alpha\} = \big\{\tilde{l}_0, l^i, l^{ij}, t^{i_1j_1}\big\}$ is equivalent to Equations (2)–(5) for all admissible values of spin, and for the field $\Phi_{\mu^1(s_1),\mu^2(s_2),\ldots,\mu^k(s_k)}$ with a fixed spin $\mathbf{s} = (s_1, s_2, \ldots, s_k)$, once, in addition to (8), (9), we have to add k more constraints with g^i_0,

$$g^i_0|\Phi\rangle = (s_i + \frac{d}{2})|\Phi\rangle. \tag{12}$$

The condition for the algebra of o_α to be closed under the [,]-multiplication leads to an enlargement of o_α by adding the operators g^i_0 and the Hermitian conjugates o^+_α,

$$\big(l^{i+},\ l^{ij+},\ t^{i_1j_1+}\big) = \big(-ia^{i+}_\mu D^\mu,\ \tfrac{1}{2}a^{i+}_\mu a^{j\mu+},\ a^{i_1}_\mu a^{j_1\mu+}\big),\ i \le j;\ i_1 < j_1, \tag{13}$$

with respect to the inner product in $\mathcal{H}$,

$$\langle\Psi|\Phi\rangle = \int d^dx\sqrt{|g|}\sum_{i=1}^{k}\sum_{s_i=0}^{s_{i-1}}\frac{(-1)^{s_1+\ldots+s_k}}{s_1!\ldots s_k!}\Psi^*_{\mu^1(s_1),\mu^2(s_2),\ldots,\mu^k(s_k)}(x)\Phi^{\mu^1(s_1),\mu^2(s_2),\ldots,\mu^k(s_k)}(x). \quad (14)$$

This fact guarantees the Hermiticity of a corresponding BFV–BRST operator, with the self-conjugate operators $(l_0^+,\ g_0^{i\,+}) = (l_0,\ g_0^i)$ taken into account (hence a real value of the Lagrangian $\mathcal{L}$) for the system of all the operators $\{o_\alpha, o_\alpha^+, g_0^i\}$. We call the algebra of these operators an *integer higher-spin (integer HS) symmetry algebra in an AdS space, with a Young tableaux with k rows*[5] and denote it as $\mathcal{A}(Y(k), AdS_d)$.

The maximal Lie subalgebra of the operators $l^{ij}, t^{i_1j_1}, g_0^i, l^{ij+},\ t^{i_1j_1+}$ is isomorphic to the symplectic algebra $sp(2k)$ (see [59] for details; later on, we refer to it as $sp(2k)$), whereas the only nontrivial quadratic commutators in $\mathcal{A}(Y(k), AdS_d))$ are due to the operators containing D_μ: $l^i, \tilde{l}_0, l^{i+}$. For the purpose of an LF construction, it is sufficient to have a simplified (having no central charge $\tilde{m}_b^2$), so-called *modified*, HS symmetry algebra $\mathcal{A}_m(Y(k), AdS_d)$ with the operator l_0 (8) instead of $\tilde{l}_0$ so that the AdS mass term $\tilde{m}_b^2 + r\big((g_0^1 - 2\beta - 2)g_0^1 - \sum_{i=2}^k g_0^i\big)$ is restored later on, as usual, within a conversion procedure and an appropriate LF construction.

The algebra $\mathcal{A}_m(Y(k), AdS_d))$ of the operators o_I, from the viewpoint of a Hamiltonian analysis for dynamical systems, contains the first-class constraint l_0, as well as the $2k$ differential l_i, l_i^+ and $2k^2$ algebraic $t^{i_1j_1}, t^+_{i_1j_1}, l^{ij}, l^+_{ij}$ second-class constraints o_a, and also the operators g_0^i, composing an invertible matrix $\Delta_{ab}(g_0^i)$ for a topological (having a zero Hamiltonian) gauge system due to

$$[o_a,\ o_b] = f^c_{ab}o_c + f^{cd}_{ab}o_co_d + \Delta_{ab}(g_0^i),\ [o_a,\ l_o] = f^c_{a[l_0]}o_c + f^{cd}_{a[l_0]}o_co_d. \quad (15)$$

Here, $f^c_{ab}, f^{cd}_{ab}, f^c_{a[l_0]}, f^{cd}_{a[l_0]}, \Delta_{ab}$ are antisymmetric with respect to the permutations of the lower indices; the constant quantities and the quantities $\Delta_{ab}(g_0^i)$ form a non-vanishing $2k(k+1)\times 2k(k+1)$ matrix $\|\Delta_{ab}\|$ in the Fock space $\mathcal{H}$ on the surface $\Sigma\subset\mathcal{H}$, $\|\Delta_{ab}\|_{|\Sigma}\neq 0$, which is determined by the equation $o_\alpha|\Phi\rangle = 0$. The set of o_I satisfies non-linear relations, being additional to those for $sp(2k)$ and given by the multiplication Table 1.

Table 1. The HS symmetry non-linear algebra $\mathcal{A}_m(Y(k), AdS_d)$.

$[\downarrow,\rightarrow\}$	$t^{i_1j_1}$	$t^+_{i_1j_1}$	l_0	l^i	l^{i+}	$l^{i_1j_1}$	$l^{i_1j_1+}$
$t^{i_2j_2}$	$A^{i_2j_2,i_1j_1}$	$B^{i_2j_2}{}_{i_1j_1}$	0	$l^{j_2}\delta^{i_2i}$	$-l^{i_2+}\delta^{j_2i}$	$l^{\{j_1j_2}\delta^{i_1\}i_2}$	$-l^{i_2\{i_1+}\delta^{j_1\}j_2}$
$t^+_{i_2j_2}$	$-B^{i_1j_1}{}_{i_2j_2}$	$A^+_{i_1j_1,i_2j_2}$	0	$l_{i_2}\delta^i_{j_2}$	$-l^+_{j_2}\delta^i_{i_2}$	$l_{i_2}{}^{\{j_1}\delta^{i_1\}}_{j_2}$	$-l_{j_2}{}^{\{j_1+}\delta^{i_1\}}_{i_2}$
l_0	0	0	0	$-r\mathcal{K}_1^{i+}$	$r\mathcal{K}_1^i$	0	0
l^j	$-l^{j_1}\delta^{i_1j}$	$-l^{i_1}\delta^{j_1j}$	$r\mathcal{K}_1^{j+}$	W_b^{ji}	X_b^{ji}	0	$-\frac{1}{2}l^{\{i_1+}\delta^{j_1\}j}$
l^{j+}	$l^{i_1+}\delta^{j_1j}$	$l^{j_1+}\delta^{i_1j}$	$-r\mathcal{K}_1^j$	$-X_b^{ij}$	$-W_b^{ji+}$	$\frac{1}{2}l^{\{i_1}\delta^{j_1\}j}$	0
$l^{i_2j_2}$	$-l^{j_1\{j_2}\delta^{i_2\}i_1}$	$-l^{i_1\{i_2}\delta^{j_2\}j_1}$	0	0	$-\frac{1}{2}l^{\{i_2}\delta^{j_2\}i}$	0	$L^{i_2j_2,i_1j_1}$
$l^+_{i_2j_2}$	$l^{i_1+}_{\{i_2}\delta^{j_1}_{j_2\}}$	$l^+_{j_1\{j_2}\delta_{i_2\}i_1}$	0	$\frac{1}{2}l^+_{\{i_2}\delta^i_{j_2\}}$	0	$-L^{i_1j_1,i_2j_2}$	0
g_0^j	$-F^{i_1j_1,j}$	$F^{i_1j_1,j+}$	0	$-l^i\delta^{ij}$	$l^{i+}\delta^{ij}$	$-l^{j\{i_1}\delta^{j_1\}j}$	$l^{j\{i_1+}\delta^{j_1\}j}$

First of all, note that Table 1 does not include any columns with $[\ ,\ \}$-products of all o_I with g_0^j, which can be observed from the rows containing g_0^j, as follows, $[b, o_I\}^+ = -[o_I^+, b\}$, with allowance for the closure of the HS algebra with respect to the Hermitian conjugation. Second, the operators $t^{i_2j_2}, t^+_{i_2j_2}$ are defined so as to satisfy the following properties:

$$(t^{i_2j_2}, t^+_{i_2j_2}) \equiv (t^{i_2j_2}, t^+_{i_2j_2})\theta^{j_2i_2},\ \theta^{j_2i_2} = (1,0)\ \texttt{for}\ (j_2 > i_2, j_2 \le i_2), \tag{16}$$

with the Heaviside θ-symbol[6] θ^{ji}. Third, the products $B^{i_2j_2}_{i_1j_1}, A^{i_2j_2,i_1j_1}, F^{i_1j_1,i}, L^{i_2j_2,i_1j_1}$ are determined by the explicit relations

$$\begin{aligned} B^{i_2j_2}{}_{i_1j_1} &= (g_0^{i_2} - g_0^{j_2})\delta^{i_2}_{i_1}\delta^{j_2}_{j_1} + (t_{j_1}{}^{j_2}\theta^{j_2}{}_{j_1} + t^{j_2+}{}_{j_1}\theta_{j_1}{}^{j_2})\delta^{i_2}_{i_1} - (t^{+}_{i_1}{}^{i_2}\theta^{i_2}{}_{i_1} + t^{i_2}{}_{i_1}\theta_{i_1}{}^{i_2})\delta^{j_2}_{j_1}, && (17)\\ A^{i_2j_2,i_1j_1} &= t^{i_1j_2}\delta^{i_2j_1} - t^{i_2j_1}\delta^{i_1j_2}, \qquad F^{i_2j_2,i} = t^{i_2j_2}(\delta^{j_2i} - \delta^{i_2i}), && (18)\\ L^{i_2j_2,i_1j_1} &= \tfrac{1}{4}\Big\{\delta^{i_2i_1}\delta^{j_2j_1}\Big[2g_0^{i_2}\delta^{i_2j_2} + g_0^{i_2} + g_0^{j_2}\Big] - \delta^{j_2\{i_1}\Big[t^{j_1\}i_2}\theta^{i_2j_1\}} + t^{i_2j_1\}+}\theta^{j_1\}i_2}\Big] \\ &\quad -\delta^{i_2\{i_1}\Big[t^{j_1\}j_2}\theta^{j_2j_1\}} + t^{j_2j_1\}+}\theta^{j_1\}j_2}\Big]\Big\}. && (19) \end{aligned}$$

They satisfy the obvious additional properties of antisymmetry and Hermitian conjugation

$$\begin{aligned} &A^{i_2j_2,i_1j_1} = -A^{i_1j_1,i_2j_2}, \qquad A^+_{i_1j_1,i_2j_2} = (A_{i_1j_1,i_2j_2})^+ = t^+_{i_2j_1}\delta_{j_2i_1} - t^+_{i_1j_2}\delta_{i_2j_1}, && (20)\\ &(L^{i_2j_2,i_1j_1})^+ = L^{i_1j_1,i_2j_2}, \qquad F^{i_2j_2,i+} = (F^{i_2j_2,i})^+ = t^{i_2j_2+}(\delta^{j_2i} - \delta^{i_2i}), && (21)\\ &B^{i_2j_2}{}_{i_1j_1}{}^+ = (g_0^{i_2} - g_0^{j_2})\delta^{i_2}_{i_1}\delta^{j_2}_{j_1} + (t^{+}_{j_1}{}^{j_2}\theta^{j_2}{}_{j_1} + t^{j_2}{}_{j_1}\theta_{j_1}{}^{j_2})\delta^{i_2}_{i_1} - (t_{i_1}{}^{i_2}\theta^{i_2}{}_{i_1} + t^{i_2+}{}_{i_1}\theta_{i_1}{}^{i_2})\delta^{j_2}_{j_1}. && (22) \end{aligned}$$

Fourth, the independent quantities $\mathcal{K}_1^k, W_b^{ki}, X_b^{ki}$ in Table 1 are quadratic in o_I:

$$W_b^{ij} = [l^i, l^j] = 2r\Big[(g_0^j - g_0^i)l^{ij} - (\sum_m t^{m[j}\theta^{[jm} + t^{[jm+}\theta^{m[j})l^{i]m}\Big], \tag{23}$$

$$\mathcal{K}_1^k = r^{-1}[l_0, l^{k+}] = \Big(4\sum_i l^{ki+}l^i + l^{k+}(2g_0^k - 1) - 2(\sum_i l^{i+}t^{+ik}\theta^{ki} + l^{i+}t^{ki}\theta^{ik})\Big), \tag{24}$$

$$\begin{aligned} X_b^{ij} &= \{l_0 + r(K_0^{0i} + \sum_{l=i+1}^{k}\mathcal{K}_0^{il} + \sum_{l=1}^{i-1}\mathcal{K}_0^{li})\}\delta^{ij} && (25)\\ &\quad -r\big[4\textstyle\sum_l l^{jl+}l^{li} - \displaystyle\sum_{l=1}^{j-1} t^{lj+}t^{li} - \sum_{l=i+1}^{k} t^{il+}t^{jl} - \sum_{l=j+1}^{i-1} t^{li}t^{jl} + (g_0^j + g_0^i - j - 1)t_{ji}\big]\theta^{ij} \\ &\quad -r\big[4\textstyle\sum_l l^{jl+}l^{li} - \displaystyle\sum_{l=1}^{i-1} t^{lj+}t^{li} - \sum_{l=j+1}^{k} t^{il+}t^{jl} - \sum_{l=i+1}^{j-1} t^{il+}t^{lj+} + t^+_{ij}(g_0^j + g_0^i - i - 1)\big]\theta^{ji}. \end{aligned}$$

In (25), we used the quantities $K_0^{0i}, i, j = 1, \ldots, k, \mathcal{K}_0^{ij}$, composing a Casimir operator $\mathcal{K}_0(k)$ for the $sp(2k)$ algebra:

$$\mathcal{K}_0(k) = \sum_i K_0^{0i} + 2\sum_{i,j}\mathcal{K}_0^{ij}\theta^{ji} = \sum_i\big((g_0^i)^2 - 2g_0^i - 4l^{ii+}l^{ii}\big) + 2\Big(\sum_{i=1}^{k}\sum_{j=i+1}^{k}(t^+_{ij}t^{ij} - 4l^+_{ij}l^{ij} - g_0^j)\Big). \tag{26}$$

The algebra $\mathcal{A}_m(Y(k), AdS_d)$ may be regarded as a non-linear deformation (in powers of r) of the integer HS symmetry algebra $\mathcal{A}(Y(k), R^{1,d-1})$ [59] in Minkowski space, namely,

$$\mathcal{A}_m(Y(k), AdS_d) = \mathcal{A}(Y(k), R^{1,d-1})(r) = \Big(T^k \oplus T^{k*} \oplus l_0\Big)(r) \supsetplus sp(2k), \tag{27}$$

for a k-dimensional commutative (in $R^{1,d-1}$) algebra $T^k = \{l_i\}$ and its dual $T^{k*} = \{l^{i+}\}$, which represents a semidirect sum of the symplectic algebra $sp(2k)$, being an algebra of internal derivations of $(T^k \oplus T^{k*})$. For HS fields with a single spin s_1, $k = 1$, and a two-component spin (s_1, s_2), $k = 2$, the algebra $\mathcal{A}_m(Y(k), AdS_d)$ coincides with the respective familiar HS symmetry algebras in AdS spaces given by [20,63].

Now, we utilize the results of a special theorem, and then proceed to a conversion of the algebra of o_I so as to obtain the algebra of O_I with first-class constraints alone.

2.2. On Additive Conversion for Polynomial Algebras

In this subsection, having in mind the problem of an additive conversion for non-linear algebras (superalgebras) with a subset of second-class constraints, we need to use an important statement based on the following (see [23] for a detailed description).

Definition 1. *A non-linear commutator superalgebra $\mathcal{A}$ of basis elements o_I, $I \in \Delta$ (with Δ being a finite or infinite set of indices) is called a* ***polynomial superalgebra of order*** n*,* $n \in \mathbf{N}$ *if the set $\{o_I\}$ is subject to the n-th order polynomial supercommutator relations*

$$[o_I, o_J\} = F^K_{IJ}(o)o_K,\ F^K_{IJ} = f^{(1)K}_{IJ} + \sum_{n=2}^{\infty} f^{(n)K_1\ldots K_{n-1}K}_{IJ} \prod_{i=1}^{n-1} o_{K_i}\,,$$
$$f^{(n)K_1\ldots K_n}_{IJ} \neq 0, \text{ and } f^{(k)K_1\ldots K_n K_{n+1}\ldots K_k}_{IJ} = 0,\ k > n, \tag{28}$$

with structure coefficients $f^{(n)K_1\ldots K_{n-1}K}_{IJ}$ of generalized antisymmetry with respect to the permutations of lower indices, $f^{(n)K_1\cdots K_n}_{IJ} = -(-1)^{\varepsilon(o_I)\varepsilon(o_J)} f^{(n)K_1\cdots K_n}_{JI}$.

Now, we are in a position to formulate our basic statement of Section 2.2, presented as the following.

Proposition 1. *Let $\mathcal{A}$ be a polynomial superalgebra of order n with basis elements o_I defined in a Hilbert space $\mathcal{H}$. Then, for a set $\mathcal{A}'$ of elements o'_I defined in a new Hilbert space $\mathcal{H}'$ ($\mathcal{H} \bigcap \mathcal{H}' = \emptyset$) and supercommuting with o_I, as well as for a direct sum of sets $\mathcal{A}_c = \mathcal{A} + \mathcal{A}'$ of the operators O_I, $O_I = o_I + o'_I$, defined in the tensor product $\mathcal{H} \otimes \mathcal{H}'$, the requirement to be in involution*

$$[O_I, O_J\} = F^K_{IJ}(o', O)O_K\,, \tag{29}$$

implies that the sets $\{o'_I\}$ and $\{O_I\}$ form the respective polynomial commutator superalgebra $\mathcal{A}'$ of order n and the non-linear commutator superalgebra $\mathcal{A}_c$ with the composition laws

$$[o'_I, o'_J\} = f^{(1)K_1}_{IJ} o'_{K_1} + \sum_{m=2}^{n} (-1)^{m-1+\varepsilon_{K_{(m)}}} f^{(m)K_m\cdots K_1}_{IJ} \prod_{s=1}^{m} o'_{K_s}, \tag{30}$$

$$[O_I, O_J\} = \Big(f^{(1)K}_{IJ} + \sum_{m=2}^{n} F^{(m)K}_{IJ}(o', O)\Big)O_K, \tag{31}$$

for the values $\varepsilon_{K_{(n)}}$, $\varepsilon_{K_{(n)}} = \sum_{s=1}^{n-1} \varepsilon_{K_s}\left(\sum_{l=s+1}^{n} \varepsilon_{K_l}\right)$. The structure functions $F^{(m)k}_{IJ}(o', O)$ in (30) are constructed with respect to the coefficients $f^{(m)K_1\ldots K_m}_{IJ} \equiv f^{K_1\ldots K_m}_{IJ}$ used in (28) as follows:

$$F^{(m)K}_{IJ} = f^{K_1\cdots K_m}_{IJ} \prod_{p=1}^{m-1} O_{K_p} + \sum_{s=1}^{m-1} (-1)^{s+\varepsilon_{K(s)}} f^{\widehat{K_s\cdots K_1}\widehat{K_{s+1}\cdots K_m}}_{ij} \prod_{p=1}^{s} o'_{K_p} \prod_{l=s+1}^{m-1} O_{K_l}, \quad \texttt{where} \tag{32}$$

$$f^{\widehat{K_s\cdots K_1}\widehat{K_{s+1}\cdots K_m}}_{ij} = f^{K_s\cdots K_1 K_{s+1}\cdots K_m}_{ij} + f^{K_s\cdots K_{s+1}K_1K_{s+2}\cdots K_m}_{ij}(-1)^{\varepsilon_{K_{s+1}}\varepsilon_{K_1}} + \cdots + f^{K_{s+1}K_s\cdots K_1K_{s+2}\cdots K_m}_{ij}(-1)^{\varepsilon_{K_{s+1}}\sum_{l=1}^{s}\varepsilon_{K_l}} + \Big(f^{K_{s+1}K_s\cdots K_{s+2}K_1K_{s+3}\cdots K_m}_{ij}(-1)^{\varepsilon_{K_{s+2}}\varepsilon_{K_1}} + \cdots + f^{K_{s+1}K_{s+2}K_s\cdots K_1K_{s+3}\cdots K_m}_{ij}(-1)^{\varepsilon_{K_{s+2}}\sum_{l=1}^{s}\varepsilon_{K_l}}\Big)(-1)^{\varepsilon_{K_{s+1}}\sum_{l=1}^{s}\varepsilon_{K_l}} + \cdots + (-1)^{\sum_{l=s+1}^{m}\varepsilon_{K_l}\sum_{p=1}^{s}\varepsilon_{K_p}} f^{K_{s+1}\cdots K_mK_s\cdots K_1}_{ij}, \tag{33}$$

where the sum in (33) contains $\frac{m!}{s!(m-s)!}$ *terms, with all the possible ways of arrangement for* $(K_{s+1},\ldots,K_m)$ *among the indices* $(K_s,\ldots,K_1)$ *in* $f^{K_s\cdots K_1K_{s+1}\cdots K_m}_{ij}$, *without changing the separate ordering of the indices*[7] $K_{s+1},\ldots,K_m$ *and* $K_s,\ldots,K_1$.

The correctness of the proposition is examined in Appendix A. Turning to the structure of the superalgebra $\mathcal{A}_c$, we emphasize that, in contrast to $\mathcal{A}$ and $\mathcal{A}'$, we call it a *non-homogeneous polynomial superalgebra of order n*, due to the form of relations (32); see Note 15 in Appendix A for remarks. In the case of a Lie superalgebra ($n = 1$), the structures of superalgebras $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}_c$ are identical, and used as the integer HS symmetry algebra $\mathcal{A}(Y(k), R^{1,d-1})$ [59]. For quadratic algebras ($n = 2$), the algebraic relations for $\mathcal{A}^{n-1}$, $\mathcal{A}'^{n-1}$ and $\mathcal{A}^{n-1}_c$ do not coincide with each other, due to the presence of structure functions $f^{(2)K_1K_2}_{IJ}$, which is originally shown for the algebra $\mathcal{A}(Y(1), AdS_d)$ with totally symmetric HS tensors in an AdS space [21], having the form

$$[o_I, o_J] = f^{(1)K_1}_{IJ} o_{K_1} + f^{(2)K_1K_2}_{IJ} o_{K_1}o_{K_2}, \quad [o'_I, o'_J] = f^{(1)K_1}_{IJ} o'_{K_1} - f^{(2)K_2K_1}_{IJ} o'_{K_1}o'_{K_2}, \tag{34}$$

$$[O_I, O_J] = \Big(f^{(1)K}_{IJ} + F^{(2)K}_{IJ}(o', O)\Big)O_K, \quad F^{(2)K}_{IJ} = f^{(2)K_1K}_{IJ} O_{K_1} - (f^{(2)K_1K}_{IJ} + f^{(2)KK_1}_{IJ})o'_{K_1}. \tag{35}$$

The relations (34) and (35) are sufficient to determine the form of multiplication laws for the algebra of the additional parts o'_I, $\mathcal{A}'(Y(k), AdS_d)$ and for the converted O_I, operator algebra $\mathcal{A}_c(Y(k), AdS_d)$.

As a new result, we present an explicit form for the cubic commutator algebras $\mathcal{A}'$, $\mathcal{A}_c$,

$$\begin{aligned} [o'_I, o'_J\} &= f^{(1)K_1}_{IJ} o'_{K_1} - f^{(2)K_2K_1}_{IJ} o'_{K_1}o'_{K_2} + f^{(3)K_3K_2K_1}_{IJ} o'_{K_1}o'_{K_2}o'_{K_3}, \\ [O_I, O_J\} &= \Big(f^{(1)K}_{IJ} + \sum_{l=2}^{n} F^{(l)K}_{IJ}(o', O)\Big)O_K, \end{aligned} \tag{36}$$

$$\begin{aligned} F^{(3)K}_{IJ} &= f^{(3)K_1K_2K}_{IJ} O_{K_1}O_{K_2} - (f^{(3)K_1K_2K}_{IJ} + f^{(3)K_2K_1K}_{IJ})o'_{K_1}O_{K_2} \\ &\quad + (f^{(3)K_2K_1K}_{IJ} + f^{(3)K_2KK_1}_{IJ} + f^{(3)KK_2K_1}_{IJ})o'_{K_1}o'_{K_2}, \end{aligned} \tag{37}$$

if the commutator relations for the initial algebra $\mathcal{A}$ are given by (28) for $n = 3$.

In Appendix B, we consider the example of a polynomial 3-parametric algebra $\mathcal{A}$ of order $n-1$, $n \in \mathbf{N}$, being a polynomial deformation of the $su(1,1)$ algebra for $n > 2$.

3. Auxiliary HS Symmetry Algebra $\mathcal{A}'(Y(k), AdS_d)$

The procedure of additive conversion for the non-linear HS symmetry algebra $\mathcal{A}(Y(k), AdS_d)$ of the operators o_I implies establishing, first of all, an explicit form of the algebra $\mathcal{A}'(Y(k), AdS_d)$ for the additional parts o'_I, and second, a representation of $\mathcal{A}'(Y(k), AdS_d)$ in terms of appropriate Heisenberg algebra elements acting in a new Fock space $\mathcal{H}'$. The structure of non-linear commutators of the initial algebra implies a need for converting all the operators o_I in order to construct an unconstrained LF for a given HS field $\Phi_{(\mu^1)_{s_1},(\mu^2)_{s_2},\ldots,(\mu^k)_{s_k}}$.

The former step is based on determining a multiplication table $\mathcal{A}'(Y(k), AdS_d)$ for the operators o'_I, following the structure of the algebra $\mathcal{A}(Y(k), AdS_d)$, given by (34) and Table 1.

As a result, the required composition law for $\mathcal{A}'(Y(k), AdS_d)$ is the same as the one for the algebra $\mathcal{A}(Y(k), AdS_d)$ in its linear Lie part, i.e., for the $sp(2k)$ subalgebra of elements $(l'^{ij}, l'^{ij+}, t'^{i_1 j_1}, t'^{+}_{i_1 j_1}, g'^{i}_0)$, and is different in the non-linear part of Table 1, determined by the isometry group elements l'_i, l'^{+}_j, l'_0. The corresponding non-linear submatrix of the multiplication matrix for $\mathcal{A}'(Y(k), AdS_d)$ has the form given by Table 2.

Table 2. The non-linear part of the algebra $\mathcal{A}'(Y(k), AdS_d)$.

$[\downarrow, \rightarrow\}$	l'_0	l'^i	l'^{i+}
l'_0	0	$r\mathcal{K}'^{bi+}_1$	$-r\mathcal{K}'^{i}_1$
l'^j	$-r\mathcal{K}'^{j+}_1$	$-W'^{ji}_b$	X'^{ji}_b
l'^{j+}	$r\mathcal{K}'^{j}_1$	$-X'^{ij}_b$	W'^{ji+}_b

Here, the functions $\mathcal{K}'^{i+}_1, \mathcal{K}'^{i}_1, W'^{ji}_b, W'^{ji+}_b$ $(X'^{ij}_b - l'_0)$ have the same definition as those in (23)–(25) for the initial operators o_I, albeit with an opposite sign for $(X'^{ij}_b - l'_0)$,

$$W'^{ij}_b = 2r\left[(g'^{j}_0 - g'^{i}_0)l'^{ij} - (\sum_m t'^{m[j}\theta^{[jm} + t'^{[jm+}\theta^{m[j})l'^{i]m}\right], \tag{38}$$

$$\mathcal{K}'^{j}_1 = \Big(4\sum_i l'^{ji+}l'^{i} + l'^{j+}(2g'^{j}_0 - 1) - 2(\sum_i l'^{i+}t'^{+ij}\theta^{ji} + l'^{i+}t'^{ji}\theta^{ij})\Big), \tag{39}$$

$$\begin{aligned} X'^{ij}_b = &\{l'_0 - r\big(K'^{0i}_0 + \sum_{l=i+1}^{k}\mathcal{K}'^{il}_0 + \sum_{l=1}^{i-1}\mathcal{K}'^{li}_0\big)\}\delta^{ij} \\ &+r\{4\textstyle\sum_l l'^{jl+}l'^{li} - \displaystyle\sum_{l=1}^{j-1}t'^{lj+}t'^{li} - \sum_{l=i+1}^{k}t'^{il+}t'^{jl} - \sum_{l=j+1}^{i-1}t'^{li}t'^{jl} + (g'^{j}_0 + g'^{i}_0 - j - 1)t'_{ji}\}\theta^{ij} \\ &+r\{4\textstyle\sum_l l'^{jl+}l'^{li} - \displaystyle\sum_{l=1}^{i-1}t'^{lj+}t'^{li} - \sum_{l=j+1}^{k}t'^{il+}t'^{jl} - \sum_{l=i+1}^{j-1}t'^{il+}t'^{lj+} + t'^{+}_{ij}(g'^{j}_0 + g'^{i}_0 - i - 1)\}\theta^{ji}. \end{aligned} \tag{40}$$

Now, we can outline the points leading to an oscillator representation for the elements of an auxiliary HS symmetry algebra.

3.1. On Representations of $\mathcal{A}'(Y(k), AdS_d)$

Here, we assume that a generalization of the Poincaré–Birkhoff–Witt theorem for the second-order algebra $\mathcal{A}'(Y(k), AdS_d)$ does indeed hold true (for a generalization of the PBW theorem involving a quadratic algebra, see [19]), and so we begin to construct a Verma module based on a Cartan-like decomposition[8] extended from that of $sp(2k)$ $(i \leq j, l < m)$,

$$\mathcal{A}'(Y(k), AdS_d) = \{l'^{+}_{ij}, t'^{+}_{lm}, l'^{+}_i\} \oplus \{g'^{i}_0, l'_0\} \oplus \{l'_{ij}, t'_{lm}, l'_i\} \equiv \mathcal{E}^{-}_k \oplus H_k \oplus \mathcal{E}^{+}_k. \tag{41}$$

Distinct from the case of a Lie algebra, the element l'_0 does not diagonalize the elements of the upper $\mathcal{E}^{-}_k$ (lower, $\mathcal{E}^{+}_k$), triangular subalgebra, due to the quadratic relations (39), as in the case of totally symmetric HS fields in AdS spaces [20,21]. Additionally, the negative-root vectors l'^{+}_i, l'^{+}_j do not commute.

Since the Verma module over a semi-simple finite-dimensional Lie algebra g (an induced module $\mathcal{U}(g) \otimes_{\mathcal{U}(b)} |0\rangle_V$ with a vacuum vector[9] $|0\rangle_V$) is isomorphic, due to the

PBW theorem, as a vector space to a polynomial algebra $\mathcal{U}(g^-) \otimes_C |0\rangle_V$, it is clear that g can be realized by first-order inhomogeneous differential operators acting on these polynomials.

We examine a generalization of a *Verma module for quadratic algebras*: $g(r)$ presents a $g(r)$-deformation of a Lie algebra g in such a way that $g(r=0) = g$. Thus, we consider a Verma module for such non-linear algebras, assuming that the PBW theorem is valid for $g(r)$. The latter will be proved later on by an explicit construction of the Verma module.

Let us consider a quadratic algebra $\mathcal{A}'(Y(k), AdS_d)$ as an r-deformation of the Lie algebra $\mathcal{A}'(Y(k), R^{1,d-1})$, with (27) taken into account (see [59] for details);

$$\mathcal{A}'(Y(k), AdS_d) = \left(T'^k \oplus T'^{k*} \oplus l'_0\right)(r) \Subset sp(2k), \quad T'^k = \{l'_i\},\ T'^{k*} = \{l'^+_i\}. \tag{42}$$

Therefore, according to (41), the basis vector of the Verma module $|\vec{N}\rangle_V$ has the form $|\vec{N}\rangle_V = |\vec{n}_{ij}, (\vec{n}, \vec{p}_m)_l\rangle_V = |\vec{n}_{ij}, n_1, p_{12}, \ldots, p_{1k}, n_2, p_{23}, \ldots, p_{2k}, \ldots, p_{k-1k}, n_k\rangle_V$,

$$|\vec{N}\rangle_V \equiv \prod_{i\leq j}^{k}\left(l'^+_{ij}\right)^{n_{ij}} \prod_{l}^{k}\left[\prod_{m,m>l}\left(\frac{l'^+_l}{m_l}\right)^{n_l}\left(t'^+_{lm}\right)^{p_{lm}}\right]|0\rangle_V,\ \mathcal{E}^+_k|0\rangle_V = 0, \tag{43}$$

with the (vacuum) highest-weight vector $|0\rangle_V$, non-negative integers n_{ij}, n_l, p_{lm}, and arbitrary constants m_l with the dimensionality of mass.

Now, we are in a position to construct a Verma module for the algebra $\mathcal{A}'(Y(2), AdS_d)$, thus leaving the general case of an arbitrary Young tableaux with $k > 2$ rows out of the scope of the paper. The solution of a similar problem for the auxiliary polynomial algebra $\mathcal{A}'$ is given by Appendix B.

3.2. Verma Module for Quadratic Algebra $\mathcal{A}'(Y(2), AdS_d)$ with $k = 2$ Rows

Let us first specify a commutator multiplication law for the algebra $\mathcal{A}'(Y(2), AdS_d)$, belonging to $\mathcal{A}'(Y(k), AdS_d)$. To do so, we can choose $\theta^{ij} = \delta^{i2}\delta^{j1}$, and therefore the only surviving operators are $t^{ij} = t^{12}$, $t^+_{ij} = t'^+_{12}$.

As a result, the Lie part of Tables 1 and 2 for $k = 2$ is the same as in the case of a bosonic Lie subalgebra in [58], once the following expressions for the non-vanishing operators $B'^{12}{}_{12}$, A'^{1212}, $F'^{12,j}$, $F'^{12,j+}$, $L'^{i_1j_1,i_2j_2}$,

$$B'^{12}{}_{12} = (g^1_0 - g^2_0), \quad A'^{12,12} = 0, \quad (F'^{12,j}, \equiv F'^j = t'^{12}(\delta^{j2} - \delta^{j1}), \tag{44}$$

$$L'^{i_1j_1,i_2j_2} = \tfrac{1}{4}\Big\{\delta^{i_2i_1}\delta^{j_2j_1}\left[2g^{i_2}_0\delta^{i_2j_2} + g^{i_2}_0 + g^{j_2}_0\right] - \delta^{j_2\{i_1}\left[t^{12}\delta^{j_1\}1}\delta^{i_22} + t^{12+}\delta^{j_1\}2}\delta^{i_21}\right] \tag{45}$$
$$-\delta^{i_2\{i_1}\left[t^{12}\delta^{j_22}\delta^{j_1\}1} + t^{12+}\delta^{j_1\}2}\delta^{j_21}\right]\Big\},$$

are identical to the same operators o_I from the initial algebra $\mathcal{A}(Y(2), AdS_d)$. Since the non-linear part of $\mathcal{A}'(Y(2), AdS_d)$ is determined by the same functions $W'^{ji}_b, \mathcal{K}'^i_1, W'^{ji+}_b, \mathcal{K}'^{i+}_1 X'^{ij}_b$ as those for arbitrary k, its manifest expression is implied directly by (38)–(40) as follows:[10]

$$W'^{ij}_b = 2r\epsilon^{ij}\left[(g'^2_0 - g'^1_0)l'^{12} - t'^{12}l'^{11} + t'^{12+}l'^{22}\right], \tag{46}$$

$$\mathcal{K}'^j_1 = \left(4\sum_i l'^{ji+}l'^i + l'^{j+}(2g'^j_0 - 1) - 2l'^{2+}t'^{+12}\delta^{j1} - 2l'^{1+}t'^{12}\delta^{j2}\right), \tag{47}$$

$$X'^{ij}_b = \{l'_0 - r(K'^{0i}_0 + \mathcal{K}'^{12}_0)\}\delta^{ij} + r\{[4\textstyle\sum_l l'^{1l+}l'^{l2} + (g'^2_0 + g'^1_0 - 2)t'_{12}]\delta^{i2}\delta^{j1} \tag{48}$$
$$+r\{[4\textstyle\sum_l l'^{l2+}l'^{1l} + t'^+_{12}(g'^2_0 + g'^1_0 - 2)]\delta^{i1}\delta^{j2}.$$

with the totally antisymmetric $sp(2)$-invariant tensor ϵ^{ij}, $\epsilon^{12} = 1$, and the operator $\mathcal{K}^{12}_0 = (t^+_{12}t^{12} - 4l^+_{12}l^{12} - g^2_0)$ deduced from a Casimir operator for the $sp(4)$ algebra in (26) for $k = 2$.

All the Lie part quantities of Table 1, except for o'_I, and those of Table 2 are specified by (44)–(48) used in the HS symmetry algebra $\mathcal{A}'(Y(2), AdS_d)$ of the additional part o'_I.

The Cartan-like decomposition (41) and the Verma module basis vector (43) of the algebra $\mathcal{A}'(Y(k), AdS_d)$ are reduced to those for $\mathcal{A}'(Y(2), AdS_d)$ in the form

$$\mathcal{A}'(Y(2), AdS_d) = \{l'^{+}_{ij}, t'^{+}_{12}, l'^{+}_{i}\} \oplus \{g'^{i}_{0}, l'_{0}\} \oplus \{l'_{ij}, t'_{12}, l'_{i}\} \equiv \mathcal{E}^{-}_{2} \oplus H_2 \oplus \mathcal{E}^{+}_{2}, \qquad (49)$$

$$|\vec{N}(2)\rangle_V = |\vec{n}_{ij}, n_1, p_{12}, n_2\rangle_V \equiv \prod_{i\leq j}^{2} (l'^{+}_{ij})^{n_{ij}} \left(\frac{l'^{+}_{1}}{m_1}\right)^{n_1} (t'^{+}_{12})^{p_{12}} \left(\frac{l'^{+}_{2}}{m_2}\right)^{n_2} |0\rangle_V, \ \mathcal{E}^{+}_{2}|0\rangle_V = 0. \quad (50)$$

Here, in contrast to the case of a Lie algebra with totally symmetric HS fields in AdS spaces [20,22,58], the negative-root vectors $l'^{+}_{1}, t'^{+}, l'^{+}_{2}$ do not commute, which means that $|\vec{N}(2)\rangle_V$ is not an eigenvector for the operators t'^{+}_{12}, l'^{+}_{2}, thereby featuring substantial peculiarities in constructing a Verma module for $\mathcal{A}'(Y(2), AdS_d)$.

By definition, the highest-weight vector $|0\rangle_V$ is an eigenvector of the Cartan-like subalgebra H_2,

$$(g'^{i}_{0}, l'_{0})|0\rangle_V = (h^i, m_0^2)|0\rangle_V, \qquad (51)$$

with certain real numbers h^1, h^2, m_0, whose values are to be determined later on, at the end of LF construction, in order for the Lagrangian equations of motion to reproduce the initial AdS group irrep conditions (2)–(5).

Then, we determine the action of negative-root vectors from the subspace $\mathcal{E}^{-}_{2}$ on the basis vector $|\vec{N}(2)\rangle_V$, which does not present the action of the raising operators in a manifest form and reads as

$$l'^{+}_{lm}\Big|\vec{N}(2)\rangle_V = \Big|\vec{n}_{ij} + \delta_{ij,lm}, \vec{n}_s\rangle_V , \qquad (52)$$

$$l'^{+}_{l}\Big|\vec{N}(2)\rangle_V = \delta^{l2}\prod_{i\leq j}^{2}(l'^{+}_{ij})^{n_{ij}} l'^{+}_{2}\Big|\vec{0}_{ij}, \vec{n}_s\rangle_V + \delta^{l1} m_1 |\vec{n}_{ij}, n_1+1, p_{12}, n_2\rangle_V , \ l = 1,2, \quad (53)$$

$$t'^{+}_{12}\Big|\vec{N}(2)\rangle_V = \prod_{i\leq j}^{2}(l'^{+}_{ij})^{n_{ij}} t'^{+}_{12}|\vec{0}_{ij}, \vec{n}_s\rangle_V - 2n_{11}|n_{11}-1, n_{12}+1, n_{22}, \vec{n}_s\rangle_V \\ -n_{12}|n_{11}, n_{12}-1, n_{22}+1, \vec{n}_s\rangle_V . \qquad (54)$$

Here, we used the following notation: first, $\vec{n}_s \equiv (n_1, p_{12}, n_2)$ and $\vec{0}_{ij} \equiv (0,0,0)$ in accordance with the definition of $|\vec{N}(2)\rangle_V$; second, $\delta_{ij,lm} = \delta_{il}\delta_{jm}$ for $i \leq j, l \leq m$ so that the vector $\Big|\vec{N} + \delta_{ij,lm}\rangle_V$ in (52) is subject to the definition (50), increasing the coordinate n_{ij} in the vector $|\vec{N}\rangle_V$, for $i = l, j = m$ by a unit while leaving intact the values of the remaining ones.

In turn, the action of Cartan-like generators on the vector $|\vec{N}(2)\rangle_V$ is given by the relations

$$g'^{l}_{0}\Big|\vec{N}(2)\rangle_V = \Big(2n_{ll} + n_{12} + n_l + (-1)^l p_{12} + h^l\Big)\Big|\vec{N}(2)\rangle_V , \ l = 1,2, \qquad (55)$$

$$l'_{0}\Big|\vec{N}(2)\rangle_V = \textstyle\prod_{i\leq j}^{2}(l'^{+}_{ij})^{n_{ij}} l'_{0}\Big|\vec{0}_{ij}, \vec{n}_s\rangle_V . \qquad (56)$$

To obtain the representations (52)–(56), we used a formula for the product of operators A, B,

$$AB^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} B^{n-k}\mathrm{ad}^k_B A , \ \mathrm{ad}^k_B A = [[\ldots[A, \overbrace{B\}\ldots\}, B}^{k \text{ times}}\}, \ \texttt{for} \ n \geq 0. \quad (57)$$

Finally, the action of positive-root vectors from the subspace $\mathcal{E}_2^+$ on the vector $|\vec{N}(2)\rangle_V$, according to the rule (57), reads as follows:

$$t'_{12}\Big|\vec{N}(2)\rangle_V = \prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}}\Big(\frac{l'^+_1}{m_1}\Big)^{n_1}(t'^+_{12})^{p_{12}}t'_{12}\Big|\vec{0}_{ij},0,0,n_2\rangle_V - \sum_l l n_{l2}\Big|\vec{n}_{ij}-\delta_{ij,l2}+\delta_{ij,1l},\vec{n}_s\rangle_V$$
$$+p_{12}(h^1-h^2-n_2-p_{12}+1)\big|\vec{n}_{ij},n_1,p_{12}-1,n_2\rangle_V\ , \tag{58}$$

$$l'^1\Big|\vec{N}(2)\rangle_V = -m_1 n_{11}\big|\vec{n}_{ij}-\delta_{ij,11},n_1+1,p_{12},n_2\rangle_V$$
$$+\Big\{\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}}l'^1 - \frac{n_{12}}{2}\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}-\delta_{ij,12}}l'^+_2\Big\}\Big|\vec{0}_{ij},\vec{n}_s\rangle_V\,, \tag{59}$$

$$l'^2\Big|\vec{N}(2)\rangle_V = -m_1\frac{n_{12}}{2}\big|\vec{n}_{ij}-\delta_{ij,12},n_1+1,p_{12},n_2\rangle_V$$
$$+\Big\{\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}}l'^2 - n_{22}\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}-\delta_{ij,22}}l'^+_2\Big\}\Big|\vec{0}_{ij},\vec{n}_s\rangle_V\,, \tag{60}$$

$$l'^{11}\Big|\vec{N}(2)\rangle_V = n_{11}(n_{11}+n_{12}+n_1-p_{12}-1+h^1)\big|\vec{n}_{ij}-\delta_{ij,11},\vec{n}_s\rangle_V$$
$$+\frac{n_{12}(n_{12}-1)}{4}\big|\vec{n}_{ij}-2\delta_{ij,12}+\delta_{ij,22},\vec{n}_s\rangle_V \tag{61}$$
$$+\Big\{\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}}l'^{11} - \frac{n_{12}}{2}\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}-\delta_{ij,12}}t'^+_{12}\Big\}\Big|\vec{0}_{ij},\vec{n}_s\rangle_V\,,$$

$$l'^{12}\Big|\vec{N}(2)\rangle_V = \frac{n_{12}}{4}\Big(n_{12}+\sum_l(2n_{ll}+n_l+h^l)-1\Big)\big|\vec{n}_{ij}-\delta_{ij,12},\vec{n}_s\rangle_V$$
$$+\frac{1}{2}p_{12}n_{11}(h^2-h^1+n_2+p_{12}-1)\big|\vec{n}_{ij}-\delta_{ij,11},n_1,p_{12}-1,n_2\rangle_V$$
$$+n_{11}n_{22}\big|\vec{n}_{ij}-\delta_{ij,11}-\delta_{ij,22}+\delta_{ij,12},\vec{n}_s\rangle_V$$
$$+\Big\{\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}}l'^{12} - \frac{n_{22}}{2}\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}-\delta_{ij,22}}t'^+_{12}\Big\}\Big|\vec{0}_{ij},\vec{n}_s\rangle_V$$
$$-\frac{n_{11}}{2}\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}-\delta_{ij,11}}\Big(\frac{l'^+_1}{m_1}\Big)^{n_1}(t'^+_{12})^{p_{12}}t'_{12}\Big|\vec{0}_{ij},0,0,n_2\rangle_V\,, \tag{62}$$

$$l'^{22}\Big|\vec{N}(2)\rangle_V = n_{22}(n_{12}+n_{22}+p_{12}+n_2-1+h^2)\big|\vec{n}_{ij}-\delta_{ij,22},\vec{n}_s\rangle_V$$
$$+\frac{n_{12}p_{12}}{2}(p_{12}-1+h^2-h^1+n_2)\big|\vec{n}_{ij}-\delta_{ij,12},n_1,p_{12}-1,n_2\rangle_V$$
$$+\frac{n_{12}(n_{12}-1)}{4}\big|\vec{n}_{ij}+\delta_{ij,11}-2\delta_{ij,12},\vec{n}_s\rangle_V$$
$$+\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}}\Big(\frac{l'^+_1}{m_1}\Big)^{n_1}l'^{22}\Big|\vec{0}_{ij},0,p_{12},n_2\rangle_V$$
$$-\frac{n_{12}}{2}\prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}-\delta_{ij,12}}\Big(\frac{l'^+_1}{m_1}\Big)^{n_1}(t'^+_{12})^{p_{12}}t'_{12}\Big|\vec{0}_{ij},0,0,n_2\rangle_V\ . \tag{63}$$

It is easy to see that, in order to complete the calculation in (53), (54), (59)–(63), we need to find the action of the positive-root vectors l'^m, l'^{1m}, $m=1,2$, the Cartan-like vector l'_0, and the negative-root vectors l'^+_2, t'^+_{12} on $|\vec{0}_{ij},\vec{n}_s\rangle_V$, and also the action of the remaining operators t'_{12}, l'^{22} on an arbitrary $|\vec{0}_{lm},0,0,n_2\rangle_V$ in terms of linear combinations of certain vectors. We solve this rather non-trivial technical problem explicitly in Appendix C, with an introduction of some auxiliary quantities, which we call a *primary block operator*, $\widehat{t}'_{12}$, (A37), (A38) and *derived* *block operators*, $\widehat{t}'^{+}_{12}, \widehat{l}'^{+}_{2}, \widehat{l}'_0, \widehat{l}'_m, \widehat{l}'_{m2}, m = 1, 2$, (A42), (A43), (A47)–(A51), (A56), presenting the result in the following form.

Theorem 1. *A Verma module for the non-linear second-order algebra $\mathcal{A}'(Y(2), AdS_d)$ does exist, is determined by the relations (52), (55), (64)–(72), and is expressed using the primary $\widehat{t}'_{12}$ and derived block operators $\widehat{t}'^{+}_{12}, \widehat{l}'^{+}_{2}, \widehat{l}'_0, \widehat{l}'_m, \widehat{l}'_{m2}, m = 1, 2$, (A42), (A43), (A47)–(A51), (A56) in the final form*

$$
\begin{aligned}
t'_{12}\Big|\vec{N}(2)\rangle_V \;&=\; p_{12}(h^1 - h^2 - n_2 - p_{12} + 1)\big|\vec{n}_{ij}, n_1, p_{12} - 1, n_2\rangle_V \\
&\quad - \sum_l l n_{l2}\Big|\vec{n}_{ij} - \delta_{ij,l2} + \delta_{ij,1l}, \vec{n}_s\rangle_V + \hat{t}'_{12}\Big|\vec{N}(2)\rangle_V\,, && (64)\\
t'^{+}_{12}\Big|\vec{N}(2)\rangle_V \;&=\; -\sum_l (3-l) n_{1l}\Big|\vec{n}_{ij} - \delta_{ij,1l} + \delta_{ij,l2}, \vec{n}_s\rangle_V + \hat{t}'^{+}_{12}\Big|\vec{N}(2)\rangle_V\,, && (65)\\
l'^{+}_l\Big|\vec{N}(2)\rangle_V \;&=\; \delta^{l1} m_1\Big|\vec{N}(2) + \delta_{s,1}\rangle_V + \delta^{l2}\hat{l}'^{+}_2\Big|\vec{N}(2)\rangle_V\,, && (66)\\
l'_0\Big|\vec{N}(2)\rangle_V \;&=\; l'_0\Big|\vec{0}_{ij}, \vec{n}_s\rangle_V\,|_{[\vec{0}_{ij}\to\vec{n}_{ij}]}, && (67)\\
l'_1\Big|\vec{N}(2)\rangle_V \;&=\; l'_1\Big|\vec{0}_{ij}, \vec{n}_s\rangle_V\,|_{[\vec{0}_{ij}\to\vec{n}_{ij}]} - m_1 n_{11}\big|\vec{n}_{ij} - \delta_{ij,11}, \vec{n}_s + \delta_{s,1}\rangle_V \\
&\quad - \frac{n_{12}}{2}\widehat{l}'^{+}_2\big|\vec{n}_{ij} - \delta_{ij,12}, \vec{n}_s\rangle_V\,, && (68)\\
l'_2\Big|\vec{N}(2)\rangle_V \;&=\; l'_2\Big|\vec{0}_{ij}, \vec{n}_s\rangle_V\,|_{[\vec{0}_{ij}\to\vec{n}_{ij}]} - m_1\frac{n_{12}}{2}\big|\vec{n}_{ij} - \delta_{ij,12}, \vec{n}_s + \delta_{s,1}\rangle_V \\
&\quad - n_{22}\widehat{l}'^{+}_2\big|\vec{n}_{ij} - \delta_{ij,22}, \vec{n}_s\rangle_V\,, && (69)\\
l'_{11}\Big|\vec{N}(2)\rangle_V \;&=\; n_{11}(n_{11} + n_{12} + n_1 - p_{12} - 1 + h^1)\big|\vec{n}_{ij} - \delta_{ij,11}, \vec{n}_s\rangle_V \\
&\quad + \frac{n_{12}(n_{12}-1)}{4}\big|\vec{n}_{ij} - 2\delta_{ij,12} + \delta_{ij,22}, \vec{n}_s\rangle_V + l'_{11}\Big|\vec{0}_{ij}, \vec{n}_s\rangle_V\,|_{[\vec{0}_{ij}\to\vec{n}_{ij}]} \\
&\quad - \frac{n_{12}}{2}\widehat{t}'^{+}_{12}\big|\vec{n}_{ij} - \delta_{ij,12}, \vec{n}_s\rangle_V\,, && (70)\\
l'_{12}\Big|\vec{N}(2)\rangle_V \;&=\; \frac{n_{12}}{4}\Big(n_{12} + \sum_l (2n_{ll} + n_l + h^l) - 1\Big)\big|\vec{n}_{ij} - \delta_{ij,12}, \vec{n}_s\rangle_V \\
&\quad + \frac{1}{2}p_{12}n_{11}(h^2 - h^1 + n_2 + p_{12} - 1)\big|\vec{n}_{ij} - \delta_{ij,11}, n_1, p_{12} - 1, n_2\rangle_V \\
&\quad + n_{11}n_{22}\big|\vec{n}_{ij} - \delta_{ij,11} - \delta_{ij,22} + \delta_{ij,12}, \vec{n}_s\rangle_V + l'_{12}\Big|\vec{0}_{ij}, \vec{n}_s\rangle_V\,|_{[\vec{0}_{ij}\to\vec{n}_{ij}]} \\
&\quad - \frac{n_{22}}{2}\widehat{t}'^{+}_{12}\big|\vec{n}_{ij} - \delta_{ij,22}, \vec{n}_s\rangle_V - \frac{n_{11}}{2}\widehat{t}'_{12}\big|\vec{n}_{ij} - \delta_{ij,11}, \vec{n}_s\rangle_V\,, && (71)\\
l'_{22}\Big|\vec{N}(2)\rangle_V \;&=\; n_{22}(n_{12} + p_{12} + n_2 + n_{22} - 1 + h^2)\big|\vec{n}_{ij} - \delta_{ij,22}, \vec{n}_s\rangle_V \\
&\quad + \frac{n_{12}p_{12}}{2}(p_{12} - 1 + h^2 - h^1 + n_2)\big|\vec{n}_{ij} - \delta_{ij,12}, \vec{n}_s - \delta_{s,12}\rangle_V \\
&\quad + \frac{n_{12}(n_{12}-1)}{4}\big|\vec{n}_{ij} + \delta_{ij,11} - 2\delta_{ij,12}, \vec{n}_s\rangle_V + \widehat{l}'_{22}\Big|\vec{N}(2)\rangle_V \\
&\quad - \frac{n_{12}}{2}\widehat{t}'_{12}\big|\vec{n}_{ij} - \delta_{ij,12}, \vec{n}_s\rangle_V\,. && (72)
\end{aligned}
$$

To obtain these relations, we applied the obvious rule

$$
\prod_{i\le j}^{2}(l'^{+}_{ij})^{n_{ij}}(l'_0, l'_n, l'_{lm})\Big|\vec{0}_{ij}, \vec{n}_s\rangle_V \equiv (l'_0, l'_n, l'_{lm})\Big|\vec{0}_{ij}, \vec{n}_s\rangle_V\,|_{[\vec{0}_{ij}\to\vec{n}_{ij}]}, \quad l, m, n = 1, 2,\ l \le m, \quad (73)
$$

when the multipliers $(l'^{+}_{ij})^{n_{ij}}$ act as the only raising operators for the vectors $(l'_0, l'_k, l'_{km})\Big|\vec{0}_{ij}, \vec{n}_s\rangle_V$.

The above result has obvious corollaries: first of all, in the case of $\mathcal{A}'(Y(2), AdS_d)$ reduced to the quadratic algebra $\mathcal{A}'(Y(1), AdS_d)$, given in [20] for the vanishing components $n_{l2} = n_2 = p_{12} = 0, l = 1, 2$, of an arbitrary VM vector $\left|\vec{N}(1)\right\rangle_V \equiv |n_{11}, 0, 0, n_1, 0, 0\rangle_V$, and, secondly, in the case of the AdS space reduced to the Minkowski space $R^{1,d-1}$, when the non-linear algebra $\mathcal{A}'(Y(2), AdS_d)$ at $r = 0$ turns to the Lie algebra $\left(T'^2 \oplus T'^{2*} \oplus l'_0\right)(0) \supset\!\!\!\!+ sp(4)$ for $k = 2$ in (42). In the former case, we obtain the Verma module [20], familiar from the results of Theorem 1, whereas in the latter case, we obtain a new Verma module (see Appendix C), with the above Lie algebra being different from that for $sp(4)$ in [55], and also in [59] for $k = 2$.

3.3. Fock Space Realization of $\mathcal{A}'(Y(2), AdS_d)$

In this section, we obtain, on a basis of the constructed Verma module, a realization of $\mathcal{A}'(Y(2), AdS_d)$ as a formal power series in the creation and annihilation operators $(B_a, B_a^+) = \left((b_i, b_{ij}, d_{12}), (b_i^+, b_{ij}^+, d_{12}^+)\right)$ in $\mathcal{H}'$, whose number coincides with that of second-class constraints among o'_I, i.e., with $\dim(\mathcal{E}_2^- \oplus \mathcal{E}_2^+)$. This task is solved by following the results of [97] and the algorithms suggested in [98,99], initially elaborated for a simple Lie algebra, and then enlarged to a non-linear quadratic algebra[11] $\mathcal{A}(Y(1), AdS_d)$ [20]. To this end, we make use of the following mapping for an arbitrary basis vector in the Verma module and the vector $\left|\vec{n}_{ij}, \vec{n}_s\right\rangle$ in a new Fock space $\mathcal{H}'$:

$$\left|\vec{n}_{ij}, \vec{n}_s\right\rangle_V \longleftrightarrow \left|\vec{n}_{ij}, \vec{n}_s\right\rangle = \prod_{i\le j}^2 (b_{ij}^+)^{n_{ij}} \prod_{l=1}^2 (b_l^+)^{n_l} (d_{12}^+)^{p_{12}} |0\rangle\,, \qquad (74)$$
$$\text{for } b_{ij}|0\rangle = b_l|0\rangle = d_{12}|0\rangle = 0, \quad |\vec{N}(2)\rangle_{[\vec{N}=\vec{0}]} \equiv |0\rangle\,.$$

Here, the vectors $\left|\vec{n}_{ij}, \vec{n}_s\right\rangle$ for non-negative integers n_{ij}, n_l, p_{12} are the basis vectors of the Fock space $\mathcal{H}'$, generated by six pairs of bosonic operators (B, B^+), being the basis elements of the Heisenberg algebra A_6 with the standard (non-vanishing) commutation relations

$$[B_a, B_b^+] = \delta_{ab} \Longleftrightarrow [b_{ij}, b_{lm}^+] = \delta_{il,jm},\ [b_i, b_m^+] = \delta_{im},\ [d_{12}, d_{12}^+] = 1. \qquad (75)$$

To present the elements of the Verma module as a formal power series in the generators of the algebra A_6, we use an additive correspondence between the special Verma module vector $\left|A_{\vec{0}_{ij},0,0,n_2}\right\rangle_V$ (A30) and the Fock space $\mathcal{H}'$ vector $\left|\vec{0}_{ij}, 0, 0, n_2\right\rangle$; see (A87) in Appendix D. Summarizing, we present our basic result in the form of Theorem A1, given in the same Appendix D as polynomials for the trivial negative-root vectors $l_1'^+$ as well as for the particle number operators $g_0'^i$ (A90), the remaining root vectors, and the Cartan-like vector l'_0 from $\mathcal{A}'(Y(2), AdS_d)$, as power series in oscillators B_a, B_b^+. The latter ones are given by (A91)–(A101). To obtain the above representations, we use the *primary block operator* $\hat{t}'_{12}$ (A92) and the *derived block operators* $\hat{t}'^+_{12}, \hat{l}'_0, \hat{l}'_m, \hat{l}'_{m2}, m = 1, 2$.

The additional parts $o'_I(B, B^+)$, as formal power series in the oscillators (B, B^+), do not obey the usual properties,

$$(l'_{lm})^+ \neq l'^+_{lm}\,, \qquad (t'_{12})^+ \neq t'^+_{12}, \qquad (l'_0)^+ \neq l'_0\,, \qquad (l'_m)^+ \neq l'^+_m\,,\ l \le m, \qquad (76)$$

if one uses the standard rules of Hermitian conjugation for the new creation and annihilation operators, $(B_a)^+ = B_a$. The standard Hermitian conjugation properties for o'_I are restored by changing the inner product in $\mathcal{H}'$ as follows:

$$\langle\Phi_1|\Phi_2\rangle_{\text{new}} = \langle\Phi_1|K'|\Phi_2\rangle\,, \qquad (77)$$

for any vectors $|\Phi_1\rangle, |\Phi_2\rangle$ with a certain non-degenerate operator K'. This operator is determined by the condition that all the operators of the algebra should have the standard Hermitian properties with respect to the new inner product:

$$\langle\Phi_1|K'E^{-\prime\alpha}|\Phi_2\rangle = \langle\Phi_2|K'E'^{\alpha}|\Phi_1\rangle^*,\ \langle\Phi_1|K'G'|\Phi_2\rangle = \langle\Phi_2|K'G'|\Phi_1\rangle^*, \qquad (78)$$

for $(E'^{\alpha}; E^{-\prime\alpha}) = (l'_{lm}, t'_{12}, l'_m; l'^{+}_{lm}, t'^{+}_{12}, l'^{+}_m)$, $G' = (g'^{i}_0, l'_0)$. The relations (78) determine an operator K' which is Hermitian with respect to the standard inner product $\langle \, | \, \rangle$,

$$K' = Z^{+}Z, \qquad Z = \sum_{(\vec{n}_{lm}, \vec{n}_s) = (\vec{0}, \vec{0})}^{\infty} \left| \vec{N}(2) \right\rangle_V \frac{1}{(\vec{n}_{lm})!(\vec{n}_s)!} \langle 0 | \prod_{r=1}^{2} b_r^{n_r} d_{12}^{p_{12}} \prod_{l,m \geq l} b_{lm}^{n_{lm}}, \tag{79}$$

where $(\vec{n}_{lm})! = n_{11}!n_{12}!n_{22}!$, $(\vec{n}_s)! = n_1!n_2!p_{12}!$, and the normalization ${}_V\langle 0|0\rangle_V = 1$ is assumed.

Theorem A1 has the same consequences as those of Theorem 1, which concerns, first of all, a flat-space limit of the algebra $\mathcal{A}'(Y(2), AdS_d)$, and therefore a new representation for the Lie algebra $\left(T'^2 \oplus T'^{2*} \oplus l'_0\right)(0) \niplus sp(4)$. Second, modulo the oscillator pairs $b_{m2}, b^{+}_{m2}, b_2, b^{+}_2, d_{12}, d^{+}_{12}, m = 1, 2$, the deduced representation coincides with the one for a totally symmetric HS field quadratic algebra $\mathcal{A}'(Y(1), AdS_d)$ in [20,21] for the Weyl ordering of quadratic combinations of O_I.

The set of equations that constitute the results of Theorems 1 and A1 presents a general solution of the second problem (mentioned in Introduction) of an LF construction for mixed-symmetry HS tensors in AdS spaces with a given mass and spin $\mathbf{s} = (s_1, s_2)$.

4. Construction of Lagrangian Actions

In order to construct a general Lagrangian formulation for an HS tensor field of fixed generalized spin $\mathbf{s} = (s_1, \ldots, s_k)$, we should explicitly determine a composition law for the deformed algebra $\mathcal{A}_c(Y(k), AdS_d)$, then we have to specify this to the case of a YT with $k = 2$ rows, find a BRST operator for the non-linear algebra $\mathcal{A}_c(Y(2), AdS_d)$, and, finally, reproduce a correct gauge-invariant Lagrangian formulation for the basic bosonic field $\Phi_{(\mu)_{s_1}, (\nu)_{s_2}}$.

4.1. Explicit Form of $\mathcal{A}_c(Y(k), AdS_d)$

As in the case of the algebra $\mathcal{A}'(Y(k), AdS_d)$ of o'_I, it is only the multiplication law for the quadratic part of the initial algebra $\mathcal{A}(Y(k), AdS_d)$ that is modified, whereas the linear part is given by the same $sp(2k)$ algebra as for the maximal Lie subalgebra, $\mathcal{A}(Y(k), AdS_d)$ and $\mathcal{A}'(Y(k), AdS_d)$, with the same form of commutators $[O_a, O_I]$, $O_a \in sp(2k)$. From (34), (35) and Table 1, the non-linear part of the algebra $\mathcal{A}_c(Y(k), AdS_d)$ can be recovered using Table 3.

Table 3. The non-linear part of the converted algebra $\mathcal{A}_c(Y(k), AdS_d)$.

$[\downarrow, \rightarrow]$	$\boldsymbol{L_0}$	$\boldsymbol{L^i}$	$\boldsymbol{L^{i+}}$
L_0	0	$-r\hat{\mathcal{K}}^{i+}_1$	$r\hat{\mathcal{K}}^{i}_1$
L^j	$r\hat{\mathcal{K}}^{j+}_1$	$\hat{W}^{ji}_b$	$\hat{X}^{ji}_b$
L^{j+}	$-r\hat{\mathcal{K}}^{j}_1$	$-\hat{X}^{ij}_b$	$-\hat{W}^{ji+}_b$

Here, the functions $\hat{\mathcal{K}}^i_1$, $\hat{W}^{ji}_b$, $\hat{X}^{ij}_b$ (hence, the Hermitian conjugate quantities $\hat{\mathcal{K}}^{i+}_1$, $\hat{W}^{ji+}_b$) are given by

$$\begin{aligned}
\hat{W}^{ij}_b &= 2r\Big\{\big[G^j_0 - G^i_0 - (g'^j_0 - g'^i_0)\big]L^{ij} - l'^{ij}(G^j_0 - G^i_0) - \textstyle\sum_m\Big[(T^{m[j} - t'^{m[j})\theta^{[jm}L^{i]m} \\
&\quad - l'^{[im}T^{mj]}\theta^{j]m} + (T^{[jm+} - t'^{[jm+})\theta^{m[j}L^{i]m} - l'^{[im}T^{j]m+}\theta^{mj]}\Big]\Big\}, && (80)\\
\hat{\mathcal{K}}^j_1 &= 4\textstyle\sum_{i=1}^k\Big\{(L^{ji+} - l'^{ji+})L^i - l'^iL^{ji+}\Big\} + 2(L^{j+} - l'^{j+})G^j_0 - 2(g'^j_0 + \frac{1}{2})L^{j+} \\
&\quad -2\textstyle\sum_{i=1}^k\Big\{\big[(L^{i+} - l'^{i+})T^{ij+} - t'^{ij+}L^{i+}\big]\theta^{ji} + \big[(L^{i+} - l'^{i+})T^{ji} - t'^{ji}L^{i+}\big]\theta^{ij}\Big\}, && (81)\\
\hat{X}^{ij}_b &= \Big\{L_0 + r\Big[\hat{\mathcal{K}}^{0i}_0 - 2g'^i_0G^i_0 + 4l'^{ii+}L^{ii} + 4l'^{ii}L^{ii+} + \sum_{l=i+1}^k\Big(\hat{\mathcal{K}}^{il}_0 - t'^{il+}T^{il} - t'^{il}T^{il+} \\
&\quad +4l'^{il+}L^{il} + 4l'^{il}L^{il+}\Big) + \sum_{l=1}^{i-1}\Big(\hat{\mathcal{K}}^{li}_0 - t'^{li+}T^{li} - t'^{li}T^{li+} + 4l'^{li+}L^{li} + 4l'^{li}L^{li+}\Big)\Big]\Big\}\delta^{ij} \\
&\quad -r\Big\{4\textstyle\sum_l\Big[(L^{jl+} - l'^{jl+})L^{li} - l'^{li}L^{jl+}\Big] - \displaystyle\sum_{l=1}^{j-1}\Big[(T^{lj+} - t'^{lj+})T^{li} - t'^{li}T^{lj+}\Big] \\
&\quad - \sum_{l=i+1}^k\Big[(T^{il+} - t'^{il+})T^{jl} - t'^{jl}T^{il+}\Big] - \sum_{l=j+1}^{i-1}\Big[(T^{li} - t'^{li})T^{jl} - t'^{jl}T^{li}\Big] \\
&\quad +\Big(G^j_0 + G^i_0 - (g'^j_0 + g'^i_0) - j - 1\Big)T^{ji} - t'^{ji}\Big(G^j_0 + G^i_0\Big)\Big\}\theta^{ij} \\
&\quad -r\Big\{4\textstyle\sum_l\Big[(L^{jl+} - l'^{jl+})L^{li} - l'^{li}L^{jl+}\Big] - \displaystyle\sum_{l=1}^{i-1}\Big[(T^{lj+} - t'^{lj+})T^{li} - t'^{li}T^{lj+}\Big] \\
&\quad - \sum_{l=j+1}^k\Big[(T^{il+} - t'^{il+})T^{jl} - t'^{jl}T^{il+}\Big] - \sum_{l=i+1}^{j-1}\Big[(T^{il+} - t'^{il+})T^{lj+} - t'^{lj+}T^{il+}\Big] \\
&\quad +\Big(T^+_{ij} - t'^+_{ij}\Big)\Big(G^j_0 + G^i_0\Big) - \Big(g'^i_0 + g'^j_0 + i + 1\Big)T^+_{ij}\Big\}\theta^{ji}. && (82)
\end{aligned}$$

Here, the quantities $\hat{\mathcal{K}}^{0i}_0$, $\hat{\mathcal{K}}^{ij}_0$, $\hat{\mathcal{K}}^{1i}_0$ are the same as those of (26), albeit expressed in terms of O_I.

Let us follow our experience in the study of the (super)algebra $\mathcal{A}_c(Y(1), AdS_d)$ [20–22], when, in order to find an exact BRST operator, we choose the Weyl (symmetric) ordering for the quadratic combinations of O_I in the r.h.s. of (80)–(82) as $O_IO_J = \frac{1}{2}(O_IO_J + O_JO_I) + \frac{1}{2}[O_I, O_J]$. As a result, for this kind of ordering, Table 3 must contain certain quantities $\hat{W}^{ij}_{bW}$, $\hat{\mathcal{K}}^j_{1W}$, $\hat{X}^{ij}_{bW}$ (and $\hat{W}^{ij+}_{bW}$, $\hat{\mathcal{K}}^{j+}_{1W}$), which have the form

$$\begin{aligned}
\hat{W}^{ij}_{bW} &= r\Big\{[\mathcal{G}^j_0 - \mathcal{G}^i_0]L^{ij} + \mathcal{L}^{ij}[G^j_0 - G^i_0] - \textstyle\sum_{m=1}^k\Big[\mathcal{T}^{m[j}\theta^{[jm}L^{i]m} + \mathcal{L}^{[im}T^{mj]}\theta^{j]m} \\
&\quad +\mathcal{T}^{[jm+}\theta^{m[j}L^{i]m} + \mathcal{L}^{[im}T^{j]m+}\theta^{mj]}\Big]\Big\}, && (83)\\
\hat{\mathcal{K}}^j_{1W} &= 2\textstyle\sum_{i=1}^k\Big\{\mathcal{L}^{ji+}L^i + \mathcal{L}^iL^{ji+}\Big\} + \mathcal{L}^{j+}G^j_0 + \mathcal{G}^j_0L^{j+} - \textstyle\sum_{i=1}^k\Big\{\big[\mathcal{L}^{i+}T^{ij+} \\
&\quad +\mathcal{T}^{ij+}L^{i+}\big]\theta^{ji} + \big[\mathcal{L}^{i+}T^{ji} + \mathcal{T}^{ji}L^{i+}\big]\theta^{ij}\Big\}, && (84)\\
\hat{X}^{ij}_{bW} &= \Big\{L_0 + r\Big[\mathcal{G}'^i_0G^i_0 - 2\mathcal{L}^{ii+}L^{ii} - 2\mathcal{L}^{ii}L^{ii+} + \frac{1}{2}\textstyle\sum_{l=i+1}^k\Big(\mathcal{T}^{il}T^{il+} + \mathcal{T}^{il+}T^{il} \\
&\quad -4\mathcal{L}^{il+}L^{il} - 4\mathcal{L}^{il}L^{il+}\Big) + \frac{1}{2}\textstyle\sum_{l=1}^{i-1}\Big(\mathcal{T}^{li}T^{li+} + \mathcal{T}^{li+}T^{li} - 4\mathcal{L}^{li+}L^{li} - 4\mathcal{L}^{li}L^{li+}\Big)\Big]\Big\}\delta^{ij}
\end{aligned}$$

$$-r\Big\{2\sum_l[\mathcal{L}^{jl+}L^{li}+\mathcal{L}^{li}L^{jl+}]-\frac{1}{2}\sum_{l=1}^{j-1}[\mathcal{T}^{lj+}T^{li}+\mathcal{T}^{li}T^{lj+}]-\frac{1}{2}\sum_{l=i+1}^{k}[\mathcal{T}^{il+}T^{jl}$$

$$+\mathcal{T}^{jl}T^{il+}]-\frac{1}{2}\sum_{l=j+1}^{i-1}[\mathcal{T}^{li}T^{jl}+\mathcal{T}^{jl}T^{li}]+\frac{1}{2}(\mathcal{G}_0^j+\mathcal{G}_0^i)T^{ji}+\frac{1}{2}\mathcal{T}^{ji}(G_0^j+G_0^i)\Big\}\theta^{ij}$$

$$-r\Big\{2\sum_l\{\mathcal{L}^{jl+}L^{li}+\mathcal{L}^{li}L^{jl+}\}-\frac{1}{2}\sum_{l=1}^{i-1}[\mathcal{T}^{lj+}T^{li}+\mathcal{T}^{li}T^{lj+}]-\frac{1}{2}\sum_{l=j+1}^{k}[\mathcal{T}^{il+}T^{jl}$$

$$+\mathcal{T}^{jl}T^{il+}]-\frac{1}{2}\sum_{l=i+1}^{j-1}[\mathcal{T}^{+}_{il}T^{+}_{lj}+\mathcal{T}'^{+}_{lj}T^{+}_{il}]+\frac{1}{2}\Big[\mathcal{T}^{+}_{ij}(G_0^j+G_0^i)+(\mathcal{G}_0^i+\mathcal{G}_0^j)T^{+}_{ij}\Big]\Big\}\theta^{ji}, \quad (85)$$

with the notation $\mathcal{O}_I$ for the quantity $\mathcal{O}_I=(O_I-2o'_I)$. Note that the ordering in (83)–(85) does not contain any linear terms (except for L_0 in the r.h.s. of the last relation) as compared to (80)–(82).

Thus, we deduce the algebra of converted operators O_I underlying an HS field subject to an arbitrary unitary irreducible AdS group representation with spin $\mathbf{s}=(s_1,\dots,s_k)$ in an AdS space so that the problem is to find a BRST operator for $\mathcal{A}_c(Y(k),AdS_d)$.

4.2. BRST Operator for Converted Algebra $\mathcal{A}_c(Y(2),AdS_d)$

In this section, we turn to HS fields subject to a YT with two rows. The non-linear algebra no longer has a closed form, due to the operator functions $F^{(2)K}_{IJ}(o',O)$ in (35), and, as shown in [63], this leads to an emergence of higher-order structure functions, due to the quadratic algebraic relations (83)–(85) and their Hermitian conjugation, corresponding to the quantities $F^{(2)K}_{IJ}(o',O)$ for $i,j=1,2$. In [63], new structure functions are found, $F^{RS}_{IJK}(O)$, being of the third order [35] and implied by a resolution of the Jacobi identities $[[O_I,O_J],O_K]+cycl.perm.(I,J,K)=0$ as follows (33):

$$\Big\{F^{M}_{IJ}F^{P}_{MK}+[F^{(2)P}_{IJ},O_K]+cycl.perm.(I,J,K)\Big\}=F^{RS}_{IJK}\Big(O_R\delta^P_S-\tfrac{1}{2}F^{P}_{RS}\Big), \quad (86)$$

for $F^{M}_{IJ}\equiv\big(f^{M}_{IJ}+F^{(2)M}_{IJ}\big)$. The structure functions $F^{RS}_{IJK}(o',O)$ are antisymmetric with respect to a permutation of any two of the lower indices (I,J,K) and the upper ones R,S, and do exist because of non-trivial Jacobi identities for the $k(2k-1)$ triples (L_i,L_j,L_0), (L^+_i,L^+_j,L_0), (L_i,L^+_j,L_0).

The construction of a BFV–BRST operator Q' for $\mathcal{A}_c(Y(2),AdS_d)$ is studied in [63] and has the general form

$$Q'=\mathcal{C}^I\big[O_I+\tfrac{1}{2}\mathcal{C}^J(f^P_{JI}+F^{(2)P}_{JI})\mathcal{P}_P+\tfrac{1}{12}\mathcal{C}^J\mathcal{C}^K F^{RP}_{KJI}\mathcal{P}_R\mathcal{P}_P\big], \quad (87)$$

with the $(\mathcal{CP})$-ordering for the ghost coordinates $\mathcal{C}^I=\{\eta_0,\,\eta^i_G\,\eta^+_i,\,\eta_i,\,\eta^+_{ij},\,\eta_{ij},\,\vartheta_{12},\,\vartheta^+_{12}\}$, and their conjugate momenta[12] $\mathcal{P}_I=\{\mathcal{P}_0,\,\mathcal{P}^i_G,\,\mathcal{P}_i,\,\mathcal{P}^+_i,\,\mathcal{P}_{ij},\,\mathcal{P}^+_{ij},\,\lambda^+_{12},\,\lambda_{12}\}$. Explicitly, Q' is given by

$$Q'=Q'_1+Q'_2+r^2\Big\{\eta_0\sum_{i,j}\eta_i\eta_j\varepsilon^{ij}\Big[\tfrac{1}{2}\sum_m\Big(G^m_0[\lambda_{12}\mathcal{P}^+_{22}-\lambda^+_{12}\mathcal{P}^+_{11}+i\mathcal{P}^+_{12}\sum_l(-1)^l\mathcal{P}^l_G]$$

$$-i(L^+_{11}\lambda^+_{12}-L^+_{22}\lambda_{12})\mathcal{P}^m_G+4L^{mm}\mathcal{P}^+_{m2}\mathcal{P}^+_{1m}\Big)-L^+_{12}\mathcal{P}^1_G\mathcal{P}^2_G+2L^{12}\mathcal{P}^+_{22}\mathcal{P}^+_{11}\Big]$$

$$+\eta_0\sum_{i,j}\eta^+_i\eta_j\Big[\sum_m\Big((-1)^m\tfrac{i}{2}G^m_0\sum_l\mathcal{P}^l_G+2(L^+_{22}\mathcal{P}^{22}-L^{11}\mathcal{P}^+_{11})\Big)\lambda_{12}\delta^{1j}\delta^{2i}$$

$$+\varepsilon^{\{1j}\delta^{2\}i}\Big(\imath\sum_m\Big[\tfrac{1}{2}T^{12}\lambda^+_{12}-2L^{12}\mathcal{P}^+_{12}(-1)^m\Big]\mathcal{P}^m_G+2\Big[L^{12}\lambda_{12}-T^{12}\mathcal{P}^{12}\Big]\mathcal{P}^+_{22}$$
$$+2\Big[T^+_{12}\mathcal{P}^{12}-L^{12}\lambda^+_{12}\Big]\mathcal{P}^+_{11}\Big)-T^{12}\Big[\mathcal{P}^1_G\mathcal{P}^2_G\delta^{2i}\delta^{1j}+2\mathcal{P}^{11}\mathcal{P}^+_{22}\delta^{1i}\delta^{2j}\Big]$$
$$-2\sum_m(-1)^m\Big[(G^m_0\mathcal{P}^{11}+\imath L^{11}\mathcal{P}^m_G)\delta^{1i}\delta^{2j}-(G^m_0\mathcal{P}^{22}+\imath L^{22}\mathcal{P}^m_G)\delta^{2i}\delta^{1j}\Big]\mathcal{P}^+_{12}\Big]+h.c.\Big\}, \quad (88)$$

for $\varepsilon^{ij}=-\varepsilon^{ji},\varepsilon^{12}=1$, with the standard form of the linear Q'_1 and quadratic Q'_2 terms in the ghosts $\mathcal{C}^I$; for details, see [63]. The Hermiticity of the nilpotent operator Q' in the total Hilbert space $\mathcal{H}_{tot}=\mathcal{H}\otimes\mathcal{H}'\otimes\mathcal{H}_{gh}$ is defined by the rule

$$Q'^+K=KQ',\,\mathrm{for}K=\hat{1}\otimes K'\otimes\hat{1}_{gh}, \quad (89)$$

with the operator K' given by (79), which provides the Hermiticity of o'_I in $\mathcal{H}'$.

4.3. Lagrangian Formulation

The construction of Lagrangians for bosonic HS fields in AdS$_d$ spaces can be achieved by partially following the algorithm of [21,37] (see also [59]), which is a particular case of our construction, corresponding to $s_2=0$. As a first step, we extract the dependence of the BRST operator Q' (88) on the ghosts $\eta^i_G,\mathcal{P}^i_G$,

$$Q' \;=\; Q+\eta^i_G(\sigma^i+h^i)+\mathcal{B}^i\mathcal{P}^i_G, \quad (90)$$

with certain (non-essential later on) operators $\mathcal{B}^i$ and a BRST operator[13] Q, which corresponds only to the converted first-class constraints $\{O_I\}\setminus\{G^i_0\}$,

$$Q=\tfrac{1}{2}\eta_0L_0+\sum_i\eta^+_iL_i+\sum_{l\le m}\eta^+_{lm}L_{lm}+\vartheta^+_{12}T_{12}+\tfrac{\imath}{2}\sum_l\eta^+_l\eta_l\mathcal{P}_0$$
$$-\vartheta^+_{12}\sum_n(1+\delta_{1n})\eta^+_{1n}\mathcal{P}_{n2}+\vartheta^+_{12}\sum_n(1+\delta_{n2})\eta_{n2}\mathcal{P}^+_{1n}+\tfrac{1}{2}\sum_n\eta^+_{n2}\eta_{1n}\lambda_{12}$$
$$-\tfrac{1}{2}\sum_{l\le m}(1+\delta_{lm})\eta_m\eta^+_{lm}\mathcal{P}_l-\big[\vartheta_{12}\eta^+_2\mathcal{P}_1+\vartheta^+_{12}\eta^+_1\mathcal{P}_2\big]$$
$$+r\Big\{\eta_0\sum_i\eta^+_i\Big[2\mathcal{L}_{ii}\mathcal{P}^+_i+2\mathcal{L}^+_i\mathcal{P}_{ii}+\mathcal{G}^i_0\mathcal{P}_i+2\big(\mathcal{L}_{12}\mathcal{P}^+_{\{1}+\mathcal{L}^+_{\{1}\mathcal{P}_{12}\big)\delta_{2\}i}$$
$$-\delta^{1i}\big(\mathcal{L}_2\lambda^+_{12}+\mathcal{T}^+_{12}\mathcal{P}_2\big)-\delta^{2i}\big(\mathcal{L}_1\lambda_{12}+\mathcal{T}_{12}\mathcal{P}_1\big)\Big]$$
$$-\tfrac{1}{2}\sum_{i,j}\eta^+_i\eta^+_j\varepsilon^{ij}\Big[\sum_m(-1)^m\mathcal{G}^m_0\mathcal{P}_{12}-\big(\mathcal{T}_{12}\mathcal{P}_{11}+\mathcal{L}_{11}\lambda_{12}\big)+\mathcal{T}^+_{12}\mathcal{P}_{22}+\mathcal{L}_{22}\lambda^+_{12}\Big]$$
$$+2\eta^+_i\eta_j\Big[\sum_m\mathcal{L}^+_{jm}\mathcal{P}_{im}-\tfrac{1}{8}\big(\mathcal{T}^+_{12}\lambda_{12}+\mathcal{T}_{12}\lambda^+_{12}\big)\delta^{ij}+\tfrac{1}{4}\sum_m\mathcal{G}^m_0\lambda_{12}\delta^{j1}\delta^{i2}\Big]\Big\}$$
$$+r^2\Big\{\eta_0\sum_{i,j}\eta_i\eta_j\varepsilon^{ij}\Big[\tfrac{1}{2}\sum_m\Big(G^m_0\big[\lambda_{12}\mathcal{P}^+_{22}-\lambda^+_{12}\mathcal{P}^+_{11}\big]+4L_{mm}\mathcal{P}^+_{m2}\mathcal{P}^+_{1m}\Big)+2L_{12}\mathcal{P}^+_{22}\mathcal{P}^+_{11}\Big]$$
$$+\eta_0\sum_{i,j}\eta^+_i\eta_j\Big[2\sum_m\big(L^+_{22}\mathcal{P}_{22}-L_{11}\mathcal{P}^+_{11}\big)\big)\lambda_{12}\delta^{1j}\delta^{2i}-2T_{12}\mathcal{P}_{11}\mathcal{P}^+_{22}\delta^{1i}\delta^{2j}$$
$$+2\varepsilon^{\{1j}\delta^{2\}i}\Big(\big(L_{12}\lambda_{12}-T_{12}\mathcal{P}_{12}\big)\mathcal{P}^+_{22}+\big(T^+_{12}\mathcal{P}_{12}-L_{12}\lambda^+_{12}\big)\mathcal{P}^+_{11}\Big)$$
$$-2\sum_m(-1)^m\Big(G^m_0\mathcal{P}_{11}\delta^{1i}\delta^{2j}-G^m_0\mathcal{P}_{22}\delta^{2i}\delta^{1j}\Big)\mathcal{P}^+_{12}\Big]\Big\}+h.c.\,. \quad (91)$$

The generalized spin operator $\vec{\sigma}_2=(\sigma^1,\sigma^2)$, extended by ghost Wick-pair variables,

$$\sigma^i=G^i_0-h^i-\eta_i\mathcal{P}^+_i+\eta^+_i\mathcal{P}_i+\sum_m(1+\delta_{im})(\eta^+_{im}\mathcal{P}_{im}-\eta_{im}\mathcal{P}^+_{im})+(\vartheta^+_{12}\lambda_{12}-\vartheta_{12}\lambda^+_{12})(-1)^i\,, \quad (92)$$

supercommutes with Q. The nilpotency $Q'^2=0$ entails the system of equations

$$Q^2=2\imath\sum_i\mathcal{B}^i\sigma^i, \qquad [Q,\,\sigma^i\}=0; \quad (93)$$
$$Q^+K=KQ, \qquad (\sigma^i)^+K=K\sigma^i. \quad (94)$$

We choose a representation for the Hilbert space $\mathcal{H}_{tot}$ to be coordinated with the decomposition (90) so that the operators $(\eta_i, \eta_{ij}, \vartheta_{12}, \mathcal{P}_0, \mathcal{P}_i, \mathcal{P}_{ij}, \lambda_{12}, \mathcal{P}^i_G)$ annihilate the vacuum vector $|0\rangle$, and suppose that the field vectors $|\chi\rangle$, as well as the gauge parameters $|\Lambda\rangle$, do not depend on the ghosts η^i_G,

$$\begin{aligned} |\chi\rangle \quad = \quad & \sum_n \prod_l^2 (b_l^+)^{n_l} \prod_{i\le j}^2 (b_{ij}^+)^{n_{ij}} (d_{12}^+)^{p_{12}} (\eta_0^+)^{n_{f0}} \prod_{i,j,l\le m,n\le o}^2 (\eta_i^+)^{n_{fi}} (\mathcal{P}_j^+)^{n_{pj}} (\eta_{lm}^+)^{n_{flm}} (\mathcal{P}_{no}^+)^{n_{pno}} \\ & \times (\vartheta_{12}^+)^{n_{f12}} (\lambda_{12}^+)^{n_{\lambda 12}} |\Phi(a_i^+)^{n_{f0}(n)_{fi}(n)_{pj}(n)_{flm}(n)_{pno}(n)_{f12}(n)_{\lambda 12}}_{(n)_l(n)_{ij}p_{12}}\rangle\,. \end{aligned} \quad (95)$$

The brackets $(n)_{fi}, (n)_{pj}, (n)_{pno}$ in (95) imply, for example, for $(n)_{pno}$, the set of indices $(n_{p11}, n_{p12}, n_{p22})$. The above sum runs over n_l, n_{ij}, p_{12}, in the range from 0 to infinity, as well as over the remaining n from 0 to 1. The Hilbert space $\mathcal{H}_{tot}$ decomposes into a direct sum of Hilbert subspaces with definite ghost numbers: $\mathcal{H}_{tot} = \bigoplus_{k=-6}^{6} \mathcal{H}_k$. Denote by $|\chi^k\rangle \in \mathcal{H}_{-k}$ the state (95) with the ghost number $-k$, i.e., $gh(|\chi^k\rangle) = -k$. Thus, the physical state with the ghost number zero is $|\chi^0\rangle$, and the gauge parameters $|\Lambda\rangle$ with the ghost number -1 are $|\chi^1\rangle$, and so on. For the vanishing of all the auxiliary creation operators B^+ and ghost variables $\eta_0, \eta_i^+, \mathcal{P}_i^+, \ldots$, the vector $|\chi^0\rangle$ must contain only the physical string-like vector $|\Phi\rangle = |\Phi(a_i^+)^{(0)_{fo}(0)_{fi}(0)_{pj}(0)_{flm}(0)_{pno}(0)_{f12}(0)_{\lambda 12}}_{(0)_l(0)_{ij}0_{12}}\rangle$,

$$|\chi^0\rangle \quad = \quad |\Phi\rangle + |\Phi_A\rangle, \quad |\Phi_A\rangle\Big|_{[B^+=\eta_0=\eta_i^+=\mathcal{P}_i^+=\eta_{lm}^+=\mathcal{P}_{no}^+=\vartheta_{12}^+=\lambda_{12}^+=0]} = 0. \quad (96)$$

One can show, using some of the equations of motion and gauge transformations, that the vector $|\Phi_A\rangle$ can be completely removed.

The equation for the physical state $Q'|\chi^0\rangle = 0$ and the tower of reducible gauge transformations $\delta|\chi\rangle = Q'|\chi^1\rangle$, $\delta|\chi^1\rangle = Q'|\chi^2\rangle, \ldots, \delta|\chi^{(s-1)}\rangle = Q'|\chi^{(s)}\rangle$ lead to the following relations:

$$\begin{aligned} Q|\chi\rangle &= 0, & (\sigma^i + h^i)|\chi\rangle = 0, && (\epsilon, gh)(|\chi\rangle) = (0,0), && (97)\\ \delta|\chi\rangle &= Q|\chi^1\rangle, & (\sigma^i + h^i)|\chi^1\rangle = 0, && (\epsilon, gh)(|\chi^1\rangle) = (1,-1), && (98)\\ \ldots & \quad \ldots & \ldots && && \\ \delta|\chi^{s-1}\rangle &= Q|\chi^s\rangle, & (\sigma^i + h^i)|\chi^s\rangle = 0, && (\epsilon, gh)(|\chi^s\rangle) = (s\, mod\, 2, -s). && (99) \end{aligned}$$

Here, $s = 5$ is the maximal stage of reducibility for a massive bosonic HS field, due to the subspaces $\mathcal{H}_k = \emptyset$, for all integer $k \le -7$. The middle set of Equations (97)–(99) determines the possible values of the parameters h^i and the eigenvectors of the operators σ^i. Solving the spectral problem, we obtain a set of eigenvectors, $|\chi^0\rangle_{\vec{n}_2}, |\chi^1\rangle_{\vec{n}_2}, \ldots, |\chi^s\rangle_{\vec{n}_2}$, $n_1 \ge n_2 \ge 0$, and a set of eigenvalues,

$$\sigma_i|\chi\rangle_{\vec{n}_2} = \left(n^i + \tfrac{d-1-4i}{2}\right)|\chi\rangle_{\vec{n}_2}, \ -h^i = n_i + \tfrac{d-1-4i}{2} \quad i = 1,2, \ n_1 \in \mathbf{Z}, \ n_2 \in \mathbf{N}_0\,. \quad (100)$$

It is easy to see that, in order to construct a Lagrangian for a field corresponding to a certain Young tableaux (1), the numbers n_i must be equal to the numbers of boxes in the i-th row of the corresponding Young tableaux, i.e., $n_i = s_i$. Thus, the state $|\chi\rangle_{\vec{s}_2}$ contains the physical field (7) and all of its auxiliary fields. Here, we fix some values of $n_i = s_i$, and after the substitution $h^i \to h^i(s_i)$, the operator $Q_{\vec{s}_2} \equiv Q_{|h^i \to h^i(s_i)}$ becomes nilpotent in each subspace $\mathcal{H}_{tot\vec{s}_2}$ whose vectors satisfy (97) for (100). Having in mind the Lagrangian equations of motion (those corresponding to (2)–(5) for $k = 2$), the sequence of reducible gauge transformations acquires the form

$$Q_{\vec{s}_2}|\chi^0\rangle_{\vec{s}_2} = 0, \quad \delta|\chi^s\rangle_{\vec{s}_2} = Q_{\vec{s}_2}|\chi^{s+1}\rangle_{\vec{s}_2}, \ s = 0, \ldots, 5. \quad (101)$$

By analogy with the totally symmetric bosonic HS fields [21,37], one can show that the Lagrangian action for a fixed spin $\vec{n}_2 = \vec{s}_2$ is defined up to an overall factor, as follows:

$$\mathcal{S}^m_{0|\vec{s}_2} = \int d\eta_0 \;_{\vec{s}_2}\langle\chi^0|K_{\vec{s}_2}Q_{\vec{s}_2}|\chi^0\rangle_{\vec{s}_2}, \text{ for } |\varnothing^0\rangle \equiv |\varnothing\rangle, \tag{102}$$

where the usual inner product for the creation and annihilation operators is assumed, with the measure $d^dx\sqrt{|g|}$ over the AdS space. The vector $|\chi^0\rangle_{(s)_2}$ and the operator $K_{\vec{s}_2}$ in (102) are, respectively, the vector $|\chi\rangle$ (95), subject to the spin distribution relations (100) for the HS tensor field $\Phi_{(\mu^1)_{s_1},(\mu^2)_{s_2}}(x)$, and the operator K (89), where the substitution $h_i \to -(n_i + \frac{d-1-4i}{2})$ has been made. The corresponding LF for the bosonic field of spin $\mathbf{s}$, subject to $Y(s_1,s_2)$, is a reducible gauge theory of $L = 6$ stage of reducibility at the most.

One can prove that the equations of motion (101) reproduce only the basic conditions (2)–(5) for HS fields of a given spin $\vec{s}_2$ and mass m. Indeed, the corresponding analysis is presented in Appendix E and repeats a similar proof of the paper [59] for arbitrary bosonic HS fields in flat spaces (with the AdS space being the only specific one), and allows one to gauge away all the auxiliary fields so that the remaining physical vector satisfies only the AdS group irreducible conditions (2)–(5). Therefore, the resulting equations of motion, due to the representation (96), acquire the form

$$L_0|\Phi\rangle_{\vec{s}_2} = (l_0 + m_0^2)|\Phi\rangle_{\vec{s}_2},\; (l_i, l_{ij}, t_{i_1j_1})|\Phi\rangle_{\vec{s}_2} = (0,0,0),\; i \le j,\; i_1 < j_1. \tag{103}$$

The above relations allow one, in a unique way, to determine the parameter m_0 in terms of $h^i(s_i)$,

$$m_0^2 = m^2 + r\Big\{\beta(\beta+1) + \tfrac{d(d-6)}{4} + \Big(h^1 - \tfrac{1}{2} + 2\beta\Big)\Big(h^1 - \tfrac{5}{2}\Big) + \Big(h^2 - \tfrac{9}{2}\Big)\Big\}, \tag{104}$$

whereas the values of parameters m_1, m_2 remain arbitrary, and can be used to provide the characteristic properties of a Lagrangian for a given HS field.

The general action (102) presents a direct recipe to obtain a Lagrangian for any component field $\Phi_{(\mu^1)_{s_1},(\mu^2)_{s_2}}(x)$, starting from a general vector $|\chi^0\rangle_{\vec{s}_2}$ since we only need to calculate the vacuum expectation values of products for certain creation and annihilation operators. Examples of applying the general action are given below in Section 6. We emphasize that Lagrangian formulations for massive bosonic HS fields described by a Young tableaux with two rows subject to traceless and Young constraints in (A)dS spaces have only been known in a frame-like form: see [72].

5. BRST Approach to Cubic Interaction Vertices

Here, we follow our results on constructing general cubic vertices for massless [38,94] and massive [96] totally symmetric fields of integer higher spin in Minkowski spaces by using two approaches. The first one is based on an ambient formalism of embedding an AdS$_d$ space in a $(d+1)$-dimensional Minkowski space [100]; see also [43] and the references therein. Now, we use the second of the above approaches to a description of AdS$_d$ spaces: with no reference to flat space-times.

For completeness, we draw the reader's attention to the results of Lagrangian cubic vertices construction in AdS$_d$ spaces [101,102], using various methods, basically for reducible interacting totally symmetric HS fields, Maxwell-like Lagrangians, and the $d = 3$ case [103], including interaction with gravity for a partially massless spin-2 field [104]. Also, cubic vertices deduced from the Fradkin–Vasiliev recipe [105,106] are shown to have a smooth flat limit for AdS vertices [107,108].

To construct a cubic vertex, we consider three copies of vectors $|\chi^{(i)}\rangle_{\vec{s}^i_2}$ and gauge parameters $|\chi^{(i)1}\rangle_{\vec{s}^i_2}, \ldots, |\chi^{(i)6}\rangle_{\vec{s}^i_2}$ from a corresponding Hilbert space $\mathcal{H}^{(i)}_{tot}$, with the respective vacuum vectors $|0\rangle^i$ and oscillators for $i = 1,2,3$. This allows one to obtain a deformed action and a sequence of deformed reducible gauge transformations for a massive triple $(m)_3 \equiv (m_1, m_2, m_3)$, with accuracy up to the first order in the parameter g:

$$S^{(m)_3}_{[1]|(\vec{s})^3}[\chi^{(1)},\chi^{(2)},\chi^{(3)}] = \sum_{i=1}^{3}\mathcal{S}^{m_i}_{0|\vec{s}^i_2} + g\int\prod_{e=1}^{3} d\eta_0^{(e)}\Big({}_{\vec{s}^e_2}\langle\chi^{(e)}K^{(e)}|V^{(3)}\rangle^{(m)_3}_{(\vec{s})^3_2} + h.c.\Big), \quad (105)$$

$$\delta_{[1]}|\chi^{(i)}\rangle_{\vec{s}^i_2} = Q^{(i)}|\chi^{(i)1}\rangle_{\vec{s}^i_2} - g\int\prod_{e=1}^{2} d\eta_0^{(i+e)}\Big({}_{\vec{s}^{i+1}_2}\langle\chi^{(i+11)}K^{(i+1)}|_{\vec{s}^{i+2}_2}\langle\chi^{(i+2)}K^{(i+1)}| + (i+1\leftrightarrow i+2)\Big)|\widetilde{V}^{(3)}_0\rangle^{(m)_3}_{(s)^3_2}, \quad (106)$$

$$\delta_{[1]}|\chi^{(i)1}\rangle_{\vec{s}^i_2} = Q^{(i)}|\chi^{(i)2}\rangle_{\vec{s}^i_2} - g\int\prod_{e=1}^{2} d\eta_0^{(i+e)}\Big({}_{\vec{s}^{i+1}_2}\langle\chi^{(i+1)2}K^{(i+1)}|_{\vec{s}^{i+2}_2}\langle\chi^{(i+2)}K^{(i+1)}| + (i+1\leftrightarrow i+2)\Big)|\widetilde{V}^{(3)}_1\rangle^{(m)_3}_{(\vec{s})^3_2}, \quad (107)$$

$$\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots$$

$$\delta_{[1]}|\chi^{(i)5}\rangle_{\vec{s}^i_2} = Q^{(i)}|\chi^{(i)6}\rangle_{\vec{s}^i_2} - g\int\prod_{e=1}^{2} d\eta_0^{(i+e)}\Big({}_{\vec{s}^{i+1}_2}\langle\chi^{(i+1)6}K^{(i+1)}|_{\vec{s}^{i+2}_2}\langle\chi^{(i+2)}K^{(i+1)}| + (i+1\leftrightarrow i+2)\Big)|\widetilde{V}^{(3)}_5\rangle^{(m)_3}_{(\vec{s})^3_2}, \quad (108)$$

with some unknown three-vectors $|V^{(3)}\rangle^{(m)_3}_{(\vec{s})^3_2}$, $|\widetilde{V}^{(3)}_l\rangle^{(m)_3}_{(\vec{s})^3_2}$, $l = 0,1,\dots,5$, defined in a total Hilbert space $\bigotimes_{i=1}^{3}\mathcal{H}^{(i)}_{tot}$. Here, $\mathcal{S}^{m_i}_{0|\vec{s}^i_2}$ is the free action (102) for the field $|\chi^{(i)}\rangle_{\vec{s}^i_2}$; $Q^{(i)}$ is the BRST charge (91) corresponding to the spin $\vec{s}^i_2$, $i = 1,2,3$; $K^{(i)}$ is the operator K (89) related to the spin $\vec{s}^i_2$, $i = 1,2,3$ for a massive field; and g is a deformation parameter, usually called a coupling constant. We also use the convention $[i+3 \simeq i]$.

The construction of a cubic interaction involves finding some 3-vectors $|V^{(3)}\rangle^{(m)_3}_{(\vec{s})^3_2}$, $|\widetilde{V}^{(3)}_l\rangle^{(m)_3}_{(\vec{s})^3_2}$. For this purpose, we can apply the set of fields, constraints, and ghost operators related to the spins $\vec{s}^1_2, \vec{s}^2_2, \vec{s}^3_2$, as well as the respective conditions of gauge invariance for the deformed action under the deformed gauge transformations, and also the preservation of the form of gauge transformations for the fields $|\chi^{(i)}\rangle_{\vec{s}^i_2}$ under the first-level gauge transformations $\delta_{[1]}|\chi^{(i)1}\rangle_{\vec{s}^i_2}$, having the form of l-th level gauge transformations for the gauge parameters $|\chi^{(i)l}\rangle_{\vec{s}^i_2}$ under the $(l+1)$-th level gauge transformations $\delta_{[1]}|\chi^{(i)l+1}\rangle_{\vec{s}^i_2}$ for $l = 1,\dots,5$ at the first power in g:

$$g\int\prod_{e=1}^{3} d\eta_0^{(e)}\,{}_{\vec{s}^j_2}\langle\chi^{(j)1}K^{(j)}|_{\vec{s}^{j+1}_2}\langle\chi^{(j+1)}K^{(j+1)}|_{\vec{s}^{j+2}_2}\langle\chi^{(j+2)}K^{(j+2)}|\mathcal{Q}(V^3,\widetilde{V}^3_0) = 0, \quad (109)$$

$$g\int\prod_{e=1}^{2} d\eta_0^{(e)}\,{}_{\vec{s}^{j+1}_2}\langle\chi^{(j+1)2}K^{(j+1)}|_{\vec{s}^{j+2}_2}\langle\chi^{(j+2)}K^{(j+2)}|\Big(\mathcal{Q}(\widetilde{V}^3_0,\widetilde{V}^3_1) - Q^{(j+2)}|\widetilde{V}^{(3)}_1\rangle\Big) = 0, \quad (110)$$

$$\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots$$

$$g\int\prod_{e=1}^{2} d\eta_0^{(e)}\,{}_{\vec{s}^{j+1}_2}\langle\chi^{(j+1)6}K^{(j+1)}|_{\vec{s}^{j+2}_2}\langle\chi^{(j+2)}K^{(j+2)}|\Big(\mathcal{Q}(\widetilde{V}^3_4,\widetilde{V}^3_5) - Q^{(j+2)}|\widetilde{V}^{(3)}_5\rangle\Big) = 0, \quad (111)$$

where

$$\mathcal{Q}(\widetilde{V}^3_l,\widetilde{V}^3_{l+1}) = \sum_{k=1}^{3} Q^{(k)}|\widetilde{V}^{(3)}_{l+1}\rangle^{(m)_3}_{(\vec{s})^3_2} + Q^{(j)}\Big(|\widetilde{V}^{(3)}_l\rangle^{(m)_3}_{(\vec{s})^3_2} - |\widetilde{V}^{(3)}_{l+1}\rangle^{(m)_3}_{(\vec{s})^3_2}\Big),\ j = 1,2,3, \quad (112)$$

for $l = -1,\dots,4$ and $\widetilde{V}^3_{-1} \equiv V^3$.

Following our results [38,94], we assume coincidence for the vertices $|V^{(3)}\rangle = |\widetilde{V}^{(3)}_0\rangle = |\widetilde{V}^{(3)}_e\rangle$, $e = 1,\dots,5$, which provides the validity of the operator generating equations at the first order in g (the highest orders being necessary to find the quartic and higher vertices),

$$Q^{tot}|V^{(3)}\rangle^{(m)_3}_{(\vec{s})^3_2} = 0, \qquad \vec{\sigma}^{(i)}_2|V^{(3)}\rangle^{(m)_3}_{(\vec{s})^3_2} \; = \; 0, \tag{113}$$

along with the spin conditions as a consequence of the generalized spin Equation (100) for each sample (with $|\chi^{(i)}\rangle_{\vec{s}^i_2}$, $|\chi^{(i)1}\rangle_{\vec{s}^i_2},\dots,|\chi^{(i)6}\rangle_{\vec{s}^i_2}$), providing the nilpotency of a total BRST operator $Q^{tot} \equiv \sum_i Q^{(i)}$ when evaluated on the vertex, due to Equation (93) and $\{Q^{(i)}, Q^{(j)}\} = 0$ for $i \neq j$.

A local dependence on the space-time coordinates in the vertices (given by Figure 1) $|V^{(3)}\rangle$, $|\widetilde{V}^{(3)}_e\rangle$ implies

$$|V^{(3)}\rangle^{(m)_3}_{(\vec{s})^3_2} = \int d^dx\sqrt{|g|}\prod_{i=1}^{3}\delta^{(d)}\big(x - x_i\big)\,V^{(3)|(m)_3}_{(\vec{s})^3_2}\prod_{j=1}^{3}\eta^{(j)}_0|0\rangle, \quad |0\rangle \equiv \otimes^3_{e=1}|0\rangle^e, \tag{114}$$

for $(\varepsilon, gh)V^{(3)|(m)_3}_{(\vec{s})^3_2} = (0,0)$.

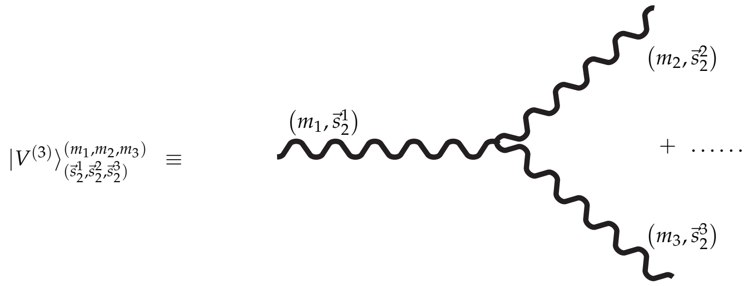

Figure 1. The interaction vertex $|V^{(3)}\rangle^{(m)_3}_{(\vec{s})^3_2}$ for massive fields $\Phi^{(i)}_{\mu^1(s^i_1)\mu^2(s^i_2)}$ of masses m_i and generalized spins $\vec{s}^i_2$ for $i = 1,2,3$. The terms in "..." correspond to the auxiliary fields of $|\Phi^{(i)}\rangle_{\vec{s}^i_2}$.

Here, we have a conservation law $\sum_{i=1}^3 p^{(i)}_\mu = 0$ for the momenta associated with all the vertices. Once again, as in the case of flat space-times [38,96], the deformed gauge transformations form a closed algebra, which implies, after a simple calculation,

$$\begin{aligned}
&[\delta^{\chi^1_1}_{[1]}, \delta^{\chi^1_2}_{[1]}]|\chi^{(i)}\rangle \;=\; -g\delta^{\chi^1_3}_{[1]}|\chi^{(i)}\rangle,\\
&|\chi^{(i)1}_3\rangle \sim \int\prod_{e=1}^{2} d\eta^{(i+e)}_0\Big(\langle\chi^{(i+1)1}_2 K|\langle\chi^{(i+2)1}_1|K + \big(i+1 \leftrightarrow i+2\big)\Big) - \big(\chi^1_1 \leftrightarrow \chi^1_2\big)\Big\}|V^{(3)}\rangle,
\end{aligned} \tag{115}$$

with the Grassmann-odd gauge parameter χ^1_3 being a function of the parameters χ^1_1, χ^1_2 : $\chi^1_3 = \chi^1_3(\chi^1_1, \chi^1_2)$.

Equation (113) for identical vertices, $V^{(3)} = \widetilde{V}^{(3)}_0 = \dots = \widetilde{V}^{(3)}_5$, along with the commutator of gauge transformations (115), determine the cubic interaction vertices for irreducible massive mixed-symmetric, $Y^{(i)}(2) \equiv Y(s^{(i)}_1, s^{(i)}_2)$, higher-spin fields in an AdS$_d$ space. To solve the generating Equation (114), we should find BRST-closed forms, e.g., having, in the flat-space limit $r \to 0$, the following expressions, in a so-called *minimal derivative scheme* for different masses $m_i \neq m_j$, $i \neq j$,

$$L_t^{(i)} = \widehat{p}_\mu^{(i)} a_t^{(i)\mu+} + \frac{m_{i+1}^2 - m_{i+2}^2}{m_i} b_t^{(i)+} - \imath \widehat{\mathcal{P}}_{0|t}^{(i)} \eta_t^{(i)+} + \mathcal{O}(r), \quad i = 1,2,3, \ t = 1,2; \tag{116}$$

$$L_{st}^{(ii+1)+} = a_s^{(i)\mu+} a_{\mu|t}^{(i+1)+} - \frac{1}{2}\mathcal{P}_s^{(i)+}\eta_t^{(i+1)+} - \frac{1}{2}\mathcal{P}_t^{(i+1)+}\eta_s^{(i)+} + \frac{b_s^{(i)+}}{2m_i} L_t^{(i+1)} - \frac{b_t^{(i+1)+}}{2m_{i+1}} L_s^{(i)} + \mathcal{O}(r); \tag{117}$$

$$L_{st}^{(ii)+} = \frac{1}{2} a_s^{(i)\mu+} a_{\mu|t}^{(i)+} - (1/2) b_s^{(i)+} b_t^{(i)+} + b_{st}^{(i)+} + \mathcal{P}_s^{(i)+}\eta_t^{(i)+} + \mathcal{P}_t^{(i)+}\eta_s^{(i)+} + \mathcal{O}(r), \tag{118}$$

for $s,t = 1,2$, $\widehat{p}_\mu^{(i)} = p_\mu^{(i+1)} - p_\mu^{(i+2)}$, $p_\mu^{(i)} = -\imath D_\mu^{(i)}$ and $\widehat{\mathcal{P}}_{0|t}^{(i)} = \mathcal{P}_{0|t}^{(i)} - \mathcal{P}_{0|t}^{(i+2)}$. The above operators are such as those suggested in the case of a Minkowski space-time for reducible mixed-symmetry integer spins [89], with an incomplete BRST operator and without the vanishing of non-dynamical algebraic (trace and mixed symmetry) constraints, as the operator is evaluated on the vertices (see [96] for details), being, at the same time, augmented by the trace and mixed-symmetry operators, according to our presentation [96] of cubic vertices for irreducible totally symmetric higher-spin fields in $\mathbb{R}^{1,d-1}$ with a complete BRST operator.

A solution for the cubic vertex (114) satisfying the generating Equation (113) requires extending the operators (116)–(118) by means of mixed-symmetric oscillators, and adding some new BRST-closed trace and mixed-symmetric operators, which poses a separate problem outside the scope of the present article.

6. Examples

Here, we apply the general prescriptions of our Lagrangian formulation to the case of free mixed-symmetry bosonic fields at the lowest values of rows and spins.

6.1. Spin-$(s_1, 0)$ Totally Symmetric Tensor Field

Let us consider a totally symmetric field $\Phi_{\mu^1(s_1)}$ corresponding to spin $(s_1, 0)$, i.e., to the Young tableaux $Y(s_1)$. We suppose that our result should reduce to the one examined in [21] for totally symmetric massive bosonic fields in an AdS space. According to our procedure, we have $n_1 = s_1$, $n_2 = 0$. It can be shown that in the case $n_2 = 0$ (95), all the components related to the second row in the Young tableaux are equal to zero, namely,

$$n_{a2} = n_{b2} = n_{b22} = n_{b21} = n_b = n_{f2} = n_{p2} = n_{f22} = n_{p22} = n_{f12} = n_{p12} = n_{fg12} = n_{pg12} = 0\,. \tag{119}$$

Therefore, the state vector reduces to

$$|\chi\rangle = \sum_n (a_1^{\mu_1+}\cdots a_n^{\mu_n+})(b_1^+)^{n_{b1}}(b_{11}^+)^{n_{b11}}(\eta_0^+)^{n_{f0}}(\eta_1^+)^{n_{f1}}(\mathcal{P}_1^+)^{n_{p1}}(\eta_{11}^+)^{n_{f11}}(\mathcal{P}_{11}^+)^{n_{p11}} \times \chi(x)^{n_{b_1}0n_{b11}000n_{f0}n_{f1}0n_{f11}000}_{\mu_1\cdots\mu_{n_{a1}}0n_{p1}0n_{p11}000}|0\rangle\,, \tag{120}$$

which corresponds to the one of [21], with the Weyl ordering in the right-hand sides of non-linear supercommutators for the HS symmetry algebra $\mathcal{A}_c(Y(1), AdS_d)$. Then the BRST and spin operators (supercommuting with each other) are reduced to $\widetilde{Q}$ and $\widetilde{\sigma}$:

$$\begin{aligned}
\widetilde{Q} &= \tfrac{1}{2}\eta_0 L_0 + \textstyle\sum_i \eta_i^+ L^i + \eta_{11}^+(L_{11} + \eta_1\mathcal{P}_1) + \tfrac{\imath}{2}\eta_1^+\eta_1\mathcal{P}_0 \\
&\quad + r\Big\{\eta_0\eta_1^+\Big[2\mathcal{L}_{11}\mathcal{P}_1^+ + 2\mathcal{L}_1^+\mathcal{P}_{11} + \mathcal{G}_0^1\mathcal{P}_1\Big] + 2\eta_1^+\eta_1\mathcal{L}_{11}^+\mathcal{P}_{11}\Big\} + h.c.; \\
\widetilde{\sigma} &= G_0^1 - h^1 - \eta_1\mathcal{P}_1^+ + \eta_1^+\mathcal{P}_1 + 2(\eta_{11}^+\mathcal{P}_{11} - \eta_{11}\mathcal{P}_{11}^+).
\end{aligned} \tag{121}$$

The corresponding Lagrangian formulation for a free massive field of higher integer spin is given by the action

$$\mathcal{S}^m_{0|s_1}[\phi, \phi_1, \ldots] = \mathcal{S}^m_{0|s_1}[|\chi\rangle_{s_1}] = \int d\eta_{0s_1} \langle\chi|K\widetilde{Q}|\chi\rangle_{s_1}, \tag{122}$$

which is invariant under the reducible gauge transformations

$$\delta|\chi\rangle_{s_1} = \widetilde{Q}|\chi^1\rangle_{s_1}, \;\; \delta|\chi^1\rangle_{s_1} = \widetilde{Q}|\chi^2\rangle_{s_1}, \;\; \delta|\chi^2\rangle_{s_1} = 0, \tag{123}$$

and thus determines, at most, a first-stage reducible abelian gauge theory.

A cubic vertex for three copies of massive interacting totally symmetric fields $(m_i; s_i)$, for $i = 1, 2, 3$, being determined as a solution of the generating equations

$$\widetilde{Q}^{tot}|V^{(3)}\rangle^{(m)_3}_{(s)^3} = 0, \qquad \widetilde{\sigma}^{(i)}|V^{(3)}\rangle^{(m)_3}_{(s)^3} \;=\; 0, \tag{124}$$

for $\widetilde{Q}^{tot} = \sum_i \widetilde{Q}^{(i)}$, may have various structures, due to various relations among the masses, and thus poses a separate problem. For non-coincident masses, however, the vertex may have a structure generated by the monomials (116)–(118), for $t = s = 1$.

We emphasize that for massless totally symmetric fields of integer helicities $\lambda^{(i)}$, the corresponding BRST and spin operators are simplified to the form presented in [37] so that, in addition to the above restrictions (119), there are no oscillators $b^{(i)}, b^{(i)+}$, and thus the additional parts o'_I become finite polynomials. The cubic vertices—for three massless fields, for one massless and two massive totally symmetric HS fields, as well as for two massless and one massive totally symmetric HS fields in an AdS space-time—can be found according to our approach [38,94,96].

6.2. *Spin-$(1,1)$ Antisymmetric Tensor Field*

Our next example is the simplest mixed-symmetry case, namely, a spin-$(1,1)$ rank-2 totally antisymmetric tensor field. Below, we denote all the gauge parameters by primed characters, and all of the gauge parameters of the first level for the gauge transformations, by double-primed characters. The values of the parameters h_i are equal to $h_i = -(\frac{d+1-4i}{2})$.

6.2.1. Action

Let us decompose the state vector (95), having a zero ghost number[14] and obeying (100) at $n_1 = n_2 = 1$, in the ghost oscillators, as well as in the auxiliary creation operators b^+_{11}, b^+_{22}, b^+_{12}, d^+_{12},

$$\begin{aligned} |\chi^0\rangle_{1,1} \;\;=\;\; & |\widetilde{\Phi}\rangle_{1,1} + \mathcal{P}^+_1\eta^+_1|\widetilde{\phi}_3\rangle_{-1,1} + \mathcal{P}^+_1\eta^+_2|\phi_4\rangle_{0,0} + \mathcal{P}^+_2\eta^+_1|\phi_5\rangle_{0,0} \\ & + \vartheta^+_{12}\mathcal{P}^+_1|A_1\rangle_{1,0} + \vartheta^+_{12}\mathcal{P}^+_{11}|\phi_6\rangle_{0,0} + \lambda^+_{12}\eta^+_{11}|A_2\rangle_{1,0} + \lambda^+_{12}\eta^+_{11}|\phi_7\rangle_{0,0} \\ & + \eta_0\Big(\mathcal{P}^+_1|\widetilde{H}\rangle_{0,1} + \mathcal{P}^+_2|A_4\rangle_{1,0} + \mathcal{P}^+_{11}|\widetilde{\phi}_8\rangle_{-1,1} + \mathcal{P}^+_{12}|\phi_{10}\rangle_{0,0} \\ & + \lambda^+_{12}|\widetilde{T}_2\rangle_{2,0} + \lambda^+_{12}\mathcal{P}^+_1\eta^+_1|\phi_{11}\rangle_{0,0}\Big), \end{aligned} \tag{125}$$

where the states $|\ldots\rangle$ in the right-hand sides depend on $a^+_{i,\mu}$, b^+_i, d^+_{12}, b^+_{1i}, and are expanded according to

$$|\widetilde{\Phi}\rangle_{1,1} \;=\; |\Phi\rangle_{1,1} + d^+_{12}|T_1\rangle_{2,0} + b^+_{12}|\phi_1\rangle_{0,0} + b^+_{11}d^+_{12}|\phi_2\rangle_{0,0}, \tag{126}$$

$$|\widetilde{H}\rangle_{0,1} \;=\; |H\rangle_{0,1} + d^+_{12}|A_3\rangle_{1,0}, \qquad |\widetilde{T}_2\rangle_{2,0} \;=\; |T_2\rangle_{2,0} + b^+_{11}|\phi_9\rangle_{0,0}, \tag{127}$$

$$|\Phi\rangle_{1,1} \;=\; \big(-a^{+\mu}_1 a^{+\nu}_2\Phi_{\mu\nu}(x) - ib^+_1 a^{+\mu}_2 H_{(2)\mu}(x) - ib^+_2 a^{+\mu}_1 A_{(7)\mu}(x) + b^+_1 b^+_2\phi_{(19)}(x)\big)|0\rangle, \tag{128}$$

$$|T_j\rangle_{2,0} \;=\; \big(-a^{+\mu}_1 a^{+\nu}_1 T_{(j)\mu\nu}(x) - ib^+_1 a^{+\mu}_1 A_{(4+j)\mu}(x) + (b^+_1)^2\phi_{(16+j)}(x)\big)|0\rangle, \tag{129}$$

$$|\widetilde{H}\rangle_{0,1} \;=\; \big(-ia^{+\mu}_2 H_\mu(x) + b^+_2\phi_{(16)}(x)\big)|0\rangle, \tag{130}$$

$$|A_j\rangle_{1,0} \;=\; \big(-ia^{+\mu}_1|0\rangle A_{(j)\mu}(x) + b^+_1\phi_{(11+j)}(x)\big)|0\rangle, \; j = 1,2,3,4, \tag{131}$$

$$|\widetilde{\phi}_k\rangle_{-1,1} \;=\; d^+_{12}|\phi_k\rangle_{0,0}, \; k = 3, 8, \tag{132}$$

$$|\phi_i\rangle_{0,0} \;=\; \phi_{(i)}(x)|0\rangle. \tag{133}$$

Then the relation (102) gives the Lagrangian action in a ghost-independent form

$$
\begin{aligned}
\mathcal{S}^m_{(1,1)}\Big[|\chi^0\rangle_{1,1}\Big] \;=\; & {}_{1,1}\langle\widetilde{\Phi}|K_{1.1}\Big\{\frac{1}{2}L_0|\widetilde{\Phi}\rangle_{1,1} - L_1^+|\widetilde{H}\rangle_{0,1} - L_2^+|A_4\rangle_{1,0} - L_{11}^+|\widetilde{\phi}_8\rangle_{-1,1} - L_{12}^+|\phi_{10}\rangle_{0,0} \\
& - T_{12}^+|\widetilde{T}_2\rangle_{2,0} + 2r\Big[(l_{11}^+ - l_{11}'^+)|\widetilde{\phi}_3\rangle_{-1,1} + (l_{12}^+ - l_{12}'^+)(|\phi_4\rangle_{0,0} + |\phi_5\rangle_{0,0}) \\
& + \frac{1}{2}(l_2^+ - l_2'^+)|A_1\rangle_{1,0}\Big]\Big\} + {}_{-1,1}\langle\widetilde{\phi}_3|K_{1.1}\Big\{\frac{1}{2}(L_0 - 2r[g_0^1 - g_0'^1])|\widetilde{\phi}_3\rangle_{-1,1} - L_1\widetilde{H}\rangle_{0,1} \\
& + T_{12}^+|\phi_{11}\rangle_{0,0} - |\widetilde{\phi}_8\rangle_{-1,1} + r(t_{12}^+ - t_{12}'^+)(|\phi_4\rangle_{0,0} + |\phi_5\rangle_{0,0})\Big\} + {}_{0,0}\langle\phi_4|K_{1.1}\Big\{(L_0 - r\sum_i(g_0^i \\
& - g_0'^i))|\phi_5\rangle_{0,0} - L_1|A_4\rangle_{1,0} - |\phi_{11}\rangle_{0,0} + \frac{1}{2}|\phi_{10}\rangle_{0,0}\Big\} + {}_{0,0}\langle\phi_5|K_{1.1}\Big\{ - L_2|\widetilde{H}\rangle_{0,1} \\
& - r(l_1 - l_1')|A_1\rangle_{1,0} + \frac{1}{2}|\phi_{10}\rangle_{0,0} - |\phi_{11}\rangle_{0,0}\Big\} + {}_{1,0}\langle A_1|K_{1.1}\Big\{(L_0 + r[g_0^1 - g_0'^1])|A_2\rangle_{1,0} + 2r(l_1^+ \\
& - l_1'^+)|\phi_7\rangle_{0,0} + L_1|\widetilde{T}\rangle_{2,0} + L_1^+|\phi_{11}\rangle_{0,0} + \frac{r}{2}(t_{12} - t_{12}')|\widetilde{H}\rangle_{0,1} - \frac{r}{2}\sum_m[g_0^m - g_0'^m])|A_4\rangle_{1,0}\Big\} \\
& + {}_{0,0}\langle\phi_6|K_{1.1}\Big\{L_0|\phi_7\rangle_{0,0} + L_{11}^+|\widetilde{T}\rangle_{2,0} - |\phi_{11}\rangle_{0,0} - \frac{1}{2}|\phi_{10}\rangle_{0,0}\Big\} \\
& - {}_{1,0}\langle A_2|K_{1.1}T_{12}|\widetilde{H}\rangle_{0,1} - {}_{0,0}\langle\phi_7|K_{1.1}|\phi_{10}\rangle_{0,0}\Big\} \\
& - \frac{1}{2}\Big({}_{0,1}\langle\widetilde{H}|K_{1.1}|\widetilde{H}\rangle_{0,1} + {}_{1,0}\langle A_4|K_{1.1}|A_4\rangle_{1,0}\Big) + h.c., \qquad (134)
\end{aligned}
$$

and a tensor form (for the leading field term),

$$
\begin{aligned}
\mathcal{S}^m_{(1,1)} \;=\; & \int d^dx\sqrt{|g|}\Big\{\Phi_{\mu\nu}(\nabla^2 - \frac{d(d-6)}{4} + m_0^2)\Phi^{\mu\nu} - 4T_{(2)(\mu\nu)}\Phi^{\mu\nu} \\
& + 2A_{(4)\mu}\nabla_\nu\Phi^{\mu\nu} + 2H_\nu\nabla_\mu\Phi^{\mu\nu} + \phi_{(10)}\Phi^\mu_\mu + \text{more}\Big\}. \qquad (135)
\end{aligned}
$$

Here, the mass parameter m_0^2 (104) equals to

$$
m_0^2 = m^2 + r\Big\{4 + \frac{d(d-10)}{2}\Big\}. \qquad (136)
$$

Let us now consider the set of gauge transformations.

6.2.2. Zero-Level Gauge Transformations

We decompose the vector (95), satisfying (100), for $n_1 = n_2 = 1$, and having the ghost number -1, as follows:

$$
\begin{aligned}
|\chi^1\rangle_{1,1} \;=\; & \mathcal{P}_1^+(|H'\rangle_{0,1} + d_{12}^+|A_1'\rangle_{1,0}) + \mathcal{P}_2^+|A_2'\rangle_{1,0} + \mathcal{P}_{11}^+d_{12}^+|\phi_1'\rangle_{0,0} + \mathcal{P}_{12}^+|\phi_2'\rangle_{0,0} \\
& + \lambda_{12}^+(|T_1'\rangle_{2,0} + b_{11}^+|\phi_3'\rangle_{0,0}) + \lambda_{12}^+\mathcal{P}_1^+\eta_1^+|\phi_4'\rangle_{0,0} \\
& + \eta_0\Big(\mathcal{P}_1^+\mathcal{P}_2^+|\phi_5'\rangle_{0,0} + \mathcal{P}_1^+\lambda_{12}^+|A_3'\rangle_{1,0} + \mathcal{P}_{11}^+\lambda_{12}^+|\phi_6'\rangle_{0,0}\Big), \qquad (137)
\end{aligned}
$$

where

$$
\begin{aligned}
|T'\rangle_{2,0} &= \big(- a_1^{+\mu}a_1^{+\nu}T'_{\mu\nu}(x) - ib_1^+a_1^{+\mu}A'_{(4)\mu}(x) + (b_1^+)^2\phi'_{(11)}(x)\big)|0\rangle, && (138) \\
|H'\rangle_{0,1} &= \big(- ia_2^{+\mu}H'_\mu(x) + b_2^+\phi'_{(10)}(x)\big)|0\rangle, && (139) \\
|A_j'\rangle_{1,0} &= \big(- ia_1^{+\mu}A'_{(j)\mu}(x) + b_1^+\phi'_{(6+j)}(x)\big)|0\rangle \; j = 1,2,3, && (140) \\
|\phi_i'\rangle_{0,0} &= \phi'_{(i)}(x)|0\rangle\,. && (141)
\end{aligned}
$$

Substituting (125), (137) into the relations (98) on the left, we obtain the following gauge transformations for the fields:

$$\delta\Phi_{\mu\nu} = -2T'_{\mu\nu} + \nabla_\mu H'_\nu + \nabla_\nu A'_{(2)\mu} - \frac{\eta_{\mu\nu}}{2}\phi'_{(2)}, \quad (142)$$

$$\delta T_{(1)\mu\nu} = T'_{\mu\nu} + \nabla_{\{\mu}A'_{(1)\nu\}} - \frac{\eta_{\mu\nu}}{2}\phi'_{(1)}, \quad (143)$$

$$\delta T_{(2)\mu\nu} = \left(\nabla^2 - r\frac{d(d-6)}{4} + m_0^2\right)T'_{\mu\nu} - \nabla_{\{\mu}A'_{(3)\nu\}} + \frac{\eta_{\mu\nu}}{2}\left(\phi'_{(6)} + 2r\phi'_{(4)} + \frac{r}{2}\phi'_{(5)}\right) - r\nabla_{\{\mu}A'_{(2)\nu\}}, \quad (144)$$

$$\delta A_{(1)\mu} = -2A'_{(1)\mu} - A'_{(2)\mu} - H'_\mu, \quad (145)$$

$$\delta A_{(2)\mu} = -2\nabla^\nu T'_{\mu\nu} - \frac{1}{m_1}(m_0^2 - r\left[\frac{d^2}{4} - \frac{17}{4}\right])A'_{(4)\mu} - \nabla_\mu\phi'_{(4)} + A'_{(3)\mu} + \frac{r}{2}\left(H'_\mu - 2A'_{(1)\mu} + A'_{(2)\mu}(1+2d)\right), \quad (146)$$

$$\delta A_{(3)\mu} = \left(\nabla^2 - r\frac{(d+4)(d-6)}{4} + r(d-\frac{3}{2}) + m_0^2\right)A'_{(1)\mu} + A'_{(3)\mu} + rA'_{(2)\mu}, \quad (147)$$

$$\delta A_{(4)\mu} = -\nabla_\mu\phi'_{(5)} + \left(\nabla^2 - r\frac{d(d-6)}{4} + m_0^2 + r(d-\frac{7}{2})\right)A'_{(2)\mu} - A'_{(3)\mu} + r(H'_\mu - 2A'_{(1)\mu}), \quad (148)$$

$$\delta A_{(5)\mu} = m_1 A'_{(1)\mu} + \nabla_\mu\phi'_{(7)} + A'_{(4)\mu}, \quad (149)$$

$$\delta A_{(6)\mu} = -mA'_{(3)\mu} - \nabla_\mu\phi'_{(9)} + \left(\nabla^2 - r\frac{d(d-6)}{4} + m_0^2\right)A'_{(4)\mu} - r(\nabla_\mu\phi'_{(8)} - A'_{(2)\mu} + (3-d)A'_{(4)\mu}), \quad (150)$$

$$\delta A_{(7)\mu} = m_2 A'_{(2)\mu} + \nabla_\mu\phi'_{(10)} - \frac{m_2}{m_1}A'_{(4)\mu}, \quad (151)$$

$$\delta H_\mu = \nabla_\mu\phi'_{(5)} + \left(\nabla^2 - r\frac{d(d-6)}{4} + m_0^2 + r(d-\frac{3}{2})\right)H'_\mu - A'_{(3)\mu} + rA'_{(2)\mu}, \quad (152)$$

$$\delta H_{(2)\mu} = m_1 H'_\mu + \nabla_\mu\phi'_{(8)} - A'_{(4)\mu}, \quad (153)$$

$$\delta\phi_{(1)} = \phi'_{(2)} - 2\phi'_{(3)}, \quad (154)$$

$$\delta\phi_{(2)} = \phi'_{(1)} + \phi'_{(3)}, \quad (155)$$

$$\delta\phi_{(3)} = -\nabla^\mu A'_{(1)\mu} - \frac{1}{m_1}(m_0^2 - r\left[\frac{d(d+4)}{4} - \frac{29}{4}\right])\phi'_{(7)} + \phi'_{(1)} + \phi'_{(4)}, \quad (156)$$

$$\delta\phi_{(4)} = -\nabla^\mu H'_\mu + \frac{1}{m_2}(m_0^2 - r\left[\frac{d(d-8}{4} + \frac{14}{4}\right])\phi'_{(10)} + 2r(3-d)\phi'_{(7)}, + \frac{1}{2}\phi'_{(2)} - \phi'_{(4)} - \phi'_{(5)}, \quad (157)$$

$$\delta\phi_{(5)} = -\nabla^\mu A'_{(2)\mu} - \frac{1}{m_1}(m_0^2 - r\left[\frac{d^2}{4} - 5\right])\phi'_{(8)} + \frac{1}{2}\phi'_{(2)} - \phi'_{(4)} + \phi'_{(5)}, \quad (158)$$

$$\delta\phi_{(6)} = -2\phi'_{(1)} - \phi'_{(2)}, \quad (159)$$

$$\delta\phi_{(7)} = \frac{d-3}{2}\phi'_{(3)} + T'^{\mu}_{\mu} + \phi'_{(4)} - \frac{1}{m_1^2}(m_0^2 - r\left[\frac{d^2-17}{4}\right])\phi'_{(11)} + \frac{1}{2}\phi'_{(2)}, \quad (160)$$

$$\delta\phi_{(8)} = \left(\nabla^2 - r\frac{d(d-6)}{4} + m_0^2\right)\phi'_{(1)} + \phi'_{(6)} + r\phi'_{(5)} - 2r\nabla^\mu A'_{(1)\mu} + \frac{2r^2}{m_2}(3-d)\phi'_{(10)} \quad (161)$$

$$-2r\left(m_0^2 - r\left[\frac{d(d+4)-29}{4}\right]\right)\phi'_{(7)}, \quad (162)$$

$$\delta\phi_{(9)} = \left(\nabla^2 - r\frac{d(d-6)}{4} + m_0^2\right)\phi'_{(3)} - \phi'_{(6)}, -2r(\phi'_{(4)} \quad (163)$$
$$+r(\phi'_{(8)} + \frac{r}{2}\phi'_{(5)} - 2(4-d)\phi'_{(11)-}),$$
$$\delta\phi_{(10)} = \left(\nabla^2 - r\frac{d(d-6)}{4} + m_0^2\right)\phi'_{(2)} - 2\phi'_{(6)} + 2r\phi'_{(5)} + 2r\Big\{\nabla^\mu A'_{(2)\mu} - \frac{2r}{m_1}(3-d)\phi'_{(7)} \quad (164)$$
$$-\frac{1}{m_1}\left(m_0^2 - r\left[\frac{d^2-17}{4}\right]\right)\phi'_{(8)} + \nabla^\mu H'_\mu + \left(m_0^2 - r\left[\frac{d(d+8)+7}{4}\right]\right)\phi'_{(10)}\Big\},$$
$$\delta\phi_{(11)} = \nabla^\mu A'_{(3)\mu} + \frac{1}{m_1}\left(m_0^2 - r\left[\frac{d^2-17}{4}\right]\right)\phi'_{(9)} + \left(\nabla^2 - r\frac{d(d-6)}{4} + m_0^2\right)\phi'_{(4)} - \phi'_{(6)} \quad (165)$$
$$+r(2d-3)\phi'_{(4)} + r\left(d - \frac{5}{2}\right)\phi'_{(5)} + 2r\left(T'^\mu{}_\mu + \frac{1}{m_1^2}\left(m_0^2 - r\left[\frac{d^2-17}{4}\right]\right)\phi'_{(11)}\right.$$
$$\left.+\frac{3-d}{2}\phi'_{(3)} + \frac{1}{2}\nabla^\mu H'\mu + \frac{1}{2m_2}\left(m_0^2 - r\left[\frac{d(d-8)+7}{4}\right]\right)\phi'_{(10)} + \frac{r}{m_1}(3-d)\phi'_{(7)}\right),$$

$$\delta\phi_{(12)} = -2\phi'_{(7)} - \phi'_{(8)} - \frac{m_1}{m_2}\phi'_{(10)}, \quad (166)$$
$$\delta\phi_{(13)} = -\nabla^\mu A'_{(4)\mu} + m_1(\phi'_{(3)} - \phi'_{(4)}) + \frac{2}{m_1}(m_0^2 - r\left[\frac{d(d-4)}{4} - \frac{13}{4}\right])\phi'_{(11)} + \phi'_{(9)} \quad (167)$$
$$+\frac{r}{2}\left(-\frac{m_2}{m_1}\phi'_{(10)} + 2\phi'_{(7)} + \phi'_{(8)}(2d-1)\right),$$
$$\delta\phi_{(14)} = \left(\nabla^2 - r\frac{d(d-10)-8}{4} + r(d-\frac{3}{2}) + m_0^2\right)\phi'_{(7)} + \phi'_{(9)} + r\phi'_{(7)} + 2r\frac{m_1}{m_2}\phi'_{(10)}, \quad (168)$$
$$\delta\phi_{(15)} = \left(\nabla^2 - r\frac{d(d-6)}{4} + m_0^2 + r(2d-\frac{9}{2})\right)\phi'_{(8)} - \phi'_{(9)} - r\left(2\phi'_{(7)} + \frac{m_1}{m_2}\phi'_{(10)}\right), \quad (169)$$
$$\delta\phi_{(16)} = m_2\phi'_{(5)} + \left(\nabla^2 - r\frac{d(d-6)}{4} + r(d-\frac{3}{2}) + m_0^2\right)\phi'_{(10)} - \frac{m_2}{m_1}(\phi'_{(9)} \quad (170)$$
$$+r\phi'_{(8)}) + r\frac{m_2}{2m_1}\phi'_{(7)},$$
$$\delta\phi_{(17)} = m_1\phi'_{(7)} + \phi'_{(11)}, \quad (171)$$
$$\delta\phi_{(18)} = -m\phi'_{(9)} + \left(\nabla^2 - r\frac{d(d-6)}{4} + m_0^2\right)\phi'_{(11)} + \frac{1}{2}\phi'_{(6)}, \quad (172)$$
$$\delta\phi_{(19)} = m_2\phi'_{(8)} + m_1\phi'_{(10)} - 2\frac{m_2}{m_1}\phi'_{(11)}. \quad (173)$$

Let us now turn to the gauge transformation for the gauge parameters.

6.2.3. First-Level Gauge Transformations

We decompose the vector (95), satisfying (100) at $n_1 = n_2 = 1$ and having the ghost number -2, according to

$$|\chi^2\rangle_{1,1} = \mathcal{P}_1^+\mathcal{P}_2^+|\phi_1''\rangle_{0,0} + \mathcal{P}_1^+\lambda_{12}^+|A''\rangle_{1,0} + \mathcal{P}_{11}^+\lambda_{12}^+|\phi_2''\rangle_{0,0}, \quad (174)$$

where

$$|A''\rangle_{1,0} = \big(-ia_1^{+\mu}A''_\mu(x) + b_1^+\phi''_{(3)}(x)\big)|0\rangle, \qquad |\phi_i''\rangle_{0,0} = \phi''_{(i)}(x)|0\rangle. \quad (175)$$

Substituting (137) and (174) into the relations (99) on the right, we find the gauge transformations for the gauge parameters:

$$\delta T'_{\mu\nu} = \nabla_{\{\mu}A''_{\nu\}} - \frac{1}{2}\eta_{\mu\nu}(\phi''_{(2)} + r\phi''_{(1)}), \qquad \delta A'_{(1)\mu} = -A''_{\mu}, \tag{176}$$

$$\delta A'_{(2)\mu} = \nabla_{\mu}\phi''_{(1)} - A''_{\mu}, \qquad \delta A'_{(3)\mu} = (\nabla^2 - r\frac{d(d-6)-34}{4} + m_0^2)A''_{\mu}, \tag{177}$$

$$\delta A'_{(4)\mu} = m_1 A''_{\mu} + \nabla_{\mu}\phi''_{(3)}, \qquad \delta H'_{\mu} = -\nabla_{\mu}\phi''_{(1)} + A''_{\mu}, \tag{178}$$

$$\delta\phi'_{(1)} = -\phi''_{(2)} + r\phi''_{(1)}, \qquad \delta\phi'_{(2)} = 2\phi''_{(2)} - 2r\phi''_{(1)}, \tag{179}$$

$$\delta\phi'_{(3)} = \phi''_{(2)} - r\phi''_{(3)}, \qquad \delta\phi'_{(5)} = (\nabla^2 - r\frac{d(d-6)-40}{4} + m_0^2)\phi''_{(1)}, \tag{180}$$

$$\begin{aligned} \delta\phi'_{(4)} &= -\nabla^{\mu}A''_{\mu} - \frac{1}{m_1}(m_0^2 - r\frac{d^2-14}{4})\phi''_{(3)} + \phi''_{(2)} + \frac{5}{2}r\phi''_{(1)}, \\ \delta\phi'_{(6)} &= (\nabla^2 - r\frac{d(d-6)-20}{4} + m_0^2)\phi''_{(2)} + 2r\nabla^{\mu}A''_{\mu} - \frac{1}{m_1}(m_0^2 - r\frac{d(d+2)-23}{2})\phi''_{(3)} - 10r^2\phi''_{(1)}, \end{aligned} \tag{181}$$

$$\delta\phi'_{(7)} = -\phi''_{(3)}, \qquad \delta\phi'_{(8)} = m_1\phi''_{(1)} + \phi''_{(3)}, \tag{182}$$

$$\delta\phi'_{(9)} = (\nabla^2 - r\frac{d(d-6)-34}{4} + m_0^2)\phi''_{(3)}, \qquad \delta\phi'_{(10)} = -m_2\phi''_{(1)} - \frac{m_2}{m_1}\phi''_{(3)}, \tag{183}$$

$$\delta\phi'_{(11)} = m_1\phi''_{(3)} - \frac{1}{2}\phi''_{(2)}. \tag{184}$$

6.2.4. Gauge-Fixing and Partial Use of Equations of Motion

We now fix the gauge symmetry completely, using the gauge-fixing conditions (A129) obtained in our general consideration. It is easy to see that the following fields are gauged away:

$$\phi_{(1)}, \phi_{(2)}, \phi_{(9)}, \phi_{(13)}, \phi_{(15)}, \phi_{(17)}, \phi_{(18)}, \phi_{(19)}, A_{(5)}, A_{(6)}, A_{(7)}, H_{(2)}, T_{(1)} \longrightarrow 0. \tag{185}$$

Then, using the equations of motion for all the fields, except the antisymmetric part of the basic field $\Phi_{[\mu\nu]}$, we find that we are left only with $\Phi_{[\mu\nu]}$ so that the corresponding Lagrangian action (134), up to a total derivative of a certain quantity f^{μ}, has the form

$$\begin{aligned} \mathcal{S}^m[\Phi_{[\mu\nu]}] &= \int d^dx\sqrt{|g|}\Big\{\Phi_{[\mu\nu]}(\nabla^2 + m^2)\Phi^{[\mu\nu]} + 2(\nabla_{\mu}\Phi^{[\mu\lambda]})(\nabla^{\nu}\Phi_{[\nu\lambda]}) \\ &= \int d^dx\sqrt{|g|}\Big\{-\frac{1}{3}F_{\mu\nu\lambda}F^{\mu\nu\lambda} + m^2\Phi_{[\mu\nu]}\Phi^{[\mu\nu]} + \nabla_{\mu}f^{\mu}\Big\}, \end{aligned} \tag{186}$$

where $F_{\mu\nu\lambda}$ stands for the field strength of $\Phi_{[\mu\nu]}$, namely,

$$F_{\mu\nu\lambda} = \nabla_{\lambda}\Phi_{[\mu\nu]} + \nabla_{\mu}\Phi_{[\nu\lambda]} + \nabla_{\nu}\Phi_{[\lambda\mu]}. \tag{187}$$

As a result, we obtained the gauge-invariant action (134) for a massive rank-2 antisymmetric higher integer spin field interacting with an AdS background field $g_{\mu\nu}$, containing a complete set of auxiliary fields and gauge parameters, as well as the action of a non-gauge formulation (186). Note that an equivalent form of the Lagrangian formulation for the field $\Phi_{[\mu\nu]}$ is constructed in [62], using an antisymmetric basis of the initial oscillators.

7. Conclusions

In this paper, we obtained quadratic non-linear HS symmetry algebras for a description of arbitrary integer HS fields defined in d-dimensional AdS space and subject to a k-row Young tableaux $Y(s_1,\ldots,s_k)$. We showed that the difference of the resulting algebras $\mathcal{A}(Y(k), AdS_d)$, $\mathcal{A}'(Y(k), AdS_d)$ and $\mathcal{A}_c(Y(k), AdS_d)$, corresponding, respectively, to the initial set of operators, their additional parts, and the converted set of operators within an additive conversion procedure, is due to purely non-linear parts, related to the AdS radius $(\sqrt{r})^{-1}$, in the set of AdS space isometry operators.

To obtain the above algebras, we start by embedding the bosonic HS fields into the vectors of an auxiliary Fock space, regarding the fields as coordinates of Fock space vectors, and reformulate the theory in these terms. We realize the conditions that determine an irreducible AdS group representation of a given mass and generalized spin in terms of differential operator constraints imposed on the Fock space vectors. These constraints generate a closed non-linear algebra of HS symmetry, which contains, with the exception of k basis generators in the Cartan subalgebra, a system of first- and second-class constraints. The above algebra coincides, modulo the isometry group generators, with its Howe-dual $sp(2k)$ symplectic algebra. The construction of a correct Lagrangian description requires the initial symmetry algebra to be deformed to the algebra $\mathcal{A}_c(Y(k), AdS_d)$, introducing the algebra $\mathcal{A}'(Y(k), AdS_d)$, different only due to the respective quadratic parts given by (83)–(85) and Tables 1 and 2, with the use of Weyl ordering for the right-hand sides.

We generalized the method of constructing a Verma module [19], starting from Lie (super)algebras [59,97–99] and the quadratic algebra $\mathcal{A}'(Y(1), AdS_d)$ for totally symmetric HS fields [20,21], to the case of a non-linear algebra underlying the mixed-symmetric HS bosonic fields in an AdS space with a two-row Young tableaux. Theorem 1 presents our basic result in this respect. We show, as a by-product of the Verma module construction, that the Poincaré–Birkhoff–Witt theorem is valid in the case of the algebra in question, thereby providing a lifting of the Verma module for the Lie algebra $\mathcal{A}(Y(2), R^{1,d-1})$—isomorphic to $\left(T^2 \oplus T^{2*}\right) \Subset sp(4)$—to the Verma module for an algebra quadratic in the deformation parameter r. The same is certainly expected to hold true in the case of the general algebra $\mathcal{A}'(Y(k), AdS_d)$, for which, we believe, it is pure machinery to obtain an explicit form of the Verma module in a recursive manner by means of some new *primary and derived block operators*, such as $\widehat{t}'_{12}$ (A92) and $\widehat{t}'^{+}_{12}$ (A94).

We obtained a representation for 15 generators of the algebra $\mathcal{A}'(Y(2), AdS_d)$ as a formal power series over the Heisenberg–Weyl algebra with six pairs of Grassmann-even oscillators, which, in the flat-space limit $r = 0$, takes a polynomial form identical with familiar results, at least for $m = 0$ [55], and appears to be novel in the massive case [56] for $k = 2$ [59]. Theorem A1 finalizes our second basic result, solving the Fock space realization problem for $\mathcal{A}'(Y(2), AdS_d)$ by using the approach (A90)–(A101) of a generalized Verma module construction.

On the basis of an exact BRST operator Q' constructed for the nonlinear algebra $\mathcal{A}_c(Y(2), AdS_d)$ of 15 converted constraints O_I (having a third order in the ghost coordinates), by analyzing the corresponding structure of Jacobi identities, we consistently develop a construction of gauge-invariant Lagrangian descriptions for bosonic HS fields of a given spin $\mathbf{s} = (s_1, s_2)$ and mass in an AdS$_d$ space. To this end, in the standard manner, we extract from Q' (88) the related BRST operator Q (91), which is associated with the converted first-class constraints alone. The latter operator is nilpotent only in those proper subspaces of the total Hilbert space which have vanishing eigenvalues of the spin operators $\vec{\sigma}_2 = (\sigma^1, \sigma^2)$ (92). The corresponding Lagrangian formulation is, at most, a 5-th-stage-reducible abelian gauge theory, and is given by (101) and (102), with a specific mass m_0 (104). These last relations may be regarded as our main result in solving the general problem of constructing a Lagrangian formulation for non-Lagrangian initial AdS group irreducible representation relations, which describe a bosonic HS field with two rows in the Young tableaux (2)–(5). It should be noted that unconstrained Lagrangians for free mixed-symmetry HS fields with two rows in the Young tableaux on AdS backgrounds, in both the metric-like and frame-like formalisms, have not been obtained until now. We emphasize that a Lagrangian description of massive bosonic HS fields described by a two-row Young tableaux in (A)dS spaces has been known only in the frame-like form [72], with off-shell traceless and Young constraints.

We used these results to study a deformation procedure in the approach with a complete BRST operator, along the lines of [38,94,96], so as to obtain an interacting theory, in particular, a system of generating Equation (112) for the cubic vertices of three copies of initial Lagrangian formulations, with a respective set of constraints and oscillators. The above system is shown to be specified by three Equation (113), in the particular case of coincident cubic vertices $V^3 = \widetilde{V}^3_l = \widetilde{V}^3_{l+1}$ for $l = 0, \dots, 5$, so that a nontrivial solution should be a BRST (Q)-closed three-vector with vanishing values of spin $\vec{\sigma}_2$. By construction, all the holonomic constraints (tracelessness and Young symmetry) are included on equal footing with the remaining differential constraints in the total BRST operator. This guarantees an equal number of physical degrees of freedom in interacting and free (undeformed) Lagrangian formulations, thus allowing the interacting model to describe an irreducible interacting triple of massive HS fields (see [109] for the recent progress in the necessary and sufficient conditions for an interacting model, obtained with the use of the constrained BRST approach so as to preserve irreducibility for interacting HS fields in flat spaces). Unfortunately, there is no familiar application of Lagrangian dynamics in an AdS space, even for a free HS field of a given mass and spin, in the approach with an incomplete BRST operator, due to the absence of off-shell BRST extended constraints.

In addition, we consider a number of simple examples of the proposed Lagrangian descriptions: we deduce one for a totally symmetric HS field obtained earlier in [21] and another one for an antisymmetric massive tensor field of spin $(1,1)$, whose first-stage reducible unconstrained Lagrangian formulation with a complete set of auxiliary fields and gauge parameters was obtained in Section 6.2. The final ungauged Lagrangian is given by (186) after an application of partial gauge-fixing and a resolution of some of the equations of motion and is shown to coincide with a Lagrangian given by a different approach [62].

As a first by-product, we obtain polynomial deformations of the $su(1,1)$ algebra related to the conversion analyzed in Appendix B. Second, we demonstrate that a BRST Lagrangian formulation for an HS field of mass m and spin (s_1, s_2) leads to equations of motions, which are equivalent to the initial irreducibility conditions (2)–(5). Third, we obtain a new Lagrangian formulation for the HS field of mass m and spin (s_1, s_2) in a d-dimensional flat space-time starting from a general description for the same field in an AdS space in the limit $r \to 0$, due to new representations for the Verma and Fock modules given, respectively, by (A63)–(A71) and (A107)–(A112), with the gauge-invariant Lagrangian action $\mathcal{S}^m_{0|\vec{s}_2} = \int d\eta_0 \,{}_{\vec{s}_2}\langle\chi|K^0_{\vec{s}_2} Q^0_{\vec{s}_2}|\chi\rangle_{\vec{s}_2}$ for the BRST operator $Q^0_{\vec{s}_2} = Q_{\vec{s}_2}|_{r=0}$ and $K^0_{\vec{s}_2} = K_{\vec{s}_2}|_{r=0}$. This Lagrangian is different from the one obtained in [56] in view of the dependence on two additional mass-like parameters m_1, m_2.

From the mathematical standpoint, the construction of a Verma module for the algebra $\mathcal{A}'(Y(2), AdS_d)$ opens a possibility of analyzing the module structure and searching for singular and subsingular vectors in it so that, in principle, this could then allow one to construct new (non-scalar) infinite-dimensional representations of the given algebra. The above results allow one, certainly, to understand the problem of (generalized) Verma module construction for the HS symmetry algebras and superalgebras underlying HS bosonic and (respectively) fermionic fields in AdS spaces subject to a multi-row Young tableaux.

We expect, first of all, to use the deformation procedure for irreducible totally symmetric massless HS fields elaborated for free fields in [37] as applied to AdS spaces, and also anticipate the concept of BRST–BV quantization as applied to deduce interacting models in the BRST approach, while adapting the algorithm developed in flat spaces [110], albeit with a complete BRST operator, as in the case of a minimal BRST–BV action [111]. As a subsquent problem to be solved, we intend to apply our results to an explicit construction of a generalized Verma module, a BRST operator, and an LF for irreducible massive half-integer HS fields of spin $(n_1 + 1/2, n_2 + 1/2)$ subject to $Y(n_1, n_2)$ in AdS spaces.

Among other directions of application and development of the suggested approach, such as finding LFs and implementing free theories and cubic vertices for irreducible massive half-integer higher-spin fields on AdS backgrounds, having in mind the possibility of taking a flat-space limit [107] for numerous cubic vertices [105,106], one should also mention the problems of constructing quartic and higher vertices, as well as the related problems of locality (see the discussions in [112–118]), which, as we believe, may be addressed using the BRST method. We also expect that (ir)reducible HS fields can be employed as composite fields in the recent approach [119,120].

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Appendix A. Proposition Proof

In this appendix, we examine the correctness of our proposition in Section 2.2. The analysis is based on deriving the multiplication laws (30), (31), (33) explicitly for the operator sets $\mathcal{A}'$ of o'_I and $\mathcal{A}_c$ of O_I. Namely, from the r.h.s. of (29), we find (due to the commutativity of o_I with o'_J) the following equations that determine the unknown structure functions $F^K_{IJ}(o', O)$:

$$[O_I, O_J\} = [o_I, o_J\} + [o'_I, o'_J\} = \sum_{m=1}^{n} f^{K_1\cdots K_m}_{ij} \prod_{l=1}^{m} o_{K_l} + [o'_I, o'_J\}. \tag{A1}$$

Expressing in (A1) the initial elements $o_{K_1}, \dots, o_{K_n}$ through the enlarged O_I and additional o'_I operators with the use of the $o'O$-ordering, we obtain a sequence of relations for each power of o_K,

$$f^{K_1}_{IJ} o_{K_1} = f^{K_1}_{IJ} O_{K_1} - f^{K_1}_{IJ} o'_{K_1}, \tag{A2}$$

$$f^{K_1K_2}_{IJ} o_{K_1} o_{K_2} = f^{K_1K_2}_{IJ} O_{K_1} O_{K_2} - (f^{K_1K_2}_{IJ} + f^{K_2K_1}_{IJ}) o'_{K_1} O_{K_2} + f^{K_2K_1}_{IJ} o'_{K_1} o'_{K_2}, \tag{A3}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots ,$$

$$f^{K_1\cdots K_n}_{IJ} \prod_{l=1}^{n} o_{K_l} = f^{K_1\cdots K_n}_{IJ} \prod\nolimits_{m=1}^{n} O_{K_m} + \sum\nolimits_{s=1}^{n-1} (-1)^s f^{\widehat{K_s\cdots K_1}\widehat{K_{s+1}\cdots K_n}}_{ij} \times \prod\nolimits_{p=1}^{s} o'_{K_p} \prod\nolimits_{m=s+1}^{n} O_{K_m} - (-1)^n f^{(n)K_n\cdots K_1}_{IJ} \prod\nolimits_{s=1}^{n} o'_{K_s}, \tag{A4}$$

where the hats in $f^{\widehat{K_s\cdots K_1}\widehat{K_{s+1}\cdots K_n}}_{ij}$ for the quantities $f^{K_s\cdots K_1K_{s+1}\cdots K_n}_{ij}$ designate a set of $C^n_s = \frac{n!}{s!(n-s)!}$ terms, obtained using $f^{\widehat{K_s\cdots K_1}\widehat{K_{s+1}\cdots K_n}}_{ij}$ by symmetrization as shown explicitly in (33). First, the above system (A2)–(A4) allows one, immediately, to establish, from the above rightmost terms in (A1)–(A4), that the set of o'_I forms a polynomial algebra $\mathcal{A}'$ of order n, subject to the algebraic relations (30). Second, the remaining terms in (A1)–(A4) completely determine the structure functions $F^{(m)K}_{IJ}(o', O)$, $m = 1, \dots, n$ in the form (32) and also show that the set of O_I actually determines the non-linear algebra[15] $\mathcal{A}_c$.

Appendix B. Polynomial Deformations of $su(1,1)$ Algebra

The polynomial algebra $\mathcal{A}^{n-1}$ of order $n-1$, as a deformation of the algebra $su(1,1)$ with three bosonic elements, $\{o_I\} = \{q_+, q_-, q_0\}$, $I = 1,2,3$, and a central extension [121], suggested in connection with quantum optics and the Bethe ansatz systems, determines the non-vanishing relations[16]

$$[q_0, q_\pm] = \pm q_\pm, \qquad [q_+, q_-] = \phi^{(n)}(q_0) - \phi^{(n)}(q_0 - 1), \tag{A5}$$
$$\phi^{(n)}(x) = -\prod_{l=1}^{n}\left(x + \frac{ln-1}{n^2}\right) + \prod_{l=1}^{n}\left(\frac{l-n}{n} - \frac{1}{n^2}\right). \tag{A6}$$

The second relation (A6) can be recast in a form similar to (28) as follows:

$$[q_+, q_-] = \sum_{l=0}^{n-1} f^{(l)} q_0^l, \quad f^{(l)} = \frac{d^{(l)}\phi^{(n)}}{dx^l}_{(x=0)} - \frac{d^{(l)}\phi^{(n)}}{dx^l}_{(x=-1)}. \tag{A7}$$

The study of [121] proposed a so-called one-mode boson realization of the algebra $\mathcal{A}^{n-1}$, in terms of a pair of creation and annihilation operators, b^+, b, $[b, b^+] = 1$,

$$q_+ = \frac{(b^+)^n}{(n)^{\frac{n}{2}}}, \qquad q_- = \frac{b^n}{(n)^{\frac{n}{2}}}, \qquad q_0 = \frac{1}{n}\left(b^+ b + \frac{1}{n}\right). \tag{A8}$$

The relations (A5), (A7) for $\mathcal{A}^{n-1}$ show that we are within the conditions of our Proposition 1, which implies the following composition law for the polynomial algebra of order $n-1$, $\mathcal{A}'^{n-1}$ with the elements q'_+, q'_-, q'_0,

$$[q'_0, q'_\pm] = \pm q'_\pm, \qquad [q'_+, q'_-] = \sum_{l=0}^{n-1} (-1)^{l-1} f^{(l)} q_0'^l, \tag{A9}$$

as we have chosen the value of the central extension for $\mathcal{A}'^{n-1}$ to be opposite to the one for $\mathcal{A}^{n-1}$, i.e., $(-f^{(0)})$, so that the central extension of the non-homogeneous polynomial algebra $\mathcal{A}_c^{n-1}$, $F^{(0)} = (f^{(0)} - f^{(0)})$ of order $n-1$ is zero. The composition law for the algebra $\mathcal{A}_c^{n-1}$ with the elements $Q_\pm = q_\pm + q'_\pm$, $Q_0 = q_0 + q'_0$, following the prescription (31) and (33), is determined by

$$[Q_0, Q_\pm] = \pm Q_\pm, \tag{A10}$$
$$[Q_+, Q_-] = \sum_{l=1}^{n-1} f^{(l)} Q_0^l + \sum_{l=2}^{n-1} f^{(l)} \sum_{s=1}^{l-1} (-1)^s \frac{l!}{s!(l-s)!} (q'_0)^s Q_0^{l-s}. \tag{A11}$$

The relations established for the polynomial algebras $\mathcal{A}_c^{n-1}$, $\mathcal{A}_c^{n-1}$ can be applied to various objectives. From the number of issues analyzed in this article, one can examine a Verma module and obtain some new representations for each of the above algebras, as well as constructing a BRST operator.

Appendix C. Verma Module Construction for $\mathcal{A}'(Y(2), AdS_d)$

Here, we describe the details of a Verma module construction [19] for the algebra $\mathcal{A}'(Y(2), AdS_d)$ so as to validate the relations (64)–(72) of Theorem 1. Starting from (53), (54), (56), (58)–(63), we formulate the following auxiliary.

Lemma A1. *The n-th power in the action of the operator* $\mathrm{ad}_{l_i'^+}$ *on the operators*[17] $W_b'^{12+}$, $X_b'^{12}$, $X_b'^{21}$ *(38)–(40) and the Casimir operator* $\mathcal{K}_0(2) \equiv \mathcal{K}_0$ *for the algebra* $sp(4)$*,*

$$\mathcal{K}_0 = \sum_i K_0^{0i} + 2\mathcal{K}_0^{12} = \sum_i \left((g_0'^i)^2 - 2g_0'^i - 4l'^{ii+}l'^{ii}\right) + 2\left(t_{12}'^+ t_{12}' - 4l_{12}'^+ l'^{12} - g_0'^2\right), \tag{A12}$$

denoted by $\left(\mathcal{K}^i_n, W'^{12+i}_{bn}, X'^{12}_{bin}, X'^{21}_{bin}\right) \equiv \mathrm{ad}^n_{l'^+_i}\left(\mathcal{K}_0, W'^{12+}_b, X'^{12}_b, X'^{21}_b\right)$*, yields the following set of relations for natural numbers n:*

$$\mathcal{K}^i_n = \left(-8rl'^{ii+}\right)^{[(n-1)/2]}\sum_{m=1}^{[(n+1)/2]}\left(\mathcal{K}^i_2\delta_{n,2m} + \mathcal{K}^i_1\delta_{n,2m-1}\right),\ n \geq 1, \quad (A13)$$

$$\mathcal{K}^{12i}_n = \left(-2rl'^{ii+}\right)^{[(n-1)/2]}\sum_{m=1}^{[(n+1)/2]}\left(\mathcal{K}^{12i}_2\delta_{n,2m} + \mathcal{K}^{12i}_1\delta_{n,2m-1}\right),\ n \geq 1, \quad (A14)$$

$$W'^{12+}_{bin} = \left(-2rl'^{ii+}\right)^{[n/2]}\sum_{m=0}^{[(n+1)/2]}\left(W'^{12+}_b\delta_{n,2m} + W'^{12+i}_b\delta_{n,2m-1}\right), \quad (A15)$$

$$X'^{12}_{b2n} = \left(-2rl'^{22+}\right)^{[n/2]}\sum_{m=0}^{[(n+1)/2]}\left(X'^{12}_b\delta_{n,2m} + X'^{12}_{b2}\delta_{n,2m-1}\right), \quad (A16)$$

$$X'^{21}_{b1n} = \left(-2rl'^{11+}\right)^{[n/2]}\sum_{m=0}^{[(n+1)/2]}\left(X'^{21}_b\delta_{n,2m} + X'^{21}_{b1}\delta_{n,2m-1}\right), \quad (A17)$$

where the symbol $[x]$ *denotes the integer part of a real-valued x; the operators* $\mathcal{K}^i_k$, $\mathcal{K}^{12i}_k = \mathrm{ad}^k_{l'^+_i}\mathcal{K}^{12}_0$, $k = 1,2$, W'^{12+i}_b, X'^{21}_{b1}, X'^{21}_{b2}*, in accordance with the rule (57) for the operation* $\mathrm{ad}^k_{l'^+_i}$*, and the action on the highest-weight vector* $|0\rangle_V$*, are defined, respectively, by (A72)–(A78) and (A79)–(A86) from Appendix C.1.*

The correctness of (A13)–(A17) follows from the calculations proceeding by induction in n.

The results of Lemma A1 imply the validity of the following rather technically.

Lemma A2. *The n-th power of the action of* $\mathrm{ad}_{l'^+_i}$ *on any* o'_I, $o'_I \in \{t'^+_{12}, l'^+_2, l'_0, \mathcal{E}^+_2\}$ *is determined by the relations (A18)–(A26),*

$$\left[l'^{ii}, \left(\tfrac{l'^{i+}}{m_i}\right)^{n_i}\right] = -\tfrac{n_i}{m_i}\left(\tfrac{l'^{i+}}{m_i}\right)^{n_i-1}l'^i - \tfrac{n_i(n_i-1)}{2m_i^2}\left(\tfrac{l'^{i+}}{m_i}\right)^{n_i-2}K'^{0i}_2 + 2r\sum_{m=1}^{[(n_i-1)/2]}\left(-8rl'^{ii+}\right)^{m-1}$$
$$\times\left(\tfrac{l'^{i+}}{m_i}\right)^{n_i-2m-2}\left(\tfrac{1}{m_i}\right)^{2m+1}\Big\{\tfrac{l'^{i+}}{m_i}C^{n_i}_{2m+1}[\mathcal{K}^i_1 - \tfrac{2}{4^m}\mathcal{K}^{12i}_1]$$
$$+ \tfrac{1}{m_i}C^{n_i}_{2m+2}[\mathcal{K}^i_2 - \tfrac{2}{4^m}\mathcal{K}^{12i}_2]\Big\}, \quad (A18)$$

$$\left[l'^i, \left(\tfrac{l'^{i+}}{m_i}\right)^{n_i}\right] = \tfrac{n_i}{m_i}\left(\tfrac{l'^{i+}}{m_i}\right)^{n_i-1}K'^{0i}_2 - 2r\sum_{m=1}^{[n_i/2]}\left(-8rl'^{ii+}\right)^{m-1}\left(\tfrac{l'^{i+}}{m_i}\right)^{n_i-2m-1}\times$$
$$\times\left(\tfrac{1}{m_i}\right)^{2m}\Big\{\tfrac{l'^{i+}}{m_i}C^{n_i}_{2m}[\mathcal{K}^i_1 - \tfrac{2}{4^m}\mathcal{K}^{12i}_1] + \tfrac{1}{m_i}C^{n_i}_{2m+1}[\mathcal{K}^i_2 - \tfrac{2}{4^m}\mathcal{K}^{12i}_2]\Big\}, \quad (A19)$$

$$\left[t'_{12}, \left(\tfrac{l'^{i+}}{m_i}\right)^{n_i}\right] = -\delta^{i2}\Big\{\tfrac{n_2}{m_2}\left(\tfrac{l'^{2+}}{m_2}\right)^{n_2-1}l'^{1+} + \sum_{m=1}^{[n_2/2]}\left(-2rl'^{22+}\right)^{m-1}\left(\tfrac{l'^{2+}}{m_2}\right)^{n_2-2m-1}\times$$
$$\times\left(\tfrac{1}{m_2}\right)^{2m}\left[\tfrac{l'^{2+}}{m_2}C^{n_2}_{2m}W'^{12+}_b + \tfrac{1}{m_2}C^{n_2}_{2m+1}W'^{12+}_{b2}\right]\Big\}, \quad (A20)$$

$$\left[l'^1, \left(\tfrac{l'^{2+}}{m_2}\right)^{n_2}\right] = \sum_{m=0}^{[n_2-1/2]}\left(-2rl'^{22+}\right)^m\left(\tfrac{l'^{2+}}{m_2}\right)^{n_2-2m-2}\left(\tfrac{1}{m_2}\right)^{2m+1}\times$$
$$\times\Big\{\tfrac{l'^{2+}}{m_2}C^{n_2}_{2m+1}X'^{12}_b + \tfrac{1}{m_2}C^{n_2}_{2m+2}X'^{12}_{b2}\Big\}, \quad (A21)$$

$$\left[l'^2, \left(\frac{l'^{1+}}{m_1}\right)^{n_1}\right] = \sum_{m=0}^{[n_1-1/2]}\left(-2rl'^{11+}\right)^m\left(\frac{l'^{1+}}{m_1}\right)^{n_1-2m-2}\left(\frac{1}{m_1}\right)^{2m+1}\times$$
$$\times\left\{\frac{l'^{1+}}{m_1}C^{n_1}_{2m+1}X'^{21}_b + \frac{1}{m_1}C^{n_1}_{2m+2}X'^{21}_{b1}\right\}, \quad \text{(A22)}$$

$$\left[l'^{12}, \left(\frac{l'^{i+}}{m_i}\right)^{n_i}\right] = -\frac{n_i}{2m_i}\left(\frac{l'^{i+}}{m_i}\right)^{n_i-1}l'^{\{1}\delta^{2\}i} - \delta^{i2}\frac{1}{2}\sum_{m=1}^{[n_2/2]}\left(-2rl'^{22+}\right)^{m-1}\left(\frac{l'^{2+}}{m_2}\right)^{n_2-2m-1}\times$$
$$\times\left(\frac{1}{m_2}\right)^{2m}\left\{\frac{l'^{2+}}{m_2}C^{n_2}_{2m}X'^{12}_b + \frac{1}{m_2}C^{n_2}_{2m+1}X'^{12}_{b2}\right\}$$
$$-\delta^{i1}\frac{1}{2}\sum_{m=1}^{[n_1/2]}\left(-2rl'^{11+}\right)^{m-1}\left(\frac{l'^{1+}}{m_1}\right)^{n_1-2m-1}\left(\frac{1}{m_1}\right)^{2m}\times$$
$$\times\left\{\frac{l'^{1+}}{m_1}C^{n_1}_{2m}X'^{21}_b + \frac{1}{m_1}C^{n_1}_{2m+1}X'^{21}_{b1}\right\}, \quad \text{(A23)}$$

$$\left[t'^+_{12}, \left(\frac{l'^{i+}}{m_i}\right)^{n_i}\right] = -\delta^{i1}\left\{\frac{n_1}{m_1}\left(\frac{l'^{1+}}{m_1}\right)^{n_1-1}l'^{2+} - \sum_{m=1}^{[n_1/2]}\left(-2rl'^{11+}\right)^{m-1}\left(\frac{l'^{1+}}{m_1}\right)^{n_1-2m-1}\times\right.$$
$$\left.\times\left(\frac{1}{m_1}\right)^{2m}\left[\frac{l'^{1+}}{m_1}C^{n_1}_{2m}W'^{12+}_b + \frac{1}{m_1}C^{n_1}_{2m+1}W'^{12+}_{b1}\right]\right\}, \quad \text{(A24)}$$

$$\varepsilon_{ij}\left[l'^{i+}, \left(\frac{l'^{j+}}{m_j}\right)^{n_j}\right] = -\sum_{m=0}^{[n_j-1/2]}\left(-2rl'^{jj+}\right)^m\left(\frac{l'^{j+}}{m_j}\right)^{n_j-2m-2}\left(\frac{1}{m_j}\right)^{2m+1}\times$$
$$\times\left\{\frac{l'^{j+}}{m_j}C^{n_j}_{2m+1}W'^{12+}_b + \frac{1}{m_j}C^{n_j}_{2m+2}W'^{12+}_{bj}\right\}, \quad \text{(A25)}$$

$$\left[l'_0, \left(\frac{l'^{i+}}{m_i}\right)^{n_i}\right] = -r\sum_{m=0}^{[(n_i-1)/2]}\left(-8rl'^{ii+}\right)^m\left(\frac{l'^{i+}}{m_i}\right)^{n_i-2m-2}\left(\frac{1}{m_i}\right)^{2m+1}\times$$
$$\times\left\{\frac{l'^{i+}}{m_i}C^{n_i}_{2m+1}\mathcal{K}^i_1 + \frac{1}{m_i}C^{n_i}_{2m+2}\mathcal{K}^i_2\right\}. \quad \text{(A26)}$$

In the above formulae, we need to take into account that $C^n_{n+k} = 0$ for any $n, k \in \mathbf{N}$, and also, in (A25), the fact that $\varepsilon_{ij}\varepsilon^{jl} = \delta^l_i$, $\varepsilon_{21} = 1$.

It is then easy to see that the undetermined action of the remaining operators from the subalgebra $\mathcal{E}^-_2$, l'^+_2, t'^+_{12} on the vector $|\vec{0}_{ij}, \vec{n}_s\rangle_V$ in (53), (54) reduces to the action of t'_{12} on the vector $|\vec{0}_{ij}, 0, 0, n_2\rangle_V = |n_2\rangle_V$. Explicitly, this is implied by the relations[18]

$$t'^+_{12}\left|\vec{0}_{ij}, \vec{n}_s\rangle_V\right. = \sum_{m=0}^{[n_1/2]}\left(\frac{-2r}{m_1^2}\right)^m\left\{C^{n_1}_{2m}\left|\vec{0}_{ij} + m\delta_{ij,11}, \vec{n}_s - 2m\delta_{s,1} + \delta_{s,12}\rangle_V\right.\right.$$
$$\left.-C^{n_1}_{2m+1}\frac{m_2}{m_1}\left|\vec{0}_{ij} + m\delta_{ij,11}, \vec{n}_s - (2m+1)\delta_{s,1} + \delta_{s,2}\rangle_V\right.\right\}$$
$$-\sum_{m=1}^{[n_1/2]}\left(\frac{-2r}{m_1^2}\right)^m\left\{\left(C^{n_1}_{2m}(h^2 - h^1 + 2p_{12} + n_2) - C^{n_1}_{2m+1}\right)\right.$$
$$\times\left|\vec{0}_{ij} + (m-1)\delta_{ij,11} + \delta_{ij,12}, \vec{n}_s - 2m\delta_{s,1}\rangle_V\right. + C^{n_1}_{2m}p_{12}(h^1 - h^2$$
$$\left.-n_2 - p_{12} + 1)\left|\vec{0}_{ij} + (m-1)\delta_{ij,11} + \delta_{ij,22}, \vec{n}_s - 2m\delta_{s,1} - \delta_{s,12}\rangle_V\right.\right\}$$
$$-\sum_{m=1}^{[n_1/2]}\left(\frac{-2r}{m_1^2}\right)^m(l'^+_{11})^{m-1}\left(\frac{l'^{1+}}{m_1}\right)^{n_1-2m}C^{n_1}_{2m}l'^{22+}(t'^+_{12})^{p_{12}}t'_{12}|n_2\rangle_V, \quad \text{(A27)}$$

$$
\begin{aligned}
l_2'^{+}|\vec{0}_{ij},\vec{n}_s\rangle_V \;=\; & m_2\sum_{m=0}^{[n_1+1/2]}\left(\frac{-2r}{m_1^2}\right)^m C^{n_1}_{2m}\Big|\vec{0}_{ij}+m\delta_{ij,11},\vec{n}_s-2m\delta_{s,1}+\delta_{s,2}\rangle_V \\
& +m_1\sum_{m=0}^{[n_1-1/2]}\left(\frac{-2r}{m_1^2}\right)^{m+1}\Big\{\Big(C^{n_1}_{2m+1}(h^2-h^1+2p_{12}+n_2)-C^{n_1}_{2m+2}\Big) \\
& \times\Big|\vec{0}_{ij}+m\delta_{ij,11}+\delta_{ij,12},\vec{n}_s-(2m+1)\delta_{s,1}\rangle_V+C^{n_1}_{2m+1}\Big(p_{12}(h^1-h^2 \\
& -n_2-p_{12}+1)\Big|\vec{0}_{ij}+m\delta_{ij,11}+\delta_{ij,22},\vec{n}_s-(2m+1)\delta_{s,1}-\delta_{s,12}\rangle_V \\
& -\Big|\vec{0}_{ij}+(m+1)\delta_{ij,11},\vec{n}_s-(2m+1)\delta_{s,1}+\delta_{s,12}\rangle_V\Big)\Big\} \\
& +m_1\sum_{m=0}^{[n_1-1/2]}\left(\frac{-2r}{m_1^2}\right)^{m+1}(l_{11}'^{+})^m l_{22}'^{+}\left(\frac{l'^{1+}}{m_1}\right)^{n_1-2m-1}C^{n_1}_{2m+1}(t_{12}'^{+})^{p_{12}}t'_{12}|n_2\rangle_V\,. \qquad \text{(A28)}
\end{aligned}
$$

In a manifest form, the action of t'_{12} on the vector $|n_2\rangle_V$ is determined by

Lemma A3. *The operator t'_{12} in its action on $|n_2\rangle_V$ satisfies the following* recurrent relations*:*

$$
\begin{aligned}
t'_{12}|n_2\rangle_V \;=\; & |A_{n_2}\rangle_V-\sum_{^1m=0}^{[n_2/2]}\;\sum_{^1l=0}^{[n_2/2-{}^1m-1/2]}\left(\frac{-2rl'^{22+}}{m_2^2}\right)^{^1m+{}^1l+1} \\
& \times C^{n_2}_{2^1m+1}C^{n_2-2^1m-1}_{2^1l+1}t'_{12}|n_2-2(^1m+{}^1l+1)\rangle_V, \qquad \text{(A29)}
\end{aligned}
$$

with a completely specified vector $|A_{n_2}\rangle_V$ from the Verma module,

$$
\begin{aligned}
|A_{n_2}\rangle_V \;=\; & \sum_{m=1}^{[n_2/2]}\left(\frac{-2r}{m_2^2}\right)^m\Big\{-C^{n_2}_{2m}|1,0,m-1,0,1,n_2-2m\rangle_V \\
& +[C^{n_2}_{2m}(h^2-h^1)+C^{n_2}_{2m+1}]|0,1,m-1,0,0,n_2-2m\rangle_V\Big\} \\
& -\frac{m_1}{m_2}\sum_{m=0}^{[n_2/2]}\left(\frac{-2r}{m_2^2}\right)^m C^{n_2}_{2m+1}|0,0,m,1,0,n_2-2m-1\rangle_V \\
& -\sum_{^1m=0}^{[n_2/2]}\;\sum_{^1l=0}^{[n_2/2-{}^1m-1]}\left(\frac{-2r}{m_2^2}\right)^{^1m+{}^1l+1}C^{n_2}_{2^1m+1}\Big\{\Big[C^{n_2-2^1m-1}_{2^1l+1}\Big(h^2-h^1+n_2 \\
& -2(^1m+{}^1l+1)\Big)-C^{n_2-2^1m-1}_{2^1l+2}\Big]\Big|0,1,{}^1m+{}^1l,0,0,n_2-2(^1m+{}^1l+1)\rangle_V \\
& -C^{n_2-2^1m-1}_{2^1l+1}\Big|1,0,{}^1m+{}^1l,0,1,n_2-2(^1m+{}^1l+1)\rangle_V \\
& +\frac{m_1}{m_2}C^{n_2-2^1m-1}_{2^1l+2}\Big|0,0,{}^1m+{}^1l+1,1,0,n_2-2(^1m+{}^1l+1)-1\rangle_V\Big\}. \qquad \text{(A30)}
\end{aligned}
$$

Considering the representation (A29), we use the commutation relation t'_{12} with $\left(\frac{l_2'^{+}}{m_2}\right)^{n_2}$ (A20), from which we obtain the first terms of $|A_{n_2}\rangle_V$ with single sums, denoted below by S.S., the remaining term being a manifest example of resolving the entanglement between the negative-root vectors $l_2'^{+}$, with $l_1'^{+}$, due to the permuted structure opposite to that of $|N(2)\rangle_V$ (50),

$$
t'_{12}|n_2\rangle_V=\text{S.S.}-\sum_{^1m=0}^{[n_2/2]}\left(\frac{-2rl'^{22+}}{m_2^2}\right)^{^1m}C^{n_2}_{2^1m+1}\underbrace{\left(\frac{l_2'^{+}}{m^2}\right)^{n_2-2^1m-1}l_1'^{+}}|0\rangle_V, \qquad \text{(A31)}
$$

In order to commute $\left(\frac{l_2'^+}{m^2}\right)^{n_2-2^1m-1}$ *in (A31) to the right through* $l_1'^+$*, one should use the following relation, which is not difficult to establish:*

$$\left[\left(\frac{l_2'^+}{m_2}\right)^{n_2-2^1m-1}, l_1'^+\right] = \sum_{^1l=0}^{[n_2-2^1m-2/2]}\left(\frac{-2rl'^{22+}}{m_2^2}\right)^{^1l}\frac{1}{m_2}\left\{C^{n_2-2^1m-1}_{2^1l+1}W_b'^{12+}\frac{l_2'^+}{m_2} - \frac{1}{m_2}C^{n_2-2^1m-1}_{2^1l+2}W_{b2}'^{12+}\right\}\left(\frac{l_2'^+}{m_2}\right)^{n_2-2(^1m+^1l+1)-1}, \tag{A32}$$

and which follows from an analogue of the commutation relation (57),

$$B^nA = \sum_{k=0}^{n}C^n_k(\widehat{\mathrm{ad}}_B^k A)B^{n-k}, \quad \widehat{\mathrm{ad}}_B A = [B, A]. \tag{A33}$$

Rearranging the terms over the figure brackets in (A31) by the rule (A33), and also using (46), (A77), and (A86), we establish the validity of the sought-for relations (A29), (A30).

From (A31), it follows that a single application of the operator t'_{12} to $|n_2\rangle_V$ implies, at least, a reduction in the last component in the above vector by the value of 2, i.e., a reduction to a vector with $|\vec{0}_{ij} + \delta_{i2}\delta_{j2}, 0, 0, n_2 - 2\rangle_V$. Therefore, applying the operator action from Lemma A3 as many as $\left[\frac{(n_2-1)}{2}\right]$ times, we resolve the recurrent relations (A29) in the form

$$\begin{aligned} t'_{12}|n_2\rangle_V = & \sum_{k=0}^{[(n_2-1)/2]}\left\{\sum_{^1m=0}^{[n_2/2]}\sum_{^1l=0}^{[n_2/2-(^1m+1)]}\cdots\sum_{^km=0}^{\left[n_2/2-\sum_{i=1}^{k-1}(^im+^il)-(k-1)\right]}\sum_{^kl=0}^{\left[n_2/2-\sum_{i=1}^{k-1}(^im+^il)-^km-k\right]}(-1)^k\right. \\ & \times\left(\frac{-2r}{m_2^2}\right)^{\sum_{i=1}^{k}(^im+^il)+k}C^{n_2}_{2^1m+1}C^{n_2-2^1m-1}_{2^1l+1}\cdots C^{n_2-2(\sum_{i=1}^{k-1}(^im+^il)+k-1)}_{2^km+1} \\ & \left.\times\, C^{n_2-2(\sum_{i=1}^{k-1}(^im+^il)+k-1)-2^km-1}_{2^kl+1}\left|A_{0,0,\sum_{i=1}^{k}(^im+^il)+k,0,0,n_2-2[\sum_{i=1}^{k}(^im+^il)+k]}\right\rangle_V\right\}, \end{aligned} \tag{A34}$$

$$\text{where } \left|A_{0,0,m,0,0,n_2}\right\rangle_V \equiv (l'^{22+})^m|A_{n_2}\rangle_V, \tag{A35}$$

so that we completely determine the action of t'_{12} on the vector $|n_2\rangle_V$.

Due to Lemmas A2 and A3, we consider the action of the operator t'_{12} on the vector $|\vec{0}_{ij}, 0, 0, n_2\rangle_V$ as a primary block operator, which can be used to obtain the action of all o'_I on the vector $|\vec{N}(2)\rangle_V$.

Note that, in the case of the flat-space limit $r = 0$, it is only the zero order in r of (A34), (A35) that survives so that we have a term which is not originated by the $sp(4)$ algebra:

$$t'_{12}|\vec{0}_{ij}, 0, 0, n_2\rangle_V|_{r=0} = -\frac{m_1}{m_2}n_2|\vec{0}_{ij}, 1, 0, n_2 - 1\rangle_V. \tag{A36}$$

Then, the problem of finding the action of t'_{12} on $|\vec{N}(2)\rangle_V$ in (58) is readily solved in the form

$$\begin{aligned} t'_{12}\Big|\vec{N}(2)\rangle_V &= p_{12}(h^1-h^2-n_2-p_{12}+1)|\vec{n}_{ij},n_1,p_{12}-1,n_2\rangle_V \\ &-\sum_l l n_{l2}\Big|\vec{n}_{ij}-\delta_{ij,l2}+\delta_{ij,1l},\vec{n}_s\rangle_V + \sum_{k=0}^{[(n_2-1)/2]}\Bigg\{\sum_{^1m=0}^{[n_2/2]}\sum_{^1l=0}^{[n_2/2-(^1m+1)]}\dots \\ &\dots\sum_{^km=0}^{\left[n_2/2-\sum\limits_{i=1}^{k-1}(^im+^il)-(k-1)\right]}\sum_{^kl=0}^{\left[n_2/2-\sum\limits_{i=1}^{k-1}(^im+^il)-^km-k\right]}(-1)^k\left(\frac{-2r}{m_2^2}\right)^{\sum_{i=1}^k(^im+^il)+k} \\ &\times C^{n_2}_{2^1m+1}C^{n_2-2^1m-1}_{2^1l+1}\dots C^{n_2-2(\sum_{i=1}^{k-1}(^im+^il)+k-1)}_{2^km+1}C^{n_2-2(\sum_{i=1}^{k-1}(^im+^il)+k-1)-2^km-1}_{2^kl+1} \\ &\times\Big|\hat{A}_{\vec{n}_{ij}+[\sum\limits_{i=1}^{k}(^im+^il)+k]\delta_{i2}\delta_{j2},n_1,n,n_2-2[\sum\limits_{i=1}^{k}(^im+^il)+k]}\Big\rangle_V\Bigg\}, \end{aligned} \tag{A37}$$

as we extracted from t'_{12} the action of the Lie part t'^{Lie}_{12}, $sp(4)$, and also the part $\hat{t}'_{12}$, being non-linear in r,

$$\begin{aligned} t'_{12}\Big|\vec{N}(2)\rangle_V &= p_{12}(h^1-h^2-n_2-p_{12}+1)|\vec{n}_{ij},n_1,p_{12}-1,n_2\rangle_V \\ &-\sum_l l n_{l2}\Big|\vec{n}_{ij}-\delta_{ij,l2}+\delta_{ij,1l},\vec{n}_s\rangle_V + \hat{t}'_{12}\Big|\vec{N}(2)\rangle_V\,, \end{aligned} \tag{A38}$$

in order to prove the validity of (64). The vector with a "hat", $|\hat{A}_{\dots}\rangle_V$, in (A37) is constructed using the vector $|A_{\dots}\rangle_V$ of (A30) by the rule

$$\begin{aligned} \Big|\hat{A}_{\vec{N}(2)}\rangle_V &= \prod_{i\le j}^2(l'^+_{ij})^{n_{ij}}\left(\frac{l'^+_1}{m_1}\right)^{n_1}(t'^+_{12})^{p_{12}}\Big|A_{\vec{0}_{ij},0,0,n_2}\rangle_V = n_2p_{12}\Big|\vec{N}(2)-\delta_{s,12}\rangle_V \\ &-\frac{m_1}{m_2}\sum_{^1m=0}^{[n_2/2]}\left(\frac{-2r}{m_2^2}\right)^{^1m}C^{n_2}_{2^1m+1}\Big|\vec{N}(2)+{}^1m\delta_{ij,22}+\delta_{s,1}-(2^1m+1)\delta_{s,2}\rangle_V \\ &-\sum_{^1m=1}^{[n_2/2]}\left(\frac{-2r}{m_2^2}\right)^{^1m}\Bigg\{C^{n_2}_{2^1m}\Big|\vec{N}(2)+\delta_{ij,11}+(^1m-1)\delta_{ij,22}+\delta_{s,12}-2^1m\delta_{s,2}\rangle_V \\ &-\left[C^{n_2}_{2^1m}(h^2-h^1+2p_{12})+C^{n_2}_{2^1m+1}\right]\Big|\vec{N}(2)+\delta_{ij,12}+(^1m-1)\delta_{ij,22}-2^1m\delta_{s,2}\rangle_V \\ &+p_{12}C^{n_2}_{2^1m}(h^2-h^1+p_{12}-1)\Big|\vec{N}(2)+{}^1m\delta_{ij,22}-\delta_{s,12}-2^1m\delta_{s,2}\rangle_V\Bigg\} \\ &-\sum_{^1m=0}^{[n_2/2]}\sum_{^1l=0}^{[n_2/2-(^1m+1)]}\left(\frac{-2r}{m_2^2}\right)^{^1m+^1l+1}C^{n_2}_{2^1m+1}\Bigg\{\Big[C^{n_2-2^1m-1}_{2^1l+1}(h^2-h^1+2p_{12}+n_2 \\ &-2(^1m+^1l+1))-C^{n_2-2^1m-1}_{2^1l+2}\Big]\Big|\vec{N}(2)+\delta_{ij,12}+(^1m+^1l)\delta_{ij,22}-2(^1m \\ &+^1l+1)\delta_{s,2}\rangle_V - p_{12}C^{n_2-2^1m-1}_{2^1l+1}(h^2-h^1+n_2-2(^1m+^1l+1)+p_{12}-1) \\ &\times\Big|\vec{N}(2)+(^1m+^1l+1)\delta_{ij,22}-\delta_{s,12}-2(^1m+^1l+1)\delta_{s,2}\rangle_V \\ &-C^{n_2-2^1m-1}_{2^1l+1}\Big|\vec{N}(2)+\delta_{ij,11}+(^1m+^1l)\delta_{ij,22}+\delta_{s,12}-2(^1m+^1l+1)\delta_{s,2}\rangle_V \\ &+\frac{m_1}{m_2}C^{n_2-2^1m-1}_{2^1l+2}\Big|\vec{N}(2)+(^1m+^1l+1)\delta_{ij,22}+\delta_{s,1}-2(^1m+^1l+\tfrac{3}{2})\delta_{s,2}\rangle_V\Bigg\}. \end{aligned} \tag{A39}$$

Thus, the action of the negative-root vectors t'^{+}_{12}, l'^{+}_{2} on $|\vec{0}_{ij}, \vec{n}_s\rangle_V$ (A27), (A28), due to Lemma A2, is completely determined, and the corresponding terms containing t'_{12} can be presented as follows, in terms of the derived block operators $\hat{t}'^{+}_{12}, \hat{l}'^{+}_{2}$ constructed from the primary ones $\hat{t}'_{12}$, namely,

$$\begin{aligned}
\hat{t}'^{+}_{12}|\vec{0}_{ij}, \vec{n}_s\rangle_V &\equiv t'^{+}_{12}|\vec{0}_{ij}, \vec{n}_s\rangle_V = \ldots - \sum_{m=1}^{[n_1/2]} \left(\frac{-2r}{m_1^2}\right)^m C^{n_1}_{2m} \hat{t}'_{12} \\
&\quad \times \Big|\vec{0}_{ij} + (m-1)\delta_{ij,11} + \delta_{ij,22}, \vec{n}_s - 2m\delta_{s,1}\rangle_V , \qquad \text{(A40)}\\
\hat{l}'^{+}_{2}|\vec{0}_{ij}, \vec{n}_s\rangle_V &\equiv l'^{+}_{2}|\vec{0}_{ij}, \vec{n}_s\rangle_V = \ldots m_1 \sum_{m=0}^{[n_1-1/2]} \left(\frac{-2r}{m_1^2}\right)^{m+1} C^{n_1}_{2m+1} \\
&\quad \times \hat{t}'_{12}\Big|\vec{0}_{ij} + m\delta_{ij,11} + \delta_{ij,22}, \vec{n}_s - (2m+1)\delta_{s,1}\rangle_V . \qquad \text{(A41)}
\end{aligned}$$

Combining the results (53), (54), (A37)–(A41), we arrive at the final representation for the action of t'^{+}_{12}, l'^{+}_{2} on an arbitrary vector $|\vec{N}(2)\rangle_V$, in terms of the primary block operator $\hat{t}'_{12}$,

$$\begin{aligned}
t'^{+}_{12}\Big|\vec{N}(2)\rangle_V &= -\sum_l (3-l)n_{1l}\Big|\vec{n}_{ij} - \delta_{ij,1l} + \delta_{ij,l2}, \vec{n}_s\rangle_V \\
&+ \sum_{m=0}^{[n_1/2]} \left(\frac{-2r}{m_1^2}\right)^m \Big\{ C^{n_1}_{2m}\Big|\vec{N}(2) + m\delta_{ij,11} - 2m\delta_{s,1} + \delta_{s,12}\rangle_V \\
&- C^{n_1}_{2m+1}\frac{m_2}{m_1}\Big|\vec{N}(2) + m\delta_{ij,11} - (2m+1)\delta_{s,1} + \delta_{s,2}\rangle_V \Big\} \\
&- \sum_{m=1}^{[n_1/2]} \left(\frac{-2r}{m_1^2}\right)^m \Big\{ \Big[C^{n_1}_{2m}(h^2 - h^1 + 2p_{12} + n_2) - C^{n_1}_{2m+1} \Big] \\
&\quad \times |N(2) + (m-1)\delta_{ij,11} + \delta_{ij,12} - 2m\delta_{s,1}\rangle_V + C^{n_1}_{2m} p_{12}(h^1 - h^2 \\
&\quad - n_2 - p_{12} + 1)|N(2) + (m-1)\delta_{ij,11} + \delta_{ij,22} - 2m\delta_{s,1} - \delta_{s,12}\rangle_V \Big\} \\
&- \sum_{m=1}^{[n_1/2]} \left(\frac{-2r}{m_1^2}\right)^m C^{n_1}_{2m}\hat{t}'_{12}\Big|\vec{N}(2) + (m-1)\delta_{ij,11} + \delta_{ij,22} - 2m\delta_{s,1}\rangle_V , \qquad \text{(A42)}\\
l'^{+}_{l}\Big|\vec{N}(2)\rangle_V &= \delta^{l1} m_1\Big|\vec{N}(2) + \delta_{s,1}\rangle_V \\
&+ \delta^{l2}\Big\{ m_2 \sum_{m=0}^{[n_1+1/2]} \left(\frac{-2r}{m_1^2}\right)^m C^{n_1}_{2m}\Big|\vec{N}(2) + m\delta_{ij,11}\vec{N}(2) - 2m\delta_{s,1} + \delta_{s,2}\rangle_V \\
&+ m_1 \sum_{m=0}^{[n_1-1/2]} \left(\frac{-2r}{m_1^2}\right)^{m+1} \Big[\Big(C^{n_1}_{2m+1}(h^2 - h^1 + 2p_{12} + n_2) - C^{n_1}_{2m+2} \Big) \\
&\quad \times \Big|\vec{N}(2) + m\delta_{ij,11} + \delta_{ij,12} - (2m+1)\delta_{s,1}\rangle_V + C^{n_1}_{2m+1}\Big(p_{12}(h^1 - h^2 \\
&\quad - n_2 - p_{12} + 1)\Big|\vec{N}(2) + m\delta_{ij,11} + \delta_{ij,22} - (2m+1)\delta_{s,1} - \delta_{s,12}\rangle_V \\
&\quad - \Big|\vec{N}(2) + (m+1)\delta_{ij,11} - (2m+1)\delta_{s,1} + \delta_{s,12}\rangle_V \Big)\Big] \\
&+ m_1 \sum_{m=0}^{[n_1-1/2]} \left(\frac{-2r}{m_1^2}\right)^{m+1} C^{n_1}_{2m+1}\hat{t}'_{12}\Big|\vec{N}(2) + m\delta_{ij,11} + \delta_{ij,22} - (2m+1)\delta_{s,1}\rangle_V \Big\}. \qquad \text{(A43)}
\end{aligned}$$

Equivalently, the above representation for t'^{+}_{12}, l'^{+}_{2} can be written in terms of the derived blocks $\hat{t}'^{+}_{12}, \hat{l}'^{+}_{2}$, as in the resulting formulae (65) and (66),

$$t'^{+}_{12}\Big|\vec{N}(2)\rangle_V = -\sum_l (3-l)n_{1l}\Big|\vec{n}_{ij} - \delta_{ij,1l} + \delta_{ij,l2}, \vec{n}_s\rangle_V + \hat{t}'^{+}_{12}\Big|\vec{N}(2)\rangle_V\,, \tag{A44}$$

$$l'^{+}_{l}\Big|\vec{N}(2)\rangle_V = \delta^{l1}m_1\Big|\vec{N}(2) + \delta_{s,1}\rangle_V + \delta^{l2}\hat{l}'^{+}_{2}\Big|\vec{N}(2)\rangle_V\,, \tag{A45}$$

with allowance for the notation

$$\hat{B}\Big|\vec{N}(2)\rangle_V = \prod_{i\leq j}^{2}(l'^{+}_{ij})^{n_{ij}}B\Big|\vec{0}_{ij}, \vec{n}_s\rangle_V\,,\quad B\in\{t'^{+}_{12}, l'^{+}_{2}\}, \tag{A46}$$

introduced by presenting the vectors l'^{+}_{ij} in a manner which does not affect the structure of $B|\vec{0}_{ij}, \vec{n}_s\rangle_V$; see (A40), (A41).

Lemma A4. *The action of the operators* l'_0, l'_l, l'_{l2}, $l = 1,2$ *on the vector* $|\vec{0}_{ij}, 0, p_{12}, n_2\rangle_V = |p_{12}, n_2\rangle_V$ *is given by*

$$\begin{aligned}
l'_0\Big|p_{12}, n_2\Big\rangle_V &= m_0^2\Big|\vec{0}_{ij}, 0, p_{12}, n_2\rangle_V - r\sum_{m=0}^{[n_2-1/2]}\left(\frac{-8r}{m_2^2}\right)^m\Big\{\Big[C^{n_2}_{2m+1}(2h^2 + 2p_{12} - 1)\\
&+2C^{n_2}_{2m+2}\Big]\,|0,0,m,0,p_{12},n_2-2m\rangle_V + 2\frac{m_1}{m_2}C^{n_2}_{2m+1}|0,0,m,1,p_{12}+1,n_2-2m-1\rangle_V\Big\}\\
&+\frac{1}{2}\sum_{m=0}^{[n_2-1/2]}\left(\frac{-8r}{m_2^2}\right)^{m+1}C^{n_2}_{2m+2}\Big\{\Big(m_0^2 - r\big[h^2(h^2-4) + h^1 + p_{12}(2h^1 - 3 - p_{12})\big]\Big)\\
&\quad\times\Big|0,0,m+1,0,p_{12},n_2-2(m+1)\rangle_V + r\Big|1,0,m,0,p_{12}+2,n_2-2(m+1)\rangle_V\\
&\quad+2r(h^1 - p_{12} - 2)\,|0,1,m,0,p_{12}+1,n_2-2(m+1)\rangle_V\Big\}\\
&+2r\sum_{m=0}^{[n_2-1/2]}\sum_{l=0}^{[n_2/2-m-1]}\left(\frac{-8r}{m_2^2}\right)^m\left(\frac{-2r}{m_2^2}\right)^{l+1}C^{n_2}_{2m+1}\Big\{\Big[C^{n_2-2m-1}_{2l+1}(h^2 - h^1 + 2p_{12} + n_2\\
&\quad-2(m+l)) - C^{n_2-2m-1}_{2l+2}\Big]\Big|0,1,m+l,0,p_{12}+1,n_2-2(m+l+1)\rangle_V\\
&-C^{n_2-2m-1}_{2l+1}\Big[\Big|1,0,m+l,0,p_{12}+2,n_2-2(m+l+1)\rangle_V - (p_{12}+1)\\
&\quad\times\big[h^1 - h^2 - n_2 - p_{12} + 2(m+l+1)\big]\Big|0,0,m+l+1,0,p_{12},n_2-2(m+l+1)\rangle_V\Big]\\
&\quad+C^{n_2-2m-1}_{2l+2}\frac{m_1}{m_2}\Big|0,0,m+l+1,1,p_{12}+1,n_2-2(m+l+1)-1\rangle_V\Big\}\\
&\quad+2r\sum_{m=0}^{[n_2-1/2]}\sum_{l=0}^{[n_2/2-m-1]}\left(\frac{-8r}{m_2^2}\right)^m\left(\frac{-2r}{m_2^2}\right)^{l+1}C^{n_2}_{2m+1}C^{n_2-2m-1}_{2l+1}\\
&\qquad\times\hat{t}'_{12}\Big|0,0,m+l+1,0,p_{12}+1,n_2-2(m+l+1)\Big\rangle_V,
\end{aligned} \tag{A47}$$

$$
\begin{aligned}
l_1'\Big|p_{12},n_2\Big\rangle_V ={}& -\frac{m_2}{2}\sum_{m=0}^{[n_2-1/2]}\left(\frac{-2r}{m_2^2}\right)^{m+1}\Big\{C^{n_2}_{2m+1}(h^1+h^2-2)+C^{n_2}_{2m+2}\Big\}\\
&\times\Big|0,0,m,0,p_{12}+1,n_2-2m-1\Big\rangle_V,
\end{aligned}
\tag{A48}
$$

$$
\begin{aligned}
l_2'\Big|p_{12},n_2\Big\rangle_V ={}& \sum_{m=0}^{[n_2/2]}\left(\frac{-8r}{m_2^2}\right)^{m}C^{n_2}_{2m+1}\frac{[m_0^2-rh^2(h^2-3)]}{m_2}\Big|0,0,m,0,p_{12},n_2-2m-1\Big\rangle_V\\
&+\frac{m_2}{4}\sum_{m=1}^{[n_2/2]}\left(\frac{-8r}{m_2^2}\right)^{m}\Big\{C^{n_2}_{2m}(2h^2-1)+2C^{n_2}_{2m+1}\Big\}\Big|0,0,m-1,0,p_{12},n_2-2m+1\Big\rangle_V\\
&-p_{12}l_1'\Big|\vec{0}_{ij},0,p_{12}-1,n_2\Big\rangle_V\\
&-\sum_{m=0}^{[n_2/2]}\left(\frac{-2r}{m_2^2}\right)^{m}(4^m-1)C^{n_2}_{2m}\Big\{\frac{m_1}{2}\Big|0,0,m-1,1,p_{12}+1,n_2-2m\rangle_V\\
&-p_{12}\frac{m_2}{2}\Big|0,0,m-1,0,p_{12},n_2-2m+1\rangle_V\Big\}\\
&+\frac{m_2}{2}\sum_{m=0}^{[n_2/2]}\left(\frac{-2r}{m_2^2}\right)^{m+1}(4^m-1)C^{n_2}_{2m+1}\Big\{\Big[(2p_{12}+1)h^1-h^2-3p_{12}-p_{12}^2\Big]\\
&\times\Big|0,0,m,0,p_{12},n_2-2m-1\rangle_V-\Big|1,0,m-1,0,p_{12}+2,n_2-2m-1\Big\rangle_V\\
&-2(h^1-p_{12}-2)\Big|0,1,m-1,0,p_{12}+1,n_2-2m-1\Big\rangle_V\Big\}\\
&-\frac{m_2}{2}\sum_{m=0}^{[n_2/2]}\sum_{l=0}^{[n_2-2m-1/2]}\left(\frac{-2r}{m_2^2}\right)^{m+l+1}(4^m-1)C^{n_2}_{2m}\\
&\times\Big\{\Big[C^{n_2-2m}_{2l+1}\big(h^2-h^1+2p_{12}+n_2-2(m+l)+1\big)-C^{n_2-2m}_{2l+2}\Big]\\
&\times\Big|0,1,m+l-1,0,p_{12}+1,n_2-2(m+l)-1\Big\rangle_V\\
&-C^{n_2-2m}_{2l+1}\Big[\Big|1,0,m+l-1,0,p_{12}+2,n_2-2(m+l)-1\Big\rangle_V+\big(h^2-h^1+n_2\\
&-2(m+l)-1\big)\Big|0,0,m+l,0,p_{12},n_2-2(m+l)-1\Big\rangle_V\Big]\\
&+\frac{m_1}{m_2}C^{n_2-2m}_{2l+2}\Big|0,0,m+l,1,p_{12}+1,n_2-2(m+l+1)\Big\rangle_V\Big\}\\
&+\sum_{m=0}^{[n_2/2]}\sum_{l=0}^{[n_2-2m-1/2]}\left(\frac{-2r}{m_2^2}\right)^{m+l+1}(4^m-1)C^{n_2}_{2m}C^{n_2-2m}_{2l+1}\Big\{p_{12}\frac{m_2}{2}[h^2-h^1\\
&+n_2+p_{12}-2(m+l)]\Big|0,0,m+l,0,p_{12},n_2-2(m+l)-1\Big\rangle_V\\
&-\frac{m_2}{2}\hat{t}_{12}'\Big|0,0,m+l,0,p_{12}+1,n_2-2(m+l)-1\Big\rangle_V\Big\},
\end{aligned}
\tag{A49}
$$

$$l'_{12}\Big|p_{12},n_2\Big\rangle_V = \frac{1}{4}\sum_{m=1}^{[n_2/2]}\left(\frac{-2r}{m_2^2}\right)^m \left\{C^{n_2}_{2m}(h^1+h^2-2)+C^{n_2}_{2m+1}\right\} \times\Big|0,0,m-1,0,p_{12}+1,n_2-2m\Big\rangle_V, \tag{A50}$$

$$\begin{aligned}
l'_{22}\Big|p_{12},n_2\Big\rangle_V &= -2p_{12}l'_{12}\Big|p_{12}-1,n_2\Big\rangle_V \\
&-\sum_{m=0}^{[(n_2-1)/2]}\left(\frac{-8r}{m_2^2}\right)^m C^{n_2}_{2m+2}\frac{\left[m_0^2-rh^2(h^2-3)\right]}{m_2^2}\Big|0,0,m,0,p_{12},n_2-2(m+1)\Big\rangle_V \\
&-\frac{1}{4}\sum_{m=1}^{[(n_2-1)/2]}\left(\frac{-8r}{m_2^2}\right)^m\left\{C^{n_2}_{2m+1}(2h^2-1)+2C^{n_2}_{2m+2}\right\}\Big|0,0,m-1,0,p_{12},n_2-2m\Big\rangle_V \\
&+\frac{1}{2}\sum_{m=0}^{[(n_2-1)/2]}\left(\frac{-2r}{m_2^2}\right)^{m+1}(4^m-1)\Big\{C^{n_2}_{2m+2}\Big[\Big|1,0,m-1,0,p_{12}+2,n_2-2(m+1)\Big\rangle_V \\
&+2\big((h^1-2)-p_{12}\big)\Big|0,1,m-1,0,p_{12}+1,n_2-2(m+1)\Big\rangle_V \\
&+\big(h^2-h^1-p_{12}(2h^1-p_{12}-3)\big)\Big|0,0,m,0,p_{12},n_2-2(m+1)\Big\rangle_V\Big] \\
&+C^{n_2}_{2m+1}\Big[\frac{m_1}{m_2}\Big|0,0,m-1,1,p_{12}+1,n_2-2m-1\Big\rangle_V \\
&-p_{12}\Big|0,0,m-1,0,p_{12},n_2-2m\Big\rangle_V\Big]\Big\} \\
&+\frac{1}{2}\sum_{m=0}^{[(n_2-1)/2]}\sum_{l=0}^{[n_2/2-m-1]}\left(\frac{-2r}{m_2^2}\right)^{m+l+1}(4^m-1)C^{n_2}_{2m+1}\Big\{\Big[C^{n_2-2m-1}_{2l+1}(h^2-h^1+2p_{12} \\
&+n_2-2(m+l))-C^{n_2-2m-1}_{2l+2}\Big]\Big|0,1,m+l-1,0,p_{12}+1,n_2-2(m+l+1)\Big\rangle_V \\
&+C^{n_2-2m-1}_{2l+1}\Big[(p_{12}+1)\Big(h^1-h^2-p_{12}-n_2+2(m+l+1)\Big)\Big] \\
&\times\Big|0,0,m+l,0,p_{12},n_2-2(m+l+1)\Big\rangle_V \\
&-C^{n_2-2m-1}_{2l+1}\Big|1,0,m+l-1,0,p_{12}+2,n_2-2(m+l+1)\Big\rangle_V \\
&+C^{n_2-2m-1}_{2l+2}\frac{m_1}{m_2}\Big|0,0,m+l,1,p_{12}+1,n_2-2(m+l+1)-1\Big\rangle_V\Big\} \\
&+\frac{1}{2}\sum_{m=0}^{[(n_2-1)/2]}\sum_{l=0}^{[n_2/2-m-1]}\left(\frac{-2r}{m_2^2}\right)^{m+l+1}(4^m-1)C^{n_2}_{2m+1}C^{n_2-2m-1}_{2l+1} \\
&\times\widehat{t}'_{12}\Big|0,0,m+l,0,p_{12}+1,n_2-2(m+l+1)\Big\rangle_V.
\end{aligned} \tag{A51}$$

The deduction of the above auxiliary formulae (A47)–(A51) is straightforward and relies on (A18), (A19), (A21)–(A26), (A79)–(A86).

Using the results of Lemma A4, we introduce a *new derived block operator* by the following rule, similar to that given by (A46), for $l = 1,2$:

$$\hat{C}\Big|\vec{N}(2)\rangle_V = \prod_{i\leq j}^{2}(l'^+_{ij})^{n_{ij}}\left(\frac{l'^+_1}{m_1}\right)^{n_1}C\Big|\vec{0}_{ij},0,p_{12},n_2\rangle_V\,,\ C\in\{l'_{l2},l'_l,l'_0\}. \tag{A52}$$

This allows one to completely establish the operator action on the vectors that remain undetermined by (67)–(72) due to the following

Lemma A5. *The action of the operators l'_0, l'_l, l'_{ij}, for $i,j,l = 1,2$ and $i \leq j$, on the vector $|\vec{0}_{ij}, \vec{n}_s\rangle_V = |\vec{n}_s\rangle_V$ is given by*

$$
\begin{aligned}
l'_0\Big|\vec{n}_s\Big\rangle_V &= \widehat{l}'_0\Big|n_s\rangle_V + \frac{m_1}{2}\sum_{m=0}^{[n_1-1/2]}\left(\frac{-8r}{m_1^2}\right)^{m+1} C^{n_1}_{2m+1}\Big\{\widehat{l}'_2\Big|(m,1,0),\vec{n}_s-(2m+1)\delta_{s,1}\Big\rangle_V \\
&+\widehat{l}'_1\Big|(m+1,0,0),\vec{n}_s-(2m+1)\delta_{s,1}\Big\rangle_V - \frac{m_2}{2}\widehat{t}'_{12}\Big|(m,0,0),\vec{n}_s-(2m+1)\delta_{s,1}+\delta_{s,2}\Big\rangle_V \\
&-\frac{m_2}{2}p_{12}(h^1-h^2-n_2-p_{12})\Big|(m,0,0),\vec{n}_s-(2m+1,1,-1)\Big\rangle_V\Big\} \\
&-r\sum_{m=0}^{[n_1-1/2]}\left(\frac{-8r}{m_1^2}\right)^{m}\Big\{C^{n_1}_{2m+1}\big[2(h^1-p_{12})-3\big]+2C^{n_1}_{2m+2}\Big\}\Big|(m,0,0),\vec{n}_s-2m\delta_{s,1}\Big\rangle_V \\
&+\frac{1}{2}\sum_{m=0}^{[n_1-1/2]}\left(\frac{-8r}{m_1^2}\right)^{m+1} C^{n_1}_{2m+2}\Big\{\widehat{l}'_0\Big|(m+1,0,0),\vec{n}_s-(2m+2)\delta_{s,1}\Big\rangle_V \\
&-r\Big([h^1-p_{12}][h^1-p_{12}-2]-h^2-p_{12}-n_2+2p_{12}(h^1-h^2-n_2-p_{12}+1)\Big) \\
&\times\Big|(m+1,0,0),\vec{n}_s-(2m+2)\delta_{s,1}\Big\rangle_V - 2r\widehat{t}'_{12}\Big|(m+1,0,0),\vec{n}_s-(2m+2,-1,0)\Big\rangle_V \\
&+8r\widehat{l}'_{12}\Big|(m+1,1,0),\vec{n}_s-(2m+2)\delta_{s,1}\Big\rangle_V + 4r\widehat{l}'_{22}\Big|(m,2,0),\vec{n}_s-(2m+2)\delta_{s,1}\Big\rangle_V \\
&+2r(h^2+n_2+p_{12}-2)\widehat{t}'_{12}\Big|(m,1,0),\vec{n}_s-(2m+2)\delta_{s,1}\Big\rangle_V \\
&+2rp_{12}(h^2+n_2+p_{12}-2)(h^1-h^2-n_2-p_{12}+1)\Big|(m,1,0),\vec{n}_s-(2m+2,1,0)\Big\rangle_V\Big\} \\
&+\frac{r}{2}\sum_{m=0}^{[n_1-1/2]}\left(\frac{-8r}{m_1^2}\right)^{m+1} C^{n_1}_{2m+2}\Big\{p_{12}(p_{12}-1)(h^1-h^2-n_2-p_{12}+2) \\
&\times(h^1-h^2-n_2-p_{12}+1)\Big|(m,0,1),\vec{n}_s-(2m+2,2,0)\Big\rangle_V \\
&+p_{12}(h^1-h^2-n_2-p_{12}+1)\widehat{t'}_{12}\Big|(m,0,1),\vec{n}_s-(2m+2,1,0)\Big\rangle_V \\
&+\sum_{k=0}^{[(n_2-1)/2]}\Bigg[\sum_{{}^1m=0}^{[n_2/2]}\sum_{{}^1l=0}^{[n_2/2-({}^1m+1)]}\cdots\cdots\sum_{{}^km=0}^{\left[n_2/2-\sum_{i=1}^{k-1}({}^im+{}^il)-(k-1)\right]}\sum_{{}^kl=0}^{\left[n_2/2-\sum_{i=1}^{k-1}({}^im+{}^il)-{}^km)-k\right]}(-1)^k \\
&\times\left(\frac{-2r}{m_2^2}\right)^{\sum_{i=1}^{k}({}^im+{}^il)+k} C^{n_2}_{2^1m+1}C^{n_2-2^1m-1}_{2^1l+1}\ldots \\
&\times C^{n_2-2(\sum_{i=1}^{k-1}({}^im+{}^il)+k-1)}_{2^km+1}C^{n_2-2(\sum_{i=1}^{k-1}({}^im+{}^il+k-1)-2^km-1}_{2^kl+1} \\
&\times\widehat{t'_{12}}\Big|A_{(m,0,\sum_{i=1}^{k}({}^im+{}^il)+k+1),\vec{n}_s-(2m+2)\delta_{s,1}-2[\sum_{i=1}^{k}({}^im+{}^il)+k]\delta_{s,2}}\Big\rangle_V\Bigg]\Bigg\},
\end{aligned} \quad \text{(A53)}
$$

$$
\begin{aligned}
l'_1\Big|\vec{n}_s\Big\rangle_V = & \sum_{m=0}^{[n_1/2]}\left(\frac{-8r}{m_1^2}\right)^m C^{n_1}_{2m}\widehat{l'}_1\Big|\vec{0}_{ij}+m\delta_{ij,11},\vec{n}_s-2m\delta_{s,1}\rangle_V \\
& +\frac{1}{m_1}\sum_{m=0}^{[n_1/2]}\left(\frac{-8r}{m_1^2}\right)^m C^{n_1}_{2m+1}\Big\{\widehat{l}'_0\Big|(m,0,0),n_s-(2m+1,0,0)\Big\rangle_V \\
& -r\Big[(h^1-p_{12})(h^1-p_{12}-2)-h^2-p_{12}-n_2\Big]\Big|(m,0,0),\vec{n}_s-(2m+1,0,0)\Big\rangle_V \\
& -rp_{12}(h^1-h^2-n_2-p_{12}+1)\Big|(m,0,0),\vec{n}_s-(2m+1,0,0)\Big\rangle_V \\
& -r\widehat{t}'_{12}\Big|(m,0,0),\vec{n}_s-(2m+1,-1,0)\Big\rangle_V\Big\} \\
-4m_1 & \sum_{m=0}^{[n_1/2]}\left(\frac{-2r}{m_1^2}\right)^{m+1} C^{n_1}_{2m+1}\Big(4^m-\tfrac{1}{2}\Big)\widehat{l}'_{12}\Big|(m,1,0),\vec{n}_s-(2m+1)\delta_{s,1}\Big\rangle_V \\
& +\frac{m_1}{4}\sum_{m=1}^{[n_1/2]}\left(\frac{-8r}{m_1^2}\right)^m\Big\{C^{n_1}_{2m}(2(h^1-p_{12})-1)+2C^{n_1}_{2m+1}\Big\} \\
& \quad\times\Big|(m-1,0,0),\vec{n}_s-(2m-1,0,0)\Big\rangle_V \\
& +\frac{1}{2}\sum_{m=1}^{[n_1/2]}\left(\frac{-2r}{m_1^2}\right)^m(4^m-1)C^{n_1}_{2m}\Big\{2\widehat{l}'_2\Big|(m-1,1,0),\vec{n}_s-(2m,0,0)\Big\rangle_V \\
& -m_2\widehat{t}'_{12}\Big|(m-1,0,0),\vec{n}_s-(2m,0,-1)\Big\rangle_V-m_1\Big|(m-1,0,0),\vec{n}_s-(2m-1)\delta_{s1}\Big\rangle_V \\
& -m_2p_{12}(h^1-h^2-n_2-p_{12})\Big|(m-1,0,0),\vec{n}_s-(2m,1,-1)\Big\rangle_V\Big\} \\
& +\frac{m_1}{2}\sum_{m=1}^{[n_1/2]}\left(\frac{-2r}{m_1^2}\right)^{m+1}(4^m-1)C^{n_1}_{2m+1}\Big\{p_{12}(h^1-h^2-n_2-p_{12}+1) \\
& \times\Big|(m,0,0),\vec{n}_s-(2m+1)\delta_{s,1}\Big\rangle_V+\widehat{t}'_{12}\Big|(m,0,0),\vec{n}_s-(2m+1)\delta_{s,1}+\delta_{s,12}\Big\rangle_V \\
& -4\widehat{l}'_{22}\Big|\vec{0}_{ij}+(m-1)\delta_{ij,11}+2\delta_{ij,12},\vec{n}_s-(2m+1)\delta_{s,1}\Big\rangle_V \\
& -2(h^2+n_2+p_{12}-2)\Big[p_{12}(h^1-h^2-n_2-p_{12}+1) \\
& \quad\times\Big|\vec{0}_{ij}+(m-1)\delta_{ij,11}+\delta_{ij,12},\vec{n}_s-(2m+1)\delta_{s,1}-\delta_{s,12}\Big\rangle_V \\
& +\widehat{t}'_{12}\Big|\vec{0}_{ij}+(m-1)\delta_{ij,11}+\delta_{ij,12},\vec{n}_s-(2m+1)\delta_{s,1}\Big\rangle_V\Big]\Big\} \\
& -\frac{m_1}{2}\sum_{m=1}^{[n_1/2]}\left(\frac{-2r}{m_1^2}\right)^{m+1}(4^m-1)C^{n_1}_{2m+1}\Big\{p_{12}(p_{12}-1)(h^1-h^2-n_2-p_{12}+2) \\
& \quad\times(h^1-h^2-n_2-p_{12}+1)\Big|(m-1,0,1),\vec{n}_s-(2m+1)\delta_{s,1}-2\delta_{s,12}\Big\rangle_V \\
& +p_{12}(h^1-h^2-n_2-p_{12}+1)\widehat{t'}_{12}\Big|(m-1,0,1),\vec{n}_s-(2m+1)\delta_{s,1}-\delta_{s,12}\Big\rangle_V \\
& +\sum_{k=0}^{[(n_2-1)/2]}\Bigg[\sum_{{}^1m=0}^{[n_2/2]}\sum_{{}^1l=0}^{[n_2/2-({}^1m+1)]}\cdots \\
& \cdots\sum_{{}^km=0}^{\left[n_2/2-\sum_{i=1}^{k-1}({}^im+{}^il)-(k-1)\right]}\sum_{{}^kl=0}^{\left[n_2/2-\sum_{i=1}^{k-1}({}^im+{}^il)-{}^km)-k\right]}(-1)^k\left(\frac{-2r}{m_2^2}\right)^{\sum_{i=1}^k({}^im+{}^il)+k} \\
& \times C^{n_2}_{2^1m+1}C^{n_2-2^1m-1}_{2^1l+1}\ldots C^{n_2-2(\sum_{i=1}^{k-1}({}^im+{}^il)+k-1)}_{2^km+1}C^{n_2-2(\sum_{i=1}^{k-1}({}^im+{}^il)+k-1)-2^km-1}_{2^kl+1} \\
& \times\widehat{t'_{12}\Big|A}_{(m-1,0,\sum_{i=1}^k({}^im+{}^il)+k+1),\vec{n}_s-(2m+1)\delta_{s,1}-2[\sum_{i=1}^k({}^im+{}^il)+k]\delta_{s,2}}\Big\rangle_V\Bigg]\Bigg\},
\end{aligned}\tag{A54}
$$

$$
\begin{aligned}
l'_2\Big|\vec{0}_{ij},\vec{n}_s\rangle_V \;=\; & \sum_{m=0}^{[n_1+1/2]}\left(\frac{-2r}{m_1^2}\right)^m C^{n_1}_{2m}\hat{l}'^2\Big|\vec{0}_{ij}+m\delta_{ij,11},\vec{n}_s-2m\delta_{s,1}\Big\rangle_V \\
& -2m_1\sum_{m=0}^{[n_1-1/2]}\left(\frac{-2r}{m_1^2}\right)^{m+1}\Big\{C^{n_1}_{2m+1}\hat{l}'_{12}\Big|\vec{0}_{ij}+(m+1)\delta_{ij,11},\vec{n}_s-(2m+1)\delta_{s,1}\Big\rangle_V \\
& +C^{n_1}_{2m+1}\hat{l}'_{22}\Big|\vec{0}_{ij}+m\delta_{ij,11}+\delta_{ij,12},\vec{n}_s-(2m+1)\delta_{s,1}\Big\rangle_V \\
& +\frac{1}{4}\Big[C^{n_1}_{2m+1}(h^1+h^2+n_2-2)+C^{n_1}_{2m+2}\Big]\hat{t}'_{12}\Big|\vec{0}_{ij}+m\delta_{ij,11},\vec{n}_s-(2m+1)\delta_{s,1}\Big\rangle_V \\
& +\frac{p_{12}}{4}\Big(C^{n_1}_{2m+1}(h^1+h^2+n_2-2)+C^{n_1}_{2m+2}\Big)(h^1-h^2-n_2-p_{12}+1) \\
& \times\Big|\vec{0}_{ij}+m\delta_{ij,11},\vec{n}_s-(2m+1)\delta_{s,1}-\delta_{s,12}\Big\rangle_V\Big\}, \qquad \text{(A55)}
\end{aligned}
$$

$$
\begin{aligned}
l'_{11}\Big|\vec{0}_{ij},\vec{n}_s\rangle_V \;=\; & \frac{1}{2}\sum_{m=0}^{[n_1-1/2]}\left(\frac{-8r}{m_1^2}\right)^{m+1} C^{n_1}_{2m+2}\hat{l}'_{12}\Big|(m,1,0),\vec{n}_s-2(m+1)\delta_{s1}\rangle_V \\
& -\frac{1}{8}\sum_{m=0}^{[n_1-1/2]}\left(\frac{-8r}{m_1^2}\right)^{m+1} C^{n_1}_{2m+2}\Big\{p_{12}(h^1-h^2-p_{12}-n_2+1)+(h^1-p_{12}) \\
& \times(h^1-p_{12}-2)-h^2-p_{12}-n_2\Big\}\Big|\vec{0}_{ij}+m\delta_{ij,11},\vec{n}_s-2(m+1)\delta_{s1}\rangle_V \\
& -\frac{1}{m_1^2}\sum_{m=0}^{[n_1-1/2]}\left(\frac{-8r}{m_1^2}\right)^{m} C^{n_1}_{2m+2}\hat{l}'_0\Big|\vec{0}_{ij}+m\delta_{ij,11},\vec{n}_s-2(m+1)\delta_{s1}\rangle_V \\
& -\frac{1}{8}\sum_{m=0}^{[n_1-1/2]}\left(\frac{-8r}{m_1^2}\right)^{m+1} C^{n_1}_{2m+2}\hat{t}'_{12}\Big|\vec{0}_{ij}+m\delta_{ij,11},\vec{n}_s-2(m+1)\delta_{s1}+\delta_{s,12}\rangle_V \\
& -\frac{1}{4}\sum_{m=1}^{[n_1-1/2]}\left(\frac{-8r}{m_1^2}\right)^{m}\Big\{C^{n_1}_{2m+1}(2(h^1-p_{12})-1)+2C^{n_1}_{2m+2}\Big\} \\
& \quad\times\Big|\vec{0}_{ij}+(m-1)\delta_{ij,11},\vec{n}_s-2m\delta_{s1}\rangle_V \\
& -\frac{1}{m_1}\sum_{m=0}^{[n_1-1/2]}\left(\frac{-8r}{m_1^2}\right)^{m} C^{n_1}_{2m+1}\hat{l}'_1\Big|\vec{0}_{ij}+m\delta_{ij,11},\vec{n}_s-(2m+1)\delta_{s1}\rangle_V \\
& -\frac{1}{m_1}\sum_{m=0}^{[n_1-1/2]}\left(\frac{-2r}{m_1^2}\right)^{m}(4^m-1)C^{n_1}_{2m+1}\hat{l}'_2\Big|(m-1,1,0),\vec{n}_s-(2m+1)\delta_{s1}\rangle_V \\
& +\frac{m_2}{2m_1}\sum_{m=0}^{[n_1-1/2]}\left(\frac{-2r}{m_1^2}\right)^{m}(4^m-1)C^{n_1}_{2m+1}\Big\{\hat{t}'_{12}\Big|(m-1,0,0),\vec{n}_s-(2m+1,0,-1)\rangle_V \\
& \quad +p_{12}(h^1-h^2-p_{12}-n_2+1)\Big|(m-1,0,0),\vec{n}_s-(2m+1,1,-1)\rangle_V\Big\} \\
& +\frac{1}{2}\sum_{m=0}^{[n_1-1/2]}\left(\frac{-2r}{m_1^2}\right)^{m}(4^m-1)C^{n_1}_{2m+1}\Big|(m-1,0,0),\vec{n}_s-2m\delta_{s1}\rangle_V \\
& +\frac{1}{2}\sum_{m=0}^{[n_1-1/2]}\left(\frac{-2r}{m_1^2}\right)^{m+1}(4^m-1)pC^{n_1}_{2m+2}p_{12}(h^1-h^2-p_{12}-n_2+1) \\
& \times\Big\{2(h^2+n_2+p_{12}-2)\Big|(m-1,1,0),\vec{n}_s-2(m+1)\delta_{s1}-\delta_{s,12}\rangle_V \\
& -\Big|\vec{0}_{ij}+m\delta_{ij,11},\vec{n}_s-2(m+1)\delta_{s1}\rangle_V\Big\} \\
& -\frac{1}{2}\sum_{m=0}^{[n_1-1/2]}\left(\frac{-2r}{m_1^2}\right)^{m+1}(4^m-1)C^{n_1}_{2m+2}\Big\{\hat{t}'_{12}\Big|(m,0,0),\vec{n}_s-2(m+1)\delta_{s1}+\delta_{s,12}\rangle_V
\end{aligned}
$$

$$
\begin{aligned}
&-4\widehat{l'}_{12}\Big|(m,1,0),\vec{n}_s-2(m+1)\delta_{s1}\rangle_V-4\widehat{l'}_{22}\Big|(m-1,2,0),\vec{n}_s-2(m+1)\delta_{s1}\rangle_V\\
&-2(h^2+n_2+p_{12}-2)\widehat{t'}_{12}\Big|(m-1,1,0),\vec{n}_s-2(m+1)\delta_{s1}\rangle_V\Big\}\\
&+\frac{1}{2}\sum_{m=0}^{[n_1-1/2]}\left(\frac{-2r}{m_1^2}\right)^{m+1}(4^m-1)C^{n_1}_{2m+2}\Big\{p_{12}(p_{12}-1)(h^1-h^2-n_2-p_{12}+2)\\
&\times(h^1-h^2-n_2-p_{12}+1)\Big|(m-1,0,1),\vec{n}_s-2(m+1)\delta_{s1}-2\delta_{s,12}\rangle_V\\
&+p_{12}\Big(h^1-h^2-n_2-p_{12}+1\Big)\widehat{t'}_{12}\Big|(m-1,0,1),\vec{n}_s-2(m+1)\delta_{s1}-\delta_{s,12}\rangle_V\\
&+\sum_{k=0}^{[(n_2-1)/2]}\Bigg[\sum_{{}^1m=0}^{[n_2/2]}\ \sum_{{}^1l=0}^{[n_2/2-({}^1m+1)]}\cdots\\
&\cdots\sum_{{}^km=0}^{\left[n_2/2-\sum_{i=1}^{k-1}({}^im+{}^il)-(k-1)\right]}\ \sum_{{}^kl=0}^{\left[n_2/2-\sum_{i=1}^{k-1}({}^im+{}^il)-{}^km)-k\right]}(-1)^k\left(\frac{-2r}{m_2^2}\right)^{\sum_{i=1}^{k}({}^im+{}^il)+k}\\
&\times C^{n_2}_{2{}^1m+1}C^{n_2-2{}^1m-1}_{2{}^1l+1}\ldots C^{n_2-2(\sum_{i=1}^{k-1}({}^im+{}^il)+k-1)}_{2{}^km+1}C^{n_2-2(\sum_{i=1}^{k-1}({}^im+{}^il)+k-1)-2{}^km-1}_{2{}^kl+1}\\
&\times\widehat{t'_{12}\Big|A}_{(m-1,0,\sum_{i=1}^{k}({}^im+{}^il)+k+1),\vec{n}_s-2(m+1)\delta_{s,1}-2[\sum_{i=1}^{k}({}^im+{}^il)+k]\delta_{s,2}}\Big\rangle_V\Bigg]\Bigg\},
\end{aligned}
\tag{A56}
$$

$$
\begin{aligned}
l'_{12}\Big|\vec{n}_s\rangle_V=&-\frac{1}{2m_1}\sum_{m=0}^{[n_1/2]}\left(\frac{-2r}{m_1^2}\right)^m C^{n_1}_{2m+1}\widehat{l'}_2\Big|\vec{0}_{ij}+m\delta_{ij,11},\vec{n}_s-(2m+1)\delta_{s1}\rangle_V\\
&+\sum_{m=0}^{[n_1/2]}\left(\frac{-2r}{m_1^2}\right)^m C^{n_1}_{2m}\widehat{l'}_{12}\Big|\vec{0}_{ij}+m\delta_{ij,11},\vec{n}_s-2m\delta_{s1}\rangle_V\\
&+\sum_{m=1}^{[n_1/2]}\left(\frac{-2r}{m_1^2}\right)^m\Big\{C^{n_1}_{2m}\widehat{l'}_{22}\Big|\vec{0}_{ij}+(m-1)\delta_{ij,11}+\delta_{ij,12},\vec{n}_s-2m\delta_{s1}\rangle_V\\
&+\frac{1}{4}\Big[C^{n_1}_{2m}(h^1+h^2+n_2-2)+C^{n_1}_{2m+1}\Big]\Big[\widehat{t'}_{12}\Big|\vec{0}_{ij}+(m-1)\delta_{ij,11},\vec{n}_s-2m\delta_{s1}\rangle_V\\
&+p_{12}(h^1-h^2-n_2-p_{12}+1)\Big|\vec{0}_{ij}+(m-1)\delta_{ij,11},\vec{n}_s-2m\delta_{s1}-\delta_{s12}\rangle_V\Big]\Big\},
\end{aligned}
\tag{A57}
$$

$$
l'_{22}\Big|\vec{n}_s\rangle_V=\widehat{l'}_{22}\Big|\vec{n}_s\rangle_V.
\tag{A58}
$$

In (A53)–(A58), we use the block operators introduced by the rule (A46), (A52) and the below convention (A62) for the vector $\widehat{t'_{12}\Big|A}_{\vec{n}_{ij},n_1,p_{12},q}\Big\rangle_V$. In establishing the validity of Lemma A5, we also intensely used (A18), (A19), (A21)–(A26), (A79)–(A86). It is only the term $(2rl'^{+}_{22}(t'_{12})^2\ \Big|\vec{0}_{ij},0,p_{12},n_2\rangle_V$, implied by the quantity $\mathcal{K}^{12i}_2$ for $i=1$ (A75) and arising in the proof of the relations (A53), (A54), and (A56), that calls for particular attention. To this end, it is sufficient to use the result of the action of t'_{12}, determined by (A38), on the vector $\Big|\widehat{A}_{\vec{0}_{ij}+q\delta_{ij,22},0,p_{12},n_2-2q}\Big\rangle_V$ (A39), for $q=[\sum_{i=1}^{k}({}^im+{}^il)+k]$, as in (A56). This is implied by the following representation, which is not difficult to establish:

$$(t'_{12})^2\Big|p_{12},n_2\rangle_V \;=\; p_{12}(p_{12}-1)(h^1-h^2-n_2-p_{12}+2)$$
$$\times(h^1-h^2-n_2-p_{12}+1)\Big|p_{12}-2,n_2\rangle_V$$
$$+p_{12}(h^1-h^2-n_2-p_{12}+1)\widehat{t}'_{12}\Big|p_{12}-1,n_2\rangle_V$$
$$+\sum_{k=0}^{[(n_2-1)/2]}\Bigg\{\sum_{{}^1m=0}^{[n_2/2]}\sum_{{}^1l=0}^{[n_2/2-({}^1m+1)]}\ldots$$
$$\ldots\sum_{{}^km=0}^{\left[n_2/2-\sum\limits_{i=1}^{k-1}({}^im+{}^il)-(k-1)\right]}\sum_{{}^kl=0}^{\left[n_2/2-\sum\limits_{i=1}^{k-1}({}^im+{}^il)-{}^km)-k\right]}(-1)^k\left(\frac{-2r}{m_2^2}\right)^{\sum_{i=1}^{k}({}^im+{}^il)+k}\times$$
$$C^{n_2}_{2^1m+1}C^{n_2-2^1m-1}_{2^1l+1}\ldots C^{n_2-2(\sum_{i=1}^{k-1}({}^im+{}^il)+k-1)}_{2^km+1}C^{n_2-2(\sum_{i=1}^{k-1}({}^im+{}^il+k-1)-2^km-1}_{2^kl+1}$$
$$\times t'_{12}\Big|\widehat{A}_{\vec{0}_{ij}+[\sum\limits_{i=1}^{k}({}^im+{}^il)+k]\delta_{ij,22},0,p_{12},n_2-2[\sum\limits_{i=1}^{k}({}^im+{}^il)+k]}\Big\rangle_V\Big\}\Bigg\}. \tag{A59}$$

To complete the proof, we present the final term explicitly, by using a linear combination of the primary block $\widehat{t}'_{12}$ as acting on specific vectors of the Verma module:

$$t'_{12}\Big|\widehat{A}_{\vec{0}_{ij}+q\delta_{ij,22},0,p_{12},n_2-2q}\rangle_V \;=\; -2q\Big|\widehat{A}_{\vec{0}_{ij}+\delta_{ij,12}+(q-1)\delta_{ij,22},0,p_{12},n_2-2q}\rangle_V$$
$$+(l'^{+}_{22})^q t'_{12}\Big|\widehat{A}_{\vec{0}_{ij},0,p_{12},n_2-2q}\rangle_V\,, \tag{A60}$$
$$t'_{12}\Big|\widehat{A}_{\vec{0}_{ij},0,p_{12},n_2-2q}\rangle_V \;=\; p_{12}(n_2-2q)\widehat{t}'_{12}\Big|\vec{0}_{ij},0,p_{12}-1,n_2-2q\rangle_V$$
$$-\sum_{m=1}^{[n_2/2-q]}\left(\frac{-2r}{m_2^2}\right)^m\Big\{C^{n_2-2q}_{2m}\widehat{t}'_{12}\Big|1,0,m-1,0,p_{12}+1,n_2-2(q+m)\rangle_V$$
$$-(C^{n_2-2q}_{2m}(h^2-h^1+2p_{12})+C^{n_2-2q}_{2m+1})\widehat{t}'_{12}\Big|0,1,m-1,0,p_{12},n_2-2(q+m)\rangle_V$$
$$+p_{12}C^{n_2-2q}_{2m}(h^2-h^1+p_{12}-1)\widehat{t}'_{12}\Big|0,0,m,0,p_{12}-1,n_2-2(q+m)\rangle_V\Big\}$$
$$-\frac{m_1}{m_2}\sum_{m=0}^{[n_2/2-q]}\left(\frac{-2r}{m_2^2}\right)^m C^{n_2-2q}_{2m+1}\widehat{t}'_{12}\Big|0,0,m,1,p_{12},n_2-2(q+m)-1\rangle_V$$
$$-\sum_{{}^1m=0}^{[n_2/2-q]}\sum_{{}^1l=0}^{[n_2/2-q-{}^1m-1]}\left(\frac{-2r}{m_2^2}\right)^{{}^1m+{}^1l+1}C^{n_2-2q}_{2^1m+1}\Big\{\Big[C^{n_2-2(q+{}^1m)-1}_{2^1l+1}\big(h^2-h^1+n_2-2q+2p_{12}$$
$$-2({}^1m+{}^1l+1)\big)-C^{n_2-2q-2^1m-1}_{2^1l+2}\Big]\widehat{t}'_{12}\Big|0,1,{}^1m+{}^1l,0,p_{12},n_2-2(q+{}^1m+{}^1l+1)\rangle_V$$
$$-p_{12}C^{n_2-2q-2^1m-1}_{2^1l+1}\big(h^2-h^1+n_2-2q+p_{12}-2({}^1m+{}^1l+1)-1\big)$$
$$\times\widehat{t}'_{12}\Big|0,0,{}^1m+{}^1l+1,0,p_{12}-1,n_2-2(q+{}^1m+{}^1l+1)\Big\rangle_V$$
$$-C^{n_2-2q-2^1m-1}_{2^1l+1}\widehat{t}'_{12}\Big|1,0,{}^1m+{}^1l,0,p_{12}+1,n_2-2(q+{}^1m+{}^1l+1)\Big\rangle_V$$

$$
\begin{aligned}
&+\frac{m_1}{m_2}C_{2^1l+2}^{n_2-2q-2^1m-1}\widetilde{t}_{12}\Big|0,0,{}^1m+{}^1l+1,1,p_{12},n_2-2(q+{}^1m+{}^1l+1)-1\rangle_V\Big\}\\
&+p_{12}(p_{12}-1)(n_2-2q)(h^1-h^2-n_2+2q-p_{12}+2)\Big|\vec{0}_{ij},0,p_{12}-2,n_2-2q\rangle_V\\
&-\sum_{m=1}^{[n_2/2-q]}\left(\frac{-2r}{m_2^2}\right)^m\Bigg\{\Big[C_{2m}^{n_2-2q}\big[(p_{12}+1)(h^1-h^2-n_2+2(q+m)-p_{12})\\
&\quad+(h^2-h^1+2p_{12})\big]+C_{2m+1}^{n_2-2q}\Big]\Big|1,0,m-1,0,p_{12},n_2-2(q+m)\rangle_V\\
&\quad-2(m-1)C_{2m}^{n_2-2q}\Big|1,1,m-2,0,p_{12}+1,n_2-2(q+m)\rangle_V\\
&\quad-\Big(C_{2m}^{n_2-2q}(h^2-h^1+2p_{12})+C_{2m+1}^{n_2-2q}\Big)\Big[p_{12}(h^1-h^2-n_2+2(q+m)\\
&\quad-p_{12}+1)\Big|0,1,m-1,0,p_{12}-1,n_2-2(q+m)\rangle_V\\
&\quad-2(m-1)\Big|0,2,m-2,0,p_{12},n_2-2(q+m)\rangle_V\Big]\\
&\quad+p_{12}C_{2m}^{n_2-2q}(h^2-h^1+p_{12}-1)\Big[(p_{12}-1)(h^1-h^2-n_2+2(q+m)\\
&\quad-p_{12}+2)\Big|0,0,m,0,p_{12}-2,n_2-2(q+m)\rangle_V\\
&\quad-2m\Big|0,1,m-1,0,p_{12}-1,n_2-2(q+m)-1\rangle_V\Big]\Bigg\}\\
&+\frac{m_1}{m_2}\sum_{m=0}^{[n_2/2-q]}\left(\frac{-2r}{m_2^2}\right)^m C_{2m+1}^{n_2-2q}\Big\{2m\Big|0,1,m-1,1,p_{12},n_2-2(q+m)-1\rangle_V\\
&-p_{12}[h^1-h^2-n_2+2(q+m)-p_{12}+2]\Big|0,0,m,1,p_{12}-1,n_2-2(q+m)-1\rangle_V\Big\}\\
&-\sum_{{}^1m=0}^{[n_2/2-q]}\sum_{{}^1l=0}^{[n_2/2-q-{}^1m-1]}\left(\frac{-2r}{m_2^2}\right)^{{}^1m+{}^1l+1}C_{2^1m+1}^{n_2-2q}\Bigg\{\Big[C_{2^1l+1}^{n_2-2(q+{}^1m)-1}\big(h^2-h^1+n_2-2q\\
&+2p_{12}-2({}^1m+{}^1l+1)\big)-C_{2^1l+2}^{n_2-2q-2^1m-1}\Big]\Big[p_{12}(h^1-h^2-n_2+2(q+{}^1m+{}^1l+1)\\
&\quad-p_{12}+1)\Big|0,1,{}^1m+{}^1l,0,p_{12}-1,n_2-2(q+{}^1m+{}^1l+1)\rangle_V\\
&\quad-\Big|1,0,{}^1m+{}^1l,0,p_{12},n_2-2(q+{}^1m+{}^1l+1)\rangle_V\\
&\quad-2({}^1m+{}^1l)\Big|0,2,{}^1m+{}^1l-1,0,p_{12},n_2-2(q+{}^1m+{}^1l+1)\rangle_V\Big]\\
&-p_{12}C_{2^1l+1}^{n_2-2q-2^1m-1}\big(h^2-h^1+n_2-2q+p_{12}-2({}^1m+{}^1l+1)-1\big)\\
&\times\Big[(p_{12}-1)(h^1-h^2-n_2+2(q+{}^1m+{}^1l+1)\\
&\quad-p_{12}+2)\Big|0,0,{}^1m+{}^1l+1,0,p_{12}-2,n_2-2(q+{}^1m+{}^1l+1)\rangle_V\\
&\quad-2({}^1m+{}^1l+1)\Big|0,1,{}^1m+{}^1l,0,p_{12}-1,n_2-2(q+{}^1m+{}^1l+1)\rangle_V\Big]\\
&-C_{2^1l+1}^{n_2-2q-2^1m-1}\Big[(p_{12}+1)(h^1-h^2-n_2+2(q+{}^1m+{}^1l+1)\\
&\quad-p_{12})\Big|1,0,{}^1m+{}^1l,0,p_{12},n_2-2(q+{}^1m+{}^1l+1)\rangle_V\\
&\quad-2({}^1m+{}^1l)\Big|1,1,{}^1m+{}^1l-1,0,p_{12}+1,n_2-2(q+{}^1m+{}^1l+1)\rangle_V\Big]
\end{aligned}
$$

$$+\frac{m_1}{m_2}C^{n_2-2q-2^1m-1}_{2^1l+2}\Big[p_{12}(h^1-h^2-n_2+2(q+{}^1m+{}^1l+1)$$
$$-p_{12}+2)\Big|0,0,{}^1m+{}^1l+1,1,p_{12}-1,n_2-2(q+{}^1m+{}^1l+1)-1\rangle_V$$
$$-2({}^1m+{}^1l+1)\Big|0,1,{}^1m+{}^1l,1,p_{12},n_2-2(q+{}^1m+{}^1l+1)-1\rangle_V\Big]\Bigg\}, \quad \text{(A61)}$$

as we have assumed $q=\sum_{i=1}^{k}({}^im+{}^il)+k$.

Finally, adopting the notation based on (A60) and (A61), and using the conventions (A46) and (A52),

$$\widehat{t'_{12}\Big|A_{\vec{n}_{ij}+q\delta_{ij,22},n_1,p_{12},n_2-2q}\Big\rangle_V} \equiv \prod_{i\le j}^{2}(l'^{+}_{ij})^{n_{ij}}\Big(\frac{l'^{+}_{1}}{m_1}\Big)^{n_1}t'_{12}\Big|\widehat{A}_{0_{ij}+q\delta_{ij,22},0,p_{12},n_2-2q}\rangle_V\,, \quad \text{(A62)}$$

we arrive at the validity of Lemma A5, and therefore also at the validity of Theorem 1.

In the flat-space limit $r=0$, we obtain a new Verma module realization for the above Lie algebra $\mathcal{A}(Y(2),R^{1,d-1})$, being different from that for $sp(4)$ in [55], as well as in [59] for $k=2$,

$$t'_{12}\Big|\vec{N}(2)\rangle_V = p_{12}(h^1-h^2-n_2-p_{12}+1)\big|\vec{n}_{ij},\vec{n}_s-\delta_{s,12}\rangle_V$$
$$-\sum_l l n_{l2}\Big|\vec{n}_{ij}-\delta_{ij,l2}+\delta_{ij,1l},\vec{n}_s\rangle_V-\frac{m_1}{m_2}n_2\Big|\vec{N}(2)+\delta_{s,1}-\delta_{s,2}\rangle_V\,, \quad \text{(A63)}$$

$$t'^{+}_{12}\Big|\vec{N}(2)\rangle_V = -\sum_l(3-l)n_{1l}\Big|\vec{n}_{ij}-\delta_{ij,1l}+\delta_{ij,l2},\vec{n}_s\rangle_V$$
$$+\Big|\vec{N}(2)+\delta_{s,12}\rangle_V-n_1\frac{m_2}{m_1}\Big|\vec{N}(2)-\delta_{s,1}+\delta_{s,2}\rangle_V\,, \quad \text{(A64)}$$

$$l'^{+}_l\Big|\vec{N}(2)\rangle_V = m_l\Big|\vec{N}(2)+\delta_{s,l}\rangle_V\,,\ l=1,2, \quad \text{(A65)}$$

$$l'_0\Big|\vec{N}(2)\rangle_V = m_0^2\Big|\vec{N}(2)\rangle_V\,, \quad \text{(A66)}$$

$$l'_1\Big|\vec{N}(2)\rangle_V = n_1\frac{m_0^2}{m_1}\Big|\vec{N}(2)-\delta_{s,1}\rangle_V-m_1n_{11}\big|\vec{n}_{ij}-\delta_{ij,11},\vec{n}_s+\delta_{s,1}\rangle_V$$
$$-\frac{m_2n_{12}}{2}\big|\vec{n}_{ij}-\delta_{ij,12},\vec{n}_s+\delta_{s,2}\rangle_V\,, \quad \text{(A67)}$$

$$l'_2\Big|\vec{N}(2)\rangle_V = n_2\frac{m_0^2}{m_2}\Big|\vec{N}(2)-\delta_{s,2}\rangle_V-m_1\frac{n_{12}}{2}\big|\vec{n}_{ij}-\delta_{ij,12},\vec{n}_s+\delta_{s,1}\rangle_V$$
$$-m_2n_{22}\big|\vec{n}_{ij}-\delta_{ij,22},\vec{n}_s+\delta_{s,2}\rangle_V\,, \quad \text{(A68)}$$

$$l'_{11}\Big|\vec{N}(2)\rangle_V = n_{11}(n_{11}+n_{12}+n_1-p_{12}-1+h^1)\big|\vec{n}_{ij}-\delta_{ij,11},\vec{n}_s\rangle_V$$
$$+\frac{n_{12}(n_{12}-1)}{4}\big|\vec{n}_{ij}-2\delta_{ij,12}+\delta_{ij,22},\vec{n}_s\rangle_V-\frac{m_0^2}{2m_1^2}n_1(n_1-1)\Big|\vec{N}(2)-2\delta_{s,1}\rangle_V$$
$$-\frac{n_{12}}{2}\Big\{\Big|\vec{N}(2)-\delta_{ij,12}+\delta_{s,12}\rangle_V-n_1\frac{m_2}{m_1}\Big|\vec{N}(2)-\delta_{ij,12}-\delta_{s,1}+\delta_{s,2}\rangle_V\Big\}, \quad \text{(A69)}$$

$$l'_{12}\Big|\vec{N}(2)\rangle_V = \frac{n_{12}}{4}\Big(n_{12}+\sum_l(2n_{ll}+n_l+h^l)-1\Big)\big|\vec{n}_{ij}-\delta_{ij,12},\vec{n}_s\rangle_V$$
$$+\frac{1}{2}p_{12}n_{11}(h^2-h^1+n_2+p_{12}-1)\big|\vec{n}_{ij}-\delta_{ij,11},n_1,p_{12}-1,n_2\rangle_V$$

$$
\begin{aligned}
&+n_{11}n_{22}\big|\vec{n}_{ij}-\delta_{ij,11}-\delta_{ij,22}+\delta_{ij,12},\vec{n}_s\rangle_V-\frac{n_1n_2m_0^2}{2m_1m_2}\Big|\vec{N}(2)-\delta_{s,1}-\delta_{s,2}\rangle_V\\
&-\frac{n_{22}}{2}\Big\{\Big|\vec{N}(2)-\delta_{ij,22}+\delta_{s,12}\rangle_V-n_1\frac{m_2}{m_1}\Big|\vec{N}(2)-\delta_{ij,22}-\delta_{s,1}+\delta_{s,2}\rangle_V\Big\}\\
&+\frac{m_1}{m_2}\frac{n_2n_{11}}{2}\Big|\vec{N}(2)-\delta_{ij,11}+\delta_{s,1}-\delta_{s,2}\rangle_V, \qquad \text{(A70)}
\end{aligned}
$$

$$
\begin{aligned}
l'_{22}\Big|\vec{N}(2)\rangle_V &= n_{22}(n_{12}+p_{12}+n_2+n_{22}-1+h^2)\big|\vec{n}_{ij}-\delta_{ij,22},\vec{n}_s\rangle_V\\
&+\frac{n_{12}p_{12}}{2}(p_{12}-1+h^2-h^1+n_2)\big|\vec{n}_{ij}-\delta_{ij,12},\vec{n}_s-\delta_{s,12}\rangle_V\\
&+\frac{n_{12}(n_{12}-1)}{4}\big|\vec{n}_{ij}+\delta_{ij,11}-2\delta_{ij,12},\vec{n}_s\rangle_V-\frac{n_2(n_2-1)}{2}\frac{m_0^2}{m_2^2}\Big|\vec{N}(2)-2\delta_{s,2}\rangle_V\\
&+\frac{m_1}{m_2}\frac{n_2n_{12}}{2}\Big|\vec{N}(2)-\delta_{ij,12}+\delta_{s,1}-\delta_{s,2}\rangle_V, \qquad \text{(A71)}
\end{aligned}
$$

whereas the action of the negative-root vectors l'^{+}_{ij} and that of the Cartan generators g'^{i}_0 on $|\vec{N}(2)\rangle_V$ remains unchanged and is given by (52), (55).

Appendix C.1. Auxiliary Calculation for $\mathcal{A}'(Y(2), AdS_d)$

Here, we list the formulae for the operator quantities from Lemma A1:

$$
\begin{aligned}
\mathcal{K}^i_n &= K^{0i}_n+2\mathcal{K}^{12i}_n, \qquad \Big(K^{0i}_n,\mathcal{K}^{12i}_n\Big)\equiv \mathrm{ad}_{l'^{i+}}\Big(K^{0i}_{n-1},\mathcal{K}^{12i}_{n-1}\Big),\ n\in\mathbf{N}, && \text{(A72)}\\
\mathcal{K}^i_1 &= K^{0i}_1+2\mathcal{K}^{12i}_1\\
&= \Big\{4l'^{ii+}l'^{i}+l'^{i+}(2g'^{i}_0-1)\Big\}+2\Big\{2l'^{+}_{12}l'^{\{1}\delta^{2\}i}-\delta^{1i}l'^{2+}t'_{12}-\delta^{2i}l'^{1+}t'^{+}_{12}\Big\}, && \text{(A73)}\\
K^{0i}_2 &= 4l'^{ii+}K'^{0i}_2+2l'^{i+2}, && \text{(A74)}\\
\mathcal{K}^{12i}_2 &= 2l'^{12+}X'^{\{1i}_b\delta^{2\}i}+W'^{21+}_b(\delta^{1i}t'_{12}-\delta^{2i}t'^{+}_{12})\\
&= 2rl'^{12+}\Big\{\delta^{i1}\Big[4l'^{12+}l'^{22}+2(g'^{2}_0-1)t'_{12}\Big]+\delta^{i2}\Big[4l'^{12+}l'^{11}+2(g'^{1}_0-1)t'^{+}_{12}\Big]\Big\}\\
&\quad -2rl'^{ii+}\Big(\mathcal{K}^{12}_0+g^{\{1}_0\delta^{2\}i}\Big)+2r\Big\{\delta^{i1}l'^{22+}(t'_{12})^2+\delta^{i2}l'^{11+}(t'^{+}_{12})^2\Big\}, && \text{(A75)}\\
K'^{0i}_2 &= X'^{ii}_b=l'_0-r\Big(K'^{0i}_0+\mathcal{K}'^{12}_0\Big), && \text{(A76)}\\
W'^{12+i}_b &= [W'^{12+}_b,l'^{i+}]=2r\Big\{\big[l'^{+}_{12}l'^{2+}-l'^{+}_{22}l'^{1+}\big]\delta^{i2}-\big[l'^{+}_{12}l'^{1+}-l'^{+}_{11}l'^{2+}\big]\delta^{i1}\Big\}, && \text{(A77)}\\
\Big(X'^{12}_{b2},X'^{21}_{b1}\Big) &= \Big([X'^{12}_b,l'^{2+}],[X'^{21}_b,l'^{1+}]\Big)=r\Big\{-2l'^{+}_{22}l'^{1}+t'^{+}_{12}l'^{2+},\,-2l'^{+}_{11}l'^{2}+l'^{1+}t'_{12}\Big\}. && \text{(A78)}
\end{aligned}
$$

The non-vanishing action of the operators (A12) and (A72)–(A76) and that of $\{-W'^{12+}_b, X'^{12}_b\}$, as acting on the highest-weight vector $|0\rangle_V$, are given by

$$
\begin{aligned}
\Big(\mathcal{K}_0,\mathcal{K}^{12i}_1\Big)|0\rangle_V &= \Big(\Big[\textstyle\sum_i h^i(h^i-2)-2h^2\Big]|0\rangle_V,-m_1\delta^{i2}\Big|\vec{0}_{lm},1,1,0\rangle_V\Big), && \text{(A79)}\\
\mathcal{K}^i_1|0\rangle_V &= 2m_ih^i\Big|\vec{0}_{lm},\delta^{i1},0,\delta^{i2}\rangle_V-2m_1\delta^{i2}\Big|\vec{0}_{lm},1,1,0\rangle_V, && \text{(A80)}\\
K'^{0i}_2|0\rangle_V &= M^2_i|0\rangle_V, \qquad M^2_i=m^2_0-rh^i(h^i-2-\delta^{i2}), && \text{(A81)}\\
K^{0i}_2|0\rangle_V &= 4M^2_i\Big|\delta^{i1},0,\delta^{i2},\vec{0}_s\rangle_V+2m^2_i\Big|\vec{0}_{lm},2\delta^{i1},0,2\delta^{i2}\rangle_V, && \text{(A82)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{K}_2^{12i}|0\rangle_V &= 2r\delta^{2i}\Big\{(h^2-h^1)\Big|0,0,\delta^{i2},\vec{0}_s\rangle_V + 2(h^1-2)\delta^{i2}|0,1,0,0,1,0\rangle_V \\
&\quad +\delta^{i2}|1,0,0,0,2,0\rangle_V\Big\}, \quad \text{(A83)}\\
\mathcal{K}_2^{i}|0\rangle_V &= \Big\{4M_i^2 + 4r\delta^{2i}(h^2-h^1)\Big\})\Big|\delta^{i1},0,\delta^{i2},\vec{0}_s\rangle_V + 2m_i^2\Big|\vec{0}_{lm},2\delta^{i1},0,2\delta^{i2}\rangle_V \\
&\quad +8r(h^1-2)\delta^{i2}|0,1,0,0,1,0\rangle_V + 4r\delta^{i2}|1,0,0,0,2,0\rangle_V, \quad \text{(A84)}\\
W_b^{\prime 12+}|0\rangle_V &= 2r\Big\{(h^2-h^1)\Big|0,1,0,\vec{0}_s\rangle_V - |1,0,0,0,1,0\rangle_V\Big\}, \quad \text{(A85)}\\
X_b^{\prime 12}|0\rangle_V &= -2r(h^2+h^1-2)\Big|\vec{0}_{lm},0,1,0\rangle_V\,. \quad \text{(A86)}
\end{aligned}
$$

Appendix D. Oscillator Realization of Additional Parts: New Fock Space

In this appendix, we examine the correctness of Theorem A1 for a bosonic realization of the non-linear algebra $\mathcal{A}'(Y(2), AdS_d)$ over the Heisenberg algebra A_6 of $(b_i, b_i^+, b_{ij}, b_{ij}^+, d_{12}, d_{12}^+)$. To this end, it is sufficient to show how the most involved of the Verma module relations can be transformed to an oscillator form. Thus, the vector $|A_{n_2}\rangle_V$ (A30) reads

$$
\begin{aligned}
|A_{n_2}\rangle_V \quad \longleftrightarrow \quad & \Bigg\{-\frac{m_1}{m_2}\sum_{m=0}\left(\frac{-2r}{m_2^2}\right)^m \frac{(b_{22}^+)^m}{(2m+1)!}b_1^+ b_2^{2m+1} \\
&+\sum_{m=1}\left(\frac{-2r}{m_2^2}\right)^m (b_{22}^+)^{m-1}\left[b_{12}^+\left\{\frac{h^2-h^1}{(2m)!}+\frac{b_2^+b_2}{(2m+1)!}\right\}-\frac{b_{11}^+d_{12}^+}{(2m)!}\right]b_2^{2m} \\
&-\sum_{m=0}\sum_{l=0}\left(\frac{-2r}{m_2^2}\right)^{m+l+1}\frac{1}{(2m+1)!}(b_{22}^+)^{m+l}\Bigg[b_{12}^+\Bigg\{\frac{h^2-h^1+b_2^+b_2}{(2l+1)!} \\
&-\frac{b_2^+b_2}{(2l+2)!}\Bigg\}-\frac{b_{11}^+d_{12}^+}{(2l+1)!}+\frac{m_1}{m_2}\frac{b_{22}^+}{(2l+2)!}b_1^+b_2\Bigg]b_2^{2(m+l+1)}\Bigg\}\Big|\vec{0}_{ij},0,0,n_2\rangle\,. \quad \text{(A87)}
\end{aligned}
$$

This relation allows one to establish the form of the primary block operator $\hat{t}'_{12}$ in (A92). Let us specify the operator related to the action of $(t'_{12})^2$ in (A61). For instance, there is an obvious one-to-one correspondence for the following term in $l'_0\Big|\vec{0}_{ij},\vec{n}_s\rangle_V$ (A53):

$$
\begin{aligned}
&\frac{r}{2}\sum_{m=0}^{[n_1-1/2]}\left(\frac{-8r}{m_1^2}\right)^{m+1} C^{n_1}_{2m+2}(l'^+_{11})^m l'^+_{22}\left(\frac{l'^+_1}{m_1}\right)^{n_1-2(m+1)}\underbrace{(t'_{12})^2}\Big|\vec{0}_{ij},0,p_{12},n_2\rangle_V \\
&\equiv \frac{r}{2}\sum_{m=0}^{[n_1-1/2]}\left(\frac{-8r}{m_1^2}\right)^{m+1} C^{n_1}_{2m+2}\Big[p_{12}(p_{12}-1)(h^1-h^2-n_2-p_{12}+2) \\
&\times(h^1-h^2-n_2-p_{12}+1)|(m,0,1),\vec{n}_s-(2m+2,2,0)\rangle_V \\
&+p_{12}(h^1-h^2-n_2-p_{12}+1)\widehat{t}'_{12}|(m,0,1),\vec{n}_s-(2m+2,1,0)\rangle_V \\
&+\sum_{k=0}^{[(n_2-1)/2]}\Bigg\{\sum_{{}^1m=0}^{[n_2/2]}\ \sum_{{}^1l=0}^{[n_2/2-({}^1m+1)]}\cdots
\end{aligned}
$$

$$\ldots \sum_{^k m=0}^{\left[n_2/2-\sum\limits_{i=1}^{k-1}(^im+^il)-(k-1)\right]} \sum_{^k l=0}^{\left[n_2/2-\sum\limits_{i=1}^{k-1}(^im+^il)-^km)-k\right]} (-1)^k\left(\frac{-2r}{m_2^2}\right)^{\sum_{i=1}^k(^im+^il)+k}$$
$$\times C^{n_2}_{2^1m+1}C^{n_2-2^1m-1}_{2^1l+1}\ldots C^{n_2-2(\sum_{i=1}^{k-1}(^im+^il)+k-1)}_{2^km+1}C^{n_2-2(\sum_{i=1}^{k-1}(^im+^il+k-1)-2^km-1}_{2^kl+1}$$
$$\times \widehat{t'_{12}\Big|A}_{(m,0,\sum_{i=1}^k(^im+^il)+k+1),\vec{n}_s-(2m+2)\delta_{s,1}-2[\sum\limits_{i=1}^k(^im+^il)+k]\delta_{s,2}}\Big\rangle_V\Big\}\Big]$$
$$\longleftrightarrow \frac{r}{2}\sum_{m\geq 0}\left(\frac{-8r}{m_1^2}\right)^{m+1}\frac{(b_{11}^+)^m b_{22}^+}{(2m+2)!}\Big[\Big\{(h^1-h^2-b_2^+b_2-d_{12}^+d_{12})d_{12}+\widehat{t'}_{12}\Big\}$$
$$\times(h^1-h^2-b_2^+b_2-d_{12}^+d_{12})d_{12}+t'_{12}\widehat{t'}_{12}\Big](b_1)^{2(m+1)}(b_1^+)^{n_1}\Big|\vec{0}_{ij},0,p_{12},n_2\rangle. \qquad \text{(A88)}$$

In particular, (A88) implies an oscillator representation for the action of $(t'_{12})^2$ on the vector $\big|\vec{0}_{ij},0,p_{12},n_2\rangle_V$, due to an associative composition of the operators t'_{12} and $\hat{t}'_{12}$ given by (A91) and (A92),

$$(t'_{12})^2\Big|\vec{0}_{ij},0,p_{12},n_2\rangle_V \longleftrightarrow \Big[\Big\{(h^1-h^2-b_2^+b_2-d_{12}^+d_{12})d_{12}+\widehat{t'}_{12}\Big\}$$
$$\times(h^1-h^2-b_2^+b_2-d_{12}^+d_{12})d_{12}+t'_{12}\widehat{t'}_{12}\Big]\Big|\vec{0}_{ij},0,p_{12},n_2\rangle\,. \qquad \text{(A89)}$$

By using the arguments on the oscillator realization of the quadratic terms in t'_{12} and the structure of the operator related to the vector $|A_{n_2}\rangle_V$, all of the operators o'_I can be restored in an oscillator form, given by (A90)–(A101), by virtue of the following.

Theorem A1. *An oscillator realization of the non-linear second-order algebra $\mathcal{A}'(Y(2),AdS_d)$ over the Heisenberg algebra A_6 does exist in terms of a formal power series in the degrees of creation and annihilation operators, is given by the relations (A90)–(A93), (A95), and (A96)–(A101), and is expressed using the* primary block operator $\widehat{t}'_{12}$ *(A92) and the* derived block operators $\widehat{t}'^+_{12},\widehat{l}'_0,\widehat{l}'_m,\widehat{l}'_{m2}, m=1,2$ *(A94), (A102)–(A106) as follows.*

First, for trivial negative-root vectors, we have

$$l'^+_1 = m_1 b_1^+\,,\quad l'^+_{ij}=b^+_{ij}\,,\quad g'^i_0 = 2b^+_{ii}b_{ii}+b^+_{12}b_{12}+(-1)^i d^+_{12}d_{12}+b^+_i b_i+h^i. \quad \text{(A90)}$$

Second, for the operator $t'_{12}(B,B^+)$ and the primary block operator $\widehat{t}'_{12}(B,B^+)$*, we obtain*[19]

$$t'_{12} = \Big(h^1-h^2-b_2^+b_2-d^+_{12}d_{12}\Big)d_{12}-b^+_{11}b_{12}-2b^+_{12}b_{22}+\widehat{t}'_{12}\,, \qquad \text{(A91)}$$
$$\widehat{t}'_{12} = \sum_{k=0}\Big[\sum_{^1m=0}\sum_{^1l=0}\ldots\sum_{^km=0}\sum_{^kl=0}(-1)^k\left(\frac{-2r}{m_2^2}\right)^{\sum_{i=1}^k(^im+^il)+k}\prod_{i=1}^k\frac{1}{(2^im+1)!}\frac{1}{(2^il+1)!}$$
$$\times(b^+_{22})^{\sum_{i=1}^k(^im+^il)+k}\Big\{b_2^+d_{12}b_2-\frac{m_1}{m_2}\sum_{m=0}\left(\frac{-2r}{m_2^2}\right)^m\frac{(b^+_{22})^m}{(2m+1)!}b_1^+b_2^{2m+1}$$
$$+\sum_{m=1}\left(\frac{-2r}{m_2^2}\right)^m(b^+_{22})^{m-1}\Big[b^+_{12}\Big\{\frac{h^2-h^1+2d^+_{12}d_{12}}{(2m)!}+\frac{b_2^+b_2}{(2m+1)!}\Big\}$$

$$
-\frac{b_{11}^{+}d_{12}^{+}}{(2m)!}-\frac{b_{22}^{+}}{(2m)!}(h^2-h^1+d_{12}^{+}d_{12})d_{12}\Big]b_2^{2m}
$$
$$
-\sum_{m=0}\sum_{l=0}\left(\frac{-2r}{m_2^2}\right)^{m+l+1}\frac{1}{(2m+1)!}(b_{22}^{+})^{m+l}\Big[b_{12}^{+}\Big\{\frac{h^2-h^1+2d_{12}^{+}d_{12}+b_2^{+}b_2}{(2l+1)!}
$$
$$
-\frac{b_2^{+}b_2}{(2l+2)!}\Big\}-\frac{b_{11}^{+}d_{12}^{+}}{(2l+1)!}-\frac{b_{22}^{+}}{(2l+1)!}(h^2-h^1+b_2^{+}b_2+d_{12}^{+}d_{12})d_{12}
$$
$$
+\frac{m_1}{m_2}\frac{b_{22}^{+}}{(2l+2)!}b_1^{+}b_2\Big]b_2^{2(m+l+1)}\Bigg\}(b_2)^{2(\sum_{i=1}^{k}({}^{i}m+{}^{i}l)+k)}\Bigg]. \tag{A92}
$$

Third, for the operators $t_{12}^{\prime+}, l_2^{\prime+}$ and the derived block operator $\widehat{t}_{12}^{\prime+}$, we have

$$
t_{12}^{\prime+} = -2b_{12}^{+}b_{11}-b_{22}^{+}b_{12}+\widehat{t}_{12}^{\prime+}, \tag{A93}
$$
$$
\begin{aligned}
\widehat{t}_{12}^{\prime+} = {} & \sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m}(b_{11}^{+})^{m}\Big\{\frac{d_{12}^{+}}{(2m)!}-\frac{m_2}{m_1}\frac{b_2^{+}b_1}{(2m+1)!}\Big\}b_1^{2m} \\
& -\sum_{m=1}\left(\frac{-2r}{m_1^2}\right)^{m}(b_{11}^{+})^{m-1}\Bigg\{b_{12}^{+}\Bigg[\frac{(h^2-h^1+2d_{12}^{+}d_{12}+b_2^{+}b_2)}{(2m)!}-\frac{b_1^{+}b_1}{(2m+1)!}\Bigg] \\
& +b_{22}^{+}\frac{(h^1-h^2-d_{12}^{+}d_{12}-b_2^{+}b_2)}{(2m)!}d_{12}+\frac{b_{22}^{+}}{(2m)!}\,\widehat{t}_{12}^{\prime}\Bigg\}b_1^{2m},
\end{aligned} \tag{A94}
$$
$$
\begin{aligned}
l_2^{\prime+} = {} & m_1\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m+1}(b_{11}^{+})^{m}\Bigg\{b_{12}^{+}\Bigg[\frac{(h^2-h^1+2d_{12}^{+}d_{12}+b_2^{+}b_2)}{(2m+1)!}-\frac{b_1^{+}b_1}{(2m+2)!}\Bigg] \\
& +b_{22}^{+}\frac{(h^1-h^2-d_{12}^{+}d_{12}-b_2^{+}b_2)}{(2m+1)!}d_{12}-b_{11}^{+}\frac{d_{12}^{+}}{(2m+1)!}\Bigg\}b_1^{2m+1} \\
& +m_2\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m}\frac{(b_{11}^{+})^{m}b_2^{+}}{(2m)!}b_1^{2m}+m_1\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m+1}\frac{(b_{11}^{+})^{m}b_{22}^{+}}{(2m+1)!}\,\widehat{t}_{12}^{\prime}\,b_1^{2m+1}.
\end{aligned} \tag{A95}
$$

Fourth, for the Cartan-like vector $l_0^{\prime}$, the following representation holds true:

$$
\begin{aligned}
l_0^{\prime} = {} & \widehat{l}_0^{\prime}+\frac{m_1}{2}\sum_{m=0}\left(\frac{-8r}{m_1^2}\right)^{m+1}\frac{(b_{11}^{+})^{m}}{(2m+1)!}\Bigg\{b_{12}^{+}\,\widehat{l}_2^{\prime}\;+\;b_{11}^{+}\,\widehat{l}_1^{\prime} \\
& -\frac{m_2}{2}[\widehat{t}_{12}^{\prime}+(h^1-h^2-b^{2+}b_2-d_{12}^{+}d_{12})d_{12}]\;b_2^{+}\Bigg\}b_1^{2m+1} \\
& -r\sum_{m=0}\left(\frac{-8r}{m_1^2}\right)^{m}(b_{11}^{+})^{m}b_1^{+}\Bigg\{\frac{2(h^1-d_{12}^{+}d_{12})-3}{(2m+1)!}+\frac{2b_1^{+}b_1}{(2m+2)!}\Bigg\}b_1^{2m+1} \\
& +\frac{1}{2}\sum_{m=0}\left(\frac{-8r}{m_1^2}\right)^{m+1}\frac{(b_{11}^{+})^{m}}{(2m+2)!}\Bigg\{b_{11}^{+}\,\widehat{l}_0^{\prime}-rb_{11}^{+}\Big([h^1-d_{12}^{+}d_{12}][h^1-d_{12}^{+}d_{12}-2] \\
& -h^2-d_{12}^{+}d_{12}-b_2^{+}b_2+2d_{12}^{+}(h^1-h^2-b_2^{+}b_2-d_{12}^{+}d_{12})d_{12}\Big)
\end{aligned}
$$

$$-2rb_{11}^+\widehat{t}'_{12}\, d_{12}^+ + 8rb_{11}^+ b_{12}^+ \widehat{l}'_{12} \; + 4r(b_{12}^+)^2\, \widehat{l}'_{22} + 2rb_{12}^+\Big[\, (h^1 - h^2$$
$$-b_2^+ b_2 - d_{12}^+ d_{12}) d_{12} + \widehat{t}'_{12}\Big] \,\Big[h^2 + b_2^+ b_2 + d_{12}^+ d_{12} - 2\Big]\Bigg\} b_1^{2m+2}$$
$$+\frac{r}{2}\sum_{m=0}\left(\frac{-8r}{m_1^2}\right)^{m+1} \frac{(b_{11}^+)^m}{(2m+2)!} b_{22}^+ \Bigg\{ \Big[(h^1 - h^2 - b_2^+ b_2 - d_{12}^+ d_{12}) d_{12} + \widehat{t'}_{12}\Big]$$
$$\times (h^1 - h^2 - b_2^+ b_2 - d_{12}^+ d_{12}) d_{12} + t'_{12}\widehat{t}'_{12}\Bigg\}\, b_1^{2m+2}. \tag{A96}$$

Fifth, the operators l'_m, l'_{lm}, $l, m = 1, 2, l \leq m$ read as follows:

$$l'_1 \;=\; -m_1 b_1^+ b_{11} - \frac{1}{2} l'^{2+} b_{12} + \sum_{m=0}\left(\frac{-8r}{m_1^2}\right)^m \frac{(b_{11}^+)^m}{(2m)!} \widehat{l}'^1 b_1^{2m}$$
$$+\frac{1}{m_1}\sum_{m=0}\left(\frac{-8r}{m_1^2}\right)^m \frac{(b_{11}^+)^m}{(2m+1)!}\Bigg\{\widehat{l}'_0 - r\Big[(h^1 - d_{12}^+ d_{12})(h^1 - d_{12}^+ d_{12} - 2) - h^2$$
$$-d_{12}^+ d_{12} - b_2^+ b_2\Big] - r d_{12}^+\Big[h^1 - h^2 - b_2^+ b_2 - d_{12}^+ d_{12}\Big] d_{12} - r\, \widehat{t}'_{12}\, d_{12}^+\Bigg\} b_1^{2m+1}$$
$$-4m_1 \sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m+1} \frac{(b_{11}^+)^m b_{12}^+}{(2m+1)!} (4^m - \tfrac{1}{2})\, \widehat{l}'_{12}\, b_1^{2m+1}$$
$$+\frac{m_1}{2}\sum_{m=1}\left(\frac{-8r}{m_1^2}\right)^m (b_{11}^+)^{m-1} b_1^+ \Bigg\{\frac{1}{2}\frac{(2(h^1 - d_{12}^+ d_{12}) - 1)}{(2m)!} + \frac{b_1^+ b_1}{(2m+1)!}\Bigg\} b_1^{2m}$$
$$+\frac{1}{2}\sum_{m=1}\left(\frac{-2r}{m_1^2}\right)^m (4^m - 1)\frac{(b_{11}^+)^{m-1}}{(2m)!}\Big\{2b_{12}^+ \widehat{l}'^2 - m_1 b_1^+$$
$$-m_2(\widehat{t}'_{12} + h^1 - h^2 - b_2^+ b_2 - d_{12}^+ d_{12}) d_{12} b_2^+\Big\} b_1^{2m}$$
$$+\frac{m_1}{2}\sum_{m=1}\left(\frac{-2r}{m_1^2}\right)^{m+1} (4^m - 1)\frac{(b_{11}^+)^{m-1}}{(2m+1)!}\Bigg\{b_{11}^+\, \widehat{t}'_{12}\, d_{12}^+ - 4(b_{12}^+)^2 \widehat{l}'_{22}$$
$$-2b_{12}^+\Big[(h^1 - h^2 - b_2^+ b_2 - d_{12}^+ d_{12}) d_{12} + \widehat{t}'_{12}\Big](h^2 + b_2^+ b_2 + d_{12}^+ d_{12} - 2)$$
$$+b_{11}^+ d_{12}^+ (h^1 - h^2 - b_2^+ b_2 - d_{12}^+ d_{12}) d_{12}\Bigg\} b_1^{2m+1}$$
$$-\frac{m_1}{2}\sum_{m=1}\left(\frac{-2r}{m_1^2}\right)^{m+1} (4^m - 1)\frac{(b_{11}^+)^{m-1}}{(2m+1)!} b_{22}^+ \Bigg\{\Big[(h^1 - h^2 - b_2^+ b_2 - d_{12}^+ d_{12}) d_{12}$$
$$+\widehat{t}'_{12}\Big](h^1 - h^2 - b_2^+ b_2 - d_{12}^+ d_{12}) d_{12} + t'_{12}\widehat{t}'_{12}\Bigg\} b_1^{2m+1}, \tag{A97}$$

$$l'_2 \;=\; \sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^m \frac{(b_{11}^+)^m}{2m!}\widehat{l}'_2\, b_1^{2m} - 2m_1 \sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m+1} \frac{(b_{11}^+)^m}{(2m+1)!}\sum_k b_{1k}^+ \widehat{l}'_{k2} b_1^{2m+1}$$

$$
\begin{aligned}
&-2m_1\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m+1}\frac{(b_{11}^+)^m}{4}\hat{t}'_{12}\left\{\frac{(h^1+h^2+b_2^+b_2-2)}{(2m+1)!}+\frac{b_1^+b_1}{(2m+2)!}\right\}b_1^{2m+1}\\
&-\frac{m_1}{2}\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m+1}(b_{11}^+)^m\left\{\frac{(h^1+h^2+b_2^+b_2-2)}{(2m+1)!}+\frac{b_1^+b_1}{(2m+2)!}\right\}\times\\
&\times(h^1-h^2-b_2^+b_2-d_{12}^+d_{12})d_{12}b_1^{2m+1}-\frac{m_1}{2}b_1^+b_{12}-l_2'^+\,b_{22},
\end{aligned}
\tag{A98}
$$

$$
\begin{aligned}
l'_{11} \;=\;& (b_{11}^+b_{11}+b_{12}^+b_{12}+b_1^+b_1-d_{12}^+d_{12}+h^1)b_{11}+\frac{b_{22}^+}{4}b_{12}^2-\frac{1}{2}\hat{t}'^+_{12}b_{12}\\
&+\frac{1}{2}\sum_{m=0}\left(\frac{-8r}{m_1^2}\right)^{m+1}\frac{(b_{11}^+)^m}{(2m+2)!}\left\{\frac{1}{4r}\hat{l}'_0\;+b_{12}^+\,\hat{l}'_{12}-\frac{1}{4}\Big[d_{12}^+(2-h^1-h^2-b_2^+b_2)d_{12}\right.\\
&\left.+h^1(h^1-2)-h^2-b_2^+b_2+\hat{t}'_{12}d_{12}^+\Big]\right\}b_1^{2m+2}\\
&-\frac{1}{2}\sum_{m=1}\left(\frac{-8r}{m_1^2}\right)^{m}(b_{11}^+)^{m-1}b_1^+\left\{\frac{h^1-d_{12}^+d_{12}-\frac{1}{2}}{(2m+1)!}+\frac{b_1^+b_1}{(2m+2)!}\right\}b_1^{2m+1}\\
&-\frac{1}{m_1}\sum_{m=0}\left\{\left(\frac{-8r}{m_1^2}\right)^{m}\frac{(b_{11}^+)^m}{(2m+1)!}\,\hat{l}'_1\;-\left(\frac{-2r}{m_1^2}\right)^{m}(4^m-1)\frac{(b_{11}^+)^{m-1}b_{12}^+}{(2m+1)!}\,\hat{l}'_2\right\}b_1^{2m+1}\\
&+\frac{1}{2}\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m}(4^m-1)(b_{11}^+)^{m-1}\frac{b_1^+b_1}{(2m+1)!}b_1^{2m}\\
&+\frac{m_2}{2m_1}\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m}(4^m-1)\frac{(b_{11}^+)^{m-1}}{(2m+1)!}\left\{t'_{12}+\sum_l l b_{1l}^+b_{l2}\right\}b_2^+b_1^{2m+1}\\
&-\frac{1}{2}\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m+1}(4^m-1)\frac{(b_{11}^+)^{m-1}}{(2m+2)!}\Big[b_{11}^+d_{12}^+-2b_{12}^+(h^2+b_2^+b_2+d_{12}^+d_{12}-1)\Big]\times\\
&\times\Big(h^1-h^2-d_{12}^+d_{12}-b_2^+b_2\Big)d_{12}b_1^{2m+2}\\
&-\frac{1}{2}\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m+1}(4^m-1)\frac{(b_{11}^+)^{m-1}}{(2m+2)!}\Big\{\,b_{11}^+\,\hat{t}'_{12}\,d_{12}^+-4b_{11}^+b_{12}^+\,\hat{l}'_{12}\\
&-4(b_{12}^+)^2\,\hat{l}'_{22}\;-2b_{12}^+\,\hat{t}'_{12}\,(h^2+b_2^+b_2+d_{12}^+d_{12}-2)\Big\}b_1^{2m+2}\\
&+\frac{1}{2}\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m+1}(4^m-1)\frac{(b_{11}^+)^{m-1}}{(2m+2)!}b_{22}^+\,\Big\{\Big[(h^1-h^2-b_2^+b_2-d_{12}^+d_{12})d_{12}+\hat{t}'_{12}\Big]\times\\
&\times(h^1-h^2-b_2^+b_2-d_{12}^+d_{12})d_{12}+t'_{12}\hat{t}'_{12}\Big\}b_1^{2m+2},
\end{aligned}
\tag{A99}
$$

$$
\begin{aligned}
l'_{12} \;=\;& \frac{1}{4}\Big(b_{12}^+b_{12}+\sum_m(2b_{mm}^+b_{mm}+b_m^+b_m+h^m)\Big)b_{12}+b_{12}^+b_{11}b_{22}-\frac{1}{2}\big(\hat{t}'^+_{12}\,b_{22}+\hat{t}'_{12}\,b_{11}\big)\\
&+\frac{1}{2}(h^2-h^1+b_2^+b_2+d_{12}^+d_{12})b_{11}d_{12}-\frac{1}{2m_1}\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m}\frac{(b_{11}^+)^m}{(2m+1)!}\hat{l}'_2b_1^{2m+1}\\
&+\sum_{m=0}\left(\frac{-2r}{m_1^2}\right)^{m}\frac{(b_{11}^+)^m}{(2m)!}\hat{l}'_{12}b_1^{2m}+\sum_{m=1}\left(\frac{-2r}{m_1^2}\right)^{m}\frac{(b_{11}^+)^{m-1}b_{12}^+}{(2m)!}\,\hat{l}'_{22}b_1^{2m}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}\sum_{m=1}\left(\frac{-2r}{m_1^2}\right)^m (b_{11}^+)^{m-1}\Bigg\{\widehat{t}'_{12}\left[\frac{(h^1+h^2+b_2^+b_2-2)}{(2m)!}+\frac{b_1^+b_1}{(2m+1)!}\right] \\
&+\left[\frac{h^1+h^2+b_2^+b_2-2}{(2m)!}+\frac{b_1^+b_1}{(2m+1)!}\right](h^1-h^2-b_2^+b_2-d_{12}^+d_{12})d_{12}\Bigg\}b_1^{2m}, \qquad \text{(A100)}\\
l'_{22} &= \widehat{l}'_{22}+(b_{12}^+b_{12}+d_{12}^+d_{12}+b_2^+b_2+b_{22}^+b_{22}+h^2)b_{22}+\frac{1}{4}b_{11}^+b_{12}^2-\frac{1}{2}\widehat{t}'_{12}\,b_{12} \\
&+\frac{1}{2}(h^2-h^1+d_{12}^+d_{12}+b_2^+b_2)d_{12}b_{12}. \qquad \text{(A101)}
\end{aligned}
$$

In (A96)–(A101), the derived block operators $\widehat{l}'_0, \widehat{l}'_m, \widehat{l}'_{m2}$, for $m=1,2$, are written as follows:[20]

$$
\begin{aligned}
\widehat{l}'_0 &= m_0^2-r\sum_{m=0}\left(\frac{-8r}{m_2^2}\right)^m (b_{22}^+)^m\Bigg\{b_2^+\left[\frac{1}{(2m+1)!}(2h^2+2d_{12}^+d_{12}-1)\right.\\
&\left.+\frac{2b_2^+b_2}{(2m+2)!}\right]b_2-2\frac{m_1}{m_2}\frac{b_1^+d_{12}^+}{(2m+1)!}b_2\Bigg\}b_2^{2m}\\
&+\frac{1}{2}\sum_{m=0}\left(\frac{-8r}{m_2^2}\right)^{m+1}\frac{(b_{22}^+)^m}{(2m+2)!}\Big\{b_{22}^+\Big(m_0^2-r\big[h^2(h^2-4)+h^1-(d_{12}^+)^2d_{12}^2\\
&+2d_{12}^+d_{12}(h^1-2)\big]\Big)+2rb_{12}^+d_{12}^+(h^1-d_{12}^+d_{12}-2)+rb_{11}^+(d_{12}^+)^2\Big\}b_2^{2m+2}\\
&+2r\sum_{m=0}\sum_{l=0}\left(\frac{-8r}{m_2^2}\right)^m\left(\frac{-2r}{m_2^2}\right)^{l+1}\frac{(b_{22}^+)^m}{(2m+1)!}\frac{(b_{22}^+)^{l+1}}{(2l+1)!}\widehat{t}'_{12}\,d_{12}^+b_2^{2(m+l+1)}\\
&+2r\sum_{m=0}\sum_{l=0}\left(\frac{-8r}{m_2^2}\right)^m\left(\frac{-2r}{m_2^2}\right)^{l+1}\frac{(b_{22}^+)^{m+l}}{(2m+1)!}\Bigg\{b_{12}^+d_{12}^+\left[\frac{[h^2-h^1+2d_{12}^+d_{12}+b_2^+b_2+2]}{(2l+1)!}\right.\\
&\left.-\frac{b_2^+b_2}{(2l+2)!}\right]-\frac{b_{11}^+(d_{12}^+)^2}{(2l+1)!}+\frac{b_{22}^+(d_{12}^+d_{12}+1)}{(2l+1)!}[h^1-h^2-b_2^+b_2-d_{12}^+d_{12}]\\
&+\frac{m_1}{m_2}\frac{b_{22}^+b_1^+d_{12}^+}{(2l+2)!}b_2\Bigg\}b_2^{2(m+l+1)}, \qquad \text{(A102)}\\
\widehat{l}'_1 &= -\frac{m_2}{2}\sum_{m=0}\left(\frac{-2r}{m_2^2}\right)^{m+1}(b_{22}^+)^m d_{12}^+\Bigg\{\frac{(h^1+h^2-2)}{(2m+1)!}+\frac{b_2^+b_2}{(2m+2)!}\Bigg\}b_2^{2m+1}, \qquad \text{(A103)}\\
\widehat{l}'_2 &= -\widehat{l}'_1d_{12}-\sum_{m=0}\left(\frac{-8r}{m_2^2}\right)^m\frac{(b_{22}^+)^m}{(2m+1)!}\frac{[m_0^2-rh^2(h^2-3)]}{m_2}b_2^{2m+1}\\
&+\frac{m_2}{4}\sum_{m=1}\left(\frac{-8r}{m_2^2}\right)^m(b_{22}^+)^{m-1}b_2^+\Bigg\{\frac{(2h^2-1)}{(2m)!}+2\frac{b_2^+b_2}{(2m+1)!}\Bigg\}b_2^{2m}\\
&+\frac{m_2}{2}\sum_{m=1}\left(\frac{-2r}{m_2^2}\right)^{m+1}(b_{22}^+)^m d_{12}^+\Bigg\{\frac{(h^1+h^2-2)}{(2m+1)!}+\frac{b_2^+b_2}{(2m+2)!}\Bigg\}d_{12}b_2^{2m+1}\\
&-\frac{1}{2}\sum_{m=0}\left(\frac{-2r}{m_2^2}\right)^m(4^m-1)(b_{22}^+)^{m-1}d_{12}^+\Bigg\{\frac{m_1b_1^+}{(2m)!}-\frac{m_2b_2^+d_{12}}{(2m)!}\Bigg\}b_2^{2m}
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{m_2}{2}\sum_{m=0}\left(\frac{-2r}{m_2^2}\right)^{m+1}(4^m-1)\frac{(b_{22}^+)^{m-1}}{(2m+1)!}\Big\{b_{22}^+\Big[2(h^1-2)d_{12}^+d_{12} \\
& \quad +h^1-h^2-(d_{12}^+)^2d_{12}^2\Big]-2b_{12}^+d_{12}^+(h^1-d_{12}^+d_{12}-2)-b_{11}^+(d_{12}^+)^2\Big\}b_2^{2m+1} \\
& -\frac{m_2}{2}\sum_{m=0}\sum_{l=0}\left(\frac{-2r}{m_2^2}\right)^{m+l+1}(4^m-1)\frac{(b_{22}^+)^{m+l-1}}{(2m)!}\Bigg\{\frac{m_1}{m_2}\frac{b_{22}^+b_1^+d_{12}^+}{(2l+2)!}b_2 \\
& \quad +b_{12}^+d_{12}^+\left[\frac{(h^2-h^1+2d_{12}^+d_{12}+b_2^+b_2+2}{(2l+1)!}-\frac{b_2^+b_2}{(2l+2)!}\right] \\
& \quad -\frac{\Big(b_{11}^+(d_{12}^+)^2+b_{22}^+\big[h^2-h^1+b_2^+b_2\big]\Big)}{(2l+1)!}\Bigg\}b_2^{2(m+l)+1} \\
& +\frac{m_2}{2}\sum_{m=0}\sum_{l=0}\left(\frac{-2r}{m_2^2}\right)^{m+l+1}(4^m-1)\frac{(b_{22}^+)^m}{(2m)!}\frac{(b_{22}^+)^l}{(2l+1)!}\times \\
& \quad \times d_{12}^+\Big\{h^2-h^1+b_2^+b_2+d_{12}^+d_{12}+2\Big\}d_{12}b_2^{2(m+l)+1} \\
& -\frac{m_2}{2}\sum_{m=0}\sum_{l=0}\left(\frac{-2r}{m_2^2}\right)^{m+l+1}(4^m-1)\frac{(b_{22}^+)^m}{(2m)!}\frac{(b_{22}^+)^l}{(2l+1)!}\,\hat{t}'_{12}\,d_{12}^+b_2^{2(m+l)+1},
\end{aligned}
\tag{A104}
$$

$$
\hat{l}'_{12} = \frac{1}{4}\sum_{m=1}\left(\frac{-2r}{m_2^2}\right)^m(b_{22}^+)^{m-1}d_{12}^+\left\{\frac{(h^1+h^2-2)}{(2m)!}+\frac{b_2^+b_2}{(2m+1)!}\right\}b_2^{2m}, \tag{A105}
$$

$$
\begin{aligned}
\hat{l}'_{22} = & -2\hat{l}'_{12}d_{12}-\sum_{m=0}\left(\frac{-8r}{m_2^2}\right)^m\frac{(b_{22}^+)^m}{m_2^2(2m+2)!}\big\{m_0^2-rh^2(h^2-3)\big\}b_2^{2(m+1)} \\
& -\frac{1}{2}\sum_{m=1}\left(\frac{-8r}{m_2^2}\right)^m(b_{22}^+)^{m-1}b_2^+\left\{\frac{1}{2}\frac{(2h^2-1)}{(2m+1)!}+\frac{b_2^+b_2}{(2m+2)!}\right\}b_2^{2m+1} \\
& +\frac{1}{2}\sum_{m=0}\left(\frac{-2r}{m_2^2}\right)^{m+1}(4^m-1)(b_{22}^+)^{m-1}\Bigg\{\frac{b_{12}^+d_{12}^+}{(2m+2)!}\big[2(h^1-2)-2d_{12}^+d_{12}\big]b_2 \\
& +\frac{b_{22}^+}{(2m+2)!}\Big[h^2-h^1-2d_{12}^+d_{12}(h^1-2)+(d_{12}^+)^2d_{12}^2\Big]b_2+\frac{b_{11}^+(d_{12}^+)^2b_2}{(2m+2)!} \\
& -\frac{b_2^+d_{12}^+d_{12}}{(2m+1)!}\Bigg\}b_2^{2m+1}+\frac{m_1}{2m_2}\sum_{m=0}\left(\frac{-2r}{m_2^2}\right)^{m+1}\frac{(b_{22}^+)^{m-1}b_1^+d_{12}^+}{(2m+1)!}(4^m-1)b_2^{2m+1} \\
& +\frac{1}{2}\sum_{m=0}\sum_{l=0}\left(\frac{-2r}{m_2^2}\right)^{m+l+1}(4^m-1)\frac{(b_{22}^+)^{m+l-1}}{(2m+1)!}\Bigg\{\frac{m_1}{m_2}\frac{b_{22}^+b_1^+d_{12}^+}{(2l+2)!}b_2-\frac{b_{11}^+(d_{12}^+)^2}{(2l+1)!} \\
& +\frac{b_{22}^+}{(2l+1)!}\big(d_{12}^+d_{12}+1\big)\big(h^1-h^2-b_2^+b_2-d_{12}^+d_{12}\big) \\
& +b_{12}^+d_{12}^+\left[\frac{(h^2-h^1+2d_{12}^+d_{12}+b_2^+b_2+2)}{(2l+1)!}-\frac{b_2^+b_2}{(2l+2)!}\right]\Bigg\}b_2^{2(m+l+1)} \\
& +\frac{1}{2}\sum_{m=0}\sum_{l=0}\left(\frac{-2r}{m_2^2}\right)^{m+l+1}(4^m-1)\frac{(b_{22}^+)^{m+l}}{(2m+1)!(2l+1)!}\,\hat{t}'_{12}\,d_{12}^+b_2^{2(m+l+1)}.
\end{aligned}
\tag{A106}
$$

As a consequence, in the flat-space case $r=0$, the oscillator realization of the second-order algebra $\mathcal{A}'(Y(2),AdS_d)$ is reduced to a 2-parametric (with the parameters $m_l, l=1,2$) polynomial realization of the Lie algebra $\mathcal{A}(Y(2),R^{1,d-1})$, with the central extension m_0^2 given by (52), (55), and the following relations:

$$t'_{12} = \left(h^1 - h^2 - d^+_{12}d_{12}\right)d_{12} - \sum_l l b^+_{1l}b_{l2} - \frac{m_1}{m_2}b^+_1 b_2, \tag{A107}$$

$$t'^+_{12} = d^+_{12} - \sum_l (3-l)b^+_{l2}b_{1l} - \frac{m_2}{m_1}b^+_2 b_1, \qquad l'_0 = m_0^2, \tag{A108}$$

$$l'^+_l = m_l b^+_l, \qquad l'_l = \frac{m_0^2}{m_l}b_l - \frac{m_1}{l}b^+_1 b_{1l} - \frac{l m_2}{2}b^+_2 b_{l2},\ l = 1,2, \tag{A109}$$

$$\begin{aligned} l'_{11} = &(b^+_{11}b_{11} + b^+_{12}b_{12} + b^+_1 b_1 - d^+_{12}d_{12} + h^1)b_{11} + \frac{b^+_{22}}{4}b^2_{12} - \frac{m_0^2}{m_1^2}b_1^2 \\ &-\frac{1}{2}\left\{d^+_{12} - \frac{m_2}{m_1}b^+_2 b_1\right\}b_{12}, \end{aligned} \tag{A110}$$

$$\begin{aligned} l'_{12} = &\frac{1}{4}\Big(b^+_{12}b_{12} + \sum_k (2b^+_{kk}b_{kk} + b^+_k b_k + h^k)\Big)b_{12} + b^+_{12}b_{11}b_{22} - \frac{m_0^2}{2m_1 m_2}b_1 b_2 \\ &+\frac{1}{2}(h^2 - h^1 + d^+_{12}d_{12})b_{11}d_{12} - \frac{1}{2}\left\{d^+_{12} - \frac{m_2}{m_1}b^+_2 b_1\right\}b_{22} + \frac{m_1}{2m_2}b^+_1 b_2 b_{11}, \end{aligned} \tag{A111}$$

$$\begin{aligned} l'_{22} = &(b^+_{12}b_{12} + d^+_{12}d_{12} + b^+_2 b_2 + b^+_{22}b_{22} + h^2)b_{22} + \frac{b^+_{11}}{4}b^2_{12} \\ &+\frac{1}{2}(h^2 - h^1 + d^+_{12}d_{12})d_{12}b_{12} - \frac{m_0^2}{2m_2^2}b_2^2 + \frac{m_1}{2m_2}b^+_1 b_2 b_{12}. \end{aligned} \tag{A112}$$

Appendix E. Reduction to Initial Irreducible Relations

Here, we demonstrate that the equations of motion (2)–(5), or, equivalently, (8) for $\tilde{l}_0 = l_0 + m^2$ and (9), can be obtained from the action (102), after gauge-fixing and removing the auxiliary fields by using part of the equations of motion. Let us start by gauge fixing.

Appendix E.1. Gauge Fixing

Let us consider the field $|\chi^l\rangle$, for $l = 0,1,\dots,6$, at certain fixed values of spin (s_1, s_2). In this section, we omit the subscripts associated with the eigenvalues of the σ^i-operators (92). Then we extract the dependence of Q (91) on the zero-mode ghosts η_0 and $\mathcal{P}_0$,

$$Q = \eta_0\widetilde{L}_0 + \imath\sum_m \eta^+_m\eta^m\mathcal{P}_0 + \Delta Q \tag{A113}$$

$$\begin{aligned} \Delta Q = &\eta^+_i L_i + \sum_{l\le m}\eta^+_{lm}L_{lm} + \vartheta^+_{12}T_{12} - \vartheta^+_{12}\textstyle\sum_n (1+\delta_{1n})\eta^+_{1n}\mathcal{P}_{n2} \\ &+\vartheta^+_{12}\textstyle\sum_n (1+\delta_{n2})\eta_{n2}\mathcal{P}^+_{1n} + \frac{1}{2}\sum_n \eta^+_{n2}\eta_{1n}\lambda_{12} \\ &-\frac{1}{2}\sum_{l\le m}(1+\delta_{lm})\eta_m\eta^+_{lm}\mathcal{P}_l - \left[\vartheta_{12}\eta^+_2\mathcal{P}_1 + \vartheta^+_{12}\eta^+_1\mathcal{P}_2\right] \\ &+r\Big\{-\tfrac{1}{2}\textstyle\sum_{i,j}\eta^+_i\eta^+_j\varepsilon^{ij}\Big[\sum_m(-1)^m\mathcal{G}^m_0\mathcal{P}_{12} - (\mathcal{T}_{12}\mathcal{P}_{11} + \mathcal{L}_{11}\lambda_{12}) + \mathcal{T}^+_{12}\mathcal{P}_{22} + \mathcal{L}_{22}\lambda^+_{12}\Big] \\ &+2\eta^+_i\eta_j\Big[\textstyle\sum_m \mathcal{L}^+_{jm}\mathcal{P}_{im} - \frac{1}{8}(\mathcal{T}^+_{12}\lambda_{12} + \mathcal{T}_{12}\lambda^+_{12})\delta^{ij} + \frac{1}{4}\sum_m \mathcal{G}^m_0\lambda_{12}\delta^{j1}\delta^{i2}\Big]\Big\} + h.c., \end{aligned} \tag{A114}$$

where

$$\widetilde{L}_0 := \tfrac{1}{2}L_0 + r\sum_i \eta_i^+\Big[2\mathcal{L}_{ii}\mathcal{P}_i^+ + 2\mathcal{L}_i^+\mathcal{P}_{ii} + \mathcal{G}_0^i\mathcal{P}_i + 2\big(\mathcal{L}_{12}\mathcal{P}^+_{\{1} + \mathcal{L}^+_{\{1}\mathcal{P}_{12}\big)\delta_{2\}i} \qquad (A115)$$
$$-\delta^{1i}\big(\mathcal{L}_2\lambda_{12}^+ + \mathcal{T}_{12}^+\mathcal{P}_2\big) - \delta^{2i}\big(\mathcal{L}_1\lambda_{12} + \mathcal{T}_{12}\mathcal{P}_1\big)\Big]$$
$$+r^2\Big\{\sum_{i,j}\eta_i\eta_j\varepsilon^{ij}\Big[\tfrac{1}{2}\sum_m\Big(G_0^m\big[\lambda_{12}\mathcal{P}_{22}^+ - \lambda_{12}^+\mathcal{P}_{11}^+\big] + 4L_{mm}\mathcal{P}_{m2}^+\mathcal{P}_{1m}^+\Big) + 2L_{12}\mathcal{P}_{22}^+\mathcal{P}_{11}^+\Big]$$
$$+\sum_{i,j}\eta_i^+\eta_j\Big[2\sum_m\big(L_{22}^+\mathcal{P}_{22} - L_{11}\mathcal{P}_{11}^+)\big)\lambda_{12}\delta^{1j}\delta^{2i} - 2T_{12}\mathcal{P}_{11}\mathcal{P}_{22}^+\delta^{1i}\delta^{2j}$$
$$+2\varepsilon^{\{1j}\delta^{2\}i}\Big(\big(L_{12}\lambda_{12} - T_{12}\mathcal{P}_{12}\big)\mathcal{P}_{22}^+ + \big(T_{12}^+\mathcal{P}_{12} - L_{12}\lambda_{12}^+\big)\mathcal{P}_{11}^+\Big)$$
$$-2\sum_m(-1)^m\Big(G_0^m\mathcal{P}_{11}\delta^{1i}\delta^{2j} - G_0^m\mathcal{P}_{22}\delta^{2i}\delta^{1j}\Big)\mathcal{P}_{12}^+\Big]\Big\} + h.c.,$$

and do the same with the fields and gauge parameters:

$$|\chi^l\rangle = |S^l\rangle + \eta_0|B^l\rangle. \qquad (A116)$$

Then the equations of motion and gauge transformations (101) can be recast as follows:

$$\delta|S^{l-1}\rangle = \Delta Q|S^l\rangle - \sum_m \eta_m^+\eta^m|B^l\rangle, \qquad (A117)$$
$$\delta|B^{l-1}\rangle = \widetilde{L}_0|S^l\rangle - \Delta Q|B^l\rangle\,, \qquad \delta|\chi^{-1}\rangle \equiv 0, \qquad (A118)$$

for $l = 0, \ldots, 6$.

As a next step, we examine the lowest-level gauge transformation

$$\delta|S^5\rangle = \Delta Q|S^6\rangle, \qquad \delta|B^5\rangle = \widetilde{L}_0|S^6\rangle, \qquad (A119)$$

where, due to the ghost number condition, we took into account $|B^6\rangle \equiv 0$. Extracting the dependence of the gauge parameters, and that of the operator ΔQ (A114), on the ghost coordinate and momentum η_{11}, $\mathcal{P}_{11}^+$,

$$|\chi^l\rangle = |\chi_0^l\rangle + P_{11}^+|\chi_1^l\rangle, \qquad \Delta Q = \Delta Q_{11} + \eta_{11}T_{11}^+ + U_{11}\mathcal{P}_{11}^+, \qquad (A120)$$

where the quantities $|\chi_0^l\rangle$, $|\chi_1^l\rangle$, T_{11}^+, U_{11}, ΔQ_{11} do not depend on η_{11}, $\mathcal{P}_{11}^+$, we obtain the gauge transformation of $|S_0^5\rangle$, with the same decomposition for $|S^5\rangle$, $|S^5\rangle = |S_0^5\rangle + \mathcal{P}_{11}^+|S_1^5\rangle$ as the one for $|\chi^l\rangle$ (A120),

$$\delta|S_0^5\rangle = T_{11}^+|S_1^6\rangle. \qquad (A121)$$

Here, we used the fact that $|S_0^6\rangle \equiv 0$, due to the ghost number condition. Since $T_{11}^+ = L_{11}^+ + O(\mathcal{C}) = b_{11}^+ + \ldots$, as implied by the structure of ΔQ in (A114), we can remove the dependence of $|S_0^5\rangle$ on the operator b_{11}^+, using all of the degrees of freedom in $|S_1^6\rangle$. Therefore, after gauge fixing at the lowest level of gauge transformations, we find the following conditions for $|S_0^5\rangle$:

$$b_{11}|S_0^5\rangle = 0 \iff b_{11}\mathcal{P}_{11}^+|\chi^5\rangle = 0, \qquad (A122)$$

so that the theory becomes a fourth-stage reducible gauge theory.

Let us turn to the next level of gauge transformations. Extracting the explicit dependence of the gauge parameters and ΔQ on η_{11}, $\mathcal{P}_{11}^+$, η_{12}, $\mathcal{P}_{12}^+$, and using similar arguments as those at the previous level of gauge transformations, we find that $|\chi^4\rangle$ can be subject to the gauge

$$b_{11}\mathcal{P}_{11}^+|\chi^4\rangle = 0, \qquad b_{12}\mathcal{P}_{11}^+\mathcal{P}_{12}^+|\chi^4\rangle = 0. \qquad (A123)$$

To obtain these gauge conditions, one has to use all the degrees of freedom in the gauge parameters $|\chi^5\rangle$, restricted by (A122).

Proceeding with the above, step by step, first of all, for $l = 3$, we obtain

$$b_{11}\mathcal{P}^+_{11}|\chi^l\rangle = 0, \quad b_{12}\mathcal{P}^+_{11}\mathcal{P}^+_{12}|\chi^l\rangle = 0, \quad b_{22}\mathcal{P}^+_{11}\mathcal{P}^+_{12}\mathcal{P}^+_{22}|\chi^l\rangle = 0. \tag{A124}$$

As we define the following set of operators used in (A122)–(A124),

$$[\mathcal{A}^l] = \Big(b_{11}\mathcal{P}^+_{11}, b_{12}\prod_i^2 \mathcal{P}^+_{1i}, b_{22}\mathcal{P}^+_{22}\prod_i^2 \mathcal{P}^+_{1i}\Big), \quad l = 1, 2, 3 \tag{A125}$$

(where, for instance, the second component of the set $[\mathcal{A}^l]$ is equal to $\mathcal{A}^2 = b_{12}\prod_i^2 \mathcal{P}^+_{1i}$), we can equivalently represent the gauge conditions (A122)–(A124), and all of the subsequent ones, which are based on decomposing the gauge parameters in all of the ghost momenta $P^+_{ij}, i \le j$, namely,

$$[\mathcal{A}^l]|\chi^{6-l}\rangle = 0, \quad \text{for} \quad l = 1, 2, 3. \tag{A126}$$

Next, we apply the same procedure as above, albeit starting from the gauge parameter $|\chi^2\rangle$, and carry out extraction from this parameter and from the operator ΔQ (A114) of the ghost coordinates and momenta η_{ij}, $\mathcal{P}^+_{ij}$, $i \le j$, as well as η_1, $\mathcal{P}^+_1$. As a result, we obtain a set of gauge conditions for the parameter $|\chi^2\rangle$:

$$\Big([\mathcal{A}^3], b_1\mathcal{P}^+_1 \prod_{i,j=1, i\le j}^2 \mathcal{P}^+_{ij}\Big)|\chi^2\rangle = 0. \tag{A127}$$

In the process of further extraction, starting with the ghosts η_1, η_2, $\mathcal{P}^+_1$, $\mathcal{P}^+_2$, we obtain two sets of gauge conditions for the parameters $|\chi^{3-m}\rangle$, $m = 1, 2$,

$$\Big([\mathcal{A}^3], b_1\mathcal{P}^+_1 \prod_{i,j=1, i\le j}^2 \mathcal{P}^+_{ij}, b_2\prod_m^2 \mathcal{P}^+_m \prod_{i,j=1, i\le j}^2 \mathcal{P}^+_{ij}\Big)|\chi^1\rangle = 0. \tag{A128}$$

Finally, implementing the same algorithm as the one above, albeit initiated by the field $|\chi^0\rangle$, and extracting from this field, and also from the operator ΔQ (A114), the ghost coordinates and momenta η_m, $\mathcal{P}^+_m$, η_{ij}, $\mathcal{P}^+_{ij}$, $i \le j$, ϑ_{12}, λ^+_{12}, we deduce a set of gauge conditions for the field $|\chi^0\rangle$:

$$\Big([\mathcal{A}^3], b_1\mathcal{P}^+_1 \prod_{i,j=1, i\le j}^2 \mathcal{P}^+_{ij}, b_2\prod_m^2 \mathcal{P}^+_m \prod_{i,j=1, i\le j}^2 \mathcal{P}^+_{ij}, d_{12}\lambda^+_{12}\prod_m^2 \mathcal{P}^+_m \prod_{i,j=1, i\le j}^2 \mathcal{P}^+_{ij}\Big)|\chi^0\rangle = 0. \tag{A129}$$

Let us now turn to removing the auxiliary fields by using the equations of motion.

Appendix E.2. Removal of Auxiliary Fields by Resolving Equations of Motion

As a first step, we decompose the field $|S^0\rangle$ in the form

$$\begin{aligned} |S^0\rangle &= |S^0_0\rangle + \mathcal{P}^+_{11}|S^0_1\rangle, & |S^0_{(0)_3}\rangle &= |S^0_{(0)_30}\rangle + \mathcal{P}^+_1|S^0_{(0)_31}\rangle, \\ |S^0_0\rangle &= |S^0_{00}\rangle + \mathcal{P}^+_{12}|S^0_{01}\rangle, & |S^0_{(0)_30}\rangle &= |S^0_{(0)_5}\rangle + \mathcal{P}^+_2|S^0_{(0)_41}\rangle, \\ |S^0_{(0)_2}\rangle &= |S^0_{(0)_20}\rangle + \mathcal{P}^+_{22}|S^0_{(0)_21}\rangle, & |S^0_{(0)_5}\rangle &= |S^0_{(0)_50}\rangle + \lambda^+_{12}|S^0_{(0)_51}\rangle, \end{aligned} \tag{A130}$$

and do the same with the vector $|B^0\rangle$,

$$|B^0\rangle = \mathcal{P}^+_{11}|B^0_1\rangle + \mathcal{P}^+_{12}|B^0_{01}\rangle + \mathcal{P}^+_{22}|B^0_{(0)_21}\rangle + \mathcal{P}^+_1|B^0_{(0)_31}\rangle + \mathcal{P}^+_2|B^0_{(0)_41}\rangle + \lambda^+_{12}|B^0_{(0)_51}\rangle,$$

where the term independent of the ghost momenta is absent due to the spin value and the ghost number condition $gh(|B^0\rangle) = -1$. Notice that, due to the spin value and the fact that $gh(|S^0\rangle) = 0$, the vector $|S^0_{(0)_6}\rangle$ does not depend on the ghost coordinates and momenta, as a consequence of the gauge conditions (A129), $|S^0_{(0)_6}\rangle = |\Phi\rangle$, with $|\Phi\rangle$ being the physical field (7) for $k = 2$.

Next, similar to the fields, we extract from ΔQ (A114) the dependence on η_{11}, $\mathcal{P}^+_{11}$, η_{12}, $\mathcal{P}^+_{12}$, η_{22}, $\mathcal{P}^+_{22}$, and then the dependence on η_l, $\mathcal{P}^+_l$, $l = 1, 2$, as well as on ϑ_{12}, λ^+_{12}, respectively.

Substituting these 6 decompositions into the equation of motion

$$\tilde{L}_0|S^0\rangle - \Delta Q|B^0\rangle = 0, \quad \text{(A131)}$$

and using the gauge conditions (A129), we can show that $|B^0_{(0)_51}\rangle = 0$, and then $|B^0_{(0)_41}\rangle = 0$, etc., until $|B^0_1\rangle = 0$, which implies

$$\tilde{L}_0|S^0\rangle = 0, \qquad |B^0\rangle = 0. \quad \text{(A132)}$$

In a similar way, we consider the second equation of motion

$$\Delta Q|S^0\rangle = 0, \quad \text{(A133)}$$

where $|B^0\rangle = 0$ is used. After the same decomposition, we deduce, step by step, the facts that

$$|S^0_{(0)_51}\rangle = |S^0_{(0)_41}\rangle = \ldots = |S^0_{01}\rangle = |S^0_1\rangle = 0. \quad \text{(A134)}$$

The relations (A132) and (A134) imply that all of the auxiliary fields are zero, and then, as a result, we have $|\chi^0\rangle_{(s)_k} = |\Phi\rangle$. The operator $\widetilde{L}_0$ reduces to L_0, with no dependence on the auxiliary oscillators when acting on $|\Phi\rangle$ so that the equations of motion (2) and (3)–(5) hold true, with allowance for the mass term m_0^2 (104). Therefore, we proved that the space of cohomologies for the BRST operator Q (91) with a vanishing ghost number is determined only by the constraints related to an irreducible AdS group representation.

Notes

1 For recent developments in the ambient flat space-time techniques of the incomplete BRST approach to totally symmetric tensor fields in AdS spaces, see [43].

2 For the AKSZ model used in higher-spin gravity, see, for example, [51].

3 This choice of oscillators corresponds to the case of a symmetric basis, whereas there also exists another realization of an auxiliary Fock space, generated by fermionic oscillators (antisymmetric basis) $\hat{a}^m_{\mu^m}(x)$, $\hat{a}^{\hat{n}+}_{\nu^n}(x)$, with the anticommutation relations $\{\hat{a}^m_{\mu^m}, \hat{a}^{n+}_{\nu^n}\} = -g_{\mu^m\nu^m}\delta^{mn}$, for $m, n = 1, \ldots, s_1$. The treatment below proceeds along the lines of [62] for totally antisymmetric tensors with $s_1 = s_2 = \ldots = s_k = 1$.

4 The operators $a^{a_i}_i, a^{b_j+}_j$ satisfy the usual (in the space $R^{1,d-1}$) commutation relations $[a^{a_i}_i, a^{b_j+}_j] = -\eta^{a_ib_i}\delta_{ij}$ for $\eta^{ab} = \mathrm{diag}(+, -, \ldots, -)$.

5 The term *higher-spin symmetry algebra*, applied here to a free HS formulation, is not to be confused with the algebraic structure known as a *higher-spin algebra* which arises in describing HS interactions; see, for example, [15].

6 There is no summation with respect to the indices i_2, j_2 in (16), and the figure brackets for the indices i_1, i_2 in the quantity $A^{\{i_1}B^{i_2\}i_3}\theta^{i_3i_2\}}$ imply symmetrization: $A^{\{i_1}B^{i_2\}i_3}\theta^{i_3i_2\}} = A^{i_1}B^{i_2i_3}\theta^{i_3i_2} + A^{i_2}B^{i_1i_3}\theta^{i_3i_1}$. These indices are raised and lowered using the Euclidean metric tensors δ^{ij}, δ_{ij}, δ^i_j.

7 This superalgebra can be equally used for both bosonic HS fields in an antisymmetric basis and fermionic HS fields in an AdS space.

8 One can choose $sp(2k)$ in a Cartan–Weyl basis for a unified description; however, without loss of generality, the basis elements and structure constants of the algebra under consideration are chosen as in Table 1.

9 Here, the symbols $\mathcal{U}(g), \mathcal{U}(b), \mathcal{U}(g^-)$ denote the universal enveloping algebras, respectively, for g, for the Borel subalgebra, and for the lower-triangular subalgebra g^-, such as $\mathcal{E}^-_k$ in (41).

10 For the first time [63], the treatment of quadratic "primed" quantities has been presented in the case of $Y(s_1.s_2)$ so that the following properties hold true: $\left(X'^{ii}_b\right)^+ = X'^{ii}_b$, $(X'^{12}_b)^+ = X'^{21}_b$.

11 In the case of a quadratic superalgebra $\mathcal{A}'^f(Y(1), AdS_d)$, elaborated for a totally symmetric fermionic HS field oscillator realization, this is established in [22,23].

12 The momenta satisfy the independent non-vanishing anticommutation relations $\{\vartheta_{12}, \lambda^+_{12}\} = 1, \{\eta_i, \mathcal{P}^+_j\} = \delta_{ij}$, $\{\eta_{lm}, \mathcal{P}^+_{ij}\} = \delta_{li}\delta_{jm}$, $\{\eta_0, \mathcal{P}_0\} = \imath, \{\eta^i_G, \mathcal{P}^j_G\} = \imath\delta^{ij}$, as well as possessing the standard ghost number distribution $gh(\mathcal{C}^I) = -gh(\mathcal{P}_I) = 1$, providing the property $gh(Q') = 1$, and have the Hermitian conjugation properties of zero-mode pairs, $\left(\eta_0, \eta^i_G, \mathcal{P}_0, \mathcal{P}^i_G\right)^+ = \left(\eta_0, \eta^i_G, -\mathcal{P}_0, -\mathcal{P}^i_G\right)$.

13 Here, in the coefficients depending on o'_I, O_I, we use the convention for $\mathcal{O}_I$ adopted in (83)–(85).

14 The states (95) in the case of spin-$(1,1)$ have the lowest ghost number -2 due to the restriction (99), (100) at $n_1 = n_2 = 1$.

15 The algebraic relations (31) for the algebra $\mathcal{A}_c$ are different from those for the polynomial algebra in view of a non-homogeneous character of the structure functions $F^{(m)K}_{IJ}(o', O)$ in O_I, due to the presence of the elements o'_I.

16 In the case $n = 1$, the corresponding algebra must be of zero order because of the relation (A6), which has the form $[q_+, q_-] = -1$; however, in this case, we can add the unity 1 to the set of o_I, as was done with the Heisenberg algebra A_1, and then remove the element q_0 from o_I, due to the possibility of the representation $q_0 = q_-q_+$.

17 Explicitly, $W'^{12+}_b = 2r[l'^+_{12}(g'^2_0 - g'^1_0) - l'^+_{11}t'^+_{12} + l'^+_{22}t'_{12}]$, which follows from W'^{12}_b by Hermitian conjugation.

18 In (A27), (A28) and later on, for the sake of convenience, we use the following notation for an arbitrary vector: $\big|\vec{n}_{ij} + m\delta_{ij,11} - 2\delta_{ij,12}, \vec{n}_s - 2m\delta_{s,1} + \delta_{s,12}\rangle_V \equiv \big|\vec{n}_{ij} + (m, -2, 0), \vec{n}_s - (2m, -1, 0)\rangle_V \equiv \big|\vec{N}(2) + (m, -2, 0, -2m, +1, 0)\rangle_V$.

19 In (A92) for $k = 0$, there are no double sums. The products $\prod^0_{i=1}\ldots$ are equal to 1, and the terms inside the internal brackets, $\big(b^+_2 d_{12} b_2 - \frac{m_1}{m_2}\sum_{m=0}\left(\frac{-2r}{m^2_2}\right)^m \frac{(b^+_{22})^m}{(2m+1)!} b^+_1 b^{2m+1}_2 + \sum_{m=1}\ldots\big)$, are the only ones to survive.

20 These operators are immediately determined by their action on the vector $|\vec{0}_{ij}, 0, p_{12}, n_2\rangle_V$ in (A47)–(A51).

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