DOUBLY PERIODIC MONOPOLE DYNAMICS AND CRYSTAL VOLUMES

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As members of the Dissertation Committee, we certify that we have read the dissertation prepared by ***Rebekah Cross***, titled ***Doubly Periodic Monopole Dynamics and Crystal Volumes*** and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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DEDICATION

To my parents Patty and Joe Cross.

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ABSTRACT

We determine the large-modulus dynamics of the U(N) doubly periodic BPS monopole in Yang-Mills-Higgs theory, called a monopole wall. We accomplish this by exploring its Higgs curve using the Newton polytope and amoeba. In particular, we prove that the monopole wall splits into subwalls when any of its moduli become large. The long-distance gauge and Higgs field interactions of these subwalls are abelian, allowing us to derive an asymptotic metric for the monopole wall moduli space via electromagnetic scattering. We carry out a generalized Legendre transform to determine complex coordinates and Kähler potential for the asymptotic metric. We prove that the Kähler potential is determined by the cut volume of a crystal associated with the Higgs curve, i.e. the volume of a region enclosed by a plane arrangement in $\mathbb{R}^3$.

CHAPTER 1

Introduction

1.1 Context

In 1931 [1], Dirac proposed a magnetic cousin to the electron in classical electromagnetism, now referred to as the Dirac magnetic monopole. Analogous to the classical electron, it is a point particle with radial magnetic field that is singular. Nearly five decades later, 't Hooft and Polyakov [2, 3] expanded the idea of the magnetic monopole by identifying non-singular solutions now called 't Hooft-Polyakov monopoles in nonabelian Yang-Mills-Higgs theory, in which the Yang-Mills gauge fields couple to a scalar field with the usual "winebottle" Higgs potential. Prasad and Sommerfield [4] found an explicit static $SU(2)$ solution for this theory in the limit that the coefficient of the Higgs potential vanishes. Under conditions of time-independence and the vanishing of the Higgs potential coefficient, Bogmolny [5] derived his eponymous equation. Solutions to the Bogomolny equation solve the Yang-Mills-Higgs field equation and minimize energy. They are called BPS (Bogomolny-Prasad-Sommerfield) monopoles.

Nonabelian magnetic monopoles are interesting in their own right, appearing as they do in many contestant grand unified field theories. They have garnered attention in recent decades, however, for their significance in relation to certain supersymmetric Yang-Mills quantum field theories and supersymmetric quantum chromodynamics. One nontrivial connection to these theories is via their moduli spaces of vacua. The moduli space of BPS Yang-Mills-Higgs monopoles (a set of solutions that share fixed boundary conditions up to gauge equivalence and which together form a manifold) is isometric to the Coulomb branch of the moduli space of vacua in the associated super Yang-Mills theory [6, 7, 8]. These moduli spaces are hyperkähler, i.e. they are kähler manifolds which are holomorphically symplectic.

In early studies of BPS monopoles, their moduli spaces were used to determine monopole dynamics. Manton established [9] that the low-energy dynamics for BPS monopoles can be approximated as geodesic motion on their moduli space. In the modern context, monopole moduli spaces have applications in quantum theories. Despite their importance, few metrics on monopole moduli spaces are known. BPS solutions in which some or all of the constituent monopoles are closely spaced represent regions in the interior of the moduli space. BPS solutions in which the monopoles are very widely-spaced are points on the moduli space in its *asymptotic region*. Long-range abelian approximations have been used to obtain the latter type of solution and metrics have been calculated for the corresponding asymptotic moduli spaces, but solutions of the former type have been mostly illusive. A notable recent contribution to solutions for monopole Higgs and gauge fields has come from Braden and Enolski [10], [11] and their prescription for analytical monopole solutions computed directly from the Hitchin spectral curve. Nevertheless, most moduli space metrics that have been produced are accurate only for the asymptotic portion of the moduli space. The following paragraph enumerates these efforts.

Atiyah and Hitchin [12] derived a metric on the *full moduli space* for two $SU(2)$ BPS monopoles on $\mathbb{R}^3$. Gibbons and Manton [13] then extended this to n BPS, *well-separated*, unit charge $SU(2)$ monopoles and found the asymptotic moduli space metric. Lee, Weinberg, and Yi derived a similar asymptotic metric for general gauge group [14]. Cherkis and Kapustin [15] used an approach echoing Gibbons and Manton's to determine the asymptotic moduli space metric for an $SU(2)$ monopole on $\mathbb{R}^2 \times S^1$ with n monopole constituents with distinct charges, as did Hamanaka, Kanno, and Muranaka [16] for an $SU(2)$ monopole on $\mathbb{R} \times T^2$ with n monopole constituents with distinct charges.

Prior to discussions of moduli spaces for widely separated doubly periodic monopoles, there is a twenty year history of exploration of doubly periodic instantons and their dimensional reduction, doubly periodic monopoles. In the case of the singly-periodic monopole, as in [17], the Nahm transform maps the monopole onto a solution of the Nahm equations [18], formulating the problem of interacting monopoles as a Nahm system and validating the abelian approximation in the asymptotic regime [19]. This approach is unsuccessful in the case of the doubly-periodic monopole, which is mapped to another doubly-periodic monopole under the Nahm transform. Instead it is sensible to study some key behaviors of the doubly-periodic monopole using the Higgs spectral curve [20, 15, 21, 22], which allows a geometrical treatment of the monopole interactions in the BPS limit. Still, the Nahm transform has been useful in the study of doubly periodic instantons and their dimensional reductions (monopoles and Hitchin systems).

In the late 1990s and early 2000s Jardim showed that one can carry out the Nahm transform to obtain finite energy doubly periodic instantons from certain singular solutions to Hitchin's equations on a two-torus and that the transform is invertible. He associated a spectral curve with the doubly periodic monopole through the Hitchin's solutions [23, 24, 25]. He and Biquard described the moduli space for SU(2) widely separated constituent instantons on a doubly periodic underlying space. They computed its dimension to be 8k-4 for instanton number k, and classified the moduli spaces based on asymptotic conditions [26, 27]. Later, Ford and Pawlowski [28, 29, 30] used analytical and numerical methods to explore the action density for doubly periodic $SU(2)$ instanton configurations with radial symmetry where "radial" refers to distance in the two non-periodic directions on the underlying space. Between 2013 and 2019, Mochizuki [31, 32, 33] has extended this work and used the Nahm transform specifically to relate doubly periodic instantons with square-integrable field strength to solutions for Hitchin's equations on a two-torus with higher order singularities, i.e. where the Higgs field has singularities of order $n > 1$. He also examined the dimensional reduction of these instantons to doubly periodic monopoles and their associated Hitchin curve.

In 2005, Lee [34] introduced the concept of doubly periodic monopoles, called "monopole sheets," as infinite square lattices of 't Hooft-Polyakov type monopoles solving the Bogomolny equation and connected these to D3 brane configurations in type IIB string theory. A couple of years later, Ward [35] published a set of numerically generated field solutions for a unit lattice of $SU(2)$ 't Hooft-Polyakov monopoles. In 2011 Ward [36] produced field solutions and energy density plots for $SU(2)$ monowalls with distinct left and right magnetic charges. Soon after this Cherkis and Ward invoked the Higgs spectral curve (introduced previously in the context of periodic monopoles [15]) to establish the monowall moduli and find the monowall moduli space dimensions. They related the spectral data for the Higgs spectral curve to the boundary conditions on the Higgs field and gauge phases which specify the monowall moduli space. They used the Newton polygon and real Log projection of the Higgs spectral curve called the *amoeba* to illustrate these conditions. Hamanaka et al. soon after publish their result for doubly periodic monopoles with distinct charges [16] which were mentioned above. From this point forward, the Higgs spectral curve becomes a standard tool used to analyze monowall behavior. In 2014, Maldonado and Ward [37] published a set of numerically generated field solutions for a pair of doubly periodic $SU(2)$ 't Hooft-Polyakov monopoles with several relative charge examples and they used these results to compute the coefficients of the metric on the moduli space of these monowalls. In [38] Cherkis explicitly associated phases of the monowall's amoeba with triangulations of the Newton polygon and explored the set of amoeba phases using the secondary polytope and fan associate with the Newton polygon. He identified all monowalls with four moduli and discussed the asymptotics of their moduli spaces. In [39] we used the Higgs spectral curve to argue for monowall breaking when monowall moduli grow large and then generalized the results from Hamanaka et al. to a metric on the asymptotic moduli space of a $U(N)$ monowall with n widely separated subwalls with distinct charges.

As mentioned earlier in this section, BPS monopoles arise in classical Yang-Mills-Higgs theory. Their moduli spaces are argued to be isometric to moduli spaces of vacua for $SU(n)$ super Yang-Mills quantum gauge theories (with boundary conditions and dimension particular to each of the monopole periodicity cases). Seiberg and Witten originally discovered the existence of these relationships in [6], following work by Seiberg and Witten [40, 41], and Intriligator and Seiberg [42, 43]. Chalmers, Hanany, and Witten [7, 8] explained these relationships using brane dualities. Later Haghighat and Vandoren [44] examined the compacitified five dimensional quantum field theory relevant to doubly periodic BPS monopoles, and the underlying theory connecting them.

For n monopoles on $\mathbb{R}^3$, this theory is related via the relative moduli space metric to the Coulomb branch of $N = 4$ $SU(n)$ quantum super Yang-Mills vacua in three dimensions [6]. For two such monopoles, the relative metric is called the Atiyah-Hitchin metric. n periodic monopoles (on $\mathbb{R}^2 \times S^1$, called "monopole chains") are related via their moduli space metric to the Coulomb branch of vacua for $N = 2$ $SU(n)$ quantum super Yang-Mills in four dimensions which has been compactified on a circle [17]. Similarly, n doubly-periodic monopoles (on $\mathbb{R} \times T^2$, called "monopole walls" or "monowalls") are related via their moduli space metric to the Coulomb branch of vacua for $N = 1$ $SU(n)$ quantum super Yang-Mills in five dimensions which has been compactified on a two-torus [22, 16, 44]. These moduli spaces of monowalls are the main subject of this thesis.

The next step in the discussion of monowall moduli spaces is to identify complex coordinates on them and explore their Kähler potentials, which are local functions of these complex coordinates or moduli and whose second derivatives give the metric coefficients. In the following work we use the generalized Legendre transform to find complex coordinates and asymptotic Kähler potentials for the asymptotic moduli spaces of monowalls. In 1987 Hitchin, Karlhede, Lindström and Roček [45, 46] introduced the generalized Legendre transform as a tool for generating hyperkähler metrics and, relevantly, for metrics of n widely-separated monopoles. Ivanov and Roček [47] further illustrated the utility of the transform for monopole metrics by reproducing the Atiyah-Hitchin moduli space metric for two $SU(2)$ monopoles. In 1997 Chalmers [48] carried out the generalized Legendre transform to derive moduli space metric for widely separated monopoles on $\mathbb{R}^3$ and generalized from $SU(2)$ to $SU(N)$ gauge symmetry. Later Houghton [49] connected the generalized Legendre transform to monopole twistor theory. He summarized the role of the generating function and generating integrand and how to compute them for a known hyperkähler metric. This dissertation compiles work which continues and elaborates on the efforts listed for monowalls: Here we 1) demonstrate that monowalls break into subwalls for large moduli and the subwall interactions become approximately abelian. 2) We compute metrics and Kähler potentials for the asymptotic moduli spaces of subwalls and 3) relate these Kähler potentials crystal volumes.

1.2 Yang-Mills-Higgs theory

In classical, 3+1-dimensional $U(N)$ Yang-Mills-Higgs theory the pure Yang-Mills action is augmented by that of a scalar with the usual symmetry-breaking potential.

$$S = \int d^4x \mathrm{Tr} \left[\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - \left(D_\mu \Phi\right) \left(D^\mu \Phi\right) - \lambda \left(\Phi^2 + v^2\right)^2 \right]. \tag{1.1}$$

We shall have both the gauge and Higgs fields antihermitian in the adjoint representation. They can be expressed as linear combinations of the antihermitian $U(N)$ generators T_b: $\Phi = \Phi^b T_b$, $A_\mu = A^b_\mu T_b$ where Φ^b and A^b_μ are real functions, $b = 1, ..., N^2$ indexes the $U(N)$ generators, and v is a real constant. The gauge covariant derivative is $D_\mu \Phi = \partial_\mu \Phi + [A_\mu, \Phi]$ and the field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$.

The action-extremizing Yang-Mills-Higgs field equations are easily derived,

$$D^2\Phi = \lambda\,(\Phi^2 + v^2)\,\Phi, \qquad\qquad D_\mu F^{\mu\nu} = [\Phi, D^\nu\Phi], \tag{1.2}$$

but we can more strongly constrain the solutions by requiring time-independence ($\partial_0 = 0$) and taking the limit $\lambda \to 0$. Under these conditions the energy is minimized when the following equation, called the Bogomolny equation, is satisfied:

$$B_i = \pm D_i\Phi, \tag{1.3}$$

where the magnetic field is found from the field strength: $B_i = -\frac{1}{2}\varepsilon_{ijk}F^{jk}$ and $i = x, \varphi, \theta$. Alternately, in the form of a complex and a real equation (we have chosen the $(-)$ case in (1.3) and will use it from now on),

$$\begin{aligned} &[D_x + iD_\theta, D_\varphi + i\Phi] = 0, \\ &[D_x + iD_\theta, (D_x + iD_\theta)^\dagger] + [D_\varphi + i\Phi, (D_\varphi + i\Phi)^\dagger] = 0. \end{aligned} \tag{1.4}$$

These conditions are collectively known as the *BPS limit* and solutions to the Bogomolny equation are static BPS magnetic monopoles [50]. Static magnetic charge configurations are possible in this limit because the magnetic field repulsion is canceled by the Higgs field attraction the fields satisfy the Bogomolny equation (1.4).

Now, a BPS solution is a static solution, i.e the Higgs and gauge field configurations are time-independent. For fixed total charge and a given set of gauge and Higgs field boundary conditions, there may be many such static solutions. A monopole (or monowall) moduli space is the set of BPS solutions for fixed total monowall charge and boundary conditions. If boundary conditions are chosen appropriately it is a manifold. Each point on the manifold represents a BPS solution, up to gauge transformations, with associated charge distribution. To identify nearby locations on the moduli space, we perturb a BPS solution $(\Phi, A_i) \rightarrow (\Phi + \delta\Phi, A_i + \delta A_i)$ and require that it still satisfy the Bogomolny equation. The linearized form of the Bogomolny equation is

$$\frac{1}{2}\varepsilon^{ijk}\left[D_j(\delta A_i) - D_k(\delta A_j)\right] = -\left(D^i(\delta\Phi) + [\delta A^i, \Phi]\right). \tag{1.5}$$

But the moduli space does not contain gauge redundancy and so these perturbations must also be orthogonal to small gauge transformations e^{Λ} which affect the fields $\Phi \rightarrow e^{\Lambda}\Phi e^{-\Lambda}$ and $A_\mu \rightarrow e^{\Lambda}(A_\mu + \partial_\mu)e^{-\Lambda}$. Expanding in Λ we can write such gauge shifts as $(\Phi, A_i) \rightarrow (\Phi + \delta_\Lambda\Phi, A_i + \delta_\Lambda A_i)$ where the small gauge effects on the field are $(\delta_\Lambda\Phi, \delta_\Lambda A_i) = ([\Lambda, \Phi], -[D_i, \Lambda])$. If our perturbed BPS fields are to lie on the moduli space adjacent to the original solutions then the perturbations $(\delta\Phi, \delta A_i)$ are *orthogonal to the small gauge shifts* $(\delta_\Lambda\Phi, \delta_\Lambda A_i)$, i.e. they must satisfy

$$\int d^3x \mathrm{Tr}\left[-(\delta A^i)D_i\Lambda + (\delta\Phi)[\Lambda, \Phi]\right] = 0 \tag{1.6}$$

for all small gauge transformations Λ. Integrating by parts reveals that this is equivalent to the following equation, which fixes the perturbations to be orthogonal to gauge slices:

$$D^i(\delta A_i) - [\delta\Phi, \Phi] = 0. \tag{1.7}$$

Perturbations of BPS solutions which satisfy equations (1.5) and (1.7) will be tangent to the moduli space and the metric on the moduli space can be expressed as the overlap of these. We establish a set of generalized complex coordinates on the moduli space indexed by p: $(\chi_p, \bar{\chi}_p)$. The moduli perturbations which satisfy these conditions must be accompanied by a small gauge transformation Ω to keep the moduli perturbations orthogonal to the gauge orbit. These perturbations take the form

$$\begin{aligned} \delta_p(A_1 + iA_3) &= \partial_p(A_1 + iA_3) - [(D_1 + iD_3), \Omega_p], \\ \delta_p(A_2 + i\Phi) &= \partial_p(A_2 + i\Phi) - [D_2 + i\Phi, \Omega_p], \end{aligned} \tag{1.8}$$

where $\partial_p = \frac{\partial}{\partial \chi_p}$. The metric on the moduli space of monopoles which describes the space of BPS solutions (Φ, A_i) which share a set of fixed boundary conditions is[1]

$$g_{p\bar{q}} = \int d^3x \mathrm{Tr}\big[\delta_p(A_1+iA_3)\delta_{\bar{q}}(A_1-iA_3)+\delta_p(A_2+i\Phi)\delta_{\bar{q}}(A_2-i\Phi)\big]+\mathrm{h.c.} \tag{1.9}$$

Moduli spaces for BPS monopoles are hyperkähler [7] and therefore once a complex structure is chosen for a moduli space its metrics can be related to a Kähler potential K[2]. A Kähler potential is a local scalar function of complex moduli whose second derivatives give the set of metric coefficients.

$$g_{p\bar{q}} = \partial_p \partial_{\bar{q}} K. \tag{1.10}$$

The metric on the moduli space is important for describing monopole dynamics. If the positions of localized charge density gain very small velocities, this motion can be approximated by geodesic motion on this moduli space [52, 51]. We allow the moduli to be time-dependent $\chi^p(t)$ and the fields to possess time dependence only through the moduli. The time component of the gauge field is no longer zero in this case and must satisfy the component of the Yang-Mills-Higgs field equations which is linear in time and is analogous to Gauss' law $D_i F^{i0} + [D^0\Phi, \Phi] = 0$. This is satisfied by setting it equal to the same

[1]This description of the moduli space of BPS monopoles follows [51].

[2]Each complex structure has its own Kähler potential.

small gauge perturbation seen above: $A_0 = \Omega_p \dot{\chi}^p$. We can then related the time derivatives of the fields to the moduli perturbations $(\delta_p \Phi, \delta_p A_i)$:

$$F_{0i} = (\delta_p A_i)\dot{\chi}^p, \qquad D_0\Phi = (\delta_p \Phi)\dot{\chi}^p. \tag{1.11}$$

The remaining fields are not affected by small time dependence up to first order in t. This implies that for small time-dependent perturbations around BPS solutions the kinetic components of the Lagrangian give precisely the metric on the moduli space:

$$\begin{aligned} L &= \int d^3x \, \mathrm{Tr}\left[F_{0i}F^{0i} + (D_0\Phi)(D^0\Phi) - F_{ij}F^{ij} - (D_i\Phi)(D^i\Phi)\right] \\ &= g_{p\bar{q}} \, \dot{\chi}^p \dot{\bar{\chi}}^q + \text{constants}. \end{aligned} \tag{1.12}$$

The monopoles are in motion now and just as a moving electric charge produces a magnetic field, so does a moving magnetic charge produce an electric field perpendicular to the direction of motion. An additional effect comes with this small time-dependence, however: these magnetic charges gain electric charge and so altogether may interact magnetically, electrically, and via the Higgs field. This effect is controlled by a periodic phase modulus τ associated with each charge [9].

This effect can be seen by allowing the gauge field component A_0 to include a term proportional to the Higgs field, which satisfies Gauss' law. We expect A_0 to be linear in time derivatives so we interpret the coefficient of this term as a small velocity along the phase direction τ: $A_0 \to A_0 + \dot{\tau}\Phi$. This term produces the following effects on the electromagnetic fields:

$$\delta_\tau F_{0i} = \dot{\tau} B_i, \tag{1.13}$$

which is parallel to the magnetic field, suggesting that the monopole is now a source of electric field as well as magnetic field.

This phenomenon can be interpreted as resulting from a type of gauge transformation. While we identify BPS solutions which are related by small gauge transformations, we can introduce this phase modulus τ explicitly to the fields through a *large* gauge transformation. A large gauge transformation is one which does not become the identity as $r \to \infty$. This function will have the form $e^{\tau\Phi}$ where the monopole's position $\tau(t)$ along this phase modulus is linearly time dependent with a very small velocity. This creates a small contribution to the zeroth gauge component $\delta_\tau A_0 = \dot{\tau}\Phi$ (note that the effect appears only when the monopole moves along the τ direction). Due to the gauge slice orthogonality condition (1.7), the spatial components are unchanged $\delta_\tau A_i = 0$ by this motion along the τ direction.

We refer to objects with both electric and magnetic charge as *dyons* [53]. Dyons arise when we allow small time-dependent perturbations around BPS field solutions and we can model their low-velocity dynamics with geodesic motion on the moduli space.

In particular, we are interested here in exploring this theory in an underlying three-space $\mathbb{R} \times T^2$ with two coordinates φ and θ compactified on a two torus, each with period 2π: $(\varphi, \theta) \sim (\varphi + 2\pi, \theta) \sim (\varphi, \theta + 2\pi)$, and $x \in \mathbb{R}$. Monopoles in such a space are referred to as monopole walls, or *monowalls*. Certain components of the gauge field gain mass in some regions of the underlying space because the Higgs field is non-vanishing, and because of the gauge field holonomies associated with the periodic directions. As x grows large, we choose the Higgs field to approach diagonal with at most linear growth, the gauge holonomies to approach diagonals which are constant in space, and the $U(N)$ symmetry to be maximally broken to $U(1)^N$ in the asymptotic region. Then only diagonal gauge field components, those representing the Cartan subalgebra of $U(N)$, remain massless. We identify the locations of magnetic charge with positions at which partial or full gauge symmetry is restored [54]. The massive gauge field components decay exponentially with distance from such locations.

1.3 Objectives

This dissertation pursues four goals. First we show that a BPS monowall that has moduli (degrees of freedom) will split into distinct, well-separated subwalls if any of its moduli becomes large. Second, we determine the moduli space metric corresponding to the gauge field and Higgs interactions of n widely separated, slow-moving subwalls with distinct charges by following Manton's method [9, 13]. Third, we use the generalized Legendre transform to identify a generating function $\tilde{G}$ from which we identify a set of complex coordinates on the moduli space and a Kähler potential which produces the asymptotic moduli space metric that we found in the second part. Last, we compute the cut volume, a function of a quarter of the moduli, and compare it with the function $\tilde{G}$ introduced above in pursuit of the third goal. We prove that the cut volume equals $\tilde{G}$.

To accomplish the first objective, in Chapter 2 we review the construction of the Higgs spectral curve and analyze its behavior for large, distinct moduli using the Newton polygon and amoeba associated with this curve. The large-moduli amoeba directly relates to the BPS monopole when its constituent subwalls are widely-spaced, so we will demonstrate that as one of the monowall's moduli becomes very large, the monowall breaks into subwalls which move apart. Note that the number and charge of the constituent subwalls depends on relative ranking and size of moduli, which will be discussed further in section 2.2. Furthermore, we show that the symmetry breaks from $U(N)$ to $U(1)^N$ beyond a determined distance from each subwall. The subwalls then behave as distinct objects and their gauge and Higgs field interactions are approximately abelian, with *exponential* precision.

We reach the second objective in Chapter 3 to calculate the moduli space metric for n well-separated subwalls by modeling the moving subwalls as planes with scalar, magnetic, and electric abelian interactions with one another. For these subwalls the Lagrangian reduces to purely kinetic in the slow-velocity limit. Lagrange's equations are now equivalent to the geodesic equation for the monowall moduli and we can read off the moduli space metric from the kinetic term.

Here are the defining parameters of the moduli space we will calculate. Note that the parameters are constant quantities specifying the boundary conditions on the fields. They distinguish monopole moduli spaces and are not themselves moduli. The Yang-Mills-Higgs abelian asymptotic field equations imply a harmonic Higgs field. Following [22], we constrain the Higgs field of the $U(N)$ monowall to diverge no more than linearly, and its eigenvalues to behave as follows when $x \to \pm\infty$:

$$\Phi_a^{\pm\infty} = -i\left(\mathbb{G}_a^{\pm}x + v_a^{\pm}\right) + \mathcal{O}(x^{-1}), \tag{1.14}$$

where $a = 1, ..., N$ indexes the N unbroken $U(1)$ factors, i.e. the N diagonal elements of the field matrices with which the Higgs eigenvalues are in one-to-one correspondence. The left and right magnetic charges of the monowall $\mathbb{G}_a^\pm$ are rational constants and the subleading terms $v_a^\pm$ are real constants. Also fixed as $x \to \pm\infty$ are the holonomy eigenvalues $e^{id_{\varphi,a}}$ and $e^{id_{\theta,a}}$ associated with the two periodic directions (φ, θ). We use the shorthand $\vec{d}_a^\pm = (0, d_{\varphi,a}^\pm, d_{\theta,a}^\pm)$, where the vector symbol indicates the three spatial directions and $d_{a,i}^\pm \in [0, 2\pi)$. Together with the locations of any singular (called Dirac) monowalls, these constants $(\mathbb{G}_a^\pm, v_a^\pm, \vec{d}_a^\pm)$ fully specify the moduli space. Cherkis and Ward [22] have established necessary conditions the above parameters must satisfy for the existence of BPS solutions to be guaranteed. These are determined using the Newton polygon construction, which will be described later in this section. They determined [22] that the number of real moduli is then four times the number of integer points in the interior of the Newton polygon, which the next subsection describes.

To reach the third objective in Chapter 4 we review the generalized Legendre transform [48, 45, 46] for monopoles and the method for computing a generating integrand $\tilde{G}$ and generating function F (from which to calculate complex coordinates on the moduli space and the Kähler potential). We apply the generalized Legencre transform to monopoles on periodic and doubly periodic underlying spaces in order to find the generating integrand G for the metric on the moduli space of widely separated, slow-velocity doubly periodic monopoles.

For the last objective, in Chapter 5 we consider a plane arrangement determined by the moduli. We compute a volume cut out by these planes called the *cut volume* (which was introduced and motivated in [38]) using the Lawrence formula for polytope volumes [55] and compare it with the generating integrand for doubly periodic monopoles and establish that they are the same function (up to an overall factor of 1/2).

CHAPTER 2

Higgs Curve and Crystal

2.1 Higgs Spectral Curve

This description of the Higgs spectral curve closely follows that given in [39, Secs. 2.3 and 3]. Label the three spatial coordinates (x, θ, φ) where $x \in \mathbb{R}$, $\varphi \sim \varphi + 2\pi$ and $\theta \sim \theta + 2\pi$. For each periodic coordinate φ and θ, define the *Higgs spectral curve* (or "monopole spectral curve") [15, 21, 22]. We will use one of these as a tool to explore behaviors of BPS solutions in $U(N)$ Yang-Mills-Higgs theory. The φ-direction Higgs curve Σ_φ, for example, is determined by the characteristic equation for the holonomy of the differential operator $D_\varphi + i\Phi$. The fields (Φ, A_i) are assumed to be BPS, i.e. satisfying the Bogomolny equation (1.4). To define the holonomy, introduce a matrix function $V(x, \varphi, \theta)$ which, given a solution (Φ, A_i) of the Bogomolny equation (1.4), solves the system of equations

$$(D_\varphi + i\Phi)V = 0, \qquad\qquad (D_x + iD_\theta)V = 0. \tag{2.1}$$

The consistency condition for this linear system is $[D_x + iD_\theta, D_\varphi + i\Phi] = 0$ which amounts to two of the three Bogomolny's equations. The holonomy of $D_\varphi + i\Phi$ is $W(x, \theta) = V(x, 2\pi, \theta)\, V^{-1}(x, 0, \theta)$, which is a holomorphic function of $x + i\theta$ thanks to the second equation in (2.1) [15, 22]. Since θ is periodic, define a more convenient coordinate $s = e^{x+i\theta} \in \mathbb{C}^*$. The eigenvalues of the holonomy $W(s)$ are finite (away from the Dirac singularities) and nonzero, the Higgs spectral curve is described by the characteristic (eigenvalue) equation of $W(s)$:

$$F(s,t) := \det(t - W(s)) = 0, \qquad \text{where} \quad F(s,t) = \sum_l^N k_l(s)t^l. \tag{2.2}$$

$F(s,t)$ is a polynomial in $t \in \mathbb{C}^*$ of degree N. Given the linear boundary conditions we have set on the fields [22] the functions $k_l(s)$ are rational functions of s. Without affecting the curve $\Sigma : F(s,t) = 0$ (i.e. without affecting the set of roots $\{(s,t)|F(s,t) = 0\}$), we can rescale by a common denominator polynomial in s to obtain a polynomial in s and t, labeled $f(s,t)$. This is referred to as the spectral polynomial [22], or *Higgs spectral polynomial*. The algebraic curve produced by $f(s,t) = 0$ is the *Higgs spectral curve* and lives in $\mathbb{C}^* \times \mathbb{C}^*$, where $\mathbb{C}^*$ is the complex plane with the origin omitted, s is the coordinate in the first $\mathbb{C}^*$ factor and t is the coordinate in the second $\mathbb{C}^*$ factor.

We now introduce the Newton polygon and amoeba for this polynomial, which can be written $f(s,t) = \sum_{i=1}^{\sigma+\sigma'} a_i s^{m_i} t^{n_i}$, where $\sigma + \sigma'$ is the number of terms in the polynomial. The *Newton polygon* $\mathcal{N}(f)$ is the minimal convex hull of the points $\{(m_i, n_i)\}$ in $\mathbb{Z}^2$ for which the coefficients are nonzero $a_i \neq 0$. The concept generalizes to arbitrary dimension [56, 22]. Let σ' be the number of integer points along the perimeter of the Newton polygon and σ the number of internal points.

To obtain the *amoeba*, project the Higgs spectral curve $\Sigma \subset \mathbb{C}^* \times \mathbb{C}^* = \mathbb{R}^2 \times S^1 \times S^1$ from two complex dimensions down to two real dimensions $\mathbb{R}^2$ by taking the modulus of each factor of $\mathbb{C}^*$ and applying the Log map

$$(s,t) \to (\log|s|, \log|t|) = (x,y). \tag{2.3}$$

This yields a more intuitive view of the nature of the curve, particularly in the large-x regime, as will be seen. As we shall see, when one or more of the moduli are large, equation (2.1) simplifies significantly when the commutator vanishes and the Higgs field becomes approximately linear in x. It is clear that x is the noncompact coordinate on the underlying space and in this region y corresponds to the x-linear Higgs eigenvalue magnitudes (multiplied by $-2\pi i$). When the Higgs curve is projected in this manner, the result is called the *amoeba* $\mathcal{A}(f) \subset \mathbb{R}^2$ of Σ for its distinctive appearance [57] (see for example Figure 2.1).

We will use the φ-direction Higgs curve but it should be noted that a different spectral curve could be found by exchanging the roles of coordinates θ and φ. These curves however share the same Newton polygon [22].

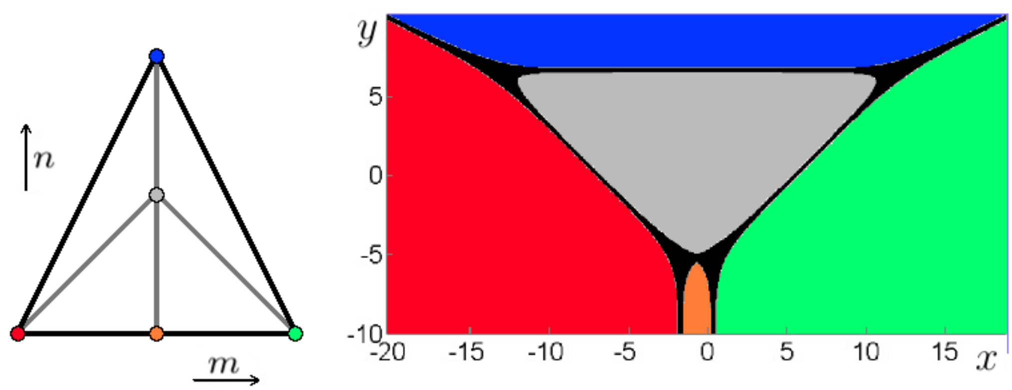

Figure 2.1: Newton polygon and amoeba for $F(s,t) = 1.3st^2 + ust + (-1 + 5s + 4s^2)$ with $u = 1000$.

2.2 Multi moduli spectral curve, relationship to monowalls

Each edge of the Newton polygon is associated with a set of external amoeba legs stretching out to infinity. Each external leg asymptotes to a normal to its associated polygon edge, and its position is determined by the monomials of $f(s,t)$ associated with that edge (their powers (m,n) and coefficients). In order to keep the boundary conditions on the fields fixed, the polynomial coefficients corresponding to edge terms must be fixed [22]. This constraint does not apply to points on the interior of the Newton polygon, and we may consider the family of polynomials with fixed perimeter coefficients and a range of values for internal coefficients. To this purpose, we begin by allowing the internal point coefficients to vary, i.e. we consider the family of polynomials for which the coefficient of the internal points take complex values, while the perimeter coefficients are each fixed in the complex plane. Rather than considering each such polynomial individually, we may look at the whole picture by treating the internal coefficients as moduli of the curve. While the external legs are fixed in position and slope by boundary conditions on the fields (Φ, A), the internal legs of the amoeba will shift in position and thickness as the long moduli $\log|a_\rho^{\text{int}}| = R_\rho^{\text{int}}$ vary. We may distinguish among the *phases* of the amoeba by specifying the relative magnitudes of the moduli. Each such phase corresponds to a regular triangulation of the Newton polygon [57, Sec 7.1.C]. This determines how many internal legs exist, their slopes, and the graph they form. We will take a moment here to expand on how the relative size of the internal coefficients relates to the phase of the amoeba and correspond to triangulations of the Newton polygon. In the following subsection we assume a fixed phase and discuss the associated amoeba, and will return to the question of phases of the amoeba at the end of the section.

2.2.1 Three dimensional Amoeba

In order to explore a given phase without shuffling the relative size ranking of the moduli, we will introduce a parameter h such that the perimeter coefficients a_ρ^{per} are h-independent, while real exponentials in the internal coefficients are scalable by modifying h. Let the internal coefficient be $\tilde{a}_\rho^{\text{int}} = e^{\frac{R_\rho}{h} + i\Theta_\rho}$. Note that $R_\rho \in \mathbb{R}_{\geq 0}$, $h \in \mathbb{R}_{\geq 0}$ and $\Theta_\rho \in [0, 2\pi)$. By taking h to be very small, we explore the regime in which all of the internal coefficients in the Higgs spectral polynomial are very large while preserving their relative ranking. This corresponds to choosing a certain direction to approach infinity on the moduli space. Let us for now treat the quantity $u := e^{\frac{1}{h}}$ as an independent variable on par with s and t. This effectively increases the number of complex coordinates of the function from two to three but demotes it from polynomial. We write the three dimensional Higgs spectral function (from now on referred to as the *Newton function*) with $\sigma + \sigma'$ terms as

$$\tilde{f}(s,t,u) = \sum_{\rho=1}^{\sigma} e^{i\Theta_\rho} s^{m_\rho} t^{n_\rho} u^{R_\rho} + \sum_{\mu=\sigma+1}^{\sigma+\sigma'} a_\mu s^{m_\mu} t^{n_\mu}. \tag{2.4}$$

The three-dimensional Newton polytope $\tilde{\mathcal{N}}(\tilde{f})$ associated with this Newton function is the minimal convex hull of the points $\{(m_\rho, n_\rho, R_\rho)_{\rho \in \text{Int}(\mathcal{N})}\} \cup \{(m_\rho, n_\rho, 0)_{\rho \in \text{Per}(\mathcal{N})}\}$ in $\mathbb{Z}^2 \times \mathbb{R}$ for which the coefficients are nonzero $a_\rho \neq 0$. We are interested in relative rankings of R_ρ such that all σ internal points appear as vertices of this hull, i.e. such that all monomials in $\tilde{f}$ are represented on the Newton polytope surface.

The three-dimensional amoeba $\tilde{\mathcal{A}}(\tilde{f}) \in \mathbb{R}^3$ for $\tilde{f}(s,t,u) = 0$ also has externalities extending to infinity, known as the asymptotic three-dimensional amoeba. According to Gelfand, Kapranov, and Zelevinski[1] [57] and Viro [58, Sec. 4], this three-dimensional amoeba asymptotically approaches the *core* of the amoeba exponentially fast. The amoeba approaches this core when one or more of its internal coefficients becomes very large. The amoeba core can be described in the following way (see Figure 2.2): Orthogonal to each edge of the three-dimensional Newton polytope for $\tilde{f}(s,t,u)$ is a continuous set of directions which form plane *wedges*. Wedges for different edges on a given face of the Newton polytope intersect at and terminate on the *leg* associated with that face. The three dimensional amoeba legs are a set of cylinders each normal to a polytope face and having two-dimensional amoeba cross-sections. Recall that $x = \log|s|$ is the non-compact spatial coordinate and that $y = \log|t|$ asymptotically corresponds to the Higgs eigenvalue magnitudes. The significance of the new, third component $z = \log|u|$ is the following. The intersection of the three-dimensional amoeba with a horizontal plane defined by a given height of $z = \frac{1}{h}$ is precisely the two-dimensional amoeba for $f(s,t)$ (e.g. Figure 2.3) for a given value of z. The Newton polygon for this two-dimensional amoeba is the projection of the three-dimensional Newton polytope onto the $(m,n,0)$ lattice. Each subwall corresponds to a face of the three-dimensional polytope.

For a horizontal plane positioned at very large z, its intersection with the three-dimensional amoeba is as follows: The plane intersections with the wedges of the three-dimensional amoeba along straight lines, called *amoeba lines*. The plane intersections with the three-dimensional amoeba legs, called *junctions*, are sections of the amoeba legs whose cross-sections are two-dimensional amoe-

[1]Proposition 1.13, Ch. 6

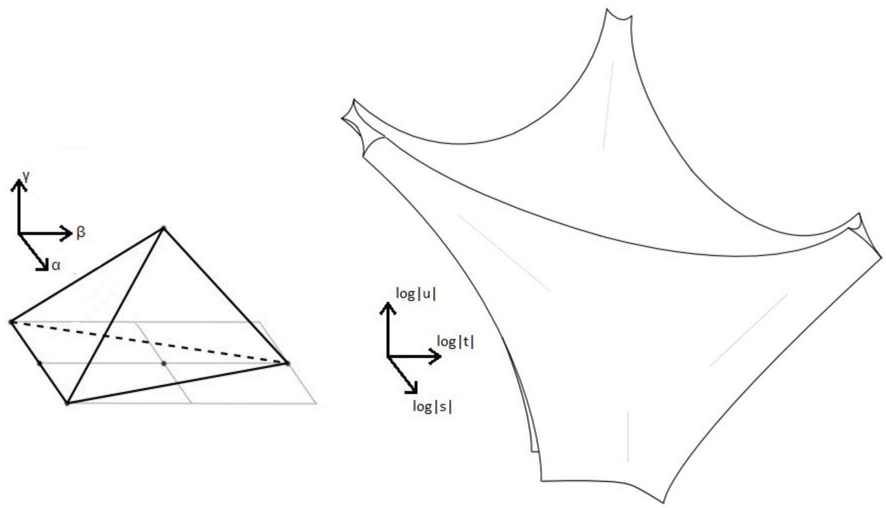

Figure 2.2: Three-dimensional Newton polygon and sketch of three-dimensional amoeba core for $\tilde{f}(s,t,u) = 1.3st^2 + ust + 4s^2 + 5s - 1$.

bas. Importantly, these sections have fixed areas asymptotically which differ from the cylinder cross-sections by a constant factor. Each subwall (which will be defined in detail in the following subsection), then, is associated with and its behavior determined by a face of the Newton polytope when one or more moduli are large. The separations/relative positions of subwalls *depend linearly on the parameter* z.

Here we will make a brief note about secondary polytopes and their fans in order to describe the phases of the amoeba and how they correspond to triangulations of the Newton polytope. A useful tool for exploring the range of values of the internal coefficients which correspond to a given triangulation of the Newton polytope is the *secondary polytope* introduced in [57, Ch. 7] and *secondary fan*. Let us describe the secondary polytope, an object defined by

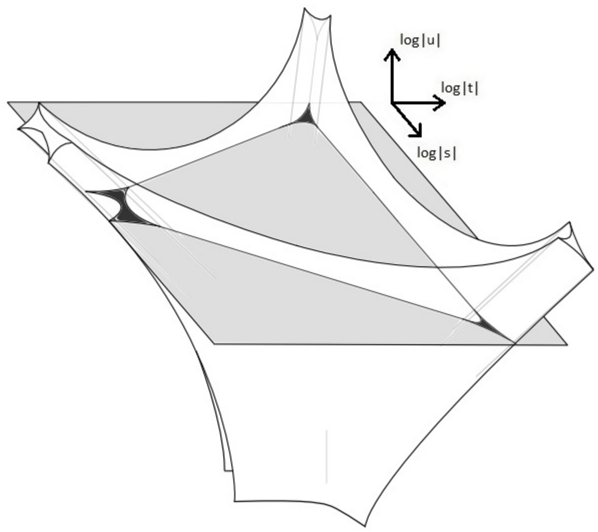

Figure 2.3: Three dimensional amoeba (white) and $\frac{R}{h}$ plane (grey). The intersection (black) gives the two-dimensional amoeba for a given value of $\frac{R}{h}$.

the set of possible *coherent triangulations* (sometimes called *regular triangulation*) of the Newton polytope [57, Sec 7.1.C], i.e. triangulations in which the vertices of the subdivisions of the triangulation are either vertices of the Newton polytope or internal points with non-zero coefficients in the Newton polytope.

For a k-dimensional Newton polytope with σ internal points and σ' perimeter points, its secondary polytope is generally a $(\sigma+\sigma'-k-1)$-dimensional object in a $(\sigma+\sigma')$ dimensional space—and each dimension corresponds to one of the $\sigma+\sigma'$ points on the Newton polytope. More specifically, each dimension corresponds to the real part of the coefficient of the associated point on the Newton polytope. Each vertex of the secondary polytope corresponds to a coherent triangulation of the Newton polytope and the coordinates of that vertex of the secondary polytope are determined by how each of the $\sigma + \sigma'$ Newton polytope points feature in that triangulation.

Given a triangulation and its associated vertex of the secondary polytope, we can calculate the coordinate of that vertex along the ρ^{th} direction. In this triangulation of the Newton polytope, the ρ^{th} point in the Newton polytope $\mathcal{N}$ is a vertex for some of the subdivisions of $\mathcal{N}$. The coordinate of the secondary polytope vertex along the $\rho^t h$ direction is equal to the sum of the volumes subdividing $\mathcal{N}$ for which point ρ is a vertex. The reason the secondary polytope is $(\sigma + \sigma' - k - 1)$ dimensional rather than $(\sigma + \sigma')$-dimensional is because shifting the Newton polytope in any of its k dimensions does not affect the secondary polytope, and the sum of the elements of any secondary polytope vertex coordinate is $k + 1$ times the total volume of the Newton polytope. This last condition is because each simplex in the Newton polytope subdivision (triangulation in the case of a Newton polygon) has $k+1$ vertices and therefore its volume gets counted $k + 1$ times in the computation of the coordinate of the secondary polytope vertex.

The fan of the secondary polytope is dual it. Each vertex of the secondary polytope has a cone, the volume spanned between the normals of all faces adjacent to that vertex. The relevance of the cone for a given vertex is that it contains the range of directions in "coefficient space" that will cause the monowall to break into the set of subwalls corresponding to that triangulation in a large moduli limit. In other words, with each cone in the secondary polytope fan is associated a phase of the three dimensional amoeba.

For example, consider a square with side length 2, four perimeter points and one internal point, as in part a of figure 2.2.1. This possesses three coherent triangulations seen in part b of figure 2.2.1. The secondary polytope is a two dimensional triangle in five dimensional space, as seen in part c of figure 2.2.1. Within the plane of the triangle, the set of directions spanning between the upward normal arrow and the normal arrow on the right give the set of directions in coefficient space along which the amoeba will be in the phase corresponding to the associated triangulation. Note that the first two triangulations given in figure 2.2.1 correspond to small values of the internal point's coefficient and so are inaccessible triangulations in the limit that the internal coefficient is large.

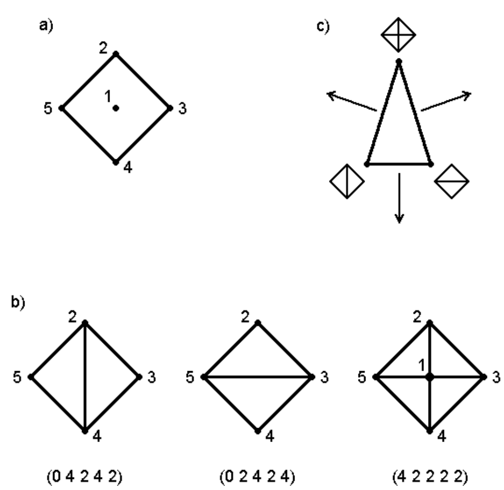

Figure 2.4: The Newton polygon in a) has three coherent triangulations, which can be seen in b). Its secondary polygon c) has vertices at (0 4 2 4 2), (0 2 4 2 4) and (4 2 2 2 2). It is a two dimensional secondary polytope in a five dimensional space for a two dimensional Newton polygon. Here, $\sigma = 1$, $\sigma' = 4$ and $k = 2$.

It is worth making explicit that these various phases of the monowall indicate that while the monowall breaks into subwalls for large moduli, these subwalls *are not fundamental constituents* in the way that, for example, quarks are fundamental constituents of a proton. The subwalls of a particular phase do not exist when the monowall is unbroken and the subwall may break into one set of constituent walls in one phase, and into a completely different set of constituent walls in a different phase.

2.2.2 Monowall breaking

We now proceed to describe the breaking of the monowall into subwalls and how we define the subwall thickness so that we may identify the conditions under which the monowall may be treated as a collection of constituent subwalls with abelian interactions. The equations giving these conditions and the width of a subwall are respectively equations (2.11) and (2.10). In Section 2.3 we define the crystal and continue on in Chapter 4 to derive a metric on the moduli space of these widely separated subwalls.

In the y versus x plane at z, the amoeba lines correspond to regions in x where the Higgs eigenvalues take values linear in x, with multiplicity equal to the denominator of the slope the Higgs eigenvalues $\mathrm{Eig}\,(i\Phi) = -\frac{m_\rho - m_\mu}{n_\rho - n_\mu} x - \frac{R_\rho - R_\mu}{n_\rho - n_\mu}$, i.e. the height $n_\rho - n_\mu$ of the associated Newton polytope edge. In such regions, it is not the minimum difference in Higgs eigenvalues which produces mass in the off-diagonal gauge fields, but the minimum difference in Holonomy eigenvalues. As we will show, the $U(N)$ symmetry in these regions is maximally broken to $U(1)^N$ by the non-vanishing gauge field holonomies for the φ and

θ directions, and the fields are exponentially close to abelian. The junctions correspond to regions in which the Higgs field eigenvalues are not linear in x and the gauge field holonomies cannot be approximated well, and so we are unable to infer fully broken symmetry; we interpret these regions as locations of magnetic charges, or subwalls. It is necessary now to define the widths of these subwalls, or the extents in x of their nonabelian interiors. We will define the subwalls to be "well-separated" when their separations are much greater than the maximal subwall width and therefore their interactions are abelian.

To accomplish this, we must quantify the decay of the off-diagonal gauge field components which mediate nonabelian field interactions. Gauge field components which do not commute with the Higgs field must decay exponentially at a rate proportional to the separation of Higgs eigenvalues [59, 10.5, Ch. IV]. Here this decay rate amounts to the Log of the ratio of eigenvalues, $\log(t_\rho/t_\mu)$, for the holonomy $\tilde{W}(s,u)$ since nonvanishing gauge field holonomies can asymptotically generate gauge field masses analogously to the Higgs mechanism. At the point where these non-commuting gauge field components have decayed by some chosen fraction, we mark the edge of a subwall. We define the subwall width as the distance at which the exponential rates for the decay of the nonabelian gauge field components are bounded from below by some small value T_0, plus the distance $1/T_0$ at which the fields will have decreased by a factor of $1/e$ (the inverse of Euler's number).

While the behavior of the real part $\log|\tilde{W}|$ of the holonomy (as a function of x) is illustrated by the amoeba, the behavior of its argument is not. We must therefore look to the Newton function to determine the various branches of $t = T(s,u)$, which locally satisfy $\tilde{f}(s, T(s,u), u) = 0$. This is done by calculating the Newton-Puiseux expansion [60, 61] for $T(s,u)$ with respect to s and u. If the two Newton polytope faces corresponding to two subwalls share an edge, then the fields between two subwalls are governed primarily by the two monomials in the Newton function that are associated with the edge $\mathfrak{e}$ joining the two faces. There are also smaller contributions from the remaining monomials. The resulting expansion will take the following form and only the first two terms in the expansion are of concern here:

$$\begin{aligned} T_\alpha(s,u) &= c_{1\alpha}\, s^{\gamma_1} u^{\gamma_3} + c_{2\alpha}\, s^{\tilde{\gamma}_{1\alpha}} u^{\tilde{\gamma}_{3\alpha}} + ... \\ &= c_{1\alpha}\, s^{\gamma_1} u^{\gamma_3} \left(1 + (c_{2\alpha}/c_{1\alpha})\, s^{\tilde{\gamma}_{1\alpha}-\gamma_1} u^{\tilde{\gamma}_{3\alpha}-\gamma_3}\right) + ... \end{aligned} \tag{2.5}$$

Briefly, for a direction $w \in \mathbb{R}^3$ within the normal cone of an edge of the Newton

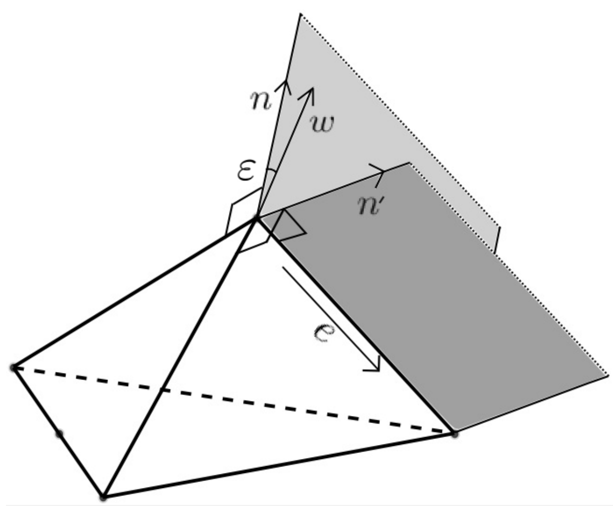

Figure 2.5: The area above the grey partial planes is the normal cone for edge e. The normal vector n' is normal to the front right face, while n is normal to the rear face. The vector w is normal to edge e and lies in the wedge bounded by n and n'. It is defined as a rotation if n through angle ε.

polytope (see Figure 2.5), the Newton-Puiseux series is constructed iteratively.

We consider first the case of an edge joining a perimeter point and an internal point. The dominant series term solves the vanishing of the edge function $e^{i\Theta_\rho}u^{R_\rho}s^{m_\rho}T^{n_\rho} + a_\mu s^{m_\mu}T^{n_\mu} = 0$, so that in the dominant term in the series, the coefficient is $c_{1\alpha} = \left(-a_\mu e^{-\Theta_\rho}\right)^{-1/(n_\rho - n_\mu)} e^{2\pi i\cdot\alpha/(n_\rho - n_\mu)}$, and the powers are $\gamma_1 = -(m_\rho - m_\mu)/(n_\rho - n_\mu)$, and $\gamma_3 = -R_\rho/(n_\rho - n_\mu)$. For an edge joining two internal points, the first series term solves the vanishing of the edge function $e^{i\Theta_\rho}u^{R_\rho}s^{m_\rho}T^{n_\rho} + e^{i\Theta_\mu}u^{R_\mu}s^{m_\mu}T^{n_\mu} = 0$, and the coefficient and powers are instead $c_{1\alpha} = (-1)^{1/(n_\rho - n_\mu)}e^{-i(\Theta_\rho - \Theta_\mu)/(n_\rho - n_\mu)}e^{2\pi i\cdot\alpha/(n_\rho - n_\mu)}$, $\gamma_1 = -(m_\rho - m_\mu)/(n_\rho - n_\mu)$, and $\gamma_3 = -(R_\rho - R_\mu)/(n_\rho - n_\mu)$.

More formally, the powers γ_1 and γ_3 are the negative of the components of the slope vector $S_\mathfrak{e} = \left(\frac{\mathfrak{e}_1}{\mathfrak{e}_2}, 0, \frac{\mathfrak{e}_3}{\mathfrak{e}_2}\right) = -(\gamma_1, 0, \gamma_3)$ associated with a perimeter-interior edge $\mathfrak{e} = (m_\rho - m_\mu, n_\rho - n_\mu, R_\rho)$ or an interior-interior edge $\mathfrak{e} = (m_\rho - m_\mu, n_\rho - n_\mu, R_\rho - R_\mu)$, and the coefficient c_1 solves the equation $\sum_{\rho\in\mathfrak{e}} a_\rho c_1^{n_\rho} = 0$, excluding trivial solutions, where here a_ρ is a stand-in for either constant perimeter coefficients or variable interior coeffients $e^{i\Theta_\rho}$. The second term, $c_2 s^{\tilde{\gamma}_1}u^{\tilde{\gamma}_3}$ is found by repeating this process for the Newton polytope for the function $\tilde{f}_1(s, T_1, u) = \tilde{f}(s, T_1 + c_1 s^{\gamma_1}u^{\gamma_3}, u) = \sum_{\rho=0}^{\sigma^1} a_\rho^1 s^{m_\rho^1} T_1^{n_\rho^1}$, choosing an edge $\tilde{\mathfrak{e}}$ which maximizes $-S_{\tilde{\mathfrak{e}}}\cdot w$ (called the *order* $\tilde{\nu}$ of edge $\tilde{\mathfrak{e}}$ with respect to w) while satisfying $-S_{\tilde{\mathfrak{e}}}\cdot w < -S_\mathfrak{e}\cdot w$. The coefficient c_2 solves the equation $\sum_{\rho\in\tilde{\mathfrak{e}}} a_\rho^1 c_2^{m_\rho^1} = 0$ and $|c_2| \leq \left(1 + \frac{\max\left(\{|a_\rho c_1^N|\}, \{|a_\rho|\}\right)}{\min\left(\{|a_\rho c_1^N|\}, \{|a_\rho|\}\right)}\right) =: C_2$ is its maximum magnitude [62]. Again, here a_ρ stands in for either the constant perimeter coefficients or the variable internal coefficient $e^{i\Theta_\rho}$ which may vary between -1 and 1.

Define $\delta = (\delta_{1\alpha}, 0, \delta_{3\alpha}) = (\tilde{\gamma}_{1\alpha} - \gamma_1, 0, \tilde{\gamma}_{3\alpha} - \gamma_3)$, which behave as follows in the asymptotic limits: For $s \to 0$ and $u \to \infty$, $\delta_{1\alpha} > 0$ and $\delta_{3\alpha} < 0$; for

$s \to \infty$ and $u \to \infty$, $\delta_{1\alpha} < 0$ and $\delta_{3\alpha} < 0$. In other words, the quantity $s^{\delta_{1\alpha}} u^{\delta_{3\alpha}}$ decays in both of these limits of s and u. Given the first two terms of the Newton-Puiseux series, the ratio of two eigenvalues T_α of the holonomy $\tilde{W}(s,u)$ is written

$$\begin{aligned} \frac{T_\alpha(s,u)}{T_\beta(s,u)} = & \frac{c_{1\alpha}}{c_{1\beta}} \left(1 + \frac{c_{2\alpha}}{c_{1\alpha}} s^{\delta_{1\alpha}} u^{\delta_{3\alpha}} - \frac{c_{2\beta}}{c_{1\beta}} s^{\delta_{1\beta}} u^{\delta_{3\beta}}\right) \\ & + \mathcal{O}\left(\min(s^{2\delta_{1a}} u^{2\delta_{3a}})\right)_{a=\alpha,\beta} . \end{aligned} \tag{2.6}$$

In this expression, every quantity but the first term $c_{1\alpha}/c_{1\beta}$ decays in the asymptotic limits. Simplifying the ratio of coefficients $c_{1\alpha}/c_{1\beta} = e^{2\pi i(\alpha-\beta)/(n_\rho - n_\mu)}$, the Log of equation (2.6) becomes

$$\begin{aligned} \log\left(\frac{T_\alpha}{T_\beta}\right)(s,u) = & \frac{2\pi i(\alpha-\beta)}{n_i - n_j} + \left(\frac{c_{2\alpha}}{c_{1\alpha}} s^{\delta_{1\alpha}} u^{\delta_{3\alpha}} - \frac{c_{2\beta}}{c_{1\beta}} s^{\delta_{1\beta}} u^{\delta_{3\beta}}\right) \\ & + \mathcal{O}\left(\min(s^{2\delta_{1a}} u^{2\delta_{3a}})\right)_{a=\alpha,\beta} . \end{aligned} \tag{2.7}$$

The first term in this series is constant, while in the asymptotic limit the quantity in the parentheses is the largest decaying term in the series.

The expansion direction $w \in \mathbb{R}^3$ comes explicitly into play when determining the relative sizes of the quantities s^{δ_1} and u^{δ_3}. Along the direction w, the variables behave as

$$(s_0, t_0, u_0) \sim (s_0 e^{\bar{w}_1}, t_0 e^{\bar{w}_2}, e^{\bar{w}_3}) \tag{2.8}$$

relative to some initial values $(s_0, t_0, 1)$ [58], where $\bar{w}$ is the vector w multiplied by a coefficient so that its third component is $\bar{w}_3 = z$: $\bar{w} = \frac{z}{w_3} w$ for $z \in \mathbb{R}_{\geq 0}$. Also define the extended face normal vector $\bar{n} = \frac{z}{n_3} n$.

We have not said very much so far about the direction vector w except that it must lie within the normal cone of the edge e. Define it relative to the nearest of the two adjacent face normal vectors n (see Figure 2.5). For angle ε, we define the expansion direction as a rotation of the normal vector n of one of the edge's adjacent faces: $w = n\cos\varepsilon + \frac{(\mathfrak{e}\times n)}{|\mathfrak{e}|}\sin\varepsilon + \frac{\mathfrak{e}(\mathfrak{e}\cdot n)}{|\mathfrak{e}|^2}(1-\cos\varepsilon)$. The third term vanishes since the face normal n is orthogonal to the edge vector and $\mathfrak{e}\cdot n = 0$. Applying this form for the vector $\bar{w} = \frac{z}{w_3}w$, the largest decaying terms in equation (2.7) are

$$s^{\delta_{1\alpha}}u^{\delta_{3\alpha}} = s_0^{\delta_{1\alpha}}e^{(\tilde{\nu}-\nu)z/w_3} = s_0^{\delta_{1\alpha}}e^{-|\bar{w}_1-\bar{n}_1|/\lambda}, \tag{2.9}$$

where the denominator $\lambda := \frac{|(\mathfrak{e}\times n)_1-(\mathfrak{e}\times n)_3 n_1/n_3|}{|(\mathfrak{e}\times n)\cdot\delta|}$ is α-independent, and the powers are $\delta = (\delta_{1\alpha}, 0, \delta_{3\alpha})$ for any $\alpha = 1, 2, ..., (n_\rho - n_\mu)$. Recall that we define the order of an edge $\mathfrak{e}$ as $\nu = \gamma_1 w_1 + \gamma_3 w_3$. The difference in orders of the secondary edge $\tilde{\mathfrak{e}}$ and the original edge $\mathfrak{e}$ is $(\tilde{\nu}-\nu) = w\cdot\delta$ and it is α-independent. The vector component $\bar{n}_1 = x_{1\ell} - x^0_{3\ell}$ of the rescaled normal vector is the x-distance between the ℓ^{th} subwall's position $x_{1\ell}$ and its reference position $x^0_{1\ell}$, i.e. the linearly extrapolated position of the wall when $z = 0$. We identify the subwall initial position for edge e with the greatest magnitude as $\max|x^0_{1\ell}|$ and that with the smallest magnitude as $\min|x^0_{1\ell}|$ for $\ell = 1, ..., n$. The rightmost term in equation (2.9) is decaying at rate λ with x-distance $|\bar{w}_1 - \bar{n}_1|$ from the subwall.

For a $U(N)$ monowall, we find that beyond a distance $D = \lambda\log\left|\frac{c_{\alpha\beta}e^{N/\lambda\pi}}{\pi/N}\right|$ from the wall's position, the exponential decay rates of the off-diagonal gauge field components are bounded by $|\log(T_\alpha/T_\beta)| \geq \pi/N$, where the mixed-index coefficient is defined $c_{\alpha\beta} := \left(\frac{c_{2\alpha}}{c_{1\alpha}}s_0^{\delta_{1\alpha}} - \frac{c_{2\beta}}{c_{1\beta}}s_0^{\delta_{1\beta}}\right)$ and the power of $s_{0\ell}$ is bounded by $1/N^2 \leq |\delta_{1\alpha}| \leq N^2$. See Figure 2.2.2. The bounded Newton-Puiseux coefficients satisfy $|c_2| \leq \left(1 + \frac{\max\left(\{|a_\rho c_1^N|\},\{|a_\rho|\}\right)}{\min\left(\{|a_\rho c_1^N|\},\{|a_\rho|\}\right)}\right) =: \mathcal{C}_2$ and $|c_1| \geq \max\{|a_\rho|\}^{-P} =: \mathcal{C}_1$, where $P = N$ if $\max\{|a_\rho|\}$ is greater than unity and $P = 1/N$ if it is less than unity. The coefficient $c_{\alpha\beta}$ is then limited by

$|c_{\alpha\beta}| \leq 2\frac{\mathcal{C}_2}{\mathcal{C}_1}e^{\max(x_\ell^0)\delta_1}$. We find the following is the maximum subwall width for a $U(N)$ monopole on $\mathbb{R} \times T^2$ where each torus period is 2π:

$$\frac{\Delta}{2} = \log\left|\left(\frac{2N\mathcal{C}_2}{\pi\mathcal{C}_1}\right)^\lambda\right| + \max|x_\ell|N^2\lambda + \frac{N}{\pi} + \left(\max(x_\ell^0) - \min(x_\ell^0)\right), \quad (2.10)$$

where $\lambda := \frac{|(e\times n)_1 - (e\times n)_3 n_1/n_3|}{|(e\times n)\cdot\delta|}$ is the minimum decay rate of the off-diagonal gauge field components. No subwall of a $U(N)$ monowall may have a width greater than Δ. Recall that the subwall separations are linear in z. We have computed the width of the subwall, i.e. the width of the region in which the off-diagonal gauge fields are too large to neglect and the fields are therefore nonabelian. In order to consider the subwalls to be well-separated and their interactions abelian, we require of the non-compact parameter

$$\boxed{z >> \Delta} \quad (2.11)$$

for each $U(N)$ subwall. Since wall separations are linear in this parameter z, when this condition is satisfied, the subwalls will be considered well-separated.

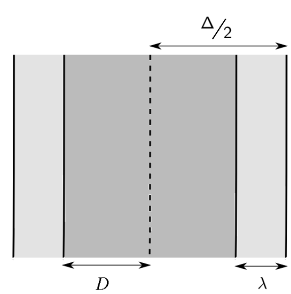

Figure 2.6: Beyond a distance D the off-diagonal gauge field components decay with a minimum decay rate λ. Beyond the distance $D + \lambda$ the off-diagonal gauge fields are neglected.

Our main goal in this section is to describe the large-moduli behavior of a BPS $U(N)$ monowall in regions away from the subwalls using the Higgs spectral curve and its Newton polygon and amoeba. By allowing a coefficient in the interior of the polygon to vary, we introduced a parameter $z := \log|u| = \frac{1}{h}$ for the monowall. Using the Newton-Puiseux series for the eigenvalues $T_\alpha(s,u)$ of the holonomy $\tilde{W}(s,u)$ to characterize the off-diagonal gauge field decay rates, we showed that when the modulus z is very large, the monowall breaks into subwalls whose separations increase with z. For $z >> \Delta$, the subwall interactions reduce to N abelian interactions (i.e. $U(N)$ breaks to $U(1)^N$) up to corrections exponentially small in z. In the following section, we will derive an asymptotic moduli space metric for these well-separated subwalls using their abelian Higgs, magnetic, and electric interactions.

2.3 Crystal

We now compute the cut volume, which was introduced in [38]. It is a three-dimensional polytope which can be helpful to study the behavior of the amoeba spine, and the amoeba spine allows us to approximate the positions of subwalls, as we will show. The amoeba is a two dimensional shape in $x = \log|s|$ and $y = \log|t|$, but it is useful here to introduce a third dimension. Associate a plane $z = m_\rho x + n_\rho y + R_\rho$ with each point (m_ρ, n_ρ) on the Newton polygon. Here, the normal vector for the ρ^{th} plane is $(m_\rho, n_\rho, -1)$ and the modulus R_ρ is the plane's z-intercept. In accordance with [38] we define the *cut crystal* as the region which is above all of these planes, those associated with perimeter and internal points of the Newton polygon $\mathcal{N}$. Define the *crystal* as the volume above *perimeter* planes. The *cut volume* is the volume of the crystal outside of the cut crystal, as shown in Figure 2.3.

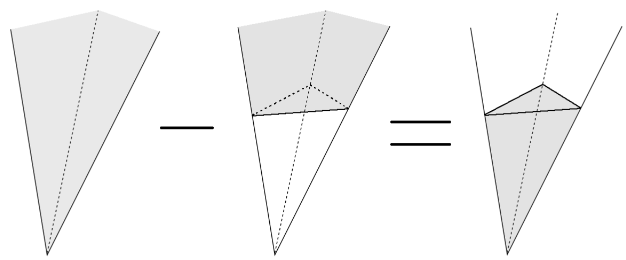

Figure 2.7: From left to right: The crystal, the cut crystal, the cut volume.

The bottom surface of the cut crystal is a piecewise planar surface whose edges projected onto the (x, y)-plane coincide with the amoeba lines of the amoeba spine and whose vertices projected onto the z=0 plane coincide with the positions of the junctions of the amoeba spine. For moduli R_i large relative to the perimeter coefficients, all the vertex positions of this bottom surface will depend on one, two or three moduli. Label the a^{th} vertex as the intersection of planes ρ, μ and ν. The associated subtriangle in the Newton polygon triangulation has vertices (ρ, μ, ν) arranged in clockwise sequence. Computing the position of the intersection of planes ρ, μ and ν gives (see figure 2.3)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_a = \frac{\begin{pmatrix} n_{\rho\mu}R_\nu + n_{\mu\nu}R_\rho + n_{\nu\rho}R_\mu \\ -m_{\rho\mu}R_\nu - m_{\mu\nu}R_\rho - m_{\nu\rho}R_\mu \\ -\begin{vmatrix} m_\rho & n_\rho \\ m_\mu & n_\mu \end{vmatrix} R_\nu - \begin{vmatrix} m_\mu & n_\mu \\ m_\nu & n_\nu \end{vmatrix} R_\rho - \begin{vmatrix} m_\nu & n_\nu \\ m_\rho & n_\rho \end{vmatrix} R_\mu \end{pmatrix}}{\begin{vmatrix} m_{\mu\nu} & n_{\mu\nu} \\ m_{\rho\nu} & n_{\rho\nu} \end{vmatrix}}. \tag{2.12}$$

The first component of this is equal to the x-position of the associated subwall, which is used to compute the metric on the moduli space of monowalls in Section 3.3.

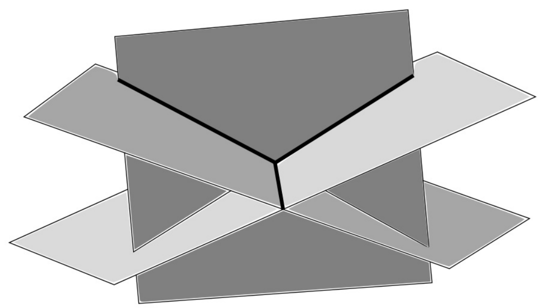

Figure 2.8: Three intersecting planes. The intersections of topmost planes are marked with dark lines. This set of lines projects down to the amoeba skeleton. The remaining lines of intersection are not significant.

CHAPTER 3

Asymptotic Moduli Space

3.1 Lagrangian and fields

We have established that a monowall splits into subwalls when a modulus becomes large. We will now consider the regime in which the monowall is split into n subwalls which have no remaining internal moduli, and the subwall Higgs and gauge interactions are abelian. In order to model these abelian long-distance interactions, we contrive a system of n abelian monowalls in a linear Higgs background which have scalar, magnetic, and electric interactions with one another and with their background. The interactions of one of these abelian monowalls with the background and the $n-1$ remaining abelian monowalls mimics the long-distance interactions of one of the nonabelian subwalls with the $n-1$ remaining subwalls. We will from here forward refer to these model abelian monowalls simply as subwalls. We describe the abelian monowall interactions using Lorentz-invariant Maxwell electromagnetism with a scalar field. We write the Lagrangian and consider only very small subwall velocities. We then Legendre transform from the electric charge q_i, which is a momentum, to its canonical coordinate given by a periodic phase modulus τ_i. The Lagrangian reduces to purely kinetic under the BPS conditions. We will then read the monowall moduli space metric off of this kinetic Lagrangian.

Recall that we choose a gauge in which the Higgs field is diagonal and we have demonstrated that it is x-linear outside of the subwalls. The off-diagonal gauge fields gain mass and are exponentially small and therefore negligible, while the diagonal gauge field components remain massless. We will represent the generators of the Cartan subalgebra as the N generators $\{H_\alpha\}$ of $N \times N$ imaginary diagonal matrices, and write the asymptotic fields as $\Phi = \Phi^\alpha H_\alpha$ and $A_\mu = A^\alpha_\mu H_\alpha$ where $(\Phi^\alpha, A^\alpha_\mu)$ are real and $\alpha = 1, ..., N$. We employ an additional, adjoint dual gauge potential $\tilde{A}_\mu = \tilde{A}^\alpha_\mu H_\alpha$ to model magnetic interactions. This dual gauge field can be related to $\vec{A}_\mu$ via the usual dual field strength $\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu + [\tilde{A}_\mu, \tilde{A}_\nu]$, which is defined as $\tilde{F}_{\mu\nu} = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. The relativistic Lagrangian for the i^{th} wall interacting with the gauge, dual gauge, and Higgs fields $(\Phi, A_\mu, \tilde{A}_\mu)$ of the $n-1$ remaining subwalls and background is

$$L_i = -i\mathrm{Tr}\Bigg[4\pi\Phi\, g_i\sqrt{1+q_i^2}\sqrt{1-\vec{V}_i^2} -4\pi q_i g_i A^0 + 4\pi q_i g_i \vec{V}_i \cdot \vec{A} - 4\pi g_i \tilde{A}^0 + 4\pi g_i \vec{V}_i \cdot \vec{\tilde{A}}\Bigg], \tag{3.1}$$

where the three-space velocity is $\vec{V}_i = \dot{\vec{x}}_i$ and we use the dotted time-derivative notation $\dot{x} = \frac{dx}{dt}$. The magnetic, electric, and scalar charges of the i^{th} subwall are interpreted as $(g_i, q_i, g_i\sqrt{1+q_i^2})$ respectively, where this form of the scalar charge follows from the BPS conditions under which the static forces cancel for well-separated subwalls [13]. Note that the electric charges q_i are momenta associated with the phase degrees of freedom τ_i for subwalls. The electric charges of the subwalls are subject to net electric charge conservation, and individual electric charge conservation when the subwalls are well-separated as they are here. While the magnetic charges may differ between factors of $U(1)$ the electric charges, which are momenta, do not.

Recall that we allow gauge field holonomies to have non-zero, spatially uniform, linearly time-dependent values between the walls. Define the linearly time-dependent terms in the asymptotic holonomies (phases) $\vec{a}(t) = \mathrm{sgn}(x)(0, a_\varphi, a_\theta)(t)$ associated with each of the periodic spatial directions, and their dual vector $\vec{\tilde{a}}(x)$ such that $\dot{\vec{a}} = \vec{\nabla} \times \vec{\tilde{a}}$. The effect of the phase velocities $\dot{\vec{a}}$ is equivalent to that of the transverse spatial velocities $\dot{\varphi}_i$ and $\dot{\theta}_i$: $\vec{V}_i = (\dot{x}_i, -\dot{a}_{\theta i}, \dot{a}_{\varphi i})$. For later use, define two dual functions $u(x)$ and $\vec{w}(x)$ such that $\vec{\nabla} u = \vec{\nabla} \times \vec{w}$.

$$
\begin{aligned}
\vec{w}(x) &= \tfrac{1}{2}\mathrm{sgn}(x)\left[-\theta\hat{\varphi} + \varphi\hat{\theta}\right], & \vec{a}(t) &= \mathrm{sgn}(x)(a(t)\hat{\varphi} + b(t)\hat{\theta}), \\
u(x) &= |x|, & \vec{\tilde{a}}(x) &= \mathrm{sgn}(x)[-\dot{a}_\theta\varphi + \dot{a}_\varphi\theta]\hat{x}.
\end{aligned} \tag{3.2}
$$

Note that $u(x) = u(-x)$, $\vec{w}(x) = -\vec{w}(x)$ and $\vec{a}(x) = \vec{a}(-x)$.

In addition to fields generated by subwalls, which we will write next, we include static field backgrounds for each factor of $U(1)$. For convenience, we split these backgrounds into constant terms $(-v^\alpha, d^\alpha_\mu, \tilde{d}^\alpha_\mu)$ and a background linear Higgs Φ_0 with the associated linear gauge fields $(A_{\mu,0}, \tilde{A}_{\mu,0})$ required by BPS conditions:

$$
\begin{aligned}
(-v, d_\mu, \tilde{d}_\mu) &= (-v^\alpha H_\alpha, d^\alpha_\mu H_\alpha, \tilde{d}^\alpha_\mu H_\alpha), & \Phi^\alpha_0(x) &= -g^\alpha_0 x, \\
A^{0,\alpha}_0(x) &= 0, & \vec{A}^\alpha_0(x) &= \tfrac{g^\alpha_0}{2}\left[-\theta\hat{\varphi} + \varphi\hat{\theta}\right], \\
\tilde{A}^{0,\alpha}_0(x) &= -g^\alpha_0 x, & \vec{\tilde{A}}^\alpha_0(x) &= 0.
\end{aligned} \tag{3.3}
$$

The gauge and Higgs fields for the j^{th} wall moving with velocity $\vec{V}_j$ are Lorentz boosted versions of those for the stationary wall. We keep only terms up to quadratic in velocities and electric charge in the Lagrangian, so in the fields $\vec{A}$

and $\vec{\tilde{A}}$ discard terms which are higher order than linear in velocities. Similarly, in the scalar expressions Φ, A^0, $\tilde{A}^0$ discard terms which are higher order than quadratic in velocities. This requires approximation of the Liénard-Wiechert denominator $(\vec{x}^2 - (\vec{x} \times \vec{V})^2)^{1/2}$ as $|\vec{x}|$ since the denominator would appear in the scalar-type quantities with coefficients linear in velocity, resulting in negligible terms cubic in velocity [17, 13]. In this approximation, a subwall moving at velocity $\vec{V}_j$ with respect to the origin generates the following fields $(\Phi_j^\alpha, A_{\mu,j}^\alpha, \tilde{A}_{\mu,j}^\alpha)$.

$$
\begin{aligned}
\Phi_j^\alpha(x) &= -g_j^\alpha u(x)\sqrt{1+q_j^2}\sqrt{1-\vec{V}_j^2}, \\
A_j^{0,\alpha}(x) &= -q_j g_j u(x) + g_j^\alpha \left(\vec{w}(x)\cdot\vec{V}_j\right) - g_j^\alpha \left(\vec{a}\cdot\vec{V}_j\right), \\
\tilde{A}_j^{0,\alpha}(x) &= -g_j^\alpha u(x) - q_j g_j \left(\vec{w}(x)\cdot\vec{V}_j\right) - g_j^\alpha \left(\vec{\tilde{a}}_j(x)\cdot\vec{V}_j\right), \\
\vec{A}_j^\alpha(x) &= g_j^\alpha \vec{w}(x) - g_j^\alpha \vec{a}, \\
\vec{\tilde{A}}_j^\alpha(x) &= -q_j g_j \vec{w}(x) - g_j^\alpha \vec{\tilde{a}}_j(x) - g_j^\alpha u(x)\vec{V}_j.
\end{aligned}
\tag{3.4}
$$

The net gauge fields must respect the periodic boundary conditions on $\mathbb{R}\times T^2$ and so we require that, for a coordinate shift in one of the periodic directions, the fields be gauge-shifted under the $U(1)$ symmetry, with gauge functions given here.

$$
\begin{aligned}
\varphi \to \varphi + 2\pi, &\qquad \tau_j \to \tau_j + \pi g_j \operatorname{sgn}(x)\theta, \\
\theta \to \theta + 2\pi, &\qquad \tau_j \to \tau_j - \pi g_j \operatorname{sgn}(x)\varphi.
\end{aligned}
\tag{3.5}
$$

3.2 Two-monowall interactions

Using these fields, we may now write the Lagrangian in a convenient form and begin by doing so for two subwalls. For a pair of walls, define the following relative position, phase, and charge quantities:

$$\begin{aligned} \vec{x} &= \vec{x}_1 - \vec{x}_2, \qquad & q &= q_1 - q_2, \qquad & g^\alpha &= g_1^\alpha - g_2^\alpha, \\ \vec{a} &= \vec{a}_1 - \vec{a}_2, \qquad & \vec{\tilde{a}} &= \vec{\tilde{a}}_1 - \vec{\tilde{a}}_2, \qquad & \mathbb{G}^\alpha &= \sum_{i=1}^{2} g_i^\alpha. \end{aligned} \tag{3.6}$$

It should be noted that the last quantity here, the total magnetic charge, is a constant which can be expressed in terms of boundary conditions alone. The remaining quantities are associated with individual pairs of subwalls. Neglecting constant terms, suppressing the index α for $U(1)$ factors, the symmetrized Lagrangian for each set of $U(1)$ interactions takes the form

$$\begin{aligned} \tfrac{L}{4\pi} = \; & \tfrac{v}{2\mathbb{G}} \left[\left(\sum_{i=1}^{2} g_i \vec{V}_i \right)^2 + g_1 g_2 \vec{V}^2 \right] - \tfrac{v}{2\mathbb{G}} \left[\left(\sum_{i=1}^{2} g_i q_i \right)^2 + g_1 g_2 q^2 \right] \\ & + \sum_{i=1}^{2} q_i g_i \vec{V}_i \cdot \vec{d} - \sum_{i=1}^{2} q_i g_i d^0 + \sum_{i=1}^{2} g_i \vec{V}_i \cdot \vec{\tilde{d}} \\ & + \sum_{i=1}^{2} \tfrac{g_0 g_i x_{3,i}}{2} \left(\vec{V}_i^2 - q_i^2 \right) + \sum_{i=1}^{2} g_0 g_i q_i \left(\vec{w}(x) \cdot \vec{V}_i \right) \\ & + \left[\tfrac{g_1 g_2 u(x)}{2} (\vec{V}^2 - q^2) + g_1 g_2 q \left(\vec{w}(x) \cdot \vec{V} \right) + g_1 g_2 [\vec{\tilde{a}} - q\vec{a}] \cdot \vec{V} \right]. \end{aligned} \tag{3.7}$$

To find the full Lagrangian, we add up all N of these $U(1)$ Lagrangians. This splits into the center of mass Lagrangian and the remainder Lagrangian, $L = L_{CM} + L_{rem}$. We integrate here over the periodic coordinates φ and θ from $-\pi$ to π. Because the terms with $\vec{w}(x) \cdot \vec{V}$ and $\vec{\tilde{a}}(x) \cdot \vec{V}$ are linear in φ

and θ positions, these terms vanish after integration.[1] Here is the result after separating the center of mass and remainder components of the Lagrangian, with implicit sum over the suppressed index α:

$$\begin{aligned} \frac{L_{CM}}{4\pi} &= \frac{v}{2\mathbb{G}}\left[\left(\sum_{i=1}^{2} g_i\vec{V}_i\right)^2 - \left(\sum_{i=1}^{2} g_i q_i\right)^2\right] \\ &\quad +\frac{1}{2}\left(\sum_{i=1}^{2} q_i\right)\left(\sum_{j=1}^{2} g_j\vec{V}_j\cdot\vec{d}\right) - \left(\sum_{i=1}^{2} g_i q_i d^0\right) + \left(\sum_{i=1}^{2} g_i\vec{V}_i\cdot\vec{\tilde{d}}\right), \\ \frac{L_{rem}}{4\pi} &= \frac{g_1 g_2}{2}\left(\frac{v}{\mathbb{G}} + u(x)\right)(\vec{V}^2 - q^2) + \sum_{i=1}^{2}\frac{g_0 g_i x_i}{2}\left(\vec{V}_i^2 - q_i^2\right) \\ &\quad -g_1 g_2(\vec{a}\cdot\vec{V})q + \frac{\vec{d}}{2}\cdot\left(g_1\vec{V}_1 - g_2\vec{V}_2\right)q. \end{aligned} \tag{3.8}$$

Maintaining the low-velocity approximation, the Lagrangian is purely kinetic since q behaves as a velocity. We now apply the fixed asymptotic boundary conditions constraint, which is equivalent to fixing the sums of the three-space and periodic positions of the subwalls (i.e. fixing the center of mass, or its analog). Incidentally, there is a physical motivation for fixing the boundary conditions. Because the fields diverge as $x \to \pm\infty$, so too does the energy. Changing the boundary conditions on the fields would require infinite kinetic energy. After fixing the center of mass, the Lagrangian reduces to the remainder Lagrangian (from now on referred to simply as the Lagrangian).

Currently, the Lagrangian is written in terms of the x positions of the subwalls, their phases $\vec{a}$, and their electric charges. The electric charge is not itself a velocity but is a momentum conjugate to a periodic direction τ. A Legendre

[1] Altering the integration bounds of φ and θ yields different but physically equivalent forms of the Lagrangian.

transform, changing coordinates from q to $\dot{\tau}$, produces the Lagrangian written explicitly in terms of subwall quantities $(x_i, a_{\varphi i}, a_{\theta i}, \tau^i)$.

$$L' = L + q_i \dot{\tau}^i. \tag{3.9}$$

After implementing the Legendre transform, we write the metric in Lee-Weinberg-Yi form [14] in terms of absolute rather than relative coordinates:

$$\boxed{\begin{aligned} \frac{ds^2}{4\pi} = & \ \frac{1}{2} U_{ij} d\vec{x}^i \cdot d\vec{x}^j \\ & + \frac{1}{2} \left[U^{-1}\right]_{ij} \left[d\tau^i + \vec{W}^{ik} \cdot d\vec{x}_k\right] \left[d\tau^j + \vec{W}^{jl} \cdot d\vec{x}_l\right] \end{aligned}} \tag{3.10}$$

with the following tensors defined for two subwalls:

$$\begin{aligned} U_{ii} &= \sum_{\alpha=1}^{N} \left[g_\alpha^0 g_{i\alpha} x_i + \sum_{\substack{j=1 \\ j\neq i}}^{2} g_{i\alpha} g_{j\alpha} \left(\frac{v_\alpha}{\mathbb{G}_a} + |x_i - x_j| \right) \right], \\ U_{ij} &= -\sum_{\alpha=1}^{N} g_{i\alpha} g_{j\alpha} \left(\frac{v_\alpha}{\mathbb{G}_\alpha} + |x_i - x_j| \right), \\ \vec{W}^{ii} &= \sum_{\alpha=1}^{N} \left[\frac{\vec{d}_\alpha}{2} g_\alpha^i - \sum_{\substack{j=1 \\ j\neq i}}^{2} g_\alpha^i g_\alpha^j (\vec{a}^i - \vec{a}^j) \right], \\ \vec{W}^{ij} &= -\sum_{\alpha=1}^{N} \left[\frac{\vec{d}_\alpha}{2} g_\alpha^j + g_\alpha^i g_\alpha^j (\vec{a}^i - \vec{a}^j) \right]. \end{aligned} \tag{3.11}$$

where the first components of the following vectors vanish $d_{1\alpha} = W_{1\alpha}^{ii} = W_{1\alpha}^{ij} = 0$ and the three-space differential is expressed $d\vec{x} = (dx, -db, da)$. The index $\alpha = 1, ..., N$ runs over the factors of $U(1)$. This metric retains the $U(1)$ symmetries, and symmetry under the $SL(2, \mathbb{Z})$ action on the φ and θ phases (a_φ, a_θ).

3.3 Multi-monowall interactions and moduli relations

In this section we first establish linear relations between the four types of monowall coordinates $(x_i, \varphi_i, \theta_i, \tau_i)$ and the four types of moduli $(R_\rho, \Psi_\rho, \Theta_\rho, T_\rho)$ for widely separated monowalls, furthering the arguments made in section 2.3, in equation (2.12). We then extend the arguments made in section 3.2 from two-monowall interactions to n monowall interactions.

3.3.1 Subwall coordinates are linear in the moduli with the same coefficients

In this brief section, we establish that the four types of subwall coordinates are linear in the four types of monowall moduli with the same coefficients (and define those coefficients), i.e. that $(dx_i, d\varphi_i, d\theta_i, dq_i) = \sum_{\rho=1}^{\sigma} c_i^\rho (dR_\rho, d\Psi_\rho, d\Theta_\rho, dQ_\rho)$. Through the Legendre transform in equation (3.9) connecting the electric charge q_i to the phase coordinate τ_i we see that the relation for the phase coordinates differs: $T^\rho = \sum_{i=1}^{n} c_i^\rho \tau_i$.

As usual, the following discussion applies to widely separated doubly periodic monopoles whose fields far from any monopole are abelian, i.e. the $U(N)$ symmetry is broken to $U(1)^N$. We can introduce a small electric charge by introducing a small component for A_0 which are zero in the case of stationary monowalls. Then up to small corrections, the fields take the following linear forms in the regions between monowalls and far to the left or right of the monowalls:

$$\begin{aligned} \Phi^I &= -(\mathbb{G}^I x + v^I), & A^I_\varphi &= 0, \\ A^I_0 &= -\left(\mathbb{Q}^I x + d^I_q\right), & A^I_\theta &= \mathbb{G}^I \varphi + d^I_\varphi, \end{aligned} \tag{3.12}$$

where I indexes the $n+1$ regions between the walls or in the $x \to \pm\infty$ regions. The static boundary conditions restrict v^1 and v^{n+1} to constants, and the center of mass position $x_{CM} = \frac{1}{\mathbb{G}} \sum_{i=1}^{n} g_i x_i$ is fixed, where $\mathbb{G} = \sum_{i=1}^{n} g_i$ is the total magnetic charge. The index α for the $U(1)$ factors is suppressed except where necessary.

At the location of wall i, the Higgs eigenvalues to which the wall couples (i.e. $g_i^\alpha \neq 0$) must coincide. See Figure 3.1. This yields a set of linear relationships

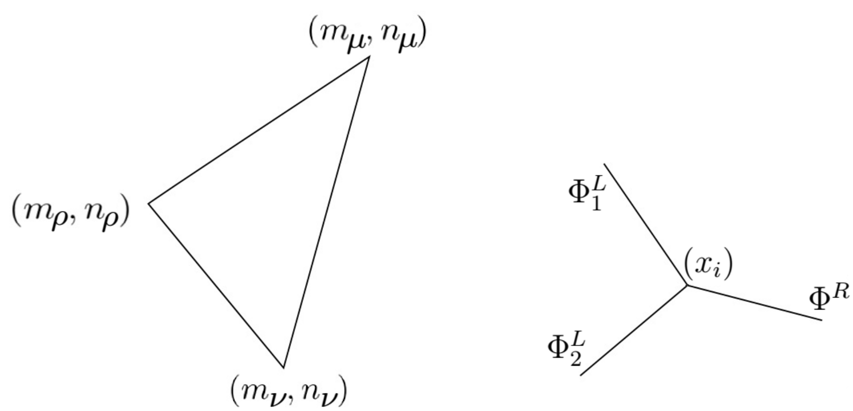

Figure 3.1: Subtriangle (left) for subwall a and the associated Higgs eigenvalues (right) near its position x_a.

between the individual wall charges and the Higgs slopes $\mathbb{G}^I$, between the wall positions x_i and the v^I terms, between the wall coordinatess φ_i and the d^I_φ terms, and so on. There are two constraints for each type of field in equation

(3.12) associated with each subtriangle. The Higgs constraint is

$$\Phi_1^L(x_i) = \Phi^R(x_i), \qquad \Phi_2^L(x_i) = \Phi^R(x_i),$$

implying

$$-G_1^L x_i - v_1^L = -G^R x_i - v^R, \qquad -G_2^L x_i - v_1^L = -G^R x_i - v^R. \tag{3.13}$$

Because the center of mass position is fixed, reducing the number of degrees of freedom for the wall positions from n to $n-1$, this represents $2(n-1)$ constraints. This reduces the degrees of freedom for the Higgs fields from the number of distinct v_I plus $(n-1)$ to the number of distinct v_I minus $n-1$. This gives the correct number of degrees of freedom for the moduli.[2]

We use these relations in equation (3.13) to express the individual wall positions in terms of $\mathbb{G}^I$ and v^I along each $U(1)$ factor, as well as relate the fixed $\mathbb{G}^I$ to the wall charges:

$$\begin{aligned} x_i &= -\tfrac{1}{2g_{i,1}}\left(v^R - v_1^L\right), & x_i, &= -\tfrac{1}{2g_{i,2}}\left(v^R - v_2^L\right), \\ g_{i,1} &= \tfrac{\mathbb{G}^R - \mathbb{G}_1^L}{2}, & g_{i,2} &= \tfrac{\mathbb{G}^R - \mathbb{G}_2^L}{2}. \end{aligned} \tag{3.14}$$

In identical fashion, we obtain the following linear relations for the φ, θ coordinates. The relations for θ are derived in a different gauge in which $A_\theta = 0$

[2]The Euler characteristic for the triangulated convex polygon is (Vertices) − (Edges) + (Subtriangles)+1 = 2. The number σ' of perimeter vertices is equal to the number of perimeter edges of the polygon, leaving (Internal Vertices) = (Internal Edges)−[(Subtriangles)−1]. Since the number of internal edges is equal to the number of distinct v_I, this relation tells us that the number of degrees of freedom for the monowall Higgs fields is equal to the number of internal points in the Newton polygon.

and A_φ depends linearly on θ.

$$\begin{aligned} \varphi_i &= -\tfrac{1}{2g_{i,1}}\left(d_\varphi^R - d_{\varphi,1}^L\right), & \varphi_i &= -\tfrac{1}{2g_{i,2}}\left(d_\varphi^R - d_{\varphi,2}^L\right), \\ \theta_i &= -\tfrac{1}{2g_{i,1}}\left(d_\theta^R - d_{\theta,1}^L\right), & \theta_i &= -\tfrac{1}{2g_{i,2}}\left(d_\theta^R - d_{\theta,2}^L\right). \end{aligned} \tag{3.15}$$

The equivalent operation for constraining the electric charges q_i of the individual walls requires expressing the field $A_0^I = -\sum_{i=1}^{n} g_i q_i |x - x_i| - d_q$ for each region explicitly in terms of individual wall contributions and comparing to the expression for A_0^I in equation (3.12): $\mathbb{Q}^I x + d_q^I = \sum_{i=1}^{n} g_i q_i |x - x_i| + d_q$. This establishes a relationship between q_i and the pairs of $\mathbb{Q}^I$ associated with the subtriangle i, as well as between the d_α^I and q_i and x_i:.

$$\begin{aligned} q_i &= \tfrac{1}{2g_{i,1}}\left(\mathbb{Q}^R - \mathbb{Q}_1^L\right), & q_i &= \tfrac{1}{2g_{i,2}}\left(\mathbb{Q}^R - \mathbb{Q}_2^L\right), \\ q_i x_i &= -\tfrac{1}{2g_{i,1}}\left(d_q^R - d_{q,1}^L\right), & q_i x_i &= -\tfrac{1}{2g_{i,2}}\left(d_q^R - d_{q,2}^L\right). \end{aligned} \tag{3.16}$$

Equivalently the second row of the above equation can be written more generally as a relation for each d_q^I for each $U(1)$ factor α:

$$d_{q,\alpha}^I = d_q - \sum_{i=1}^{I} g_i q_i x_i + \sum_{i=I+1}^{n} g_i q_i x_i. \tag{3.17}$$

This establishes that the d_q^I are not independent variables but are fully determined by the v^I through x_i and the $\mathbb{Q}^I$ through q_i such that A_0 is continuous at the wall positions x_i. In these relations, the following are further restricted to constants by the boundary and center of mass constraints: d_φ^1, d_φ^{n+1}, d_θ^1, d_θ^{n+1}, $\mathbb{Q}^1$, $\mathbb{Q}^{n+1}$, $\varphi_{CM} = \frac{1}{\mathbb{G}} \sum_{i=1}^{n} g_i \varphi_i$, $\theta_{CM} = \frac{1}{\mathbb{G}} \sum_{i=1}^{n} g_i \theta_i$ and $\mathbb{Q}_\alpha = \sum_{i=1}^{n} g_{i,\alpha} q_i$ where $\mathbb{Q}$ is the total electric charge along the α $U(1)$ factor of the constituent monowalls.

We can linearly relate these moduli to the choice of moduli in which each set of four moduli $(R_\rho, \Psi_\rho, \Theta_\rho, Q_\rho)$ are associated with a vertex in the triangulation of the Newton polynomial for a monowall. Let ρ index the subtriangle vertices (m_ρ, n_ρ) for the subtriangle associated with the i^{th} wall. As discussed in Chapter 2, the coordinates are linearly related to these moduli by

$$(x_i, \varphi_i, \theta_i, q_i) = c_i^\rho (R_\rho, \Psi_\rho, \Theta_\rho, Q_\rho) + (x_{0,i}, \varphi_{0,i}, \theta_{0,i}, q_{0,i}), \tag{3.18}$$

with a sum over $\rho = 1, 2, ..., \sigma$ where σ is the number of internal points in the Newton polygon. The constants $x_{0,i}$ etc are the coordinates of the constituent monopoles when all of the corresponding moduli are zero, $R_\rho = 0$ etc. We further introduce the phase moduli T^ρ which is the coordinate dual to the electric charge (which acts as a momentum) moduli Q_ρ. These phase moduli are linearly related to the phases τ^i of the subwalls by $dT^\rho = c_i^\rho d\tau_i$ such that the following holds with summation over i and ρ: $q_i \dot{\tau}^i = Q_\rho \dot{T}^\rho$.

These constants c_i^ρ can be related to the coordinates of the subtriangle vertices (m_ρ, n_ρ) for the subtriangle associated with the i^{th} wall and can be read off from Equation (2.12) in Section 2.3. The associated subtriangle in the Newton polygon triangulation has vertices (ρ, μ, ν) arranged in clockwise sequence. With no sum over ρ, μ, or ν, the coefficients are:

$$c_i^\rho = \frac{n_{\mu\nu}}{\begin{vmatrix} m_{\nu\rho} & n_{\nu\rho} \\ m_{\mu\rho} & n_{\mu\rho} \end{vmatrix}}, \qquad c_i^\mu = \frac{n_{\nu\rho}}{\begin{vmatrix} m_{\nu\rho} & n_{\nu\rho} \\ m_{\mu\rho} & n_{\mu\rho} \end{vmatrix}}, \qquad c_i^\nu = \frac{n_{\rho\mu}}{\begin{vmatrix} m_{\nu\rho} & n_{\nu\rho} \\ m_{\mu\rho} & n_{\mu\rho} \end{vmatrix}}, \tag{3.19}$$

where the determinant in the denominator is equal to twice the area of the associated subtriangle.

We have established in this subsection that the four types of coordinates of the monopoles share the same n constants of proportionality, i.e. $(dx_i, d\varphi_i, d\theta_i, dq_i) = c_i^\rho (dR_\rho, d\Psi_\rho, d\Theta_\rho, dQ_\rho)$ with sum over $\rho = 1, 2, ..., \sigma$ and

$dT^{\lambda} = c^{\lambda}_{j} d\tau^{j}$ with sum over $j = 1, 2, ..., n$ for the phases dual to the electric charge (which behaves as a momentum).

3.3.2 Multi-monowall interactions

The metric (3.10) holds for the extension to n subwalls. The n-subwall tensors are

$$\begin{aligned}
U_{ii} &= \sum_{\alpha=1}^{N} \left[g^0_{\alpha} g_{i\alpha} x_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} g_{i\alpha} g_{j\alpha} \left(\frac{v_\alpha}{\mathbb{G}_\alpha} + |x_i - x_j| \right) \right], \\
U_{ij} &= -\sum_{\alpha=1}^{N} g_{i\alpha} g_{j\alpha} \left(\frac{v_\alpha}{\mathbb{G}_\alpha} + |x_i - x_j| \right), \\
\vec{W}^{ii} &= \sum_{\alpha=1}^{N} \left[\frac{\vec{d}_\alpha}{n} g^i_\alpha - \sum_{\substack{j=1 \\ j \neq i}}^{n} g^i_\alpha g^j_\alpha (\vec{a}^i - \vec{a}^j) \right], \\
\vec{W}^{ij} &= \sum_{\alpha=1}^{N} \left[-\frac{\vec{d}_\alpha}{n} g^j_\alpha + g^i_\alpha g^j_\alpha (\vec{a}^i - \vec{a}^j) \right].
\end{aligned} \tag{3.20}$$

where $d^1_\alpha = W^{ii}_{1,\alpha} = W^{ij}_{1,\alpha} = 0$ and $d\vec{x} = (dx, -db, da)$. The index $\alpha = 1, ..., N$ again runs over the factors of $U(1)$. Note that we have chosen the U_{ij} functions to have a $1/\mathbb{G}$ term. This does not present an issue in cases where $\mathbb{G} = 0$, which can be seen by making the substitution $\sum_{i>j}^{n,n} g_i g_j x^2_{ij} = \mathbb{G} \left[\sum_{i=0}^{n} g_i x^2_i \right] - (\mathbb{G} x_{CM})^2$ and using that the position of the monowall center of mass is fixed and at the origin. We have made the choice to include the $1/\mathbb{G}$ term to mimic the Yi-Weinberg-Lee form of metric seen in [17, 16] for periodic and doubly periodic monopoles.

When the boundary conditions of [22] are satisfied, the number of independent moduli reduces from $4n$ to 4σ, where σ is the number of internal points in the Newton polygon. We use the variable z to parameterize the values of x and Φ in the (x, y)-plane corresponding to each of the n amoeba junctions. This is done by finding the lines in $\mathbb{R}^3$ along which two adjacent three-dimensional amoeba wedges intersect and using these to define subwall positions for each value of z. We will from here forward refer to the two-dimensional amoeba, which is the amoeba for the projection of the three-dimensional Newton polytope onto the $(m, n, 0)$ lattice. For each value of z, there is a different two-dimensional amoeba. Recall that each subwall corresponds to a face of the three-dimensional polytope and therefore each subwall now corresponds to an edge of this two-dimensional polygon. The relationships between the moduli R_ρ and (x, Φ) at the junctions are linear, for a given value of the parameter $z = \frac{1}{h}$:

$$dx_i = \sum_{\rho=1}^{\sigma} c_i^\rho \, dR_\rho, \qquad\qquad d\Phi_i = \tilde{c}_i^\rho \, dR_\rho. \tag{3.21}$$

We find the coefficients $(c_i^\rho, \tilde{c}_i^\rho)$ by direct examination of the amoeba spine i.e. we express them in terms of the points in the Newton polygon, as in equation (2.12). Here we will elaborate on the process of computing these coefficients. Let the lattice coordinates of the ν^{th} vertex in the Newton polygon be written $(m_\nu, n_\nu) \in \mathbb{Z}^2$, with $\nu \in \mathbb{Z}/n$ running over the $\sigma + \sigma'$ vertices of the Newton polygon. For large moduli, each sub-triangle in the Newton polygon triangulation represents a subwall and, as we will show, can be used to determine its motion. We choose a triangulation such that each sub-triangle contains at least one internal point (m_ρ, n_ρ), and label the remaining two vertices (m_μ, n_μ) and (m_ν, n_ν) for the ρ^{th} sub-triangle, and (ρ, μ, ν) label the three vertices of the i^{th} subtriangle in clockwise order.

Under the Log map, the set of solutions to the polynomials for each of this sub-triangle form wedges which intersect along a line. This line represents the set of positions the associated subwall may occupy in the $x - y$ plane for all values of the parameter $z \in \mathbb{R}^+$. The set of common solutions can be found by simultaneously solving the polynomials for the subtriangle's edges. This is equivalent to solving the following linear equation in which each line on the right corresponds to a plane $mx + ny + R = z$ associated with each vertex of the subtriangle and with each associated monomial $e^{i\Psi}e^{\frac{R}{h}}s^m t^n$. The equation describes the location $(x, y, z)_{\rho\mu\nu} = (x_i, y_i, z_i)$ at which the three planes intersect. The projection of this point onto the (x, y)-plane is the position of the i^{th} two-dimensional amoeba junction. We define a reference point (x_i^0, y_i^0) by solving this equation for the case $z = 0$. Then, we can relate x^i and y^i to each modulus $\frac{R_\rho}{h}$ in the following way:

$$
\begin{aligned}
-\begin{pmatrix} dR_\rho \\ dR_\mu \\ dR_\nu \end{pmatrix} &= \begin{pmatrix} m_\rho & n_\rho & -1 \\ m_\mu & n_\nu & -1 \\ m_\nu & n_\nu & -1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}, \\
dx_i &= \tfrac{n_\mu - n_\nu}{det^i} dR_\rho + \tfrac{n_\nu - n_\rho}{det^i} dR_\mu + \tfrac{n_\rho - n_\mu}{det^i} dR_\nu, \\
dy_i &= -\tfrac{m_\mu - m_\nu}{det^i} dR_\rho - \tfrac{m_\nu - m_\rho}{det^i} dR_\mu - \tfrac{m_\rho - m_\mu}{det^i} dR_\nu,
\end{aligned}
\tag{3.22}
$$

where $\det^i = (n_\nu - n_\rho)(m_\mu - m_\rho) - (n_\mu - n_\rho)(m_\nu - m_\rho)$. The monowall has σ independent non-compact moduli, one for each internal point of the Newton polygon, and the junction positions x^i asymptotically depend linearly on the three non-compact moduli R corresponding to the vertices of the i^{th}

subtriangle:

$$x_i - x_{0,i} = \sum_{\lambda=1}^{\sigma} c_i^\lambda R_\lambda, \qquad \Phi_i - \Phi_{0,i} = \sum_{\lambda=1}^{\sigma} \tilde{c}_i^\lambda R_\lambda,$$

$$c_i^\rho = \frac{n_\mu - n_\nu}{\det_i}, \qquad c_i^\mu = \frac{n_\nu - n_\rho}{\det_i}, \qquad c_i^\nu = \frac{n_\rho - n_\mu}{det^i}, \tag{3.23}$$

$$\tilde{c}_i^\rho = -\frac{m_\mu - m_\nu}{\det_i}, \qquad \tilde{c}_i^\mu = -\frac{m_\nu - m_\rho}{\det_i}, \qquad \tilde{c}_i^\nu = -\frac{m_\rho - m_\mu}{det^i},$$

and the remaining coefficients are zero $c_i^\lambda = 0$ for $\lambda \neq \rho, \mu, \nu$. The same arguments and linear relationships extend to the remaining types of moduli and subwall coordinates, as we will discuss later in section 4.3.1.

The i^{th} subwall has a set of N magnetic charges g_α^i which are determined by the Newton polygon and its triangulation. The charge of a subwall is determined by the difference in slope of the Higgs eigenvalues (which correspond to non-vertical amoeba lines) to either side of the subwall. The magnetic field due to a single, stationary subwall is $\vec{B}_i(x) = -\vec{\nabla}\Phi_i = g_i \, \mathrm{sgn}(x - x^i)$. External amoeba lines have slopes $\left(-\frac{m_\rho - m_{\tilde{\rho}}}{n_\rho - n_{\tilde{\rho}}}\right)$ normal to the corresponding Newton polygon edge and the slopes are triangulation-independent. Internal amoeba lines have slopes normal to lines of triangulation and are therefore triangulation-dependent. A subwall which has no effect on the α^{th} eigenvalue has zero charge $g_\alpha^i = 0$ with respect to the α^{th} factor of $U(1)$. A subwall which alters the slope of the α^{th} Higgs eigenvalue $\Phi_\alpha(x)$ has charge g_α^i equal to half the change in slope. Through the amoeba, the Newton polygon and its triangulation yield precise information about all Higgs eigenvalues $\{\Phi_\alpha(x)\}$. The lattice height N of the Newton polygon is the number of $U(1)$ factors from the maximally broken $U(N)$, and each horizontal strip of the lattice is associated with a $U(1)$ factor (see Figure (3.2)). A subwall whose sub-triangle has lattice height H and occupies H horizontal strips is magnetically charged with respect to each of those H factors of $U(1)$. A Higgs eigenvalue with slope k/ℓ in some region actually represents ℓ degenerate Higgs eigenvalues. To see this illustrated, see Figure (3.2). For example, in Figure (3.2), the charges for subwall 1 are $g_1^1 = -\frac{1}{4}$ and $g_2^1 = \frac{1}{4}$. For contrast, subwall 2 has charges $g_1^2 = 0$ and $g_2^2 = -\frac{1}{2}$.

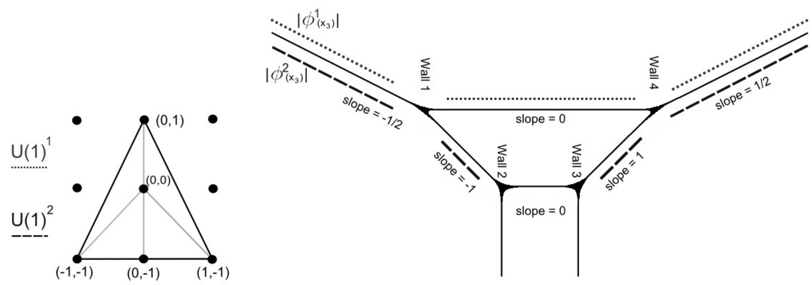

Figure 3.2: *Left:* Newton polygon (black lines) with a regular triangulation (grey lines). *Right:* Sketch of the amoeba for the associated $U(2)$ monowall, with the two Higgs eigenvalues shown in dotted and dashed lines over a range in x.

Now that we have related the subwall coordinates to the moduli, we apply this the metric we derived on the moduli space of monowalls. Define the vector $d\vec{X} = (dR, -d\Theta, d\Psi)$. In terms of the four types of moduli (R, Ψ, Θ, T) which correspond respectively to $(x, a_\theta, -a_\varphi, \tau)$, the metric may be written in the Lee-Weinberg-Yi form (to restate equations (3.10), (3.20) for completeness):

$$\boxed{\begin{aligned}\frac{ds^2}{4\pi} = &\Bigg[\ \frac{1}{2}U_{ij}d\vec{x}^i \cdot d\vec{x}^j + \\ &\frac{1}{2}\left[U^{-1}\right]_{ij}\left[d\tau^i + \vec{W}^{ik}\cdot d\vec{x}_k\right]\left[d\tau^j + \vec{W}^{jl}\cdot d\vec{x}_l\right]\Bigg]\end{aligned}} \tag{3.24}$$

with the following tensors defined:

$$
\begin{aligned}
U_{ii} &= \sum_{\alpha=1}^{N}\left[g_{\alpha}^{0} g_{i\alpha} x_i + \sum_{\substack{j=1\\ j\neq i}}^{n} g_{i\alpha} g_{j\alpha}\left(\tfrac{v_\alpha}{\mathbb{G}_\alpha} + |x_i - x_j|\right)\right],\\
U_{ij} &= -\sum_{\alpha=1}^{N} g_{i\alpha} g_{j\alpha}\left(\tfrac{v_\alpha}{\mathbb{G}_\alpha} + |x_i - x_j|\right),\\
\vec{W}^{ii} &= \sum_{\alpha=1}^{N}\left[\tfrac{\vec{d}_\alpha}{n} g_\alpha^i - \sum_{\substack{j=1\\ j\neq i}}^{n} g_\alpha^i g_\alpha^j (\vec{a}^i - \vec{a}^j)\right],\\
\vec{W}^{ij} &= \sum_{\alpha=1}^{N}\left[-\tfrac{\vec{d}_\alpha}{n} g_\alpha^j + g_\alpha^i g_\alpha^j (\vec{a}^i - \vec{a}^j)\right],
\end{aligned} \tag{3.25}
$$

$$
\vec{x}_i = \sum_{\rho=1}^{\sigma} c_i^\rho \vec{X}_\rho + \vec{x}_{i,0}, \qquad (a_{\varphi,i}, a_{\theta,i}) = \sum_{\rho=1}^{\sigma} c_i^\rho (-\Theta_\rho, \Psi_\rho),
$$

$$
q_i = \sum_{\rho=1}^{\sigma} c_i^\rho Q_\rho, \qquad T^\rho = \sum_{i=1}^{n} c_i^\rho \tau^i.
$$

Here, α indexes the N $U(1)$ factors. The asymptotic parameters of the metric and the monowall itself, the constant background Higgs, constant background gauge holonomies, the x and phase centers of mass, the total magnetic charge, and the slope of the linear background Higgs $\{(v_\alpha, \vec{d}_\alpha, (x)_\alpha^{cm}, \vec{a}_\alpha^{cm}, \mathbb{G}_\alpha, g_\alpha^0)\}$ relate to the boundary conditions $\{(\mathbb{G}_\alpha^\pm, v_\alpha^\pm, \vec{d}_\alpha^\pm)\}$, which are the left and right charges, Higgs background and holonomy background. The relations are as follows:

$$
\begin{aligned}
g_\alpha^0 &= \tfrac{1}{2}\left(\mathbb{G}_\alpha^+ + \mathbb{G}_\alpha^-\right), & \mathbb{G}_\alpha &= \tfrac{1}{2}\left(\mathbb{G}_\alpha^+ - \mathbb{G}_\alpha^-\right),\\
v_\alpha &= \tfrac{1}{2}\left(v_\alpha^+ + v_\alpha^-\right), & x_\alpha^{cm} &= -\tfrac{1}{2\mathbb{G}_\alpha}\left(v_\alpha^+ - v_\alpha^-\right),\\
\vec{d}_\alpha &= \tfrac{1}{2}\left(d_\alpha^+ + d_\alpha^-\right), & \vec{a}_\alpha^{cm} &= -\tfrac{1}{2\mathbb{G}_\alpha}\left(d_\alpha^+ - d_\alpha^-\right).
\end{aligned} \tag{3.26}
$$

In summary, we have in this section approximated the asymptotic BPS monowall moduli space metric by modeling it as abelian interactions of well-separated sub-monowalls in a linear Higgs background. Rather than the general $4n$ moduli, boundary conditions reduce the number of moduli to four times the number of internal points in the Newton polygon 4σ. We gave the explicit example for one internal point, in which the BPS monowall has but four moduli (R, Ψ, Θ, T). Each regular Newton polygon triangulation [57] yields a set of subwall magnetic charges and therefore each regular triangulation corresponds to a different sector of the moduli space. With the parameters listed above, the monowall asymptotics in each of the N factors of $U(1)$ are determined by the Higgs spectral curve and Newton polygon. The metric (3.24) gives the dynamics of well-separated sub-monowalls in terms of the moduli of the monowall.

CHAPTER 4

Generalized Legendre Transform

4.1 Generalized Legendre transform background

The following discussion of the generalized Legendre transform mostly follows those presented in [45, 48]. The generalized Legendre transform is the complex generalization of the more common real Legendre transform. Given a metric explicitly in terms of real variables, the generalized Legendre transform may allow us to determine a set of complex combinations of the real variables and the Kähler potential for the metric which is a function of those complex variables. This transform relates the *generating function* F with the Kähler potential K and relates the real variables of the monopole with the complex variables of the Kähler potential. A Kähler potential for a $4n$-dimensional hyperkähler metric is a function of $2n$ complex variables which we will label $(u^i, \bar{u}^i, z^i, \bar{z}^i)$ where $i = 1, 2, ..., n$. The metric is written $\frac{ds^2}{8\pi} = K_{p_i \bar{p}_j} d\chi^{p_i} d\bar{\chi}^{\bar{p}_j}$ where $p_i = u_i$ or z_i and $\bar{p}_j = \bar{u}_j$ or $\bar{z}_j$. We introduce a function F of a set of $2n+1$ variables y_k, some of which will be identified later with $(z_i, \bar{z}_i)$ and some of which will be auxiliary. Important to this construction is a relationship between the generating function F and a family $\eta^{(2n)}$ of order-$2n$ polynomials over $\mathbb{C}P^1$, where n is the monopole charge [48].

$$\eta^{(2n)}(\zeta) = \sum_{k=0}^{2n} y_k \zeta^k, \tag{4.1}$$

where $\mathbb{C}P^1$ is represented as the Riemman sphere and ζ is the coordinate away from the south pole. The coordinate away from the north pole is ξ and can be written in terms ζ on the region of overlap $\xi = \frac{1}{\zeta}$. Similarly, the polynomial $\eta_S^{(2n)}$ is valid on the southern hemisphere and can be written in the region of overlap as $\eta_S^{(2n)}(\xi) = \frac{\eta_N^{(2n)}(\zeta)}{\zeta^{2n}}$. This family of polynomials is chosen to obey the condition of invariance under the antipodal map, i.e.

$$\eta^{(2n)}(\zeta) = (-1)^n \zeta^{2n} \, \overline{\eta^{(2n)}\left(-\frac{1}{\bar{\zeta}}\right)}. \tag{4.2}$$

This condition has consequences for the coefficients y_k: $\bar{y}_k = (-1)^{k+n} y_{2n-k}$. The generating function F is related to the polynomials $\eta^{(2n)}$ by

$$F(y_i) = \frac{1}{2\pi i} \oint G\left(\zeta, \eta^{(2n)}(\zeta)\right) \frac{d\zeta}{\zeta^2}, \tag{4.3}$$

for some holomorphic function G of $\eta^{(2n)}$ and ζ. We will return to the topic of the choosing of this function G. This along with equation (4.2) implies that second derivatives of the generating function satisfy $F_{y_k y_\ell} = F_{\eta\eta} \eta_{y_k} \eta_{y_\ell}$ and therefore

$$F_{y_k y_\ell} - F_{y_{k+a} y_{\ell-a}} = 0, \tag{4.4}$$

for $k, \ell = 0, 1, 2, ..., 2n$ and values of $a = 0, 1, 2, ...$ such that $k + a$ and $\ell - a$ are within the interval $[0, 2n]$.

We make identifications $z^i = y_0^i$ and $v^i = y_1^i$ and label the remainder w_i where $i = 1, 2, ..., n$ indexes the n polynomials $\eta^{(2n)}$ and y_0^i is the zeroth coefficient in $\eta^{(2n)}$, y_1^i is the coefficient of ζ in $\eta^{(2n)}$, etc. The Kähler potential K is a function of $(z_i, \bar{z}_i, u_i, \bar{u}_i)$ where $\frac{u_i + \bar{u}_i}{2} := \frac{\partial F}{\partial v_i}$. We conduct a Legendre transform with respect to the variables $(v_i, \bar{v}_i)$.

The Kähler potential K and the generating function F are related by [48]

$$K(u^i, \bar{u}^i, z^i, \bar{z}^i) = -F(z^i, \bar{z}^i, v^i, \bar{v}^i, w^j) + \frac{u^k v_k + \bar{u}^k \bar{v}_k}{2}. \tag{4.5}$$

with a sum from $k = 1$ to n over repeated indices. The generating function is also extremized with respect to the remaining variables w^j, and related to the variables $(u^i, \bar{u}^i)$ through its first derivative

$$F_{w^j} = 0, \qquad F_{v^i} = u^i, \qquad F_{\bar{v}^i} = \bar{u}^i. \tag{4.6}$$

The complex variables for the Kähler potential are then $(z_i, \bar{z}_i, u_i, \bar{u}_i)$, and the relations $F_{v_i} = u_i$ provide implicit relationships between the variables of the Kähler potential and the remaining auxiliary variables v_i and w_i. These are used to express v_i as functions of the z and u variables on the right hand side of (4.5).

In the limit where monopoles are widely separated and interact like point objects, the polynomial η degenerates to a product of n polynomials of the form

$$\eta_i = z_i + 2x_i\zeta - \bar{z}_i\zeta^2. \tag{4.7}$$

In this case, v_i is real and we relabel it x_i, and $x_i = x_{1i}$, $z_i = x_{2i} + ix_{3i}$ where $(x_1, x_2, x_3)_i$ is the position of the i^{th} constituent subwall in three-space. The relation $F_{v_i} = u_i$ and its complex conjugate simplify to $F_{x_i} = \frac{u_i + \bar{u}_i}{2}$, and the Kähler potential simplifies accordingly. Recall that the electric charge of a BPS monopole is interpreted as a canonical momentum that is conjugate to a periodic, phase coordinate τ. The imaginary part of the complex variable $u_i = u_{1i} + i\tau_i$ is this phase variable corresponding to the i^{th} monopole charge.

In this limit, the relation (4.4) simplifies to something very like the Laplace equation for the generating function,

$$\frac{F_{x_i x_j}}{4} + F_{z_i \bar{z}_j} = 0. \tag{4.8}$$

Given this relation and the Legendre transform in equation (4.5), the coefficients $g_{p_i \bar{q}_i} = K_{p_i \bar{q}_i}$ of the metric in terms of second derivatives of the generating function are

$$\begin{aligned} K_{z_i \bar{z}_j} &= \tfrac{F_{x_i x_j}}{4} + F_{z_i x_k} \left(F_{x_\ell x_k}\right)^{-1} F_{x_\ell \bar{z}_j}, \\ K_{u_i \bar{z}_j} &= -\left(F_{x_k x_i}\right)^{-1} F_{x_k \bar{z}_j}, \qquad K_{u_i \bar{u}_j} = \frac{\left(F_{x_i x_j}\right)^{-1}}{4}. \end{aligned} \tag{4.9}$$

The metric in terms of second derivatives of the generating function F has Lee-Weinberg-Yi form. Implicitly summing over repeated indices, we write the metric

$$\begin{aligned} ds^2 = \;& \tfrac{1}{4} F_{x_i x_j} \left(dx_i dx_j + \tfrac{1}{2} dz_i d\bar{z}_j + \tfrac{1}{2} dz_j d\bar{z}_i\right) \\ & + \tfrac{F^{-1}_{x_i x_j}}{4} \left(d\tau_i + i F_{x_i z_k} dz_k - i F_{x_i \bar{z}_k} d\bar{z}_k\right) \times \\ & \left(d\tau_j + i F_{x_j z_\ell} dz_\ell - i F_{x_j \bar{z}_\ell} d\bar{z}_\ell\right). \end{aligned} \tag{4.10}$$

Given the relationships between the metric coefficients and the second derivatives of F in (4.9) and our knowledge of the form this type of hyperkähler metric takes and our treatment of the monopoles as point objects, we expect discontinuities in F at the locations of these individual monopoles. For the abelian interactions of these monopoles, the problem of solving for $F_{x_i x_j}$ given boundary conditions is a familiar one from classical electricity and magnetism. Once a solution for $F_{x_i x_j}$ has been identified, one can simply integrate it twice to determine the generating function F for a given charge distribution (and thereby determine the previously described generating integrand $G(\zeta, \eta)$, or simply calculate F directly), up to a function linear in the x_i variables and a function independent of x_i. The latter function does not enter into the metric or the implicit relation $F_{x_i} = \frac{u_i+\bar{u}_i}{2}$. If the metric coefficients in equation (4.9) are known, then the function linear in x_i can be determined up to initial positions of the monopoles [45, 48].

We now discuss application of this method to widely separated monopoles on $\mathbb{R}^3$. We will begin with the case of two such monopoles and will work in terms of relative coordinates $(x, z, \bar{z}) = (x_1 - x_2, z_1 - z_2, \bar{z}_1 - \bar{z}_2)$. We know that away from any of its singularities the second derivative F_{xx} of the generating function solves the Laplace equation, as does $F_{z\bar{z}}$. This can be seen taking by the Laplace equation for F, $F_{xx} + 4F_{z\bar{z}} = 0$ and operating on it from the left with $\partial_x^2 + 4\partial_z\partial_{\bar{z}} = \partial_1^2 + \partial_2^2 + \partial_3^2 = \nabla^2$. We also know by comparing the metric in terms of second derivatives of the generating function (4.9) to the metric for two monopoles on $\mathbb{R}^3$ (4.13) (seen later in this chapter) that F_{xx} contains a singularity when the two monopoles occupy the same location i.e. when $r = 0$, where r is the distance between the monopoles. Near $r = 0$, F_{xx} behaves like $\frac{g_1 g_2}{r}$. These behaviors imply that F_{xx} satisfies a relation using the Dirac delta function:

$$\nabla^2 F_{xx} = -4\pi g_1 g_2 \delta(\vec{r}), \tag{4.11}$$

where $\vec{r} = (x, \text{Re}z, \text{Im}z)$ is the separation vector between the monopoles. This is the Poisson equation and the right hand side of (4.11) is the source term. Its solutions are familiar from classical electricity and magnetism. Now, if the generating function is defined as a contour integral over some path γ of some function of η and ζ as in

$$F = \frac{1}{2\pi i} \oint_\gamma d\zeta G(\eta, \zeta),$$

then the G can only depend on the monopole separation through η and therefore we can write the relation as

$$F_{xx} = \frac{1}{2\pi i}\oint_\gamma d\zeta \eta_x^2 G_{\eta\eta}(\eta,\zeta) = g_1 g_2\left[v + \frac{1}{r}\right],$$

where g_i are the magnetic charges of the constituent monopoles and v is the constant component of the Higgs field. Substituting in $\eta_x = 2\zeta$ using equation (4.7), we can find an expression for $G(\eta,\zeta)$. First we solve for the component of $G_{\eta\eta}$ that corresponds to the constant v term:

$$\oint_0 d\zeta 4\zeta^2 G_{1,\eta\eta}(\eta,\zeta) = g_1 g_2 v,$$

where $G_{1,\eta\eta} = \frac{g_1 g_2 v}{4\zeta^3}$ and the integrating contour is a counterclockwise loop around the origin of the ζ plane. The component of G that corresponds to the $1/r$ term requires a different integrand and contour. To obtain the $1/r$ term, we integrate $1/\eta$ one that encloses either of the two roots $\zeta_\pm = \frac{x\mp r}{\bar{z}}$ of $\eta = z + 2x\zeta - \bar{z}\zeta^2$ [48]. To treat roots equivalently we choose a loop γ enclosing both poles but in opposite directions, which is accomplished with a figure-eight shape that runs counterclockwise around ζ_+ and clockwise around ζ_- as seen in Figure (4.1):

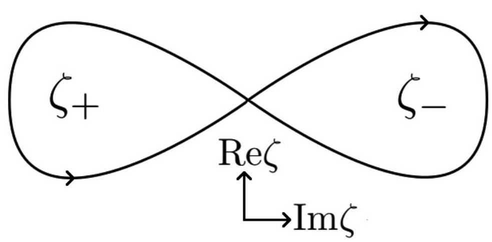

Figure 4.1: Figure eight integration contour.

$$\oint_\gamma d\zeta 4\zeta^2 G_{2,\eta\eta}(\eta,\zeta) = \frac{g_1 g_2}{r},$$

where $G_{2,\eta\eta} = \frac{g_1 g_2}{\zeta^2 \eta}$. Now that we have both components of $G_{\eta\eta}(\eta, \zeta)$, we can integrate it twice with respect to η to obtain G for two monopoles:

$$G(\eta, \zeta) = g_1 g_2 \left[\frac{\mu}{8} \frac{\eta^2}{2! \zeta^2} + \frac{\eta (\log \eta - 1)}{\zeta^2} \right].$$

This approach to computing the generating integrand $G(\eta, \zeta)$ provides a framework for determining the generating function for known metrics for widely separated monopoles on periodic and doubly periodic underlying spaces.

In the following sections, we will present two examples of applying this approach to widely separated monopoles, first for periodic magnetic monopoles, those for which one of the spatial dimensions is periodic, $\mathbb{R}^2 \times S^1$. We will next carry out the generalized Legendre transform for widely separated doubly periodic magnetic monopoles, those for which two of the spatial dimensions form a torus. These are also called monopole walls or *monowalls*. In these examples, we wish to determine the generating functions and Kähler potentials for n interacting, widely separated monopoles with distinct magnetic charges. It is helpful to first consider the simplest case of two widely separated monopoles interacting, determine their metric, generating function and Kähler potential in relative coordinates, e.g. $x = x_1 - x_2$ is the relative position in x of a monopole at x_1 interacting with a monopole at x_2. Once the metric, generating function and Kähler potential have been established for two monopoles, one can then generalize straightforwardly to n interacting monopoles. For two widely separated monopoles on $\mathbb{R}^3$, their metric is the familiar Gibbons-Hawking metric [13] with two complex variables. The metrics of two widely separated periodic and doubly periodic monopoles follow a very similar pattern to the $\mathbb{R}^3$ case and so I refer to these metrics respectively as the periodic Gibbons-Hawking and the doubly periodic Gibbons-Hawking metrics.

First let us briefly review the Gibbons-Hawking metric, which gives the monopole moduli space for two widely separated monopoles of arbitrary magnetic charges in $\mathbb{R}^3$ [13]. Let the monopoles have magnetic charges g_1 and g_2 and be positioned at $\vec{x}_1$ and $\vec{x}_2$, with relative positions $\vec{x} = \vec{x}_1 - \vec{x}_2$. We define the complex variable $z = x_2 + ix_3$ and the real variable $x = x_1$. Then the metric has Gibbons-Hawking form:

$$\frac{ds^2}{4\pi} = \frac{U}{2}(dx^2 + dzd\bar{z}) + \frac{1}{2U}\left[d\tau - iW\right]^2, \tag{4.12}$$

where

$$W = \tfrac{g_1 g_2}{2r(x+r)}\left(zd\bar{z} - \bar{z}dz\right), \qquad U = g_1 g_2\left(\tfrac{v}{\mathbb{G}} + \tfrac{1}{r}\right) \tag{4.13}$$

with $r = \sqrt{x^2 + z\bar{z}}$ is the distance between the two constituent monopoles and $\mathbb{G} = \sum_{i=1}^{2} g_i$ the total monopole charge and v the constant value of the Higgs field far from either monopole. For widely separated monopoles with $U(N)$ fields, the gauge symmetry breaks maximally to N $U(1)$ factors when the Higgs field eigenvalues are all different. The two monopoles may have different charges g_i with respect to each $U(1)$ factor. The metric for N $U(1)$ factors with such interactions between widely separated monopoles will be a sum over N contributions like that in equation (4.13). We suppress the explicit summation over $U(1)$ factors for simplicity of notation.

The metric in (4.12) appears in the Lagrangian for widely separated monopoles on $\mathbb{R}^3$ describing their kinetics and interactions [48]. As we see, this metric can be reproduced using the generalized Legendre transform with a generating function $F = F_1 + F_2$ in which F_1 corresponds to the individual monopole kinetic energies and F_2 describes the monopole interactions. As described above, the components of the generating function can be written as integrals over a generating integrand $G(\eta, \zeta)$ which is a function of ζ and η. The relative monopole positions on the underlying space are encoded as coefficients of a quadratic polynomial η in ζ. For a monopole at coordinates $(x_i, z_i) \in \mathbb{R} \times \mathbb{C} = \mathbb{R}^3$, we define the polynomial $\eta_i = z_i + 2x_i\zeta - \bar{z}\zeta^2$, the relative polynomial $\eta = \eta_1 - \eta_2$, and $\tilde{\eta} = \frac{\eta}{\zeta}$. The analogous polynomial corresponding to the constant component of the monopole fields is $\mathcal{V}(\zeta) = 2v\zeta$ and $\tilde{\mathcal{V}} = \mathcal{V}(\zeta)/\zeta$ (later we will see a more general case of this).

The kinetic component of the generating function is

$$\begin{aligned} F_1 &= \tfrac{1}{2\pi i} \oint_0 d\zeta \left(1 - \tfrac{1}{\zeta^2}\right) \left[\tfrac{g_1 g_2}{8} \tilde{\mathcal{V}} \tfrac{\tilde{\eta}^2}{2!}\right] \\ &= g_1 g_2 \tfrac{v}{2} \left(x^2 - \tfrac{z\bar{z}}{2}\right). \end{aligned} \tag{4.14}$$

The kinetic component of the generating function will take this form for periodic and doubly periodic monopoles as well, though the position of the center of mass is fixed in both of those cases as we will see.

The second piece of the generating function corresponds to the pairwise interactions between monopoles in that its derivatives show up in the monopole moduli space metric coefficients corresponding to interaction terms. When we extend the generalized Legendre transform to periodic and doubly periodic monopoles, the kinetic generating function F_1 is unchanged, while the interaction piece F_2 must reflect monopole interactions. Because these interactions are mediated by the monopole fields and the form of the fields at large distances from the monopoles is determined by the underlying space on which the monopoles move, this piece of the generating function must reflect the magnetic Coulomb interaction potential on the underlying space. For instance the second derivative of this piece of the generating function $F_{2,xx}$ must depend inversely on the monopole separations for widely separated monopoles on $\mathbb{R}^3$, it must be logarithmic in monopole separations for widely separated periodic monopoles and linear in monopole separation for widely separated doubly periodic monopoles. This is because the interaction piece, which contains singularities when the monopoles occupy the same position, must solve the Poisson equation with source terms representing the monopoles. For two widely separated monopoles on $\mathbb{R}^3$ the interaction part of the generating function is

$$\begin{aligned} F_2 &= -\tfrac{1}{2\pi i}\oint_\gamma \tfrac{d\zeta}{\zeta^2} g_1 g_2 \eta\left(\log(\eta)-1\right) \\ &= g_1 g_2\left[-r+\tfrac{x}{2}\log\left(\tfrac{r+x}{r-x}\right)\right], \end{aligned} \tag{4.15}$$

where γ is the figure-eight contour on the ζ plane seen in Figure (4.1). Applying relation (4.5) between the Kähler potential K and the generating function F, and relations (4.9) between the metric coefficients $K_{z\bar{z}}$, $K_{z\bar{u}}$ etc. and second derivatives of the generating function, we can express the metric exclusively in terms of second derivatives of F. We can eliminate explicit u_1-dependence from the metric and rearrange it to reflect its Gibbons-Hawking-like form. This is done by expanding $u = u_1 + i\tau$, completing the square in $d\tau$, and applying the implicit variable relation $dF_x = \frac{du+d\bar{u}}{2}$ in a rearranged form $du_1\left(du_1 - 2dzF_{xz} - 2d\bar{z}F_{x\bar{z}}\right) = dx^2F_{xx}^2 - \left(dzF_{xz} + d\bar{z}F_{x\bar{z}}\right)^2$. This metric in Gibbons-Hawking-like form is

$$\frac{ds^2}{8\pi} = \frac{F_{xx}}{4}\left(dzd\bar{z}+dx^2\right) + \frac{1}{4F_{xx}}\left(d\tau + idzF_{xz} - id\bar{z}F_{x\bar{z}}\right)^2. \tag{4.16}$$

The relevant second derivatives of the generating function are

$$F_{xx} = g_1 g_2\left[v + \tfrac{1}{r}\right], \qquad F_{x\bar{z}} = g_1 g_2 \tfrac{z}{2r(r+x)}. \tag{4.17}$$

Using equation (4.5) the Kähler potential for this metric is a function of the complex relative moduli $(z,\bar{z},u,\bar{u}) = (z_1-z_2, \bar{z}_1-\bar{z}_2, u_1-u_2, \bar{u}_1-\bar{u}_2)$. We write it explicitly in terms of the monopole relative position $(x, z, \bar{z})$ and include the implicit relationship between $(u, \bar{u})$ and $(x, z, \bar{z})$.

$$K(z, \bar{z}, u, \bar{u}) = g_1 g_2 \left[v \left(x^2 + \frac{z\bar{z}}{2} \right) + r \right], \tag{4.18}$$

where the implicit coordinate relation as in equation (4.6) is

$$F_x(x, z, \bar{z}) = g_1 g_2 \left[vx + \frac{1}{2} \log \left(\frac{r + x}{r - x} \right) \right]. \tag{4.19}$$

We have identified a set of complex coordinates on the monopole moduli space and a corresponding Kähler potential, from which all coefficients are determined for the metric on the moduli space. Extending this to n monopoles is intuitive.

The generalization to n monopoles promotes the derivatives of the generating function to tensors in the Legendre transform, but the result has nearly identical form. The generating function is

$$F = \sum_{\alpha}^{N} \sum_{\substack{i,j=1 \\ j<i}} g_i^\alpha g_j^\alpha \left[\frac{v^\alpha}{2} \left(x_{ij}^2 - \frac{z_{ij}\bar{z}_{ij}}{2} \right) - r_{ij} + \frac{x_{ij}}{2} \log \left(\frac{r_{ij} + x_{ij}}{r_{ij} - x_{ij}} \right) \right], \tag{4.20}$$

where $\alpha = 1, 2, ..., N$ indexes the N factors of $U(1)$. The metric has Lee-Weinberg-Yi form,

$$\begin{aligned} \tfrac{ds^2}{8\pi} = \quad & \tfrac{F_{x_i x_j}}{4} \left(\tfrac{dz_i d\bar{z}_j}{2} + \tfrac{dz_j d\bar{z}_i}{2} + dx_i dx_j \right) \\ & + \tfrac{\left(F_{x_i x_j}\right)^{-1}}{4} \left(d\tau_i + idz_k F_{x_i z_k} - id\bar{z}_k F_{x_i \bar{z}_k} \right) \\ & \times \left(d\tau_j + idz_\ell F_{x_j z_\ell} - id\bar{z}_\ell F_{x_j \bar{z}_\ell} \right). \end{aligned} \tag{4.21}$$

Including explicit summation over the N $U(1)$ factors, the relevant second derivatives of the generating function are

$$F_{x_ix_i} = \sum_{\alpha=1}^{N}\sum_{j,j\neq i}^{n} g_i^\alpha g_j^\alpha \left[v^\alpha + \tfrac{1}{r_{ij}}\right], \qquad F_{x_ix_j} = -\sum_{\alpha=1}^{N} g_i^\alpha g_j^\alpha \left[v^\alpha + \tfrac{1}{r_{ij}}\right],$$
$$F_{x_i\bar{z}_i} = \sum_{\alpha=1}^{N}\sum_{j,j\neq i}^{n} g_i^\alpha g_j^\alpha \tfrac{z_{ij}}{2r_{ij}(r_{ij}+x_{ij})}, \qquad F_{x_i\bar{z}_j} = -\sum_{\alpha=1}^{N} g_i^\alpha g_j^\alpha \tfrac{z_{ij}}{2r_{ij}(r_{ij}+x_{ij})}. \tag{4.22}$$

Using equation (4.5) we express the Kähler potential for this metric as a function of the complex moduli $(z_i, \bar{z}_i, u_i, \bar{u}_i)$, which coincide with the monopole positions for monopoles on $\mathbb{R}^3$. This is done by writing the Kähler potential explicitly in terms of $(x_i, z_i, \bar{z}_i)$ and giving the complex relationship between $(u_i, \bar{u}_i)$ and $(x_i, z_i, \bar{z}_i)$.

$$K(z_1, \bar{z}_1, u_1, \bar{u}_1, ...) = \sum_{\alpha=1}^{N}\sum_{\substack{i,j \\ j\leq i}}^{n,n} g_i^\alpha g_j^\alpha \left[v^\alpha\left(x_{ij}^2 + \frac{z_{ij}\bar{z}_{ij}}{2}\right) + r_{ij}\right]. \tag{4.23}$$

Lastly, the complex coordinate relation as in equation (4.6) is

$$F_{x_i}(x_1, z_1, \bar{z}_1, ...) = \sum_{\alpha=1}^{N}\sum_{j,j\neq i}^{n} g_i^\alpha g_j^\alpha \left[v^\alpha x_{ij} + \frac{1}{2}\log\left(\frac{r_{ij}+x_{ij}}{r_{ij}-x_{ij}}\right)\right]. \tag{4.24}$$

We now address periodic and doubly periodic monopoles. As with the $U(N)$ monopole on $\mathbb{R}^3$, we discuss the case for widely separated monopoles and give the metric, generating function and Kähler potential for a single $U(1)$ factor of the maximally broken gauge symmetry.

4.2 The Generalized Legendre transform for n widely separated periodic monopoles

Here we will use the generalized Legendre transform to derive a set of complex variables and Kähler potential for periodic Gibbons-Hawking metric, the metric for two widely separated periodic monopoles. The center of mass position is fixed. Unlike for doubly periodic monopoles, we have not determined a generating integrand $G(\eta, \zeta)$ which can produce the moduli space metric for periodic monopoles described in [17]. Instead, we determine the generating function directly and use it to obtain the Kähler potential and implicit coordinate relation for periodic monopoles. The monopoles move in a three-dimensional space $\mathbb{R}^2 \times S^1$ for which we label the coordinates $\theta \in S^1$, $x_2 \in \mathbb{R}$ and $x_3 \in \mathbb{R}$. We will choose one of the complex variables of the Kähler potential to be the complex combination of the relative positions of the monopoles along the two long directions $z = x_2 + ix_3 \in \mathbb{C}$. The periodic spatial coordinate θ will serve as auxiliary coordinate in the following. We are seeking a second complex variable u, the imaginary part of which is the relative variable τ along the coordinate which is conjugate to the electric charges of the monopoles (recall that when BPS monopoles move they develop an electric charge and that this charge is interpreted as momentum).

We can represent a periodic monopole with period 2π by a string of identical monopoles on $\mathbb{R}^3$ with uniform 2π separation. At large distances in x_2 and x_3 from this monopole, its fields mimic those of a uniformly charged wire, and so we seek solutions to the Laplace equation (4.8) which correspond in form to those of a charged wire. This solution is logarithmic and the argument of the Log function is the radial distance $\rho(z, \bar{z}) = \sqrt{z\bar{z}}$ from the wire, or

$F_{2,\theta\theta} = \log\rho = \frac{1}{2}\log(z\bar{z})$. Integrating this once with respect to θ gives us the implicit relation $F_\theta = \frac{u+\bar{u}}{2} + g(z,\bar{z})$. One can then differentiate with respect to the complex coordinate z and compare to the metric in order to determine $g(z,\bar{z})$. We find that $g(z,\bar{z}) = 0$, so the full generating function has form

$$F = g_1 g_2 \frac{v}{2\mathbb{G}}\left(\theta^2 - \frac{z\bar{z}}{2}\right) + g_1 g_2 \frac{\theta^2}{4}\log(z\bar{z}) + h(z\bar{z}), \tag{4.25}$$

where the total charge is $\mathbb{G} = \sum_{i=1}^{2} g_i$. The metric depends only on $F_{\theta\theta}$, and $F_{\theta z}$ and $F_{\bar{z}\theta}$ so the function $h(z,\bar{z})$ does not affect the metric. It must simply be chosen so that this generating function satisfies $F_{\theta\theta} + 4F_{z\bar{z}} = 0$. The full generating function for two periodic monopoles interacting at large separation is

$$F = g_1 g_2 \frac{v}{2\mathbb{G}}\left(\theta^2 - \frac{z\bar{z}}{2}\right) + g_1 g_2 \left[\frac{\theta^2}{4}\log(z\bar{z}) - \frac{z\bar{z}}{4}\left(\frac{1}{2}\log(z\bar{z}) - 1\right)\right]. \tag{4.26}$$

This generating function has second derivatives

$$F_{\theta\theta} = g_1 g_2 \left[\tfrac{v}{\mathbb{G}} + \tfrac{1}{2}\log(z\bar{z})\right], \qquad F_{\theta z} = \tfrac{\theta\bar{z}}{2z\bar{z}}. \tag{4.27}$$

This metric is invariant under $U(1)$ gauge transformations acting on the fields (Φ, A_i) of the periodic monopole. Since this models monopoles in a periodic space, we would expect the metric (but not necessarily its coefficients) to be invariant under integer shifts along the periodic direction of the form $\theta \to \theta + 2\pi m$ for $m \in \mathbb{Z}$. Such shifts are equivalent to holomorphic translations of the complex coordinate u, analogous to that mentioned in [45].

$$u \to u + \frac{4\pi m + (2\pi m)^2}{4}\log z, \tag{4.28}$$

where $\frac{u-\bar{u}}{2i} = \tau$, and the periodic monopole's electric charge is associated with translations along the τ direction. The generalization from two periodic monopoles to n periodic monopoles does not present complications since it is essentially a sum of pairwise interactions. For completeness, we present these results.

The generating function for n widely separated periodic monopoles expressed in terms of coordinates (θ_i, z_i) for each monopole is

$$\boxed{\begin{aligned} F = \sum_{\substack{i,j=1 \\ i<j}}^{n,n} g_i g_j \Bigg[& \tfrac{v}{2\mathbb{G}} \left(\theta_{ij}^2 - \tfrac{z_{ij}\bar{z}_{ij}}{2}\right) + \\ & \tfrac{\theta_{ij}^2}{4} \log\left(z_{ij}\bar{z}_{ij}\right) - \tfrac{z_{ij}\bar{z}_{ij}}{4} \left(\tfrac{1}{2}\log(z_{ij}\bar{z}_{ij}) - 1\right) \Bigg] \end{aligned}} \tag{4.29}$$

where the relative coordinates are $\theta_{ij} = \theta_i - \theta_j$ etc, the total charge is $\mathbb{G} = \sum_{i=1}^{n} g_i$, and v is the constant component of the Higgs field. As with monopoles on $\mathbb{R}^3$, we have suppressed the explicit overall summation over the $U(1)$ factors. Each $U(1)$ factor has a different set of monopole charges g_i and constant Higgs components v.

The set of necessary second derivatives of the generating function F which appear in the metric coefficients are

$$\begin{aligned} F_{\theta_i\theta_i} &= \sum_{\substack{j=1 \\ n\neq i}} g_i g_j \left[\tfrac{v}{\mathbb{G}} + \tfrac{1}{2}\log\left(z_{ij}\bar{z}_{ij}\right)\right], \\ F_{\theta_i\theta_j} &= -g_i g_j \left[\tfrac{v}{\mathbb{G}} + \tfrac{1}{2}\log\left(z_{ij}\bar{z}_{ij}\right)\right], \\ F_{\theta_i z_i} &= \sum_{\substack{j=1 \\ j\neq i}}^{n} \tfrac{\theta_{ij}}{2(z_{ij}\bar{z}_{ij})}\bar{z}_{ij}, \qquad F_{\theta_i z_j} = -\tfrac{\theta_{ij}}{2(z_{ij}\bar{z}_{ij})}\bar{z}_{ij}. \end{aligned} \tag{4.30}$$

The implicit coordinate relation between the complex coordinates $(u_i, \bar{u}_i)$ and the monopole coordinates $(\theta_i, z_i, \bar{z}_i)$ is

$$\frac{u_i + \bar{u}_i}{2} = \sum_{\substack{j=1 \\ j \neq i}}^{n} g_i g_j \theta_{ij} \left[\frac{v}{\mathbb{G}} + \frac{1}{2} \log \left(z_{ij} \bar{z}_{ij} \right) \right], \tag{4.31}$$

which follows from the definition $\frac{u_i + \bar{u}_i}{2} := \partial_{\theta_i} F$.

Shifts along the periodic direction θ of the form $\theta_k \to \theta_k + 2\pi m$ correspond to shifts in the complex coordinates

$$\begin{aligned} u_i &\to u_i - (2\pi m) g_i g_k \left[\tfrac{v}{\mathbb{G}} + \log z_{ik} \right] \qquad\qquad \text{for } i \neq k, \\ u_k &\to u_k + (2\pi m) \sum_{\substack{j=1 \\ j \neq k}}^{n} g_k g_j \left[\tfrac{v}{\mathbb{G}} + \log z_{kj} \right]. \end{aligned} \tag{4.32}$$

The Kähler potential is computed from this generating function via the Legendre transform $K(z_1, \bar{z}_1, u_1, \bar{u}_1, ...) = \theta^i \frac{u_i + \bar{u}_i}{2} - F(\theta_1, z_1, \bar{z}_1, ...)$ with implicit sum over the repeated indices.

$$\begin{aligned} K = & \sum_{\substack{i,j=1 \\ i<j}}^{n,n} g_i g_j \Bigg[\tfrac{v}{2\mathbb{G}} \left(\theta_{ij}^2 + \tfrac{z_{ij} \bar{z}_{ij}}{2} \right) \\ & + \tfrac{\theta_{ij}^2}{4} \log \left(z_{ij} \bar{z}_{ij} \right) - \tfrac{z_{ij} \bar{z}_{ij}}{4} \left(\tfrac{1}{2} \log (z_{ij} \bar{z}_{ij}) - 1 \right) \Bigg]. \end{aligned} \tag{4.33}$$

Under shifts along the periodic direction $\theta_k \to \theta_k + 2\pi m$ with $m \in \bar{Z}$, the Kähler potential transforms as

$$\begin{aligned} K \to \; & K + \sum_{\substack{j=1 \\ j \neq k}}^{n} \Big[(4\pi m (\theta_k - \theta_j) + (2\pi m)^2) \\ & \times g_k g_j \left[\tfrac{v}{2\mathbb{G}} + \tfrac{1}{4} \left(\log z_{kj} + \log \bar{z}_{kj} \right) \right] \Big]. \end{aligned} \tag{4.34}$$

The shift term that is proportional to m^2 is of the form $f(z,u)+\overline{f(z,u)}$ but the term proportional to m is linear in the auxiliary coordinates θ_i. In order to see that this term is also of the holomorphic plus antiholomorphic form, we use equation (4.31) for the implicit relation for widely separated periodic monopoles to re-express the above shifts:

$$\begin{aligned} K \to K \quad & +(2\pi m)\tfrac{u_k+\bar{u}_k}{2} \\ & + \sum_{\substack{j=1 \\ j\neq k}}^{n} (2\pi m)^2\, g_k g_j \left[\tfrac{v}{2\mathbb{G}} + \tfrac{1}{4}\left(\log z_{kj} + \log \bar{z}_{kj}\right)\right], \end{aligned} \tag{4.35}$$

which explicitly of form $K \to K + f(z,u) + \overline{f(z,u)}$.

Explicitly, with a sum over repeated indices $i,j = 1,2,...,n$, and with an implicit sum over all $U(1)$ factors, the metric can be expressed as

$$\frac{ds^2}{8\pi} = \frac{U_{ij}}{2} d\vec{x}^i d\vec{x}^j + \frac{(U_{ij})^{-1}}{2}\left[d\theta_j + \vec{W}^{j\ell}\cdot d\vec{x}_\ell\right]\left[d\theta_i + \vec{W}^{ik}\cdot d\vec{x}_k\right], \tag{4.36}$$

with

$$\begin{aligned} U_{ii} &= \sum_{\substack{k=1 \\ k\neq i}}^{n} g_i g_k \left[\tfrac{v}{\mathbb{G}} + \tfrac{1}{2}\log\left(z_{ik}\bar{z}_{ik}\right)\right], \\ U_{ij} &= -g_i g_j \left[\tfrac{v}{\mathbb{G}} + \tfrac{1}{2}\log\left(z_{ik}\bar{z}_{ik}\right)\right], \\ \vec{W}_{ii} &= -\sum_{\substack{j=1 \\ j\neq i}} g_i g_j \tfrac{\theta_{ij}}{2 z_{ij}\bar{z}_{ij}} \left(x_3^{ij}\hat{x}_2 - x_2^{ij}\hat{x}_3\right), \\ \vec{W}_{ij} &= g_i g_j \tfrac{\theta_{ij}}{2 z_{ij}\bar{z}_{ij}} \left(x_3^{ij}\hat{x}_2 - x_2^{ij}\hat{x}_3\right). \end{aligned} \tag{4.37}$$

Now one can carry out a transformation of the differential of the internal modulus τ without having to modify the generating function F explicitly. For the periodic monopole, the generalized Legendre transform we have carried out here produces a term like $a = -\frac{i}{2}\theta\left(\frac{zd\bar{z}-\bar{z}dz}{z\bar{z}}\right) = -\theta d\varphi$ where $\varphi = \tan^{-1}\left(\frac{z-\bar{z}}{z+\bar{z}}\right)$ is the argument of $z = |z|e^{i\varphi}$, when what we want in order to achieve the Cherkis-Kapustin metric [17] is $\tilde{a} = \varphi d\theta$. This is accomplished by introducing the coordinate change $\tilde{\tau} = \tau + \theta\varphi$.

Let us introduce the constant phase b associated with the periodic direction θ as in the Cherkis-Kapustin description [17] and the function $\varphi(z,\bar{z}) = \tan^{-1}\left(\frac{z-\bar{z}}{z+\bar{z}}\right)$. Generalizing these principles to the case of n distinct, interacting periodic monopoles, we can carry out the following transformation.

$$\tilde{\tau}_i = \tau_i + \sum_{\substack{j=1\\ j\neq i}} g_i g_j \left[\frac{b}{\mathbb{G}}\theta + ij + (\theta_i - \theta_j)(\varphi_i - \varphi_j)\right].$$

In the metric described in (4.36) and (4.37), we transform

$$\begin{aligned} \sum_{\substack{j=1\\ j\neq i}}^{n} \vec{W}^{ij}\cdot d\vec{x}_j &= -\sum_{\substack{j=1\\ j\neq i}}^{n} g_i g_j\, \theta_{ij}\, d\varphi(z_{ij},\bar{z}_{ij}) \\ &\to \sum_{\substack{j=1\\ j\neq i}}^{n} g_i g_j \left[\tfrac{b}{\mathbb{G}} + \varphi(z_{ij},\bar{z}_{ij})\right] d\theta_{ij}. \end{aligned} \tag{4.38}$$

With this transformation of the internal moduli τ_i, we have obtained a Kähler potential (4.33) and implicit relation (4.31) for n widely separated periodic monopoles. This is the description of periodic monopoles for a single $U(1)$ interaction. Recall that the gauge symmetry of the fields for a monopole with $U(N)$ gauge symmetry breaks fully to $U(1)^N$ when all constituent subwalls are sufficiently well separated [17] and where Higgs field has N distinct eigenvalues. This means that the interactions of widely separated periodic monopoles in $U(N)$ gauge theory can be obtained by summing over the N $U(1)$ interactions in a fashion analogous to that in equation (4.20) and those following it. The above metric contains an implicit sum over the N independent $U(1)$ interactions which yields the full metric for widely separated $U(N)$ periodic monopoles. This is the $U(N)$ generalization of the Cherkis-Kapustin metric [17] for widely-separated periodic monopoles with distinct charges. The Cherkis-Kapustin metric describes interactions of n widely separated, unit-charge periodic monopoles in $SU(2)$ gauge theory; that paper points out the simplicity with which the gauge symmetry is generalized to $SU(N)$ or $U(N)$ as is done here.

4.3 The Generalized Legendre transform for n widely separated doubly periodic monopoles

In this chapter we will extend the generalized Legendre transform to widely seperated doubly periodic monopoles, which are the main subject of this dissertation. First we will discuss the choice of carrying out the generalized Legendre transform with respect to the subwall coordinates or the monowall moduli and show that the two transforms are physically equivalent. Then we will discuss gauge choices made in Chapter 3 and here. Next we will apply the generalized Legendre transform to the comparatively simple case of two widely separated constituent monowalls. Last, we will carry out the generalized Legendre transform with respect to each of the monopole moduli for n widely separated monopoles and the result will reproduce our metric on the moduli space of monowalls in equation (3.24).

4.3.1 The Generalized Legendre transform with respect to subwall coordinates versus moduli

We wish to determine that carrying out the generalized Legendre transform for widely separated subwalls produces the same complex coordinates and metric whether the transform is carried out with respect to the subwall coordinates $(x_i, \varphi_i, \theta_i, \tau_i)$ (as if they are independent variables) or the monowall moduli $(R_\rho, \Psi_\rho, \Theta_\rho, T_\rho)$. We will show that the Kähler potentials produced in these two approaches are equal up to shifts of the form $f(u, z) + \overline{f(u, z)}$ where f is holomorphic. After these arguments, we will briefly discuss gauge choices in Section 4.3.2 and go on to apply the generalized Legendre transform with respect to the moduli in Section 4.3.3 to compute a set of complex coordinates and a Kähler potential for the metric on the moduli space of monowalls in equation (3.24).

We established in subsection 3.3.1 that the four types of coordinates of the monopoles share the same n constants of proportionality, i.e. $(dx_i, d\varphi_i, d\theta_i, dq_i) = c_i^\rho(dR_\rho, d\Psi_\rho, d\Theta_\rho, dQ_\rho)$ with sum over $\rho = 1, 2, ..., \sigma$ and $dT^\lambda = c_j^\lambda d\tau^j$ with sum over $j = 1, 2, ..., n$ for the phases dual to the electric charge (which behaves as a momentum). We now have the choice of carrying out the generalized Legendre transform for doubly periodic monopoles with respect to the constituent monopole coordinates (as we demonstrated here for monopoles on $\mathbb{R}^3$ in section 4.1 and periodic monopoles in section 4.2) or with respect to the monopole moduli now that we have established linear relationships between coordinates of widely separated constituent monopoles and the moduli. Perhaps the most straightforward approach would be to work exclusively with the moduli going forward and discard the constituent monopole coordinates entirely from our computations. But given that the subwall coordinates are the physical observables for our monowall on the underlying space, and also for simplicity of notation, we choose here to express our various functions explicitly in terms of the subwall coordinates and therefore implicitly in terms of the moduli. We carry out the generalized Legendre transform with respect to the moduli, however.

Given the linear relation between coordinates and moduli, it is simple to show with the chain rule that whether we conduct the generalized Legendre transform with respect to each of the monopole coordinates or the moduli, we obtain the same metric and implicit relation between Ψ_ρ and $(U_\rho, \mathcal{Z}_\rho)$. To establish the equivalence of taking the generalized Legendre transform for widely separated monowalls with respect to the coordinates $(x_i, \theta_i, \varphi_i)$ or the moduli $(R_\rho, \Psi_\rho, \Theta_\rho)$, we consider the Kähler potentials resulting from each approach, $K(\mathcal{U}_1, ..., \mathcal{Z}_1, ..., \mathcal{Z}_\sigma)$ and $\tilde{K}(u_1, ..., z_1, ...z_n)$, given a generating function $F\big(x(R), z(\mathcal{Z})\big)$. The Kähler potential for the coordinates:

$$\tilde{K}(u_1, ..., z_1, ...z_n) = -F + \sum_{i=1}^{n} \varphi_i F_{\varphi_i},$$

where $\frac{u_i+\bar{u}_i}{2} = F_{\varphi_i}$. The Kähler potential for the moduli:

$$K(\mathcal{U}_1, ..., \mathcal{Z}_1, ..., \mathcal{Z}_\sigma) = -F + \sum_{\rho=1}^{\sigma} \Psi_\rho F_{\Psi_\rho},$$

where $\frac{\mathcal{U}+\bar{\mathcal{U}}}{2} = F_\Psi$ and the moduli are related linearly to the coordinates by $\mathcal{U}^\rho = \sum\limits_{i=1}^{n} c_i^\rho u^i$. The difference between the Kähler potentials is

$$\begin{aligned} \tilde{K} - K \quad &= \sum_{i=1}^{n} \varphi_i F_{\varphi_i} - \sum_{\rho=1}^{\sigma} \Psi_\rho F_{\Psi_\rho} \\ &= \sum_{i=1}^{n} \varphi_{0,i} \tfrac{u_i+\bar{u}_i}{2}, \end{aligned}$$

where $\varphi_{0,i}$ is the coordinate of the i^{th} constituent monopole when the relevant moduli are all zero $\Phi_\rho = 0$. This is of the form $f(u)+\overline{f(u)}$ with f holomorphic with respect to the coordinates u and therefore does not affect the metric and nor would it affect the implicit relation between u and (φ, z). While we will carry out the generalized Legendre transform with respect to the constituent coordinates in our two-monopole example for simplicity, for the general n-monopole case we will deal directly with the moduli.

4.3.2 Gauge differences in doubly periodic monopole metrics

In this chapter, we will apply the generalized Legendre transform to widely seperated doubly periodic monopoles. The metric this construction corresponds to is that given at the end of Chapter 3. The generating function we have computed here, however, corresponds to a different gauge choice than was made in Chapter 3, which presents material published previously in [39]. We will elaborate on the relationship between the phases $\vec{a}$ associated with doubly periodic subwalls and the motion of subwalls along the periodic spatial directions φ and θ for slowly moving abelian subwalls. Recall that we assume that the subwall velocities are very small and the subwall accelerations are vanishingly small. We are interested in keeping only quantities up to quadratic in subwall velocities in the Lagrangian, which corresponds to keeping only up to terms quadratic in velocities in the fields Φ and A^0 and keeping only up to terms linear in velocities in the fields $\vec{A}$ for each subwall. In this discussion all velocities are presumed small.

We now compare the fields of a wall moving with small velocities along the periodic directions to a wall stationary along the periodic directions but with small time-dependent phases (a_φ, a_θ) and relate these two cases by gauge transform. In other words, we compare the gauge field $\vec{\underline{A}}_i$ and electric field $\vec{\underline{E}}_i$ produced by the i^{th} electrically and magnetically charged wall with velocities $\vec{v}^i = (v^i_1, v^i_\varphi, v^i_\theta)$ with the electric field $\vec{E}_i$ produced by a wall with velocity $\vec{v}^i = (v^i_1, 0, 0)$ and small gauge phases $\vec{a}^i = \text{sgn}(x - x^i)(0, a^i_\varphi, a^i_\theta)$. Up to quantities cubic in time derivatives the magnetic fields are not affected by small velocities in the φ and θ directions. Recall that $\vec{w}$ is a function of the spatial coordinates (x, φ, θ) and its form depends on our choice of gauge. In [39] we chose its form to be $\vec{w}(x, \varphi, \theta) = \frac{\text{sgn}(x)}{2}(0, -\theta, \varphi)$ while the form which produces the metric we have generated using the generalized Legendre transform (4.56) is $\vec{w}(x, \varphi) = \text{sgn}(x)(0, 0, \varphi)$.

$$\underline{A}^0_i = -g_i q_i |x - x_i| + g_i(\vec{w} \cdot \vec{v}_i), \qquad A^0 = -g_i q_i |x - x_i|,$$

$$\vec{\underline{A}}_i = g_i \vec{w}(x - x_i), \qquad \vec{A}_i = g_i \vec{w}(x - x_i) - g_i \vec{a},$$

$$\vec{\underline{E}}_i = \text{sgn}(x - x_i)\left(g_i q_i,\, g_i \partial_\varphi(\vec{w} \cdot \vec{v}),\, g_i \partial_\theta(\vec{w} \cdot \vec{v})\right),$$

$$\vec{E}_i = \text{sgn}(x - x_i)\left(g_i q_i,\, g_i \dot{a}_\varphi,\, g_i \dot{a}_\theta\right).$$

Comparing the last two lines of the above equation we see that for slow moving subwalls including a small time dependent gauge phase $\vec{a}$ in the definitions of the gauge fields A^i_φ and A^i_θ affects the electomagnetic fields in the same way as velocities along the φ and θ directions if we make the following identifications between these periodic direction velocities and the gauge phases:

$$\partial_\varphi(\vec{w} \cdot \vec{v}) = \dot{a}_\varphi, \qquad \partial_\theta(\vec{w} \cdot \vec{v}) = \dot{a}_\theta. \tag{4.39}$$

In [39] this corresponded to identifying up to constants $\frac{\varphi^i}{2}$ with a_θ and $\frac{\theta^i}{2}$ with a_φ. In the case discussed here, this corresponds to identifying (up to a constant) φ^i with a_θ and setting a_φ to zero. For simplicity we will refer to a_θ as simply a. Then we can rewrite the metric coefficients in (4.56) as

$$\vec{W}_{ii} = -\sum_{\substack{j=1 \\ j\neq i}} g_i g_j \left[\tfrac{d_\theta}{\mathbb{G}} + a_{ij}\right]\hat{\theta}, \qquad \vec{W}_{ij} = g_i g_j \left[\tfrac{d_\theta}{\mathbb{G}} + a_{ij}\right]\hat{\theta}.$$

4.3.3 The Generalized Legendre transform for monowalls

Now that we have addressed these matters, we get down to the business of computing the generating function and integrand for monowalls, our main goal for this chapter. As with monopoles on $\mathbb{R}^3$, we begin the story of the generalized Legendre transform and computing the generating function for doubly periodic monopoles (i.e. monopoles on a $\mathbb{R}\times T^2$ underlying space) by examining the Gibbons-Hawking-like metric and Kähler potential for two widely separated doubly periodic monopoles in equation (4.40) and comparing it to the metric for two monopoles in terms of second derivatives of the generating function F in equation (4.21).

In this case the monopole relative coordinates are encoded in a quadratic function of ζ with complex coefficient $z = x + i\theta$ and real coefficient φ where $\varphi \sim \varphi + 2\pi$ and $\theta \sim \theta + 2\pi$ are periodic while $x \in \mathbb{R}$. For simplicity at this point, we will also assume the monopole has symmetric left- and right-charges $\mathbb{G}^\pm = \pm(g_1 + g_2)$ where g_1 and g_2 are the charges of the constituent subwalls. After this simple case is discussed we will address doubly periodic monopoles (monowalls) with multiple constituent subwalls and asymmetric left- and right-charges. We assign the coordinates as real φ and complex $z = x + i\theta$ where $x \in \mathbb{R}$ is the long spatial coordinate along which the monopoles move and φ and θ are coordinates on a spatial torus, each with period 2π. In [39], I demonstrated that when $U(N)$ doubly periodic monopoles become widely separated, symmetry breaking occurs between the monopoles and as x approaches $\pm\infty$. Under maximal symmetry breaking the full $U(N)$ interaction breaks to N sets of $U(1)$ interactions. We will now apply the generalized Legendre transform to obtain a set of complex coordinates $(u, \bar{u}, z, \bar{z})$ and a Kähler potential for this metric. The Gibbons-Hawking-like metric for two widely separated doubly periodic monopoles in terms of their relative coordinates is

$$\begin{aligned} \frac{ds^2}{8\pi} = \quad & \frac{g_1 g_2}{2}\left[\frac{v}{\mathbb{G}} + \frac{z+\bar{z}}{2}\right](dz d\bar{z} + d\varphi^2) \\ & + \frac{1}{2g_1 g_2\left[\frac{v}{\mathbb{G}} + \frac{z+\bar{z}}{2}\right]}\left[d\tau - g_1 g_2\left(\frac{d_3}{\mathbb{G}} + \varphi\right)\frac{dz - d\bar{z}}{2i}\right]^2 . \end{aligned} \tag{4.40}$$

Note that a shift of $\theta \to \theta + 2\pi$ does not affect the metric. A shift of $\varphi \to \varphi + 2\pi$ corresponds to a holomorphic shift of $u \to u - 2\pi g_1 g_2 z$ which does not meaningfully change our monowall description [48].

We can see that the kinetic component of the generating function satisfies

$$F_{\varphi\varphi} = \frac{1}{2\pi i}\oint_\gamma d\zeta \eta_x^2 G_{\eta\eta}(\eta, \zeta) = g_1 g_2\left[\frac{v}{\mathbb{G}} + \frac{z+\bar{z}}{2}\right],$$

for $\mathrm{Re} z = \mathrm{Re}(z_1 - z_2)$ positive. Substituting in $\eta_x = 2\zeta$ as before, we can find an expression for the generating integrand $G(\eta, \zeta)$. We first solve for the kinetic component of $G_{\eta\eta}$ which corresponds to the constant v term:

$$\oint_0 d\zeta 4\zeta^2 G_{1,\eta\eta}(\eta, \zeta) = g_1 g_2 \frac{v}{\mathbb{G}},$$

where $G_{1,\eta\eta} = \frac{g_1 g_2 v}{4\mathbb{G}\zeta^3}$ and the integrating contour is a counterclockwise loop around the origin of the ζ plane. We next must look at the second derivative of the generating function with respect to φ and z and see how it incorporates the constant φ and θ phases $d_\theta^\pm$ and $d_\varphi^\pm$ associated with the doubly periodic monopole far to the left and right of the monopole.

$$F_{\varphi z} = \frac{g_1 g_2}{2}\left[\frac{d_\theta}{\mathbb{G}} + \varphi\right],$$

where $d_\theta = \frac{1}{2}(d_\theta^+ + d_\theta^-)$ is the average of the left and right constant phases for the φ direction. The φ-linear term on the right in this expression results from the interaction term in the generating function which will be dealt with next. It may be replaced with the phase $\frac{g_1 g_2}{2}a$ to give the metric coefficient explicitly in terms of phases. The d_θ term can be introduced in the following way:

$$\oint_0 d\zeta \eta_x \eta_z G_{1',\eta\eta}(\eta,\zeta) = \frac{g_1 g_2}{2}\frac{d_\theta}{\mathbb{G}},$$

where $\eta_z = 1$ and $G_{1',\eta\eta} = \frac{g_1 g_2 d_3}{8\mathbb{G}\zeta^2}$. These terms can be conveniently combined by defining a function of ζ with constant coefficients $\mathcal{V}(\zeta) = (v + id_\varphi) + 2d_\theta\zeta - (v - id_\varphi)\zeta^2$ (compare with $\eta = (x + i\theta) + 2\varphi\zeta - (x - i\theta)\zeta^2$) so that $G_{\mathrm{kin},\eta\eta} = \frac{g_1 g_2}{8\mathbb{G}}\frac{\mathcal{V}(\zeta)}{\zeta}$. This modification changes the generating function by quantities of the form $f(z) + \overline{f(z)}$ with holomorphic f. Then we write the full kinetic component of the generating integrand as

$$G_{kin} = \frac{1}{8\zeta}\left(\zeta - \frac{1}{\zeta}\right)\frac{\mathcal{V}(\zeta)}{\mathbb{G}\zeta}\frac{\eta^2}{2!\zeta^2}.$$

The component of the function G that corresponds to the $\frac{z+\bar{z}}{2}$ term requires a different integrand but will have the same contour. To obtain the $\frac{z+\bar{z}}{2}$ term, we integrate $\frac{1}{\zeta}\left(\zeta - \frac{1}{\zeta}\right)\frac{\eta}{\zeta}$ counterclockwise around the origin.

$$\oint_\gamma d\zeta 4\zeta^2 G_{2,\eta\eta}(\eta,\zeta) = g_1 g_2 \frac{z+\bar{z}}{2},$$

where $G_{2,\eta\eta} = \frac{1}{\zeta}\left(\zeta - \frac{1}{\zeta}\right)\frac{\eta}{8\zeta^3}$. Now that we have all components of $G_{\eta\eta}(\eta,\zeta)$, we can integrate it twice with respect to η to obtain G:

$$
\begin{aligned}
G(\eta,\zeta) &= \left(\zeta - \tfrac{1}{\zeta}\right)\tfrac{g_1 g_2}{8\zeta}\left[\tfrac{\mathcal{V}(\zeta)}{\mathbb{G}\zeta}\tfrac{\eta^2}{2!\zeta^2} + \tfrac{\eta^3}{3!\zeta^3}\right] \\
&= \left(\zeta - \tfrac{1}{\zeta}\right)\tfrac{g_1 g_2}{8\zeta}\left[\tfrac{\tilde{\mathcal{V}}(\zeta)}{\mathbb{G}}\tfrac{\tilde{\eta}^2}{2!} + \tfrac{\tilde{\eta}^3}{3!}\right],
\end{aligned}
$$

where $\mathcal{V}(\zeta) = (v + id_\varphi) + 2d_\theta\zeta - (v - id_\varphi)\zeta^2$, $\tilde{\eta} = \frac{\eta}{\zeta}$, and $\tilde{\mathcal{V}}(\zeta) = \frac{\mathcal{V}(\zeta)}{\zeta}$. Later, when we compare this function to the crystal volume in Chapter 5, we will compare the cut volume with the portion of the above function of $\tilde{\eta}$ only:

$$
\tilde{G}(\tilde{\eta}) = \frac{g_1 g_2}{8}\left[\frac{\tilde{\mathcal{V}}(\zeta)}{\mathbb{G}}\frac{\tilde{\eta}^2}{2!} + \frac{\tilde{\eta}^3}{3!}\right]. \tag{4.41}
$$

4.3.4 The Generalized Legendre transform for n Doubly Periodic Monopoles

Generalizing the above discussion from two doubly periodic monopoles to n such monopoles and computing the generalized Legendre transform with respect to the moduli $(\Psi_\rho, R_\rho + i\Theta_\rho) = (\Psi_\rho, \mathcal{Z}_\rho)$ rather than coefficients (φ_i, z_i) yields the following. To simplify our expressions without losing generality, we choose the coordinate system such that the center of mass of the monopole has position $x_{CM} = 0$. The generating integrand including the possibility of asymmetric left and right charges is

$$
\boxed{8\,\tilde{G}(\tilde{\eta}_1, ..., \tilde{\eta}_n) = \sum_{\alpha=1}^{N}\left[\tilde{\mathcal{V}}^\alpha(\zeta)\sum_{i=1}^{n} g_i^\alpha \frac{\tilde{\eta}_i^2}{2!} + \sum_{i=1}^{n} g_0^\alpha g_i^\alpha \frac{\tilde{\eta}_i^3}{3!} + \sum_{j<i}^{n,n} g_i^\alpha g_j^\alpha \frac{\tilde{\eta}_{ij}^3}{3!}\right].} \tag{4.42}
$$

The generating function is

$$
F(\Psi_1, \mathcal{Z}_1, ...\Psi_\sigma, \mathcal{Z}_\sigma) = \frac{1}{2\pi i}\oint_0 \frac{d\zeta}{\zeta}\left(\zeta - \frac{1}{\zeta}\right)\tilde{G}(\tilde{\eta}_1, \tilde{\eta}_2, ..., \tilde{\eta}_n) \tag{4.43}
$$

in terms of the coefficients (φ_i, z_i), which are linear functions of the moduli $(\varphi_i, z_i) = c_i^\rho(\Psi_\rho, \mathcal{Z}_\rho)$ with constants c_i^ρ. It can be computed explicitly:

$$\begin{aligned} F(\Psi_1, ..., \Psi_\sigma, \mathcal{Z}_1, ..., \mathcal{Z}_\sigma) = & \sum_{\alpha=1}^{N} \Bigg[d_\theta^\alpha \sum_{i=1}^{n} g_i^\alpha \varphi_i \left(\tfrac{z_i+\bar{z}_i}{2}\right) \\ & + \tfrac{v^\alpha}{2} \sum_{i=1}^{n} g_i^\alpha \left(\varphi_i^2 - \tfrac{z_i \bar{z}_i}{2}\right) \\ & + \sum_{i=1}^{n} \tfrac{g_0^\alpha g_i^\alpha}{2} \left(\tfrac{z_i+\bar{z}_i}{2}\right) \left(\varphi_i^2 - \tfrac{z_i \bar{z}_i}{4}\right) \\ & + \sum_{\substack{i,j \\ j<i}} \tfrac{g_i^\alpha g_j^\alpha}{2} \left(\tfrac{z_{ij}+\bar{z}_{ij}}{2}\right) \left(\varphi_{ij}^2 - \tfrac{z_{ij} \bar{z}_{ij}}{4}\right) \Bigg]. \end{aligned} \tag{4.44}$$

where $x_{ij} = x_i - x_j$ etc, $\mathbb{G} = \sum_{i=1}^{n} g_i$, v is the constant background Higgs field for this particular factor of $U(1)$, and d_θ is the φ-direction phase for this $U(1)$ factor and α indexes the $U(1)$ factors. We have omitted $-\sum_{i=1}^{n} \frac{(v+id_\varphi)\bar{z}_i^2+(v-id_\varphi)z_i^2}{16}$ which originates from the $\mathcal{V}(\zeta)$ term since it contributes to the Kähler potential only a term of the form $f(\mathcal{Z}) + \overline{f(\mathcal{Z})}$ with f holomorphic, which contributes nothing to the metric.

The implicit relation between the auxiliary moduli Ψ_ρ, and the complex moduli $(\mathcal{U}_\rho, \mathcal{Z}_\rho)$ is

$$\begin{aligned} \tfrac{\mathcal{U}_\rho+\bar{\mathcal{U}}_\rho}{2} = & \sum_{\alpha=1}^{N} \sum_{i=1}^{n} \Bigg[d_\theta^\alpha g_i^\alpha \tfrac{z_i+\bar{z}_i}{2} + v g_i^\alpha \varphi_i \\ & + g_0^\alpha g_i^\alpha \varphi_i \tfrac{z_i+\bar{z}_i}{2} + g_i^\alpha g_j^\alpha \varphi_{ij} \tfrac{z_{ij}+\bar{z}_{ij}}{2} \Bigg] c_\rho^i. \end{aligned} \tag{4.45}$$

which follows from $\frac{\mathcal{U}_\rho+\bar{\mathcal{U}}_\rho}{2} := \partial_{\Psi_\rho} F$. Shifts along the periodic directions of $\varphi_i \to \varphi_i + 2\pi m$ and $\theta_j \to \theta_j + 2\pi n$ respectively correspond to the following shift for complex subwall coordinates $(u_i, \bar{u}_i)$ introduced earlier in this chapter. Because the wall coordinates along the periodic direction θ_i do not appear in the implicit relation, the u_i are unaltered by θ shifts

$$\begin{aligned}
u_i &\to u_i + 2\pi m \left[v g_i + g_0 g_i \tfrac{z_i+\bar{z}_i}{2} + \sum_{\substack{j=1\\ j\neq i}}^{n} g_i g_j \tfrac{z_{ij}+\bar{z}_{ij}}{2} \right], \\
u_k &\to u_k - 2\pi m g_i g_k \tfrac{z_{ik}+\bar{z}_{ik}}{2},
\end{aligned}$$

for $k \neq i$ and with implicit summation over the $U(1)$ factors on the right-hand side. These complex coordinates are related to the complex moduli $\mathcal{U}^\rho$ via $\mathcal{U}\rho = c^\rho_j u^j$ with summation over $j = 1, 2, ..., n$. Shifts in the coordinates u^i correspond to the following shifts in the moduli $\mathcal{U}^\rho$:

$$\begin{aligned}
\mathcal{U}^\rho = \;& 2\pi m c^\rho_i \left[v g_i + g_0 g_i \tfrac{z_i+\bar{z}_i}{2} + \sum_{\substack{j=1\\ j\neq i}}^{n} g_i g_j \tfrac{z_{ij}+\bar{z}_{ij}}{2} \right] \\
& -2\pi m c^\rho_i \sum_{\substack{k=1\\ k\neq i}}^{n} \left[g_i g_k \tfrac{z_{ik}+\bar{z}_{ik}}{2} \right],
\end{aligned} \tag{4.46}$$

with implicit summation over the $U(1)$ factors on the right-hand side.

The set of second derivatives of the generating function F which appear in the metric are

$$F_{\Psi_\rho \Psi_\lambda} = \sum_{\substack{i=1\\ j=1}}^{n,n} c^\rho_i c^\lambda_j F_{\varphi_i \varphi_j}, \qquad F_{\Psi_\rho \mathcal{Z}_\lambda} = \sum_{\substack{i=1\\ j=1}}^{n,n} c^\rho_i c^\lambda_j F_{\varphi_i z_j}, \tag{4.47}$$

where

$$\begin{aligned}
F_{\varphi_i \varphi_i} &= g_0 g_i \tfrac{z_i+\bar{z}_i}{2} + \sum_{\substack{j=1\\ n\neq i}} g_i g_j \left[\tfrac{v}{\mathbb{G}} + \tfrac{z_{ij}+\bar{z}_{ij}}{2} \right], \\
F_{\varphi_i \varphi_j} &= -g_i g_j \left[\tfrac{v}{\mathbb{G}} + \tfrac{z_{ij}+\bar{z}_{ij}}{2} \right], \\
F_{\varphi_i z_i} &= \sum_{\substack{j=1\\ j\neq i}}^{n} \tfrac{g_i g_j}{2} \left[\tfrac{d_\theta}{\mathbb{G}} + \varphi_{ij} \right], \qquad F_{\varphi_i z_j} = -\tfrac{g_i g_j}{2} \left[\tfrac{d_\theta}{\mathbb{G}} + \varphi_{ij} \right],
\end{aligned} \tag{4.48}$$

with implicit summation over the $U(1)$ factors on the right-hand side of each equation in (4.48).

The Kähler potential is related to the generating function by $K = -F + \Psi_\rho \partial_{\Psi_\rho} F$. Thus,

$$K_{\mathcal{Z}_\rho \bar{\mathcal{Z}}_\lambda} = \frac{F_{\Psi_\rho \Psi_\lambda}}{4} + F_{\mathcal{Z}_\rho \Psi_\mu} \left(F_{\Psi_\nu \Psi_\mu}\right)^{-1} F_{\Psi_\nu \bar{\mathcal{Z}}_\lambda},$$

$$K_{\mathcal{U}_\rho \bar{\mathcal{Z}}_\lambda} = -\left(F_{\Psi_\mu \Psi_\rho}\right)^{-1} F_{\Psi_\mu \bar{\mathcal{Z}}_\lambda}, \qquad K_{\mathcal{U}_\rho \mathcal{U}_\lambda} = \frac{\left(F_{\Psi_\lambda \Psi_\rho}\right)^{-1}}{4}. \tag{4.49}$$

The Kähler potential up to terms of the form $f(\mathcal{Z},\mathcal{U}) + \overline{f(\mathcal{Z},\mathcal{U})}$ with f holomorphic is

$$\boxed{\begin{aligned} K(\mathcal{U}_1, \mathcal{Z}_1, \bar{\mathcal{U}}_1, \bar{\mathcal{Z}}_1, ..., \mathcal{U}_\sigma, ...) = \sum_{\alpha=1}^{N} \Bigg[& \frac{v^\alpha}{2} \sum_{i=1}^{n} g_i^\alpha \left(\varphi_i^2 + \frac{z_i \bar{z}_i}{2}\right) \\ & + \sum_{i=1}^{n} \frac{g_0^\alpha g_i^\alpha}{2} \left(\frac{z_i + \bar{z}_i}{2}\right) \left(\varphi_i^2 + \frac{z_i \bar{z}_i}{4}\right) \\ & + \sum_{\substack{i,j \\ j<i}} \frac{g_i^\alpha g_j^\alpha}{2} \left(\frac{z_{ij} + \bar{z}_{ij}}{2}\right) \left(\varphi_{ij}^2 + \frac{z_{ij} \bar{z}_{ij}}{4}\right) \Bigg]. \end{aligned}} \tag{4.50}$$

From here forward in this chapter we will leave the summation over the $U(1)$ factors implicit. Under shifts along the periodic directions of the form

$\varphi_i \to \varphi_i + 2\pi m$ and $\theta_j \to \theta_j + 2\pi n$, the Kähler potential transforms as

$$\begin{aligned} K \to \quad & K + \tfrac{v}{2} g_i \left(2\pi m \varphi_i + (2\pi m)^2 + \tfrac{2\pi n (i\bar{z}_j - i z_j) + (2\pi n)^2}{2} \right) \\ & + \tfrac{g_0 g_i}{2} \left[\left(2\pi m \varphi_i + (2\pi m)^2 \right) \left(\tfrac{z_i + \bar{z}_i}{2} \right) + \tfrac{2\pi n (i\bar{z}_j - i z_j) + (2\pi n)^2}{4} \right] \\ & + \sum_{\substack{k=1 \\ k \neq i}}^{n} \tfrac{g_i g_k}{2} \left(2\pi m \varphi_{ik} + (2\pi m)^2 \right) \left(\tfrac{z_{ik} + \bar{z}_{ik}}{2} \right) \\ & + \sum_{\substack{\ell=1 \\ \ell \neq j}}^{n} g_j g_\ell \left(\tfrac{2\pi n (i\bar{z}_{j\ell} - i z_{j\ell}) + (2\pi n)^2}{4} \right), \end{aligned} \tag{4.51}$$

which are linear or constant with respect to moduli and so the metric is unchanged by integer shifts along the periodic directions. The auxiliary variables φ_i are neither holomorphic nor antiholomorphic but the terms in the above shift which are linear in φ_i can be expressed as being holomorphic/antiholomorphic, i.e. of the form $K \to K + f(\mathcal{Z},\mathcal{U}) + \overline{f(\mathcal{Z},\mathcal{U})}$, using the implicit relation among the monowall coordinates as in Equations (3.18) and (4.45). Re-expressed like this, the Kähler potential shifts in the following way, where $\mathcal{U}_\rho = c^i_\rho u_i$ and $z_i = c^\rho_i \mathcal{Z}_\rho$,

$$\begin{aligned} K \to \quad & K + \tfrac{v}{2} g_i \left((2\pi m)^2 + \tfrac{2\pi n (i\bar{z}_j - i z_j) + (2\pi n)^2}{2} \right) \\ & + \tfrac{g_0 g_i}{2} \left[\tfrac{2\pi n (i\bar{z}_j - i z_j) + (2\pi n)^2}{4} \right] + \sum_{\substack{k=1 \\ k \neq i}}^{n} \tfrac{g_i g_k}{2} (2\pi m)^2 \left(\tfrac{z_{ik} + \bar{z}_{ik}}{2} \right) \\ & + \sum_{\substack{\ell=1 \\ \ell \neq j}}^{n} g_j g_\ell \left(\tfrac{2\pi n (i\bar{z}_{j\ell} - i z_{j\ell}) + (2\pi n)^2}{4} \right) + 2\pi \left[\tfrac{u_i + \bar{u}_i}{2} - d_\theta \tfrac{z_i + \bar{z}_i}{2} \right]. \end{aligned} \tag{4.52}$$

The generating function for n doubly periodic monopoles reproduces the Lee-Weinberg-Yi type metric on the moduli space of monowalls given in [39] which is derived from the dynamics of widely separated monowalls.

$$\frac{g_{p_\rho \bar{p}_\lambda}}{8\pi} = \sum_{\rho=1}^{\sigma}\sum_{\lambda=1}^{\sigma} \partial_{p_\rho}\partial_{\bar{p}_\lambda} K, \tag{4.53}$$

where $\rho = 1, 2, ..., \sigma$, $p_\rho = \mathcal{U}_\rho, \mathcal{Z}_\rho$ and $\bar{q}_\rho = \bar{\mathcal{U}}_\rho, \bar{\mathcal{Z}}_\rho$. We now substitute in the phase a_i for φ_i as described in Section 4.3.1. Explicitly, with a sum over repeated indices $i, j = 1, 2, ..., n$, and with an implicit sum over all $U(1)$ factors, the metric can be expressed as

$$\boxed{\begin{aligned} \frac{ds^2}{8\pi} = \ & \frac{\left(U_{ij} c^i_\rho c^j_\lambda\right)}{2} d\vec{X}^\rho d\vec{X}^\lambda \\ & + \frac{(Ucc)^{-1}_{\rho\lambda}}{2}\left[dT_\lambda + \left(\vec{W}_{j\ell} c^j_\lambda c^\ell_\nu\right)\cdot d\vec{X}^\nu\right] \\ & \times \left[dT_\rho + \left(\vec{W}_{ik} c^i_\rho c^k_\rho\right)\cdot d\vec{X}^\rho\right] \end{aligned}} \tag{4.54}$$

with

$$U_{ii} = g_0 g_i x_i + \sum_{\substack{k=1 \\ k\neq i}}^{n} g_i g_k \left[\tfrac{v}{\mathbb{G}} + \tfrac{z_{ij}+\bar{z}_{ij}}{2}\right], \qquad U_{ij} = -g_i g_j \left[\tfrac{v}{\mathbb{G}} + \tfrac{z_{ij}+\bar{z}_{ij}}{2}\right], \tag{4.55}$$

$$\begin{aligned} \vec{W}_{ii}\cdot d\vec{X}^\mu &= -\sum_{\substack{j=1 \\ j\neq i}} g_i g_j \left[\tfrac{d_\theta}{\mathbb{G}} + a_{ij}\right] d\Theta^\mu, \\ \vec{W}_{ij}\cdot \vec{X}^\mu &= g_i g_j \left[\tfrac{d_\theta}{\mathbb{G}} + a_{ij}\right] d\Theta^\mu. \end{aligned} \tag{4.56}$$

This is the Kähler potential (4.50) and implicit relation (4.45) which produce the doubly periodic monopole metric of [39] up to gauge differences, which is a generalization of the metric derived in [16] from n identical monopoles to n distinct monopoles and from $SU(2)$ gauge symmetry to $U(N)$ symmetry. As in the previous chapter, note that we have chosen the U_{ij} functions to have a $1/\mathbb{G}$ term. This does not present an issue in cases where $\mathbb{G} = 0$, which can be seen by making the substitution $\sum_{i>j}^{n,n} g_i g_j x_{ij}^2 = \mathbb{G}\left[\sum_{i=0}^{n} g_i x_i^2\right] - (\mathbb{G}x_{CM})^2$ and using that the position of the monowall center of mass is fixed and at the origin. We have made the choice to include the $1/\mathbb{G}$ term to mimic the Yi-Weinberg-Lee form of metric seen in [17, 16] for periodic and doubly periodic monopoles.

CHAPTER 5

Generating Integrand is Crystal Volume

5.1 General case (without compound walls)

We will relate the expression for the generating integrand as a function of the subwall positions along the noncompact spatial direction to the cut volume. Using the Lawrence polytope volume formula [55] we give an expression for the cut volume which is given in terms of the x coordinates of the crystal vertices. Asymptotically (i.e. when one or more of the moduli are very large), the $x = \log|s|$ coordinate of the cut crystal's bottom vertices can be identified with the positions of subwalls along the noncompact direction x in $\mathbb{R} \times T^2$. This identification is covered in Chapter 2.2 in which the amoeba and cut volume are introduced.

5.1.1 Lawrence Polytope Volume Formula

We first briefly describe the Lawrence volume formula for convex polytopes in three dimensions. Identify a plane $\vec{a}_\rho \cdot (\vec{x} - \vec{x}_{0,\rho}) = 0$ with every face of the polytope and define the volume of the polytope as the region simultaneously satisfying $\vec{a}_\rho \cdot (\vec{x} - \vec{x}_{0,\rho}) < 0$ for ρ indexing all polytope faces (recall, each face of this polytope which is the cut volume corresponds to a point on the Newton polytope). The normal to the plane is given by $\vec{a}_\rho$ and $\vec{x}_{0,\rho}$ shifts the plane. Now define a reference plane $\vec{c}\cdot(\vec{x}-\vec{x}_c)=0$ which is not part of the polytope and which is not parallel to any of the polytope's faces, i.e. $\vec{c}\times\vec{a}_\rho \neq 0$. The Lawrence formula for the volume of the polytope is written as the sum of a set of signed sub-volumes where each sub-volume corresponds to one and only one of the vertices of the polytope. The sub-volumes are simplices (in three dimensions, these are tetrahedra) which are bounded by the three polytope planes intersecting at the vertex in question and by the reference plane. The sign of the sub-volume is positive if an even number of normal vectors for the three planes which bound it are pointing inward in the sub-volume, and the sub-volume is negative if an odd number of normal vectors point inward. See figure (5.1) for a two-dimensional analogue of this.

The volume of a simplex in three dimensions with vertices at $\vec{X}_{i=1,2,3,4}$ is $\frac{1}{3!}$ times the determinant of the 3×3 matrix whose columns are the displacement vectors between three of the vertices and the fourth.

$$\mathrm{Vol} = \frac{1}{3!}\det\begin{pmatrix}(X_2-X_1) & (X_3-X_1) & (X_4-X_1)\\ (Y_2-Y_1) & (Y_3-Y_1) & (Y_4-Y_1)\\ (Z_2-Z_1) & (Z_3-Z_1) & (Z_4-Z_1)\end{pmatrix}.$$

Associate a 3×3 matrix M_i with each polytope vertex i whose columns are the normal vectors $\vec{a}_\rho$, $\vec{a}_\mu$ and $\vec{a}_\nu$ of the three polytope planes intersecting at that vertex. Define the vector $\vec{\gamma}=(\gamma^x,\gamma^y,\gamma^z)$ whose elements solve the linear equation $c = M_i\gamma_i$ (no sum over i). Then the Lawrence volume formula for convex polytopes can be expressed concisely as

$$\mathrm{Vol}_{\mathrm{Lawrence}} = \sum_{i=1}^{n}\frac{1}{3!}\frac{[\vec{c}\cdot(\vec{x}_i-\vec{x}_c)]^3}{\gamma_i^x\gamma_i^y\gamma_i^z\det M_i}.$$

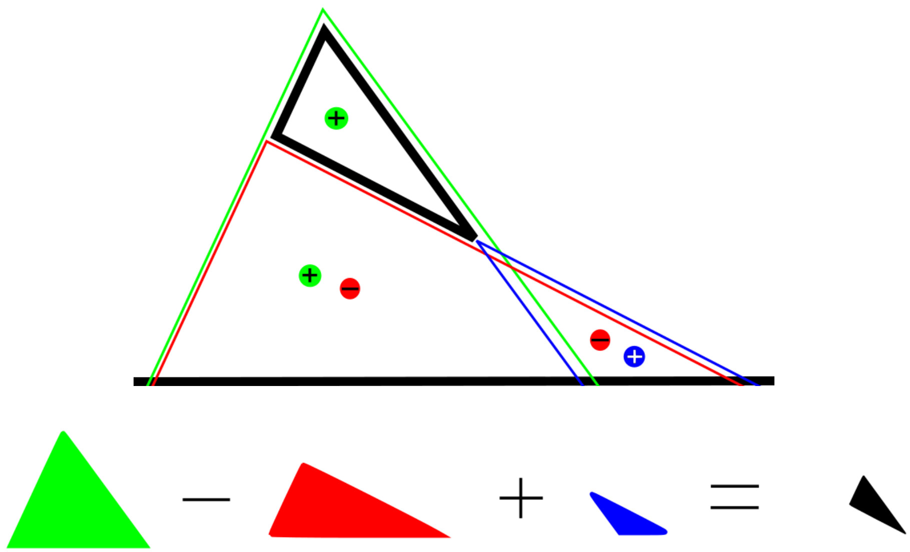

Figure 5.1: The triangular polygon and reference plane are both shown in thick black. The sub-triangles associated with the polygon's top, left and right vertices are shown respectively in green, red and blue. The normal vectors for these planes are all outward pointing for the black polygon. The green and blue sub-volumes have positive sign since they have even numbers of inward pointing normals (zero and two, respectively). The red sub-volume has negative sign since it has an odd number (one) of inward pointing normals.

5.1.2 Generating Integrand

The *generating integrand* $\tilde{G}\left(\tilde{\eta}_1, \tilde{\eta}_2, ..., \tilde{\eta}_n\right)$ is a function of $\tilde{\eta}_i = \frac{\eta_i}{\zeta}$ where $\eta_i = z_i + 2x_i\zeta - \bar{z}_i\zeta^2$. In the polynomial η_i, the coefficient x_i is one of the three real spatial positions of the i^{th} subwall and z_i is a linear complex combination of the remaining two spatial positions. We also introduce the function η_v in which the coefficients are constants and are a real and complex combination of the constant component v of the Higgs field and the constant phases (d_θ, d_φ)

associated with the two periodic directions (φ, θ). The noncompact direction has coordinate x. For example, expressions for η_i and η_v might be $\eta_i = (x_i + i\theta_i) + 2\varphi_i\zeta - (x_i - i\theta_i)\zeta^2$ and $\eta_v = (v + id_\varphi) + 2d_\theta\zeta - (v - id_\varphi)\zeta^2$. Consider n subwalls with spatial positions given by (x_i, z_i) and subwall charges g_i with $i = 1, ..., n$. We write the total charge for the α^{th} U(1) factor as $\mathbb{G}^\alpha = \sum_{i=1}^{n} g_i^\alpha$. We define the difference of η polynomials as $\eta_{ij} = (z_i - z_j) + 2(x_i - x_j)\zeta + (\bar{z}_i - \bar{z}_j)\zeta^2$, and the function of ζ encoding the constant Higgs v and gauge field (d_φ, d_θ) contributions $\mathcal{V} = (v + id_\varphi) + 2d_\theta\zeta - (v - id_\varphi)\zeta^2$, and further define $(\tilde{\eta}, \tilde{\mathcal{V}}) = \left(\frac{\eta}{\zeta}, \frac{\mathcal{V}}{\zeta}\right)$. We write the generating function for this monowall with widely separated subwalls. With an implicit sum over the $U(1)$ factors it is

$$\tilde{G}\left(\tilde{\eta}_1, \tilde{\eta}_2, ..., \tilde{\eta}_n\right) = \sum_{i=1}^{n} \left[\tilde{\mathcal{V}} g_i \frac{\tilde{\eta}_i^2}{2!} + g_0 g_i \frac{\tilde{\eta}_i^3}{3!}\right] + \sum_{\substack{i=1 \\ j=1 \\ i>j}}^{n,n} g_i g_j \frac{\tilde{\eta}_{ij}^3}{3!}. \tag{5.1}$$

To compare this expression with the cut volume we write the integrand $\tilde{G}$ as a function of the real coordinates x_i of the n subwalls along and suppress the superscript for simplicity. We substitute x_i for $\tilde{\eta}_i$ and substitute the constant component v of the Higgs for $\tilde{\eta}_v$. Assume the walls are ordered such that $x_{i+1} > x_i$.

$$\tilde{G}(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} \left[v g_i \frac{x_i^2}{2!} + g_0 g_i \frac{x_i^3}{3!}\right] + \sum_{\substack{i=1 \\ j=1 \\ i>j}}^{n,n} g_i g_j \frac{x_{ij}^3}{3!}. \tag{5.2}$$

The subwalls may have different charges g_i^α in each $U(1)$ factor (indexed by α) and the parameters $(v^\alpha, d_\varphi^\alpha, d_\theta^\alpha)$ differ as well. The summation over $U(1)$ factors is implicit in the above expression and the following expressions until (5.4). We will express $\tilde{G}(x_1, ..., x_n)$ as a sum of terms in which each term contains information about a single subwall and its corresponding subtriangle in the Newton polygon triangulation. Once the generating integrand is written in these terms we will make the $U(1)$ sum explicit in order to finally relate it to the cut volume.

To write the generating integrand in this manner we eliminate the ordering in the double sum over walls and take into account that the position of the monowall center of mass $x_{cm} = \frac{\sum_i^n g_i x_i}{\mathbb{G}}$ is fixed *in each* $U(1)$ *factor*. An identity is useful to apply to the first term in $\tilde{G}(x)$:

$$\sum_{i>j}^{n,n} g_i g_j x_{ij}^2 = \mathbb{G}\left[\sum_{i=0}^{n} g_i x_i^2\right] - \left(\mathbb{G} x_{CM}\right)^2 .$$

Using this identity we write the generating integrand as the following, (up to constant terms which will not appear in the monowall metric)

$$\tilde{G}(x_1, ..., x_n) = \sum_{i=1}^{n}\left[v g_i \frac{x_i^2}{2!} + g_0 g_i \frac{x_i^3}{3!} + \frac{1}{2}\sum_{j=1}^{n} g_i g_j \frac{|x_{ij}|^3}{3!}\right].$$

But this may be rewritten in terms of the total Higgs field at the location of the i^{th} subwall:

$$\begin{aligned} \Phi(x) &= -g_0 x - \sum_{j=1}^{n} g_j |x - x_j| - v, \\ \Phi(x_i) &= -g_0 x_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} g_j |x_{ij}| - v. \end{aligned} \tag{5.3}$$

We label $\Phi(x_i) = \Phi_i$ and express the generating integrand as the following with the sum over the N $U(1)$ factors indexed by $\alpha = 1, 2, .., N$ made explicit,

$$\tilde{G}(x_1, ..., x_n) = -\frac{1}{3!}\sum_{\alpha=1}^{N}\sum_{i=1}^{n}\left[3 g_i^\alpha x_i^2 \Phi_i + 2 g_i^\alpha x_i^3 \sum_{j\neq i}^{n} g_j^\alpha \operatorname{sign}(x_{ij})\right]. \tag{5.4}$$

Note that for a particular subwall while it may interact along several $U(1)$ factors and have different charges along them, its position is the same in each

$U(1)$ factor. For all subtriangles that *do not* have a horizontal leg, the value of the net Higgs field is the same for each of the $U(1)$ factors (at the location of that subwall) in which the subwall has nonzero charge. The sum over $U(1)$ factors of the charges associated with the corresponding subwall vanishes (refer to figure 5.2):

$$\begin{aligned}\sum_{\alpha=1}^{N} g_i^\alpha &= \sum_{\alpha\in\text{bottom}} g_i^\alpha + \sum_{\alpha\in\text{top}} g_i^\alpha \\ &= n_{\rho\nu}\ \tfrac{1}{2}\left(\frac{m_{\mu\nu}}{n_{\mu\nu}} - \frac{m_{\rho\nu}}{n_{\rho\nu}}\right) + n_{\mu\rho}\ \tfrac{1}{2}\left(\frac{m_{\mu\nu}}{n_{\mu\nu}} - \frac{m_{\mu\rho}}{n_{\mu\rho}}\right) \\ &= m_{\mu\nu} + m_{\rho\mu} + m_{\nu\rho} = 0.\end{aligned} \tag{5.5}$$

In such cases, the first term in equation (5.4) vanishes since the summation over the $U(1)$ factors indexed by α only applies to the magnetic charges in this term and the other quantities can be pulled out of the sum. (The cases in which the i^{th} subtriangle does have a horizontal leg will be dealt with in the following subsection.) In the second term in the above equation, the sum $\sum_j g_j^\alpha\,\text{sign}(x_{ij})$ is equal to the half the negative of the slope of the Higgs field S_i^α in the α^{th} $U(1)$ factor immediately to the left of the i^{th} wall plus the slope S_{i+1}^α immediately to the right of that wall:

$$\begin{aligned} g_0 - \sum_{j=1}^{i-1} g_j + \sum_{j=i+1}^{n} g_j = \ & \tfrac{1}{2}[(S_{n+1} + S_1) - (S_2 - S_1) - (S_3 - S_2) - \\ & ... - (S_i - S_{i-1}) + (S_{i+2} - S_{i+1}) + \\ & ... + (S_{n+1} - S_n)] \\ = \ & -\tfrac{1}{2}(S_{i+1} + S_i). \end{aligned} \tag{5.6}$$

The charge of the i^{th} wall is the half the difference of these slopes $g_i = (S_R^{i,\alpha} - S_L^{i,\alpha})/2$. We see then that the second term in (5.4) is $x_i^3/3$ multiplied by the

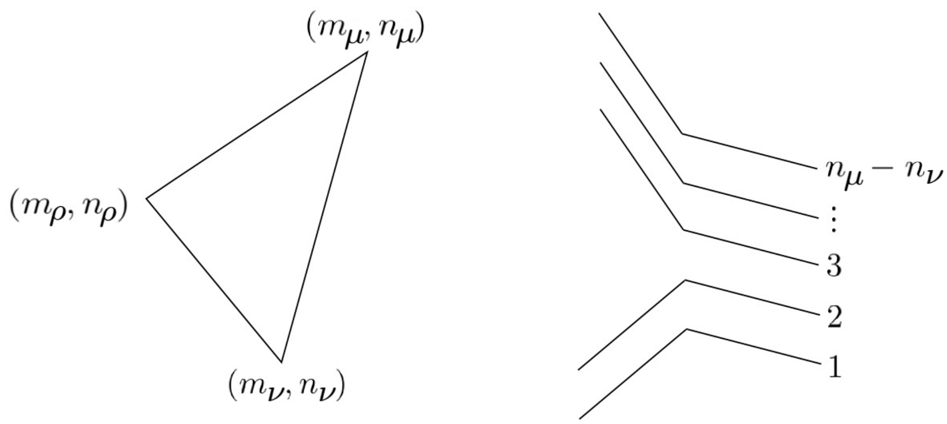

Figure 5.2: A subtriangle and the set of Higgs eigenvalues associated with it.

difference of squares of the slopes on either side of the i^{th} wall. When this is summed over the $U(1)$ factors, we obtain the following expression:

$$\sum_{\alpha=1}^{N}(S_R^\alpha)_i^2 - (S_L^\alpha)_i^2 = \frac{\delta_i^2}{(n_{\rho\mu}n_{\mu\nu}n_{\nu\rho})_i},$$

where the vertices of the i^{th} subtriangle in the triangulated Newton polygon are labeled (ρ, μ, ν) in a clockwise fashion, δ_i is twice the area of the i^{th} subtriangle and $n_{\rho\mu}$ is the height of the leg of the subtriangle with endpoints at the ρ and μ vertices. We then obtain our final form for the generating integrand in terms of the subwall positions,

$$\tilde{G}(x_1, ..., x_n) = \frac{1}{2}\sum_{i=1}^{n}\frac{(x_i)^3\delta_i^2}{3!(n_{\rho\mu}n_{\mu\nu}n_{\nu\rho})_i}. \tag{5.7}$$

Using the Lawrence polytope volume formula as in [55] with the $x = 0$ plane as the reference plane, we obtain the formula for the cut volume up to constant terms,[1] provided no edges of the crystal along which two faces of the crystal intersect are parallel with the x=0 plane:

$$\text{Vol}(x_1, ..., x_n) = \sum_{i=1}^{n} \frac{(x_i)^3 \delta_i^2}{3!(n_{\rho\mu} n_{\mu\nu} n_{\nu\rho})_i}. \tag{5.8}$$

We have proved that the generating integrand $\tilde{G}(x)$ in equation (5.7) is the cut volume $\text{Vol}(x)$ in equation (5.8) up to an overall factor of 2.

$$\boxed{\tilde{G}(x) = \frac{\text{Vol}(x)}{2}} \tag{5.9}$$

5.2 The case for compound subwalls

We now prove the same relation for crystal edges parallel to the $x = 0$ plane. We consider the special cases when an edge, say the edge joining points (m_ρ, n_ρ) and $m_\mu, n_\mu)$, in the Newton polygon triangulation is horizonal. In the monowall, we refer to these as *compound subwalls* since in these cases two subwalls will share a location along the x-direction and will move together as the monowall moduli vary. Unlike before, the first term in equation (5.4) for the contribution from subtriangle i does not vanish, and the second term omits the quantity with $n_{\mu\rho}$ in the denominator. If the horizontal edge in the Newton polygon triangulation is on the polygon perimeter, then the position

[1] The Lawrence polytope volume formula applies to convex polytopes, which the cut volume is generally not. The cut volume is the difference between the crystal and the cut crystal, both of which are convex and it is in this way that we apply the Lawrence formula. The Lawrence formula sums over vertices of the polytope in question and here we are only interested in contributions from vertices whose positions vary with the moduli R and we can neglect contributions from vertices whose positions are constant.

of the associated subwall depends on the two perimeter coefficients and does not depend on any moduli. In this case, the contribution from subwall i is linear in a modulus and will not contribute to the monowall metric and may therefore be ignored.

If the horizontal edge in the Newton polygon triangulation is on the polygon interior, the edge is shared by two subtriangles and we address the associated terms in the generating integrand together.

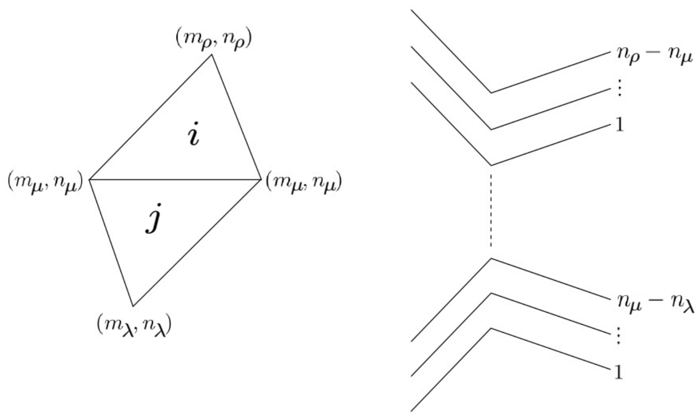

Figure 5.3: Two subtriangles sharing a horizontal edge and the set of Higgs eigenvalues associated with them.

$$
\begin{aligned}
\tilde{G}_i(x_i) + \tilde{G}_j(x_j) = \;& \tfrac{1}{3!}\left[-3\left(\sum_{\alpha=1}^{N} g_i^\alpha\right) x_i^2 \Phi_i + \tfrac{1}{2} x_i^3 \left(\tfrac{m_{\rho\mu}^2}{n_{\rho\mu}} + \tfrac{m_{\nu\rho}^2}{n_{\nu\rho}}\right)\right] \\
& + \tfrac{1}{3!}\left[-3\left(\sum_{\alpha=1}^{N} g_j^\alpha\right) x_j^2 \Phi_j + \tfrac{1}{2} x_j^3 \left(\tfrac{m_{\mu\lambda}^2}{n_{\mu\lambda}} + \tfrac{m_{\lambda\nu}^2}{n_{\lambda\nu}}\right)\right] \\
= \;& \tfrac{1}{2}\tfrac{1}{3!}\Bigg[3 m_{\mu\nu} x_i^2 (\Phi_i - \Phi_j) \\
& + x_i^3 \left(\tfrac{m_{\rho\mu}^2}{n_{\rho\mu}} + \tfrac{m_{\nu\rho}^2}{n_{\nu\rho}} + \tfrac{m_{\mu\lambda}^2}{n_{\mu\lambda}} + \tfrac{m_{\lambda\nu}^2}{n_{\lambda\nu}}\right)\Bigg].
\end{aligned} \tag{5.10}
$$

Note that we have used here that the two subwalls in question share the same x-position $x_i = x_j$. The corresponding terms in the Lawrence volume formula as we express it are not defined, so we raise the vertex at (m_ν, n_ν) by a small amount such that the edge joining (m_μ, n_μ) and (m_ν, n_ν) is tilted at small angle ε. We leave the horizontal coordinates m unaltered. Expanding the term for the i^{th} crystal vertex to first order in ε and taking the limit $\varepsilon \to 0$ results in the following terms:

$$\begin{aligned}\lim_{\varepsilon \to 0} \left(N_i(x_i, \varepsilon) + N_j(x_j, \varepsilon) \right) = \; & \tfrac{1}{3!} \Big[3 m_{\mu\nu} x_i^2 (y_i - y_j) \\ & + x_i^3 \left(\tfrac{m_{\rho\mu}^2}{n_{\rho\mu}} + \tfrac{m_{\nu\rho}^2}{n_{\nu\rho}} + \tfrac{m_{j\mu\lambda}^2}{n_{\mu\lambda}} + \tfrac{m_{\lambda\nu}^2}{n_{\lambda\nu}} \right) \Big]\end{aligned} \tag{5.11}$$

where, recall, y is the $\log |t|$ direction in the crystal space. Equations (5.10) and (5.11) can be identified with one another up to a factor of 2 and the correspondence between the generating integrand and the cut volume includes monowalls with compound subwalls.

CHAPTER 6

Conclusions

A doubly periodic magnetic monopole, known as a monowall, is a magnetic monopole on $\mathbb{R} \times T^2$. It has certain phase degrees of freedom; motion along these phase directions generates electric charge in the monowall. In $U(N)$ classical Yang-Mills-Higgs gauge theory we employ the Higgs spectral curve, its Newton polygon, and its amoeba to establish the asymptotic behavior of a monowall that has moduli. These tools give us an intuitive picture of the monowall in terms of its constituent charges, or subwalls, and of the symmetries when the charges are widely spread apart. When a modulus of the monowall becomes large, the monowall breaks up into subwalls whose separations vary linearly with respect to the modulus. The subwalls are positioned at locations of partially or fully restored gauge symmetry, a condition that can be inferred from the amoeba. The size of a subwall is the width of the region outside of which gauge and Higgs field interactions can be effectively approximated as abelian.

Once the walls are widely separated with respect to this width, their gauge and Higgs interactions are approximated as N $U(1)$ interactions and emulate classical electromagnetism with a massless Higgs. We proceed to treat the subwalls as uniformly electrically, magnetically and scalar charged planes and write the relativistic Lagrangian, including background gauge and Higgs fields (which satisfy prescribed boundary conditions).

For small velocities, this Lagrangian reduces to purely kinetic and we can read off the monowall moduli space metric. We write this metric for widely separated subwalls in terms of real subwall coordinates and phases, which we have shown are linearly dependent on the real moduli. To write the metric in terms of complex coordinates on the moduli space and compute a Kähler potential, we derive a generating integrand G and generating function F using the generalized Legendre transform. We then Legendre transform from the generating function, which is a function of the real moduli to the Kähler potential for the metric which is a local scalar function of the complex moduli and whose second derivatives are metric coefficients.

We introduce the cut volume, which is a polytope whose volume is a function of the Newton polynomial coefficients that define the Higgs spectral curve. Using the Lawrence polytope volume formula [55], we express the volume function in terms of its vertex coordinates. We express the generating function G as a sum over contributions from individual subwalls. Using these pieces of information we show that the generating function, which is a differential geometric object, is equal to the cut volume, an algebraic object. In other words, we show that the asymptotic Kähler potential and complex coordinates can be computed from the cut volume, indicating that the Higgs spectral curve determines asymptotic monowall dynamics and perhaps general monowall dynamics.

Subwall interactions yield hyperkähler moduli space metrics and hyperkähler *asymptotic* moduli space metrics in the limit that the subwalls are well-separated. The moduli space of a monowall is important in its own right:

for small velocities, the subwall dynamics can be approximated as geodesic motion on this moduli space. It has additional importance to supersymmetric Yang-Mills quantum gauge theory, since moduli spaces of monowalls in Yang-Mills-Higgs theories are isometric to the Coulomb branch moduli spaces of vacua in the associated five-dimensional quantum field theories.

The asymptotic moduli space of well-separated doubly-periodic monopoles has been addressed previously [16], and we expand on this work. In [16], the asymptotic monowall moduli space metric was determined for subwalls of identical magnetic charge in $SU(2)$ theory with spatially uniform background fields. We generalized from $SU(2)$ theory to $U(N)$ for subwalls of arbitrary magnetic charge and linear background Higgs field, and additionally justify the abelian long-distance approximation by analyzing the Higgs curve and amoeba for large values of a modulus. We then apply the generalized Legendre transform [45, 46] to widely separated monopoles which we have adapted from its application to $SU(N)$ monopoles on $\mathbb{R}^3$ [48] to doubly periodic monopoles in order to obtain a set of complex coordinates on the moduli space and a Kähler potential for the metric with those coordinates. We then show that the generating function for the metric is equal to the cut volume for the Higgs spectral curve.[1]

Still, the approach used in this dissertation is limited to well-separated subwalls. While asymptotic moduli spaces have been derived for a variety of monopoles, periodic and non-periodic, finding the interior metric on such mod-

[1]This is similar to how in [54] the asymptotic Kähler potential for BPS vortices in five-dimensional supersymmetric gauge theory on $R \times T^2$ with fundamental Higgs was related to a tetrahedron volume associated with an algebraic curve analogous to our Higgs spectral curve.

uli spaces remain an open problem. The corresponding supersymmetric systems, and in the case of periodic monopoles, the Higgs curve construction may play important roles in future efforts to derive the full moduli space metrics of monopoles. In particular, explorations of the Higgs spectral curve for moduli values corresponding to phase transitions of the amoeba may shed light on the regions of the moduli space associated with subwalls interacting at close distances and merging. The secondary polytope described in Section 2.2.1 emerges from the coherent triangulations of the Newton polygon and the phases of the amoeba and monowall and its faces correspond to moduli values for these phase transitions that are of interest.

APPENDIX A

Yang-Mills-Higgs Moduli Space Metric

This appendix is dedicated to deriving the Yang-Mills-Higgs monopole moduli space metric in terms of a Hermitian matrix function which we will label Ω. We will re-express our field equations in terms of a complex and a real coordinate and use the Bogomolny equation to write the gauge and Higgs fields in terms of a matrix function S. Next we will show that while the function S possesses the freedom in the Bogomolny equation to be rescaled by an antiholomorphic function, the fields and the metric are invariant under such rescalings.

A.1 Complex coordinates, Bogomolny equation, and kinetic terms

In terms of a real coordinate φ and a complex coordinate $z = x + i\theta$, $\bar{z} = x - i\theta$, define $\partial_w = \frac{1}{2}\partial_\varphi$, $\partial_{\bar{w}} = \frac{1}{2}\partial_\varphi$ and $\partial_z = \frac{1}{2}(\partial_x - i\partial_\theta)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_\theta)$. The Bogomolny equation can be written, as in equation (1.4),

$$[D_w, D_z] = 0, \qquad [D_z, D_{\bar{z}}] + [D_w, D_{\bar{w}}] = 0. \tag{A.1}$$

by further defining $A_w = \frac{1}{2}(A_\varphi + i\Phi)$, $A_z = \frac{1}{2}(A_x + iA_\theta)$, $\sigma = z, w$, $\bar{\sigma} = \bar{z}, \bar{w}$, $D_\sigma = \partial_\sigma + A_\sigma$, and $F_{0\sigma} = [D_0, D_\sigma]$, the Energy including kinetic terms can be expressed

$$\begin{aligned} E &= \text{mass } + \int d^3x \mathrm{Tr}[F_{0i}F^{0i} + (D_0\Phi)^2] \\ &= \text{mass } + \int d^3x Tr[F_{0z}F_{0\bar{z}} + F_{0w}F_{0\bar{w}}]. \end{aligned} \tag{A.2}$$

Note that we have not yet written A_0 and D_0. In the static BPS limit, the kinetic terms $F_{0\sigma}F_{0\bar{\sigma}}$ vanish. If we allow slow variations of the solutions A_σ with time, then small kinetic terms appear. With these small time perturbations, the zeroth component of the gauge field A_0, as well as A_σ, must satisfy the component of the Yang-Mills-Higgs gauge field equation (1.2)b which is linear in the time parameter. The fields A_σ still satisfy the Bogomolny equation (1.4). In terms of the complex and real coordinates, this component of the field equations is

$$D_\sigma F_{0\bar{\sigma}} + D_{\bar{\sigma}} F_{0\sigma} = 0, \tag{A.3}$$

with a sum over σ, $\bar{\sigma}$ implied by the repeated index. It is the component of the Yang-Mills-Higgs field equations which is equivalent to Gauss' law in electrodynamics (abelian Yang-Mills theory), and so we will refer to it this way here. We will return to the question of A_0 shortly.

The Bogomolny equation (A.1) implies there is a matrix function S such that $D_\sigma S = 0$, which allows us to solve for the fields (A_z, A_w) in terms of S:

$$A_\sigma = -\partial_\sigma S \cdot S^{-1}, \qquad\qquad A_{\bar{\sigma}} = -A_\sigma^\dagger. \tag{A.4}$$

The gauge transformations are $S \to US$ for unitary U, which gives

$$A_\sigma \to -\partial_\sigma(US)\cdot(US)^{-1} = UA_\sigma U^{-1} - \partial_\sigma U \cdot U^{-1},$$

$$A_{\bar{\sigma}} \to \left(\partial_{\bar{\sigma}}(US)\cdot(US)^{-1}\right)^\dagger = UA_{\bar{\sigma}}U^{-1} - \partial_{\bar{\sigma}} U \cdot U^{-1},$$

which is the usual gauge transformation for the vector field. The gauge transformed fields still satisfy $A_{\bar{\sigma}} = -A_\sigma^\dagger$.

Note that the Bogomolny equation constrains the matrix function S only up to antiholomorphic rescalings $S \to S\beta(\bar{z})$ (holomorphic rescalings $S^\dagger \to \beta^\dagger(z)S^\dagger$) under which the field itself A_σ ($A_{\bar{\sigma}}$) is invariant. It is necessary now to show that the Yang-Mills-Higgs monopole metric described in the following section is invariant under such rescalings and so we omit this $\beta(\bar{z})$ ($\beta^\dagger(z)$) factor in the remainder of our discussion since they are not affected.

First of all, it is clear from equation (A.4), which expresses the gauge fields in terms of the functions S and $S^\dagger$, that the gauge fields are invariant under these antiholomorphic rescalings of S. Because the metric is formed by taking the overlap of two variations of the fields $(A_\sigma, A_{\bar{\sigma}})$, the metric is necessarily invariant under such rescalings as well. We will take it one step further, however, and verify that the kinetic term in the Lagrangian which yields the metric is itself invariant under antiholomorphic rescalings of S.

These fields and functions will be functions of time only through the moduli $(\chi^p(t), \bar{\chi}^p(t))$, i.e. the chain rule gives

$$\partial_0 = \dot{\chi}^p \delta_p + \dot{\bar{\chi}}^p \delta_{\bar{p}}, \tag{A.5}$$

where $\dot{\chi}$ is the time derivative of χ and δ_p and $\delta_{\bar{p}}$ are variations with respect to moduli. Let us write $S = V\beta(\bar{z})$, $\lambda = -\delta U \cdot U^{-1}$, $\mu = -\delta V \cdot V^{-1}$, and label the moduli variation in terms of the antiholomorphic rescaling factor as $\rho = -\delta\beta \cdot \beta^{-1}$. We may work in a specific gauge for simplicity, and choose $U = 1$. In this case, the time-linear components of the field strength are written as covariant derivatives of the variations (λ, μ, ρ).

$$F_{0\sigma} = D_\sigma \left(\mu_{\bar{p}} + \mu_{\bar{p}}^\dagger + \left(V^\dagger\right)^{-1} \rho_{\bar{p}}^\dagger \, V^\dagger \right) \dot{\bar{\chi}}^p,$$

$$F_{0\bar{\sigma}} = D_{\bar{\sigma}} \left(\mu_p + \mu_p^\dagger + V \rho_p \, V^{-1} \right) \dot{\chi}^p.$$

We also observe that the gauge fields and covariant derivatives are β-independent. Since $\beta(\bar{z})$ and $\rho = -\delta\beta \cdot \beta^{-1}$ are antiholomorphic, one can apply the chain rule to show that the quantity vanishes $D_\sigma(V\rho V^{-1}) = 0$. From there it is straightforward to apply integration by parts to $\int d^3x \mathrm{Tr}[F_{0\sigma}F_{0\bar{\sigma}}]$ with static boundary conditions and use $[D_\sigma, D_{\bar{\sigma}}] = 0$ to eliminate the ρ terms from the kinetic term in the Lagrangian.

A.2 Metric and gauge perturbations

In this section, we address the gauge freedom of the fields which satisfy the Bogomolny equation and add a new constraint which will eliminate gauge redundancies from the monopole metric. For fixed boundary conditions on the fields (Φ, A_i), there is a continuum of static solutions to the Bogomolny equation which minimize the energy. The set of distinct gauge classes of solutions forms a manifold, referred to as a *moduli space*. One of our primary goals is to describe this surface in a way that omits gauge redundancies and to compute its metric.

Generally, the metric on the Yang-Mills-Higgs moduli space is written

$$\begin{aligned} g_{p\bar{q}} &= \int d^3x \mathrm{Tr}[(\delta_p A_i)(\delta_{\bar{q}} A^i) + (\delta_p \Phi)(\delta_{\bar{q}} \Phi)] \\ &= \int d^3x \mathrm{Tr}[(\delta_p A_{\bar{\sigma}})(\delta_{\bar{q}} A_\sigma) + \mathrm{h.c.}], \end{aligned} \tag{A.6}$$

where δ_p represent small variations with respect to the complex moduli χ_p. These field variations must not only satisfy the Bogmolny equation (or the linearized version of it, for small variations, to be given in equation (A.10)), but must also be orthogonal to gauge slices (to be given in equation (A.11)).

This condition can be restated as the requirement that these perturbations δA_σ, which satisfy a linearized form of the Bogomolny equation, be orthogonal to small gauge perturbations whose effects vanish as $x_1 \to \pm\infty$. In terms of the manifestly gauge-dependent field $U(-\partial_\sigma S \cdot S^{-1})U^{-1} - \partial_\sigma U \cdot U^{-1}$, the modulus variation is

$$\begin{aligned} \delta A_\sigma = \; & U^{-1} D_\sigma \left(-\delta S \cdot S^{-1}\right) U^{-1} + \partial_\sigma \left(-\delta U \cdot U^{-1}\right) + \\ & + \left[U^{-1} D_\sigma (-\delta S \cdot S^{-1}) U^{-1} - \partial_\sigma U \cdot U^{-1}, -\delta U \cdot U^{-1}\right]. \end{aligned}$$

More concisely,

$$\delta A_\sigma = D_\sigma(\nu + \lambda), \tag{A.7}$$

where $\nu = -\delta S \cdot S^{-1}$ corresponds to the physical component of the perturbation and $\lambda = -\delta U \cdot U^{-1}$ corresponds to the gauge component. In order to require that δA_σ be orthogonal to gauge perturbations, we might imagine we can simply set $\lambda = 0$. This constraint does not hold in any gauge, however. We must include a general λ in the expression for δA_σ in order to allow us the freedom to orient δA_σ orthogonal to all other small gauge perturbations. These take the form $D_\sigma \tilde{\lambda}$ which vanishes as $x_1 \to \pm\infty$:

$$\int d^3x \mathrm{Tr}[(\delta A_\sigma)(D_{\bar{\sigma}} \tilde{\lambda}) + (\delta A_{\bar{\sigma}})(D_\sigma \tilde{\lambda})] = 0,$$

for arbitrary $\tilde{\lambda}$. Using integration by parts, the gauge-fixing condition can be re-expressed compactly as

$$D_\sigma \delta A_{\bar{\sigma}} + D_{\bar{\sigma}} \delta A_\sigma = 0. \tag{A.8}$$

This constraint holds importance comparable to that of the Bogomolny equation (A.1).

A.3 Time dependence and the moduli

Here we introduce the zeroth component of the gauge fields, which must satisfy Gauss' law (A.3). We then show that the kinetic term in the Yang-Mills-Higgs Lagrangian yields the metric on the YMH monopole moduli space. When we allow small time variations of the fields (Φ, A_σ) but restrict these variations to be orthogonal to gauge slices, the fields depend on the small parameter time only through the moduli $\chi_p(x_0)$ [50]. In other words, time derivatives of the fields are $\partial_0 A_\sigma = (\delta_p A_\sigma)\dot{\chi}^p$. We can now address the relationship between the metric and the kinetic terms $F_{0\sigma}F_{0\bar{\sigma}}$ in the energy. We first need an expression for the zeroth gauge field A_0, which must solve the component of the Yang-Mills-Higgs field equation $D_\mu F^{\mu\nu} = [\Phi, D^\nu \Phi]$ which is linear in time derivatives, referred to here as Gauss' law (A.3). The solution, which is analogous to those found in [63], is

$$A_0 = \left[-\delta_p(US) \cdot (US)^{-1}\right] \dot{\chi}^p + \left[-\delta_p(US) \cdot (US)^{-1}\right]^\dagger \dot{\bar{\chi}}^p. \tag{A.9}$$

Showing that this A_0 satisfies the Yang-Mills-Higgs field equations to lowest order in time derivatives (A.3) requires the Master equation (real Bogomolny equation) $[D_\sigma, D_{\bar{\sigma}}] = 0$ and its linearized form, which is found by taking the first variation of the Bogomolny equation,

$$D_\sigma \delta A_{\bar{\sigma}} - D_{\bar{\sigma}} \delta A_\sigma = D_\sigma D_{\bar{\sigma}}(-\nu^\dagger + \lambda) - D_{\bar{\sigma}} D_\sigma(\nu + \lambda) = 0, \tag{A.10}$$

and the gauge-fixing equation, i.e. the constraint which guarantees that the moduli variations $\delta A_{\bar{\sigma}}$ and δA_σ are orthogonal to gauge variations, thereby prohibiting gauge redundancy from the metric we are constructing:

$$D_\sigma \delta A_{\bar{\sigma}} + D_{\bar{\sigma}} \delta A_\sigma = D_\sigma D_{\bar{\sigma}}(-\nu^\dagger + \lambda) + D_{\bar{\sigma}} D_\sigma(\nu + \lambda) = 0, \tag{A.11}$$

with sums over σ in both expressions, where $\nu = -\delta S \cdot S^{-1}$ and $\lambda = -\delta U \cdot U^{-1}$. Together, the linearized Bogomolny (A.10) and gauge slice orthogonality condition (A.11) imply each of the following equalities:

$$D_\sigma D_{\bar\sigma}\nu = -D_\sigma D_{\bar\sigma}\nu^\dagger, \quad D_\sigma D_{\bar\sigma}\nu = -D_\sigma D_{\bar\sigma}\lambda, \quad D_\sigma D_{\bar\sigma}\nu^\dagger = D_\sigma D_{\bar\sigma}\lambda. \tag{A.12}$$

The components of the field strength tensor that are linear in time derivatives can be expressed in terms of the variations $\nu = -(\delta V)V^{-1}$, $\mu = -\nu^\dagger = \left(V^\dagger\right)^{-1}\delta\bar{V}$, $\lambda = -(\delta U)U^{-1}$.

$$\begin{aligned} F_{0\sigma} &= \delta_{\bar p}\left[-\partial_\sigma(US)\cdot(US)^{-1}\right]\dot{\bar\chi}^p - D_\sigma\left[US^{-1\dagger}\delta_p(S^\dagger U^{-1})\right]\dot\chi^p \\ &= D_\sigma(\nu-\mu). \end{aligned}$$

Similarly, its negative Hermitian conjugate can be expressed $F_{0\bar\sigma} = D_{\bar\sigma}(\mu-\nu)$. The components of Yang-Mills-Higgs gauge field equations that are linear in time derivatives, i.e. Gauss' law, are

$$D_\sigma F_{0\bar\sigma} + D_{\bar\sigma}F_{0\sigma} = D_\sigma D_{\bar\sigma}(\mu-\nu) + D_{\bar\sigma}D_\sigma(\nu-\mu).$$

Both terms in this expression are zero (see equation (A.12)) and so Gauss' law is satisfied by the solution given in equation (A.9) for A_0.

It now remains to show first that the kinetic terms yield the metric up to total derivatives $\int d^3x \mathrm{Tr}\left[F_{0\sigma}F_{0\bar\sigma}\right] = \int d^3x \mathrm{Tr}\left[(\delta_p A_{\bar\sigma})(\delta_{\bar q}A_\sigma)\right]\dot\chi^p\dot{\bar\chi}^{\bar q} + ...$ which is given here, and second that the metric can be written in terms of the $U(N)$-invariant Hermitian matrix function $\Omega = S^\dagger S$ which is shown in the following subsection.

We wish to relate the kinetic term in the energy

$$\int d^3x \mathrm{Tr}\left[F_{0\sigma}F_{0\bar\sigma}\right] = \int d^3x \mathrm{Tr}\left[D_\sigma(\nu_{\bar q}+\lambda_{\bar q})\cdot D_{\bar\sigma}(-\nu_p^\dagger+\lambda)_p\right]\dot\chi^p\dot{\bar\chi}^q$$

to the metric on the Yang-Mills-Higgs monopole moduli space

$$g_{p\bar{q}} = \int d^3x \mathrm{Tr}\left[\delta_p A_{\bar{\sigma}} \cdot \bar{\delta}_q A_\sigma\right] = \int d^3x \mathrm{Tr}\left[D_\sigma(-\nu^\dagger_{\bar{q}} + \lambda_{\bar{q}}) \cdot D_{\bar{\sigma}}(\nu_p + \lambda)_p\right].$$

The second equality here results from the form of the variation δA_σ given in equation (A.7). In other words, we are asking whether the kinetic term in the Lagrangian yields the overlap integral of tangent vectors on the monopole moduli space.

Using the results of combining the linearized Bogomolny equation (A.10) with the gauge-fixing conditions (A.11) (see equation (A.12) for the useful forms), and four instances of integration by parts, we see that the kinetic term in the Lagrangian is equal to the metric up to total spacetime derivatives, i.e. boundary terms:

$$\int d^3x \mathrm{Tr}\left[F_{0\sigma}F_{0\bar{\sigma}}\right] = g_{p\bar{q}}\dot{\chi}^p\dot{\bar{\chi}}^q + \text{ boundary terms.}$$

Since we have specified that we are constraining ourselves to the case where the boundary conditions on the Yang-Mills-Higgs fields (Φ, A_i) are static in time, variations with respect to the time-dependent moduli $\chi_p(x_0)$ vanish on the boundaries of the space and therefore so should these boundary terms.

A.4 The Yang-Mills-Higgs monopole metric in terms of Ω

Given that the gauge field is written in terms of the matrix function S as $A_\sigma = -\partial S{\cdot}S^{-1}$, one can straightforwardly show that the complex field strength tensor $F_{\sigma\bar{\rho}} = [D_\sigma, D_{\bar{\rho}}]$ (where $\rho = z, u$) can be expressed in terms of S and $\Omega = S^\dagger S$ as $F_{\sigma\bar{\rho}} = -S\partial_{\bar{\rho}}(\Omega^{-1}\partial_\sigma\Omega)S^{-1}$ by computing $F_{\sigma\bar{\rho}}S$. This is outlined in the Appendix at the end of this section. Using the solution of equation (A.3) for the zeroth gauge field A_0 in equation (A.9), we see that $F_{0\sigma} = \left[-S\bar{\delta}_p(\Omega^{-1}\partial_\sigma\Omega)S^{-1}\right]\dot{\chi}^p$. Plugging this into the kinetic term in the energy, we obtain

$$E = \text{mass } + \int d^3x \text{Tr}\left[\Omega^{-1}\,\delta_p(\partial_{\bar{\sigma}}\Omega\cdot\Omega^{-1})\,\Omega\,\bar{\delta}_q(\Omega^{-1}\partial_\sigma\Omega)\right]\dot{\chi}^p\dot{\bar{\chi}}^q. \tag{A.13}$$

The metric on the moduli space is

$$ds^2 = g_{p\bar{q}}d\chi^p d\bar{\chi}^{\bar{q}}, \tag{A.14}$$

with metric tensor in terms of the Hermitian matrix function $\Omega = S^\dagger S$

$$\boxed{g_{p\bar{q}} = \int d^3x \text{Tr}\left[\Omega^{-1}\,\delta_p(\partial_{\bar{\sigma}}\Omega\cdot\Omega^{-1})\,\Omega\,\bar{\delta}_q(\Omega^{-1}\partial_\sigma\Omega)\right]} \tag{A.15}$$

where $\sigma = w, z$ and $\bar{\sigma} = \bar{w}, \bar{z}$ are implicitly summed over. This metric is demonstrably real.

REFERENCES

[1] P. A. M. Dirac, "Quantised singularities in the electromagnetic field,," *Proc. Roy. Soc. Lond.*, vol. A133, no. 821, pp. 60–72, 1931.

[2] G. 't Hooft, "Magnetic Monopoles in Unified Gauge Theories," *Nucl. Phys.*, vol. B79, pp. 276–284, 1974. [,291(1974)].

[3] A. M. Polyakov, "Particle Spectrum in the Quantum Field Theory," *JETP Lett.*, vol. 20, pp. 194–195, 1974. [,300(1974)].

[4] M. K. Prasad and C. M. Sommerfield, "An Exact Classical Solution for the 't Hooft Monopole and the Julia-Zee Dyon," *Phys. Rev. Lett.*, vol. 35, pp. 760–762, 1975.

[5] E. B. Bogomolny, "Stability of Classical Solutions," *Sov. J. Nucl. Phys.*, vol. 24, p. 449, 1976. [Yad. Fiz.24,861(1976)].

[6] N. Seiberg and E. Witten, "Gauge dynamics and compactification to three-dimensions," in *The mathematical beauty of physics: A memorial volume for Claude Itzykson. Proceedings, Conference, Saclay, France, June 5-7, 1996*, pp. 333–366, 1996.

[7] G. Chalmers and A. Hanany, "Three-dimensional gauge theories and monopoles," *Nucl. Phys.*, vol. B489, pp. 223–244, 1997.

[8] A. Hanany and E. Witten, "Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics," *Nucl. Phys.*, vol. B492, pp. 152–190, 1997.

[9] N. S. Manton, "Monopole Interactions at Long Range," *Phys. Lett.*, vol. 154B, p. 397, 1985. [Erratum: Phys. Lett.157B,475(1985)].

[10] H. W. Braden and V. Z. Enolski, "The Construction of Monopoles," *Commun. Math. Phys.*, vol. 362, no. 2, pp. 547–570, 2018.

[11] H. W. Braden and V. Z. Enolski, "The Charge 2 Monopole via the ADHMN construction," 2019.

[12] M. F. Atiyah and N. J. Hitchin, "Low-Energy Scattering of Nonabelian Monopoles," *Phys. Lett.*, vol. A107, pp. 21–25, 1985.

[13] G. W. Gibbons and N. S. Manton, "The Moduli space metric for well separated BPS monopoles," *Phys. Lett.*, vol. B356, pp. 32–38, 1995.

[14] K.-M. Lee, E. J. Weinberg, and P. Yi, "The Moduli space of many BPS monopoles for arbitrary gauge groups," *Phys. Rev.*, vol. D54, pp. 1633–1643, 1996.

[15] S. A. Cherkis and A. Kapustin, "Nahm transform for periodic monopoles and N=2 superYang-Mills theory," *Commun. Math. Phys.*, vol. 218, pp. 333–371, 2001.

[16] M. Hamanaka, H. Kanno, and D. Muranaka, "Hyper-Khler metrics from monopole walls," *Phys. Rev.*, vol. D89, no. 6, p. 065033, 2014.

[17] S. A. Cherkis and A. Kapustin, "Hyperkahler metrics from periodic monopoles," *Phys. Rev.*, vol. D65, p. 084015, 2002.

[18] W. Nahm, "ALL SELFDUAL MULTI - MONOPOLES FOR ARBITRARY GAUGE GROUPS," in *12th NATO Advanced Summer Institute on Theoretical Physics: Structural Elements in Particle Physics and Statistical Mechanics Freiburg, Germany, August 31-September 11, 1981*, p. 301, 1981.

[19] R. Bielawski, "Monopoles, particles and rational functions," *Annals of Global Analysis and Geometry*, vol. 14, pp. 123–145, May 1996.

[20] N. J. Hitchin, "MONOPOLES AND GEODESICS," *Commun. Math. Phys.*, vol. 83, pp. 579–602, 1982.

[21] S. A. Cherkis and A. Kapustin, "Periodic monopoles with singularities and N=2 super QCD," *Commun. Math. Phys.*, vol. 234, pp. 1–35, 2003.

[22] S. A. Cherkis and R. S. Ward, "Moduli of Monopole Walls and Amoebas," *JHEP*, vol. 05, p. 090, 2012.

[23] M. Jardim, "Construction of doubly periodic instantons," *Commun. Math. Phys.*, vol. 216, pp. 1–15, 2001.

[24] M. B. Jardim, *Nahm transform of doubly periodic instantons.* PhD thesis, Oxford U., Inst. Math., 1999.

[25] M. Jardim, "Spectral curves and Nahm transform for doubly periodic instantons," 1999.

[26] O. Biquard and M. Jardim, "Asymptotic behavior and the moduli space of doubly periodic instantons," 2000.

[27] M. Jardim, "Classification and existence of doubly periodic instantons," 2001.

[28] C. Ford and J. M. Pawlowski, "Constituents of doubly periodic instantons," *Phys. Lett.*, vol. B540, pp. 153–158, 2002.

[29] C. Ford and J. M. Pawlowski, "Doubly periodic instantons and their constituents," *Phys. Rev.*, vol. D69, p. 065006, 2004.

[30] C. Ford and J. M. Pawlowski, "Doubly periodic instanton zero modes," *Phys. Lett.*, vol. B626, pp. 139–146, 2005.

[31] T. Mochizuki, "Asymptotic behaviour and the nahm transform of doubly periodic instantons with square integrable curvature," *Geom. Topol.*, vol. 18, no. 5, pp. 2823–2949, 2014.

[32] T. Mochizuki, "Kobayashi-Hitchin correspondence for analytically stable bundles," 2017.

[33] T. Mochizuki, "Doubly periodic monopoles and q-difference modules," 2019.

[34] K.-M. Lee, "Sheets of BPS monopoles and instantons with arbitrary simple gauge group," *Phys. Lett.*, vol. B445, pp. 387–393, 1999.

[35] R. S. Ward, "A Monopole Wall," *Phys. Rev.*, vol. D75, p. 021701, 2007.

[36] R. S. Ward, "Skyrmions and monopoles: Isolated and arrayed," *J. Phys. Conf. Ser.*, vol. 284, p. 012005, 2011.

[37] R. Maldonado and R. S. Ward, "Dynamics of monopole walls," *Phys. Lett.*, vol. B734, pp. 328–332, 2014.

[38] S. A. Cherkis, "Phases of Five-dimensional Theories, Monopole Walls, and Melting Crystals," *JHEP*, vol. 06, p. 027, 2014.

[39] R. Cross, "Asymptotic Dynamics of Monopole Walls," *Phys. Rev.*, vol. D92, no. 4, p. 045029, 2015.

[40] N. Seiberg and E. Witten, "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory," *Nucl. Phys.*, vol. B426, pp. 19–52, 1994. [Erratum: Nucl. Phys.B430,485(1994)].

[41] N. Seiberg and E. Witten, "Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD," *Nucl. Phys.*, vol. B431, pp. 484–550, 1994.

[42] K. A. Intriligator and N. Seiberg, "Phases of N=1 supersymmetric gauge theories in four-dimensions," *Nucl. Phys.*, vol. B431, pp. 551–568, 1994.

[43] K. A. Intriligator and N. Seiberg, "Duality, monopoles, dyons, confinement and oblique confinement in supersymmetric SO(N(c)) gauge theories," *Nucl. Phys.*, vol. B444, pp. 125–160, 1995.

[44] B. Haghighat and S. Vandoren, "Five-dimensional gauge theory and compactification on a torus," *JHEP*, vol. 09, p. 060, 2011.

[45] N. J. Hitchin, A. Karlhede, U. Lindstrom, and M. Rocek, "Hyperkahler Metrics and Supersymmetry," *Commun. Math. Phys.*, vol. 108, p. 535, 1987.

[46] U. Lindstrom and M. Rocek, "New Hyperkahler Metrics and New Supermultiplets," *Commun. Math. Phys.*, vol. 115, p. 21, 1988.

[47] I. T. Ivanov and M. Rocek, "Supersymmetric sigma models, twistors, and the Atiyah-Hitchin metric," *Commun. Math. Phys.*, vol. 182, pp. 291–302, 1996.

[48] G. Chalmers, "Multi - monopole moduli spaces for SU(N) gauge group," 1996.

[49] C. J. Houghton, "On the generalized Legendre transform and monopole metrics," *JHEP*, vol. 02, p. 042, 2000.

[50] N. S. Manton and P. Sutcliffe, *Topological solitons*. Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2004.

[51] D. Tong, "TASI Lectures on Solitons," 2005.

[52] N. S. Manton, "A Remark on the Scattering of BPS Monopoles," *Phys. Lett.*, vol. 110B, pp. 54–56, 1982.

[53] B. Julia and A. Zee, "Poles with Both Magnetic and Electric Charges in Nonabelian Gauge Theory," *Phys. Rev.*, vol. D11, pp. 2227–2232, 1975.

[54] T. Fujimori, M. Nitta, K. Ohta, N. Sakai, and M. Yamazaki, "Intersecting Solitons, Amoeba and Tropical Geometry," *Phys. Rev.*, vol. D78, p. 105004, 2008.

[55] J. Lawrence, "Polytope volume computation," *Mathematics of Computation*, vol. 57, no. 195, pp. 259–271, 1991.

[56] D. Gaiotto, G. W. Moore, and A. Neitzke, "Wall-crossing, Hitchin Systems, and the WKB Approximation," 2009.

[57] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*. Boston, MA: Birkhäuser Boston, 1994.

[58] O. Viro, "From the sixteenth hilbert problem to tropical geometry," *Japanese Journal of Mathematics*, vol. 3, pp. 185–214, Dec. 2008.

[59] A. Jaffe and C. Taubes, *Vortices and Monopoles: Structure of Static Gauge Theories*. Progress in physics, Birkhäuser, 1980.

[60] J. McDonald, "Fractional power series solutions for systems of equations," *Discrete & Computational Geometry*, vol. 27, pp. 501–529, Jan. 2002.

[61] F. Beringer and F. Richard-Jung, "Multi-variate polynomials and newton-puiseux expansions," in *Symbolic and Numerical Scientific Computation* (F. Winkler and U. Langer, eds.), (Berlin, Heidelberg), pp. 240–254, Springer Berlin Heidelberg, 2003.

[62] T. Estermann, *Complex numbers and functions*. University of London, Athlone Press, 1962.

[63] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, "Manifestly supersymmetric effective Lagrangians on BPS solitons," *Phys. Rev.*, vol. D73, p. 125008, 2006.