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Lorentz and diffeomorphism violations in linearized gravity

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ABSTRACT

Lorentz and diffeomorphism violations are studied in linearized gravity using effective field theory. A classification of all gauge-invariant and gauge-violating terms is given. The exact covariant dispersion relation for gravitational modes involving operators of arbitrary mass dimension is constructed, and various special limits are discussed.

The foundational symmetries of General Relativity (GR) include diffeomorphisms and local Lorentz transformations. The former act on the spacetime manifold, while the latter act in the tangent space. These two types of transformations are partially linked through the vierbein, which provides a tool for moving objects between the manifold and the tangent space. The proposal that Lorentz invariance might be broken in an underlying theory of gravity and quantum physics such as strings [1,2] naturally raises various questions about the relationship between diffeomorphism violation and Lorentz violation and about the associated phenomenological signals. These questions can be studied independently of specific models using gravitational effective field theory [3]. Here, following a brief summary of the current status and results, we develop a model-independent framework for studying these issues in linearized gravity. This limit provides a comparatively simple arena for exploration, and it is crucial for experimental analyses of gravitational waves and of gravitation in the Newton and post-Newton limits.

A generic treatment of Lorentz violation in Minkowski spacetime in the absence of gravity is comparatively straightforward using effective field theory [4]. In this context, the role of diffeomorphisms and local Lorentz transformations is played by translations and Lorentz transformations that act globally and combine to form the Poincaré group. The two symmetries can be broken independently, and a physical breaking of either one can be represented in terms of nonzero background fields in an effective field theory. The breaking of either can be spontaneous or explicit. Spontaneous breaking occurs when the background is dynamical, which means that it must satisfy the equations of motion and that it comes with fluctuations in the form of Nambu–Goldstone modes [5] and possibly also massive modes. In most applications of spontaneous breaking, the background satisfies the equations of motion in vacuum and can therefore be viewed as the vacuum expectation value. In contrast, explicit breaking is a consequence of a prescribed background, which is typically off shell and has no associated fluctuations. Much of the phenomenological literature investigating Lorentz violation in Minkowski spacetime assumes for simplicity that global spacetime translations are preserved in an approximately local inertial frame, canonically taken to be the Sun-centered frame [6]. This guarantees conservation of energy and momentum, so phenomenological signals are restricted to violations of the conservation laws for generalized angular momenta. A large body of experimental studies constrains this type of Lorentz violation [7].

In the presence of gravity, the situation becomes more involved. One complication arises because diffeomorphisms and local Lorentz transformations act on objects in different spaces that can be linked via the vierbein, which can relate the corresponding violations. In the case of spontaneous breaking, for example, the vacuum expectation values are on shell and a nonzero background on the spacetime manifold implies one in the tangent space and vice versa. As a result, diffeomorphism violation occurs if and only if local Lorentz violation does [8]. More intuitively, local Lorentz violation can be understood as a background direction dependence in a local freely falling frame [3]. Transporting this to the spacetime manifold via the vierbein then guarantees the existence of a direction dependence on the spacetime manifold and hence diffeomorphism violation.

Another complication for gravity concerns conservation laws and arises from the difference between spontaneous and explicit breaking. In general, a theory invariant under local transforma-

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tions comes with covariantly conserved currents [9]. In spontaneous breaking, the full theory remains invariant under the transformations and the symmetry is only hidden [10]. The currents remain conserved even though the background is unchanged by the transformations because the background fluctuations transform in a nonstandard way to compensate. This contrasts with explicit breaking, when the current conservation laws fail to hold.

In GR, local Lorentz invariance implies symmetry of the energy–momentum tensor while diffeomorphism invariance implies its covariant conservation [11]. In theories with spontaneous diffeomorphism and local Lorentz violation, these current-conservation laws are unaffected: an energy–momentum tensor for the full theory remains covariantly conserved and it is always possible to make it symmetric [3]. However, if explicit breaking occurs, then there is no guarantee that the energy–momentum tensor is explicitly conserved or symmetric, and as a result a theory with explicit breaking can be inconsistent or require reformulation within Finsler geometry [3,12]. For sufficiently involved models, this situation can be rescued by the additional modes that appear in theories with explicit diffeomorphism and local Lorentz violation [13]. These additional modes arise because in explicit breaking it becomes impossible to remove all four diffeomorphism degrees of freedom and six local Lorentz degrees of freedom from the vierbein. In some models, these additional modes can be constrained to restore the covariant conservation and symmetry of the energy–momentum tensor. The additional modes are the counterparts in explicit breaking of the Nambu–Goldstone modes appearing in spontaneous breaking. Indeed, they can be understood as Nambu–Goldstone excitations of Stueckelberg fields [14,15].

The above results have several implications for the phenomenology of diffeomorphism and local Lorentz violations in gravity. If the breaking is explicit, the challenge lies in establishing the consistency of theory and, if achieved, then in determining the effects of the additional modes on observational signals. In contrast, if the breaking is spontaneous, the Nambu–Goldstone and massive fluctuations can play the role of new forces affecting the phenomenology and so must be taken into account in analyzing experimental signals. Model-independent techniques for this have been developed both in the pure-gravity and in the matter-gravity sectors [16–26] and applied to obtain model-independent constraints on diffeomorphism and local Lorentz violation in gravity from a variety of experimental tests [7,27–56].

An alternative model-independent approach to studying both spontaneous and explicit diffeomorphism and local Lorentz violation uses linearized effective field theory for gravity, formulated to incorporate gauge and Lorentz violation [46]. In this context, gauge transformations are linearized diffeomorphisms of the metric fluctuation. This technique yields an explicit construction and classification of the general quadratic Lagrange density in effective field theory with gauge invariance at linearized level. It also permits construction of the general covariant dispersion relation and investigation of the properties of the corresponding gravitational modes. These results have been applied to obtain model-independent constraints on linearized coefficients for Lorentz violation using gravitational waves [46,51] and tests of gravity at short range [52,53]. In the present work, we extend this approach to explicit gauge breaking. We construct and classify all terms for the quadratic Lagrange density in gravitational effective field theory with explicit gauge violation, and we derive the corresponding covariant dispersion relation required for experimental applications. Throughout the work, we adopt the conventions of Ref. [3]: the metric signature is $+2$, the Levi-Civita tensor satisfies $\epsilon_{0123} = +1$, and parentheses or brackets about indices indicate symmetrization or antisymmetrization without numerical factors.

To perform the linearization, we expand the dynamical metric $g_{\mu\nu}$ in a flat-spacetime background with Minkowski metric, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. A generic term of mass dimension $d \geq 2$ in the Lagrange density for the linearized gravitational effective field theory can then be written as

$$\mathcal{L}_{\mathcal{K}^{(d)}} = \tfrac{1}{4} h_{\mu\nu} \hat{\mathcal{K}}^{(d)\mu\nu\rho\sigma} h_{\rho\sigma}, \tag{1}$$

where $\hat{\mathcal{K}}^{(d)\mu\nu\rho\sigma}$ is the product of a coefficient $\mathcal{K}^{(d)\mu\nu\rho\sigma\varepsilon_1\varepsilon_2\ldots\varepsilon_{d-2}}$ with $d-2$ derivatives $\partial_{\varepsilon_1}\partial_{\varepsilon_2}\ldots\partial_{\varepsilon_{d-2}}$. The coefficients $\mathcal{K}^{(d)\mu\nu\rho\sigma\varepsilon_1\varepsilon_2\ldots\varepsilon_{d-2}}$ have mass dimension $4-d$ and are assumed constant and small. The complete traces of these coefficients control Lorentz-invariant terms in $\mathcal{L}_{\mathcal{K}^{(d)}}$, while the other components govern Lorentz violation. To contribute nontrivially to the equations of motion, the operator $\hat{\mathcal{K}}^{(d)\mu\nu\rho\sigma}$ must satisfy the requirement $\hat{\mathcal{K}}^{(d)(\mu\nu)(\rho\sigma)} \neq \pm\hat{\mathcal{K}}^{(d)(\rho\sigma)(\mu\nu)}$, where the upper sign holds for odd d and the lower one for even d.

The action is invariant under the usual gauge transformations $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ when the condition $\hat{\mathcal{K}}^{(d)(\mu\nu)(\rho\sigma)}\partial_\nu = \pm\hat{\mathcal{K}}^{(d)(\rho\sigma)(\mu\nu)}\partial_\nu$ holds. Assuming this condition, the operators $\hat{\mathcal{K}}^{(d)\mu\nu\rho\sigma}$ can be constructed explicitly, using standard methods in group theory [57]. They are found to span three representation classes [46]. For the present work, we have extended this construction by relaxing the requirement of gauge invariance. Decomposing the operator $\hat{\mathcal{K}}^{(d)\mu\nu\rho\sigma}$ into irreducible pieces then yields another 11 representation classes. This shows that a total of only 14 independent classes of operators can appear in any linearized gravitational effective field theory, whether or not the Lorentz and gauge invariances hold. These 14 classes therefore characterize all phenomenological effects in linearized gravity, including effects on the propagation of gravitational waves and in the Newton and post-Newton limits.

To simplify the notation in what follows, we denote indices contracted into a derivative as a circle index $\circ$, with n-fold contractions denoted as $\circ^n$. With this convention, the generic operator $\hat{\mathcal{K}}^{(d)\mu\nu\rho\sigma}$ can be written as $\hat{\mathcal{K}}^{(d)\mu\nu\rho\sigma} = \mathcal{K}^{(d)\mu\nu\rho\sigma\circ^{d-2}}$. Also, we denote the 14 representation classes as indicated in the first column of Table 1. To obtain the term in the Lagrange density (1) associated to a given class, it suffices to replace $\hat{\mathcal{K}}^{(d)\mu\nu\rho\sigma}$ with the operator listed. The second column displays the index symmetries of each class using Young tableaux. The Table also lists some properties of each class. The third column indicates whether the operator is fully gauge invariant, and the fourth column displays the handedness under CPT of the associated term in the Lagrange density. Each class can occur only for even or for odd d and for d above a minimal value, as shown in the next column. The final column lists the total number of independent components appearing in the coefficient $\mathcal{K}^{(d)\mu\nu\rho\sigma\varepsilon_1\varepsilon_2\ldots\varepsilon_{d-2}}$ for fixed d.

The quadratic approximation $\mathcal{L}_0$ to the Lagrange density for the Einstein–Hilbert action can conveniently be written in the form

$$\mathcal{L}_0 = \tfrac{1}{4}\epsilon^{\mu\rho\alpha\kappa}\epsilon^{\nu\sigma\beta\lambda}\eta_{\kappa\lambda}h_{\mu\nu}\partial_\alpha\partial_\beta h_{\rho\sigma}. \tag{2}$$

This gauge- and Lorentz-invariant term is constructed from a piece of the coefficient $s^{(4)\mu\rho\alpha\nu\sigma\beta}$ in the first line of Table 1. The complete Lagrange density incorporating all the operators in Table 1 can then be expressed as

$$\mathcal{L} = \mathcal{L}_0 + \tfrac{1}{4} h_{\mu\nu} \sum_{\mathcal{K},d} \hat{\mathcal{K}}^{(d)\mu\nu\rho\sigma} h_{\rho\sigma}, \tag{3}$$

where the sum is over all the representation classes $\hat{\mathcal{K}}^{(d)\mu\nu\rho\sigma}$ shown in Table 1 and also over all allowed dimensions d for each class. Larger values of d introduce higher powers of momenta and

Table 1
Operators in the quadratic action for linearized gravity.

Operator $\hat{\mathcal{K}}^{(d)\mu\nu\rho\sigma}$	Tableau	Gauge invariant	CPT	d	Number
$s^{(d)\mu\rho\circ\nu\sigma\circ\circ^{d-4}}$	$\mu\ \nu\ \cdots$ $\rho\ \sigma$ $\circ\ \circ$	yes	even	even, ≥ 4	$(d-3)(d-2)(d+1)$
$s^{(d,1)\mu\rho\nu\sigma\circ^{d-2}}$	$\mu\ \nu\ \cdots$ $\rho\ \sigma$	no	even	even, ≥ 2	$(d-1)(d+2)(d+3)$
$s^{(d,2)\mu\rho\circ\nu\sigma\circ\circ^{d-4}}$	$\mu\ \nu\ \circ\ \cdots$ $\rho\ \sigma$ $\circ$	no	even	even, ≥ 4	$\frac{4}{3}(d-2)d(d+2)$
$q^{(d)\mu\rho\circ\nu\circ\sigma\circ\circ^{d-5}}$	$\mu\ \nu\ \sigma\ \cdots$ $\rho\ \circ\ \circ$ $\circ$	yes	odd	odd, ≥ 5	$\frac{5}{2}(d-4)(d-1)(d+1)$
$q^{(d,1)\mu\rho\nu\sigma\circ\circ^{d-3}}$	$\mu\ \nu\ \sigma\ \circ\ \cdots$ ρ	no	odd	odd, ≥ 3	$\frac{1}{2}(d+1)(d+3)(d+4)$
$q^{(d,2)\mu\rho\nu\circ\sigma\circ^{d-3}}$	$\mu\ \nu\ \sigma\ \cdots$ $\rho\ \circ$	no	odd	odd, ≥ 3	$(d-1)(d+2)(d+3)$
$q^{(d,3)\mu\rho\circ\nu\sigma\circ^{d-3}}$	$\mu\ \nu\ \sigma\ \cdots$ ρ $\circ$	no	odd	odd, ≥ 3	$\frac{1}{2}d(d+1)(d+3)$
$q^{(d,4)\mu\rho\nu\circ\sigma\circ\circ\circ^{d-5}}$	$\mu\ \nu\ \sigma\ \circ\ \cdots$ $\rho\ \circ\ \circ$	no	odd	odd, ≥ 5	$\frac{5}{3}(d-3)(d+1)(d+2)$
$q^{(d,5)\mu\rho\circ\nu\circ\sigma\circ\circ^{d-5}}$	$\mu\ \nu\ \sigma\ \circ\ \cdots$ $\rho\ \circ$ $\circ$	no	odd	odd, ≥ 5	$\frac{4}{3}(d-2)d(d+2)$
$k^{(d)\mu\circ\nu\circ\rho\circ\sigma\circ\circ^{d-6}}$	$\mu\ \nu\ \rho\ \sigma\ \cdots$ $\circ\ \circ\ \circ\ \circ$	yes	even	even, ≥ 6	$\frac{5}{2}(d-5)d(d+1)$
$k^{(d,1)\mu\nu\rho\sigma\circ^{d-2}}$	$\mu\ \nu\ \rho\ \sigma\ \cdots$	no	even	even, ≥ 2	$\frac{1}{6}(d+3)(d+4)(d+5)$
$k^{(d,2)\mu\circ\nu\rho\sigma\circ\circ^{d-4}}$	$\mu\ \nu\ \rho\ \sigma\ \circ\ \cdots$ $\circ$	no	even	even, ≥ 4	$\frac{1}{2}(d+1)(d+3)(d+4)$
$k^{(d,3)\mu\circ\nu\circ\rho\sigma\circ^{d-4}}$	$\mu\ \nu\ \rho\ \sigma\ \cdots$ $\circ\ \circ$	no	even	even, ≥ 4	$(d-1)(d+2)(d+3)$
$k^{(d,4)\mu\circ\nu\circ\rho\circ\sigma\circ\circ^{d-6}}$	$\mu\ \nu\ \rho\ \sigma\ \circ\ \cdots$ $\circ\ \circ\ \circ$	no	even	even, ≥ 6	$\frac{5}{3}(d-3)(d+1)(d+2)$

so the corresponding terms in the effective field theory are expected to be more suppressed. In practice, to avoid possible issues with interpretation of the infinite sum of terms, the sum over d can be truncated at some value or restricted to specific choices of d.

The equations of motion for the metric fluctuation $h_{\mu\nu}$ can be found from the Lagrange density. Performing Fourier transforms to convert to momentum space, where $\partial_\mu \to ip_\mu$, the equations of motion can be written in the form

$$M_{\kappa\lambda}{}^{\mu\nu} h_{\mu\nu} = 0. \tag{4}$$

The operator $M_{\kappa\lambda}{}^{\mu\nu} = M_{\kappa\lambda}{}^{\mu\nu}(p)$ can be understood as a square 10×10 matrix that acts on a 10-component vector $h_{\mu\nu}$. If no gauge invariances are present, the dispersion relation for the gravitational modes is obtained by setting the determinant of the matrix to zero. However, in the presence of partial or full gauge invariance, finding a dispersion relation for the physical modes is more complicated because $M_{\kappa\lambda}{}^{\mu\nu}$ contains a null space. Nonetheless, a covariant dispersion relation can be found using methods from exterior algebra, as we show next. The technique presented here is a generalization of the method developed by us for the study of the photon sector of the SME [58] and independently by Itin for studies of premetric electrodynamics [59].

The key idea is to treat $h_{\mu\nu}$ as an element of a 10-dimensional complex vector space and $M_{\kappa\lambda}{}^{\mu\nu}$ as a linear map on the space. To keep the discussion general, we work with an N-dimensional complex vector space $\mathcal{V}$ and the exterior algebra $\wedge\mathcal{V}$ over $\mathcal{V}$. In terms of an arbitrary set of basis vectors $\{v^a\}$, $a = 1, 2, \ldots, N$, we write an n-vector ω as

$$\omega = \tfrac{1}{n!}\omega_{a_1 a_2 \ldots a_n} v^{a_1} \wedge v^{a_2} \wedge \cdots \wedge v^{a_n}, \tag{5}$$

and take its Hodge dual $*\omega$ to have components given by

$$(*\omega)^{a_1 \ldots a_{N-n}} = \tfrac{1}{n!}\epsilon^{a_1 \ldots a_{N-n} b_1 \ldots b_n}\omega_{b_1 \ldots b_n}. \tag{6}$$

Given a linear map $M : \mathcal{V} \to \mathcal{V}$ taking an arbitrary vector $x \in \mathcal{V}$ to a vector $y = M \cdot x$, we can construct a natural linear map $\wedge^n M$ between n-vectors. For n arbitrary vectors $\{x^1, x^2, \ldots, x^n\}$, we define

$$y^1 \wedge y^2 \wedge \cdots \wedge y^n = \wedge^n M(x^1 \wedge x^2 \wedge \cdots \wedge x^n). \tag{7}$$

In components this gives

$$(\wedge^n M)_{a_1 a_2 \ldots a_n}{}^{b_1 b_2 \ldots b_n} = \tfrac{1}{n!} M_{[a_1}{}^{b_1} M_{a_2}{}^{b_2} \ldots M_{a_n]}{}^{b_n}. \tag{8}$$

It is useful to define the wedge product of n different maps M^1, M^2, ..., M^n through

$$(M^1 \wedge M^2 \wedge \cdots \wedge M^n)_{a_1 a_2 \ldots a_n}{}^{b_1 b_2 \ldots b_n} = \left(\tfrac{1}{n!}\right)^2 (M^1)_{[a_1}{}^{[b_1} (M^2)_{a_2}{}^{b_2} \ldots (M^n)_{a_n]}{}^{b_n]}. \tag{9}$$

This is a map between n-vectors. Note that the product of maps is commutative and that $\wedge^n M$ is the product of n identical maps M.

Let r be the rank of M and let s be the dimension of the null space, so $r + s = N$. Denote the null space of M by $\mathcal{S}$ and its complement by $\mathcal{R}$, so that $\mathcal{V} = \mathcal{R} \oplus \mathcal{S}$. Let $\{z^1, \ldots, z^s\}$ be a set of vectors spanning $\mathcal{S}$, and let $\{x^1, \ldots, x^r, z^1, \ldots, z^s\}$ span $\mathcal{V}$. Any n-vector can then be expressed as a linear combinations of wedge products of n of these vectors. This implies that $\wedge^n M = 0$ for $r < n$. Also, the rank of $\wedge^n M$ for $n \le r$ is $r!/n!(r-n)!$ because the dimension of the image of $\wedge^n M$ matches the number of n-vectors constructed from the vectors $\{x^1, \ldots, x^r\}$. In particular, while the map M is rank r, the map $\wedge^r M$ is rank 1.

The dual map $*\wedge^r M$ can be constructed using the Hodge dual (6). However, both $\wedge^r M$ and $*\wedge^r M$ incorporate a null space, which complicates the derivation of the dispersion relation. To account explicitly for the null space of $\wedge^n M$, we can work instead with a modified dual $\star\wedge^n M$. Introducing $\zeta = *(z^1 \wedge \cdots \wedge z^s)$, some consideration reveals that we can write

$$(\wedge^n M)_{a_1 \ldots a_n}{}^{b_1 \ldots b_n} = \tfrac{1}{(r-n)!^2}\zeta^*_{a_1 \ldots a_n c_1 \ldots c_{r-n}} \zeta^{b_1 \ldots b_n d_1 \ldots d_{r-n}} \times (\star\wedge^n M)_{d_1 \ldots d_{r-n}}{}^{c_1 \ldots c_{r-n}}. \tag{10}$$

In the special case where $n = r$, we see that $\star\wedge^r M$ is a scalar obeying $\wedge^r M = \star\wedge^r M\, \zeta^* \otimes \zeta$, which implies

$$\wedge^r M = \mathrm{tr}(\wedge^r M)\, \frac{\zeta^* \otimes \zeta}{\zeta^* \cdot \zeta}. \tag{11}$$

The inverse relation for the modified dual can be obtained from Eq. (10). After some manipulation, we find

$$(\star\wedge^n M)_{b_1 \ldots b_{r-n}}{}^{a_1 \ldots a_{r-n}} = \left(\tfrac{r!}{n!\,\zeta^* \cdot \zeta}\right)^2 \times \zeta^{a_1 \ldots a_{r-n} c_1 \ldots c_n} \zeta^*_{b_1 \ldots b_{r-n} d_1 \ldots d_n} (\wedge^n M)_{c_1 \ldots c_n}{}^{d_1 \ldots d_n}. \tag{12}$$

For the case $\mathcal{S} = \emptyset$ so that $\mathcal{V} = \mathcal{R}$, the modified dual $\star\wedge^n M$ reduces to the usual dual $*\wedge^n M$. Taking instead $r = n$ yields the modified scalar dual

$$\star\wedge^r M = (\zeta^* \cdot \zeta)^{-1}\mathrm{tr}(\wedge^r M) = |\zeta \cdot \zeta|^{-2} \zeta^* \cdot \wedge^r M \cdot \zeta. \tag{13}$$

Note that since the modified scalar dual $\star\wedge^r M$ contains inverse powers of $\zeta^* \cdot \zeta$, one might naively expect a singularity when $\zeta^* \cdot \zeta = 0$. However, continuity and the relation $\wedge^r M = \star\wedge^r M\, \zeta^* \otimes \zeta$ guarantee that $\star\wedge^r M$ remains finite for nonzero ζ. Similarly, although $\star\wedge^n M$ may contain divergences, they cannot contribute to $\wedge^n M$ in Eq. (10).

Our goal is to obtain the exact covariant dispersion relation from equations of motion of the form $M \cdot x = 0$, where M depends on the momentum p^μ, while allowing for a possible nontrivial null space of M. Typically, we are interested in situations where it is convenient to split M into two pieces, $M = M_0 + \delta M$. This is useful, for instance, when M_0 is a standard expression or when calculations with M_0 can be performed in closed form. In many applications the Lorentz violation can be taken small, in which case the Lorentz-violating terms can be placed in δM and treated perturbatively. Note that the null spaces of M, M_0, and δM may all differ, but the null space of any one must contain the intersection of the null spaces of the other two.

To fix notation, suppose M_0 has rank r and M has rank r'. Let the null space $\mathcal{S}$ of M_0 be spanned by vectors $\{z^1, \ldots, z^s\}$, and let the null space $\mathcal{S}'$ of M be spanned by vectors $\{z'^1, \ldots, z'^{s'}\}$. Define $\zeta = *(z^1 \wedge \cdots \wedge z^s)$ and $\zeta' = *(z'^1 \wedge \cdots \wedge z'^{s'})$, and let $\star\wedge^n M$ be the modified dual (12) constructed with ζ and $\star'\wedge^n M$ be the modified dual constructed with ζ'. Vectors in the null space $\mathcal{S}$ represent trivial pure-gauge solutions of $M_0 \cdot x = 0$, while vectors in $\mathcal{S}'$ represent trivial solutions of $M \cdot x = 0$.

We seek nontrivial solutions to $M \cdot x = 0$, which exist if we can find p_μ that reduce the rank of M by at least one. The exact covariant dispersion relation arising from $M \cdot x = 0$ can therefore be expressed as

$$\star'\wedge^{r'} M = 0. \tag{14}$$

After expanding $\star'\wedge^{r'} M$ in terms of M_0 and δM, some calculation reveals that this equation can be written as

$$\sum_n \frac{n!}{r'!\,(r'-n)!}\mathrm{tr}\left[(\star'\wedge^n M_0) \cdot (\wedge^{(r'-n)}\delta M)\right] = 0. \tag{15}$$

The sum in this expression is understood to be limited to nonnegative wedge powers and restricted by the ranks of M_0 and δM. For example, $n \le r$ because larger values of n produce $\wedge^n M_0 = 0$ and so cannot contribute.

The expression (15) for the covariant dispersion relation is general and exact, and it is convenient for applications where the unbroken gauge vectors $\{z'^1, \ldots, z'^{s'}\}$ spanning S' and thus the form of ζ' are known. However, in some scenarios it is easier to work with the broken gauge vectors instead. In particular, in many situations of interest the null space $\mathcal{S}$ of M_0 contains the null space $\mathcal{S}'$ of M, $\mathcal{S} \supseteq \mathcal{S}'$, so that δM acts to remove a subset of null vectors in $\mathcal{S}$. Then, the ranks r of M_0 and r' of M satisfy $r \le r'$, and the coranks satisfy $s' \le s$. Null vectors in $\mathcal{S}$ can then be split into those spanning $\mathcal{S}'$ and those spanning the complement $\hat{\mathcal{S}} = \mathcal{S} - \mathcal{S}'$. Choosing a canonical ordering for definiteness, we write $\{z^1, \ldots, z^s\} = \{\hat{z}^1, \ldots, \hat{z}^{\hat{s}}, z'^1, \ldots, z'^{s'}\}$, where $\hat{s} = s - s' = r' - r$ is the number of gauge symmetries in M_0 that are broken in M. Introducing $\xi = (\hat{z}^1 \otimes \hat{z}^{1*}) \wedge \cdots \wedge (\hat{z}^{\hat{s}} \otimes \hat{z}^{\hat{s}*})$, calculation then yields an alternative form for the dispersion relation,

$$\sum_n \frac{n!(r'-n)!}{r'!\,(r-n)!^2}\mathrm{tr}\left[\left((\star\wedge^n M_0) \wedge \xi\right) \cdot (\wedge^{(r'-n)}\delta M)\right] = 0, \tag{16}$$

which holds for $\mathcal{S} \supseteq \mathcal{S}'$ and is convenient when the form of the broken gauge vectors $\{\hat{z}^1, \ldots, \hat{z}^{\hat{s}}\}$ is known.

A comparatively simple special case of the above arises when M and M_0 have the same null space $\mathcal{S}$, so that $\zeta = \zeta'$. In this case, δM preserves the gauge invariance of M_0 and the null space of δM must contain $\mathcal{S}$. The exact covariant dispersion relation then reduces to

$$\sum_n \frac{n!}{r!\,(r-n)!}\mathrm{tr}\Big[(\star\wedge^n M_0)\cdot(\wedge^{(r-n)}\delta M)\Big] = 0. \tag{17}$$

For example, with M_0 corresponding to the usual linearized Einstein–Hilbert term in the Lagrange density and taking $r = 6$, Eq. (17) provides the exact covariant dispersion relation for the general linearized gravitational theory formed by extending linearized GR with arbitrary gauge-invariant terms. As another example, if neither M nor M_0 has any gauge symmetry then $r = 10$, and Eq. (17) provides the covariant dispersion relation for any δM with or without gauge invariance.

As an application of the above results, consider linearized GR. The linearized Einstein field equations take the form ${M_0}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} = 0$. The operator ${M_0}_{\mu\nu}{}^{\rho\sigma}$ can be expressed as

$${M_0}_{\mu\nu}{}^{\rho\sigma} = \tfrac{1}{2}p^2(\pi_{\mu\nu}{}^{\rho\sigma} - P_{\mu\nu}P^{\rho\sigma}), \tag{18}$$

where the projections $P^{\mu\nu}$ and $\pi_{\mu\nu}{}^{\rho\sigma}$ given by

$$P^{\mu\nu} = \eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}, \quad \pi_{\mu\nu}{}^{\rho\sigma} = \tfrac{1}{2}P_\mu{}^{(\rho}P_\nu{}^{\sigma)} \tag{19}$$

are a subset of the standard spin-2 projection operators [60,61]. Note that the trace of P is 3, while that of π is 6. The wedge products are found to be

$$\wedge^n M_0 = \frac{p^{2n}}{2^n}\big[\wedge^n \pi - n(\wedge^{(n-1)}\pi)\wedge(P\otimes P)\big], \tag{20}$$

and the scalar dual is

$$\star\wedge^6 M_0 = \frac{p^{12}}{2^6(\zeta^*\cdot\zeta)^2}\big[\zeta\cdot\wedge^r\pi\cdot\zeta^* \\ -6\,\zeta\cdot\big((\wedge^5\pi)\wedge(P\otimes P)\big)\cdot\zeta^*\big]. \tag{21}$$

The first term in the brackets is just $\zeta^*\cdot\zeta$. The second term gives $-\zeta^*\cdot\zeta P\cdot P = -\zeta^*\cdot\zeta\,\mathrm{tr}P = -3\zeta^*\cdot\zeta$. The four gauge vectors can be written as $(z^\kappa)^{\mu\nu} = (Z^\kappa)^{(\mu}p^{\nu)}$, $\kappa = 1,\ldots,4$, where $(Z^\kappa)^\mu$ are four independent vectors. This implies $\zeta^*\cdot\zeta = 6!\,2^5\,|Z|^2 p^8$, where $Z = \det(Z^\kappa{}_\alpha)$. Putting together the pieces yields

$$\star\wedge^6 M_0 = -\frac{p^4}{6!\,2^{10}|Z|^2}. \tag{22}$$

The dispersion relation for linearized GR is thus found to be $p^4 = 0$, matching the standard result.

Next, consider the case where linearized GR is corrected by generic gauge-invariant terms. This case has been explicitly treated in Ref. [46]. We write $M = M_0 + \delta M$, where M_0 is given by Eq. (18) and δM has at least the gauge invariances of M_0 but is otherwise arbitrary. We can simplify this case by noting that the factor $\pi - P\otimes P$ of projection operators appearing in M_0 obeys $(\pi - P\otimes P)\cdot(\pi - \frac{1}{2}P\otimes P) = \pi$. The combination $\varpi = \pi - \frac{1}{2}P\otimes P$ can therefore be viewed as the gauge-invariant inverse of $\pi - P\otimes P$, which suggests defining

$$\breve{M} = M\cdot\varpi = \big(\tfrac{1}{2}p^2\pi + \delta\breve{M}\big), \\ \delta\breve{M} = \delta M\cdot\varpi = \delta M\cdot(I - \tfrac{1}{2}\eta\otimes\eta)\cdot\pi\,, \tag{23}$$

where $I - \frac{1}{2}\eta\otimes\eta$ is the trace-reversal operator. A short derivation then reveals that

$$\star\wedge^6 M = \star\big(\wedge^6\breve{M}\cdot\wedge^6(\pi - P\otimes P)\big) = -2\star\wedge^6\breve{M}, \tag{24}$$

so the dispersion relation for M is the same as that for $\breve{M}$. Direct calculation gives

$$\star\wedge^6 M = -\frac{1}{6!2^{10}|Z|^2}\sum_{n=0}^{6}2^n p^{4-2n}\mathrm{tr}(\wedge^n\delta\breve{M}). \tag{25}$$

With the reasonable assumption that higher-order terms remain finite as $p^2 \to 0$, this implies the leading-order covariant dispersion relation

$$p^4 + 2p^2\delta\breve{M}_1 + 2(\delta\breve{M}_1^2 - \delta\breve{M}_2) = 0, \tag{26}$$

where $\delta\breve{M}_n = \mathrm{tr}(\delta\breve{M}^n)$. The solution for the two perturbative modes is

$$p^2 = -\delta\breve{M}_1 \mp \sqrt{2\delta\breve{M}_2 - \delta\breve{M}_1^2}. \tag{27}$$

Upon explicit evaluation of the traces for the general gauge-invariant terms listed in Table 1, this equation reduces correctly to the results (5) and (6) for the dispersion relation given in Ref. [46].

With these previously known examples reproduced, we turn to the case where linearized GR is instead corrected by arbitrary gauge-violating terms. The covariant dispersion relation takes the form (14) with $\mathcal{S} \supseteq \mathcal{S}'$, so to evaluate it explicitly we must determine the modified duals that appear in Eq. (16). They are

$$\star\wedge^n M_0 = \frac{p^{2n}}{2^n}\star\big(\wedge^n\pi - n(\wedge^{(n-1)}\pi)\wedge(P\otimes P)\big). \tag{28}$$

After some calculation, we find the covariant dispersion relation to be

$$\sum_n \frac{(n+\hat{s})!}{n!}2^n p^{4-2n}\mathrm{tr}\Big[\big((\wedge^n\varpi)\wedge\xi\big)\cdot(\wedge^{(n+\hat{s})}\delta M)\Big] = 0. \tag{29}$$

The number of terms in the sum is restricted by the rank δr of δM, which may be less than the total rank r'. The limits on the sum are thus $0 \le n \le \delta r - \hat{s}$.

The result (29) is the exact covariant dispersion relation for any linearized model of gravity, with or without gauge violation and with or without Lorentz violation. For example, the gauge-invariant scenario (25) is contained as the special case when $r' = 6$. The framework developed here therefore realizes the desired goal of a model-independent approach to modifications of linearized gravity. It provides calculational tools for arbitrary models and also permits identifying generic features. As an example of the latter, we can see that the complexity of the exact covariant dispersion relation is determined by the rank δr of δM and the number $\hat{s}$ of symmetries of M_0 broken by δM. In particular, the number of terms in the exact dispersion polynomial is $\delta r - \hat{s} + 1$, which can constrain physical aspects of the solutions. Consider, for instance, the case where rank of δM matches the number of gauge symmetries broken by δM, $\delta r = \hat{s}$. Only one term then survives in the dispersion relation. Since that term is proportional to p^4, we find the striking result that all models of this type must leave unaffected the conventional dispersion relation $p^2 = 0$.

As an explicit illustration, consider any model in which only a single gauge symmetry associated with a vector $\hat{z}$ is broken. This implies $r' = 7$ and $\xi = \hat{z}\otimes\hat{z}^*$, and the first few terms of the exact covariant dispersion relation are calculated to be

$$0 = p^4\hat{z}^*\delta M\hat{z} + 2p^2\Big[\mathrm{tr}(\delta\breve{M})\hat{z}^*\delta M\hat{z} - \hat{z}^*\delta\breve{M}\delta M\hat{z}\Big] \\ + 2\Big[\mathrm{tr}(\delta\breve{M})^2 - \mathrm{tr}(\delta\breve{M}\delta\breve{M})\Big]\hat{z}^*\delta M\hat{z} \\ + 4\hat{z}^*\delta\breve{M}\delta\breve{M}\delta M\hat{z} - 4\mathrm{tr}(\delta\breve{M})\,\hat{z}^*\delta\breve{M}\delta M\hat{z} + \ldots \tag{30}$$

When the rank δr of δM is one, this result reduces to the monomial $p^4 \hat{z}^* \delta M \hat{z} = 0$ and so gives the usual dispersion relation $p^2 = 0$, in agreement with the conclusion above. However, when $\delta r > 1$, the dispersion relation becomes a polynomial, and the solutions can describe modifications to the behavior of the gravitational modes.

Since the covariant dispersion relation (29) is exact, it governs all gravitational modes. However, inspection reveals that terms having larger values of n involve traces of higher powers of δM, so Eq. (29) is naturally configured for perturbative reasoning concerning the usual modes of GR. Although the terms with $n \geq 3$ naively contain inverse powers of p^2, in fact this behavior is excluded by continuity and so any contributions from them are perturbatively small. The three terms with $n = 0, 1, 2$ are therefore the ones of perturbative relevance, and they yield a quadratic dispersion relation whose solution describes perturbative effects on the usual gravitational modes of GR. This structure can be seen explicitly in the example (30) with only a single broken gauge symmetry. The solution in this case takes the generic form (27), with the factors of $\delta \check{M}_1$ and $\delta \check{M}_2$ replaced by factors of the polynomial coefficients in Eq. (30).

Another interesting special case is the set of models with coefficients having only purely temporal components, which is a subset of the isotropic limit. All gauge-invariant models of this type are discussed in Ref. [46]. For example, the operator $s^{(d)\mu\rho\circ\nu\sigma\circ\circ^{d-4}}$ contains models with purely temporal components that can be isolated by working with its double dual and restricting attention to the components $(\bar{s}^{(d)})^{00\ldots}$ with $d-2$ temporal indices. Observational constraints on these coefficients have been placed using gravitational waves and other techniques [35,36,38,44,45,49,51,52].

Many of the gauge-violating representations listed in Table 1 also contain coefficients with purely temporal components. Typically, these generate nontrivial effects on the gravitational modes. The observational implications of these lie beyond our present scope and offer an interesting open direction for future investigation. Note, however, that some of these cases may produce no measurable effects. Consider, for example, the operator $k^{(d,1)\mu\nu\rho\sigma\circ^{d-2}}$ restricted to the components $k^{(d,1)00\ldots}$ with $d+2$ temporal indices. The matrix δM is then of rank $\delta r = 1$ and has nonzero component $\delta M_{00}{}^{00} = \sum_d k^{(d,1)00\ldots} E^{d-2}$. A single broken gauge vector exists, which can be taken as $\hat{z}^{\mu\nu} = \eta^{0(\mu} p^{\nu)}$, so $\hat{s} = 1$ as well. The dispersion relation becomes $p^4 \sum_d k^{(d,1)00\ldots} E^d = 0$ and reduces to $p^2 = 0$, in agreement with the general result for $\delta r = \hat{s}$ discussed above. These coefficients therefore have no effect on the behavior of the usual gravitational modes.

More involved cases exist that also leave unaffected the usual gravitational modes. One example with $\delta r = \hat{s} = 2$ involves the CPT-odd operator $q^{(d,3)\mu\rho\circ\nu\sigma\circ^{d-3}}$ for $d = 3$. Taking the dual of this and restricting to the purely temporal coefficient $\widetilde{q}^{(3,3)000}$ produces a rank-two matrix δM with nonvanishing components $\delta M^{0j0k} = -i\widetilde{q}^{(3,3)000} \epsilon_{jkl} p^l / 8$ that is symmetric under interchange of the first or second pair of indices and is antisymmetric under interchange of the pairs. Two unbroken gauge vectors exist, which can be taken as $(z'^0)^{\mu\nu} = \eta^{0(\mu} p^{\nu)}$ and $(z'^1)^{\mu\nu} = p^\mu p^\nu$. Calculation shows the dispersion relation is $p^4 (\widetilde{q}^{(3,3)000})^2 E^4 = 0$ and so again yields $p^2 = 0$, as expected. Other components of $q^{(3,3)\mu\rho\lambda\nu\sigma}$ can, however, modify gravitational propagation [62].

To summarize, we have provided in this work a framework for studying diffeomorphism and Lorentz violations in linearized gravity theories. The techniques developed here yield the classification and enumeration of all gauge-invariant and gauge-violating terms in the general effective field theory for the metric fluctuation. For the various possible scenarios, we have obtained the exact covariant dispersion relations for the gravitational modes. The expressions hold for operators of arbitrary mass dimension, and reduce to known results in suitable special limits. Results for any specific model of linearized gravity can be extracted as a special limit. The work opens the path to model-independent phenomenological studies of arbitrary gauge and Lorentz violation in nature, representing a broad arena for search and discovery.

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