Pions versus Magnons: From QCD to Antiferromagnets and Quantum Hall Ferromagnets

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Abstract

The low energy dynamics of pions in (3+1) dimensions and magnons in (2+1) dimensions, which are the Goldstone bosons of the strong interactions and of magnetism, respectively, are analogous in many ways. The electroweak interactions of pions result from gauging an $SU(2)_L \otimes U(1)_Y$ symmetry which then breaks down to the $U(1)_{em}$ gauge symmetry of electromagnetism. The electromagnetic interactions of magnons are described by gauging not only $U(1)_{em}$ but also the $SU(2)_s$ spin rotational symmetry. The electromagnetic fields $\vec{E}$ and $\vec{B}$ appear as non-Abelian vector potentials.

In the theory of pions and photons a Goldstone-Wilczek current represents the baryon number of Skyrmions and gives rise to the $\pi^0 \to \gamma\gamma$ decay. Likewise, if baby-Skyrmions have electron quantum numbers, magnons couple to the analogue of a Goldstone-Wilczek current for baby-Skyrmions. This also includes a vertex for the decay of a magnon into two photons. Analogous to axion-photon conversion in a strong magnetic field, magnon-photon conversion is also possible.

Electroweak instantons give rise to baryon number violating processes caused by the 't Hooft anomaly. This provides a decay channel for Skyrmions. There is no corresponding decay cannel for baby-Skyrmions. Baryons may also decay (transform into leptons) in the presence of a magnetic monopole. The analogue of the magnetic monopole in magnetism is a discharging wire that thereby causes baby-Skyrmion decay.

In the pion theory the number of flavors can be increased resulting in additional massless mesons. This leads to the topological Wess-Zumino-Witten term that contributes to the $\pi^0 \to \gamma\gamma$ decay channel. There is a mathematical analogue for magnons with larger symmetry groups. Again an analogue of the Wess-Zumino-Witten term arises. In the pion theory the prefactor of the Wess-Zumino-Witten term is quantized and equal to the number of quark colors. In the magnon theory, on the other hand, the prefactor represents the anyon statistics parameter θ and is thus not quantized.

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Chapter 1

Introduction

In this work the analogies of pions and magnons, the Goldstone bosons of QCD and of antiferromagnets and ferromagnets respectively, will be exemplified. Astonishing resemblances exist beyond what one might have expected. Chiral perturbation theory is used to describe low-energy interactions of both theories. Topological properties however also include extraordinary analogies that allow for anomalous contributions in both particle physics and solid state physics. The results offered are well known for QCD pions and serve as a guide and mathematical framework for investigating magnons in antiferromagnets and ferromagnets.

When only concerned with lowest energy contributions, the dynamics of the lightest particles dominates the theory [1]. In QCD these are the three pions, which in comparison with the QCD scale can be regarded as massless particles. They appear as Goldstone bosons, resulting from the spontaneous symmetry breaking of the chiral symmetry group $SU(2)_L \otimes SU(2)_R \otimes U(1)_B$ to the subgroup $SU(2)_{L=R} \otimes U(1)_B$ (3+1-dimensions) [2]. The Goldstone theorem ensures that there are three massless excitations (π^-, π^0, π^+), resulting from symmetry considerations only. Goldstone bosons are always massless, if the symmetry from which they originate is exact (in the QCD case this is not the case due to the finite mass of the quarks, hence pions are pseudo-Goldstone bosons and carry a small mass). In antiferromagnets and ferromagnets the spontaneous symmetry breaking of the global $SU(2)_s$ symmetry to the subgroup $U(1)_s$ results in two magnon polarization states [3]. These excitations are indeed massless, again only if the symmetry is exact. Even a crystal without structural defects will generally not have an exact $SU(2)_s$ symmetry. These effects will not be considered, since massless particles provide a simpler model without the loss of interesting phenomena. Chiral perturbation theory was developed for QCD to determine the dynamics of such particles [1]. Since in both theories Goldstone bosons are to be considered, it is not surprising that chiral perturbation theory can also be applied to magnons in antiferromagnets and ferromagnets.

Additional extraordinary analogies exist beyond what can be found in perturbation theory. It will be shown that in both theories there are topologically non-trivial solutions. These solitons describe heavier particles not accessible to chiral perturbation theory. In the QCD case, due to the contributions of Skyrme [4, 5, 6], it is known, or at least generally accepted, that the Skyrmions carry baryonic charge, and are indeed baryons themselves. The baryon current, known as the Goldstone-Wilczek current, enters the Lagrangian, and thereby couples the neutral pion to the electromagnetic field. (This only happens, of course, once electromagnetism has been gauged). The topological term provides a decay channel for the anomalous $\pi^0 \rightarrow \gamma\gamma$ decay. In a two spacial dimensional magnon theory there are also non-trivial homotopy groups that characterize the topological charge of solitons. It may be disputed what particle property this charge represents. It will be argued that the baby-Skyrmions are indeed electrons or constructed thereof, and the topological charge correspond in this case to the electric charge. Experiments should, at least in principle, be able to clarify this question. It has been shown by Fröhlich and Studer [7] that the Pauli equation has a hidden local $SU(2)_s$ symmetry. As a result, in order to include electromagnetism in the magnon theory, the $SU(2)_s$ symmetry must be gauged. The result of such a symmetry is that the electromagnetic fields $\vec{E}$ and $\vec{B}$ appear as non-Abelian vector potentials. This is, of course, only a mathematical description so there are no new physical gauge degrees of freedom. The analog of the Goldstone-Wilczek current can be constructed. Again, it is this term that enters the Lagrangian and allows for the decay of a neutral magnon into two photons. Experimentally it may be easier to detect the transformation of a photon into a magnon, a conversion process that occurs if the magnet is placed in an external magnetic field. This is analogous to the infamous photon-axion conversion mechanism [8], a method developed to detect axions, however, with no positive results to this day. Topological considerations also provide information about the statistics of Skyrmions and baby-Skyrmions. The Skyrmions in QCD are either bosons or fermions, while the baby-Skyrmions are anyons, and hence their statistics is determined by a free parameter θ, the anyon angle, which is the prefactor of the Hopf term.

In QCD there are two decay mechanisms for the Skyrmion, despite their topological stability. Either an electroweak instanton violates baryon number via the 't Hooft anomaly [9], or in an $SU(5)$ grand unified theory a magnetic monopole changes the baryon number of a given system [10]. In the magnon theory there is no analog of the electroweak instanton, however, analogous to the magnetic monopole a discharging wire can cause baby-Skyrmion decay (this time in two space- and one time-dimension). The wire discharges and thereby changes the number of charge carriers in the magnet. Thus the number of baby-Skyrmions, representing the charge carriers in the model, is altered.

In QCD the chiral symmetry group was chosen to include the two lightest quarks. This can be expanded to an arbitrary number of flavors when considering the global chiral symmetry group of the form $SU(N_f)_L \otimes SU(N_f)_R \otimes U(1)_B$ which breaks to the subgroup $SU(N_f)_{L=R} \otimes U(1)_B$ ($N_f \geq 3$). This leads to new non-trivial topological contributions, that are accounted for in the Wess-Zumino-Witten term [11, 12, 13]. The prefactor of the Wess-Zumino-Witten term is the number of colors N_c which is quantized. Likewise the symmetry group of the magnon theory can be expanded to the global $SU(N_f)$ symmetry which is broken to $U(N_f - 1)$ (for $N_f \geq 3$). In a Quantum Hall ferromagnet this corresponds to the layer index of the two-dimensional lattice structure [14]. The homotopy groups in the magnon theory contribute in an analogous manner, where just like in the case of the Hopf term for $SU(2)$, the prefactor of the Wess-Zumino-Witten term is not quantized in the magnon theory.

Two types of magnets are considered for which such a formalism may provide insight. Two-dimensional antiferromagnets are the precursors of high-temperature superconductors and therefore are of great interest [15, 16]. A phase diagram of such a material is sketched in fig.1.1. Upon doping, baby-Skyrmions carrying an electric charge destroy the antiferromagnetic structure of a magnet. The role of the baby-Skyrmions in the superconducting phase is not clear, however, their influence may be crucial in the formation of Cooper pairs and hence the mechanism of high-temperature superconductivity. Single or multi-layer quantum Hall ferromagnets are known to harbor electrons as topological baby-Skyrmions [14, 17, 18]. The formalism presented here verifies what is already known about these material and provides an opportunity to investigate ferromagnetic magnons.

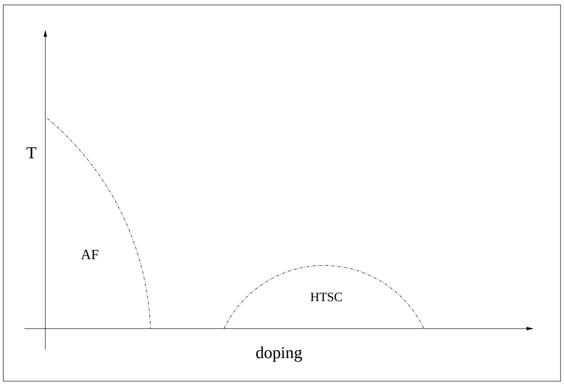

Figure 1.1: *Phase diagram of an antiferromagnetic precursor and the resulting high-temperature-superconductor.*

The setup of this work is as follows. Chapter 2 gives a brief overview of the theory that is essential to understand this work. This includes the fundamentals of spontaneous symmetry breaking, the Goldstone theorem, the linear and non-linear sigma models, and the topological nature of solitons and instantons. The Goldstone bosons and their low-energy dynamics are introduced in chapter 3. Chapter 4 illustrates the emerging homotopy groups and their relevance for this work. Electromagnetic fields are included and the decay channels of pions and magnons are investigated in chapter 5, concluding with the conversion of a photon into a magnon in an external magnetic field. Following this is a discussion of the decay of Skyrmions and baby-Skyrmions. Skyrmions decay due to the 't Hooft anomaly or a magnetic monopole, while baby-Skyrmions decay only via a discharging wire, which is the analogue of the magnetic monopole. The second to last chapter entails a comparison between the larger number of flavors in QCD and a larger symmetry group for the magnons. In both cases this leads to a Wess-Zumino-Witten term. Finally the 'Discussion and Conclusions' chapter summarizes the results.

Chapter 2

Symmetry Breaking, Massless Bosons, and Topological Configurations

2.1 Spontaneous Symmetry Breaking and the Goldstone Theorem

By 1960 Nambu and Goldstone noticed the significance of spontaneous symmetry breaking (SSB) in condensed matter physics [3]. The occurrence of SSB is especially eminent in ferro- and antiferromagnets (FM, AF). Nambu also recognized the importance of SSB in particle physics. In 1964 Higgs investigated the effect of gauging the symmetry on SSB [19]. He noticed that the spin 1 gauge field acquires a mass if a coupling between the conserved current associated with the symmetry and the gauge field exists. Weinberg and Salam used the principles of non-Abelian SSB in $SU(2)_L \times U(1)_Y$ gauge theory which unified electromagnetism and the weak nuclear force. By 1971 't Hooft proved that massless Yang-Mills fields are renormalisable [20] and thereby made electroweak theory generally accepted.

It is best to introduce spontaneous symmetry breaking with examples. Imagine a perfect cone, balanced on its tip. It is possible that the cone could remain in this position indefinitely, provided that there are no external forces whatsoever acting upon it. Yet an infinitesimal disturbance will cause the cone to fall to its side (fig.2.1).

There are three important traits to this situation: 1. The original situation displays a perfect symmetry (rotations around the z-axis). 2. The cone falls in a given direction if it experiences an infinitesimal disturbance. 3. The final state is degenerate and the cone can rotate into all other ground states.

An example more relevant to this work comes from condensed matter physics. Imagine a crystal. Each ion in the crystal carries spin and there is a coupling between two adjacent spins. This is what happens in a ferro- or antiferromagnet which are described by the

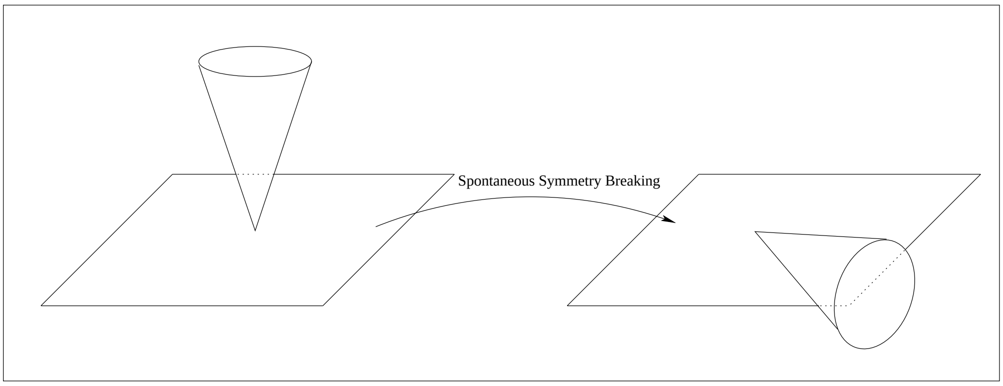

Figure 2.1: *Example of macroscopic spontaneous symmetry breaking.*

Heisenberg model. The corresponding Hamiltonian takes the form

$$H = -J \sum_{<i,j>} \vec{S}_i \cdot \vec{S}_j, \tag{2.1}$$

where $\vec{S}_i$ is the spin operator and J is a material dependent parameter whose sign determines if the crystal is ferro- or antiferromagnetic below a critical temperature. At high enough temperatures the coupling becomes negligible when compared to the dominating thermal fluctuations and the spins point in arbitrary directions. Below a critical temperature, however, the coupling takes effect and there is a preferred orientation of the spins (fig.2.2).

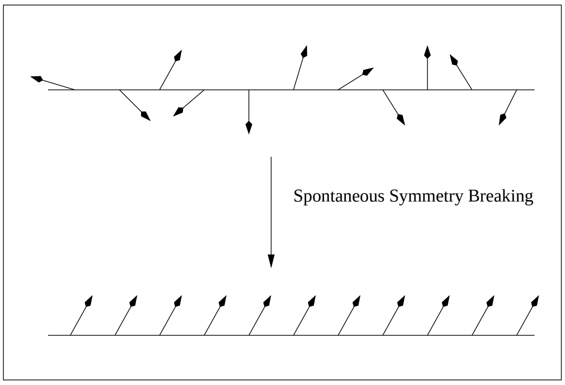

Figure 2.2: *Spontaneous symmetry breaking in a spin system.*

Likewise to the previous example there are three traits, the symmetric (disordered) state which is then broken to a final degenerate state, which can be continuously rotated into all other ground states. Which final state is chosen can not be predicted unless there is an external field which explicitly breaks the symmetry. In the solid state case there is a phase transition at a critical temperature.

The most common, and probably simplest theoretical example of SSB is provided by the linear σ-model where the Lagrangian takes the Euclidean form [21]

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^a\partial_\mu\phi^a + \frac{1}{2}m^2\phi^a\phi^a + \frac{\lambda}{4}(\phi^a\phi^a)^2. \tag{2.2}$$

The fields ϕ^a, $a \in \{1, 2, \ldots N\}$ are the components of an N-component real vector ϕ. The Lagrangian has a global $O(N)$ symmetry. The simplest example with a continuous symmetry is given by $N = 2$. This corresponds to an $O(2) \simeq U(1)$ symmetry. In this case ϕ can be written in the form $\phi = \phi^1 + i\phi^2$ and the Lagrangian can be rewritten in a complex form. This Lagrangian is invariant under the global transformation $\phi' = e^{i\Lambda}\phi$ where ϕ is the complex scalar field. If one chooses $m^2 < 0$ one finds for the vacuum expectation value $|\phi|^2 = -\frac{m^2}{2\lambda} = v^2$. By replacing the Cartesian coordinates with polar coordinates, $\phi = \rho(x)e^{i\theta(x)}$ the vacuum state becomes $(\sigma(x)+v)e^{i\theta(x)}$. It is easily seen that what previously looked like two massive fields ϕ_1 and ϕ_2 now becomes one massive field $\sigma(x)$ and one massless field $\theta(x)$. The phenomenon of new massless particles is explained by the Goldstone theorem.

The Goldstone theorem, formulated by Goldstone in 1962 [3], states that when a continuous global symmetry G is spontaneously broken to a remaining subgroup H then there are massless boson fields that live in the coset space G/H. The number of the so-called Goldstone bosons is given by the dimension of the coset space $dim(G/H) = dim(G) - dim(H)$.

The Goldstone theorem will now be illustrated with examples relevant to the models considered in later chapters. A general proof can be found in [3, 22, 21, 23].

For the magnon theory the relevant global symmetry group is given by $G = O(3)$, corresponding to $N = 3$. This is isomorphic to the $SU(2)_s$ symmetry. The Lagrangian takes the form

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial_\mu\phi^a\partial_\mu\phi^a + V(\phi), & (2.3)\\ V(\phi) &= \frac{\lambda}{4}(\phi^a\phi^a - v^2)^2, & (2.4)\\ \phi(x) &= \rho(x)(e_1(x), e_2(x), e_3(x)). & (2.5) \end{aligned}$$

The vacuum configuration minimizes the action and takes the form

$$\phi = \phi_0 = (0, 0, v). \tag{2.6}$$

This configuration is no longer $O(3)$ invariant, but there is a remaining $O(2) \simeq U(1)$ symmetry H. The coset space becomes $G/H = S^2 \simeq CP(1)$. The dimension of the coset space is 2, so one expects two Goldstone bosons (the two massless magnon degrees of freedom). Given a vacuum field with small fluctuations

$$\phi = \phi_0 + (m_1(x), m_2(x), \sigma(x)) = (m_1(x), m_2(x), v + \sigma(x)) \tag{2.7}$$

inserted into the Lagrangian eq.(2.3) a short calculations reveals

$$\mathcal{L} = \mathcal{L}_0 + \lambda v^2\sigma^2 + \mathcal{O}(m_i{}^4, m_i^2\sigma^2, m_i^2\sigma). \tag{2.8}$$

There are no quadratic terms for the fields m_i, while there is a quadratic term in σ. This means that the σ field obtains a mass given by $m^2 = 2\lambda v^2$ and there are two massless Goldstone boson fields. The Goldstone boson fields live in the coset space S^2. In later chapters the directions of ϕ will be denoted by $\vec{e}$ which is normalized to one.

For the pion theory the relevant symmetry group is given by the chiral symmetry $G = SU(2)_L \otimes SU(2)_R$ which is isomorphic to $O(4)$. This means that the linear sigma model with $N = 4$ provides a model for the pions. Now the corresponding field takes the form

$$\phi \quad = \quad \rho(x)U_\mu = \rho(x)(U_0, \vec{U}) \tag{2.9}$$

In this case, the global symmetry group G is broken to the remaining subgroup $H = SU(2)_{L=R}$. The coset space $G/H = SU(2)$ is of dimension 3, where the generators construct the three massless pion fields. These pion fields are contained in the matrix $U = U_0 + i\vec{U} \cdot \vec{\sigma} \in SU(2)$.

For any given N it can be determined that the Lagrangian of the linear sigma model has a global symmetry ($G = O(N)$). The vacuum state can be defined as

$$\phi \quad = \quad \phi_0 = (0, \ldots, 0, v). \tag{2.10}$$

This configuration is not invariant under transformations in G, but only has an $H = O(N-1)$ symmetry. The Goldstone boson fields live in the coset space $G/H = O(N)/O(N-1)$, of dimension $N - 1$.

The Goldstone theorem proves that, given any symmetry configuration, the number of massless fields is given by $dim(G/H) = dim(G) - dim(H)$. This principle is very general and works in both extremes: In some cases the remaining symmetry H is equal to G so the coset space contains only the identity. In this case there are no Goldstone bosons. On the other hand, one can contemplate a theory where there is no remaining symmetry so H is the identity and the coset space is equal to G. In this case the number of massless Goldstone bosons is the dimension of the symmetry group G.

It must be noted that the theorem only works for continuous symmetries such as the ones of the linear σ-model, $N \geq 2$, not however with discrete symmetries such as for $N = 1$. Theorem and its results are very general, and depend entirely on symmetry considerations. The form of the potential V is irrelevant (as long as it fulfills the symmetry conditions). The role of the Goldstone theorem in effective field theories is illustrated in [2].

2.2 Solitons and Instantons

In this section the terms soliton and instanton are defined. It will be discussed how and where these concepts enter the following work, and the consequences thereof. To conclude this introduction a few useful references are given that include a tangible example [24] to exemplify in what way Skyrmions carry various quantum numbers.

Solitons appear in non-linear field theories as non-perturbative stable configuration. They are a result of the topological properties of the theory and are not simply related to the quanta of the original field. Non-Abelian theories often fulfill these requirements, hence these solutions are expected to appear in particle physics[1] and, as will be seen in chapter 4.2, also in condensed matter physics. Solitons may form when the vacuum state is degenerate, as is the case after spontaneous symmetry breaking. In this case it is possible for a given field to have finite energy and at the same time the field can be in different potential minima at spacial infinity (where the energy is normalized to zero). This means that it is impossible to continuously deform a soliton solution into a vacuum state (as it is possible with all fields in perturbation theory), since an infinite amount of energy is required to shift the fields at spacial infinity from one potential well to the other. Although it may be easier to visualize a Skyrmion emerging from a degenerate vacuum, in this work they will be derived from purely topological considerations (chapter 4). The homotopy group defines how many topologically different non-trivial solutions can be found (so called Chern-Pontryagin (or homotopy) classes), and the boundary conditions of this homotopy determine the conserved topological charge.

The topological charge defines the magnitude of a 'kink', resulting from the change from one degenerated vacuum state to another. The simplest, and for this work relevant, example is again given by the linear sigma model. For the sake of illustration this will now be presented for the $N = 1$ case $O(1) \simeq \mathbb{Z}_2$. The models considered, $O(3)$ and $O(4)$, will have analogous homotopic properties. In the $\mathbb{Z}_2$ case the potential is given by fig.2.3.

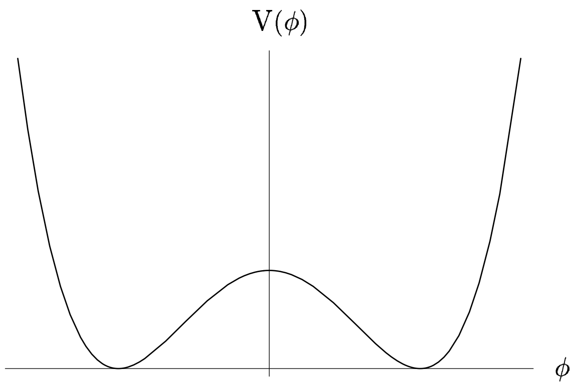

Figure 2.3: $N = 1$ *sigma model. This potential allows for a trivial and a non-trivial Chern-Pontryagin class, corresponding to one possible kink.*

For $t \to -\infty$ and $t \to +\infty$ the expectation value is in either one of the two potential minima. As time passes the field can travel from one to the other and back again, finding itself at temporal infinity either in the same or opposite minima as initially. If it is still (or again) in the initial minimum then this configuration corresponds to the trivial homotopy group. Any such configuration can be continuously deformed into the vacuum state. All perturbative configurations are of this form. If there is a net switch from one minimum to the other then the configuration is topologically non-trivial.

[1]The Standard Model does not have any solitons, but GUTs generally contain such topological configurations

For the (0+1)-dimensional σ model the equations of motion result in the solution

$$\phi(t) = \tanh(\sqrt{\frac{\lambda}{2}}(t - t_0)). \tag{2.11}$$

This result is an instanton, containing only the time degree of freedom. It is a field configuration that begins in one well at $t \to -\infty$ and crosses over to the second potential minimum for $t \to +\infty$. For the (1+1)-dimensional σ model the static solution of the equations of motion is given by

$$\phi(x) = \tanh(\alpha(x - x_0)). \tag{2.12}$$

This situation corresponds to a soliton which, for every instant in time, is characterized by the homotopy group $\Pi_0[\mathbb{Z}_2] = \mathbb{Z}_2$. (Note that $G = O(1)$ is a discrete symmetry so, after spontaneous symmetry breaking, there are no Goldstone bosons.)

The stability of the soliton solution is a consequence of topology and can be associated with a conserved topological charge and a corresponding topological current

$$j_\mu = \frac{1}{2}\epsilon_{\mu\nu}\partial_\nu\phi, \tag{2.13}$$

$$Q = \int_{-\infty}^{\infty} dx\; j_0 = \frac{1}{2}\int_{-\infty}^{\infty} dx\; \partial_1\phi = \phi(+\infty) - \phi(-\infty) \in \{0, 1\}. \tag{2.14}$$

It should be noted the the current does not follow from invariance of the Lagrangian under any symmetry transformations and hence is not a Noether current.

Further examples of solitons are vortex lines [25] (2 space-dimensions), Dirac monopoles [26] (will be discussed in chapter 6.1.2), and 't Hooft-Polyakov monopoles which necessarily appear in $SU(5)$ models [27].

Instantons are classical solutions of the Euclidean field equation and, unlike the solitons, are localized in both space and time. There is no spontaneous symmetry breaking requirement as in the soliton case. For solitons only the spacial boundary conditions were of importance, for instantons the boundary conditions for $t = \pm\infty$ also play a vital role. Solitons in $d+1$ dimensions are instantons in d dimensions. This will be used in chapter 6 when the Dirac monopole (soliton) in the pion model becomes a discharging wire (instanton) in the magnon model. In particle physics instantons allow for new decay channels such as the proton decay (experiments such as the Super-Kamiokande measured the half life of the proton to be greater than 1.610^{33} years. Theoretical calculations tent to result in a shorter half life) or as the $n + p \to e^+ + \bar{\nu}_\mu$ vertex (resulting in a half life for the deuteron of 10^{218} years!). The 't Hooft anomaly is an example where an instanton causes a baryon number violating process and will be discussed in chapter 6.

What is most interesting and exciting is when these topological solitons are relevant in descriptions of nature. In the QCD case, Skyrme argued successfully that the topological charge is, in fact, the baryon number and hence the topological current is the baryon current. This means that the solitons, which from now on will be referred to as Skyrmions, are in fact baryons. He also suggested that a topologically non-trivial solution would be in the form of a 'hedgehog' ansatz. This concept will become useful when discussing the importance of baby-Skyrmions in the destruction of antiferromagnetic order in section 4.2.

Skyrme also argued that (at least in (3+1) dimensional QCD) the Skyrmions must obey Fermi-Dirac statistics, which is essential if they are to be used to describe baryons, which for $N_c = 3$ are indeed fermions. In 2+1 dimensions, in which case the Skyrmions are referred to as baby-Skyrmions, there is no such fermion-boson quantization restriction. This will be discussed in chapter 4.2. In chapter 4.1 of this paper Skyrmions will appear in QCD representing baryons and it will be argued in chapter 4.2 that in ferro- and antiferromagnetism baby-Skyrmions are indeed related to electrons. The decay of the Skyrmion can by catalyzed by a Dirac monopole which is also realized through a soliton, and the discharge of a charged wire will be implemented via an instanton event (see chapter 6).

All this and more can be found in [22] and [21]. For a deeper understanding of the subject the reader is referred to [28] and [6]. Ref. [24] provides a simple example of how Skyrmions carry fractional quantum numbers (such as electric charge $\frac{e}{2}$).

2.3 The Non-Linear Sigma Model

The linear sigma model contains both fields, the massive fields and the massless Goldstone boson fields. In the following work only the massless excitations are considered. This is described within the nonlinear sigma model. The nonlinear sigma model is the limit of the linear sigma model, where the mass of the σ field is sent to infinity. The expectation value of the ground state ϕ_0 is held constant.

Take the scalar field ϕ to form an N-component unit vector field $\vec{e}(x)$ so that $\vec{e} \cdot \vec{e} = 1$. The vector $\vec{n}$ has N components and is subject to one constraint. Hence, the field has $N-1$ degrees of freedom. With the restriction of a global $O(N)$ symmetry to lowest order (neglecting constant terms) the most general Lagrangian takes the form

$$\mathcal{L} = \partial_\mu \vec{n} \cdot \partial_\mu \vec{n}. \tag{2.15}$$

This is essentially what will be done later to describe magnons with an $O(3)$ symmetry and pions with a symmetry isomorphic to $O(4)$.

Chapter 3

Chiral Perturbation Theory

This chapter illustrates a low-energy description of Goldstone bosons in both (3+1) dimensions for QCD and (2+1) dimensions for magnetism. This sets the framework for further developments. Skyrmions and baby-Skyrmions will be introduced in chapter 4. At this stage the similarities between the two theories are not surprising, since they originate from the same mathematical formalism. However, there are already some interesting and promising features, such as the quantization of the magnetization into integer or half-integer units. Such results, of course, do not prove the validity of a theory, however, they do give confidence that what is being done may have significance.

3.1 Pions in QCD

QCD, with massless up and down quarks, has a chiral symmetry group $G = SU(2)_R \otimes SU(2)_L \otimes U(1)_B$. This symmetry is spontaneously broken to a remaining symmetry group $H = SU(2)_{L=R} \otimes U(1)_B$. As illustrated in chapter 2, Goldstone has shown [3] that the spontaneous breaking of a continuous symmetry gives rise to massless bosons described by fields that live in the coset space $G/H = SU(2)$. The number of Goldstone boson fields is given by $dim(G) - dim(H) = 3$. The energy scale of QCD lies around 1 GeV, the mass of the proton. Compared to this scale the mass of pions is negligible, hence the massless Goldstone bosons are an adequate approximation when describing low-energy effects related to pions. In reality the symmetry is explicitly broken due to the small quark masses, giving the pions a small mass and reducing them to pseudo-Goldstone bosons. For the considerations of the model presented in this work only massless pions are considered. It is possible to add mass terms. These, however, do not change the phenomenology of the model. For accurate numerical calculations the mass terms must be included.

3.2 Magnons in Ferro- and Antiferromagnets

The magnetization (or staggered magnetization) caused through spin alignment is characterized by an $SO(3)$ symmetry. This is equivalent to an $SU(2)_s$ symmetry. The spins are coupled, either in a ferro- or antiferromagnetic manner, resulting in an ordered magnetization or staggered magnetization, respectively. The remaining, unbroken symmetry is a $U(1)$ symmetry. The presented model has a continuous $G = SU(2)_s$ symmetry which is then spontaneously broken into an $H = U(1)$ symmetry subgroup. Again, this results in $dim(G) - dim(H) = 2$ massless Goldstone bosons, which live in the coset space $G/H = CP(1)$. This time the symmetry is exact and the resulting Goldstone bosons describe massless magnons. It should be noted that there are no perfect ferromagnetic or antiferromagnetic materials. Anisotropies in the lattice structure result in a small explicit breaking of the spin symmetry. As a result the Goldstone bosons, just like the pions in QCD, obtain a small mass. This effect will not be considered explicitly. It is, however, essential that it happens. Otherwise effects such as the decay of a magnon into two photons, which will be discussed in chapter 5, would not be possible.

3.3 Effective Euclidean Pion Action

The Goldstone boson fields live in the coset space $G/H = SU(2)$, and the three resulting pions π^-, π^0, π^+ are described by the fields

$$U(x) \quad = \quad \exp(\frac{2i}{F_\pi}\pi^a(x)T^a), \tag{3.1}$$

where F_π is the pion decay constant and T^a are the generators of $SU(2)$ such that $\mathrm{Tr}\,(T^aT^b) = \frac{1}{2}\delta_{ab}$. This relation is fulfilled by the Pauli matrices $T^a = \frac{1}{2}\sigma^a$, with a, b $\in \{1,2,3\}$ and $\pi^- = \frac{1}{\sqrt{2}}(\pi^1 - i\pi^2), \pi^0 = \pi^3, \pi^+ = \frac{1}{\sqrt{2}}(\pi^1 + i\pi^2)$.

The lowest order Euclidean action is obtained using chiral perturbation theory, which is a derivative expansion as in [29]. This describes the low-energy dynamics of massless pions

$$S[U] \quad = \quad \int d^4x\, \frac{F_\pi^2}{4}\mathrm{Tr}\,[\partial_\mu U^\dagger \partial_\mu U]. \tag{3.2}$$

In the Goldstone boson approximation there is no rest energy contribution (the mass terms have been neglected, which will not change any qualitative results). This action contains all terms up to two derivatives that are invariant under global rotations of $G = SU(2)_L \otimes SU(2)_R \otimes U(1)_B$,

$$U'(x) = L^\dagger U(x) R \tag{3.3}$$

The constant vacuum field configuration, which is spontaneously selected as $U(x) = 1$ is only invariant under the subgroup $H = SU(2)_{L=R}$ as can be seen since $U' = L^\dagger \mathbb{1} R = \mathbb{1}$.

For small perturbations an expansion of eq.(3.1) can be made which takes the form

$$U(x) \approx \mathbb{1} + \frac{2i}{F_\pi}\pi^a(x)\frac{\sigma^a}{2} + \dots \tag{3.4}$$

This will be used in chapter 5 when electromagnetism is introduced.

3.4 Effective Euclidean Magnon Action

3.4.1 Magnons in Antiferromagnets

Antiferromagnets are important magnetic systems in solid state physics. It has been found that two-dimensional layered antiferromagnets arise as un-doped precursors of high-temperature superconductors. An example being $La_{2-x}Sr_xCuO_4$, an antiferromagnetic material which for high enough doping x loses its staggered magnetization and enters a high-temperature superconducting phase [30]. The staggered magnetization, defining the order of the magnet, is not a conserved quantity. Such antiferromagnets support excitations of the staggered magnetization, known as spin waves or magnons, the term used in this work. It is believed that magnons may contribute to the formation of Cooper pairs, hence the dynamics of such excitations are of great interest.

For antiferromagnetic magnons the action to first order is relatively simple. The vector $\vec{e}$ will be used to denote the magnon field which contains two modes (polarizations) corresponding to the dimension of the coset space, i.e.

$$\vec{e} = (e_1(x), e_2(x), e_3(x)), \quad \vec{e} \in G/H = S^2 \;\Rightarrow\; |\vec{e}| = 1. \tag{3.5}$$

The vector $\vec{e}$ itself describes the staggered magnetization. The magnons arise as small fluctuations around the ground state. Again, using the methods developed for chiral perturbation theory, a derivative expansion of the Euclidean action for the staggered magnetization field (which happens to be Lorentz invariant at this order of expansion) results in [1, 31, 32]

$$S[\vec{e}] \;=\; \int d^2x \int_{S^1} dt\, \frac{\varrho_s}{2}[\partial_i\vec{e}\cdot\partial_i\vec{e} + \frac{1}{c^2}\partial_t\vec{e}\cdot\partial_t\vec{e}]. \tag{3.6}$$

Here the index $i \in \{1,2\}$ labels the spatial dimensions and the convention of summing over repeated indices is used. The parameter c is the magnon velocity, an additional material-dependent constant like the previously introduced spin stiffness ϱ_s. With periodic boundary conditions it is possible to compactify Euclidean time onto a circle S^1 with the circumference $\beta = \frac{1}{T}$, thereby introducing a finite temperature T. It is easy to show that this action is invariant under $SO(3)$ transformations of the form

$$\vec{e}'(x) = O\vec{e}(x), \;\; O \in SO(3). \tag{3.7}$$

Again the spontaneously chosen vacuum state $\vec{e} = (0,0,1)$ is only invariant under transformations in the remaining symmetry group H, for which $O \in U(1)_s = SO(2)_s$.

Just as in the pion case, a vacuum state can be chosen with small perturbations added to account for the magnon fields m_a with $a \in \{1,2\}$. In this case the third axis has been chosen for the vacuum state,

$$\vec{e}(x) \;\approx\; (0,0,1) + \frac{1}{\sqrt{\varrho_s}}(m_1(x), m_2(x), -\frac{1}{2\sqrt{\varrho_s}}(m_1(x)^2 + m_2(x)^2)), \tag{3.8}$$

and ϱ_s is the spin stiffness, a material-dependant parameter. Antiferromagnetic magnons have a relativistic energy-momentum dispersion relation

$$E = |\vec{p}|c. \tag{3.9}$$

It should be noted that, even though to the lowest order the action is in a relativistic form, the underlying electron dynamics is non-relativistic, and the spin wave velocity c is, naturally, much smaller than the speed of light.

3.4.2 Magnons in Ferromagnets

Quantum Hall ferromagnets are two-dimensional (single- or multi-layered) structures with a uniform magnetization that results in an order parameter which is a conserved quantity. A microscopic model is given by the quantum Heisenberg model. These types of magnets also support fluctuations of the uniform magnetization in the form of magnons. The dynamics differ from the antiferromagnetic case and a topological term arises that forces the quantization of the total spin to an integer or half-integer value.

It is known that the dispersion relation, unlike in the antiferromagnetic case, is non-relativistic [1, 14, 31]

$$E = \frac{\varrho_s}{m}|\vec{p}|^2. \tag{3.10}$$

For the low-energy Euclidean action a non-relativistic derivative expansion is constructed resulting in

$$S[\vec{e}] = \int d^2x[\int_{S^1} dt\ \frac{\varrho_s}{2}\partial_i\vec{e}\cdot\partial_i\vec{e} - im\int_{H^2} dtd\tau\ \vec{e}\cdot(\partial_t\vec{e}\times\partial_\tau\vec{e})]. \tag{3.11}$$

Here the time-derivatives (kinetic contribution) are constructed considerably differently compared to the antiferromagnetic case. In order to find an invariant non-relativistic term, the kinetic contribution is deformed into an additional dimension τ where $\tau \in \{0,1\}$ and $\vec{e}(x)$ becomes $\vec{e}(x,\tau)$. Here τ is a deformation parameter that is not allowed to have an effect on the physics of the ferromagnetic magnon. The new extra dimension and the time-dimension form a hemisphere H^2. On the boundary $\partial H^2 = S^1$ only physical Euclidean time causes a contribution to the path integral. The boundary conditions are defined such that $\vec{e}(x,1)$ represents physical space time ($\tau = 1$ on the boundary ∂H^2) and $\vec{e}(x,0)$ is set to $(0,0,1)$. The extra dimension is illustrated in fig.3.1.

Now the model must be constrained in a fashion that ensures that the deformation parameter does not contribute to the physics of the system. Consider the relationship

$$\vec{e}\cdot(\partial_t\vec{e}\times\partial_\tau\vec{e}) \ = \ \partial_\tau[\frac{e_1\partial_t e_2 - e_2\partial_t e_1}{1+e_3}] \ = \ 2\partial_\tau(\sin^2\frac{\theta}{2}\partial_t\phi). \tag{3.12}$$

Here θ and ϕ are polar coordinates, that parameterize $\vec{e}$. Using the theorem of Gauss we can integrate over S^1 instead over H^2 with the condition that the function $\vec{e}(x,\tau)$ is analytic on H^2. It is clear however, that for example if $e_3 = -1$, the function contains a pole. In this case one must integrate along a path around the pole resulting in an additional contribution to the action.

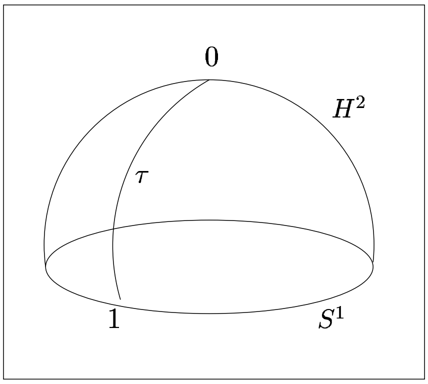

Figure 3.1: *Excursion to an extra dimension* τ.

The entire effect of this additional contribution takes the form of a topological term

$$n = \frac{1}{4\pi} \int_{S^2} dt\ d\tau\ \vec{e} \cdot (\partial_t \vec{e} \times \partial_\tau \vec{e}) \ \in \Pi_s[S^2] = \mathbb{Z}, \tag{3.13}$$

which is an integer winding number. This shows that with the exception of contributions from the boundary all additional contributions are given by $n \in \mathbb{Z}$. The topological term in the action is accompanied with the prefactor

$$\int d^2x\ 4\pi i m \ = \ 4\pi i M, \tag{3.14}$$

where m is the magnetization density and M the total spin of the magnet.

$$M \ = \ \int d^2x\ m. \tag{3.15}$$

In order for the topological term not to contribute to the physics, M must be either an integer or a half-integer, corresponding to the total spin of the entire magnet. The bulk ambiguity $4\pi i M n$ appears as a term in the action ($S[\vec{e}]$). The action enters the path integral in the form of $\exp(-S[\vec{e}])$, hence the bulk term contributes in the from $\exp(-4\pi i M n)$. For integer or half-integer values of M this factor is equal to one. Like this all ambiguities from the hemisphere are made invisible. It is, of course, known that the electron spin comes in half-integer units, but it is surprising that this property is demanded by the low-energy effective description of magnons, which is constructed in terms of the magnetization configurations of electrons in a ferromagnetic lattice.

Chapter 4

Skyrmions and Baby-Skyrmions

Topological considerations in an effective field theory can reveal new excitations and interesting contributions to the action of the theory. Information from the microscopic theory prove to be helpful in the physical interpretation of topological objects. Physical phenomena are thus revealed, which can not be derived from perturbation theory alone. The topological solutions themselves are not accessible with perturbation theory. Although such configurations are included in the theory, and certain properties such as decay channels will be discussed, the dynamics of these objects can not be derived in a systematic low-energy expansion. It is important to determine whether or not these new solutions contribute in a meaningful way to the investigation of a low-energy system. In this section we will consider the different homotopy groups and their meaning in QCD and their possible meaning in our effective magnon models. A homotopy group is defined for the maps from one space into another. There are different classes of such mappings which define certain characteristics of the possible field configurations. Parallels between the two models are drawn, with emphasis on the magnon model.

4.1 Skyrmions in QCD

In the effective theory the pion fields are described by the function $U(x) : \mathbb{R}^4 \rightarrow S^3$. This is a map from four-dimensional space-time to the three-dimensional surface of a hypersphere. Provided we have appropriate boundary conditions, it is possible to compactify space-time onto the four-dimensional surface of a sphere ($\mathbb{R}^4 \rightarrow S^4$) as is illustrated for the mapping of a two-dimensional Euclidean space onto the surface of a sphere (see fig.4.1). A stereographic projection is an example of a possible transformation, where all points at infinity are projected onto the pole.

A mapping from one space into another, such as it is defined by $U(x)$, is characterized by the homotopy groups

$$\Pi_3[S^3] = \mathbb{Z}, \quad \Pi_4[S^3] = \mathbb{Z}_2. \tag{4.1}$$

The first expression is a homotopy group that characterizes a pion field configuration at any given fixed time. In this case, only the spacial components are compactified. The

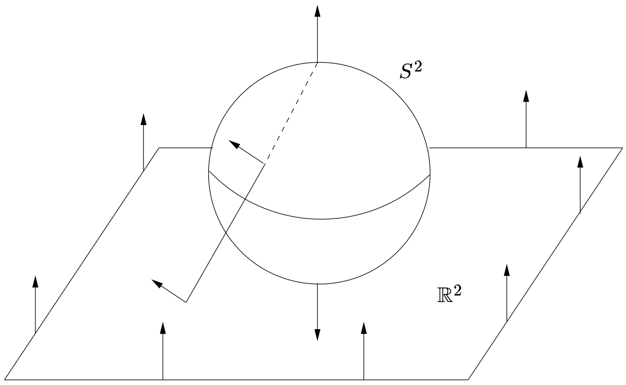

Figure 4.1: *Two-dimensional example of a topologically trivial mapping from Euclidean space to the surface of a sphere with appropriate boundary conditions.*

second expression is the fourth homotopy class of S^3. In this case, space and time have been compactified onto a sphere and then mapped with the function $U(x)$ onto S^3. The homotopy group which is equal to $\mathbb{Z}_2$ corresponds to two classes of configurations. There are trivial configurations that can always be transformed into the vacuum (such as the fields contributing to perturbation theory) and non-trivial fields. As will be seen later, this is related to the two possible statistics that a particle can have, i.e. if it is a fermion or a boson.

This takes us to the topological current and charge. In our effective pion theory the topological charge (winding number of $U(x)$) is defined as

$$B = \frac{1}{24\pi^2} \int_{S^3} d^3x \; \epsilon_{ijk} \mathrm{Tr}\, [U^\dagger \partial_i U U^\dagger \partial_j U U^\dagger \partial_k U] \in \mathbb{Z}. \tag{4.2}$$

As was shown by Skyrme [4], in QCD the topological charge can be identified as the baryon number. Hence, pion field configurations may describe effective protons and neutrons or other baryons (in this case only consisting of up and down quarks) through their topological solitons. We also have a topological current, the baryon current, defined as

$$j_\mu = \frac{1}{24\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\, [U^\dagger \partial_\nu U U^\dagger \partial_\rho U U^\dagger \partial_\sigma U] \in \mathbb{Z}, \tag{4.3}$$

which is conserved ($\partial_\mu j_\mu = 0$). Thereby the topological charge $B = \int d^3x \; j_0$ is also conserved. Topologically non-trivial solutions are called solitons, in this specific case they are named Skyrmions, after their discoverer Skyrme [6].

The fourth homotopy group also contributes to the action in an interesting way. In this case the time-dimension is included in the map and changes the homotopy group to $\Pi_4[S^3] = \mathbb{Z}_2$. The elements of this homotopy group take the form

$$\mathrm{Sign}[U] = \pm 1. \tag{4.4}$$

contributing to the path integral as an additional factor

$$Z = \int \mathcal{D}U \exp(-S[U]) \mathrm{Sign}[U]^{N_c}. \tag{4.5}$$

This corresponds to the statistical properties of the Skyrmions, in this case fermions or bosons. For three colors, $N_c = 3$, there are three quarks in every baryon which form a color singlet. Three spin $\frac{1}{2}$ quarks form a fermionic state. Any configuration with Sign$[U]$ $= 1$ is in the same homotopy class as the vacuum state and can therefore be continuously deformed into the vacuum. The Pauli principle requires that the sign term enters the path integral. While a bosonic Skyrmion (for an odd number of colors), does not receive a contribution from this term, a fermion can contribute to the path integral with a negative prefactor, hence, since baryons are fermions N_c must be odd for Skyrmions. The statistics and spin of a Skyrmion is illustrated in fig.4.2.

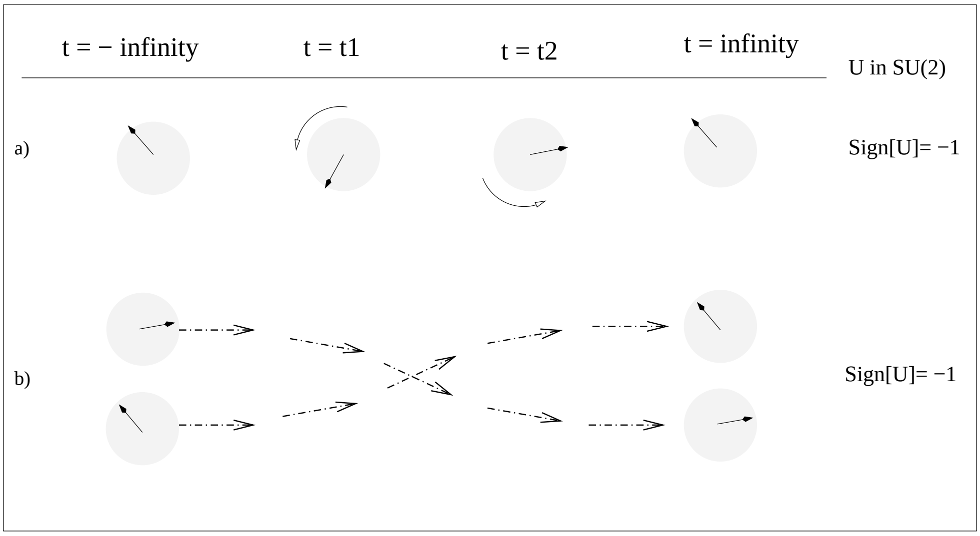

Figure 4.2: *Statistics and spin of a Skyrmion. Part (a) shows how a single Skyrmion behaves under rotation. Part (b) illustrates the statistics behavior of two Skyrmions interchanging their position.*

The magnon partition function includes the effective Euclidean action and the statistics factor with the statistics parameter N_c. If the field $U(x)$ describes a Skyrmion which rotates by 2π then Sign$[U]= -1$. This means that Skyrmions must have half-integer spin for odd N_c and integer spin for even N_c.

4.2 Baby-Skyrmions in Ferro- and Antiferromagnets

What has just been explained for the QCD case can be repeated in an analogous fashion for the magnon theories. In this case the topological excitations are called baby-Skyrmions, which also carry a topological charge [33]. At any given time the homotopy group $\Pi_2[S^2] =$

$\mathbb{Z}$ ensures that the magnon field configurations are characterized by an integer winding number

$$B = \frac{1}{8\pi} \int_{S^2} d^2x \, \epsilon_{ij} \vec{e} \cdot (\partial_i \vec{e} \times \partial_j \vec{e}) \in \mathbb{Z}. \qquad (4.6)$$

and the corresponding conserved topological current

$$j_\mu = \frac{1}{8\pi} \epsilon_{\mu\nu\rho} \vec{e} \cdot (\partial_\nu \vec{e} \times \partial_\rho \vec{e}). \qquad (4.7)$$

In the QCD case the topological charge can be identified with the baryon number. In QCD, pions consist of quarks that are confined and can not be observed in isolation. A special pion field configuration can support topological solitons, the Skyrmions which are, in fact, baryons and in a microscopic model also made up of quarks. In the magnon model it will be argued that the baby-Skyrmions, which in a microscopic theory (for example the Heisenberg model) are made of electrons, carry an electric charge. Depending on the specific magnetic material at hand, the topological charge may in fact be the electric charge. This has defining implications on the nature of baby-Skyrmions and the interaction of magnons with electromagnetism as will be seen in chapter 5.

There exists an additional important non-trivial homotopy group, which characterizes the braiding of baby-Skyrmion paths in time,

$$\Pi_3[S^2] = \mathbb{Z}. \qquad (4.8)$$

Again, just like in the QCD case, this homotopy group includes compactified time and characterizes the statistical behavior of baby-Skyrmions. This leads to the Hopf term $H[\vec{e}]$ which enters the partition function in the form

$$Z = \int \mathcal{D}\vec{e} \exp(-S[\vec{e}]) \exp(i\theta H[\vec{e}]), \qquad (4.9)$$

where $\theta \in [-\pi, \pi]$ is a material property just like ϱ_s. The parameter θ is the so-called anyon angle and it determines the statistics of the baby-Skyrmion [34]. The value $\theta = 0$ corresponds to bosons and $\theta = \pi$ to fermions ($e^{i\pi} = -1 \Rightarrow$ fermion, $e^0 = 1 \Rightarrow$ boson). As was seen in the QCD case, there are only fermions or bosons. In the magnon theory the Hopf term replaces Sign$[U]$, and the statistics parameter N_c for QCD is replaced with the anyon angle θ., which allows for any statistics. Through this it is possible to define the fermion number F as

$$F = \frac{\theta}{\pi}. \qquad (4.10)$$

Up until now the arguments have been completely general, and in what way the spins enter the model has not been discussed. From now on however, the electrons are the spin carriers. This will be explored in more detail in the following chapter. Since electrons carry a fermion number and an electric charge it is perfectly natural that the electric charge is related to the anyon angle. In this case the charge of a given baby-Skyrmion is given by

$$q = -\frac{\theta}{\pi} \cdot e \qquad (4.11)$$

where $-e$ is the charge of a single electron. For $\theta = \pi$ the topological charge is the total electric charge of a magnon field configuration. The anyon angle also defines the spin and statistics of baby-Skyrmions and thereby its fermion number, which is of course the number of electrons a given baby-Skyrmion may contain. In an analogous manner to the QCD case the electrons may be confined inside a baby-Skyrmion. This means that in the low-energy approximation the electrons themselves are not accessible inside the Skyrmions. A single Skyrmion may contain any fractional number of electrons.

This result is very general and applies to both ferromagnets and antiferromagnets. The importance of baby-Skyrmions in antiferromagnets has been discussed in [35, 36]. If the topological charge corresponds to the electric charge then the baby-Skyrmions are the cause for the destruction of the antiferromagnetic order prior to the superconducting phase (see fig.1.1) as is discussed in [15, 30, 16]. In the case of quantum Hall ferromagnets it has been argued that the baby-Skyrmions are, in fact, quasi particles with quantum numbers of single electrons [37, 38, 39]. Antiferromagnets described by the Heisenberg model at half-filling contain baby-Skyrmions which appear to be bosons [34, 40]. This may mean that $\theta = 0$, in which case baby-Skyrmions carry no electric charge. However, it is also possible that $\theta = 2\pi$ in which case there are two electrons confined in each baby-Skyrmion (Cooper pair), resulting in an electric charge of $-2e$. These situations are illustrated in fig.4.3. From this model itself it is not apparent why Skyrmions in antiferromagnets are bosons. This restriction is purely based on the works of [34, 40].

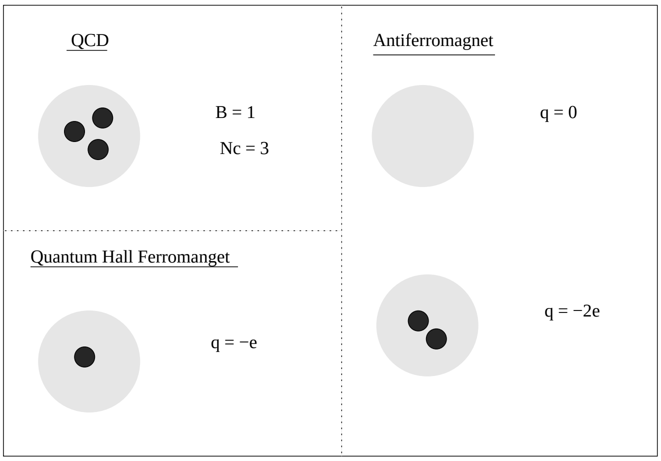

Figure 4.3: *a) QCD Skyrmion (baryon made of three quarks) b) Baby-Skyrmion in Antiferromagnet containing zero ($\theta = 0$) or two electrons ($\theta = 2\pi$) c) Baby-Skyrmion in Quantum Hall Ferromagnet containing one electron ($\theta = \pi$).*

An additional consequence for $\theta = \pi$ or 2π is the influence of doping on the antiferromagnet. If the baby-Skyrmions are indeed related to electrons then by doping an antiferromagnet additional baby-Skyrmions are added to the system. The 'hedgehog' structure (fig.4.4), resulting from a non-trivial rotation of the spin orientations, conflicts with the ordered orientation of the staggered magnetization of the un-doped antiferromagnet. It is intuitively clear that if sufficiently many baby-Skyrmions are added to the system, the entire antiferromagnetic structure will collapse [30, 41].

Finally, the magnon partition function takes the form

$$Z = \int \mathcal{D}\vec{e} \exp(-S[\vec{e}]) \exp(i\theta H[\vec{e}]). \tag{4.12}$$

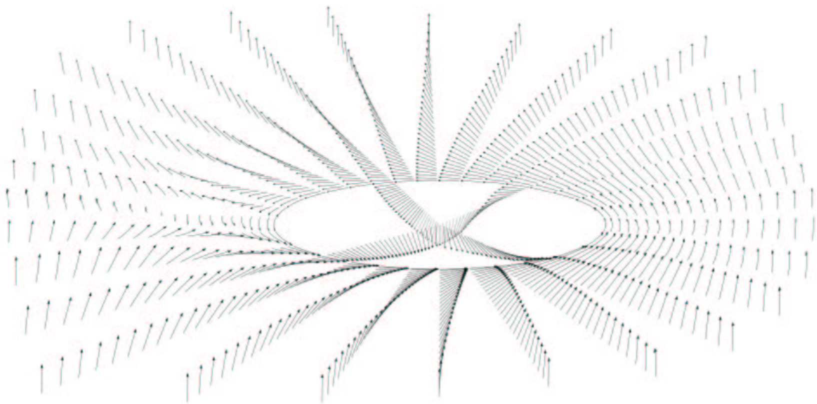

Figure 4.4: *Skyrmion spin configuration, at the center the magnetization faces down and then slowly rotates by* 180^o *by the time it reaches spacial infinity. The structure is rotationally invariant but there is no symmetry of reflection. There is also no singularity at the center (figure taken from [14]).*

Chapter 5

Introducing Electromagnetism

5.1 Pions and Photons

The most general possibility is to gauge the entire $SU(2)_L \otimes SU(2)_R$ symmetry. This is done by introducing the covariant derivatives and the gauge fields W_μ and B_μ that couple from the left or right respectively to the pion field (additional details for gauging $SU(2)$ symmetries is given in Appendix A.2). An $SU(2)_L \otimes SU(2)_R$ gauge transformation takes the form

$$\begin{aligned} U(x)' &= L^\dagger(x)U(x)R(x), && (5.1)\\ D_\mu U &\rightarrow \partial_\mu + W_\mu U - UB_\mu, && (5.2)\\ W'_\mu &= L^\dagger(x)(W_\mu + \partial_\mu)L(x), \\ B'_\mu &= R^\dagger(x)(B_\mu + \partial_\mu)R(x). \end{aligned}$$

The pion field $U(x)$ is transformed by the matrices $L \in SU(2)_L$ and $R \in SU(2)_R$.

From the Standard Model it is known that at the quark level the charge is given by

$$Q = T_L^3 + T_R^3 + \frac{1}{2}B, \tag{5.3}$$

where T_L^3 and T_R^3 are the generators of $SU(2)_L$ and $SU(2)_R$ respectively and B is the baryon number. For example the charge of the right-handed up quark is given by

$$Q_{u_L} = \frac{1}{2} + 0 + \frac{1}{2}\frac{1}{3} = \frac{2}{3}.$$

As one will see later, the coupling of the photon to the neutral pion occurs due to the charge generated by the baryon number.

In the case of the electromagnetic field there is no distinction between left- and right-handed coupling. This makes the situation easier. The fields $W_\mu^3(x)$ and $B_\mu^3(x)$ can then be replaced by $ieA_\mu T^3$. This is because $U(1)_{em}$ is a subgroup of $SU(2)_{L=R}$. The local transformations are in $U(1)_{em}$ and the following relations apply

$$
\begin{aligned}
U(x)' &= D(x)^\dagger U(x) D(x), &(5.4)\\
D_\mu U(x) &= \partial_\mu U(x) + ieA_\mu(x)[T^3, U(x)], &(5.5)\\
(D_\mu U)' &= D(x)^\dagger D_\mu U(x) D(x), &(5.6)
\end{aligned}
$$

where $D_(x)$ are diagonal matrices from the $U(1)$ subgroup of $SU(2)$.

The baryon number in our effective model is only present through the topological solitons. We must therefore concentrate on the topological current. The new, gauge invariant and conserved current is the so-called Goldstone-Wilczek current, a modification of the Skyrme current eq.(4.3) and takes the form

$$
\begin{aligned}
j_\mu^{GW} &= \frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\,[(U^\dagger D_\nu U)(U^\dagger D_\rho U)(U^\dagger D_\sigma U)]\\
&\quad -\frac{ie}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}F_{\nu\rho}\mathrm{Tr}\,[T^3(D_\sigma U U^\dagger + U^\dagger D_\sigma U)]. \qquad (5.7)
\end{aligned}
$$

It was shown in a paper by Callan and Witten [10] that the topological charge remains unaffected after gauging the symmetries. This can be seen by writing the topological current in the form

$$
\begin{aligned}
j_\mu^{GW} &= \frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\,[(U^\dagger \partial_\nu U)(U^\dagger \partial_\rho U)(U^\dagger \partial_\sigma U)]\\
&\quad +\frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}\partial_\nu(3ieA_\rho\mathrm{Tr}\,[Q(U^\dagger\partial_\sigma U + \partial_\sigma U U^\dagger)]),
\end{aligned}
$$

where $Q = \frac{1}{6}I + T^3$. The second term is a total divergence. Therefore, as long as there are no singularities and if the surface terms vanish at infinity, there is no additional contribution to the topological charge. Hence, the baryon number of a particular configuration is not affected by gauging the symmetry.

Using this new invariant and conserved topological current it is easy to couple the pions to the electromagnetic field. This adds to a new contribution to the action, the Goldstone-Wilczek term

$$
S_{GW}[U, A_\mu] = \frac{e}{2}\int d^4x\; A_\mu j_\mu^{GW}. \qquad (5.8)
$$

The factor $\frac{1}{2}$ comes from the factor $\frac{1}{2}$ in front of the baryon number that contributes to the charge of the baryons at the quark level as was shown in eq.(5.3). The new (more) complete path integral takes the from

$$
Z[A_\mu] = \int \mathcal{D}U\,\exp(-S[U])\mathrm{Sign}[U]^{N_c}\exp(iS_{GW}[U, A_\mu]). \qquad (5.9)
$$

5.1.1 Pion-Photon coupling

Now it is possible to investigate the action and determine interesting vertices. Especially surprising is the vertex for the anomalous $\pi^0 \to \gamma\gamma$ decay. This the the preferred decay channel for the neutral pion and the dominant (at this stage total) contribution comes from the Goldstone-Wilczek term. Consider the general form of the pion field

$$U(x) = \exp(2i\pi^a(x)\frac{T^a}{F_\pi}) \approx 1 + 2i\pi^a(x)\frac{T^a}{F_\pi}, \quad (5.10)$$

where $T^a = \frac{\sigma^a}{2}$ are the generators of the $SU(2)$ algebra. If we choose $\pi^1 = \pi^2 = 0$ and write $\pi^3 = \pi^0$ the resulting pion field is of the form

$$U \approx 1 + 2i\pi^0\frac{T^3}{F_\pi}. \quad (5.11)$$

After a little algebra and discarding terms $\mathcal{O}(\pi^{0\,2})$ (only terms with two photons and one π^0 are of interest here) one obtains the vertex

$$\mathcal{L}_{\pi^0\gamma\gamma} = \frac{-ie^2}{32\pi^2 F_\pi}\epsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F^{\rho\sigma}\pi^0, \quad (5.12)$$

where $F_{\mu\nu}$ is the field strength tensor of electromagnetism. This term accounts for the decay of a neutral pion into two photons ($\pi^0 \to \gamma\gamma$). The process is well know and the decay parameters have been measured to very high accuracy. Therefore it is very important that this term appears in our effective theory. Looking at other terms in the action, further experimentally observed vertices can be determined. An additional important example is

$$\mathcal{L}_{\pi^0\pi^+\pi^-\gamma} = \frac{ie}{4\pi^2 F_\pi^3}\epsilon_{\mu\nu\rho\sigma}A_\mu\partial_\nu\pi^0\partial_\rho\pi^+\partial_\sigma\pi^-, \quad (5.13)$$

(with $\pi^\pm = \frac{1}{\sqrt{2}}(\pi^1 \pm i\pi^2)$) which describes the vertex of all three pions and a photon (box diagram). For example if a π^+ collides with a π^- it is possible that a π^0 and a photon emerge (see fig.5.1).

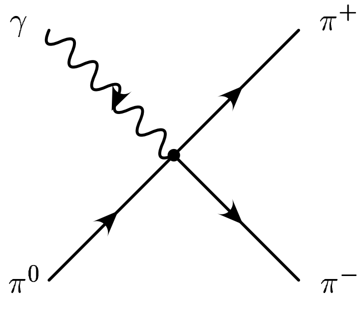

Figure 5.1: *Vertex for the three pions and a photon.*

5.2 Magnons and Photons

It has just been exemplified how a neutral pion couples directly to electromagnetism via the Goldstone-Wilczek current of the pion field. In the same way it can be shown that the coupling of magnons (which are also neutral) to the electromagnetic field is also possible and occurs through the same mechanism as in the QCD case. This section will demonstrate how the analogue of the Goldstone-Wilczek current gives rise to the anomalous decay of a magnon into two photons. This will only occur if the topological charge turns out to be equal to the electric charge as was conceived in chapter 4. Finally, the conversion process of a photon into a magnon in an external magnetic field will be illustrated. This process may be experimentally verifiable and thereby solve the mystery of the nature of the topological charge.

5.2.1 The Pauli Equation and the $SU(2)_s$ Symmetry

A complete, relativistic description of charged fermions interacting with an electromagnetic field is given by the Dirac equation

$$i\hbar\partial_t\Psi \;=\; (\alpha(i\vec{\nabla}+e\vec{A})+\beta m_0-e\phi)\Psi, \tag{5.14}$$

where e and m_0 are the charge and mass of the electron, respectively, α and β denote the usual 4×4 Dirac matrices. Here c is the speed of light, ϕ and $\vec{A}$ are the usual electromagnetic potentials forming the 4-vector A_μ. The Dirac 4-spinor Ψ contains two 2-spinors for the electron and anti-electron ($\Psi=(\psi,\chi)$). An expansion of eq.(5.14) according to the scheme of Foldy and Wouthuysen, for the ψ spinor up to $\mathcal{O}(\frac{1}{m_0^3})$, takes the form

$$\begin{aligned} i\hbar\partial_t\psi \;=\; & m_0\psi-e\phi\psi+\frac{e}{2m_0}\vec{B}\cdot\vec{\sigma}\psi+\frac{1}{2m_0}\Pi^2\psi \\ & \frac{e}{8m_0^2}[\vec{\Pi}\cdot(\vec{\sigma}\times\vec{E})+(\vec{\sigma}\times\vec{E})\cdot\vec{\Pi}]\psi+\frac{e}{8m_0^2}\vec{\nabla}\cdot\vec{E}\psi+\mathcal{O}(\frac{1}{m_0^3}), \end{aligned} \tag{5.15}$$

where $\vec{\Pi}$ is the canonical momentum operator defined as

$$\vec{\Pi} \;=\; \frac{\hbar}{i}\vec{\nabla}+\frac{e}{c}\vec{A}. \tag{5.16}$$

The first term on the right-hand side of eq.(5.15) is the rest energy of the electron, followed by the potential energy in the electrostatic potential ϕ. The third term describes the Zeeman splitting in a magnetic field $\vec{B}=\vec{\nabla}\times\vec{A}$ and $\vec{\sigma}$ are the Pauli matrices. The fourth and fifth terms are the kinetic energy of the electron and the spin-orbit interactions in the electric field $\vec{E}$, respectively. Finally, one finds the Darwin term, a higher relativistic term proportional to $\vec{\nabla}\cdot\vec{E}=4\pi\rho$ which will be absorbed into a one-body potential term. The above equation is the standard Pauli equation plus the rest mass term, the spin-orbit coupling term and the Darwin term, the latter two being relativistic corrections.

A little algebra and again dropping all terms of $\mathcal{O}(\frac{1}{m_0^3})$ and higher, it is possible to write eq.(5.15) in the form

$$i(\partial_t-ie\phi+\frac{ie}{2m}\vec{B}\cdot\vec{\sigma}+\frac{ie}{8m^2}\vec{\nabla}\cdot\vec{E})\psi \;=\; -\frac{1}{2m}(\vec{\nabla}+ie\vec{A}-\frac{ie}{4m}\vec{E}\times\vec{\sigma})^2\psi. \tag{5.17}$$

Using the $U(1)_{em}\otimes SU(2)_s$ covariant derivatives this takes the form

$$iD_t\psi \;=\; -\frac{1}{2m}D_iD_i\psi, \tag{5.18}$$

where

$$D_t = \partial_t + ieA_0 + W_0, \quad D_i = \partial_i + ieA_i + W_i, \quad W_\mu = i\vec{W}_\mu \cdot \frac{\vec{\sigma}}{2}. \tag{5.19}$$

In this case A_μ is the usual $U(1)_{em}$ vector potential with the exception of $A_0 = -\phi + \frac{e}{8m^2}\vec{\nabla} \cdot \vec{E}$ which contains the Darwin term in addition to ϕ.

Using this notation the correspondence between the $SU(2)_s$ non-Abelian gauge fields W_μ and the electromagnetic fields $\vec{E}$ and $\vec{B}$ becomes apparent (where $\mu = \frac{ge}{2m}$ is the anomalous magnetic moment. Up to QED corrections $g = 2$)

$$W_t^a = \mu B^a \quad W_i^a = \frac{\mu}{2}\epsilon_{iab}E^b. \tag{5.20}$$

Now it is clear that in order to describe all phenomena of electromagnetism in ferro- and antiferromagnets the $SU(2)_s$ symmetry must be gauged. This will be done in the following section and will lead to the anomalous decay of the magnon into two photons via the Goldstone-Wilczek current. A further example where as similar formalism is used is given by Anandan who has shown that if a weak electromagnetic field is probed using a low-energy neutral dipole, then it would also appear as an $SU(2)_s$ gauge field [42].

5.2.2 How Electromagnetic Fields Manifest themselves as non-Abelian Vector Potentials

Fröhlich and Studer [43] have shown that the Pauli equation, which describes non-relativistic charged fermions interacting with electromagnetic fields, has a hidden $SU(2)_s$ symmetry when considering all terms up to order $\frac{1}{m_e^3}$ (where m_e is the mass of an electron). As a result the physical fields $\vec{E}$ and $\vec{B}$ appear in the form of non-Abelian vector potentials, and are described by an $SU(2)_s$ gauge field. This appears to be a coincidence of nature, with very useful consequences. Usually the underlying symmetry is an $U(1)_{em}$ symmetry. This would mean there are many more terms allowed (when only considering symmetry constraints) than if there is an underlying $SU(2)_s$ symmetry. This gives our $SU(2)_s$ non-Abelian gauge theory much more predictive power. The Pauli equation can be viewed as the analogue of the Standard Model for ferro- and antiferromagnets. Although it is not possible in practice to derive phenomena such as high-temperature superconductivity and the quantum Hall effect from this model, it is still useful to consider which symmetries are implemented since they will also be respected in an effective description of these materials.

There are, however, some complications when describing electromagnetism with $SU(2)_s$ symmetry constraints. For the fields $\vec{E}$ and $\vec{B}$ to be dynamical a $F_{\mu\nu}F^{\mu\nu}$ term is required. This term is not $SU(2)_s$ invariant and is therefore not included in our low-energy effective theory. One concludes that the electromagnetic fields must be split into two categories. The high-frequency contributions are responsible for the inner construction of the lattice and are always present through the ionic bonds of the crystal. These will not be considered explicitly in our model. The second 'type' are low frequency photons. These can be interpreted as external, almost static fields (wavelength λ must be greater than the microscopic crystal structure) which do not contribute to the dynamics of the crystal. The self-interaction of the external electromagnetic field does not effect the dynamics of the magnons or baby-Skyrmions inside the magnet. For this reason the self-interaction terms are not explicitly included in the formalism, but are however contained within the field $\vec{e}$. In this limit the $SU(2)_s$ symmetry is present. The $SU(2)_s$ symmetry can be gauged and the gauge field W_μ can be associated with $\vec{E}$ and $\vec{B}$. How this can be done has been demonstrated by Fröhlich and Studer [43]. The following section illustrates this procedure and applies it specifically to this work.

5.2.3 Magnons and Gauge Fields

When gauging the symmetry of the magnon fields it is often easier to use the $SU(2)_s$ representation instead of the $SO(3)_s$ representation. The following expressions show the transformation from $SO(3)_s$ to $SU(2)_s$, which then can be applied to the terms found in section 3.4. The $SU(2)_s$ (or $SO(3)_s$) symmetry is gauged by introducing covariant derivatives (in this section $SU(2)_s$ is gauged, for a full description of electromagnetism $U(1)_{em}$ must also be gauged, since there are no charged particles, gauging $U(1)_{em}$ does not involve a covariant derivative). To gauge the action that has been constructed in section 3.4 one replaces the derivatives by $SU(2)_s$ covariant derivatives that take the form

$$\begin{gathered}D_\mu \vec{e} = \partial_\mu \vec{e} + \vec{e} \times \vec{W}_\mu, \quad D_\mu P = \partial_\mu P + [W_\mu, P], \\ \vec{W}_{\mu\nu} = \partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu - \vec{W}_\mu \times \vec{W}_\nu, \quad W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + [W_\mu, W_\nu], \qquad (5.21)\end{gathered}$$

where W_μ and $W_{\mu\nu}$ are defined as

$$W_\mu = i W_\mu^a T^a = \frac{i}{2} \vec{W}_\mu \cdot \vec{\sigma}, \quad W_{\mu\nu} = \frac{i}{2} \vec{W}_{\mu\nu} \cdot \vec{\sigma}, \qquad (5.22)$$

and $P = \frac{1+\vec{e}\cdot\vec{\sigma}}{2} \in \mathbb{C}\mathrm{P}(1)$ parameterizes the magnon fields. The transformation properties of the magnon field $P(x)$ and the gauge field W_μ are given by

$$P' = g^\dagger P g, \quad W'_\mu \ = \ g^\dagger (W_\mu + \partial_\mu) g. \qquad (5.23)$$

These are the tools needed to transform the terms in the action for the antiferromagnetic and ferromagnetic magnons. With a little algebra one obtains

$$\begin{aligned}\partial_\mu \vec{e} \cdot \partial_\nu \vec{e} &= 2\mathrm{Tr}\,[\partial_\mu P \partial_\nu P], \\ D_\mu \vec{e} \cdot D_\nu \vec{e} &= 2\mathrm{Tr}\,[D_\mu P D_\nu P], \\ \vec{e} \cdot \vec{W}_t &= \frac{2}{i}\mathrm{Tr}\,[P W_t], \\ \vec{e} \cdot \vec{W}_{\nu\rho} &= \frac{2}{i}\mathrm{Tr}\,[P W_{\nu\rho}], \\ \epsilon_{\mu\nu} \vec{e} \cdot (\partial_\mu \vec{e} \times \partial_\nu \vec{e}) &= \frac{4}{i}\epsilon_{\mu\nu}\mathrm{Tr}\,[P \partial_\mu P \partial_\nu P], \\ \epsilon_{\mu\nu} \vec{e} \cdot (D_\mu \vec{e} \times D_\nu \vec{e}) &= \frac{4}{i}\epsilon_{\mu\nu}\mathrm{Tr}\,[P D_\mu P D_\nu P]. \qquad (5.24)\end{aligned}$$

Using this notation, for the antiferromagnetic case the gauged action with the newly introduced covariant derivatives takes the equivalent forms

$$S[\vec{e},\vec{W}_\mu] = \int d^2x \int_{S^1} dt\, \frac{\varrho_s}{2}[D_i\vec{e}\cdot D_i\vec{e} + \frac{1}{c^2}D_t\vec{e}\cdot D_t\vec{e}],$$
$$S[P,W_\mu] = \int d^2x \int_{S^1} dt\, \varrho_s \mathrm{Tr}\,[D_iP\cdot D_iP + \frac{1}{c^2}D_tP\cdot D_tP]. \quad (5.25)$$

All these terms are clearly gauge invariant as was shown earlier. Now we turn to the ferromagnetic case. This is slightly more complicated because of the additional deformation parameter that must be included. As a consequence, an additional term must be added to make the action gauge invariant.

$$\begin{aligned} S[\vec{e},\vec{W}_\mu] &= \int d^2x[\int_{S^1} dt\, \frac{\varrho_s}{2}D_i\vec{e}\cdot D_i\vec{e} \\ &\quad -im\int_{H^2} dtd\tau\, \vec{e}\cdot(\partial_t\vec{e}\times\partial_\tau\vec{e}) + im\int_{S^1} dt\, \vec{e}\cdot\vec{W}_t], \\ S[P,W_\mu] &= \int d^2x[\int_{S^1} dt\, \varrho_s\mathrm{Tr}\,[D_iP\cdot D_iP] \\ &\quad -4m\int_{H^2} dtd\tau\, \mathrm{Tr}\,[P\partial_tP\partial_\tau P] + 4m\int_{S^1} dt\, \mathrm{Tr}\,[PW_t]]. \end{aligned} \quad (5.26)$$

A short calculation reveals that after an $SU(2)_s$ transformation the second term is not invariant but leaves an additional term

$$-\int_{H^2} dtd\tau\, \mathrm{Tr}\,[\partial_\tau(Pg\partial_tg^\dagger)] = -\int_{S^1} dt\, \mathrm{Tr}\,[Pg\partial_tg^\dagger], \quad (5.27)$$

which is clearly the negative of the remainder of the $SU(2)_s$ transformed third term. Therefore all three terms together are gauge invariant.

When gauging the $SU(2)_s$ symmetry, adjustments to the topological current must be made. The current need not only be gauge invariant but must also remain a conserved quantity. The new current is analogous to the Goldstone-Wilczek current of QCD, and hence will be named accordingly,

$$\begin{aligned} j_\mu^{GW} &= \frac{1}{8\pi}\epsilon_{\mu\nu\rho}[\vec{e}\cdot(D_\nu\vec{e}\times D_\rho\vec{e}) + \vec{e}\cdot\vec{W}_{\nu\rho}], \\ j_\mu^{GW} &= \frac{1}{2\pi i}\epsilon_{\mu\nu\rho}\mathrm{Tr}\,[PD_\nu PD_\rho P + \frac{1}{2}PW_{\nu\rho}]. \end{aligned} \quad (5.28)$$

Again the invariance was verified for the $SU(2)_s$ representation. Each term is gauge invariant by itself. However, both terms are needed so that the condition $\partial_\mu j_\mu^{GW} = 0$ is fulfilled.

It was verified in $SO(3)_s$ space that the topological charge is still the same as before gauging $SO(3)_s$. This is important because gauging the $SO(3)_s$ symmetry should not influence the charge of the Skyrmion,

$$B^{GW} = \int d^2x\, j_0^{GW} = \frac{1}{8\pi}\int d^2x\, \epsilon_{ij}\vec{e}\cdot(\partial_i\vec{e}\times\partial_j\vec{e}), \quad (5.29)$$

which is identical to eq.(4.6). The topological current contributes to the action in the form

$$S_{GW}[P, A_\mu, W_\mu] = q \int d^3x \; A_\mu j_\mu^{GW}. \tag{5.30}$$

Here it is assumed that the topological charge is indeed the electric charge, so $q = -\frac{\theta e}{\pi}$ from eq.(4.11). As mentioned earlier, since there are no charged fields, the U_{em} gauge field A_μ enters the action not through a covariant derivative, but couples directly to the topological current.

It should also be noted that the Hopf term $H[\vec{e}]$ is not gauge invariant. In the $SU(2)_s$ representation the Hopf term takes the form

$$H[U] = \frac{1}{24\pi^2} \int d^3x \; \epsilon_{\mu\nu\rho} \mathrm{Tr}\,[(U^\dagger \partial_\mu U)(U^\dagger \partial_\nu U)(U^\dagger \partial_\rho U)]. \tag{5.31}$$

Since $U \in SU(2)$ the result of this expression is an integer, as was shown in the pion case eq.(4.2). For $U, g \in SU(2)$ the relation

$$\begin{aligned} & \epsilon_{\nu\rho\sigma} \mathrm{Tr}\,[(Ug)^\dagger \partial_\nu (Ug)(Ug)^\dagger \partial_\rho (Ug)(Ug)^\dagger \partial_\sigma (Ug)] \\ = \; & \epsilon_{\nu\rho\sigma} \mathrm{Tr}\,[U^\dagger \partial_\nu U U^\dagger \partial_\rho U U^\dagger \partial_\sigma U] + \epsilon_{\nu\rho\sigma} \mathrm{Tr}\,[g^\dagger \partial_\nu g g^\dagger \partial_\rho g g^\dagger \partial_\sigma g] \\ & -3\epsilon_{\nu\rho\sigma} \partial_\nu \mathrm{Tr}\,[(g\partial_\rho g^\dagger)(U\partial_\sigma U^\dagger)] \end{aligned} \tag{5.32}$$

can be verified [44].

The last term is a total derivative which is then integrated over the whole three-dimensional surface of a sphere. Using Stokes theorem and the fact that ∂S^3 is an empty set, it becomes clear that the last term does not contribute to the Hopf term. Hence

$$H[Ug] = H[U] + H[g]. \tag{5.33}$$

The field U transforms as $U' = h^\dagger U g$ and $U^{\dagger\prime} = g^\dagger U^\dagger h$ where $g \in SU(2)_s$ and $h \in U(1)_s \subset SU(2)_s$. This means that

$$H[U'] = H[h^\dagger U g] = H[h^\dagger] + H[U] + H[g]. \tag{5.34}$$

It is easily verified that $H[h^\dagger] = 0$. It has been demonstrated that under a gauge transformation the Hopf term is not invariant but is shifted by an integer. This anomaly can be removed by introducing an additional Chern-Simons (CS) term,

$$\begin{aligned} S_{CS}[W_\mu] &= \frac{1}{8\pi^2} \int d^3x \; \epsilon_{\mu\nu\rho} \mathrm{Tr}\,[W_\mu(\partial_\nu W_\rho + \frac{2}{3} W_\nu W_\rho)] \\ &= -\frac{1}{16\pi^2} \int d^3x \; \epsilon_{\mu\nu\rho} [\vec{W}_\mu \cdot (\partial_\nu \vec{W}_\rho - \frac{1}{3} \vec{W}_\nu \times \vec{W}_\rho)], \end{aligned} \tag{5.35}$$

which transforms as

$$S_{CS}[W'_\mu] = S_{CS}[W_\mu] - H[g], \tag{5.36}$$

with $W'_\mu = g^\dagger (W_\mu + \partial_\mu) g$. It is clear that both the Hopf term and the CS term together are gauge invariant. The CS term will not be included in the action because it only depends on the gauge field and not on the magnon field. The magnon partition function takes the form

$$Z[A_\mu, W_\mu] = \int \mathcal{D}P \exp(-S[P, W_\mu]) \exp(i\theta H[U]) \exp(iS_{GW}[P, A_\mu, W_\mu]). \quad (5.37)$$

It is important to remember that the non-Abelian gauge field is only a mathematical construction. In reality there is only a dynamical $U(1)_{em}$ gauge field. This means that no additional degrees of freedom are introduced when the gauge potential W_μ is included. As a result there are no Goldstone boson eating fields that could acquire a mass. The Higgs mechanism does not come into play. Since this is not a true non-Abelian gauge theory it may not be necessary to remove the anomaly created by the Hopf term. However, as has just been demonstrated, it is possible to make the theory anomaly free with the Chern-Simons term.

5.2.4 The Magnon-Two-Photon Vertex

Through the Goldstone-Wilczek term the Lagrange density obtains an important contribution that allows for the anomalous decay of a (neutral)magnon into two photons,

$$\begin{aligned} \mathcal{L}_{GW} &= -i\frac{\theta e}{\pi} A_\mu j_\mu^{GW} \\ &= -\frac{i\theta e}{8\pi^2} \epsilon_{\mu\nu\rho} A_\mu \vec{e} \cdot (D_\nu \vec{e} \times D_\rho \vec{e} + \vec{W}_{\nu\rho}). \end{aligned} \quad (5.38)$$

Dropping all terms that do not describe one magnon and two photons one obtains

$$\mathcal{L}_{m\gamma\gamma} = -\frac{i\theta e}{8\pi^2} \epsilon_{\mu\nu\rho} A_\mu \partial_\nu (m^a W_\rho^a). \quad (5.39)$$

Remember that $\vec{e} \in S^2$ is the magnetization (or staggered magnetization), not the magnon field itself. Just like in the pion case where $U(x) \in SU(2)$ contains the pion fields, $\vec{e}$ contains the magnon fields. By introducing small spin fluctuation, the magnons $m_a(x)$, $a \in \{1, 2\}$ can be introduced in the (staggered) magnetization field as

$$\vec{e}(x) \approx (0, 0, 1) + \frac{1}{\sqrt{\varrho_s}}(m_1(x), m_2(x), -\frac{1}{2\sqrt{\varrho_s}}(m_1(x)^2 + m_2(x)^2)). \quad (5.40)$$

Substituting this in the magnon-two-photon vertex, after a few additional transformations one obtains

$$\mathcal{L}_{m\gamma\gamma} = \frac{iq}{8\pi\sqrt{\varrho_s}} \epsilon_{\mu\nu\rho} F_{\mu\nu}(x) m_a(x) W_\rho^a(x), \quad (5.41)$$

where $a \in \{1, 2\}$ and $\mu, \nu, \rho \in \{0, 1, 2\}$.

This vertex describes the interaction of a magnon with two photons as illustrated in fig.5.2 and should be experimentally verifiable. If observed it would be an extremely important step to proving that the baby-Skyrmions do indeed carry electric charge. The spontaneous decay of a single magnon into two photons my be experimentally difficult to verify. What might be easier to observe is the conversion of a photon into a magnon in an external magnetic field. This process is described in the following section.

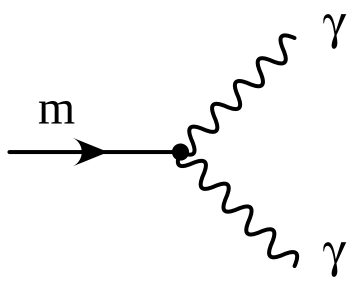

Figure 5.2: *Magnon-photon-photon vertex.*

5.2.5 Photon-Magnon Conversion in an External Magnetic Field

Analogies with Axions

Axions are particles that appear in some theoretical models, but have not been identified in experiments [8, 45, 46]. An important property (also for verification purposes) of these axions is that photons will convert into axions in the presence of a static, external magnetic field. It can easily be shown that magnons in an antiferromagnetic or ferromagnetic material can also oscillate into photons. This may be experimentally easier to verify than magnon to two photon decay.

Quantum Hall Ferromagnet in a Magnetic Field

Considering eq.(5.26) the last term of eq.(5.42) shows that for static fields the action is minimized if $\vec{e}$ is parallel to $\vec{B}$. Using eq.(5.40) for the magnon field it is clear that the magnetic field should be of the form $\vec{B} = B\vec{e}_z$ as can be seen in the following equation

$$m\vec{e}(x) \cdot \vec{W}_t(x) \quad = \quad m\mu\vec{e}(x) \cdot \vec{B} = m\mu B e_z. \tag{5.42}$$

For photon conversion into a magnon in an external magnetic field there must be a description of both a static external field and a fluctuating field. Considering all possible combinations, of external fields and fluctuating fields, one obtains that most terms vanish. What remains is the vertex responsible for a photon magnon conversion in a magnetic field,

$$\mathcal{L}_{Bm\gamma} \quad = \quad \frac{iq\mu}{4\pi\sqrt{\varrho_s}} B(B_1 m_1 + B_2 m_2). \tag{5.43}$$

This illustrates that the magnons couple to their corresponding magnetic field components, and there is no preferred coupling to either magnon polarization m_1 or m_2.

There are additional effects caused by the introduction of a magnetic field. An external magnetic field explicitly breaks the global symmetry, so the Goldstone theorem no longer holds. There are still two magnons, which are the low-energy excitations. However, these now acquire a mass that is proportional to the magnetic field. To determine the mass it is necessary to solve the Euler-Lagrange equation in Minkowski space. To leading order, the Goldstone-Wilczek term does not contribute to the equations of motion and is therefore no longer included. This comes from considerations in the power counting scheme. The gauge fields must be counted in the same manner as the derivatives. As a result the Goldstone-Wilczek term is of higher order than other included terms. If the Goldstone-Wilczek term were to be included then there may be many others that contribute with the same order of magnitude. This case was not investigated. The relevant electromagnetic field contains only a static magnetic field in the z-direction and there are no fluctuations. This means $W_t^3 = \mu B$ and $W_i^a = 0$. The Lagrangian takes the form

$$\mathcal{L}_{Minkowski} = \frac{\varrho_s}{2}\partial_i\vec{e}\cdot\partial_i\vec{e} - m\int_0^1 d\tau\ \vec{e}\cdot(\partial_t\vec{e}\times\partial_\tau\vec{e}) + m\mu B e^3 + \lambda(\vec{e}^2 - 1). \quad (5.44)$$

The first two terms on the right-hand side are the kinetic terms, the third term is minus the potential and the new last term contains a Lagrange multiplier λ which ensures the constraint ($\vec{e}^2 = 1$).

As has been discussed in chapter 3.4, the dispersion relation for magnons in a ferromagnet without an external magnetic field takes the form

$$E = \frac{\varrho_s}{m}|\vec{p}|^2. \quad (5.45)$$

Ignoring the Goldstone-Wilczek term ($\theta = 0$) it is clear that the magnetization $\vec{e}$ is parallel to the magnetic field $\vec{B}$. Taking the results from Leutwyler [1] one finds for the dispersion relation

$$E = \frac{\varrho_s}{m}|\vec{p}|^2 + \mu B. \quad (5.46)$$

This shows that there is a shift of the energy proportional to the magnetic field strength.

Antiferromagnet in a Magnetic Field

The staggered magnetization is obtained by adding every second spin and subtracting the remaining spins of every lattice point. In fig.5.3 it is illustrated how an antiferromagnet is affected by an external magnetic field. Provided the external field is not strong enough to destroy the antiferromagnetism there is always a remaining staggered magnetization. Note that an antiferromagnet can be turned into a ferromagnet with a finite external magnetic field, in which case the staggered magnetization vanishes.

It is clear that there is no $O(3)_s$ symmetry that remains. However, there remains a symmetry in the xy-plane. This explicit symmetry breaking has the effect that one of the two magnons obtains a mass and there is only one remaining (massless) Goldstone boson.

Investing an arbitrarily small amount of energy, the spin orientation can be rotated in the xy-plane (using the remaining one-dimensional continuous symmetry), while excitations in the z-direction require a finite energy. As a result, one can define the vacuum state for the magnon theory and expand around this as was done for pions in QCD.

The contribution of static fields $\vec{e}$ to the action define the orientation of a so-called canted state. It can be shown that using the previous expansion of the magnon field (see

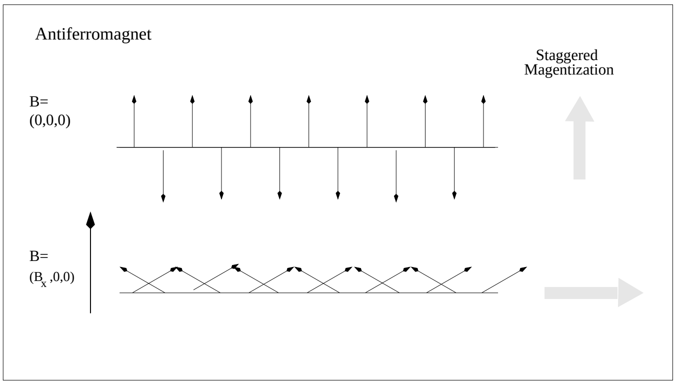

Figure 5.3: *Antiferromagnet in an external magnetic field.*

eq.(5.40)) and inserting it into the action eq.(5.25) the remaining term is of the form

$$\frac{\varrho_s}{2c^2} D_t\vec{e}(x)\cdot D_t\vec{e}(x) = \frac{\varrho_s}{2c^2}(\vec{e}(x)\times\vec{W}_t)\times(\vec{e}(x)\times\vec{W}_t) = \frac{\mu^2 B^2}{2c^2}(1-m_1(x)^2). \qquad (5.47)$$

In order to use the previous form of $\vec{e}(x)$, the magnetic field was chosen to take the form $\vec{B} = B\vec{e}_x$, which is perpendicular to the canted state and lying on the x-axis.

The term $-\frac{\mu^2 B^2}{2} m_1(x)^2$ arises (after multiplication with c^2), which is clearly a mass term, however, only for the magnon mode $m_1(x)$. Hence, in a magnetic field $B\vec{e}_x$ the magnon m_1 acquires a mass $M_m = \mu B$ and the relativistic dispersion relation takes the form

$$E = \sqrt{M_m^2 + |\vec{p}|^2 c^2}. \qquad (5.48)$$

The magnon mode m_2 remains massless (a true Goldstone boson).

Inserting the magnetic field and the canted magnon field into the magnon-two-photon vertex found earlier (eq.(5.41)) one obtains the conversion vertex for the antiferromagnetic case,

$$\mathcal{L}_{Bm\gamma}(x) = i\frac{q\mu B}{4\pi\sqrt{\varrho_s}} m_1(x) B^3(x). \qquad (5.49)$$

Here the same arguments have been used for the static and fluctuating field components as in the ferromagnetic case.

Chapter 6

Skyrmion and Baby-Skyrmion Decay

Skyrmions and baby-Skyrmions are topologically stable. Through interactions with gauge fields, however, it is possible that these topological objects decay. The Skyrmion in the pion effective field theory (which, as was shown, is associated with a baryon) decays through two channels. First, a baryon number violating electroweak instanton causes the 't Hooft anomaly. Second, in grand unified theories (GUT) magnetic monopoles exist which induce baryon decay.

For baby-Skyrmions there is no analogue of the 't Hooft anomaly. Analogous to the magnetic monopole, it can be shown that a charged wire catalyzes the decay of the baby-Skyrmion. The decay occurs as electrons, which are the building blocks of baby-Skyrmions, leave the magnet. A baby-Skyrmion can only exist within the two-dimensional structure and will decay when its electrons are extracted. Considering the entire (3+1)-dimensional world there is of course no fermion number violation.

6.1 Skyrmion Decay

6.1.1 Pions, Skyrmions, and W-Bosons

For the magnon theory a local $SU(2)_s$ symmetry appeared unexpectedly. For this there is an analogy in the pion theory. Also for the weak interactions to be included the $SU(2)_L$ symmetry must be gauged, in which case the non-Abelian W_μ-boson field is introduced. The symmetry $U(1)_Y \subset SU(2)_R$ is gauged by coupling pions to the Abelian B-bosons. Through the Higgs mechanism the $SU(2)_L \otimes U(1)_Y$ symmetry breaks to the $U(1)_{em}$ symmetry of electromagnetism. The photon field A_μ emerges as a combination of W_μ^3 and B_μ fields. This chapter concentrates on the W_μ-boson, so only $SU(2)_L$ is gauged.

As usual the derivatives are replaced by covariant derivatives,

$$\partial_\mu U \to D_\mu U = \partial_\mu U + W_\mu U. \tag{6.1}$$

Here $W_\mu = igW_\mu^a T^a$, T^a are again the generators of $SU(2)_L$, and g is the coupling strength of the gauge field. The action is given by

$$S[U, W_\mu] = \int d^4x \, \frac{F_\pi^2}{4} \mathrm{Tr}\,[D_\mu U^\dagger D_\mu U], \tag{6.2}$$

which is invariant under local $SU(2)_L$ gauge transformations $U(x)' = L(x)^\dagger U(x)$ and $W_\mu(x)' = L(x)^\dagger (W_\mu(x) + \partial_\mu) L(x)$. The partition function takes the form

$$Z = \int \mathcal{D}U \exp(-S[U]) \mathrm{Sign}[U]^{N_c}. \tag{6.3}$$

This path integral is, however, not invariant under $SU(2)_L$ transformations, due to the transformation properties of the Sign$[U]$ term

$$\mathrm{Sign}[U'] = \mathrm{Sign}[L^\dagger U] = \mathrm{Sign}[L^\dagger]\mathrm{Sign}[U] \neq \mathrm{Sign}[U]. \tag{6.4}$$

The $SU(2)_L$ gauge variation of the fermion permutation sign of the Skyrmion is a manifestation of Witten's global anomaly. For an odd number of colors N_c the theory is not consistent. Therefore additional fields are necessary. In the Standard Model the global anomaly is cancelled by the neutrino and electron left-handed doublet.

When $SU(2)_L$ is gauged, a conservation violation of the baryon number is caused by the 't Hooft anomaly through electroweak instantons. This can be observed with the appropriate Goldstone-Wilczek current,

$$\begin{aligned} j_\mu^{GW} &= \frac{1}{24\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\,[(U^\dagger D_\nu U)(U^\dagger D_\rho U)(U^\dagger D_\sigma U)] \\ &\quad - \frac{1}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\,[W_{\nu\rho}(D_\sigma U U^\dagger)]. \end{aligned} \tag{6.5}$$

It is quickly shown that this topological current is gauge invariant. It is not possible, however, to make the Goldstone-Wilczek current a conserved quantity, i.e. $\partial_\mu j_\mu^{GW} \neq 0$. This anomaly, as mentioned before, is know as the 't Hooft anomaly. In the Standard Model the baryon number is not conserved. This process has never been observed experimentally and hence must be very strongly suppressed. The Goldstone-Wilczek current has been constructed in a way that it is gauge invariant and so that the divergence leads to a new topological charge. It can be verified that

$$\partial_\mu j_\mu^{GW} = -\frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\,[W_{\mu\nu} W_{\rho\sigma}], \tag{6.6}$$

and

$$n = -\frac{1}{32\pi^2} \int d^4x \, \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\,[W_{\mu\nu} W_{\rho\sigma}] \ \in \Pi_3[SU(2)_L] = \mathbb{Z} \tag{6.7}$$

is the Pontryagin-density.

The baryon number, which is the winding number of the pion field U $\in G/H$, becomes

$$B = \frac{1}{24\pi^2} \int d^3x \, \epsilon_{ijk} \mathrm{Tr}\,[(U^\dagger \partial_i U)(U^\dagger \partial_j U)(U^\dagger \partial_k U)] - S_{CS} \notin \mathbb{Z}. \tag{6.8}$$

The first term in this equation is the usual topological charge, and always an integer. The second term is given by the Chern-Simons term, which is generally not an integer. This means that the topological charge which is associated with the baryon number is no longer an integer. To investigate this, a closer look is taken at the new topological charge which is defined through the derivative of the current. This is the topological charge of the electroweak gauge field eq.(6.7)

To simplify the notation one can write

$$\begin{aligned} \tilde{W}_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}W_{\rho\sigma}, \\ W_{\mu\nu}\tilde{W}_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}W_{\mu\nu}W_{\rho\sigma}, \end{aligned} \tag{6.9}$$

which makes

$$n = \int d^4x \; \partial_\mu j_\mu^{GW} = -\frac{1}{16\pi^2}\int d^4x \; \mathrm{Tr}\,[W_{\mu\nu}\tilde{W}_{\mu\nu}], \tag{6.10}$$

where $\frac{1}{16\pi^2}\mathrm{Tr}\,[W_{\mu\nu}\tilde{W}_{\mu\nu}]$ is known as the Pontryagin density.

The $SU(2)_L$ field strength $W_{\mu\nu}$ contains new interesting topological configurations, the instantons. An instanton is a solution of the self-duality equation

$$W_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}W_{\rho\sigma}, \tag{6.11}$$

and an anti-instanton is a solution of the anti-self duality equation

$$W_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}W_{\rho\sigma}. \tag{6.12}$$

It is an interesting property of the instantons that their action is the minimum in a given Pontryagin class. They are therefore classical solutions of the Euclidean field theory. This can be shown using the equation

$$\begin{aligned} 0 &\leq \int d^4x \; \mathrm{Tr}\,[(W_{\mu\nu} \pm \tilde{W}_{\mu\nu})(W_{\mu\nu} \pm \tilde{W}_{\mu\nu})] \\ &= \int d^4x \; \mathrm{Tr}\,[W_{\mu\nu}W_{\mu\nu} + \tilde{W}_{\mu\nu}\tilde{W}_{\mu\nu} \pm W_{\mu\nu}\tilde{W}_{\mu\nu} \pm \tilde{W}_{\mu\nu}W_{\mu\nu}]. \end{aligned} \tag{6.13}$$

Taking the minus option gives the non-trivial solution for the instanton,

$$\begin{aligned} 0 &\leq \underbrace{2\int d^4x \; \mathrm{Tr}\,[W_{\mu\nu}W_{\mu\nu}]}_{4g^2 S[W_\mu]} - \underbrace{2\int d^4x \; \mathrm{Tr}\,[W_{\mu\nu}\tilde{W}_{\mu\nu}]}_{32\pi^2 n[W_\mu]}, \\ S[W_\mu] &\geq \frac{8\pi^2}{g^2}|n[W_\mu]|. \end{aligned} \tag{6.14}$$

This solution reveals two important properties. First of all, for a given number of instantons $n[W_\mu]$ there is a minimal contribution to the action from the W_μ field unequal to zero. Second, if the field W_μ describes an instanton then the inequality becomes an equality. This means that the action is minimized for a given number of instantons, and this, of course, is the classical solution of the field theory, with

$$S_{instanton}[W_\mu] \;=\; \frac{8\pi^2}{g^2}|n[W_\mu]|. \tag{6.15}$$

Until now the Goldstone-Wilczek current was always conserved. As a consequence, the corresponding topological charge was also conserved. By gauging the $SU(2)_L$ symmetry the situation has changed, for now the topological current is no longer conserved. Imagine the universe in the shape of a cylinder, with the three spacial-dimensions wrapped up onto S^3 and the time-dimension stretching from $-\infty$ to $+\infty$. Introducing the Chern-Simons density

$$\Omega_\mu \;=\; -\frac{1}{8\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\,[(W_\nu\partial_\rho W_\sigma + \frac{2}{3}W_\nu W_\rho W_\sigma)], \tag{6.16}$$

and using

$$-\frac{1}{16\pi^2}\mathrm{Tr}\,[W_{\mu\nu}\tilde{W}_{\mu\nu}] \;=\; \partial_\mu\Omega_\mu, \tag{6.17}$$

one can write

$$n \;=\; \int_V d^4x\;\partial_\mu j^{GW}_\mu = -\frac{1}{16\pi^2}\int_V d^4x\;\mathrm{Tr}\,[W_{\mu\nu}\tilde{W}_{\mu\nu}] = \int_{\partial V} d^3\;\sigma_\mu\Omega_\mu. \tag{6.18}$$

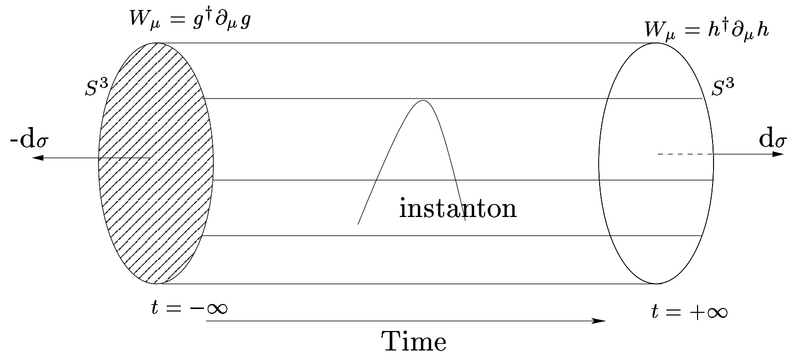

Figure 6.1: *4-dimensional space-time wrapped into a cylinder.*

In this manner n is an integral no longer over the entire space-time but over the boundary of space-time. Writing V as $S^3\otimes\mathbb{R}^1$, ∂V becomes $\partial(S^3\otimes\mathbb{R}^1) = S^3_\infty\bigcup S^3_{-\infty}$ (see fig.6.1). For all physical configurations with a finite action, for $x\in\partial V$ one finds that the field strength $W_{\mu\nu}$ vanishes. This does not mean that $W_\mu = 0$. It only means that W_μ is gauge equivalent to 0. Given

$$W'_\mu = L^\dagger(W_\mu + \partial_\mu)L, \tag{6.19}$$

for $W_\mu = 0$ one obtains a pure gauge

$$W'_\mu = L^\dagger\partial_\mu L \tag{6.20}$$

at the boundary S^3. A short calculation shows that for a pure gauge

$$\Omega_\mu = \frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\,[(L^\dagger\partial_\nu L)(L^\dagger\partial_\rho L)(L^\dagger\partial_\sigma L)]. \tag{6.21}$$

Since S_{CS} is the spacial integral of Ω_μ, it becomes clear that at the beginning and at the end of time the baryon number, given by eq.(6.8) is an integer value. The change of the topological charge is given by the change of S_{CS} from $t = -\infty$ to $t = +\infty$

$$\begin{aligned} n &= \int_V d^4x\,\partial_\mu j_\mu^{GW} = \int_{\partial V} d\sigma_\mu\, j_\mu^{GW} = \int_{S^3(t=\infty)} d^3x\, j_0^{GW} - \int_{S^3(t=-\infty)} d^3x\, j_0^{GW} \\ &= B(t=\infty) - B(t=-\infty) \ = \Delta B \ \in \mathbb{Z}. \end{aligned} \tag{6.22}$$

Since the CS-term is an integer at $t = -\infty$ and $t = +\infty$ the baryon number is automatically also an integer. This shows that the topological charge of the $SU(2)_L$ gauge field is a measure of baryon number violation.

The baryon number is violated by an instanton described through the CS term (see figs.6.2,6.3). This instanton 'eats' a baryon. It is apparent that the number of instantons is given by the topological charge of the $SU(2)_L$ gauge field.

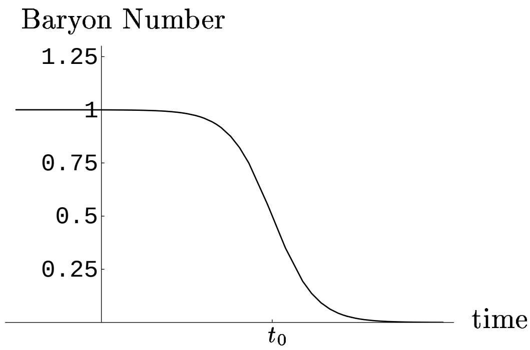

Figure 6.2: *A baryon decays at* $t = t_0$.

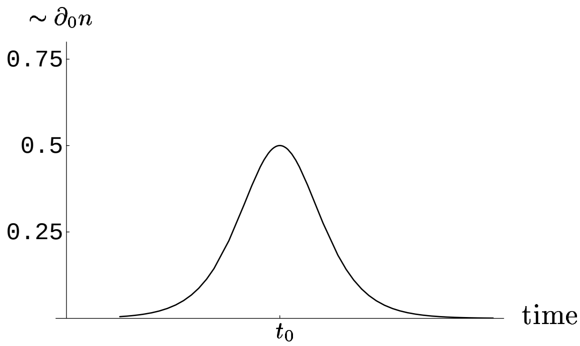

Figure 6.3: *An instanton causes baryon decay at* $t = t_0$.

6.1.2 Magnetic Monopoles and Baryon Decay

Dirac first postulated magnetic monopoles in 1931 [26]. Even though the Maxwell equations don't allow for this phenomenon, magnetic monopoles do appear in GUT theories, in the form of 't Hooft-Polyakov monopoles. These are extremely heavy, yet stable particles. When a baryon interacts with a 't Hooft-Polyakov monopole it may transform into a lepton [10]. In the $SU(5)$ GUT there still remains a global symmetry of the difference of the baryon and of the lepton number $(B-L)$, which is a conserved quantity. This symmetry disappears if neutrinos are not massless.

The magnetic current of a monopole is given by

$$m_\sigma = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial_\mu F_{\nu\rho}. \tag{6.23}$$

This is a violation of the Bianchi identity which would imply that m_σ vanishes. As a consequence of the Maxwell equations

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \Rightarrow \epsilon_{\mu\nu\rho\sigma}\partial_\mu F_{\nu\rho} = 0. \tag{6.24}$$

The existence of a magnetic monopole renders this false. Take a situation where $\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial_\mu F_{\nu\rho} = m_\sigma$. Consider the Goldstone-Wilczek current defined for the Abelian $U(1)_{em}$ gauge field in section 5

$$\begin{aligned} j_\mu^{GW} &= \frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\,[(U^\dagger D_\nu U)(U^\dagger D_\rho U)(U^\dagger D_\sigma U)] \\ &\quad -\frac{ie}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}F_{\nu\rho}\mathrm{Tr}\,[T^3(D_\sigma U U^\dagger + U^\dagger D_\sigma U)], \end{aligned} \tag{6.25}$$

with

$$D_\mu U = \partial_\mu U + ieA_\mu[T^3, U]. \tag{6.26}$$

Now that $m_\sigma \neq 0$ this current is no longer conserved, i.e.

$$\begin{aligned} \partial_\mu j_\mu^{GW} &= -\frac{ie}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}\partial_\mu F_{\mu\nu}\mathrm{Tr}\,[T^3(D_\sigma U U^\dagger + U^\dagger D_\sigma U)] \\ &= -\frac{ie}{8\pi^2}m_\sigma \mathrm{Tr}\,[T^3(D_\sigma U U^\dagger + U^\dagger D_\sigma U)]. \end{aligned} \tag{6.27}$$

Given a magnetic monopole at rest with the magnetic charge g,

$$m_0(\vec{x},t) = 4\pi g\delta(\vec{x}) = \frac{1}{2}\epsilon_{\mu\nu\rho 0}\partial_\mu F_{\nu\rho} = \vec{\nabla}\cdot\vec{B}, \quad m_i(\vec{x},t) = 0. \tag{6.28}$$

It is easy so see that the magnetic monopole only contributes to the magnetic field, where g is the magnetic charge. A vector potential describing this arrangement is best presented using spherical coordinates $(r,\ \theta,\ \phi)$

$$\vec{A}(\vec{x}) = g\frac{1-\cos\theta}{r\sin\theta}\vec{e}_\phi. \tag{6.29}$$

This field configuration only violates the Bianchi identity eq.(6.23) at the magnetic monopole. In order to obtain a magnetic monopole it is necessary to have a singular gauge field.

Introducing this into the current leads to

$$\partial_\mu j_\mu^{GW} = \partial_0 j_0^{GW} = -\frac{4\pi i e}{8\pi^2}\, g\, \delta(\vec{x})\mathrm{Tr}\,[T^3(D_0 U U^\dagger + U^\dagger D_0 U)]. \tag{6.30}$$

Setting $A_0 = \phi = 0$ (no electric field present) and using the approximation $U \cong 1 + \frac{2i}{F_\pi}\pi^a T^a$ one obtains

$$\begin{aligned}
\partial_0 j_0^{GW} &= \frac{4\pi e\, g}{4\pi^2 F_\pi}\delta(\vec{x})\partial_0\pi^0(\vec{x},t), \\
B &= \int d^3x\; j_0^{GW}, \\
\partial_0 B &= \int d^3x\; \partial_0 j_0^{GW} = \frac{e\, g}{\pi F_\pi}\partial_0\pi^0(\vec{0},t).
\end{aligned} \tag{6.31}$$

From this the change of the baryon number can be determined as

$$\int_{-\infty}^{\infty} \partial_0 B(t) dt = B(\infty) - B(-\infty) = \frac{e\, g}{\pi F_\pi}[\pi^0(\vec{0},\infty) - \pi^0(\vec{0},-\infty)] \tag{6.32}$$

Using $e\, g = \frac{1}{2}$ which is the Dirac quantization condition it is demonstrated that if $\frac{\pi^0(\vec{0})}{F_\pi}$ at the location of the monopole rotates by $2\pi\, n$ then there is a baryon number violation by n units.

6.2 Charged Wires and Baby-Skyrmion Decay

If a wire (charged or uncharged) is placed perpendicular to the 2-d crystal lattice, it allows electrons, which are described by baby-Skyrmions to leave the plane of the crystal. For an observer living on the two-dimensional lattice this looks like a violation of charge conservation.

Just like in the QCD case, in the presence of a monopole, the Goldstone-Wilczek current is no longer conserved. In two dimensions a charged wire is similar to a point charge. To construct a charged wire that violates current conservation an analogous procedure will be pursued as was done in the QCD case with a magnetic monopole. There is a slight complication, however, since the gauge field is now non-Abelian.

In the QCD case the Abelian Bianchi identity was violated (see eq.(6.23)). Now, the non-Abelian Bianchi identity must be violated

$$\vec{m} = \epsilon_{\mu\nu\rho} D_\mu \vec{W}_{\nu\rho}. \tag{6.33}$$

Using this the topological current is no longer conserved, and is violated by

$$\partial_\mu j_\mu^{GW} = \frac{1}{8\pi}\vec{m}\cdot\vec{e}. \tag{6.34}$$

In the case of the charged wire there is a point-like violation of the Bianchi identity both in space and time, which results in an event-like process (instanton) opposed to the particle-like process that was observed in the QCD case. Given a charged wire that discharges around $t = 0$ the violation of the Bianchi identity takes the form

$$m_a(x) = 4\pi g \delta^{a3} \delta(x), \tag{6.35}$$

where $a \in \{1, 2, 3\}$ and $x = (x_0, x_1, x_2)$. The gauge field can be constructed similarly to what was done before. What was previously the third spacial dimension has now become the time-dimension of the model. The gauge field takes the form

$$W_i^a = g\delta^{a3} \frac{1 - \cos\theta}{r \sin\theta} e_{\phi,i}. \tag{6.36}$$

Again spherical coordinates are the preferred manner of description.

Using the relationship between the non-Abelian gauge field and the electromagnetic field (as was derived in chapter 5.2.1) and switching to cylindrical space-time coordinates ($\rho = r\sin\theta,\ t = r\cos\theta$) one finds the electric field

$$\vec{E}(\rho, t) = \frac{2g}{\mu\rho}(1 - \frac{t}{\sqrt{t^2 + \rho^2}})\vec{e}_\rho. \tag{6.37}$$

At the beginning of time and at the end of time the electric field takes the form

$$\vec{E}(\rho, t) = \frac{4g}{\mu\rho}\vec{e}_\rho \quad \text{for } t \to -\infty, \quad \vec{E}(\rho, t) = 0 \quad \text{for } t \to +\infty \tag{6.38}$$

The instanton event describes the discharge of a charged wire (of charge $\frac{4g}{\mu}$ per unit length) perpendicular to the magnet. Plots of the electric field are shown in fig.6.4. During the actual discharge the field configuration is somewhat more complicated than that of a discharging wire, and involves a complicated charge distribution. It has not been considered how such a field may be created in a laboratory, or if it is at all possible. The charged wire is an exact analog to the Dirac string. The Dirac string describes a magnetic monopole at its spacial end $\vec{x} = 0$. This monopole catalyzes the Skyrmion decay. The string itself is made invisible through the quantization condition $eg = \frac{1}{2}$. The charged wire in the magnet is also implemented using a Dirac string. In this case there is a discharge around the temporal end of the Dirac string at $t = 0$. This catalyzes the decay of a baby-Skyrmion

$$\begin{aligned}
\partial_\mu j_\mu^{GW} &= \partial_0 j_0^{GW} + \vec{\nabla}\vec{j}^{GW}, \\
\int d^2x\, \partial_0 j_0^{GW} &= \partial_0 B = \int d^2x\, \partial_\mu j_\mu^{GW}, \\
\partial_0 B &= \int d^2x\, \frac{g}{4}\delta(x) e^3(x) = \frac{g}{4}\delta(t) e^3(t), \\
\Delta B &= \int dt\, \partial_0 B = B(\infty) - B(-\infty) = \frac{g}{4} e^3(0).
\end{aligned} \tag{6.39}$$

Since this Dirac string is a charged wire there is no need for a Dirac quantization that hides the string and reveals only the monopole.

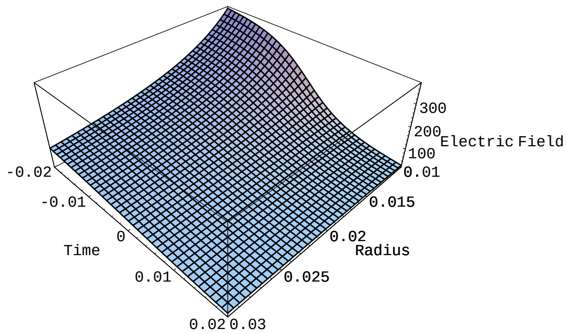

Figure 6.4: *Electric field of a charged wire that discharges around $t = 0$. The electric field is plotted vs. time and radius.*

Chapter 7

Generalization to Several Flavors

Until now an effective theory for two flavors, the up and down quarks, was considered. It is known that there are (at least) 6 flavors. This chapter investigates the influence of three or more flavors on the effective field theory. This leads to the Wess-Zumino-Witten (WZW) term, which is of topological nature and contributes to important decay channels such as $\pi^0 \to \gamma\gamma$. The prefactor of the WZW-term is quantized and equal to N_c, the number of colors in the theory. Again, there are strong analogies between QCD and the magnon theory. One can possibly replace the number of flavors N_f with the layer index N_f of weakly coupled two-dimensional ferro- or antiferromagnetic lattices. There exists a corresponding WZW-term which also contributes to the decay of a magnon into two photons. The most striking difference is that the prefactor of the WZW-term for magnons is not quantized.

7.1 Pions, Kaons, η-Mesons

The space in which the pion, kaon and η-meson fields live is

$$\begin{aligned} G &= SU(N_f)_L \otimes SU(N_f)_R \otimes U(1)_B, \\ H &= SU(N_f)_{L=R} \otimes U(1)_B, \\ G/H &= SU(N_f) \end{aligned} \tag{7.1}$$

Like always the number of Goldstone bosons is given by the dimension of the coset space G/H: $dim(G/H) = N_f^2 - 1$. For $N_f = 3$ this results in the three pions, four kaons and one η-meson, which are the eight lightest mesons in the Standard Model constructed out of up, down, and strange quarks. The symmetry group with two flavors was not an exact symmetry, and since the mass of the strange quark is higher than the mass of the up and down quarks the approximation of massless quarks is now less accurate. With more than three flavors the approximation of massless quarks is no longer appropriate.

The leading order contributions to the action are of the identical form to what was done in chapter 5, but now $U \in SU(N_f)$. The Goldstone-Wilczek current also remains unaltered as can be made plausible with topological considerations, since $\Pi_3[SU(2)] = \Pi_3[SU(N_f)] = \mathbb{Z}$. Therefore the topological charge B is also unaffected.

There is, however, an important difference in the higher order homotopy groups. While $\Pi_4[SU(2)] = \mathbb{Z}_2$ the homotopy group $\Pi_4[SU(N_f))] = \{0\}$, but $\Pi_5[SU(N_f))] = \mathbb{Z}$. This has the consequence that if $N_f \geq 3$ the time histories of the Goldstone boson field $U(x) \in SU(N_f)$ are topologically trivial and can be continuously deformed into the vacuum field $U(x) = \mathbb{1}$. Since the homotopy group $\Pi_4[SU(2)] = \mathbb{Z}_2$ insured the fermionic nature of the Skyrmions, one must find a new term that restores this topological property. The new fermionic condition is restored with the idea of Witten to add an additional un-physical dimension, $U(x) \to U(x, x_5)$, where $x_5 \in [0, 1]$. The five dimensions form a hemisphere H^5 with the boundary of the hemisphere $\partial H^5 = S^4$ being the compactified four-dimensional space-time. The interpolated Goldstone boson field is given by $U(x, 0) = \mathbb{1}$, $U(x, 1) = U(x)$. The Wess-Zumino-Witten term takes the form [11, 12, 13]

$$S_{WZW} = \frac{1}{480\pi^3 i}\int_{H^5} d^5x \; \epsilon_{\mu\nu\rho\sigma\lambda}\mathrm{Tr}\,[(U^\dagger\partial_\mu U)(U^\dagger\partial_\nu U)(U^\dagger\partial_\rho U)(U^\dagger\partial_\sigma U)(U^\dagger\partial_\lambda U)] \tag{7.2}$$

The un-physical dimension is analogous to the two-dimensional hemisphere that was constructed for the ferromagnet in section 3.4.2.

The Goldstone boson field is physical only on the boundary of the hemisphere. It can be shown that if H^5 were replaced with S^5, then the WZW-term would result in $n \in \mathbb{Z}$. This results from the homotopy group $\Pi_5[SU(N_f)] = \mathbb{Z}$. Hence, modulo n there are only contributions from ∂H^5. The path integral now takes the form

$$Z = \int \mathcal{D}U \exp(-S[U]) \exp(2\pi i N_c S_{WZW}[U]), \tag{7.3}$$

where the quantized prefactor N_c is the number of colors. The quantization condition of $N_c \in \mathbb{Z}$ ensures that the WZW-term only contributes on the boundary. Singularities from the hemisphere do not have an effect on the path integral. It should be noted that eq.(7.3) is a natural extension of what has already been derived in eq.(6.3). For $U(x, 1) \in SU(2)$ the WZW-term corresponds to the Sign$[U]$ term that was presented in chapter 4,

$$\exp(2\pi i N_c S_{WZW}[U]) = \mathrm{Sign}[U]^{N_c}. \tag{7.4}$$

Just like the $\mathrm{Sign}[U]^{N_c}$ term the WZW-term ensures hat the Skyrmions are quantized as fermions for odd N_c and as bosons for even N_c.

The WZW-term also breaks the intrinsic parity symmetry P_0. This symmetry is not present in QCD. The parity transformation acts on pseudo-scalar Goldstone bosons $\pi^a(\vec{x}, t)$ in the form

$$\begin{aligned} \text{full parity} \;\; {}^P\pi^a(\vec{x}, t) &= -\pi^a(-\vec{x}, t), \;\; {}^PU(\vec{x}, t) = U^\dagger(-\vec{x}, t), \\ \text{intrinsic parity} \;\; {}^{P_0}\pi^a(\vec{x}, t) &= -\pi^a(\vec{x}, t), \;\; {}^{P_0}U(\vec{x}, t) = U^\dagger(\vec{x}, t). \end{aligned} \tag{7.5}$$

For $N_f = 2$ intrinsic parity is a good symmetry (also known as G-parity), while for $N_f \geq 3$ it is broken. If it were not broken then this would mean that the number of Goldstone bosons is conserved modulo two. It is known, however, that, for example, the ϕ-meson can decay onto two kaons or into three pions, and therefore there is no such symmetry.

The leading order terms in the effective theory posses an intrinsic parity symmetry, which is then removed by the WZW-term,

$$\begin{aligned} S[{}^{P_0}U] &= S[U^\dagger] = S[U], \\ S_{WZW}[{}^{P_0}U] &= S_{WZW}[U^\dagger] = -S_{WZW}[U]. \end{aligned} \tag{7.6}$$

In the presence of the WZW-term the effective theory has the same symmetry as QCD.

7.2 Magnons with Several Flavors

Just like in the QCD case, the space in which the magnon fields live can be expanded to a larger symmetry group

$$\begin{aligned} G &= SU(N_f), \\ H &= U(N_f - 1), \\ G/H &= CP(N_f - 1). \end{aligned} \tag{7.7}$$

Again the number of Goldstone bosons is given by the dimension of the coset space, $dim(CP(N_f - 1)) = 2N_f - 2$.

It was shown in section 5.2.3 how the $SO(3)_s$ representation can be replaced with the $SU(2)_s$ representation. For a multi-layer theory it is best to work with the $SU(N_f)$ representation. The Goldstone bosons of $CP(N_f - 1)$ are naturally described by $N_f \times N_f$ Hermitian projection matrices $P(x)$ that obey

$$P(x)^\dagger = P(x), \quad \mathrm{Tr}\,[P(x)] = 1, \quad P(x)^2 = P(x). \tag{7.8}$$

For $N_f = 2$, P can be written as $P = \frac{1}{2}(1 + \vec{e}(x) \cdot \vec{\sigma})$.

In QCD it was clear that by increasing the number of flavors, hence adding additional quarks to the theory, the additional Goldstone bosons are additional mesons made up from the new pairing possibilities. In quantum Hall ferromagnets magnons in multi-layered systems have been observed [14]. In this case, magnons propagate as usual on a given two-dimensional layer. There is the possibility, however, of communication between neighboring layers. In principle, similar effects should be possible in antiferromagnets.

7.2.1 Antiferromagnetism

The form of the lowest order terms in chiral perturbation theory is unaltered when additional flavors are added to the theory. The action is given by

$$S[P] = \int d^2x \int_{S^1} dt\ \varrho_s(\mathrm{Tr}\,[\partial_i P \partial_i P] + \frac{1}{c^2}\mathrm{Tr}\,[\partial_t P \partial_t P]), \tag{7.9}$$

which is invariant under global special unitary transformations $g \in G = SU(N_f)$

$$P'(x) = g^\dagger P(x) g. \tag{7.10}$$

7.2.2 Ferromagnetism

The leading orders for the ferromagnet also remain of the same form

$$S[P] \;=\; \int d^2x \, (\int_{S^1} dt \; \varrho_s \mathrm{Tr}\,[\partial_i P \partial_i P] - 4m \int_{H^2} dt d\tau \; \mathrm{Tr}\,[P \partial_t P \partial_\tau P]). \qquad (7.11)$$

The second term is of topological nature. In order to ensure that the bulk ambiguity of the extra, non-physical dimension cancels, the prefactor

$$\int d^2x \; m = M \qquad (7.12)$$

must be quantized in integer or half-integer units. This is identical to the $N_f = 2$ case in section 3.4.2.

7.3 Baby-Skyrmions with Several Flavors

In the magnon theory there exists an analog of the Wess-Zumino-Witten term. In the effective theory for QCD this term arises from the homotopy group $\Pi_5[SU(N_f)] = \mathbb{Z}$, which leads to the quantization condition for N_c. For magnons the time histories are also topologically trivial as shown by the homotopy group $\Pi_3[CP(N_f-1)] = 0$. Unlike the QCD case, however, the next higher homotopy group $\Pi_4[CP(N_f-1)] = 0$ is also trivial. This means that the magnon WZW-term does not have a quantized prefactor but a continuous parameter θ, which has already been introduced as the anyon angle. The second homotopy group $\Pi_2[CP(N_f-1)] = \mathbb{Z}$ remains unchanged. It was this homotopy group that defined the topological charge and conserved current. These can now be written as

$$B = \frac{1}{2\pi i} \int d^2x \; \epsilon_{ij} \mathrm{Tr}\,(P \partial_i P \partial_j P), \quad j_\mu = \frac{1}{2\pi i} \epsilon_{\mu\nu\rho} \mathrm{Tr}\,(P \partial_\nu P \partial_\rho P). \qquad (7.13)$$

The analog of the WZW-term takes the form

$$S_{WZW} = \frac{1}{4\pi^2} \int_{H^4} d^4x \; \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\,[P \partial_\mu P \partial_\nu P \partial_\rho P \partial_\sigma P]. \qquad (7.14)$$

The factor $\frac{1}{4\pi^2}$ is derived from the condition that the WZW-term must reduce to the Hopf term for $N_f = 2$. Since $\Pi_4[CP(N_f-1)] = 0$, if the WZW-term would be integrated over the whole space S^4 it would vanish. Hence there are only contributions from the boundary, which represents the physical space-time. This can be confirmed by writing the WZW-term as a total divergence

$$\begin{aligned} & \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\,[P \partial_\mu P \partial_\nu P \partial_\rho P \partial_\sigma P] \\ = \; & \epsilon_{\mu\nu\rho\sigma} \partial_\mu \mathrm{Tr}\,[D \partial_\nu U U^\dagger D \partial_\rho U U^\dagger \partial_\sigma U U^\dagger - \frac{4}{3} D \partial_\nu U U^\dagger D \partial_\rho U U^\dagger D \partial_\sigma U U^\dagger]. \qquad (7.15) \end{aligned}$$

Here P was replaced by $U^\dagger D U$ where $D \in CP(N_f-1)$ is a $N_f \times N_f$ matrix with all elements equal to 0 with the exception of the first element equal to one and $U \in SU(N_f)$.

It must be noted that since the eigenvalues of P are degenerate U is not unique. Since the WZW-term can be written as a total divergence it is clear that if H^4 were to be replaced by the entire S^4 hypersphere the term would not contribute to the action.

The prefactor of the WZW-term is no longer quantized, and the path integral takes the form

$$Z = \int \mathcal{D}P \exp(-S[P]) \exp(i\theta S_{WZW}[U]). \tag{7.16}$$

When gauging $SU(N_f)$, the Goldstone-Wilczek current

$$j_\mu^{GW} = \frac{1}{2\pi i}\epsilon_{\mu\nu\rho}\mathrm{Tr}\,(PD_\nu PD_\rho P + \frac{1}{2}PW_{\nu\rho}) \tag{7.17}$$

remains conserved. This mean that also for $N_f \geq 3$ there is no 't Hooft anomaly in the magnon theory.

It has been verified that the WZW-term is not gauge invariant, but varies by a total divergence. Integrating over the hemisphere H^4 the total divergence resulting in

$$\begin{aligned} S_{WZW}[P'] - S_{WZW}[P] &= \frac{1}{4\pi^2}\int_{\partial H^4 = S^3} d^3x\,\epsilon_{\nu\rho\sigma}\mathrm{Tr}\,[-2\partial_\nu g g^\dagger \partial_\rho P \partial_\sigma P\,P \\ &\quad -2\partial_\nu g g^\dagger P \partial_\rho g g^\dagger \partial_\sigma P\,P - \frac{2}{3}\partial_\nu g g^\dagger P \partial_\rho g g^\dagger P \partial_\sigma g g^\dagger P \\ &\quad +\partial_\nu g g^\dagger P \partial_\rho g g^\dagger \partial_\sigma g g^\dagger P + \partial_\nu g g^\dagger \partial_\rho P \partial_\sigma g g^\dagger P]. \end{aligned} \tag{7.18}$$

It can be shown that for $N_f = 2$ this is equal to the variation of the Hopf term, which results in an integer.

In order for the theory to remain gauge invariant there must exist an additional term with the same gauge variance. For $N_f = 2$ this was the Chern-Simons term. For $N_f \geq 3$ this term takes the form

$$\begin{aligned} S_C = \frac{1}{4\pi^2}\int_{S^3} d^3x\,\epsilon_{\nu\rho\sigma}\ \mathrm{Tr}\,[&\partial_\nu P P W_\rho P W_\sigma - P\partial_\nu P W_\rho P W_\sigma - 2P\partial_\nu P \partial_\rho P W_\sigma \\ &-\frac{2}{3}PW_\nu P W_\rho P W_\sigma - PW_\nu P \partial_\rho W_\sigma]. \end{aligned} \tag{7.19}$$

Details of this calculations are given in appendix A.3. This term does not reduce to the Chern-Simons term, which countered the Hopf term gauge variance for $N_f = 2$.

The WZW term and the Chern-Simons term analog together are gauge invariant, i.e.

$$\begin{aligned} S_{WZW}[P'] + S_C[P', W'_\mu] &= S_{WZW}[P] + S_C[P, W_\mu] \\ &= \frac{1}{4\pi^2}\int_{H^4} d^4x\,\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\,[P\partial_\mu P \partial_\nu P \partial_\rho P \partial_\sigma P] \\ &+ \frac{1}{4\pi^2}\int_{S^3} d^3x\,\epsilon_{\nu\rho\sigma}\mathrm{Tr}\,[\partial_\nu P P W_\rho P W_\sigma - P\partial_\nu P W_\rho P W_\sigma \\ &- 2P\partial_\nu P \partial_\rho P W_\sigma - \frac{2}{3}PW_\nu P W_\rho P W_\sigma - PW_\nu P \partial_\rho W_\sigma]. \end{aligned} \tag{7.20}$$

To introduce electromagnetism it was necessary to gauge $SU(2)_s$. This will still be the case, even if the symmetry group is $SU(N_f)$, $N_f \geq 3$. Similarly in the Standard Model not the entire symmetry group $SU(2)_R \otimes SU(2)_L \otimes U(1)_Y$ is gauged in order to include electromagnetism, but only the subgroup $SU(2)_L \otimes U(1)_Y$.

7.3.1 Statistics and the WZW-Term

If a baby-Skyrmion is identified with an electron, it is important that it carries all electron quantum numbers. Two important properties are the spin and statistics of the particle. In an effective QCD theory the Wess-Zumino-Witten term is responsible for enforcing both that the spin and the statistical behavior of the Skyrmion agrees with the ones of observed hadrons. This vital aspect needs an analogy if the baby-Skyrmion is to be attributed to an electron. Remarkably, it has been possible to do just that. There exists an exact analogy to the WZW-term in the three-dimensional solid state theory. This term ensures that the baby-Skyrmion behaves just like an electron when considering its spin (it must rotate by 4π before it returns to the original state) or statistical behavior (when interchanging its position with another baby-Skyrmion there is a minus sign). This is illustrated for Skyrmions in fig.4.2, but also applies for baby-Skyrmions.

Chapter 8

Discussion and Conclusions

A systematic scrutiny of Goldstone bosons in QCD and magnetism has proven to be extraordinarily fruitful. There exist aesthetic analogies between the two low-energy dynamics, derived from chiral Lagrangians, that describe the behavior of pions and magnons. Topological considerations allow for a wider analysis of the model and introduce new phenomena that are not accessible with perturbation theory alone. In this respect there is a strong correspondence between the two theories, where QCD gives clues of what one might expect to find in magnetism.

When a continuous global symmetry is spontaneously broken to a subgroup the Goldstone theorem predicts the appearance of massless bosons. In QCD (in 3+1-dimensions) the global chiral symmetry group $G = SU(2)_R \otimes SU(2)_L \otimes U(1)_B$ is spontaneously broken, leaving a remaining symmetry $H = SU(2)_{L=R} \otimes U(1)_B$. The Goldstone bosons are the lightest particles of QCD, which in this case are the three pions. These Goldstone boson fields live in the coset space $G/H = SU(2)$.

In magnetism (in 2+1-dimensions) there is a continuous global symmetry $G = SU(2)_s$ which is spontaneously broken to the subgroup $H = U(1)_s$. In this case the Goldstone bosons are two magnon fields that live in the coset space $G/H = \mathbb{C}P(1)$. Goldstone bosons are massless and hence dominate at low-energies. Chiral perturbation theory provides the formalism for a derivative expansion that is used to find the leading order terms. Antiferromagnetic magnons have a relativistic dispersion relation while ferromagnetic magnons are non-relativistic. The non-relativistic action contains a topological term, resulting in the constraint that the total spin is quantized in integer or half-integer units.

The analogies between QCD and magnetism become all the more fascinating when topological considerations are included. In QCD topologically non-trivial pion fields carry a topological charge that has been identified by Skyrme to be the baryon number. This means that a pion field, which can not be continuously deformed into the vacuum configuration, carries baryon quantum numbers. A further homotopy group $\Pi_4[SU(2) \simeq S^3] = \mathbb{Z}_2$ restricts the statistics of the Skyrmion to that of bosons or fermions with the number of colors N_c as a statistics parameter. Odd N_c corresponds to fermions and even N_c corresponds to bosons. In nature $N_c = 3$, corresponding to the three quark colors.

A topological charge with a corresponding conserved topological current can also be defined in the magnon theory. The corresponding topological solitons are known as baby-

Skyrmions. It is not entirely clear what quantum numbers such objects may carry. Again a microscopic theory can provide some insight. In the QCD case non-trivial pion fields (where pions consist of quarks), result in baryons that, of course, also consist of quarks. The magnons are created through special configurations of electrons. Therefore it is not far fetched that the baby-Skyrmions are related to the electrons. In quantum Hall ferromagnets it is known that the topological charge is indeed the electron charge. There may be additional variations of ferromagnets and also antiferromagnets where the topological charge is related to the electric charge. The number of electrons that contribute to each baby-Skyrmion is defined through a parameter θ which is material-dependent. The additional homotopy group $\Pi_3[S^2] = \mathbb{Z}$ characterizes the braiding of baby-Skyrmion paths in time. There are a continuous variety of statistics, unlike in the QCD case where the corresponding homotopy group $\mathbb{Z}_2$ allows for the field configurations to be classified into either trivial (0) or non-trivial (1) classes. Hence, the only statistics allowed in QCD is either bosonic or fermionic. The baby-Skyrmions can be quantized as anyons with the corresponding anyon angle θ parameter. The baby-Skyrmions can carry fractional electron quantum numbers. It has been argued that in antiferromagnets baby-Skyrmions are bosons, in which case θ is an integer multiple of 2π. In this case the topological charge is an even multiple of $-e$ (or zero). Ferromagnetic baby-Skyrmions are known to be fermions as is the case of quantum Hall ferromagnets, where they carry the charge $-e$. In the low energy limit one may view the electrons as 'confined' inside the baby-Skyrmions. This view point makes most sense for the case where the baby-Skyrmions contain two electrons. This represents a Cooper pair, which upon condensation could lead to superconductivity, as it is the case with certain doped antiferromagnets. Doping antiferromagnets eventually destroys the staggered magnetization. The baby-Skyrmions appear as a hedgehog structure in the spin orientation. If adding electrons means increasing the number of baby-Skyrmions then it is not surprising that the hedgehog structure will eventually destroy the antiferromagnetism.

Electromagnetism is included in QCD by replacing the derivatives with $U(1)_{em}$ covariant derivatives. The topological current must also be adapted so that it is gauge invariant and conserved. Since at the microscopic level the charge of the Skyrmion is tied to the baryon number, there is a coupling between the neutral pion field and electromagnetism. This mechanism, which is realized through the Goldstone-Wilczek current, allows for a neutral pion to decay into two photons ($\pi^0 \to \gamma\gamma$).

It was shown that up to the order $\frac{1}{m_0^3}$ the Pauli equation has a local $SU(2)_s$ symmetry. The Pauli equation can be viewed as the microscopic theory for magnons analogous to the Standard Model which is the microscopic theory for pions. All symmetries of the underlying microscopic theory must be realized in the effective theory. In the Pauli equation the electric and magnetic fields appear as non-Abelian vector potentials. To include electromagnetism in the magnon theory it is necessary not only to gauge the $U(1)_{em}$ symmetry but also $SU(2)_s$. If the topological charge is related to the electric charge then the new $SU(2)_s$ gauge invariant Goldstone-Wilczek current leads to a coupling of magnons (which are neutral) to the electromagnetic field. This lead to the decay vertex $m \to \gamma\gamma$. Most likely it is experimentally easier to detect the conversion of a photon into a magnon. This conversion process occurs in the presence of an external static magnetic field, in which case the magnons acquire a mass proportional to the magnetic field. This is analogous to the photon-axion conversion process, experimentally sought after.

It was further investigated what decay processes exist for the Skyrmions and baby-Skyrmions. In the Standard Model a baryon number violating electroweak instanton allows for the decay of a baryon. This is the result of 't Hooft's global anomaly caused when $SU(2)_L$ is gauged. As a result, the Goldstone-Wilczek current is gauge invariant but no longer conserved. There is no analogy of this in the magnon model. It was also shown that, in the presence of a magnetic monopole, baryons can convert into leptons. Such magnetic monopoles appear in $SU(5)$ GUT. The monopole is located at the spacial end of a Dirac string and appears as a soliton.

In the magnon theory, with one spacial dimension less, it can be shown that a Dirac string analog represents a charged wire. A discharging wire catalyzes a baby-Skyrmion decay. In this case the event represents an instanton. The baby-Skyrmion 'decay' is just an electron leaving the two-dimensional plane of the magnet via a charge carrying wire. Baby-Skyrmions can only exist in the magnet and hence, when the electrons move onto the wire, the baby-Skyrmion decay.

It is known that there are more than just two quark flavors. The global chiral symmetry group can be expanded to include all quarks. The symmetry $G = SU(N_f)_R \otimes SU(N_f)_L \otimes U(1)_B$ then breaks to $H = SU(N_f)_{L=R} \otimes U(1)_B$. For $N_f = 3$ this results in the three pions, the four kaons, and the η-mesons. Since the quarks with flavor other than up or down have a non-negligible mass the approximation of massless mesons becomes less appropriate. This is because the global symmetry is no longer exact. For $N_f \geq 3$ the relevant homotopy groups change. Since $\Pi_4[SU(N_f)] = 0$ the homotopy group $\Pi_5[SU(N_f)] = \mathbb{Z}$ is used to restore the desired statistic behavior of the Skyrmions. For this an excursion is made into a fifth dimension using a deformation parameter. The Wess-Zumino-Witten term is then included in the theory. In order for the dynamics of the mesons to be affected only by the physical pion field, the integer prefactor N_c must be introduced. The Wess-Zumino-Witten term ensures that all parity symmetry properties present in QCD also hold in the effective theory. This term also contributes to the $\pi^0 \to \gamma\gamma$ decay.

For the magnon theory the symmetry group can be expanded to $G = SU(N_f)$ which then breaks spontaneously to $H = U(N_f - 1)$. The Goldstone boson fields live in the coset space $G/H = CP(N_f - 1)$. For quantum Hall ferromagnets this corresponds to a layer index. Following the mathematical procedures of the QCD case, a Wess-Zumino-Witten term analog has been defined and constructed from the homotopy group $\Pi_4[CP(N-1)] = \{0\}$ in an $SU(N_f)$ invariant form. For $N_f = 2$ this reduces to the Hopf term (before gauging). To include electromagnetism it is only necessary to gauge $SU(2)_s$, a subgroup of $SU(N)_f$. The Wess-Zumino-Witten term also affects the statistics of the baby-Skyrmions and contributes to the $m \to \gamma\gamma$ decay. Since the homotopy group responsible for the Wess-Zumino-Witten term is trivial, there is no quantization condition on the prefactor, which is now the anyon statistics angle θ.

The results presented in this work are condensed in a paper by Oliver Bär, Matthias Imboden, and Uwe-Jens Wiese [47].

So, what is left for the future? Now that the formalism has been developed it would be interesting to see what it can tell us about magnons. This model makes it possible to investigate magnon and baby-Skyrmion interactions in a new way. It may be possible to construct a mechanism for baby-Skyrmion coupling. In an antiferromagnet this would correspond to Cooper pair formation. For verification that the baby-Skyrmions do indeed carry electric charge experiments may be necessary. The observation of a photon-magnon conversion would be a strong verification of the electronic nature of the baby-Skyrmions. Some results obtained for the magnons and baby-Skyrmions in magnetism are analogues derived from what is already known about pions and Skyrmions in QCD. Reversely, what is known in magnetism may help explain, or at least mathematically describe, what is happening in particle physics. For example it was found, that there is an analogue between the number of flavors (or better quark-lepton generation number) in particle physics and the index number of a multi-layer quantum Hall ferromagnet. It is clear what the layer index means for a quantum Hall ferromagnet, whereas the origin of the generation number in particle physics is still a mystery. It is believed that there is still much to be learned from the relationships between particle and condensed matter physics.

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Appendix A

A.1 Important Spaces

For a two flavor QCD theory and the corresponding single-layer magnon theory the groups $G = SU(2)_L \otimes SU(2)_R \otimes U(1)_Y$ and $G = SU(2)_s$, are broken to the subgroups $H = SU(2)_{R=L} \otimes U(1)_Y$ and $H = U(1)_s$ respectively. To determine what happens when there are more than two flavors we consider the following spaces with $N_f \geq 3$.

For the QCD effective field theory we have

$$\begin{aligned} G &= SU(N_f)_L \otimes SU(N_f)_R \otimes U(1)_B, \\ H &= SU(N_f)_{R=L} \otimes U(1)_B, \\ G/H &= SU(N_f). \end{aligned} \tag{A.1}$$

While in the magnon effective theory

$$G = SU(N_f), \quad H = U(N_f - 1), \quad G/H = CP(N_f - 1), \tag{A.2}$$

where $CP(N_f - 1)$ is defined as

$$CP(N_f - 1) = \frac{SU(N_f)}{U(N_f - 1)} = \frac{S^{2N_f-1} \otimes S^{2N_f-3} \otimes \ldots \otimes S^5 \otimes S^3}{S^{2N_f-3} \otimes \ldots \otimes S^1} = \frac{S^{2N_f-1}}{S^1}. \tag{A.3}$$

For example, one has

$$\begin{aligned} CP(2) &\simeq \frac{SU(3)}{U(2)} = \frac{S^5 \cdot S^3}{S^3 \cdot S^1} = \frac{S^5}{S^1}, \\ CP(1) &\simeq \frac{SU(2)}{U(1)} = \frac{S^3}{S^1} = S^2. \end{aligned} \tag{A.4}$$

A.2 Gauging $SU(2)$

There are standard procedures for constructing a gauge theory. Some details of gauging an $SU(2)$ symmetry are illustrated below.

The covariant derivative $D_\mu P$ is given by

$$D_\mu P = \partial_\mu P + [W_\mu, P], \tag{A.5}$$

where W_μ is a non-Abelian gauge field.

The covariant derivative is constructed so that the terms in the Lagrangian are invariant under local $SU(2)$ symmetry transformations. The kinematic term takes the form $\partial_\mu P\,\partial_\mu P \;\rightarrow\; D_\mu P\, D_\mu P$. Using the gauge transformation properties of P and W_μ,

$$P' = g^\dagger P g, \qquad W'_\mu = g^\dagger (W_\mu + \partial_\mu) g, \tag{A.6}$$

the covariant term transforms as

$$(D_\mu P\, D_\mu P)' \;=\; g^\dagger (D_\mu P\, D_\mu P) g. \tag{A.7}$$

The gauge field leads to an additional term -the field strength. Since W_μ is a non-Abelian gauge field, the corresponding field strength takes the form

$$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + [W_\mu, W_\nu]. \tag{A.8}$$

A.3 Calculating the WZW-Term

Some additional information is given here to clarify the calculations used to find the gauge variation of the WZW term and the corresponding term that counters the gauge variation for the magnon theory.

The gauge variation ($S_{WZW}[P'] - S_{WZW}[P]$, $P \rightarrow P' = g^\dagger P g$) results in 81 terms that can eventually be simplified to ten terms. These can be written as a total divergence so the integration over the hemisphere H^4 is reduced to the boundary of the hemisphere which is S^3, the surface of a four-dimensional hypersphere.

To find the Chern-Simons term analog it was necessary to try out various terms containing only the magnon field P and the gauge potential W_μ and/or corresponding derivatives thereof. The new terms must have the same gauge variation as the original Wess-Zumino-Witten term. These terms can then be subtracted, making the entire conglomerate gauge invariant.

Here are some useful formulas:

$$\begin{aligned}
\partial_\nu P' &= g^\dagger (P \partial_\nu g g^\dagger + \partial_\nu P - \partial_\nu g g^\dagger) g, \\
W'_\rho &= g^\dagger (W_\rho - g \partial_\rho g^\dagger) g, \\
\partial_\nu W'_\rho &= g^\dagger (-\partial_\nu g g^\dagger + \partial_\nu W_\rho + W_\rho \partial_\nu g g^\dagger + \partial_\rho g g^\dagger \partial_\nu g g^\dagger) g, \\
\partial_\nu (\partial_\rho g g^\dagger) &= \partial_\nu g g^\dagger \partial_\rho g g^\dagger.
\end{aligned} \tag{A.9}$$

When the dust settles, the gauge variation of this term cancels all terms of the gauge variation of the WZW-term. There are additional terms that appear, some of them cancel, some remain. It can be shown that the remaining terms (that should vanish in a gauge invariant theory) form a total divergence. Since the expression is an integral over a boundary (the boundary of a four-dimensional hypersphere) the total divergence has no contribution, i.e.

$$\begin{aligned} &\int_{S^3} d^3x\ \epsilon_{\nu\rho\sigma}\partial_\nu(-P\partial_\rho g g^\dagger P W_\sigma) \\ = &\int_{S^3} d^3x\ \epsilon_{\nu\rho\sigma}(P\partial_\nu g g^\dagger \partial_\rho P W_\sigma - \partial_\nu P \partial_\rho g g^\dagger P W_\sigma \\ &+P\partial_\nu g g^\dagger P \partial_\rho W_\sigma - P\partial_\nu g g^\dagger \partial_\rho g g^\dagger P W_\sigma) = 0. \end{aligned} \tag{A.10}$$

A.4 Summary of the analogies of the pion and magnons theories

QUANTITY	MAGNONS	PIONS
Global symmetry	$SO(3)_s \simeq SU(2)_s$	$SU(2)_L \otimes SU(2)_R \otimes U(1)_B$
Unbroken subgroup H	$SO(2)_S \simeq U(1)_s$	$SU(2)_{L=R}$
Goldstone boson field in G/H	$\vec{e}(x) \in S^2,\ P \in CP(1)$	$U(x) \in S^3$
Coupling strength	spin stiffness ϱ_s	π-decay constant F_π
Propagation speed	spin wave velocity	velocity of light
Topological soliton	baby-Skyrmion	Skyrmion
Soliton homotopy	$\Pi_2[S^2] = \mathbb{Z}$	$\Pi_3[S^3] = \mathbb{Z}$
Topological charge	electric charge Q	baryon number B
Soliton statistics	anyons	bosons or fermions
Statistics homotopy	$\Pi_3[S^2] = \mathbb{Z}$	$\Pi_4[S^3] = \mathbb{Z}_2$
Statistics factor	$\exp(i\theta H[\vec{e}])$	$\mathsf{Sign}[U]^{N_c}$
Statistics parameter	anyon angle θ	number of colors N_c
Electromagnetic decay	magnon $\to \gamma\gamma$	pion $\to \gamma\gamma$
Soliton decay catalyzer	charged wire	magnetic monopole

Table A.1: *Analogies between pions and magnons.*

QUANTITY	N_f MAGNON FLAVORS	PIONS, KAONS, and η-MESONS
Global symmetry G	$SU(N_f)$	$SU(N_f)_L \otimes SU(N_f)_R$
Unbroken subgroup H	$U(N_f - 1)$	$SU(N_f)_{L=R}$
Goldstone field in G/H	$P(x) \in CP(N_f - 1)$	$U(x) \in SU(N_f)$
Additional label	layer index	generation index
Soliton homotopy	$\Pi_2[CP(N_f - 1)] = \mathbb{Z}$	$\Pi_3[SU(N_f)] = \mathbb{Z}$
Soliton statistics	anyons	bosons or fermions
Statistics homotopy	$\Pi_3[CP(N_f - 1)] = \{0\}$	$\Pi_4[SU(N_f)] = \{0\}$
WZW term	$\exp(i\theta S_{WZW}[P])$	$\exp(2\pi i N_c S_{WZW}[U])$
WZW homotopy	$\Pi_4[CP(N_f - 1)] = \{0\}$	$\Pi_5[SU(N_f)] = \mathbb{Z}$
Statistics parameter	unquantized θ	quantized N_c

Table A.2: *Analogies between pion, kaon, and η-meson physics and the physics of magnons with several flavors.*

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