Cartan Connections in Conformal Gauge Theories

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Abstract

Typical difficulties in the formulation of conformal gauge theories concerning the role of special conformal transformations and Cartan connections are discussed within the theory of second order G-structures.

1 Introduction

The basic building block in the construction of gauge theories of *internal* symmetry groups G, the minimal coupling principle, has got its natural geometrical expression via the interpretation of gauge potentials as connection forms on principal fibre bundles P with structure group G [1,2]. Accordingly, defining a connection form ω on P means for each point x_0 of spacetime M to distinguish a certain class of linearly related local (better: infinitesimal) frame fields $e(x)$ in P around x_0. These are precisely the 'horizontal subspaces' of ω [3]. Invariant matter Lagrangians, constructed by equivariant functions on P and their covariant derivatives, then locally reduce to their 'free' form ($\nabla_\mu \to \partial_\mu$) when expressed in these local frames ('G-equivalence principle' [4,5]). This is in geometrical analogy with the central role played by the Levy-Civita connections in General Relativity, the local 'horizontal' frame fields being in this case the usual local 'inertial' frames. To take this analogy seriously and to incorporate gravity into the general and successful scheme of gauge theories was the motivation to gauge also *external*, i.e. space-time groups, in particular the Poincaré group $T^4 \otimes L$, $L := SO_0(3,1)$ (see e.g. [6,7]).

In bundle language, gravitational gauge theories then are constructed on $T^4 \otimes L$-bundles P with symmetry breaking to Lorentzsubbundles P_L [4,5]. Correspondingly, gauge potentials are introduced via pull backs to P_L of connection forms on P, the translational part $\theta_\nu = \theta^i_\nu T_i$ being identified with the usual tetrad fields, the Lorentz part $\mu_\nu = \frac{1}{2}\mu^{ik}_\nu \mathcal{M}_{ik}$ carrying torsion in general. Thus, $\theta_\nu + \mu_\nu$ is to be considered as the pull back of a generalized affine connection [3], a special case of a Cartan connection [8].

The renewed interest in the conformal group C (here: $C := SO_0(4,2)$) as a spacetime group generalizing the Poincaré group [9] may give rise to investigate gauge aspects of C, again using the mathematical language applied in the case of $T^4 \otimes L$. However, as constrasted with $T^4 \otimes L$, the geometrical and algebraical structure of C generates typical additional problems in a gauge formulation. We mention the following interrelated properties and indicate a general framework for their treatment using notions and notations from [10]; a more detailed version of our discussion will be published elsewhere.

(i) The Lie algebra $c = t^4 + l + d + k^4$ of C is *not reductive* with respect to its stability subalgebra $w^{11} = (l \oplus d) \oplus k^4$, the algebra of the 'prolonged' group $W^{11} = (L \otimes D) \otimes K^4$ (D denoting dilatations and K^4 special conformal transformations). This means $[t^4, (l + d + k^4)] \not\subset t^4$ and consequently a connection form on a C-bundle in general does not induce a connection form ω on a given W^{11}-subbundle via the w^{11}-part of ω.

(ii) The maximal 'holonomic unfolding' of a Lorentz structure g on M is already given by a first order structure, i.e. by a Lorentz subbundle $L_g M$ of the bundle LM of linear (hence first order) frames on M; whereas the corresponding bundle for a conformal structure $[g]$ on M is of *second order*, namely a W^{11}-subbundle $L^2_{[g]}M$ of the bundle L^2M of second order frames on M [8,10]. Since each point in L^2M corresponds to a 'holonomic' (i.e. induced by a local coordinate system on M) horizontal subspace in LM, $L^2_{[g]}M$ precisely consists of all 'conformal inertial frames' of $[g]$ (local inertial frames of metrics in the conformal class $[g]$).

(iii) C contains the *nonlinear*, special conformal transformations; this is the geometrical reason for the nontrivial 'prolongation' of the first order conformal structure $L_{[g]}M \subset LM$ (structure group $W^7 := L \otimes D$) to $L^2_{[g]}M \subset L^2M$ and the algebraic reason for the nonreductivity of C.

On the other hand, there exist natural common structures on those two bundles which should serve as 'background models' for gauge theories of each of the groups $T^4 \otimes L$ and C. These (maximal symmetric) backgrounds of course are the groups themselves, considered as L and W^{11}-bundles, resp., over the homogeneous spaces $(T^4 \otimes L)/L = T^4 = \mathbf{R}^4$ [1] (Minkowski space) and $C/W^{11} =: \widetilde{\mathbf{R}^4}$ ('conformal space' := double covering of compactified conformal Minkowski space). The common structures are the *canonical Cartan connections* or equivalently, from a group theoretical point of view, the canonical left invariant 1-forms on $T^4 \otimes L$ and C. Since this type of 'connections' played, at least implicitly, the

[1] We don't distinguish between canonically isomorphic objects, structures, etc. .

crucial part in Poincaré gauge theories, we intend to use it consequently as the central structure element in the conformal case, too. In particular, our aim here is to explain the role of this notion in a Lagrangian formulation of conformal theory.

2 Conformal space and the Canonical Cartan Connection

Following similar lines of reasoning as in the $T^4 \otimes L$ case, a starting point in the construction of a conformal gauge theory could be a conformally invariant 'free' Lagrangian Λ of those (massless) matter which shall serve as a 'source' of gravitation. Conformal invariance then means that Λ is defined as a 4-density on conformal space $\overline{\mathbf{R}^4}$, which 'takes the same form' in each conformal coordinate system on $\overline{\mathbf{R}^4}$. Let us make this definition precise, using both the bundle and the equivalent group picture.

A *conformal coordinate system* (ccs) on $\overline{\mathbf{R}^4}$ is given by a mapping

$$(g): \ \mathbf{R}^4 = T^4 \ni (x^i) = x \mapsto gxW^{11} =: [gx] \in C/W^{11} = \overline{\mathbf{R}^4} \tag{1}$$

where $g \in C$ is fixed. Then (g) induces a (second order) local *holonomic frame field* (hff)

$$\sigma_g : \ \overline{\mathbf{R}^4} \ni [gx] \mapsto gx \in C, \qquad x \in T^4 = \mathbf{R}^4 \tag{2}$$

which equally well may be understood as a local section in the second order conformal bundle $L^2_{[\eta]}\overline{\mathbf{R}^4} = C$, η denoting the Minkowski metric on $\mathbf{R}^4$ and $[\eta]$ the conformal structure on $\overline{\mathbf{R}^4}$.

Observe that a correct $1-1$ bundle description of the set of all ccs $(g), g \in C$, is *not* given via the set of all $\sigma_g, g \in C$, because two ccs, differing by a translation in $\mathbf{R}^4$ are represented by the *same* section in C; $(g) \neq (g+x)$, but $\sigma_g = \sigma_{g+x}$, for $0 \neq x \in T^4$. However, such a description can be realized, if we introduce the 'affine' extension $A^2_{[\eta]}\overline{\mathbf{R}^4}$ of $L^2_{[\eta]}\overline{\mathbf{R}^4}$, i.e. the C-bundle over $\overline{\mathbf{R}^4}$ of 'affine second order frames' on $\overline{\mathbf{R}^4}$, as described in [10]. It is associated to $L^2_{[\eta]}\overline{\mathbf{R}^4}$, $A^2_{[\eta]}\overline{\mathbf{R}^4} := L^2_{[\eta]}\overline{\mathbf{R}^4} \times_{W^{11}} C$, with standard left group action of W^{11} on C. (Its geometric interpretation becomes clear via the definition of the (double covering of the) 'compactified conformal tangent bundle' of $\overline{\mathbf{R}^4}$, $T^2_{[\eta]}\overline{\mathbf{R}^4} := L^2_{[\eta]}\overline{\mathbf{R}^4} \times_{W^{11}} \overline{\mathbf{R}^4}$. A point in the fibre $A^2_{[\eta],z}\overline{\mathbf{R}^4} \subset A^2_{[\eta]}\overline{\mathbf{R}^4}$ over $z \in \overline{\mathbf{R}^4}$ then is a second order conformal frame in a conformal manifold isomorphic to $\overline{\mathbf{R}^4}$, namely in the fibre $T^2_{[\eta],z}\overline{\mathbf{R}^4} \subset T^2_{[\eta]}\overline{\mathbf{R}^4}$.)

The *affine holonomic frame field* (ahff) corresponding to the ccs (g) now is defined by the local section

$$\widehat{\sigma}_g : \ \overline{\mathbf{R}^4} \ni [gx] \mapsto [gx, x^{-1}] \in A^2_{[\eta]}\overline{\mathbf{R}^4}, \qquad x \in T^4. \tag{3}$$

The correspondence $g \to \widehat{\sigma}_g$ is bijective and the natural geometric meaning of $\widehat{\sigma}_g$ is obtained with the help of $T^2_{[\eta]}\overline{\mathbf{R}^4}$. The point $\widehat{\sigma}_g(z) \in A^2_{[\eta]}\overline{\mathbf{R}^4}$ thus contains the information of the coordinates (x^μ) of $z \in \overline{\mathbf{R}^4}$ with respect to the ccs (g) (cf. also (6)).

The distinction between the notions of hff and ahff is crucial

(i) for a direct description of the canonical Cartan connection $\mathring{\omega}$ on $L^2_{[\eta]}\overline{\mathbf{R}^4}$ via parallel translations,

and, correlated with this, (see sects.3, 4)

(ii) for the translation of Λ, expressed in ccs on $\overline{\mathbf{R}^4}$, to an equivalent Lagrangian defined on an appropriate principal fibre bundle over $\overline{\mathbf{R}^4}$.

Concerning (i), the tangent planes of the ahff $\widehat{\sigma}_g$ (which are compatible with right C-action on $A^2_{[\eta]}\overline{\mathbf{R}^4}$) determine a flat connection form $\hat{\mathring{\omega}}$ on $A^2_{[\eta]}\overline{\mathbf{R}^4}$. Its pull back $\mathring{\omega}$ to $L^2_{[\eta]}\overline{\mathbf{R}^4} \subset A^2_{[\eta]}\overline{\mathbf{R}^4}$ as a c-valued 1-form coincides with the *canonical Cartan-connection* on $L^2_{[\eta]}\overline{\mathbf{R}^4}$, while the kernels of the w^{11} part of $\mathring{\omega}$ in each point of $L^2_{[\eta]}\overline{\mathbf{R}^4}$ are just given by the tangent planes of the hff σ_g.

3 Affine Matter Fields

The natural background for a bundle description of Λ turns out to be the C-bundle $A^2_{[\eta]}\overline{\mathbf{R}^4}$. To introduce matter fields on $A^2_{[\eta]}\overline{\mathbf{R}^4}$, choose

(α) a linear representation R of C on $\mathbf{K}^m$, $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, restricted to $W^{11} \subset C$, and

(β) a one-dim. real linear representation $V : e^\alpha \mapsto e^{n\alpha}$, $n \in \mathbf{Z}$, of the dilatation group $D \subset W^{11}$, trivially extended to W^{11}.

The linear representation $S = R \otimes V$ of W^{11} on $\mathbf{K}^m$ then defines 'matter fields' φ as W^{11}-equivariant functions on $L^2_{[\eta]}\overline{\mathbf{R}^4} \subset A^2_{[\eta]}\overline{\mathbf{R}^4}$.

We now apply the idea of induced representations to extend S by 'trivial translations' to a *non linear* representation $\widehat{S}$ of C in order to extend each φ to a C-equivariant function $\widehat{\phi}$ on $A^2_{[\eta]}\overline{\mathbf{R}^4}$, the *affine matter field* corresponding to φ.

Take the vector bundle $E := L^2_{[\eta]}\overline{\mathbf{R}^4} \times_{W^{11}} \mathbf{K}^m$, associated with respect to S, moreover the corresponding left group action $\widehat{S}$ of C on E, induced by the bundle automorphisms $g' \mapsto g \cdot g'$, $g \in C$ fixed, of $C = L^2_{[\eta]}\overline{\mathbf{R}^4}$. Identifying $\mathbf{K}^m$ canonically with the fibre in E over $0 := W^{11} \in C/W^{11} = \overline{\mathbf{R}^4}$, $C \times_{W^{11}} \mathbf{K}^m \ni [\mathbf{1}, \xi] \mapsto \xi \in \mathbf{K}^m$, the $\mathbf{K}^m$-valued functions φ on $L^2_{[\eta]}\overline{\mathbf{R}^4}$ can be considered as W^{11}-equivariant E-valued mappings. Hence it is possible to extend them to C-equivariant E-valued mappings $\widehat{\phi}$ on $A^2_{[\eta]}\overline{\mathbf{R}^4}$ in a unique way, i.e. the 'linear' φ canonically determines the 'nonlinear' $\widehat{\phi}$. The latter may be thought of as composed by φ and the canonical 'zero section' $\mathring{Y} : \overline{\mathbf{R}^4} \to T^2_{[\eta]}\overline{\mathbf{R}^4} = L^2_{[\eta]}\overline{\mathbf{R}^4} \times_{W^{11}} \overline{\mathbf{R}^4} = A^2_{[\eta]}\overline{\mathbf{R}^4} \times_C \overline{\mathbf{R}^4}$ (cf. Eq.(5)).

To be able to define derivations of the fields $\widehat{\phi}$, we have to realize these fields by 'numbers'. This can be done with the help of the natural embedding of $\mathbf{R}^4$ into $\overline{\mathbf{R}^4}$, $\mathbf{R}^4 = T^4 \ni x \mapsto [x] \in C/W^{11} = \overline{\mathbf{R}^4}$. Consider $\mathring{Y}$ as an $\overline{\mathbf{R}^4}$ valued C-equivariant function on $A^2_{[\eta]}\overline{\mathbf{R}^4}$, define $\vartheta := \mathring{Y}^{-1}(\mathbf{R}^4)$ and restrict each field $\widehat{\phi}$ to this open domain $\vartheta \subset A^2_{[\eta]}\overline{\mathbf{R}^4}$, containing $L^2_{[\eta]}\overline{\mathbf{R}^4}$. We get

$$\begin{array}{ccc} A^2_{[\eta]}\overline{\mathbf{R}^4} \supset \vartheta & \xrightarrow{\widehat{\phi}} & \mathbf{R}^4 \times \mathbf{K}^m \subset E \\ \uparrow & & \uparrow \\ L^2_{[\eta]}\overline{\mathbf{R}^4} & \xrightarrow{\varphi} & \{0\} \times \mathbf{K}^m = \mathbf{K}^m \end{array} \tag{4}$$

where the $\mathbf{R}^4$ component of $\widehat{\phi}$ coincides with $\mathring{Y}$, and the $\mathbf{K}^m$ component $\widehat{\varphi}$ is invariant under translations. Hence

$$\widehat{\phi}(\widehat{\sigma}(z)) = (\mathring{Y}(\widehat{\sigma}(z)), \widehat{\varphi}(\widehat{\sigma}(z))), \quad z \in \overline{\mathbf{R}^4} \tag{5}$$

$$\widehat{\phi}(\widehat{\sigma}(z) \cdot t) = (\mathring{Y}(\widehat{\sigma}(z)) - t, \widehat{\varphi}(\widehat{\sigma}(z))), \quad t \in T^4. \tag{6}$$

for any section $\widehat{\sigma}$ in ϑ. If σ denotes here the unique section in $L^2_{[\eta]}\overline{\mathbf{R}^4}$ determined by $\widehat{\sigma}$ via $\widehat{\sigma}(z) = \sigma(z) \cdot t(z)$, $t(z) \in T^4$, we find

$$\widehat{\varphi}(\widehat{\sigma}(z)) = \widehat{\varphi}(\sigma(z)) = \varphi(\sigma(z)). \tag{7}$$

This is compatible with the notations $\widehat{\sigma}_g$, σ_g.

Remark 1 The domain ϑ may be thought of as a disjoint union of a continuous set of Poincaré bundles, each of them obtained via (the affine extension of the second order prolongation of) a flat Lorentz bundle $L_{g\eta}g\mathbf{R}^4$, $g\mathbf{R}^4 = \{[gx] \mid x \in T^4\} \in \overline{\mathbf{R}^4}$ being Minkowski space 'shifted' by a conformal transformation $g \in K^4 \otimes D \subset C$.
By construction ϑ contains each ahff $\widehat{\sigma}_g$, and, with $z := [gx]$, $x \in \mathbf{R}^4$, we get

$$\mathring{Y}(\widehat{\sigma}_g(z)) = x \tag{8}$$

$$\widehat{\phi}(\widehat{\sigma}_g(z)) = (x, \varphi(\sigma_g(z))) \in \mathbf{R}^4 \times \mathbf{K}^m, \tag{9}$$

which connects the fields in ahff and hff. To get standard coordinate expressions, write

$$\widehat{\Psi}(x^i) := \widehat{\phi}(\widehat{\sigma}_g(z)), \quad \psi(x^i) := \varphi(\sigma_g(z)), \quad z = [gx]. \tag{10}$$

Eq.(9) then simply reads

$$\widehat{\Psi}(x^i) = (x^i, \psi(x^i)). \tag{11}$$

Eventually, we obtain the usual transformation rules for ψ under conformal transformations $f \in C$ [11], $x \mapsto fx$ denoting the local action of C on $\mathbf{R}^4$,

$$\begin{array}{ccc} (x^i) & \mapsto & (x'^i = (fx)^i) \\ \psi(x^i) & \mapsto & \psi'(x'^i) = S(f, x^i)\psi(x^i), \end{array} \tag{12}$$

with $S(f_2 f_1, x) = S(f_2, f_1 x)S(f_1 x, x)$ for $f_1, f_2 \in C$, and $S(h, 0) = S(h) = (R \otimes V)(h)$ for $h \in W^{11}$.

4 Conformally invariant Lagrangians

We are in a position now to define a Lagrangian $\widehat{\mathcal{L}}$ on $\vartheta \subset A^2_{[\eta]}\overline{\mathbf{R}^4}$ corresponding to the given Λ on $\overline{\mathbf{R}^4}$. We sketch the steps of construction here, which run similar lines as in our treatment of Poincaré gauge theories (cf. Remark 1 and [5]).

(i) Identify $\widehat{\mathcal{L}}$ in each ahff $\widehat{\sigma}_g$ with Λ in the ccs (g), using notations (10)

$$\widehat{\mathcal{L}}(\widehat{\varphi}(\widehat{\sigma}_g(z)), \partial_i\widehat{\varphi}(\widehat{\sigma}_g(z))) := \Lambda(\psi(x^k), \partial_i\psi(x^k)), \quad z = [gx]. \tag{13}$$

(ii) To introduce arbitrary coordinates (x^μ) on $\overline{\mathbf{R}^4}$, and observing $\mathring{Y}^i(\widehat{\sigma}_g(z)) = (x^i)$, write $\partial_\mu = \frac{\partial \mathring{Y}^i}{\partial x^\mu}\partial_i$.

(iii) For a covariant formulation, i.e. to get an expression for $\widehat{\mathcal{L}}$ in arbitrary sections $\widehat{\sigma}$ in ϑ, apply the canonical connection $\widehat{\mathring{\omega}}$ to define 'covariant' derivatives $\widehat{\mathring{\nabla}}\widehat{\varphi}$. This can be done in a rigorous way using the infinitesimal expressions for the non linear action of C on $\mathbf{R}^4\times\mathbf{K}^m \subset E$. One gets a Lagrangian $\widehat{\mathcal{L}}(\widehat{\varphi}(\widehat{\sigma}(z)), [\widehat{\mathring{\nabla}}_\nu \mathring{Y}^k(\widehat{\sigma}(z))]^{-1}{}^{\mu}_{i}\widehat{\mathring{\nabla}}_\mu\widehat{\varphi}(\widehat{\sigma}(z)))$ [2], which coincides with (13) for each $\widehat{\sigma} = \widehat{\sigma}_g$.

(iv) The restriction to arbitrary sections $\widehat{\sigma} = \sigma$ in $L^2_{[\eta]}\overline{\mathbf{R}^4}$ yields

$$\mathring{Y}^i(\sigma(z)) = 0, \quad \widehat{\mathring{\nabla}}_\nu \mathring{Y}^i(\sigma(z)) = \mathring{\theta}^i_\nu(\sigma(z)) \tag{14}$$

where the t^4-part $\mathring{\theta}$ of $\mathring{\omega}$ coincides with the pullback of the canonical 1-form [3] on $L_{[\eta]}\overline{\mathbf{R}^4}$ with respect to the projection $\pi : L^2_{[\eta]}\overline{\mathbf{R}^4} \to L_{[\eta]}\overline{\mathbf{R}^4}$. Setting

$$\mathring{\nabla}_\nu\varphi(\sigma(z)) := \widehat{\mathring{\nabla}}_\nu\varphi(\sigma(z)) = \widehat{\mathring{\nabla}}_\nu\widehat{\varphi}(\sigma(z)) \tag{15}$$

for $\widehat{\sigma} = \sigma$, the 'covariant' derivatives become

$$\mathring{\nabla}\varphi = d\varphi + (\mathring{\mu}^{ik}s(\mathcal{M}_{ik}) + \mathring{\gamma}s(\mathcal{D}) + \mathring{\kappa}^i s(\mathcal{K}_i))\varphi, \tag{16}$$

with decomposition $\mathring{\omega} = \mathring{\mu} + \mathring{\gamma} + \mathring{\kappa} + \mathring{\theta}$ of the canonical Cartan connection $\mathring{\omega}$ as a $c = l + d + k^4 + t^4$-valued 1-form on $L^2_{[\eta]}\overline{\mathbf{R}^4}$; s is the Lie algebra representation of w^{11} given by S, and $\mathcal{M}_i{}^k, \mathcal{D}, \mathcal{K}^i$ are the generators of l, d, k^4. Consequently, one defines on $L^2_{[\eta]}\overline{\mathbf{R}^4}$, with $\mathring{\theta}^\mu_i\mathring{\theta}^k_\mu := \delta^k_i$:

$$\mathcal{L}(\varphi(\sigma(z)), \mathring{\theta}^\mu_i(\sigma(z))\mathring{\nabla}_\mu\varphi(\sigma(z))) := \widehat{\mathcal{L}}(\widehat{\varphi}(\sigma(z)), [\widehat{\mathring{\nabla}}_\nu\mathring{Y}^k(\sigma(z))]^{-1}{}^{\mu}_{i}\widehat{\mathring{\nabla}}_\mu\varphi(\sigma(z))) \tag{17}$$

[2] Given a function φ on a section σ of a fibre bundle over M, differentiation with respect to a coordinate system $(x^\mu) : \mathbf{R}^4 \ni x \mapsto z(x) \in M$ is always understood as $\partial_\nu\varphi(\sigma(z)) := \partial_\nu\psi(x)$, where $\psi(x) := \varphi(\sigma(z(x)))$.

[3] We use the notation $\mathring{\theta}^i_\mu$ also for the canonical 1-form (on $L_{[\eta]}\overline{\mathbf{R}^4}$); i.e. $\mathring{\theta}^i_\mu(z)e_i(z) = \partial_\mu(z)$, with $(e_i(z)) = \pi(\sigma(z))$.

(v) To obtain 'dynamical geometry', eventually, one has to replace (minimal coupling, G-equivariance principle) the flat connection form $\hat{\mathring{\omega}}$ by an arbitrary connection form $\hat{\omega}$ on $A^2_{[\eta]}\overline{\mathbf{R}^4}$ or, equivalently, to pass from $\mathring{\omega}$ to the pull back $\omega =: \mu + \gamma + \kappa + \theta$ of $\hat{\omega}$ on $L^2_{[\eta]}\overline{\mathbf{R}^4}$. As usual in gravitational gauge theories, $\theta^i = \theta^i_\mu dx^\mu$ will be interpreted as 'tetrad fields', i.e. we require $\det \theta_\mu^{\ i} \neq 0$. Hence (including this constraint) ω is an arbitrary *Cartan connection* with respect to the embedding $L^2_{[\eta]}\overline{\mathbf{R}^4} \hookrightarrow A^2_{[\eta]}\overline{\mathbf{R}^4}$ [10]. The final matter Lagrangian then is

$$\mathcal{L}(\varphi(\sigma(z)), \theta^\mu_i(\sigma(z))\nabla_\mu\varphi(\sigma(z))), \tag{18}$$

$\nabla\varphi$ defined via ω as in (16) and θ^μ_i by $\theta^\mu_i\theta^k_\mu := \delta^k_i$.

Remark 2

(a) Observe, that because of the special non linear transformation properties of $\hat{\varphi}$, no translation term occurs in (16).

(b) By construction, $\mathcal{L}(\varphi, \mathring{\nabla}\varphi)$ is independent of the chosen section σ in $L^2_{[\eta]}\overline{\mathbf{R}^4}$ iff Λ is conformally invariant in the usual sense. As a consequence (which has to be proven) this is also valid for $\mathcal{L}(\varphi, \nabla\varphi)$ whith abitrary Cartan connection ω.

(c) The 'covariant' derivatives $\mathring{\nabla}\varphi$ and $\nabla\varphi$ do *not* transform in a tensorial way. I.e., the w^{11}-valued 1-form $(\mu^i{}_k, \gamma, \kappa^i)$ on $L^2_{[\eta]}\overline{\mathbf{R}^4}$ fails to be a connection form (cf. Introduction); the horizontal subspaces of this form are not compatible with right translation for special conformal transformations, because of the non linear character of the latter. However (!), for suitable degree n in the representation V of D in $S = R \otimes V$, there may be 'operators' K^i on $\mathbf{K}^m$, such that the contraction $K^i\nabla_i\varphi$ has the correct tensorial transformation property to serve for the construction of invariant Lagrange 4-forms on $L^2_{[\eta]}\overline{\mathbf{R}^4}$. This effect is in complete agreement with the commonly used '$\mathbf{R}^6$-formalism' of Dirac [12] to find conformally invariant wave equations (special case $\nabla_i = \mathring{\nabla}_i$), based on homogeneous wave functions on $\mathbf{R}^6$, n being their *degree of homogeneity* (see example below).

(d) The generalization of the formalism to the two fold covering groups $SL(2,\mathbf{C})$, $T^4 \otimes SL(2,\mathbf{C})$, $SL(2,\mathbf{C}) \otimes K^4$, $SU(2,2)$ of L, $T^4 \otimes L$, $L \otimes K^4$, $SO_0(4,2)$ is straightforward.

(e) Similar as in the case of Poincaré gauge theories, the 'affine' extension of the formalism is redundant in the following sense. Eq.(16) may be taken directly as the definition of $\mathring{\nabla}\varphi$ for the fields φ on $L^2_{[\eta]}\overline{\mathbf{R}^4}$, and $\mathcal{L}$ in (17) may be defined by replacing $\psi(x^i)$ and $\partial_i\psi(x^k)$ in Λ by $\varphi(\sigma(z))$ and $\mathring{\theta}^\mu_i(\sigma(z))\mathring{\nabla}_\mu\varphi(\sigma(z))$ $(\mathring{\theta}^\mu_i(\sigma(z))\mathring{\nabla}_\mu\varphi(\sigma(z)) =$

$\partial_i\varphi(\sigma(z))$ for $\sigma = \sigma_g$). Correspondingly for $\nabla\varphi$ and $\mathcal{L}$ in (18). Thus, the final version of the formalism of course takes place on $L^2_{[\eta]}\overline{\mathbf{R}^4}$ only. The purpose to discuss the affine aspects here was just to clarify the natural geometric role of the canonical Cartan connection in the construction of a covariant formalism.

5 An example: The Weyl Equation

To give an explicit application, let us consider the conformally invariant Weyl equation, constructed on $L^2_{[\eta]}\overline{\mathbf{R}^4}$. For that, choose the standard 4-dimensional representation R of $SU(2,2)$, with Lie algebra representation

$$\begin{aligned} r(\mathcal{M}_{ik}) &:= \tfrac{1}{2}\begin{pmatrix} \sigma_i\tilde{\sigma}_k & 0 \\ 0 & \tilde{\sigma}_i\sigma_k \end{pmatrix}, \qquad r(\mathcal{D}) := \tfrac{1}{2}\begin{pmatrix} -\mathbf{1}_2 & 0 \\ 0 & +\mathbf{1}_2 \end{pmatrix} \\ r(\mathcal{K}_k) &:= i\begin{pmatrix} 0 & 0 \\ \tilde{\sigma}_k & 0 \end{pmatrix}, \qquad r(\mathcal{T}_k) := i\begin{pmatrix} 0 & \sigma_k \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{19}$$

($\sigma^0 =: \tilde{\sigma}_0 =: \mathbf{1}_2$; $\sigma^\alpha = -\tilde{\sigma}_\alpha$, $\alpha = 1,2,3$, Pauli matrices) and $SU(2,2)$ invariant scalar product $(\xi_1, \xi_2) := \xi_1{}^+\Gamma\xi_2$, $\Gamma = \begin{pmatrix} & \mathbf{1}_2 \\ \mathbf{1}_2 & \end{pmatrix}$ on $\mathbf{C}^4$. Set $v(\mathcal{D}) := -n\mathbf{1}_4$, i.e. for $s := r + v$, $s(\mathcal{D}) = -\frac{1}{2}\begin{pmatrix} (2n+1)\mathbf{1}_2 & 0 \\ 0 & (2n-1)\mathbf{1}_2 \end{pmatrix}$ and $s = r$ for $\mathcal{M}_{ik}, \mathcal{K}_i$. In an arbitrary section σ of $L^2_{[\eta]}\overline{\mathbf{R}^4}$ the 'Weyl Lagrangian' 4-form then looks $\mathcal{L}(\varphi, \nabla\varphi) = L(\varphi, \nabla\varphi)(\det\theta^i_\mu)d^4x$ with

$$L(\varphi, \nabla\varphi) = \tfrac{1}{2}[\overline{\varphi}s(\mathcal{K}^i)\theta^\nu_i\nabla_\nu\varphi - (\nabla_\nu\overline{\varphi})s(\mathcal{K}^i)\theta^\nu_i\varphi], \tag{20}$$

$\overline{\varphi} := \varphi^+\Gamma$, $\nabla_\nu\varphi = \partial_\nu\varphi + [\frac{1}{2}\mu^{ik}_\nu s(\mathcal{M}_{ik}) + \gamma_\nu(\mathcal{D}) + \kappa^k_\nu s(\mathcal{K}_k)]\varphi$ and Cartan connection $\omega = \mu + \gamma + \kappa + \theta$. Taking into account the special properties of Cartan connections one shows for a (passive) gauge transformation, i.e. for a change $\sigma(z) \mapsto \sigma'(z) = \sigma(z)\cdot g(z)$, $g(z) \in W^{11}$, of sections in $L^2_{[\eta]}\overline{\mathbf{R}^4}$

$$\nabla_\nu\varphi'(z) = S(g(z)^{-1})\partial_\nu\varphi(z) + (g(z)^{-1}\omega_\nu g(z))^w s(\mathcal{L}_w)S(g(z)^{-1})\varphi(z), \tag{21}$$

where summation runs over the generators $\{\mathcal{L}_w\} = \{\mathcal{M}_{ik}, \mathcal{D}, \mathcal{K}_i\}$ of W^{11}.

Local special conformal transformations e.g. then yield $g^{-1}\omega g = g^{-1}(\mu + \gamma + \kappa)g + g^{-1}\theta g$, the second term being of the form $\theta + [\theta, \varepsilon^j\mathcal{K}_j] + X$, X k-valued. The only part of $g^{-1}\theta g$ contributing to $s(\mathcal{K}^i)\theta^\nu_i\nabla_\nu\varphi$ is $[\theta, \varepsilon^j\mathcal{K}_j] = [\theta^k\mathcal{T}_k, \varepsilon^j\mathcal{K}_j] = \theta^k\varepsilon^j(\mathcal{M}_{jk} + \eta_{jk}\mathcal{D})$. We find $s(\mathcal{K}^i)\theta^\nu_i(g^{-1}\theta_\nu g)^w s(\mathcal{L}_w) = \varepsilon^j s(\mathcal{K}^i)\theta^\nu_i\theta^k_\nu s(\mathcal{M}_{jk} + \eta_{jk}\mathcal{D}) = \varepsilon^j s(\mathcal{K}^k)s(\mathcal{M}_{jk} + \eta_{jk}\mathcal{D}) = \frac{i}{2}\varepsilon^j\begin{pmatrix} 0 & 0 \\ \alpha^k_{jk} & 0 \end{pmatrix}$ with $\alpha^k_{jk} = \tilde{\sigma}^k\sigma_j\tilde{\sigma}_k - (2n+1)\tilde{\sigma}_j = -3\tilde{\sigma}_j - (2n+1)\tilde{\sigma}_j$. Hence for $n = -2$ (the usual value for the degree of homogeneity of φ [12,13]) we get

$$\begin{aligned} & s(\mathcal{K}^i)\theta^\nu_i\nabla_\nu\varphi'(z) = \\ & = s(\mathcal{K}^i)\theta^\nu_i[S(g(z)^{-1})\partial_\nu\varphi(z) + (g(z)^{-1}(\mu_\nu + \gamma_\nu + \kappa_\nu)g(z))^w s(\mathcal{L}_w)S(g(z)^{-1})\varphi(z)] \end{aligned}$$

$$= s(\mathcal{K}^i)\theta_i^\nu[S(g(z)^{-1})\partial_\nu\varphi(z) + S(g(z)^{-1})(\mu_\nu + \gamma_\nu + \kappa_\nu)S(g(z))S(g(z)^{-1})\varphi(z)] \quad (22)$$
$$= s(\mathcal{K}^i)\theta_i^\nu S(g(z)^{-1})\nabla_\nu\varphi(z)$$
$$= S(g(z)^{-1})s(\mathcal{K}^i)\theta_i^\nu\nabla_\nu\varphi(z),$$

which proves invariance of $\mathcal{L}$ under local special conformal transformations. Similarly for the remaining gauge transformations in $L^2_{[\eta]}\overline{\mathbf{R}^4}$.

We finally get the *Weyl equation*

$$s(\mathcal{K}^i)\theta_i^\nu\nabla_\nu\varphi = 0 \quad (23)$$

or, setting $\varphi =: \binom{\xi}{\chi}$, its 2-component form

$$\tilde{\sigma}^i\theta_i^\nu\nabla_\nu\xi = 0, \quad (24)$$

with arbitrary gauge degree of freedom χ. The interpretation of (24) requires some attention and is as follows. Since ξ does not transform under special conformal transformations, $e^{\varepsilon^j s(\mathcal{K}_j)}\binom{\xi}{\chi} = (1+\varepsilon^j s(\mathcal{K}_j))\binom{\xi}{\chi} = \binom{\xi}{\chi + i\varepsilon^j\tilde{\sigma}_j\xi}$, ξ might be considered as a usual *first* order 2-component spin $\frac{1}{2}$ field, i.e. as a W^7-equivariant function on $L_{[\eta]}\overline{\mathbf{R}^4}$. However, Eq.(24), explicitly written as

$$\tilde{\sigma}^i\theta_i^\nu[\partial_\nu + \frac{1}{4}\mu_\nu^{jk}\sigma_j\tilde{\sigma}_k + \frac{3}{2}\gamma_\nu]\xi = 0, \quad (25)$$

has to be seen as an equation on the *second* order bundle $L^2_{[\eta]}\overline{\mathbf{R}^4}$, because μ_ν and γ_ν in (25) transform with respect to W^{11}! Accordingly, ξ must be interpreted as the pull back to $L^2_{[\eta]}\overline{\mathbf{R}^4}$ of a first order spinor field.

In particular, for $\omega = \mathring{\omega}$, Eq.(24) just gives the usual conformally invariant Weyl equation in a covariant form. It reduces to $\tilde{\sigma}^i\mathring{\theta}_i^\nu\partial_\nu\xi = \tilde{\sigma}^i\partial_i\xi = 0$ in each hff $\sigma = \sigma_g$, $g \in C$, of $L^2_{[\eta]}\overline{\mathbf{R}^4}$ (this is the same form it would have in an ahff $\widehat{\sigma}_g$ of $A^2_{[\eta]}\overline{\mathbf{R}^4}$ because of the trivially represented translations in the affine extension $\widehat{S}$ of S).

6 Concluding Remarks

There are convincing arguments to regard the notions of second order conformal structures and Cartan connections as appropriate tools for the treatment of conformally invariant equations (cf. also [14]) and conformal gauge theories. Concerning invariant equations we summarize

- Conformal transformations in conformal Minkowski space $(\mathbf{R}^4, [\eta])$ determine and are determined by their *second* order jets in $0 \in \mathbf{R}^4$. The natural fibre bundle taking into account this fact is the second order frame bundle $L^2_{[\eta]}\overline{\mathbf{R}^4}$. Global and singularity free formulations are possible by the extension to conformal space $(\overline{\mathbf{R}^4}, [\eta])$ and to $L^2_{[\eta]}\overline{\mathbf{R}^4}$, respectively.

- Conformal coordinate systems on $\overline{\mathbf{R}^4}$ induce 'holonomic' sections σ_g, $g \in C$, in $L^2_{[\eta]}\overline{\mathbf{R}^4}$. Conformally invariant equations or Lagrangians on $\overline{\mathbf{R}^4}$ then can be expressed on $L^2_{[\eta]}\overline{\mathbf{R}^4}$ with the help of these sections. The notion of the *canonical Cartan connection* $\mathring{\omega}$ on $L^2_{[\eta]}\overline{\mathbf{R}^4}$ enables a 'covariant' formulation.

To sketch roughly some steps in a possible gauge scheme for the group C, we add

- The replacement of $\mathring{\omega}$ by *arbitrary Cartan connections* ω in a conformally invariant matter Lagrangian on $L^2_{[\eta]}\overline{\mathbf{R}^4}$ and the addition of an appropriate gauge field Lagrangian $\bar{\mathcal{L}}(\omega, \nabla\omega)$ ($\nabla\omega$ = pull back to $L^2_{[\eta]}\overline{\mathbf{R}^4}$ of the curvature form $\widehat{\nabla}\widehat{\omega}$ on $A^2_{[\eta]}\overline{\mathbf{R}^4}$) lead to a framework compatible with the usual conceptions in gravitational gauge theories. In particular, the resulting field equations yield solutions for the translational part θ of ω which (more eactly its projection to $L_{[\eta]}\overline{\mathbf{R}^4}$) can be given the interpretation of 'deformation potentials' or 'tetrad fields' responsible for curving the flat background conformal structure (cf. [5,10]): θ_μ generate a bundle automorphism $L\overline{\mathbf{R}^4} \xrightarrow{\tau} L\overline{\mathbf{R}^4}$ via $T_z\overline{\mathbf{R}^4} \ni v^\mu\partial_\mu \mapsto \theta^i_\mu v^\mu e_i \in T_z\overline{\mathbf{R}^4}$, hence a conformal structure $L_{[\eta]'}\overline{\mathbf{R}^4} := \tau(L_{[\eta]}\overline{\mathbf{R}^4})$ with prolongation $L^2_{[\eta]'}\overline{\mathbf{R}^4}$ (the induced Cartan connection $\tau_*\omega$ on $L^2_{[\eta]'}\overline{\mathbf{R}^4}$ then fulfills $\tau_*\theta = \mathring{\theta}$).

- A suitable symmetry breaking mechanism from conformal to Lorentz structure, integrated into the gauge procedure on $L^2_{[\eta]}\overline{\mathbf{R}^4}$ (e.g. generalizing the discussion in [14] of the conformally invariant scalar wave equation $(\Box + \frac{R}{6})\varphi = 0$) finally would result in a Lorentz structure $L_{\eta'}\overline{\mathbf{R}^4}$, $\eta' \in [\eta]'$, and its prolongation $L^2_{\eta'}\overline{\mathbf{R}^4} \subset L^2_{[\eta]'}\overline{\mathbf{R}^4}$. The interpretation of $\tau_*\mu$ and $\tau_*\gamma$ on $L^2_{\eta'}\overline{\mathbf{R}^4}$ via their projections μ' and γ' on $L_{\eta'}\overline{\mathbf{R}^4}$ then is obvious. Together with η' they define a *Weyl-Cartan spacetime with gauge fixing* or, equivalently, a Lorentz metric η' on $\overline{\mathbf{R}^4}$ with nonsymmetric η'-compatible connection μ' and Weyl 1-form γ', the latter describing the change of η'-length under infinitesimal μ'-parallel displacements [15].

- The generalization from the compact, hence unphysical, conformal space $(\overline{\mathbf{R}^4}, [\eta])$ to its non compact, infinite covering is straightforward. Even, the generalization to an arbitrary flat or non flat conformal background manifold $(M, [\xi])$ is possible using Cartan connections on $L^2_{[\xi]}M$ and, in particular, the canonical Cartan connection (which exists for each $[\xi]$ on M [8]).

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