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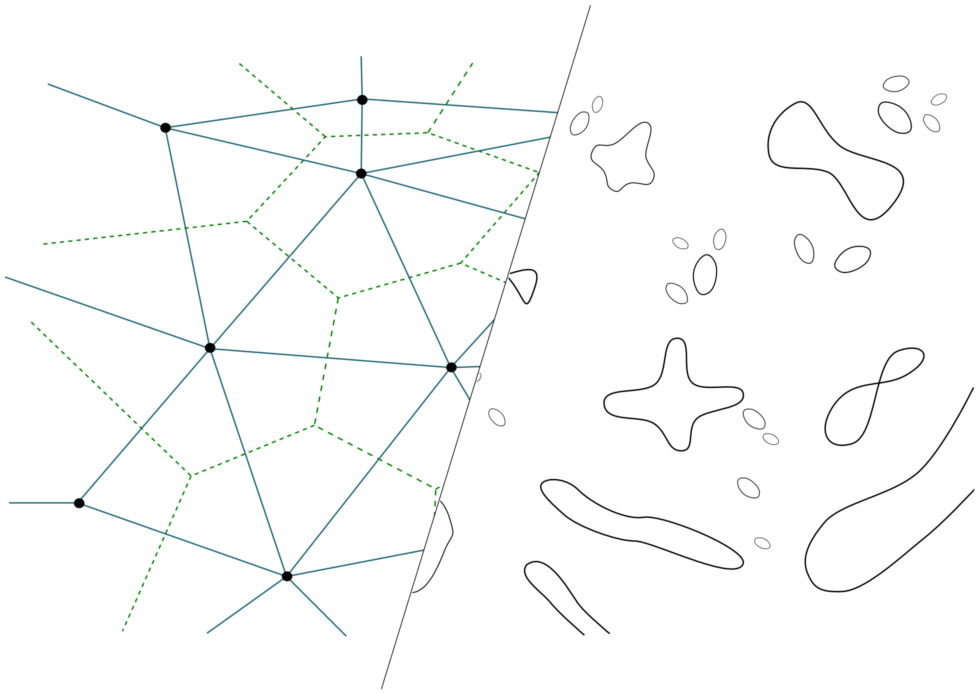
STRANGE HORIZONS

A holographic exploration into strongly coupled physics



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Dissertation presented in partial fulfillment of the requirements for the degree of Doctor of Science (PhD): Physics

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Abstract

The gauge/gravity duality, also known as the AdS/CFT correspondence or holography, relates quantum field theories to theories of quantum gravity in one dimension higher. When one theory is strongly coupled, and therefore very difficult to study directly, the other one is weakly coupled. In this thesis, we will use this duality to study a variety of physical phenomena in strongly coupled quantum field theories by performing computations in their weakly coupled gravitational duals.

After reviewing the gauge/gravity duality in the first part, we discuss various aspects of non-conformal holography in part II. We construct new solutions of ten-dimensional type II supergravity which describe the back-reaction of Dp-branes, for $1 \leq p \leq 6$, with a spherical worldvolume. These solutions are holographically dual to maximally supersymmetric Yang-Mills (SYM) theory on a $(p+1)$-dimensional sphere, S^{p+1}. The finite size of the sphere provides a IR cut-off in the gauge theory, which is reflected in the dual geometry as a smooth cap-off. In the UV, the size of the sphere plays no role and our solutions asymptote to the well-known supergravity backgrounds describing the near-horizon limit of flat Dp-branes. Using these solutions we can holographically compute the free energy and 1/2—BPS Wilson loop vacuum expectation values in the dual gauge theory. Using supersymmetric localization, we can perform the same computation directly in planar maximal SYM theory on S^{p+1} at strong coupling. We find excellent agreement between the two sets of results for all values of p. This constitutes a highly non-trivial precision test of holography in a non-conformal setting.

In part III we study a class of AdS$_p$ solutions of type II supergravity which describe the IR dynamics of p-branes wrapped on a Riemann surface. Such solutions are classified by solutions of the Liouville equation. Regular solutions lead to well-known wrapped brane supergravity solutions with the constant curvature metric on the Riemann surface. We show that singular solutions of the Liouville also have a physical interpretation as explicit point-like brane sources on the Riemann surface. These supergravity solutions are dual to strongly coupled $(p-1)$-dimensional conformal field theories obtained as the IR fixed point of a renormalization group flow across dimensions starting in $p+1$ dimensions. These theories are strongly coupled and extremely hard to access using purely field theoretical methods. We test the details of our proposal by focusing on $\mathcal{N} = 1$ superconformal field theories of class $\mathcal{S}$, arising from M5-branes wrapped on a punctured Riemann surface. We present explicitly the dual AdS_5 solutions and check the proposed duality by finding non-trivial agreement of the ’t Hooft anomalies, the dimension of the conformal manifold and the conformal dimensions of various operators in the theory with their holographic duals.

Beknopte samenvatting

De ijk/zwaartekracht dualiteit, ook gekend als de AdS/CFT correspondentie of simpelweg holografie, relateert een kwantumvelden theorie aan een theorie van kwantumzwaartekracht in één dimensie hoger. Wanneer de ene theorie sterke interacties bevat en daardoor bijzonder moeilijk te bestuderen is, zal de andere theorie zwak gekoppeld zijn. In deze thesis bestuderen we verschillende fysische fenomenen in sterk interagerende kwantumveldentheorieën door gebruik te maken van hun duale gravitationele representatie.

Nadat we in het eerste deel van deze thesis de ijk/zwaartekracht dualiteit ingeleid hebben, bestuderen we in het tweede deel verschillende aspecten van niet conforme holografie. We construeren nieuwe oplossingen van type II superzwaartekracht die de gravitationele vervorming, veroorzaakt door een stapel Dp-branen, $1 \leq p \leq 6$, met een sferisch wereldvolume, beschrijven. Deze oplossingen zijn holografisch duaal aan maximaal supersymmetrische Yang-Mills (SYM) theorie op een $(p+1)$-dimensionale sfeer, S^{p+1}. De eindige grootte van de sfeer fungeert als een lage energie cut-off in de ijktheorie. In de duale theorie is deze cut-off gerealiseerd als een gladde afkapping van de geometrie. Bij hoge energieën speelt de grootte van de sfeer geen rol en bijgevolg benaderen onze oplossingen in dit regime de welgekende superzwaartekracht oplossingen die de regio dicht bij de horizon, gecreëerd door vlakke Dp-branen, beschrijven. Met behulp van deze oplossingen kunnen we holografisch de vrije energie en de vacuüm verwachtingswaarde van 1/2-BPS Wilson lussen berekenen. Dankzij de techniek van supersymmetrische lokalisatie kunnen we dezelfde berekening ook uitvoeren in de sterk interagerende duale ijktheorieën. Voor alle waarden van p vinden we uitstekende overeenkomst tussen de resultaten. Dit is een erg niet triviale test van holografie buiten het conforme regime.

In deel III bestuderen we een klasse van AdS$_p$ oplossingen van type II superzwaartekracht die de IR dynamica van p-branen gewikkeld rond een Riemann oppervlak beschrijven. Zulke oplossingen zijn geclassificeerd door oplossingen van de Liouville vergelijking. Reguliere oplossingen van deze vergelijking resulteren in welgekende superzwaartekracht oplossingen die branen gewikkeld rond een glad Riemann oppervlak, met een metriek met constante kromming, beschrijven. We tonen aan

dat singuliere oplossingen van de Liouville vergelijking ook een fysische interpretatie hebben als expliciete punt-bronnen van branen op het Riemann oppervlak. Deze zwaartekracht oplossingen zijn duaal aan sterk interagerende $(p-1)$-dimensionale kwantumveldentheorieën die verkregen kunnen worden als het IR vast punt van een renormalisatie groep stroom vertrekkende van een $(p+1)$-dimensionale theorie. Deze theorieën zijn sterk gekoppeld en extreem moeilijk te bestuderen zijn met puur veldtheoretische methodes. We bestuderen de details van deze dualiteit door ons toe te leggen op het voorbeeld van $\mathcal{N}=1$ superconforme veldentheorieën van de $\mathcal{S}$ klasse die ontstaan als de lage energie limiet van M5-branen gewikkeld rond een Riemann oppervlak. We presenteren de expliciete duale AdS_5 oplossingen en testen de holografische dualiteit door de gelijkheid van de 't Hooft anomalieën, de dimensie van de conforme variëteit en de conforme dimensies van verschillende operatoren aan beide zijden van de dualiteit aan te tonen.

List of abbreviations

ABJM	Aharony-Bergman-Jafferis-Maldacena
ADM	Arnowitt-Deser-Misner
AdS	Anti-de Sitter space
AGT	Alday-Gaiotto-Tachikawa
AlAdS	Asymptotically locally anti-de Sitter space
ALE	Asymptotically locally euclidean space
BF	Breitenlohner-Freedman
BPS	Bogomol'nyi-Prasad-Sommerfeld
BTZ	Bañados-Teitelboim-Zanelli
CERN	Conseil européen pour la recherche nucléaire
CFT	Conformal field theory
dS	de Sitter space
GR	General relativity
IR	Infrared
KK	Kaluza-Klein
LHC	Large hadron collider

LST Little string theory

MSYM Maximally supersymmetric Yang-Mills theory

NS Neveu-Schwarz

ODE Ordinary differential equation

OPE Operator product expansion

PDE Partial differential equation

QCD Quantum chromodynamics

QED Quantum electrodynamics

QFT Quantum field theory

R Ramond

RG Renormalization group

SCFT Superconformal field theory

SYM Super Yang-Mills theory

UV Ultraviolet

VEV Vacuum expectation value

WZW Wess-Zumino-Witten

WKB Wentzel-Kramers-Brillouin

Contents

Chapter 1

Introduction

Quantum field theories (QFTs), and in particular gauge theories, play an extremely important role in theoretical particle and condensed matter physics. Apart from gravity they describe all the fundamental forces of nature. However, in all but the simplest cases, exact calculations are impossible. To make progress, the traditional tool is perturbation theory. The idea behind this technique is to consider an expansion in the coupling constants around a simple theory. This technique works extremely well for many theories. Some theories, however, have large coupling constants, in which case one cannot obtain good results by truncating the theory after a few orders in the perturbation. Such QFTs are called strongly coupled.

1.1 Strongly coupled field theories

As an example, consider the strong nuclear force, the interaction that bind the quarks into hadrons, despite their electromagnetic repulsion. At very high energies this force becomes weakly coupled, the theory is asymptotically free, so we can use perturbation theory to predict scattering amplitudes for particle accelerators. This method works extremely well and has been corroborated to incredible accuracy in particle accelerators such as the LHC at CERN. However, the interaction strength of this force depends heavily on the length or energy scale at which interactions take place. This behavior is encoded in the renormalization group flow (RG flow) and it turns out that the strong force at low energies becomes strongly coupled. From experiments it is evident that in this regime quarks are always bound into hadrons and never appear solitary. From a theoretical point of view, an understanding of this phenomenon remains elusive. Further examples of strongly coupled systems are abundant in condensed matter physics and include ultra-cold Fermi gases and cuprate high-temperature superconductors. To study such systems, new tools are needed.

Even if the interaction strength is weak and perturbation theory gives good results, we run into problems. The method of perturbation theory is conceptually straightforward but as we want to compute higher and higher order effects it quickly becomes cumbersome. Furthermore, it is well-known that this approach does not capture all the relevant effects. Non-perturbative effects, such as instantons, arising from non-trivial solutions to the equations of motion are typically exponentially suppressed. Such effects are completely missed in a perturbative approach. Using only perturbation theory we would never see the tunneling of a quantum mechanical particle!

The task to understand strongly coupled gauge theories seems daunting. We will have to resum all terms in the perturbative expansion and on top of that also include all non-perturbative effects. Understanding this strongly coupled regime in quantum field theory remains a very important challenge. However, not all is lost. Throughout the last decades, substantial progress has been made to understand aspects of this problem. One approach is to use numerical techniques, in particular lattice Monte Carlo simulations. In lattice QCD for example this has been used successfully to compute for example the spectrum of light hadronic particles [90] or to compute hadron decay constants [13]. However, these simulations are computationally very expensive and due to the sign problem[1] of limited use for strongly interacting many-body systems. Apart from these numerical methods, various analytic techniques have evolved and many complementary tools have been developed.

1.2 Supersymmetric localization

Certain observables, for example specific Wilson loops, can be computed exactly using supersymmetric localization [243]. What makes these examples more tractable is the adjective supersymmetric. This additional symmetry describes a relation between the bosonic and fermionic degrees of freedom in the theory and often implies that certain contribution from the bosons cancel against contributions from the fermions. The part that remains might then be (a lot) easier to compute. Although this method does not work for all observables, it does provide the full non-perturbative answer for those it can be applied to.

Our world might not be supersymmetric but, nonetheless, important lessons can be learned by analyzing supersymmetric theories. Many of the characteristic phenomena observed in supersymmetric theories are believed to hold in much larger generality. The study of supersymmetric field theories can offer us a promising path

[1] The sign problem is a problem arising when numerically evaluation highly oscillatory integrals, such as path integrals in quantum field theory. It poses one of the major limitations in numerical many-body physics. See [1] for a review.

to enter in the largely unexplored world of non-perturbative and strongly coupled phenomena in quantum field theory.

1.3 The power of dualities

Another important strategy that has been pursued to learn more about strongly coupled theories is inspired by various dualities. In the most interesting cases, there exists a strongly coupled regime in the gauge theory parameter space in which there is a hidden weakly coupled, perturbative description which can be recovered by introducing a suitable new set of field variables. Whenever this occurs, we might be able to obtain non-trivial information about a strongly coupled gauge theory by studying a dual weakly coupled formulation. We can divide such dualities roughly in two classes: IR dualities and UV dualities. An IR duality arises as an equivalence of certain low energy effective actions, where the UV physics might be completely different. This type of dualities are omnipresent when considering RG flows towards an IR fixed point. The second class of dualities is much more dramatic and states that the dual theories are completely equivalent up to arbitrarily high energies.

The idea behind the first class of dualities is most easily described for theories which have a low energy description in terms of an abelian gauge field A, coupled to charged matter fields q. The effective action $S[A, q; \tau_{\mathrm{IR}}]$ for this theory will depend on the effective IR coupling constant τ_{IR}. Due to the well-known phenomenon of electro-magnetic duality we can represent the theory at strong coupling by a dual action $S[A', q', \tau'_{\mathrm{IR}}]$ that depends on a dual abelian gauge field A' which is related to A through the following relation

$$F'_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\,. \tag{1.1}$$

The relation between the couplings is then given by

$$\tau'_{\mathrm{IR}} = -\frac{1}{\tau_{\mathrm{IR}}}\,, \tag{1.2}$$

therefore, we see that the dual theory is weakly coupled. The relation between q and q' in general is not so simple. Very often the dual matter fields are related to solitons, localized particle-like excitations, such as magnetic monopoles for example, in the original theory. Such solitons are usually very heavy at weak coupling but may become light at strong coupling and can be identified with the dual fundamental particles.

For other theories there exist even deeper dualities, relating full UV quantum field theories. Such dualities between QFTs or string theories are often referred to as

S-dualities. The earliest examples ([112, 190], and further refined in [200, 252]) conjecture that $\mathcal{N} = 4$ super Yang-Mills (SYM) with gauge group G and coupling τ is completely equivalent to the $\mathcal{N} = 4$ SYM with gauge group ${}^{L}G$ and coupling $\frac{-1}{n_G \tau}$, where ${}^{L}G$ is the Langlands dual of the original gauge group and n_G is the lacing number of the Lie algebra corresponding to G. This duality is very powerful and has survived an extensive series of tests [236]. In [101] this type of dualities has been substantially generalized to apply to a large class of $\mathcal{N} = 2$ supersymmetric gauge theories giving rise to a completely new strongly coupled class of non-Lagrangian gauge theories known as class $\mathcal{S}$. Finally, for $\mathcal{N} = 1$ theories a wide range of IR dualities has been discovered, which collectively go under the name of Seiberg dualities [224].

1.4 The gauge/gravity correspondence

Another particular set of dualities, especially relevant for my work, are gauge/gravity dualities. The assertion behind this type of dualities is that hidden within every non-abelian gauge theory, with strong or weak coupling, there is a theory of quantum gravity. This duality is rather different from the other ones considered so far in that it relates a quantum field theory to a theory of gravity. As it turns out this gravity theory does not even live in the same number of spacetime dimensions. In some cases, where the duality is most powerful, it represents the equivalence between a strongly coupled gauge theory and a weakly coupled supergravity theory. The original formulation of this duality relates $\mathcal{N} = 4$ SU(N) SYM at large N with type IIB string theory in $\mathrm{AdS}_5 \times S^5$, for this reason it is also known as the AdS/CFT correspondence. Although at present there is no rigorous proof for this conjecture, it has withstood impressive quantitative checks and continues to improve our understanding of strongly coupled QFTs as well as quantum gravity.

When little or no supersymmetry is present it is very hard to perform such tests. However, using the tools of supersymmetric localization, we are now able to compute various supersymmetric observables such as partition functions and Wilson loop expectation values exactly which open the way to perform more precision tests of the gauge/gravity duality. Whenever we know the dual supergravity or string theory description we will be able to compare various observables holographically. These exact results can then be used to study the behavior and phase structure of the corresponding gauge theories as a function of the 't Hooft coupling, see for example [216, 218, 251]. This correspondence can and has given many hints leading to the discovery of new strongly coupled quantum field theories. The most striking example is without a doubt the six dimensional $\mathcal{N} = (2, 0)$ theory which was conjectured to exist from M-theory [226, 246, 248] as the worldvolume theory on a stack of coincident M5-branes. This theory has attracted a great deal of attention over the last two decades but remains largely uncharted. Nevertheless, the mere existence of this theory leads to a plethora of highly non-trivial predictions. For example, by studying this theory on manifolds of the form $\mathcal{M}_4 \times \Sigma$, where Σ is a Riemann surface, we can obtain an effective IR description as a four-dimensional quantum field theories on $\mathcal{M}_4$. Playing with the choice of Σ and the choice of Lie algebra $\mathfrak{g}$ one obtains a large new set of four dimensional strongly coupled gauge theories. These gauge theories are exactly the theories of class $\mathcal{S}$ discussed above. Using this new geometric intuition we have a new tool to study these theories and various dualities between them. These discoveries subsequently led to a whole new set of dualities such as the AGT duality. More generally, we are not restricted to two-manifolds but we can wrap this six-dimensional theory on a whole set of other manifolds and as such obtain a whole new set of theories in dimensions $d < 6$. These theories are extremely difficult to study using field theoretical methods alone, and often, the only way to access them is through holography.

There are many cases where strongly coupled supersymmetric field theories are known to exist but no Lagrangian of any form is known. The existence of these theories on itself is extremely interesting and as we have seen from the example of the $6d$ $\mathcal{N} = (2,0)$ theory they can be used as building blocks to construct new quantum field theories. The gauge/gravity duality gives us a new tool, often the only one, to study these theories in more detail and provides us with a new vantage point to study the space of possible (supersymmetric) quantum field theories.

1.5 What's in this thesis

The goal of this thesis is to further investigate the gauge/gravity duality and explore its range of validity, test it and use it to probe the space of consistent QFTs. Understanding this duality better will allow us to compute a larger range of observables in a larger class of (possibly strongly coupled) quantum field theories and ultimately can provide us with a consistent definition of quantum gravity through quantum field theory.

This thesis will be organized in three parts:

I Introduction to the gauge/gravity correspondence.

II Spherical branes, localization and holography.

III Wrapped branes and punctured horizons.

In the first part we introduce the gauge/gravity duality and the necessary tools to successfully apply it in the following parts. The best understood class of gauge/gravity dualities describe an equivalence between a conformal field theory (CFT) and a theory of gravity in anti-de Sitter space. Before defining the duality in detail in Chapter 5 we will review various aspects of the two sides separately, starting with CFTs in Chapter 2, followed by a discussion of anti-de Sitter space in Chapter 3. To introduce this duality in full detail would take a book on its own so we will skip many interesting topics and focus mainly on the objects that will appear later in this thesis. The original conjecture of the gauge/gravity correspondence originated from string theory and many more concrete examples were provided by string and M-theory. In Chapter 4 we review some aspects of these theories relevant to this thesis.

In part II (which contains material from [50, 52]) we will discuss the extension of the holographic correspondence to non-conformal field theories and develop an explicit model realizing such a duality. We construct ten-dimensional supergravity solutions with sixteen preserved supercharges which describe the back-reaction of Dp-branes with a spherical worldvolume. These solutions are holographically dual to $(p+1)$-dimensional maximally supersymmetric Yang-Mills (SYM) theory on a $(p+1)$-dimensional sphere, S^{p+1}. The finite size of the sphere acts as an IR cut-off for the gauge theory which is manifested in the supergravity as a smooth cap-off of the geometry. At high energies, the size of the sphere plays no role and our solutions asymptote to the near-horizon limit of flat Dp-branes. These spherical branes provide us with an ideal testbed to study the holographic correspondence away from conformality. Once we have constructed the dual pairs, we carefully compute the free energies and supersymmetric Wilson loop expectation values of maximal super Yang-Mills theory on S^{p+1} using supersymmetric localization and successfully compare with the results obtained from the supergravity solutions. In this way we provide a non-trivial precision test of holography in a non-conformal setting.

Finally, in part III of this thesis, which can be read independently of part II, we use holography to study a set of strongly coupled $(p-1)$-dimensional CFTs realized as the low energy dynamics of a Dp-brane wrapped on a punctured Riemann surface. In [101], Gaiotto identified a large class of four-dimensional $\mathcal{N}=2$ superconformal field theories (SCFTs) by wrapping M5-branes on a punctured Riemann surface and together with Maldacena constructed the holographically dual geometry in eleven-dimensional supergravity [102]. This class of SCFTs was dubbed class $\mathcal{S}$ and its study has led to many new developments in supersymmetric quantum field theory. The $\mathcal{N}=1$ generalization of this class of theories is even richer but far less explored. In [51] we study this enlarged class of theories from the viewpoint of seven-dimensional supergravity. The branes wrapped on the punctured Riemann surface are realized in seven dimensions as solutions of the form $AdS_5 \times \Sigma_{g,\xi_i}$ where Σ_{g,ξ_i} is a Riemann surface of genus g with a number of conical singularities with opening angles ξ_i specifying the type of singularity. We show that the supergravity

BPS equations reduce to a single partial differential equation (PDE) – the Liouville equation for the conformal factor of the metric – and that all other fields are determined in terms of this conformal factor. We subsequently lift these solutions up to eleven dimensions and compute the conformal anomalies and the dimensions of various operators. By integrating the anomaly polynomial over the punctured Riemann surface and by explicitly constructing the dual quiver gauge theories we can reproduce the quantities obtained from our supergravity analysis. We furthermore enlarged the scope of this project and studied the low energy dynamics of M2-,D3-,and D4-D8-branes wrapped around a singular Riemann surface and constructed the corresponding supergravity solutions. This will provide a window to study new interacting SCFTs in $d = 1$, 2 and 3 dimensions.

The explorations made in this thesis open up many new and interesting questions which will be further discussed in the concluding Chapters 10, 16 and 17.

Part I

Introduction to the gauge/gravity correspondence

Chapter 2

Conformal field theory

Conformal field theories are ubiquitous in physics. They generically appear when considering the long distance limit of physical systems. For this reason they are present in a wide variety of physical systems. More abstractly, we can think of UV complete QFTs as an RG flow between two CFTs. Studying CFTs allows us to classify the possible end points of RG flows and thus understand the space of possible phases of QFTs.

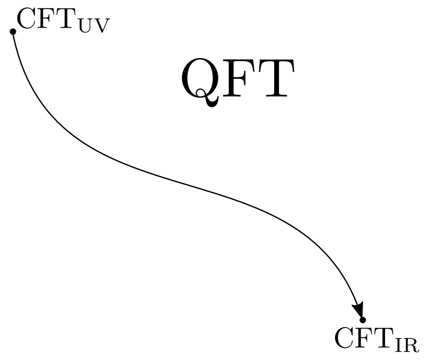


Figure 2.1: A UV complete QFT can be represented as an RG flow starting at a CFT fixed point in the UV towards a second fixed point in the IR.

CFTs can arise as the large distance limit of continuum QFTs but more generally they arise from a large variety of microscopic systems [211]. An example is the three-dimensional Ising model, which is a lattice of classical spins with nearest-neighbor interactions. In the IR this system at its critical temperature is described by a CFT, and in fact it is described by the same IR CFT as the ϕ^4 scalar field theory. Even more surprisingly the Ising CFT appears as well when describing the critical point on the phase diagram of water (and various other liquids) and the critical point of a uniaxial magnet. All these theories are described by the same IR CFT.

This phenomenon of IR equivalences is called critical universality and is another reason why CFTs are so interesting to study.

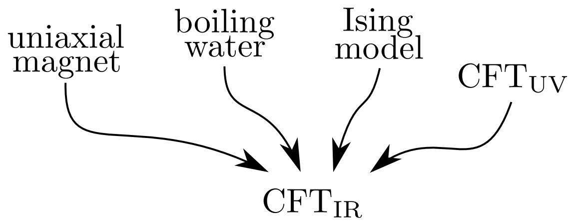


Figure 2.2: One CFT can describe the IR dynamics of very different microscopic theories. This phenomenon is called critical universality.

CFTs are quantum field theories invariant under the conformal group. We will start this lightning review by introducing the conformal group and then discuss the implications this additional symmetry has on the dynamics of a QFT.

2.1 The conformal group and algebra

The conformal group of d-dimensional spacetime is the set of invertible maps $x \to x'$ which map the line element to itself

$$x \to x', \qquad \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \to \eta_{\mu\nu}\mathrm{d}x'^\mu \mathrm{d}x'^\nu = \Omega(x)^2 \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{2.1}$$

up to an arbitrary function $\Omega(x)$ where $\mu, \nu = 0, \dots d-1$ and $\eta_{\mu\nu}$ is the Minkowski metric. Such transformations include:

- Translations, $x'^\mu = x^\mu + a^\mu$ along a constant vector a^μ,
- Lorentz transformations, $x'^\mu = \{\Lambda^\mu\}_\nu x^\nu$ with $\Lambda \in \mathrm{SO}(1, d-1)$,
- Dilatations or scale transformations, $x'^\mu = \lambda x^\mu$,
- Special conformal transformations, $x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b_\mu x^\mu + b^2 x^2}$.

The generators of these symmetries are given by respectively P_μ for translations, $J_{\mu\nu}$ for Lorentz transformations, D for dilatations and K_μ for special conformal transformations. These generators satisfy the following commutation relations:

$$\begin{aligned}
[J_{\mu\nu}, J_{\rho\sigma}] &= -i\eta_{\mu\rho}J_{\nu\sigma} \pm \text{ permutations}\,, & [J_{\mu\nu}, P_\rho] &= -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu)\,,\\
[J_{\mu\nu}, K_\rho] &= -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu)\,, & [J_{\mu\nu}, D] &= 0\,, \qquad (2.2)\\
[D, P_\mu] &= iP_\mu\,, & [D, K_\mu] &= -iK_\mu\,,\\
[P_\mu, K_\nu] &= 2i\left(J_{\mu\nu} - \eta_{\mu\nu}D\right)\,,
\end{aligned}$$

with all other commutators vanishing. The first line of relations is familiar and represent the algebra of the usual Poincaré symmetries. The remaining lines complete it to form the full conformal algebra. This algebra is isomorphic $\mathfrak{so}(d,2)$[1] and can be brought to the standard form of the $\mathfrak{so}(d,2)$ algebra by defining the generators L_{ab}, $a, b = 0, \ldots d+1$

$$L_{\mu\nu} = J_{\mu\nu}\,, \qquad L_{\mu(d)} = \frac{1}{2}(K_\mu - P_\mu)\,, \tag{2.4}$$

$$L_{(d+1)d} = D\,, \qquad L_{\mu(d+1)} = \frac{1}{2}(K_\mu + P_\mu)\,. \tag{2.5}$$

In this thesis we will also encounter Euclidean CFTs, in this case the conformal group takes the form $\mathrm{SO}(d+1,1)$. Up to some well-placed i's and minus signs, the commutation relations for the generators of the Euclidean conformal algebra are identical to (2.2)

In the presence of supersymmetry the conformal group is enhanced to a supergroup, the superconformal group. This supergroup is obtained by adding fermionic supercharges Q^a and the R-symmetries rotating them to the algebra. Additionally, we need to add a so-called superconformal charges S^a which are required to close the algebra. The full classification of superconformal algebras was performed in [191] and several examples can be found in Appendix K for various dimensions and numbers of supercharges. Schematically, the commutation relations for the

[1] There is an extra discrete symmetry that acts as a conformal transformation, namely the inversion

$$x'^\mu = \frac{x^\mu}{x^2}\,, \qquad \eta_{\mu\nu}\mathrm{d}x'^\mu \mathrm{d}x'^\nu = x^2 \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\,. \tag{2.3}$$

Including this transformation enlarges the conformal group to $\mathrm{O}(d,2)$.

superconformal algebra (in addition to (2.2)) are given by

$$\begin{aligned} &[D,Q]=\frac{i}{2}Q\,, && [D,S]=\frac{-i}{2}S\,, && [K,Q]\sim S\,, \\ &[P,S]\sim Q\,, && \{Q,Q\}\sim P\,, && \{S,S\}\sim K\,, \\ &\{Q,S\}\sim M+D+R\,. \end{aligned} \tag{2.6}$$

The exact form of the commutation relations depends on the dimension and R-symmetry of the theory under consideration.

2.2 Conformal quantum field theories

In a Poincaré invariant quantum field theory particles are identified by their mass and Lorentz quantum numbers, corresponding to the Casimir operators of the Poincaré algebra. When dilatations and special conformal transformations are added, the mass operator $P_\mu P^\mu$ does no longer commute with all operators in the spectrum. If a representation of the conformal group contains a state with a given mass or energy, it will contain states with all energies by applying dilatations. For this reason the usual machinery of the S-matrix can no longer be applied and we need a new way of labeling states.

In a Lorentz invariant QFT, local operators at the origin naturally fall into representations of the Lorentz group $\mathrm{SO}(1,d-1)$. In a scale-invariant theory it is also useful to diagonalize the dilatation operator acting on operators at the origin

$$[D,\mathcal{O}(0)]=\Delta\mathcal{O}(0)\,. \tag{2.7}$$

We will call the eigenvalue of the dilatation operator, Δ, the conformal dimension of a field. From the commutation relations (2.2) we see that P and K act as raising and lowering operators for the eigenvalues of D. Because in any sensible physical theory dimensions are bounded from below every operator will be annihilated by acting on it with K_μ a finite number of times. The lowest dimensional operators, which are subsequently annihilated by a single application of K_μ are called primary operators. Such primary operators are classified according to their conformal dimension and Lorentz quantum numbers. Given a primary operator we can obtain operators of higher dimension, so-called descendants, by successively applying the raising operator P_μ. Since the conformal algebra is a subalgebra of the superconformal algebra, representations of the superconformal algebra will split up in several representations of the conformal algebra. Generically, a superconformal primary (a field annihilated by both K and S) will decompose in several primaries of the conformal algebra which arise by applying Q to the superconformal primary.

Similar to usual spacetime symmetries, we can construct a conserved current for every conformal transformation

$$J_\mu = T_{\mu\nu}\delta x^\nu , \tag{2.8}$$

where δx^μ represents an infinitesimal coordinate transformation. Conservation of this current applied to translations implies conservation of the energy-momentum tensor, $\partial^\mu T_{\mu\nu} = 0$. The Lorentz symmetry of the theory forces the energy-momentum tensor to be symmetric. The current for dilatations $J_\mu = T_{\mu\nu}x^\nu$ on the other hand is conserved if

$$\partial^\mu (T_{\mu\nu}x^\nu) = T^\nu_\nu = 0 . \tag{2.9}$$

Scale invariance requires the energy-momentum tensor to be traceless. Finally, it is easily seen that the current for special conformal transformations is automatically conserved if the above criteria hold.

Conformal invariance gives rise to many constraints on a quantum field theory. It has been shown [170] that the Green's functions can be analytically continued from Euclidean to Lorentzian signature and that the Hilbert space of the resulting theory carries a unitary representation of the conformal group. The constraints we discuss below thus hold both in Lorentzian and Euclidean theories.

The Ward identities of the conformal group strongly constrain the correlation functions. The tracelessness of the energy-momentum tensor requires that all one-point functions vanish

$$\langle \mathcal{O}(x) \rangle = 0 . \tag{2.10}$$

Because of translation and Lorentz invariance, two point functions can only depend on the distance between the two points, $|x-y|$. Furthermore, demanding covariance on dilatations requires them to take the form

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y) \rangle = \frac{c\,\delta_{\Delta_1,\Delta_2}}{|x-y|^{2\Delta_1}} \tag{2.11}$$

where c is a constant determined by normalization. By also demanding covariance under the special conformal transformations one can fully fix the three-point functions to be

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3) \rangle = \frac{f_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}} . \tag{2.12}$$

Similar expressions can be found for non-scalar operators. Using the operator product expansion (OPE) one can reduce all higher-point functions to sums of two- and three-point functions [241,242]. Therefore, once the full spectrum of operators, $(\mathcal{O}_i, \Delta_i)$, and the full set of three point coefficients f_{ijk} is known all higher point functions are fully determined in terms of these data. The field content of any CFT includes a stress tensor $T_{\mu\nu}$, with dimension $\Delta = d$. Similarly, whenever the theory possesses a global symmetry, there is a conserved current with dimension $\Delta = d-1$. The scaling dimensions of the other fields are not fixed by conformal symmetry and often receive quantum corrections.

The hermiticity of the generators of the conformal algebra imply certain unitarity bounds on the possible spectra of conformal field theories. For example, in a d-dimensional CFT the dimension of a scalar field is restricted to be either $\Delta = 0$, which is the identity operator, or $\Delta \geq \frac{d-2}{2}$. When this bound is satisfied the scalar field is free. Similar bounds can be found for spin-s traceless symmetric field, $\Delta \geq s + d - 2$ [189].[2] Various further bounds on the dimension of operators follow from energy positivity [131], dispersion relations [161, 162] and causality [126, 171].

2.3 The Weyl anomaly

We have seen above that in a classical CFT, the trace of the stress tensor vanishes when the equations of motion are satisfied. Indeed, by definition the action of a CFT is invariant under such transformations. To make this more concrete consider an infinitesimal Weyl transformation,

$$g_{\mu\nu} \to (1+2\epsilon) g_{\mu\nu}\,, \qquad \Phi_a \to (1-\epsilon\Delta_a)\Phi_a\,, \tag{2.13}$$

where Δ_a is the conformal dimension of the field Φ_a. The change in the action S is

$$\begin{aligned} \delta S &= \int \mathrm{d}^d x \left(2\frac{\delta S}{\delta g_{\mu\nu}} g_{\mu\nu} - \sum_a \frac{\delta S}{\delta \Phi_a} \Delta_a \Phi_a \right) \epsilon(x) \\ &= \int \mathrm{d}^d x \sqrt{-g}\, T^{\mu\nu} g_{\mu\nu}\,. \end{aligned} \tag{2.14}$$

In the second line we have used the equation of motion for Φ_a and the definition of the energy-momentum tensor. In a CFT this quantity vanishes, implying that

$$T^{\mu\nu} g_{\mu\nu} = T^{\mu}_{\mu} = 0\,. \tag{2.15}$$

[2] These bounds can be obtained by demanding that the norm of ϕ, $P_\mu\phi$ and $P_\mu P^\mu \phi$ are positive. Similarly, for spinning operators one can obtain unitarity bounds by demanding that the norm of all descendants should be positive.

In a QFT, the vanishing of the trace of the energy-momentum tensor is encoded in the Ward identity for dilatations. Upon quantization the conformal invariance generically becomes anomalous. If any coupling has a non-vanishing β-function then scale invariance is broken by the renormalization scale. For example, $4d$ Yang-Mills theory coupled to massless charged particles is classically a CFT. However, in the quantum theory one finds (see [206] for example)

$$\langle T^{\mu}_{\mu} \rangle = \frac{\beta(e)}{2e^3} F_{\mu\nu} F^{\mu\nu}, \tag{2.16}$$

where $\beta(e)$ is the β-function for the gauge coupling e and $F_{\mu\nu}$ is the background value of the gauge field strength.

On the other hand, even when conformal invariance is not broken upon quantization, $\langle T^{\mu}_{\mu} \rangle$ generically does not vanish when the theory is put on a curved manifold. In this case the non-vanishing of the trace of the energy-momentum tensor is known as the Weyl anomaly. By dimensional analysis one can immediately see that the Weyl anomaly has to vanish in odd dimensions. In even dimensions, the Weyl anomaly is determined by solving the Wess-Zumino consistency condition [240] which states that the commutator of two Weyl transformations acting on the generating function has to vanish. On flat space we have $\langle T^{\mu}_{\mu} \rangle = 0$, while in curved space we find [86]

$$\langle T^{\mu}_{\mu} \rangle = a E_d + \sum_n c_n I_n , \tag{2.17}$$

where E_d is the Euler class which can be integrated to obtain the Euler characteristic of the curved manifold. The I_n are contractions of products of Riemann tensors and the Laplacian that transform homogeneously under conformal transformations, i.e. $I_n \to \Omega^{-d} I_n$ as $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$. The anomaly is then parameterized by the theory dependent coefficients a and c_n.

For example, in two dimensions E_2 is proportional to the Ricci scalar, R, and there are no I_n. In this case, the Weyl anomaly is given by

$$\langle T^{\mu}_{\mu} \rangle = \frac{aR}{4\pi} , \tag{2.18}$$

where $c = -6a$ is the central charge of the 2d CFT. c roughly counts the degrees of freedom of the CFT and decreases monotonically along the RG flow [254] between two fixed points. Similarly in four dimensions it was shown in [162] that a decreases along the RG flow. Such monotonicity arguments are very useful for ruling out possible IR fixed point as end points of an RG flow.

Chapter 3

Anti-de Sitter space

The next acronym we need to discuss is AdS, or anti-de Sitter space. In particular we want to study gravity in an Anti-de Sitter space. Anti-de Sitter space is a solution to Einstein's equations with a negative cosmological constant Λ. It is a maximally symmetric spacetime, meaning that in any coordinate system its Riemann curvature tensor is given in terms of the metric by

$$R_{\mu\nu\rho\sigma} = -\frac{1}{L_{\mathrm{AdS}}^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \qquad \Lambda = -\frac{d(d-1)}{2L_{\mathrm{AdS}}^2} \tag{3.1}$$

where d is the space dimension and L_{AdS} is the characteristic length scale of the anti-de Sitter space.

3.1 Global coordinates

There are many different ways of representing Anti-de Sitter space. Depending on the situation one representation might be beneficial over another. A first representation, which makes most of the global structure and symmetries manifest can be obtained by considering the embedding of AdS_{d+1} in $(d+2)$-dimensional flat space

$$\mathrm{d}s_{d+2}^2 = \eta_{MN}\mathrm{d}X^M\mathrm{d}X^N\,, \qquad \eta_{MN} = \mathrm{diag}(-1,1,\ldots,1,-1)\,, \tag{3.2}$$

where the indices M and N run from 0 to $d+1$. Anti-de Sitter space is defined as the $(d+1)$-dimensional hyperboloid,

$$\eta_{MN}X^MX^N = -L_{\mathrm{AdS}}^2\,. \tag{3.3}$$

The symmetries of this space are given by the rotations of flat space with signature defined by η_{MN}. By construction, this space has isometry group SO$(d,2)$, which we recognize as the conformal group in d dimensions. The generators of this group are

$$L_{MN} = X_M \partial_N - X_N \partial_M \,. \tag{3.4}$$

We can also give an intrinsic definition using the $d+1$ coordinates, (t, ρ, θ^a),

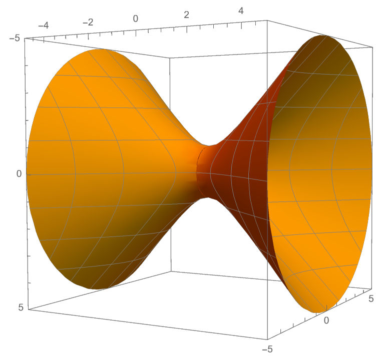


Figure 3.1: AdS_2 embedded in three-dimensional space.

without making reference to the specific embedding. These coordinates are called global coordinates and are defined as

$$\begin{aligned} X^0 &= L_{\text{AdS}} \cosh\rho \cos t\,, \qquad X^{d+1} = L_{\text{AdS}} \cosh\rho \sin t\,, \\ X^i &= L_{\text{AdS}} \sinh\rho\, \omega_i\,, \end{aligned} \tag{3.5}$$

where i runs from 1 to d and the ω^i represent the standard embedding of the unit $(d-1)$-dimensional sphere, with coordinates θ^a, in d-dimensional flat space, given by the constraint

$$\sum_{i=1}^{d} \left(\omega^i\right)^2 = 1\,. \tag{3.6}$$

The resulting induced metric on the hyperboloid is given by

$$\mathrm{d}s^2 = L_{\text{AdS}}^2 \left(-\cosh^2\rho\, \mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\rho\, \mathrm{d}\Omega_{d-1}^2\right)\,, \tag{3.7}$$

where $\mathrm{d}\Omega_{d-1}^2$ is the metric on S^{d-1}. By taking $\rho \geq 0$ and $0 \leq t < 2\pi$, we cover the entire hyperboloid once. However, since the t direction has the topology of an S^1, this space however contains closed timelike curves. To restore the causality we can simply unwrap the S^1, i.e. take $-\infty < t < \infty$ in (3.7) to obtain the universal covering of the hyperboloid without closed timelike curves. This coordinate system can teach us several interesting facts about the geometry of anti de-Sitter space.

For example, we can see that any light ray will reach spatial infinity in finite time t. Indeed, if we consider a light ray at constant position on the $(d-1)$-sphere we find

$$\Delta t = L_{\mathrm{Ads}} \int \mathrm{d}t = L_{\mathrm{Ads}} \int_0^\infty \frac{\mathrm{d}\rho}{\cosh\rho} = \frac{\pi L_{\mathrm{AdS}}}{2} . \tag{3.8}$$

This has the important implication that all observers in AdS space can communicate with each other from any point in space within finite time.

Notice that when $\rho \to \infty$, the metric blows up. The locus $\rho = \infty$ is strictly not a part of AdS. However, it is possible to define a conformal compactification of AdS including this point. $\rho = \infty$ is called a conformal boundary.[1]

3.2 Poincaré coordinates

Another useful set of coordinates we will often encounter in this thesis are called Poincaré coordinates (z, x^α). They do not cover all of AdS but have the advantage that the metric in these coordinates is conformal to flat space. More precisely, the metric in these coordinates is given by

$$\mathrm{d}s^2 = \frac{L_{\mathrm{AdS}}^2}{z^2}\left(\mathrm{d}z^2 + \eta_{ab}\mathrm{d}x^a \mathrm{d}x^b\right) , \tag{3.9}$$

where a, b run from 1 to $d-1$ and $\eta = \mathrm{diag}(-1, 1, \cdots, 1)$ and $z > 0$. Alternatively this system of coordinates can be obtained from the embedding space coordinates as

$$X^0 = \frac{L_{\mathrm{AdS}} x^0}{z} , \qquad X^d = \frac{L_{\mathrm{AdS}}^2 - z^2 - \eta_{ab} x^a x^b}{2z} , \tag{3.10}$$

$$X^a = \frac{L_{\mathrm{AdS}} x^a}{z} , \qquad X^{d+1} = \frac{L_{\mathrm{AdS}}^2 + z^2 + \eta_{ab} x^a x^b}{2z} . \tag{3.11}$$

From this one can see that indeed the Poincaré coordinates only cover the part of AdS for which $X^d + X^{d+1} > 0$. This part is called the Poincaré patch. The limit $z \to 0$ corresponds to the conformal boundary of AdS while the limit $z \to \infty$ is a horizon. At this point the killing vector $\partial/\partial t$ vanishes. This point however is not a true singularity and the metric can be analytically continued past the horizon.[2]

[1] More precisely, the conformal boundary is defined as the conformal equivalence class of metrics $\mathrm{d}\tilde{s}^2 = \mathrm{e}^{-2\rho}\mathrm{d}s^2$ with boundary $\mathbf{R}^{d-1,1}$ at $\rho = \infty$.

[2] This is clear by going back to the global coordinate system for example.

There are many equivalent forms of the metric in the Poincaré patch and we will often switch between different forms. They all differ by a redefinition of the radial coordinate z.

3.3 Conformal structure of AdS

Particularly interesting for our purposes is the relation between the conformal compactification of AdS and flat space. It is well-known that Euclidean flat space can be compactified to the d-sphere, S^d by adding a point at infinity. On the other hand, Euclidean AdS_{d+1}, which is simply the hyperbolic space, can be conformally mapped to the $(d+1)$-dimensional disk. Therefore the boundary of the compactified Euclidean AdS space is the compactified Euclidean plane.

Similarly, in Lorentzian signature, by changing coordinates $\sinh\rho = \tan\theta$ of AdS_{d+1} in global coordinates we obtain

$$\mathrm{d}s^2 = \frac{L_{\mathrm{AdS}}^2}{\cos^2\theta}\left(-\mathrm{d}t^2 + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\Omega_{d-1}^2\right), \tag{3.12}$$

which after a conformal rescaling becomes the metric of the Einstein static universe. However, it is only half of the Einstein static universe since θ is restricted to the range $[0, \pi/2)$ rather then $[0, \pi)$. The boundary of this space is at $\theta = \pi/2$ and is given by $\mathbf{R} \times S^{d-1}$. This is identical to the conformal compactification of d-dimensional Minkowski space. This identification will play an essential role in the AdS/CFT correspondence.

3.4 Matter fields in AdS

Often, we want to consider gravity in AdS spaces coupled to various matter fields. As a first step, which already allows us to describe the most salient features, let us consider matter fields fluctuating in a fixed AdS background. For simplicity, we will here only consider a massive scalar field, but similar analyses could be performed for fields with non-zero spin. The bulk action is given by

$$S = -\frac{L_{\mathrm{AdS}}^{d+1}}{2}\int \mathrm{d}^{d+1}x\,\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + m^2\phi^2\right), \tag{3.13}$$

and the corresponding equations of motion read

$$z^{d+1}\partial^z\left(\frac{\partial_z\phi}{z^{d-1}}\right) + z^2\partial^a\partial_a\phi = m^2L_{\mathrm{AdS}}^2\phi\,. \tag{3.14}$$

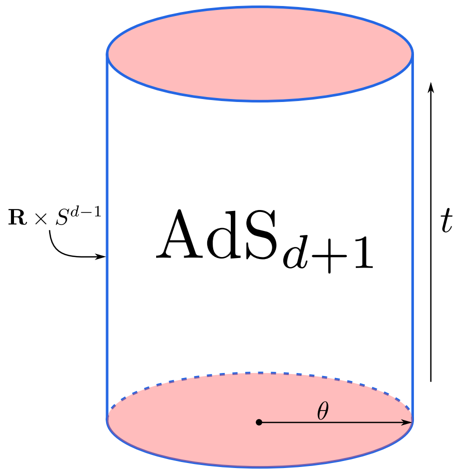


Figure 3.2: AdS$_{d+1}$ can be conformally mapped into one half the Einstein static universe. This space has boundary $\mathbf{R} \times S^{d-1}$ which is exactly the conformal compactification of Minkowski space.

The eigenstates of the flat space Laplacian $\partial^a \partial_a$ are Fourier modes. Therefore, it will be useful to consider solutions of the form $\phi(z,x) = \phi_k(z) e^{ik_a x^a}$. After inserting this ansatz, the equations of motion reduce to a Bessel equation for the z dependent Fourier coefficients

$$z^{d+1}\partial^z \left(\frac{\partial_z \phi_k}{z^{d-1}} \right) - (z^2 k^2 + m^2 L^2_{\text{AdS}})\phi_k = 0 \,. \tag{3.15}$$

where $k^2 = \eta^{ab} k_a k_b$. The exact solutions to this equation are indeed Bessel functions but we are more interested in the behavior near the conformal boundary $z \to 0$. Indeed, the dynamics in the bulk of AdS will depend heavily on the specific model under consideration, while the boundary behavior, in a sense, is universal. In this limit we can neglect the term $z^2 k^2$ and to leading order the solutions behave like

$$\phi \sim \phi_0 z^{\Delta_+} + \phi_1 z^{\Delta_-} \,, \tag{3.16}$$

where $\Delta_\pm$ solve the equation

$$\Delta(\Delta - d) = m^2 L^2_{\text{AdS}} \,, \tag{3.17}$$

and are given by $\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 L^2_{\text{AdS}}}$. In order for the $\Delta_\pm$'s to be real we demand that $m^2 L^2_{\text{AdS}} \geq -\frac{d^2}{4}$. We see that a range of tachyonic masses is allowed. In Minkowski space this would lead to an instability of the perturbative vacuum. In AdS space, whenever the mass-squared lies above this bound the free energy of the field is bounded from below and no instabilities arise. This bound is called the Breitenlohner-Freedman (BF) bound [63].

The coefficients $\phi_{0,1}(x)$ correspond to linearly independent solutions of the equations of motion. They are distinguished by the fact that ϕ_1 is not normalizable near the boundary while the solution corresponding to ϕ_0 is. On the other hand, ϕ_0 will blow up in the center of the bulk, where ϕ_1 remains regular. Therefore ϕ_1 will be the key player in holography and will act as a source for a dual CFT operator.[3]

3.5 Asymptotically locally AdS spaces

When we consider matter fields coupled to gravity they will back-react on the metric and we will no longer have an exact AdS space. However, a lot of the machinery developed for AdS spaces will still be valid. By adding matter to the theory the bulk of AdS will change but the conformal boundary is a rather robust characteristic of spaces with a negative cosmological constant. It takes an infinite energy to change the asymptotics of such spaces. Therefore we will be able to extend the analysis of pure AdS to the class of asymptotically locally AdS (AlAdS) spaces. In particular they all have the same conformal boundary.

In Poincaré coordinates this statement implies that the metric of any AlAdS space near the conformal boundary is of the form

$$\mathrm{d}s^2 = \frac{L_{\mathrm{AdS}}^2}{z^2}\left(\mathrm{d}z^2 + g_{ab}(z,x)\mathrm{d}x^a\mathrm{d}x^b\right), \tag{3.18}$$

where $g_{ab}(z,x)$ is smooth and finite as $z \to 0$. This function can be expanded in powers of the radial coordinate near the conformal boundary as

$$g_{ab}(z,x) = \sum_{n=0}^{\infty} z^n g_{ab}^{(n)}(x). \tag{3.19}$$

Similarly, the matter fields coupled to gravity can be expanded near the conformal boundary. This expansion is called the Fefferman-Graham expansion and is instrumental in defining a precise dictionary for holographically dual observables.

[3]When the mass is sufficiently small, $-d^2/4 < m^2 L_{\mathrm{AdS}}^2 < -d^2/4+1$ both modes are admissible. Thus for these masses there are two possible fall-off behaviors for a scalar wave function. This possibility was initially overlooked but as pointed out in [159] is necessary to account for certain low dimension operators in the dual CFT picture.

Chapter 4

Strings, branes and supergravity

String theory was first developed as a possible theory of the strong nuclear force before being abandoned in favor of QCD. However, the very properties making it unsuited to describe the strong force made it a promising candidate for a consistent theory of quantum gravity. The first string theories were bosonic string theories and only contained bosons. Later, they developed into superstring theory, which provides a connection between the bosons and fermions in the theory through supersymmetry. In the mid 90s, it was realized that the various string theories are all limiting cases of another, even more fundamental theory, called M-theory [133, 247].

4.1 Superstring theory and supergravity

The thesis of string theory is that fundamental particles are not point-like but instead correspond to tiny one-dimensional strings. The fluctuations of such strings are quantized, and the fluctuation modes correspond to the different particles in the theory. The spectrum of fluctuations always includes a spin-2 particle, the graviton, so string theory automatically includes gravity!

There are five types of supersymmetric string theories, corresponding to the different ways of implementing supersymmetry: type I, type IIA and type IIB and two flavors of heterotic string theory, SO(32) and $E_8 \times E_8$. Each of these theories must be formulated in ten dimensions in order to avoid the presence of a conformal anomaly, which would render the theory inconsistent.[1] The different theories allow different types of strings and the particles arising at low energies exhibit different

[1]Formulating the theory in ten dimensions is one way to assure the vanishing of the conformal anomaly. However, there are many alternatives to construct systems with $c = 26$. These theories usually go under the name of non-critical string theories and are an interesting topic on there own.

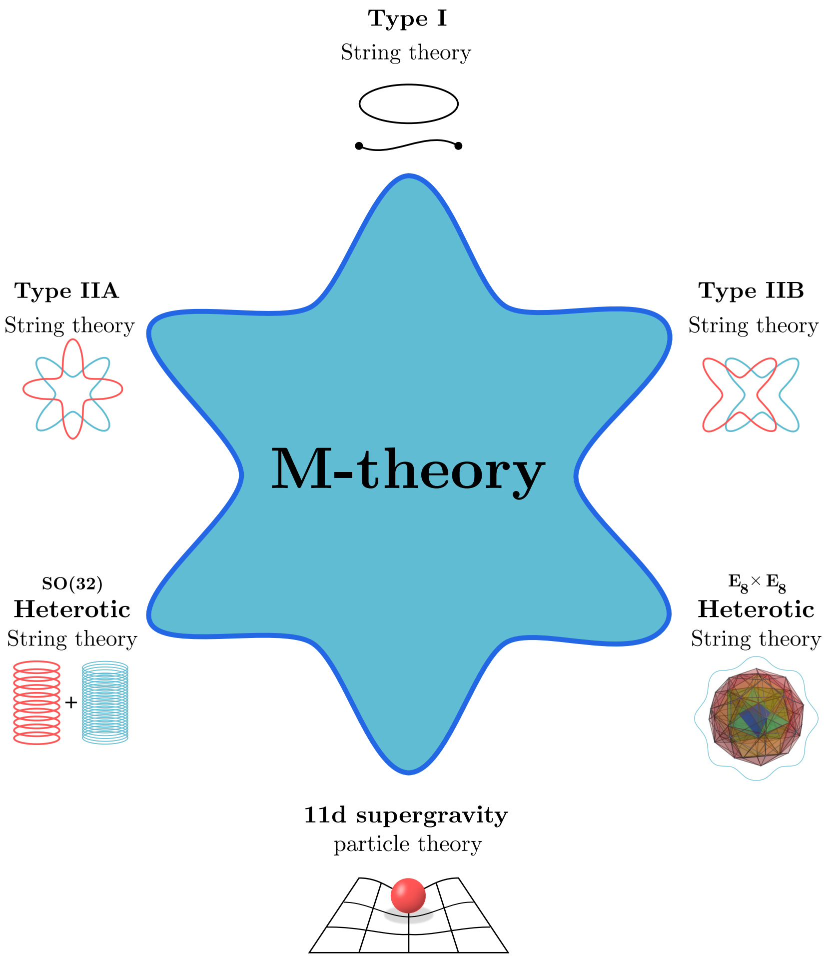


Figure 4.1: All five string theories are linked to each other by various dualities. They can all be obtained as a limiting case of an even more fundamental theory called M-theory.

symmetries. For example, the perturbative spectrum of type I string theory around flat ten-dimensional Minkowski space contains open and closed strings, while the other superstring theories only contain closed strings.[2]

One particularly interesting example is $c = 1$ string theory which consists of one timelike free boson X^0 together with $c = 25$ Liouville theory and a b, c ghost system, see for example [30, 107, 177].

[2]Of course open strings are also essential ingredients in type II string theory. However, they appear as non-perturbative objects in string perturbation theory around the vacuum. When one considers perturbation theory around the background sourced by a D-brane, open strings will appear in the spectrum.

String theory contains one dimensionful parameter, namely the string length ℓ_s. Massive string fluctuations have masses of the order $1/\ell_s$.[3] When we consider the theory at energies scales small compared to ℓ_s^{-1}, we can ignore all massive modes and obtain an effective low-energy description by only retaining the massless modes. The resulting theory is ten-dimensional supergravity. There are multiple variants of this theory corresponding to the different superstring theories. In this thesis we will mostly be interested in type IIA and type IIB supergravity theories, both contain two supercharges. In type IIB these have the same chirality, while in type IIA the supercharges have opposite chirality. In the low-energy limit these theories contain the following massless bosonic fields:

- a metric $g_{\mu\nu}$,
- a two-form gauge field, $B_{\mu\nu}$,
- a scalar field ϕ, called the dilaton, and
- a collection of n-form gauge field C_n, with n odd (even) in type IIA(B) supergravity.

In addition they contain a collection of fermionic $\mathcal{N} = 2$ superpartners. In the second string revolution, it was realized that string theory does not only contain strings but also a collection of dynamical extended objects called Dp-branes [209]. A Dp-brane is a $(p + 1)$-dimensional object, p space dimensions and one time, and is electrically charged under the gauge field C_{p+1}. Open strings can end on D-branes, explaining their name, i.e. they provide Dirichlet boundary conditions for the open strings. The massless open strings moving on a single D-brane constitute the degrees of freedom of a U(1) gauge theory. When N branes are coincident this gauge symmetry is enhanced to U(N) under which the endpoints of the strings transform in the fundamental representation.

4.2 M-theory and eleven-dimensional supergravity

In [247], partly based on observations made in [140], it was realized that at strong coupling type IIA superstring theory looks like a compactification of an eleven-dimensional theory on a circle, where the radius of the circle is parameterized by the string length. Similarly, applying various perturbative and non-perturbative dualities one can connect all five superstring theories. We are thus led to suspect that there exists an underlying theory whose various limits of its moduli space

[3] Often, one uses the Regge slope α' as the worldsheet coupling parameter of superstring theory. This parameter, in our conventions, is related to the string length as $\alpha' = \ell_s^2$.

reproduce the different weakly coupled ten-dimensional superstring theories. This fundamental eleven-dimensional theory is called M-theory.

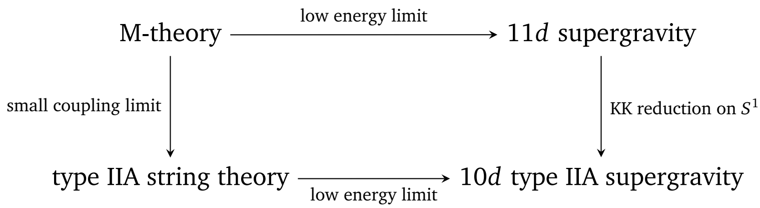


Figure 4.2: A defining characteristic of M-theory is that it reduces to IIA superstring theory at small coupling. In the low energy limit M-theory reduces to eleven-dimensional supergravity, which after a KK reduction on a sphere reduces to ten-dimensional type IIA supergravity.

M-theory is a theory of membranes, which come in two types, M2- and M5-branes. After compactification to ten dimensions the M2-brane corresponds to the fundamental string. Similarly, all other branes can be obtained as compactifications of various configurations of M-branes. In the low-energy limit, M-theory is described by eleven-dimensional supergravity. This theory contains an eleven-dimensional metric, a three-form gauge field A_3 and a gravitino. The M2-brane is electrically charged under A_3, while M5-branes are electrically charged under a six-form gauge field A_6 defined by $\mathrm{d}A_6 - \frac{1}{2}\mathrm{d}A_3 \wedge A_3 = G_7 = \star G_4$ where G_4 is the field strength associated to A_3 and $\star$ denotes the eleven-dimensional Hodge dual.

M-theory remains mysterious, we still have no fundamental formulation of it. The reason for this is that in contrast to the superstring, the M2-brane does not support perturbation theory. Because of this, the upper left corner in Figure 4.2 remains largely unknown. However, evidence that interesting dynamics takes place there arises from all other corners of the diagram. Using their low-energy descriptions, a bunch of consistency checks have been performed corroborating the relation between string and M-theory. One notable example is that type IIA string theory with a D0-brane condensate behaves like a ten-dimensional theory which develops a further circular dimension with radius scaling with the density of D0-branes [31]. Not a lot is known, but what little is known is powerful enough to lead us to new and interesting phenomena. In particular the M5-brane is believed to support a self-dual non-local theory on its worldvolume, known as little string theory. At low energies this theory is described by the six dimensional $(2,0)$ theory which will be a key player in part III of this thesis.

Chapter 5

The gauge/gravity correspondence

The AdS/CFT correspondence [120, 173, 250] is one of the most active areas of research in string theory. AdS/CFT stands for anti-de Sitter/conformal field theory, an expression that's not particularly elucidating. This correspondence will have a central role in the remainder of this thesis so let us have a closer look at what it represents.

AdS/CFT is a particular, and deeply surprising, example of a duality. It relates two very different theories and at first sight seems obviously wrong. It states that there is a duality between string theories in a particular background with a $(d+1)$-dimensional anti-de Sitter component and a conformal field theory in d dimensions. For this difference in dimension, the correspondence is also sometimes dubbed the holographic correspondence, in analogy with the more familiar holograms where three-dimensional information is encoded in a two-dimensional screen. This correspondence was first formulated by Juan Maldacena in 1997 [173], and is generally thought to be the single most important result in string theory in the last two decades.

In this chapter we will mainly discuss the best understood examples of the correspondence. In the original example the gravitational side consists of ten-dimensional type IIB string theory in a particular geometry, namely $\mathrm{AdS}_5 \times S^5$. The QFT on the other hand is the unique[1] four dimensional theory with the largest possible amount of supersymmetry, namely $\mathcal{N}=4$ super Yang-Mills theory. The duality has been extended to many other cases, and AdS/CFT is more generally referred to as the gauge/gravity correspondence. Formally, this is the statement that gravitational theories in $(d+1)$ dimensions can be completely equivalent to non-gravitational quantum field theories in d dimensions. The class of dual pairs

[1] Up to the choice of gauge group and complex coupling.

we will focus on in this chapter is the set of gravity theories in asymptotically locally AdS spaces dual to conformal field theories. However, the duality is much more general then only CFTs and AdS spaces. As we will see in the following parts of this thesis, it can be used to describe non-conformal field theories as well as RG flows in QFTs. So far there is no formal mathematical proof of this relationship. However, a large number of checks have been performed. These checks involve two calculations, using different techniques and methods, of quantities related by the holographic dictionary. Continual agreement of these calculations constitutes strong evidence for the correspondence.

In this chapter I will try to answer some very basic questions about the duality. How can a QFT be a theory of quantum gravity? When is this gravity theory classical? And hopefully, along the way I will convince you that it is interesting. For this introduction I have liberally borrowed ideas and arguments from various review articles including [8, 134, 180, 193, 210, 253] complemented with my own imagination.

5.1 Holographic hints

The assertion that hidden within every non-abelian gauge theory, with strong or weak interactions, there is a theory of quantum gravity at first seems absurd. We will start this short review with some facts, often using the benefit of hindsight, hinting that this assertion might not be so unreasonable after all. These hints come from:

1. The Weinberg-Witten theorem
2. The holographic principle
3. RG flows
4. Large N gauge theories

5.1.1 The Weinberg-Witten theorem

A theory of quantum gravity is a quantum theory with a dynamical metric. This means that the spectrum of particles contains a spin-2 massless particle which we call the graviton. For the gauge/gravity correspondence to be true we should thus be able to construct a composite object made out of gauge theory degrees of freedom representing the graviton. However this seems to go directly against the Weinberg-Witten no-go theorem [239]:

In a QFT with a Poincaré-covariant stress-energy tensor $T_{\mu\nu}$, any particle, elementary or composite, with spin $j > 1$ which carries momentum (i.e. $P_\mu = \int \mathrm{d}^d x T_{0\mu} \neq 0$) is forbidden.

This theorem immediately seems to contradict the assertion of the gauge/gravity correspondence. However, it is better looked at as a sign indicating that we are not looking in the right place. Indeed, in hindsight the loophole is evident. The graviton does not need to live in the same spacetime as the QFT!

5.1.2 The holographic principle

A further hint that the gravity theory might not live in the same spacetime comes from black hole physics. More precisely, from the holographic principle [?,228,230]. Naively, the degrees of freedom both in a theory of gravity as well as in a quantum field theory should grow as the volume of the space they live in. However, as pointed out by 't Hooft and Susskind, this is not the case. Let us illustrate this by counting states.

Consider a region of space Γ, with volume V, and for simplicity let us assume it is a sphere. Now let us compare for various systems the space of states describing an arbitrary system that fits in Γ with the region outside Γ empty. As a first example consider a three-dimensional lattice of spins with lattice spacing a. The number of spins in this system is V/a^3 and the number of possible states in Γ is

$$N = 2^{V/a^3} . \tag{5.1}$$

Next consider a QFT, in this case the number of states obviously diverges but let us regulate it by introducing a UV cutoff on the energy density $\rho_{\max}$. In this case the states can be counted by the thermodynamic entropy S of the system. The total entropy is given by

$$S = s(\rho)V \tag{5.2}$$

The total number of states can then be approximated as

$$N \sim \mathrm{e}^{S_{\max}} = \mathrm{e}^{s(\rho_{\max})V} . \tag{5.3}$$

In both cases we see that the number of states is exponential in the volume. This is a general property of local field theories, the number of degrees of freedom is additive in the volume. In counting the states we used the concept of entropy, this quantity is not really a property of the system but requires the knowledge of a particular state. The maximum entropy on the other hand is a property of the system and is given by the logarithm of the total number of states.

Let us now describe a similar system which includes gravity. For definiteness we will take spacetime to be four-dimensional and again focus on a spherical region of space Γ with boundary $\partial\Gamma$, which has an area A. Consider a system with thermodynamic entropy S. The total mass of this system cannot exceed the mass of a black hole with horizon area A, otherwise it would not fit in the region Γ. Thus we are let to consider the black hole with exactly the right energy to completely fill the region Γ. Black holes are regions in spacetime from which no timelike trajectory can escape. They are formed when an enormous amount of mass is squeezed together in a sufficiently small volume and are bounded by an event horizon of size $A = 4\pi R^2$. Depending on the mass, charge and angular momentum there is a unique black hole solution in general relativity describing the resulting black hole. However, black holes are not entirely black. As was noticed by Stephen Hawking they have a certain thermodynamic temperature and evaporate by radiating thermal radiation. Since they have a temperature one can attribute further thermodynamic properties to black holes, such as entropy. When the mass of the black hole is large, it evaporates very slowly and can be treated semi-classically. A computation in the background of the black hole shows that this entropy is given by [33, 128]

$$S_{BH} = \frac{A}{4G_N}\,, \tag{5.4}$$

where G_N is Newton's constant. The Bekenstein-Hawking entropy was furthermore argued to be the maximal possible entropy that can be assigned to a region of space in any gravitational theory. In other words, the maximal entropy of a region of space is proportional to its area! This is the content of the holographic principle. We conclude from this that a theory of quantum gravity must have a number of degrees of freedom which scale like those of a QFT in one dimension lower. This property is believed to be fundamental for any theory of quantum gravity and has inspired theorists to come up with the holographic duality.

In fact, we already know an example of this, namely gravity in three dimensions. As is well-known this is a topological theory and can be rewritten as a Chern-Simons gauge theory with action [85]

$$S_{CS} \sim \frac{k}{4\pi}\int \mathrm{Tr} A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\,. \tag{5.5}$$

If you put this theory on a space with boundary, there are local degrees of freedom which live on the boundary and constitute a WZW model, a 2d CFT [244]! This can be considered as a first example of the gauge/gravity correspondence, however, the examples we will discuss below are much more dramatic. In contrast to this simple case, there will be additional dynamics in the bulk.

5.1.3 RG flows and the extra dimension

A hint to what this extra dimension might be comes from the renormalization group flow. Wilson taught us that the best way to think of a QFT is as a renormalization group flow, sliced up by energy scale. The couplings of the theory at different length scales are related by the RG equations as a function of the RG scale u

$$u\partial_u g = \beta(g(u)) \tag{5.6}$$

This equation is local with respect to the RG scale. It is a non-linear evolution equation determining the evolution of the coupling constants at a given energy scale u without the need to know the full IR or UV behavior of the theory. Because of this locality we can entertain the idea of interpreting this RG scale as the extra dimension suggested by the holographic principle.

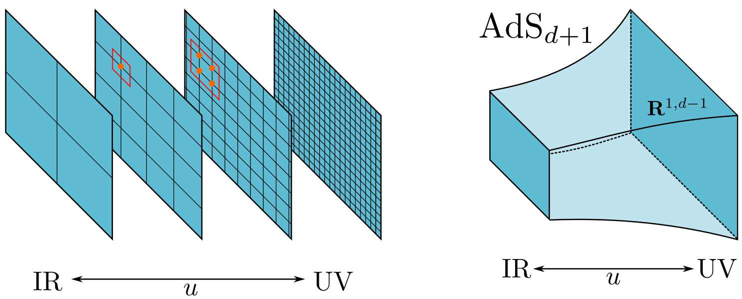


Figure 5.1: The RG scale can be thought of as the extra dimension of a theory. This dimension can be seen as the resolution scale of the field theory. The left hand side illustrates a “block spin”-like decimation procedure à la Kadanoff. The orange dots denote degrees of freedom which are averaged over a block to obtain a coarser set of degrees of freedom. On the right hand side is a picture of anti-de Sitter space which organizes field theory information in the same way. In this sense the AdS picture is a hologram, excitations at different wavelengths are put in different places in the bulk image.

Let us consider an example to make this more concrete. The simplest example of a RG flow is when $\beta = 0$ as a result of which the flow is self-similar. In a Lorentz-invariant theory this implies that scale transformations $x^\mu \to \lambda x^\mu$, and typically the enlarged set of all conformal transformations, are symmetries of the system. In other words, this is a CFT. If we think of the extra coordinate u as an energy scale, by dimensional analysis, it should transform as $u \to \frac{u}{\lambda}$ under these transformations. The most general $(d+1)$-dimensional metric with this symmetry and Poincaré invariance is of the form

$$\mathrm{d}s^2 = \left(\frac{L_{\mathrm{AdS}}}{u}\right)\mathrm{d}u^2 + \left(\frac{u}{L_{\mathrm{AdS}}}\right)^2 \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \,. \tag{5.7}$$

This is exactly AdS_{d+1} space![2] The parameter L_{AdS} has the dimension of length and is called the AdS length. This space consists of a family of copies of Minkowski space, parameterized by u whose size varies with u. This hint from the RG flow suggests that a d-dimensional CFT should be related to a theory of gravity on AdS_{d+1}. This kind of geometry indeed arises from a bulk action of the form

$$S_{\text{bulk}} = \frac{1}{16\pi G_N} \int \mathrm{d}^{d+1}x\,\sqrt{-g}(R - 2\Lambda + \cdots), \tag{5.8}$$

where the cosmological constant $\Lambda = -\frac{d(d-1)}{2L_{\mathrm{AdS}}^2}$. The dots denote additional terms which can include higher order curvature terms or other bulk fields (that vanish in the pure AdS solution). In many applications in this thesis this type of action will precisely arise from string theory at low energies, when the curvature is gentle.

5.1.4 The large N limit for gauge theories

A last clue comes from gauge theory and suggests that the dual theory might have something to do with strings. In the 't Hooft limit [132] the relevant degrees of freedom in the gauge theory seem to describe closed strings, and thus include gravity.

To see this let us consider a U(N) gauge theory in the limit of large number of colors N. The Lagrangian of this theory is given by

$$\mathcal{L} = \frac{1}{g_{\mathrm{YM}}^2} \mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu} + \cdots\right) \tag{5.9}$$

where the dots denote additional matter fields which we ignore for now. g_{YM} is the Yang-Mills coupling constant and the gauge fields transform in the adjoint representation of the gauge group so they can be written as $A_{\mu ij}$, where $i, j = 1 \dots N$. The 't Hooft limit consist of the limit $N \to \infty$ while at the same time keeping $\lambda = g_{\mathrm{YM}}^2 N$ constant. This limit greatly simplifies the perturbative analysis but it is general enough to still contain a lot of interesting dynamics. Even though the Yang-Mills coupling goes to zero in this limit, the theory does not become free since the number of modes diverges as well.

A very useful way to draw Feynman diagrams, which allows us to keep track of the N-dependence of various quantities, is illustrated in Figure 5.2. These diagrams use double line notation which makes the adjoint indices explicit. We can draw a propagator and vertex as

[2] Indeed, by defining $z = \frac{L_{\mathrm{AdS}}^2}{u}$ we obtain the metric of AdS in Poincaré coordinates as defined in (3.9).

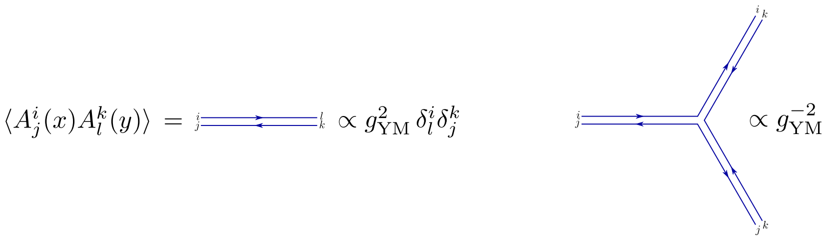


Figure 5.2: Left: The gluon propagator for U(N) in double line notation. Right: A cubic vertex in double line notation.

A general diagram built from these elements consists of a number of propagators, interaction vertices and index loops. It therefore gives a contribution

$$\text{diagram} \propto \left(\frac{\lambda}{N}\right)^{\#\text{prop.}} \left(\frac{N}{\lambda}\right)^{\#\text{vert.}} N^{\#\text{loops}}\,. \tag{5.10}$$

To see the implications of this new way of drawing diagrams, let us consider the free energy, or zero-point function. Figure 5.3 shows three diagrams contributing to this zero-point function. All these diagrams are planar, meaning that one can draw them on a piece of paper without intersecting lines.

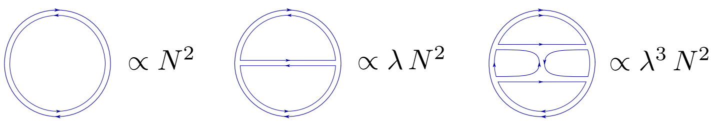


Figure 5.3: Three planar graphs contributing to the free vacuum to vacuum amplitude. All these diagrams can be drawn on a flat sheet, or a sphere, without crossing lines.

As can be seen from (5.10), they all scale as $\lambda^n N^2$. As the graphs becomes more complicated, i.e. the more loops they contain, the power of λ becomes higher. The diagram in Figure 5.4 on the other hand is non-planar. It can not be drawn on a piece of paper, however, it can be drawn on a torus. This diagram contributes as λN^0.

We can generalize this preliminary set of examples and see that every possible graph can be drawn on some Riemann surface with Euler characteristic $\chi = 2-2g = F - V + E$ where F, E and V are the number of faces (or loops), edges and vertices of the graph. Applying the Feynman rules to these graphs we see that a genus g graph scales as $N^{2-2g}\lambda^{E-V}$. The Euler characteristic is a topological invariant of a Riemann surface so all graphs with the same topology will scale in the same way with N. The exponent of λ on the other hand is not a topological invariant

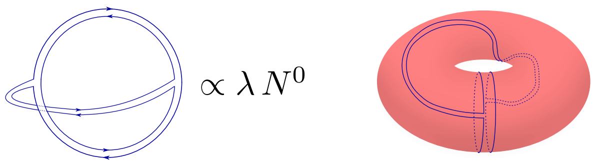


Figure 5.4: An example of a non-planar graph. This graph can not be drawn on a flat sheet or sphere without crossing lines. It can however be drawn on a torus as shown on the right.

and depends on the triangulation (or Feynman diagram) of the Riemann surface. Therefore when expanding the free energy in powers of N the expansion nicely organizes itself according to the genus

$$F = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda) . \tag{5.11}$$

Similarly, we can write a genus expansion for all the other n-point functions in the theory. This expansion is very similar to a perturbative world-sheet expansion of a string theory and the expression for the free energy is exactly analogous to the loop expansion of a closed string with coupling $g_s = \mathrm{e}^{\phi} = \frac{1}{N}$. In the large N limit the string coupling is very small so all stringy interactions are heavily suppressed. In this limit the closed string theory reproduces gravity with Newton's constant $G_N \propto N^{-2}$. We can push this analogy a bit further and ask what plays the role of the worldsheet coupling? We can think of λ as a type of chemical potential for edges in our triangulations. As can be seen from the scaling relation (5.10) diagrams with lots of edges are important when λ is large. Thus at large values one gets a smoother world-sheet which we can interpret as less quantum fluctuations. From this we expect a relation of the form $\lambda^{-c} \propto \alpha'$, with c some positive constant. This is indeed what is found in various well-understood examples of the gauge/gravity duality.

So far we only considered gauge fields. However, often we would like to include matter fields. We can easily extend the analysis above to also include fundamental matter fields such as quarks. These fields ψ_i have only one SU(N) index so can be represented as additional single lines in the diagrams above. Because of this there will also appear odd powers of N, which can be incorporated in the genus expansion by summing over Riemann surfaces with boundaries. The addition of boundaries allows the Euler characteristic to take odd values, $\chi = 2 - 2g + b$ and thus further pursuing the analogy with string theory we can associate fundamental matter fields with open strings.

5.2 Putting things together

From the early 70s hints were piling up that indeed hidden within a non-abelian gauge theory there might be theory of quantum gravity. These hints come from the Weinberg-Witten theorem, the holographic principle, RG flows and large N gauge theories. They all suggest a deep connection between gauge theory and string theory. Clearly, the dual to a weakly coupled field theory must be strongly coupled. Indeed, classical Yang-Mills is certainly not the same as classical general relativity (GR). If gravity is to emerge from a gauge theory we should expect this to happen in a strongly coupled regime where the gravitational fields emerge as effective classical fields. The large N discussion above indeed suggested that in the regime of large N and large λ classical gravity might emerge.

In [173] all these hints were finally put together resulting in our modern understanding of the gauge/gravity duality. The need for strings was already anticipated by 't Hooft so it can not be too surprising that the "cleanest" way to outline the correspondence is through string theory. Let us therefore review the original argument of Maldacena. This example concerns a duality between the $\mathcal{N} = 4$ super Yang-Mills theory and type IIB string theory on $\text{AdS}_5 \times S^5$. The former turns out to be a CFT and for this reason one often talks about the AdS/CFT correspondence.

5.2.1 Open versus closed strings

Consider a stack of N coincident D3-branes in type IIB superstring theory in $\mathbf{R}^{1,9}$. The worldvolume of these branes extends along the directions x^μ with $\mu = 0, \ldots, 3$ where x^0 plays the role of time. In the six internal dimensions we can choose spherical coordinates so that they are spanned by a five-sphere S^5 together with the radial direction r.

The type IIB geometry sourced by this configuration is uniquely fixed by the translational and rotational symmetries, the energy density and the RR-fluxes of the D3-brane. The target space metric takes the following form

$$\begin{aligned} \mathrm{d}s^2 &= H^{-1/2}(r)\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + H^{1/2}(r)\left(\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_5^2\right), \\ H(r) &= 1 + \frac{L_{\text{AdS}}^4}{r^4}, \qquad L_{\text{AdS}}^4 = 4\pi g_s N {\ell_s}^4 . \end{aligned} \tag{5.12}$$

Each of the branes couples to gravity with a strength proportional to the string coupling g_s. So the distortion of metric caused by the stack of branes is proportional to $g_s N$. Furthermore, there are a constant dilaton $g_s = \mathrm{e}^\Phi$ and axion C_0 and a five-form flux

$$F_5 = (1 + \star_{10}) \mathrm{d}(g_s H)^{-1} \wedge \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^3 \,. \tag{5.13}$$

In the metric above the branes are located at $r = 0$ but they are completely redshifted away and have disappeared in this picture.

Let us now consider the low-energy limit of this system. We can obtain this limit by sending $\ell_s \to 0$ while keeping the energies E of all physical processes fixed. In this way the dimensionless energy $\epsilon = \ell_s E$ indeed goes to zero. In the metric (5.12) this implies that $H(r) \to 1$ at $r \neq 0$. Away from the branes we therefore recover ten-dimensional flat space on which we have to consider low-energy excitations, described by type IIB supergravity.

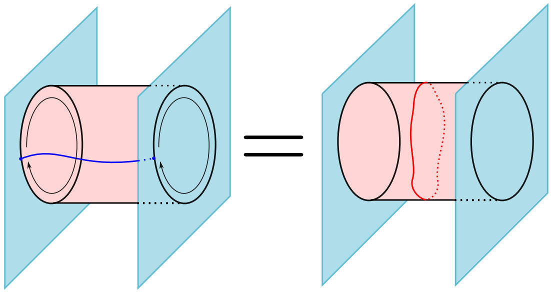


Figure 5.5: In string theory a given worldsheet with boundaries can typically be read in at least two different ways as describing either a closed string or an open string.

On the other hand, by the open-closed string duality we see that the branes themselves can be described in two ways, see Figure 5.5. Either in terms of open strings or in terms of closed strings. For any process, the string scattering amplitudes are the same. The fact that we have two different ways of describing this physical system will ultimately lead us towards the AdS/CFT correspondence.

Open string description

The first way to describe the D3-branes is using open strings attached to the stack of branes. At low energies the effective degrees of freedom of these string modes describe the adjoint gauge fields of a U(N) gauge group together with their fermionic counterparts. This gauge theory is constrained to live on the $(3+1)$-dimensional worldvolume of the stack of $D3$-branes and for this reason we often refer to it as the worldvolume theory. It is given by the $\mathcal{N} = 4$, SU(N) super Yang-Mills theory.[3] In addition to the gauge fields, this theory contains four chiral fermions and six

[3] Since all fields of the gauge theory transform in the adjoint of the gauge group $\mathrm{U}(N) = \mathrm{SU}(N) \rtimes U(1)$, the U(1) factor becomes free. This U(1) multiplet described the center of mass motion of the stack of

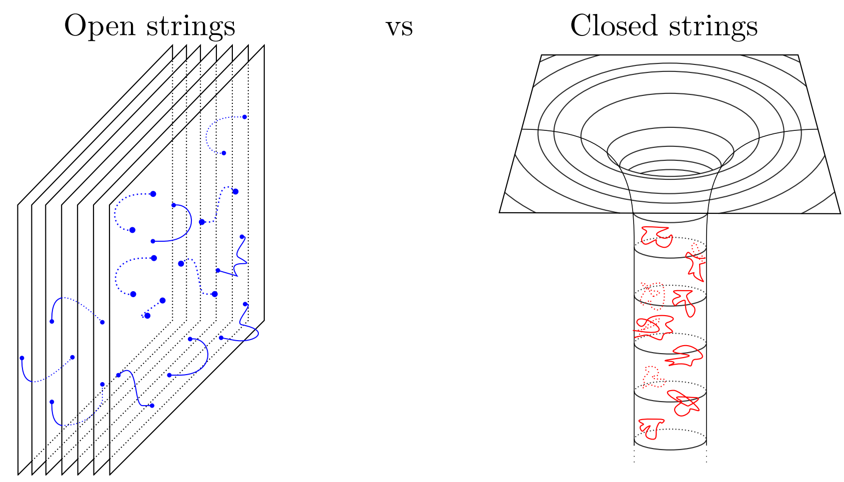


Figure 5.6: The D3-branes can be described in two different ways. Either by open strings attached to the stack of branes or as closed strings moving in the infinite throat created as the back-reaction of the branes on the geometry.

real scalars, all transforming in the adjoint representation of the gauge group. The R-symmetry is given by Spin(6) $\simeq$ SU(4) and the fermions transform in the fundamental representation of SU(4). The scalars on the other hand transform in the fundamental of SO(6). The Lagrangian is given by

$$\mathcal{L}_{\text{SYM}} = \frac{1}{g_{\text{YM}}^2}\text{Tr}\bigg(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\Phi D^\mu\Phi - i\bar{\psi}\bar{\sigma}^\mu D_\mu\psi + \psi[\Phi,\psi] + \bar{\psi}[\Phi,\bar{\psi}] - \frac{1}{4}([\Phi,\Phi])^2\bigg) + \frac{\theta}{2\pi}\text{Tr}\, F\wedge F\,, \quad (5.14)$$

where D_μ is the gauge covariant derivative and we have suppressed all spinor and R-symmetry indices. The Yang-Mills coupling constant is given in terms of the string coupling as $g_{\text{YM}}^2 = 2\pi g_s$ and the θ-angle is given in terms of the string theory axion as $\theta = 2\pi C_0$. In [223] it was shown that the β-function for the Yang-Mills coupling vanishes to all orders in perturbation theory and this is believed to hold non-perturbatively as well. This implies that g_{YM} is a physical parameter of the theory and that the theory remains scale invariant also on the quantum level. This scale invariance can be extended to the full conformal group which makes $\mathcal{N}=4$ SYM a conformal field theory.

In addition to these open strings there are the aforementioned closed strings away

branes. In the low energy limit we can safely assume the branes are at rest so what one is left with is a SU(N) gauge theory.

from the brane. In principle these could also interact with the open strings on the brane. However, as $\ell_s \to 0$ these modes completely decouple, since the interaction strength scales as $G_N \sim \ell_s^2$.

Closed string description

The second way to describe the brane system is in terms of closed strings. In this description we have to take into account the gravitational redshift of the excitation in the geometry (5.12). The energy E of an excitation at a certain value of the radial coordinate r as seen by an observer at infinity is redshifted to

$$E_\infty = H(r)^{-1/4} E\,. \tag{5.15}$$

To take the low energy limit in the closed string theory we can again send the dimensionless energies $\ell_s E_\infty$ to zero by taking the limit $\ell_s \to 0$. However, as can be seen from (5.15) this does not send all energies $\ell_s E$ to zero. Indeed, as $r \to 0$, $H(r) \to \infty$ so the modes localized at $r = 0$ survive. In this limit,

$$H(r) \to \frac{4\pi g_s N z^4}{\ell_s^4}\,, \tag{5.16}$$

where we have defined the new coordinate $z = L_{\mathrm{AdS}}^2 / r$, so the metric becomes

$$\mathrm{d}s^2 = L_{\mathrm{AdS}}^2 \left(\frac{1}{z^2} \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + \frac{\mathrm{d}z^2}{z^2} + \mathrm{d}\Omega_5^2 \right). \tag{5.17}$$

This is precisely the metric of $\mathrm{AdS}_5 \times S^5$ where both factors have a radius of curvature L_{AdS}.[4][5] In this limit, the five-form flux becomes

$$F_5 = -(1 + \star_{10}) \frac{4 g_s}{z^5} \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^3 \wedge \mathrm{d}z\,. \tag{5.18}$$

When we take the low energy limit, from the viewpoint of an asymptotic observer, two types of modes will survive: massless strings propagating in the bulk with very large wavelengths and massive excitations in the throat region near $r = 0$. Just as in the open string picture, the low energy limit is a decoupling limit, meaning that the interactions of the massless excitations away from the brane with the near-horizon modes vanish. This can be seen from the fact that the redshift factor diverges as $r \to 0$, i.e. the throat region is infinitely deep and only modes with infinite energy

[4]This is directly analogous to the well-known four-dimensional Reissner-Nordström black hole where the near horizon metric takes the form $\mathrm{AdS}_2 \times S^2$.

[5]The factors of ℓ_s in the definition of the AdS length L_{AdS} make the metric very small. However, in the worldsheet sigma model this factor cancels against a prefactor in the worldsheet string action. By absorbing this prefactor in the metric we can safely forget about its smallness.

can escape out of it. Furthermore the closed strings in the bulk cannot probe the AdS region since the absorption cross section vanishes in the low energy limit [173].

The AdS/CFT correspondence

We have observed that in the low energy limit of string theory, a stack of $D3$-branes can be described by two very different theories. In one description we recover the $\mathcal{N}=4$ SU(N) SYM theory, while in the other we find type IIB closed string theory on an $\text{AdS}_5 \times S^5$ background. We are therefore led to the conjecture that these two theories are entirely equivalent. This is the statement of the AdS/CFT correspondence.

$$\mathcal{N} = 4 \text{ SU}(N) \text{ SYM } = \text{Type IIB string theory on AdS}_5 \times S^5 . \tag{5.19}$$

This heuristic derivation makes the statement of the duality more concrete, but it does not provide a rigorous proof. However, it has been put through various tests and so far has passed all of them.

Now that we have an equivalence between two theories, we can ask how exactly they are related? To answer this question, let us first have a look at the different parameters on both sides. On the gravity side we have three parameters, the string coupling g_s, the length scale of AdS L_{AdS} and the string length ℓ_s.[6] On the gauge theory side there are two parameters, the rank of the gauge group, N and the 't Hooft coupling $\lambda = g_{\text{YM}}^2 N$. The AdS/CFT correspondence maps these parameters as follows

$$g_s = g_{\text{YM}}^2 = \frac{\lambda}{N}, \qquad \frac{L_{\text{AdS}}}{\ell_s} = \lambda^{1/4} . \tag{5.20}$$

Therefore, we immediately notice that when λ is large the stringy corrections are small, and when N is large, the quantum corrections are small. The AdS/CFT correspondence is a weak-strong duality meaning that it maps a weakly coupled theory to a strongly coupled one. By studying a weakly coupled gravity solution we can thus learn about a strongly coupled field theory and vice versa! When we furthermore consider both $N \gg \lambda \gg 1$ we have a weakly coupled classical theory of gravity on $\text{AdS}_5 \times S^5$.

[6] A well-known fact states that "In string theory there are no adjustable couplings" but what does the string coupling represent then? In string theory the string coupling g_s is determined by the expectation value of the dilaton Φ, $g_s \sim e^{\langle \Phi \rangle}$. Therefore, it is not an adjustable parameter of the theory but rather a dynamical parameter, a field in fact.

The supergravity limit

In this thesis we will be working predominantly with the (semi-)classical limit of the closed string theory, which in this case reduces to type IIB supergravity on $\mathrm{AdS}_5 \times S^5$. This approximation is only valid in the limit when the radius of curvature L_{AdS} is large in units of ℓ_s. This implies that

$$g_s N \gg 1\,. \tag{5.21}$$

Furthermore, we want to suppress string loops so we require $g_s \ll 1$ (so $N \gg 1$). In this limit the higher curvature terms can be neglected and the low energy effective action becomes that of type IIB supergravity.[7]

In the dual field theory the same limit reduces to the 't Hooft limit introduced in Section 5.1.4. In this limit the planar Feynman diagrams are dominant and the effective loop counting parameter for the gauge theory is given by $\lambda = g_{\mathrm{YM}}^2 N$. In the supergravity limit (5.21) this coupling is very big so the field theory is strongly coupled. Therefore we can not use a perturbative approach to compute observables in the quantum field theory making it very hard to explicitly test the correspondence. On the other hand, it offers us a valuable opportunity to compute new observables in the strongly coupled field theory by performing relatively straightforward computations in the supergravity theory.

Symmetries

As a first check of this duality we can compare the symmetries of the two systems. The isometries of the $\mathrm{AdS}_5 \times S^5$ supergravity background are given by $\mathrm{SO}(4,2) \times \mathrm{SO}(6)$. On the gauge theory side the Lorentz symmetry is enhanced to the full conformal symmetry $\mathrm{SO}(4,2)$. The $\mathrm{SO}(6)$ symmetry of the sphere on the other hand is realized as the R-symmetry of the $\mathcal{N}=4$ SYM theory. On both sides of the duality this bosonic symmetry is enhanced by supersymmetry to the superconformal group $\mathrm{PSU}(2,2|4)$. Additionally, both the four dimensional $\mathcal{N}=4$ gauge theory and type IIB string theory enjoy a weak-strong $\mathrm{SL}(2,\mathbf{Z})$ duality.

Generalizations

By describing the low energy excitations of a system of D3-branes in two different ways we have encountered a spectacular equivalence between a theory of gravity and a gauge theory. We discussed the relation between the various parameters on both sides of the duality and matched the symmetries. Additionally, we discussed

[7] The action together with our conventions can be found in Appendix A.

how the supergravity limit of string theory reduces to the 't Hooft limit of the gauge theory. In the next section, we will continue to explore this duality and discuss how to match various observables.

This correspondence belongs to a much wider class of dualities. Over the years there have been discovered more and more examples by considering other D- or M-branes, brane intersections or even more exotic string theory backgrounds. One can study the near-horizon limit of a variety of brane systems to uncover new $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ dualities for general d. For each of these dualities there is a internal compact manifold analogous to the S^5 in the example above. Deforming this internal manifold breaks part of the supersymmetry and opens the door to many more generalizations. In the dual field theories these deformations generally correspond to switching on sources or vacuum expectation values (VEVs) for specific operators, which can trigger an RG flow to an IR (S)CFT. Finally, one can also consider theories where the dual field theory is no longer conformal even in the UV. In these examples the dual background consequently will not be asymptotic to AdS but some more general space. Therefore it is more appropriate to refer to these dualities in general as gauge/gravity dualities.

All these dualities have a very similar structure. In particular, we can introduce a dictionary to translate between observables on both sides of the duality.

5.3 The dictionary

In the preliminary analysis above we checked that the global symmetry group agrees on both sides of the duality. A conformal field theory in d dimensions and a gravity theory in AdS_{d+1} both are invariant under $\mathrm{SO}(d,2)$ transformations. The additional isometries of the internal space of the string theory are matched to the R-symmetry of the supersymmetric gauge theory. Since the objects in these two theories live in different dimensions it is very hard to imagine a connection between them. However, in the seminal work [173] (leading to the more concrete conjecture formulated in [120, 250]) and a multitude of works after that, a precise dictionary to map observables between the two sides was formulated. In the following we introduce how to use this dictionary. The main focus will be on the objects we will encounter in part II and III of this thesis while some other very interesting observables will not be discussed.

We will often refer to the fields in AdS as bulk fields and the CFT fields as boundary fields. We assume that the interactions of the bulk theory are described by an action $S_{\mathrm{AdS}}(g, A, \phi, \dots)$ which has an AdS vacuum. The fields appearing in this action are the metric, gauge fields, scalar fields and all other possible matter fields, both fermionic and bosonic. In most applications in this thesis this action will be obtained as a consistent truncation of some higher dimensional supergravity. Similarly, we assume that the boundary CFT has a Lagrangian, given by L_{CFT}; These assumptions do certainly not encapsulate all interesting examples of the gauge/gravity correspondence but for pedagogical reasons we restrict to this class. Many phenomena observed for this restricted subset of dualities hold in much greater generality.

The spectrum of the gauge theory is defined by the set of primary operators in the CFT. An operator on the boundary is related to a field in the bulk by having the same quantum numbers and they can communicate through boundary couplings. In the CFT we can associate to every operator $\mathcal{O}$ a source h and add the following coupling to the Lagrangian

$$\mathcal{L}_{\text{CFT}} + \int \mathrm{d}^d x\, h\, \mathcal{O}\,. \tag{5.22}$$

In this formulation $h(x)$ is a d-dimensional background field obtained by evaluating the bulk field $h(x,z)$ on the conformal boundary at $z = 0$. This background field can then be used to compute connected correlation functions for the operator $\mathcal{O}$ in the usual way by defining the functional generator

$$\mathrm{e}^{W(h)} = \left\langle \mathrm{e}^{\int h\mathcal{O}} \right\rangle_{\text{CFT}} \tag{5.23}$$

and differentiating W with respect to $h(x)$

$$\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle = \frac{\delta^n W}{\delta h(x_1)\cdots\delta h(x_n)}\,. \tag{5.24}$$

By demanding that the bulk field $h(x,z)$ satisfies the $(d+1)$-dimensional equations of motion one can prove that the extension from the boundary to bulk is unique when appropriate boundary and regularity conditions are imposed. The equations of motion in AdS are second order so we need to specify two boundary conditions in order to find a unique solution. Both boundary conditions require some care. First, at the conformal boundary, we cannot simply put $h(x,0) = h(x)$ since solutions to the equations of motion either diverge or vanish at this location. The correct boundary condition at the AdS boundary is thus of the form $h(x,z) = f(z)h(x)$ for some function f of the radial coordinate. To fully fix the solution a second boundary condition has to be imposed in the bulk of AdS where we shall require the solution to be regular.

The fundamental statement of the AdS/CFT correspondence is then

$$\mathrm{e}^{W[h(x)]} = \mathrm{e}^{S_{\text{AdS}}[(h(x,z))]}\,. \tag{5.25}$$

On the left hand side we have an arbitrary d-dimensional field configuration h while on the right hand side we have the on-shell value of the $(d+1)$-dimensional gravitational action, evaluated on the unique regular solution of the equation of motion that reduces to $h(x)$ on the boundary. Since the knowledge of $W[h]$ on the field theory side completely determines the CFT this formula indeed states the equivalence between this CFT and the $(d+1)$-dimensional bulk theory. So far, the statement (5.25) uses the effective AdS action. When we furthermore have a UV completion of the theory we can also consider quantum effects on the right hand side.

So far we have not specified how to exactly match CFT states to bulk fields. In general this will depend on the details of the theory. In the remainder of this chapter we will discuss various examples. However, a lot can already be learned by looking at the symmetries. Indeed, the bulk field h and the CFT operator $\mathcal{O}$ have the same $O(2,d)$ quantum numbers. For example, the boundary value of the bulk metric is the boundary metric which is the source for the stress tensor $T_{\mu\nu}$ which has dimension $\Delta = d$. As a second example we can consider a gauge field A_M in the bulk. The dual operator must then be a vector J^μ coupled to the source as

$$\int \mathrm{d}^d x A_\mu J^\mu \,. \tag{5.26}$$

Since the bulk is invariant under gauge transformations, the boundary coupling should be invariant as well.[8] Therefore, J^μ should be a conserved current in the boundary theory with dimension $\Delta = d-1$. From this we can extract a general rule, "gauge symmetry in the bulk is mapped to global symmetry on the boundary". By covariantizing the CFT action by adding background field to the Lagrangian we have found the following natural couplings

$$\mathcal{L}_{\mathrm{CFT}} + \int \mathrm{d}^d x \sqrt{-g}\left(g_{\mu\nu}T^{\mu\nu} + A_\mu J^\mu + \Phi F_{\mu\nu}F^{\mu\nu} + \cdots\right) \tag{5.27}$$

The CFT field associated to the graviton is the stress tensor, the field associated to a gauge field in AdS is a current. The last operator we added appears in all gauge theories. In many applications, the scalar operator source Φ corresponds to the string theory or supergravity dilaton.

5.3.1 **Correlation functions in AdS/CFT**

As a first holographic observable let us briefly discuss correlation functions. As was already suggested by the notation in Chapter 3, the conformal dimension of a CFT field is mapped to the exponent Δ appearing in Equation (3.17). For example, for

[8] In the absence of anomalies

a massless scalar in AdS we find that the corresponding scalar operator in the dual CFT has dimension $\Delta = d$.

For simplicity, we will focus on scalar fields ϕ_i with masses m_i interacting through some Lagrangian $\mathcal{L}_{\text{AdS}}(\phi_i)$. We denote by ϕ_i^0 the boundary value of the scalar field as determined in Section 3.4. On the CFT side these fields are identified with sources for a dual operator $\mathcal{O}_i$ with dimension Δ_i determined by (3.17). The generating function of the CFT can then be obtained by evaluating the bulk on-shell action with the prescribed boundary conditions. A n-point function can be obtained by differentiating this on-shell action with respect to the sources ϕ_i^0 evaluated $\phi_i^0 = 0$,

$$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \rangle = \frac{\delta^n S_{\text{on-shell}}}{\delta \phi_1^0 \delta \phi_2^0 \cdots \delta \phi_n^0} \bigg|_{\phi_i^0 = 0} . \tag{5.28}$$

To go beyond the classical contribution one should consider loop contributions which extend in the bulk. These can be computed elegantly using so-called Witten diagrams. For details we refer to the reader to [250].

5.3.2 **Holographic renormalization**

In quantum field theory correlation functions suffer from various UV divergences and one needs to renormalize the theory to obtain sensible results. It turns out that all diverging quantities on the field theory side similarly diverge on the gravity side. Indeed, since the volume of an AlAdS spaces is infinite, the naive on-shell action will always be infinite as well. Therefore, we should introduce a holographic renormalization procedure.

In the QFT the divergences arise from the UV of the theory and are not affected by the IR. According to the UV/IR connection [229], UV phenomena in the field theory should be related to IR phenomena in gravity and vice-versa. Therefore the holographic renormalization procedure should only care about the near-boundary region, $z \approx 0$. The full correlation functions, however, capture the dynamics of the theory and thus should depend on the full bulk solutions. A holographic renormalization method was introduced in [83, 129, 130] and proceeds very similar to the renormalization method in QFT. In particular, the metric blows up near the conformal boundary $z = 0$ so to regularize one can apply a cutoff at $z = \epsilon$, close to the boundary, i.e. $\epsilon \ll 1$. Next, the divergences can be removed by adding appropriate boundary counterterms evaluated at $z = \epsilon$. After all divergences have been removed one can take the limit $\epsilon \to 0$ to obtain the holographically renormalized result. This analysis heavily relies on the conformal structure at the boundary and the associated Fefferman-Graham expansion of fields near the boundary. This structure allows us to identify the relevant sources and vacuum expectation values for all the fields needed to properly carry out this renormalization procedure.

The precise counterterm action is far from arbitrary. First of all, the counterterms have to be boundary terms in order for them not to affect the bulk equations of motion. In addition they should be covariant under boundary coordinate transformations and should also be local functionals of the induced fields on the slice $z = \epsilon$ in order not to change the dynamical part of the on-shell action. Finally, in order to respect the variational principle all counterterms should be defined in terms of the fields themselves and should not include conjugate momenta or radial derivatives. The precise counterterm action for AlAdS spacetimes depends on the bulk theory under consideration. However, it is independent of the particular solution to the equations of motion. Given a specific theory, the counterterms are universal and make the on-shell action finite for any solution to the equations of motion. This is dual to the QFT statement that the renormalizability of the field theory is a UV property of the theory independent of the correlation functions.

5.3.3 Holographic Weyl anomaly

Consider now a $(d+1)$-dimensional gravity theory in a AlAdS space. After introducing a cut-off and adding the appropriate counterterms to cancel the divergences we are left with a renormalized on-shell action

$$S_{\text{ren.}} = \frac{1}{16\pi G_N^{d+1}} \lim_{\epsilon\to 0} \int_\epsilon^\infty \mathrm{d}z \int \mathrm{d}^d x \mathcal{L}_{\text{fin.}} \,, \tag{5.29}$$

with a finite limit as ϵ goes to zero. The variation of the renormalized Lagrangian under an infinitesimal Weyl transformation, $\delta g^{(0)} = 2\delta\sigma g^{(0)}$, is of the form

$$\delta S_{\text{ren.}} = -\int \mathrm{d}^d x \sqrt{g^{(0)}} \delta\sigma \mathcal{A} \,, \tag{5.30}$$

As was shown in [129] the quantity $\mathcal{A}$ vanishes for odd d, while in even d it reduces to [59, 86]

$$\mathcal{A} = -\frac{d L_{\text{AdS}}^{d-1}}{8\pi G_N^{d+1}} \left(E_d + \sum_n I_n \right) , \tag{5.31}$$

where E_d is the d-dimensional Euler density of the boundary and I_n are conformal invariants. If we compare this to the two-dimensional CFT result in equation (2.18) we see that the central charge of a holographic CFT is given by

$$c = \frac{3 L_{\text{AdS}}}{2 G_N^3} \,. \tag{5.32}$$

On the other hand, when comparing (5.30) to the four dimensional CFT result, we find that every 4d CFT with a weakly coupled supergravity dual has $c = a$.

5.3.4 Wilson loops

Another natural observable that appears in every gauge theory is the Wilson loop. For every closed contour C and every representation R we can define this non-local operator as the path ordered exponential of the holonomy of the gauge field around the loop C,

$$W_R(C) = \mathrm{Tr}_R \mathcal{P} \exp\left(i \int_C A^a_\mu T^a \mathrm{d}x^\mu \right), \tag{5.33}$$

where T^a are the generators of the gauge group in the representation R.

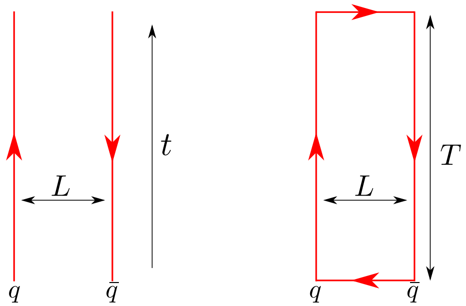


Figure 5.7: Left: A heavy quark anti-quark pair at fixed distance L. Right: The Wilson loop contour C for the computation of the quark-anti-quark potential.

Given a pure gauge theory we can add a term $\int j_\mu A^\mu$ corresponding to adding background quarks. Under a gauge transformation the quark field q transforms in the fundamental representation of the gauge group. When we translate the quark field around the closed loop C, it picks up a phase given exactly by the Wilson loop in the fundamental representation,

$$q(x + C) = W_\square(C) q(x)\,. \tag{5.34}$$

The vacuum expectation value of the Wilson loop encodes interesting information about the gauge theory. It provides an order parameter for confinement and deconfinement. As an example we consider the contour C to be a rectangle in the (t, x^1)-plane as shown in Figure 5.7. In the limit when $L \ll T$ this corresponds to a static quark–anti-quark pair at fixed distance L. For this rectangular Wilson loop only the potential energy term contributes and we have

$$W_\square(C) \propto \mathrm{e}^{-TE(L)}, \tag{5.35}$$

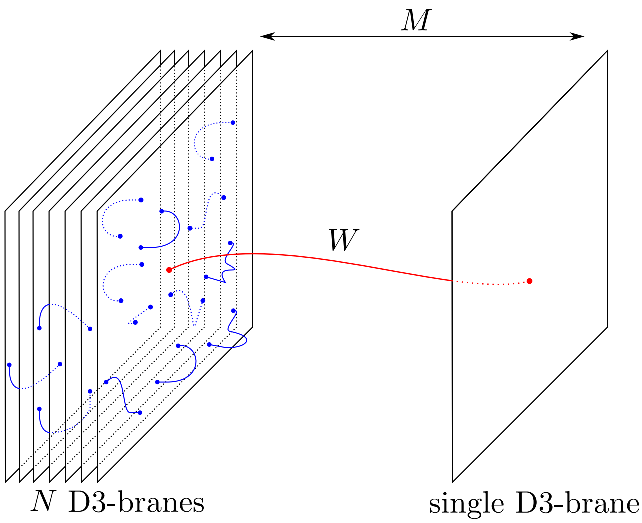


Figure 5.8: Heavy fundamental particles can be introduced in $\mathcal{N} = 4$ SYM by higgsing a $U(1)$ of the SU$(N + 1)$ gauge group. Geometrically, this corresponds to separating one brane from stack of $N + 1$ D3-branes. The separation parameterizes the mass of the fundamental particle.

where $E(L)$ is the energy of the quark–anti-quark pair at distance L. The Wilson loop vacuum expectation value can thus be used to determine the inter-quark potential and signal confinement in the gauge theory. If the Wilson line grows with the area of the loop the inter-quark energy grows linearly with the length so the theory is confining. On the other hand when the theory is asymptotically free the energy of the quark pair has the form of a Coulomb potential $E \propto 1/L$ and therefore the Wilson loop will depend on the scale invariant quantity T/L. This is the behavior expected in a conformal field theory.

But how can we describe this operator holographically? To answer this question let us first take a look at the Wilson loop operator in $\mathcal{N} = 4$ SYM. Since all the degrees of freedom in this theory are massless and transform in the adjoint representation of the gauge group we must find a natural way to introduce very heavy particles transforming in the fundamental representation. In order to do so it proves useful to go back to the worldvolume description of $\mathcal{N} = 4$ SYM as the theory living on a stack of D3-branes.

To introduce a fundamental particle consider a stack of $N + 1$ branes instead, where a single brane is separated from the N remaining ones in at least one of the six transverse directions. We can parameterize this separation by a vector Mn^i, where n^i is a unit vector indicating the direction while $M \gg 1$ is the modulus of the distance between the stack and the separated brane. This separation corresponds to giving a VEV to the six scalars $\hat{\Phi}_i$. We can view this system as an SU($N+1$) $\mathcal{N}=4$ SYM theory, where the six scalars can be expressed as

$$\hat{\Phi}^i = \begin{pmatrix} \Phi^i & W^i \\ W^{\dagger i} & Mn^i \end{pmatrix} . \tag{5.36}$$

Φ^i are the remaining massless scalars of an SU(N) theory, while the fields W and $W^\dagger$ transform in the fundamental and anti-fundamental representation of SU(N). The component Mn^i corresponds to the VEV which will, through the Higgs mechanism, give rise to masses, of the order of M, for the W and $W^\dagger$ fields. These massive fundamental particles correspond to the ground states of the open strings stretching between the stack of $D3$-branes and the single separated brane and are often called W-bosons. The trajectories of such W-bosons around a closed loop C gives rise to a phase factor, given by the vacuum expectation value of

$$W(C) = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\mathrm{exp}\left(i \oint_C \mathrm{d}s(A_\mu \dot{x}^\mu + |\dot{x}|\Phi_i n^i) \right) . \tag{5.37}$$

Next, we want to determine the corresponding observable in the dual gravity theory [172]. Let us go back to the system of $N+1$ D3-branes but now consider a closed string description of this system. As we have discussed in Section 5.2, this consists in replacing the stack of N D3-branes by their back-reaction on the geometry, i.e. AdS$_5 \times S^5$. The single separated D3-brane is located at $z=0$, on the conformal boundary of AdS$_5$. The fundamental particles now move on a closed loop C on the single D3-brane. Therefore, the expectation value of the Wilson loop operator is dual to the semi-classical partition function of a macroscopic probe string in AdS$_5 \times S^5$ whose worldsheet ends on the path C of the Wilson loop at the boundary, see Figure 5.9. The above discussion suggests that the Wilson loop should be described as the partition function of a string with boundary C,

$$\langle W(C) \rangle = \mathrm{e}^{-S_{\mathrm{string}}[C]} , \tag{5.38}$$

as we will show now.

The string has tension, so it wants to have minimum area. In flat space, this minimal surface would be entirely located on the boundary such that $W(C) = LT$, indicating a confining theory. In AdS however, things are different. Namely, in AdS the metric diverges near the boundary so it will be energetically favorable for the string to extend in the bulk. We can make this more precise by introducing the Nambu-Goto action for the string

$$S_{\mathrm{string}} = \frac{1}{2\pi\ell_s^2} \int_\Sigma \mathrm{d}\sigma \mathrm{d}\tau \sqrt{\det(g_{MN} \partial_a X^M \partial_b X_N)} \tag{5.39}$$

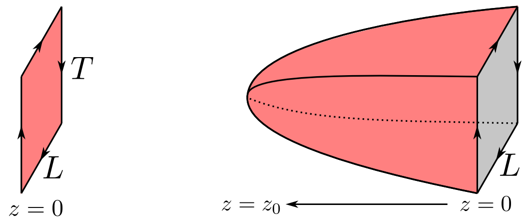


Figure 5.9: The Wilson loop contour is located on the conformal boundary at $z = 0$. A string ending on this contour will extend in the bulk spacetime and stretch down to $z = z_0$. In flat space (left) the string remains on the boundary, $z_0 = 0$, and the string worldsheet will be flat. In AdS on the other hand (right), gravity will pull the string into the bulk of AdS.

where Σ is the string worldsheet which ends on the boundary contour C and the $X_M(\sigma,\tau)$ parameterize the embedding of the string in spacetime and $a = \tau, \sigma$. For the string ending on the contour as in Figure 5.9 we can define $\tau = t$ and $\sigma = x^1$. Furthermore, we can assume that the worldsheet is translation invariant along the time direction. Since we have a static configuration we are thus left with a single variable $z(x^1) = z(x)$. The string action in terms of this variable reduces to

$$S_{\text{string}} = \frac{T L_{\text{AdS}}^2}{2\pi \ell_s^2} \int \mathrm{d}x \frac{\sqrt{1 + (\partial_x z)^2}}{z^2} . \tag{5.40}$$

Finding the minimal area is now a simple one-dimensional classical mechanics problem. Since the action does not depend on x explicitly, we immediately find that the solution satisfies

$$z^2 \sqrt{(\partial_x z)^2 + 1} = z_0^2 . \tag{5.41}$$

where z_0 is the maximal value of $z(x)$, which by symmetry occurs at $x = 0$. The solution can then be found as

$$x = \pm z_0 \int_{z/z_0}^{1} \frac{y^2 \, \mathrm{d}y}{\sqrt{1 - y^4}} , \tag{5.42}$$

where the two signs correspond to the two sides of the hanging string. To find z_0 we first note that at the boundary, $z = 0$, we have $x = \pm \frac{L}{2}$ and therefore

$$\frac{L}{2} = z_0 \int_0^1 \frac{y^2 \, \mathrm{d}y}{\sqrt{1 - y^4}} = \frac{\sqrt{2}\pi^{3/2} z_0}{\Gamma(1/4)^2} . \tag{5.43}$$

We can now compute the energy of this configuration by inserting the solution (5.42) in the string action. Noting that $z(x)$ is double valued we find,

$$S_{\text{string}} = 2\frac{TL_{\text{AdS}}^2}{2\pi\ell_s^2 z_0}\int_0^1 \frac{\mathrm{d}y}{y^2\sqrt{1-y^4}} . \tag{5.44}$$

Naively integrating this, however, results in an infinite answer. Indeed, this infinity arises from the fact that the string is stretching all the way to the boundary at $z = 0$, where the AdS metric diverges. This infinite result has a very natural interpretation in the field theory discussion and corresponds to the mass of the W-boson which corresponds to the string stretched between the separated branes. We can regularize this expression by integrating the on-shell action up to a cut-off $z = \epsilon$.

$$S_{\text{string}} = 2\frac{TL_{\text{AdS}}^2}{2\pi\ell_s^2 z_0}\int_\epsilon^1 \frac{\mathrm{d}y}{y^2\sqrt{1-y^4}} = \frac{TL_{\text{AdS}}^2}{\pi\ell_s^2 z_0}\left(-\frac{\sqrt{2}\pi^{3/2}}{\Gamma\left(\frac{1}{4}\right)^2} + \frac{z_0}{\epsilon}\right). \tag{5.45}$$

By adding a counterterm

$$\mathcal{L}_{ct} = -lM \approx -\frac{2T}{\epsilon} , \tag{5.46}$$

we can subtract the diverging term. Here $l \approx 2T$ is the length of the Wilson loop and $M \propto 1/\epsilon$ represents the diverging mass. After Subtracting the diverging contribution we find

$$S_{\text{string}} = -\frac{\sqrt{2}\pi^{3/2}TL_{\text{AdS}}^2}{\pi\ell_s^2 z_0\Gamma\left(\frac{1}{4}\right)^2} . \tag{5.47}$$

This action now should correspond to $TE(L)$ where $E(L)$ represents the inter-quark potential,

$$E(L) \propto \frac{T}{z_0} \propto \frac{T}{L} . \tag{5.48}$$

As predicted, we see that the holographic Wilson loop, in type IIB string theory in $\text{AdS}_5 \times S^5$, exhibits the behavior of a Wilson loop in a conformal field theory, such as $\mathcal{N} = 4$ SYM. In part II we will discuss further examples that are no longer conformal and holographically realize confinement. In this example we considered a square contour, however, this analysis can easily be generalized to more general contours by arranging infinitely thin squares next to each other, see Figure 5.10.

5.4 Non-conformal theories and RG flows

Holography works very well for conformal theories, however, realistic gauge theories are often not conformal. For example, pure $\mathcal{N} = 1$ or non-supersymmetric SU(N) Yang-Mills theory confines, has a mass gap and a discrete spectrum of glueballs. To

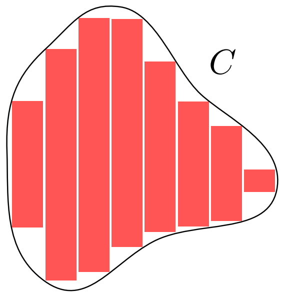


Figure 5.10: One can approximate a loop C to arbitrary accuracy by thin rectangles.

finish this chapter, we will briefly introduce some simple and universal properties of non-conformal gauge theories and RG flows from a holographic viewpoint. In our examples so far we focused largely on pure AdS spaces or zoomed in on the asymptotic structure of the solutions only. However, as already alluded to in Section 5.1.3 we can interpret the radial coordinate of AdS as energy scale and the radial evolution as an RG flow in the dual field theory.

One way to obtain a non-conformal theory, which we already encountered before is by turning on sources in the CFT. If the source is associated to a relevant operator, this induces an RG flow to some other fixed point. Another way to break conformal invariance, which is available when the CFT has a moduli space of vacua, is to turn on a VEV for some operator without sources.

Let us for now restrict to examples in which the bulk spacetime is asymptotic to AdS space. The metric is still Poincaré invariant and we impose the following domain wall ansatz

$$\mathrm{d}s^2 = \mathrm{d}y^2 + \mathrm{e}^{2A(y)}\mathrm{d}x_\mu \mathrm{d}x^\mu \,, \tag{5.49}$$

where $\mathrm{e}^{2A(y)}$ is the warp factor, which at the boundary ($y \to \infty$) reduces to[9]

$$A(y) \to \frac{y}{L_{\text{AdS}}}\,, \qquad y \to \infty \,. \tag{5.50}$$

To make this more concrete, let us start with a simple $(d+1)$-dimensional gravitational theory coupled to a number of scalar fields described by the action

$$S = \int \mathrm{d}^{d+1}x\sqrt{-g}\left(\frac{R}{4} - \frac{1}{2}G_{ab}\partial_\mu\Phi^a\partial^\mu\Phi^b - V(\Phi)\right) . \tag{5.51}$$

where G_{ab} is a matrix. This theory could for example arise as the bosonic part of the

[9]This is the metric of anti-de Sitter space in the Poincaré patch (3.9) with radial coordinate $\frac{y}{L_{\text{AdS}}} = \log\frac{L_{\text{AdS}}}{z}$. In these coordinates the conformal boundary of AdS is located at $y = \infty$.

dimensional reduction of some higher-dimensional supergravity theory. Consider now domain wall solutions to the equations of motion of this system such as (5.49). When $A(y) = \frac{y}{L_{\text{AdS}}}$ we recover AdS_{d+1} with AdS length L_{AdS}. An obvious solution to the equations of motion is then given by the critical points of the scalar potential $V(\Phi)$, if the potential at the critical point is negative. In this case we can set all scalar fields to be constant

$$\frac{\partial V}{\partial \Phi_a} = 0\,, \qquad \frac{\partial \Phi_a}{\partial y} = 0\,, \qquad \frac{\partial (A^2)}{\partial y} = -\frac{V_{\text{crit.}}}{d-1}\,. \tag{5.52}$$

As a solution we find AdS space with the AdS length determined by the critical point of the potential, $L^2_{\text{AdS}} = -(d-1)/V_{\text{crit.}}$. From our general discussion we expect these solutions to be described by a dual CFT.

However, we can also consider more general solutions which start from a critical point, say at $\Phi_a = 0$ and have a nontrivial profile for the scalars in the bulk. By expanding the action around $\Phi_a = 0$ we can find the masses m_a of the scalars from which we can determine the dimensions of the dual operators $\mathcal{O}_a$

$$m_a^2 L^2_{\text{AdS}} = \Delta_a(\Delta_a - d)\,. \tag{5.53}$$

Since asymptotically the solutions for the scalar behave as

$$\Phi_a(y) \simeq s_a \mathrm{e}^{(\Delta_a - d)y} + v_a \mathrm{e}^{-\Delta_a y} \tag{5.54}$$

we can associate s_a and v_a respectively with a source and a vacuum expectation value in the dual UV CFT. When $s_a \neq 0$ for one of the scalars, this solution describes a deformation of the dual CFT by an operator $\mathcal{O}_a$,

$$\mathcal{L}_{\text{CFT}} \to \mathcal{L}_{\text{CFT}} + \int \mathrm{d}^d x s_a \mathcal{O}_a\,. \tag{5.55}$$

On the other hand, when all sources s_a vanish the solutions describe a different vacuum of the CFT where $\langle \mathcal{O}_a \rangle = v_a$. In both cases however, the UV CFT is deformed and the conformal invariance is broken. This deformation triggers an RG flow in the CFT to which the domain wall gravity solution is dual. The profile of the scalars and the warp factor A describe the running of the deformation parameters. For generic values of the VEVs and sources in equation (5.54), these solutions are singular in the IR. These singularities reflect the fact that the dual field theory for these values of v_a and s_a flow towards a free or gapped theory. However, as we will explore in more depth in part II, by finely tuning the sources and vacuum expectation values one can nonetheless land on a regular IR fixed point. Particularly interesting is the case when the original CFT is perturbed by a relevant operator, $\Delta_a < d$ and the QFT flows to another fixed point of the potential. In this case the gravity solution describes a kink interpolating between two fixed points.

The search for such domain wall solutions is often simplified in the presence of supersymmetry. In that case the potential can usually be rewritten in terms of a superpotential and the second order equations of motion consequently reduce to first order BPS equations. In part II and III of this thesis we will discuss particular examples of holographic RG flows and study their strongly coupled dual QFTs. However, in both parts the RG flows will differ from the basic ones discussed here in various aspects. In part II we will study QFTs on a curved manifold which do not necessarily have a UV CFT as the fixed point of the RG flow. In part III on the other hand, both fixed points of the flow will be CFTs but the dimension of the IR and UV CFT will not be the same.

Part II

Spherical branes, localization and holography

Chapter 6

Introduction to part II

This second part is adapted from [50, 52].

Brane solutions in supergravity have offered multiple important insights into the structure of string theory, supergravity, and holography. They were first constructed as extremal black branes in ten- and eleven-dimensional supergravity which preserve half of the maximal supersymmetry [135]. An important insight came from the realization that they should be thought of as sourced by the D- and M-branes of string and M-theory [209], based on the earlier work [82, 165]. The dichotomy between the gauge theory living on the worldvolume of the D-branes and their back-reacted p-brane solutions ultimately led to the development of the gauge/gravity duality [120, 173, 250] and to important insights into black hole physics in string theory [227].

6.1 Curved branes

In the standard treatment of Dp-brane supergravity solutions the world-volume of the brane is $(p+1)$-dimensional flat space, $\mathbf{R}^{1,p}$. For general values of p the near-horizon limit of the supergravity background exhibits a singularity where the running dilaton diverges. This bodes well for the holographic interpretation of these backgrounds as dual to $(p+1)$-dimensional maximally supersymmetric Yang-Mills (SYM) theory on $\mathbf{R}^{1,p}$. For general values of p this theory is not conformal and it is expected that the weakly curved region of the supergravity solution is dual to the regime of strong gauge coupling while for small values of the running coupling the supergravity description is not valid and the background develops a singularity [146].

Given the importance of flat Dp-branes and their supergravity description it is natural to explore the more general situation when the worldvolume of the brane is curved. Since supersymmetry offers a great deal of computational control which often elucidates the underlying physics, one is led to look for curved supersymmetric Dp-branes. Indeed, this question was addressed in [47] where Dp-branes with worldvolumes of the form $\mathbf{R}^{1,m} \times \mathcal{M}_q$, with $m+q=p$ and $\mathcal{M}_q$ a general Euclidean manifold, were studied. To preserve supersymmetry the worldvolume theory on the brane is partially topologically twisted on $\mathcal{M}_q$ [243]. This setup has a beautiful extension into the arena of holography as emphasized in [174]. However it is well-known that the topological twist is not the only way by which a supersymmetric gauge theory can be placed on a curved manifold, see for example [48,93,207,208]. A particularly simple example of a curved manifold on which a SYM theory can be placed in a supersymmetric way is offered by the sphere equipped with an Einstein metric [48]. Thus it is natural to ask whether this gauge theory construction admits a realization in string theory on the worldvolume of spherical Dp-branes and how to construct the supergravity solutions describing the back-reaction of these branes. The goal of this part the of the thesis is to address this question from the point of view of supergravity and holography.

The spherical Dp-brane solutions can also be interpreted from a different vantage point. It is standard in the context of non-conformal holography to encounter supergravity solutions that exhibit IR singularities, see for example [60, 97, 109, 119, 146, 158]. Whenever these singularities are physically acceptable they are interpreted in the dual field theory as arising from a free or gapped phase of the IR dynamics [119]. The singularity is usually remedied by replacing the singular background by a black hole solution with the same asymptotics and a regular horizon. In the dual gauge theory this corresponds to turning on finite temperature, which in turn introduces a finite IR cut-off in the gauge theory. In the context of the flat Dp-brane solutions this is discussed in some detail in [146] and summarized in Appendix E. Our spherical brane solutions provide an alternative way to excise the singularity of flat Dp-brane supergravity backgrounds. Due to the finite length scale introduced by the sphere one finds a smooth cap-off of the metric instead of a singularity in the IR region of the geometry. We interpret this as a gravitational manifestation of the the IR cut-off for the gauge theory on S^{p+1}. The difference with the more common finite temperature cut-off is that spherical branes preserve sixteen supercharges which provides better computational control. We believe that this is the unique IR cut-off compatible with the maximal number of supercharges for a non-conformal SYM theory.

6.2 Spherical branes and SYM

Our approach to construct the supergravity solutions describing spherical Dp-branes is informed by the knowledge of the Lagrangian of the maximally supersymmetric

Yang-Mills theory on S^{p+1} for $p \leq 6$ [48, 186]. For general values of $p \neq 3$ the maximal SYM theory is not conformal and coupling it to the curvature of the sphere while preserving sixteen supercharges necessitates certain couplings in the Lagrangian. These couplings in turn break the R-symmetry of the SYM theory from $SO(1, 8-p)$ to $SO(1, 2) \times SO(6-p)$.[1] It is natural to assume that the worldvolume theory for spherical Dp-branes at low energies is the same as this maximal SYM on S^{p+1}. The symmetry breaking pattern combined with the presence of sixteen supercharges then leads to a very restrictive ansatz for the type II supergravity backgrounds describing the spherical branes. Nevertheless it is still difficult to solve the supergravity BPS equations and find the explicit solutions directly in ten dimensions. We circumvent this impasse by employing the well-known technique of reducing the ten-dimensional supergravity theory to an effective gauged supergravity in $p+2$ dimensions. The spherical brane solutions of interest are then found as supersymmetric domain walls in this gauged supergravity with non-trivial profiles for the metric as well as three scalar fields.[2] These scalar fields are the supergravity manifestation of the running gauge coupling of SYM theory and the couplings in the Lagrangian on S^{p+1} that need to be turned on to preserve supersymmetry. Working with this $(p+2)$-dimensional gauged supergravity we are able to construct explicitly the supersymmetric spherical domain wall solutions of interest and then use standard uplift formulae from the literature to convert them to solutions of type II* supergravity.

Our spherical brane solutions exhibit some common features which are in harmony with the physics of the SYM theory. In the IR region of the geometry the solution is regular and the radial coordinate combines with the metric on S^{p+1} to produce a smooth cap-off that locally looks like $\mathbf{R}^{p+2}$. This behavior reflects the fact that in the dual SYM theory the length scale associated with the sphere provides and IR cut-off for the dynamics and one cannot probe energies smaller than this scale. This type of smooth cap-off of supergravity solutions dual to non-conformal gauge theories on a sphere is a familiar feature from recent holographic studies of mass deformations of three-dimensional ABJM, four-dimensional $\mathcal{N} = 4$, and five-dimensional D4-D8 SCFTs [54, 55, 57, 98, 123]. In the UV region of the spherical brane solutions the background is the same as the near-horizon limit of the usual flat Dp-brane backgrounds [135], albeit with Euclidean worldvolume. This behavior is also in line with the dual gauge theory where at high energies the radius of the sphere should not affect the dynamics and one expects to recover the physics of SYM theory in flat space.

[1] The R-symmetry group is non-compact due the fact that the SYM theory is defined in Euclidean signature.

[2] For $p = 6$ there are only two scalar fields needed in the eight-dimensional supergravity theory. This is related to the fact that the R-symmetry in this case remains unbroken upon putting the theory on a sphere.

6.3 Supersymmetric localisation

Supersymmetric localization is a powerful tool to study the dynamics of strongly coupled supersymmetric QFTs which has been efficiently exploited in a variety of examples [208]. We can use this tool to study the correspondence between gauge theories and their gravity duals. In many situations the calculation of supersymmetric observables in the field theory reduces to an evaluation of a matrix integral which can then be studied in the planar limit using saddle point techniques. In the cases when the supersymmetric theory has a known gravitational dual this provides a fruitful avenue to quantitatively test the details of the AdS/CFT correspondence.

It is natural to consider questions on the interface of holography and supersymmetric localization for conformal theories with maximal supersymmetry, like four-dimensional $\mathcal{N} = 4$ SYM and the three-dimensional ABJM theory, on the round sphere. Indeed this was pursued extensively and many important developments are summarized in [208]. These two examples also offer the possibility to break conformal invariance and part of the supersymmetry while still maintaining computational control both in the field theory [65, 217–219] and the supergravity side [54, 55, 57, 58, 98, 155–157]. This collection of results provides a non-trivial precision test of holography away from the conformal limit. Our goal is to extend this success to maximally supersymmetric Yang-Mills on the d-dimensional round sphere, S^d.

Supersymmetric localization reduces the path integral of the theory to an ordinary matrix integral. Despite this drastic simplification the explicit evaluation of this integral is still non-trivial due to the presence of non-perturbative effects like instantons. However, when the rank of the gauge group is large it is believed that these non-perturbative effects are suppressed and the matrix integral becomes more tractable. As we discuss in detail below, for all values of d it is possible to compute the free energy and the vacuum expectation values (VEV) of a supersymmetric Wilson loop using this matrix model.[3] A further simplification occurs in the limit where the dimensionless 't Hooft coupling, defined as

$$\lambda \equiv \mathcal{R}^{4-d} g_{\text{YM}}^2 N \,, \tag{6.1}$$

where $\mathcal{R}$ is the radius of S^d, is large. In this case the results can be written in analytic form and can be formally analytically continued even to non-integer values of d.

[3] See [108] for calculations of the free energy on S^d of QFTs without gauge fields.

6.4 Non-conformal precision holography

To study these MSYM theories on S^d we will use their dual formulation as spherical branes.[4] Equipped with these supergravity backgrounds we can apply the tools of holography and compute the free energy and Wilson loop VEV at large λ. As introduced in Chapter 5, the holographic free energy is calculated by evaluating the on-shell action of the supergravity solution while the Wilson loop VEV is computed by first finding an appropriately embedded probe string and then computing the Nambu-Goto action on-shell. Both of these calculations can be performed explicitly and the results are in agreement with the ones obtained by supersymmetric localization.

The analysis on the gravity side for all $d \neq 4$ goes beyond the realm of the usual holographic dictionary. The spherical brane solutions for $d \neq 4$ are not asymptotically locally AdS and therefore there is no generally established holographic renormalization procedure. Despite this obstacle we are able to adapt the results in [149, 150] to our setting and construct appropriate counterterms in supergravity which lead to a finite on-shell action for the spherical brane backgrounds and the probe strings. The approach of [149, 150] is however not applicable for $d = 6$ due to the linear dilaton characteristic of the little string theory. Inspired by the regularization procedure in the matrix model analysis and the results in [7, 77, 176] we are able to propose a way to cancel the divergences appearing in the spherical D5-brane solution and obtain an agreement with the results from supersymmetric localization.

6.5 Outline

In Chapter 7, we start with a discussion of maximally supersymmetric Yang-Mills theory on S^{p+1}. We explain how one can put this theory on a sphere, and we use supersymmetric localization to compute the free energy and $1/2$-BPS Wilson loop vacuum expectation value. Next, in Chapter 8 we present the dual spherical brane solutions in a unified manner and show how they can be obtained by uplifting supersymmetric domain wall solutions of lower-dimensional gauged supergravity. Finally, in Chapter 9, we show how the usual holographic renormalization procedure can be generalized to apply to our solutions and subsequently compute the holographic free energy and Wilson loop VEVs for $2 \leq d \leq 7$. In the appendices A-G we clarify our conventions and elaborate on many technical results used throughout this part.

[4] See also [179, 187, 188, 234] for other constructions of supersymmetric solutions sourced by curved Euclidean branes.

Chapter 7

SYM on a sphere

An important guiding principle for constructing supersymmetric spherical Dp-brane solutions is the fact that the low-energy dynamics on the worldvolume of D-branes in flat space is governed by a maximally supersymmetric Yang-Mills theory. Thus, it is natural to expect that for spherical Dp-branes the low-energy physics is the same as that of maximal SYM theory on S^{p+1}. Since for general values of p maximal SYM is not conformal it is non-trivial to couple it to the curvature of the sphere. Therefore, let us briefly review the construction of the Lagrangian of maximal SYM on S^{p+1}.

7.1 Maximal SYM on a sphere

Maximally supersymmetric Yang-Mills theory in $d = p + 1$ dimensions has 16 real supercharges and consists of a vector multiplet transforming in the adjoint representation of the gauge group G. The fields in this multiplet are the gauge field A_μ, $9 - p$ real scalar fields, Φ_m, and 16 fermionic degrees of freedom, or gaugini, collectively denoted by Ψ. Depending on the dimension and signature of spacetime the fermionic degrees of freedom are arranged into spacetime spinors as summarized in Table 7.1.

The index m, which labels the scalar fields, transforms in the fundamental representation of the R-symmetry group, which is SO($9-p$) for Lorentzian[1] theories and SO($1, 8 - p$) for the Euclidean ones. We are mostly interested in Euclidean theories, as we intend to study SYM on S^{p+1}, but for the moment we keep the discussion general and discuss both cases.

[1] We work with a "mostly +" signature.

Dimensions	Lorentzian	Euclidean
7	1	2 (Majorana)
6	2 (Weyl)	2 (Majorana)
5	2	2
4	4 (Majorana)	4 (Weyl)
3	8 (Majorana)	4
2	16 (Majorana-Weyl)	8 (Majorana)

Table 7.1: Number of minimal spinors in each dimension used in Lorentzian and Euclidean field theories. The conditions the spinors satisfy are indicated in brackets. In all cases we denote the collective 16-component fermion with the symbol Ψ.

The classical action for the $(p+1)$-dimensional maximal SYM theory on flat space can be derived by dimensionally reducing the unique SYM action in ten dimensions [64],

$$\mathcal{L}_{\rm SYM}^{10d} = \frac{1}{2g_{\rm YM}^2}\mathrm{Tr}\Big[-F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}\Gamma^\mu D_\mu \Psi\Big]\,. \tag{7.1}$$

Here $\bar{\Psi} = \Psi^\dagger\Gamma_0$ is the Dirac adjoint with Γ_0 the ten-dimensional gamma matrix along the time direction. Note that to obtain a lower-dimensional Euclidean theory one must perform a timelike dimensional reduction. Explicitly the Lagrangian of the d-dimensional SYM theory on flat space reads[2]

$$\begin{aligned}\mathcal{L}_{\rm SYM} = \frac{1}{2g_{\rm YM}^2}\mathrm{Tr}\Big[& -F_{\mu\nu}F^{\mu\nu} - D_\mu\Phi_m D^\mu\Phi^m + \bar{\Psi}\gamma^\mu D_\mu\Psi \\ & - \frac{1}{2}[\Phi_m,\Phi_n][\Phi^m,\Phi^n] + \bar{\Psi}\Gamma^m[\Phi_m,\Psi]\Big]\,. \end{aligned}\tag{7.2}$$

In our conventions the $(9-p)$-dimensional internal gamma matrices are denoted by Γ_m and the $(p+1)$-dimensional spacetime ones are γ_μ. The Clifford algebra is $\{\gamma_\mu,\gamma_\nu\} = 2g_{\mu\nu}$, with $g_{\mu\nu}$ the spacetime metric. The Yang-Mills field strength is given by $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} + [A_\mu, A_\nu]$, and the gauge covariant derivatives are

$$D_\mu\Phi_m = \partial_\mu\Phi_m + [A_\mu,\Phi_m]\,, \qquad D_\mu\Psi = \partial_\mu\Psi + [A_\mu,\Psi]\,. \tag{7.3}$$

The m,n indices are raised and lowered with the flat metric on $\mathbf{R}^{9-p}$ for Lorentzian theories and $\mathbf{R}^{1,8-p}$ for Euclidean theories. For Euclidean theories this implies that one of the scalar fields, Φ^0, has the 'wrong sign' kinetic term.[3] Notice also that the

[2]Since we are interested in QFTs we take $p \geq 1$. We also set the θ-term in the four-dimensional SYM action to zero.

[3]At first sight one might worry that the energy is not bounded from below. However, as the sphere does not have a preferred Killing vector, there is no natural notion of a Hamiltonian in this theory and therefore this 'wrong sign' kinetic term does not cause any trouble.

scalar-fermion interaction term involves internal gamma matrices, Γ^m, associated with the R-symmetry.

The Yang-Mills coupling constant, $g_{\rm YM}$, is dimensionful for $d = p+1 \neq 4$. Its mass dimension is given by $[g_{\rm YM}^2] = 3-p$. This means that maximal SYM theories are non-renormalizable for $p > 3$ and have to be incorporated in an appropriate UV complete theory at high energies. Indeed, maximal SYM in five dimensions, i.e. $p = 4$, is conjectured to grow an extra dimension at high energies and flow towards the (2,0) CFT in six dimensions [88, 164]. Maximal SYM in six dimensions is expected to be UV completed by a non-gravitational but non-local theory called little string theory. Little string theory comes in two flavors, depending on the chirality of the supercharges in six dimensions. Six-dimensional SYM has $(1,1)$ supersymmetry and therefore flows towards the corresponding $(1,1)$ little string theory in the UV, see [4, 163] for reviews and further references. For $p > 5$ maximal SYM has a UV completion within string theory as the worldvolume theory on Dp-branes. For $p \leq 3$ the UV physics is under better control. When $p = 3$ it is well-known that maximal SYM is conformally invariant and thus UV finite. For $p < 3$ the YM coupling is asymptotically free and the physics at low-energies is strongly coupled. The theory in three dimensions, i.e. $p = 2$, is believed to flow to the interacting ABJM CFT which has maximal supersymmetry and describes the low-energy dynamics of M2-branes [5]. The IR dynamics of the maximal two-dimensional SYM is somewhat more involved, see [160] for a recent discussion.

Placing SYM on curved backgrounds such as S^{p+1} via a minimal coupling, i.e. replacing $\eta_{\mu\nu}$ with $g_{\mu\nu}$ and partial with covariant derivatives in (7.2), results in an action that in general does not preserve any supersymmetry. This is because constant supersymmetry transformation parameters, ϵ, do not exist on a general curved manifold. The supersymmetry transformation of the action is proportional to the derivative of ϵ which in general does not vanish. Understanding which supersymmetric QFTs can be placed on which curved manifolds while preserving some amount of supersymmetry can be done systematically using the formalism described in [93]. For maximal SYM on S^{p+1} this question was addressed in the earlier work [48], see also [186, 207].

In this paper we are interested in the maximal SYM theory placed on S^{p+1} with metric $\mathcal{R}^2 \mathrm{d}\Omega_{p+1}^2$ where $\mathcal{R}$ is the radius of the sphere and $\mathrm{d}\Omega_{p+1}^2$ is the unit radius Einstein metric on S^{p+1}.[4] It was shown in [48] that these Euclidean theories can preserve 16 real supercharges and the supersymmetry parameter obeys the equation

$$\nabla_\mu \epsilon = \frac{1}{2\mathcal{R}} \gamma_\mu \Lambda \epsilon \quad \text{with} \quad \Lambda = \Gamma^{012}, \tag{7.4}$$

This construction requires at least three internal Γ-matrices and we are therefore

[4]The Ricci scalar of the sphere with metric $\mathcal{R}^2 \mathrm{d}\Omega_{p+1}^2$ is equal to $\frac{p(p+1)}{\mathcal{R}^2}$.

restricted to $p \leq 6$. This is closely related to the fact that superconformal algebras only exist in six or fewer dimensions. The action for maximal SYM on S^{p+1} is explicitly given as a deformation of the action in (7.2). First we have to introduce a minimal coupling of the Lagrangian in (7.2) to the metric on the sphere. In addition, to ensure that the the supersymmetry generated by the spinor in (7.4) is preserved, we have to add the following terms to the Lagrangian

$$\begin{aligned} \delta\mathcal{L} = &-\frac{1}{\mathcal{R}^2}\mathrm{Tr}\left[(p-1)\Phi_m\Phi^m + (p-3)\Phi_a\Phi^a\right] \\ &+ \frac{1}{2\mathcal{R}}(p-3)\mathrm{Tr}\left[\bar{\Psi}\Lambda\Psi - 8\Phi_0[\Phi_1,\Phi_2]\right] , \end{aligned} \tag{7.5}$$

where the index a only runs from 0 to 2 and is contracted using the Lorentzian metric just like the m, n indices [48]. The Lagrangian in (7.5) for $p \notin \{3,6\}$ breaks the $\mathrm{SO}(1,8-p)$ R-symmetry group of the maximal theory in Euclidean space to the subgroup[5]

$$\mathrm{SU}(1,1) \times \mathrm{SO}(6-p) \, . \tag{7.6}$$

This symmetry breaking pattern is an important guiding principle for constructing the spherical brane solutions of ten-dimensional supergravity. The full superalgebras realized by these theories are given in Table 7.2 and are equal to a superconformal algebra in p Euclidean dimensions with eight real supercharges Q and eight superconformal charges S. The lack of superconformal algebras in more then six dimensions gives an additional reason why we are limited to $p \leq 6$.

p	$\mathfrak{g}$
1	$\mathfrak{osp}(4\|2)$
2	$D(2\|1;\alpha) \ltimes D(2\|1;\alpha)$
3	$\mathfrak{psu}(4\|4)$
4	$\mathfrak{su}(4\|2)$
5	$\mathfrak{f}_4$
6	$\mathfrak{osp}(8\|1,\mathbf{R})$

Table 7.2: The superalgebras corresponding to $(p+1)$-dimensional MSYM theories. For all $p \neq 2$ these are the unique choices consistent with the symmetries of SYM on a sphere. For $p = 2$ there is an infinite family of possibilities parameterized by $\alpha \in \mathbf{C}\backslash\{0,-1\}$. To determine the precise value of α one should match the full algebra.

There are at least three important reasons to study SYM theories on S^{p+1}. First, this is a maximally symmetric space which is also the unique curved manifold on which one can preserve 16 supercharges. Placing a supersymmetric theory on a sphere is an essential ingredient in the context of supersymmetric localization and indeed it

[5]For $p = 6$ the R-symmetry group is $\mathrm{SO}(1,2) \simeq \mathrm{SU}(1,1)$ and is preserved by the Lagrangian in (7.5).

was recently shown in [186], following the seminal work [207], how to study the partition function of maximal SYM using this method for any $1 \leq p \leq 6$. This in turn paves the way to compute exactly certain supersymmetric correlation functions of the SYM theory. Finally, the radius, $\mathcal{R}$, of the sphere provides a natural IR cut-off for the dynamics of the SYM theory which is compatible with supersymmetry. This is especially important in the holographic context where the IR physics of SYM theories for $p \neq 3$ results in singularities of the dual supergravity solutions. Alternatively, these can be resolved by introducing finite temperature in the form of a black hole horizon [146], see Appendix E for more details and a comparison to the spherical branes. The finite temperature is a convenient IR regulator which, however, breaks supersymmetry completely. As we show below, our spherical Dp-brane solutions, which are holographically dual to the maximal SYM on S^{p+1}, are regular in the IR while preserving all 16 supercharges.

7.2 Supersymmetric localization

Now that we have formulated a Lagrangian for maximal SYM, the full information on these quantum field theories is encoded in various path integrals. These are infinite-dimensional integrals on the space of Euclidean field configurations with a weight $\exp(-S/\hbar)$ determined by the Euclidean action. Ideally, we could obtain any correlation function by computing such path integrals. However, in practice this is very hard because of the infinite range of integration. In our first course on quantum field theory we learn how to compute path integrals perturbatively using Feynman calculus. This method is however only valid at weak coupling and even if we are able to compute the full perturbative series the result typically is only an asymptotic series with vanishing radius of convergence. Only after accounting for various non-perturbative corrections we can obtain well-defined results for finite coupling.

In light of these limitations it is tempting to look for situations where the path integral can be computed exactly. Apart from free theories where the path integral is a Gaussian integral, the only known examples until recently were topological and cohomological theories defined on compact manifolds [243, 245]. However, in 2007 a wealth of new exact results emerged in the context of supersymmetric QFTs on curved backgrounds following the work of Pestun [207]. The key technique that allowed both for the computation of path integrals in topologically twisted theories as well as more general rigid supersymmetric theories on curved spaces is called supersymmetric localization. This technique relies on supersymmetry to prove that the path integral only receives contributions from the locus of fixed points of the supersymmetry transformations. Supersymmetric localization is a natural generalization of equivariant localization.

The power of localization is that it reduces the infinite-dimensional path integral of a D dimensional theory to a path integral of a lower dimensional field theory. A particularly favorable situation arises when the localization locus consists of constant field configurations only and consequently reduces the full path integral to a finite-dimensional matrix integral. Physically the localization formulae can be viewed as instances where the semi-classical one-loop (WKB) approximation is exact. A crucial point is that this saddle point approximation is not for the original action with parameter $\hbar$ but rather for a modified action with a deformation term weighted by an auxiliary parameter t which does not change the result of the path integral.

In the remainder of this section we present a lightning introduction to supersymmetric localization from a physicists point of view. For a more complete and rigorous review we refer the reader to [208].

Let us start by considering a supersymmetric quantum field theory. In such theories there exists a conserved fermionic supercharge, Q, generating supersymmetry transformations, which squares to a bosonic charge B,

$$Q^2 = B\,, \tag{7.7}$$

which may generate a linear combination of spacetime symmetries, global internal symmetries or gauge symmetries. We can divide the fields in our theory into bosonic and fermionic fields, or equivalently Grassmann even and odd fields. The objects we will be able to compute using localization are supersymmetric or BPS observables $\mathcal{O}_{\text{BPS}}$ which are defined by being annihilated by the supercharge Q, and therefore by B [6]

$$Q\mathcal{O}_{\text{BPS}} = 0\,. \tag{7.8}$$

$\mathcal{O}_{\text{BPS}}$ may be a local operator or a combination of local operators and more generally it can also be a non-local operator such as a supersymmetric Wilson or 't Hooft loop or a surface operator for example. Our present aim is to compute the expectation value $\langle\mathcal{O}_{\text{BPS}}\rangle$ of such operators exactly using supersymmetric localization. Written as a path integral, this expectation value is given by

$$\langle\mathcal{O}_{\text{BPS}}\rangle = \int [\mathcal{D}X]\, \mathcal{O}_{\text{BPS}} \mathrm{e}^{-S[X]}\,. \tag{7.9}$$

First of all, by Stokes' theorem, the expectation value of a Q-exact observable vanishes

$$\langle Q\mathcal{O}\rangle = \int [\mathcal{D}X]\, Q\mathcal{O}\, \mathrm{e}^{-S[X]} = \int [\mathcal{D}X]\, Q\left(\mathcal{O}\mathrm{e}^{-S[X]}\right) = 0\,, \tag{7.10}$$

where we have used the fact that the action of a supersymmetric theory is invariant

[6]Here and in the following $Q\mathcal{O}$ represents the adjoint action of Q on an operator, i.e. $Q\mathcal{O} \equiv [Q,\mathcal{O}]$.

under Q, i.e. $QS = 0$. We thus end up with an integral of a total derivative in field space, which vanishes provided there are no boundary terms. In the following we always assume that the integrand decays fast enough so that we can ignore all possible boundary terms. As a result, path integrals of BPS observables depend only on the Q-cohomology class of the inserted operator,

$$\langle \mathcal{O}_{\text{BPS}} \rangle = \langle \mathcal{O}_{\text{BPS}} + Q\mathcal{O} \rangle \,. \tag{7.11}$$

Next, we show how these path integrals of BPS observables localize to the BPS locus $\mathcal{M}_Q$ of Q-supersymmetric field configurations. In the following we require that the path integral is well-defined and free of IR divergences. This is guaranteed by placing the supersymmetric theory on a compact manifold or Ω-background [198]. Since we have shown that the expectation value $\mathcal{O}_{\text{BPS}}$ only depends on the Q-cohomology class we can consider a different Q-cohomology representative obtained by adding the Q-variation of a B-invariant fermionic functional V to the action.

$$\langle \mathcal{O}_{\text{BPS}} \rangle = \int [\mathcal{D}X]\, \mathcal{O}_{\text{BPS}} \mathrm{e}^{-S[X]-tQV} \,. \tag{7.12}$$

The B-invariance of the operator V ensures that the deformed observable is Q-cohomologous to the original one. This equality is valid for all t and all functionals V that do not change the asymptotics of the integrand. It is easy to see that indeed the right hand side of (7.12) is independent of t,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int [\mathcal{D}X]\, \mathcal{O}_{\text{BPS}} \mathrm{e}^{-S[X]-tQV} &= - \int [\mathcal{D}X]\, (QV) \mathcal{O}_{\text{BPS}} \mathrm{e}^{-S[X]-tQV} \\ &= - \int [\mathcal{D}X]\, Q\left(V \mathcal{O}_{\text{BPS}} \mathrm{e}^{-S[X]-tQV}\right) \\ &= 0 \,. \end{aligned} \tag{7.13}$$

Assuming that the deformation term QV is positive (semi-)definite we can evaluate the expectation value $\langle \mathcal{O}_{\text{BPS}} \rangle$ by taking the $t \to \infty$ limit in (7.12). In this limit the integrand is dominated by the saddle points of the localizing action $S_{\text{loc}} = QV$. The canonical choice of localizing Lagrangian is

$$\mathcal{L}_{\text{loc}} = Q \sum_{\Psi} \left((Q\Psi)^\dagger \Psi + \Psi^\dagger (Q\Psi^\dagger)^\dagger \right) , \tag{7.14}$$

where the sum runs over all fermions in the theory. In this case, the saddle points of the localizing action are given precisely by the BPS configurations

$$\Psi = \Psi^\dagger = 0 \,, \qquad Q\Psi = Q\Psi^\dagger = 0 \,. \tag{7.15}$$

In the following we denote this BPS locus by X_0. To evaluate the deformed path integral (7.12) we first expand the fields, collectively denoted by X, around the saddle point configurations

$$X = X_0 + \frac{1}{\sqrt{t}}\delta X \tag{7.16}$$

and take the limit $t \to \infty$. The semi-classical expansion of the action in $\frac{1}{t}$ reduces to

$$S[X] = S[X_0] + \frac{1}{2}\int \left.\frac{\delta^2 S_{\text{loc}}}{\delta X^2}\right|_{X=X_0} \delta X^2 \tag{7.17}$$

which is exact, since higher order terms in the expansion involve negative powers of t and thus vanish in the $t \to \infty$ limit. Integrating out the fluctuations δX normal to the localization locus we obtain schematically the localization formula

$$\langle \mathcal{O}_{\text{BPS}} \rangle = \int [\mathcal{D}X_0]\, \mathcal{O}_{\text{BPS}}\big|_{X=X_0} \mathrm{e}^{-S[X_0]} \text{SDet}\left[\frac{\delta^2 S_{\text{loc}}[X_0]}{\delta X_0^2}\right]^{-1} + \cdots, \tag{7.18}$$

where SDet is the superdeterminant or Berezinian, a generalization of the determinant to supermatrices. The original path integral has localized to a lower-dimensional integral over the BPS locus where in addition to the classical action on the BPS locus there is a one-loop correction due to integration out the field fluctuations δX given by the superdeterminant of $\frac{\delta^2 S_{\text{loc}}[X_0]}{\delta X_0^2}$. Note that in deriving this localization formula there is some freedom if there are multiple conserved supercharges Q_i since any of them can be used to define a set of BPS observables. Secondly, at fixed localization charge Q we still have a choice of B-invariant operator V which is not necessarily the canonical one. These choices affect the localization locus and one-loop determinants so different localization schemes typically lead to very different looking results. However, the answers must eventually agree since they only differ by Q-exact observables.

Unfortunately, this is not yet the complete story. Generically, the result (7.18) should be completed by non-perturbative terms denoted by the dots. In general these contributions are largely unknown and very hard to compute explicitly. In the case of interest to us, namely SYM on a sphere, some results are known in the literature. In the case of SYM on S^2 the dots correspond to non-perturbative contributions coming from other localization loci with non-trivial magnetic fluxes [39]. For SYM on S^3 there are no non-perturbative corrections [124, 125, 147, 151]. For S^4, the non-perturbative contributions come from instantons localized at the north and south pole of the sphere and can are computed by the Nekrasov instanton partition function [198]. For S^5 and S^7, there are non-perturbative contributions from additional localization loci around non-trivial connections satisfying certain non-linear PDEs [153, 186, 212]. There are some natural guesses about these corrections but no systematic derivation or understanding, especially for the case of S^7. Finally, for S^6 the nature of the dots remains a mystery. When the rank of the gauge group is large, these contributions are believed to be exponentially suppressed and can therefore be ignored. In general, the k-instanton contribution is of the form

$$Z_{\text{inst.}}^{(k)} \sim \mathrm{e}^{-\frac{8\pi^2 k N^2}{g_{YM}^2}} . \tag{7.19}$$

However, to obtain the instanton partition function one has to integrate over the instanton moduli space, which can considerably change the instanton weight and even overcome the exponential suppression. For $\mathcal{N}=2$ and $\mathcal{N}=2^*$ SYM it was argued in [219] that this does not happen and that the instanton contribution remains exponentially suppressed. Further evidence for this suppression has been supplied by holography, see for example [57], where such contributions are absent to leading order in N. However, in two-dimensional QCD it was shown in [115] that at large coupling the moduli space integration can overcome the exponential suppression and lead to instanton induced phase transitions. By constructing dual supergravity solutions we therefore not only test the holographic correspondence but also show that at leading order in N such instanton contributions are indeed absent.

7.3 Localising SYM

By exploiting the supersymmetric localization technique we can compute various observables in maximal SYM on a sphere. Our starting point is the MSYM Lagrangian on S^d with radius $\mathcal{R}$ from (7.2) and (7.5),[7]

$$\mathcal{L} = -\frac{1}{2g_{\text{YM}}^2}\text{Tr}\bigg(F_{MN}F^{MN} - \bar{\Psi}\displaystyle{\not}D\Psi - \frac{(d-4)}{2\mathcal{R}}\bar{\Psi}\Lambda\Psi + \frac{(d-4)}{\mathcal{R}^2}\Phi^a\Phi_a + \frac{(d-2)}{\mathcal{R}^2}\Phi^m\Phi_m + \frac{2}{3\mathcal{R}}(d-4)\Phi^a[\Phi^b,\Phi^c]\varepsilon_{abc} - K_iK^i \bigg) . \tag{7.20}$$

where $a=0,1,2$, $m=3,\dots,8-p$ and $M,N=0,\dots 9$. Furthermore, we have introduced 7 additional auxiliary fields K_i which allow for an off-shell formulation of supersymmetry. An off-shell formulation does not exist for all 16 supercharges, but for the purpose of localization we only need an off-shell formulation for one particular ϵ. By introducing the vector field v^M such that

$$v^M \equiv \epsilon\Gamma^M\epsilon , \tag{7.21}$$

[7]For the sake of brevity we use the full $10d$ gauge fields. Their kinetic term includes the gauge kinetic term as well as the kinetic terms for the scalars in the lower dimensional SYM theory.

we can elegantly encode this choice of ϵ. The Lagrangian (7.20) is invariant under the off-shell supersymmetry transformations

$$\delta_\epsilon A_M = \epsilon\,\Gamma_M \Psi\,, \tag{7.22}$$

$$\delta_\epsilon \Psi = \frac{1}{2}\Gamma^{MN} F_{MN}\epsilon + \frac{(d-3)}{d}\Gamma^{\mu A}\Phi_a \nabla_\mu\,\epsilon + \frac{1}{d}\Gamma^{\mu m}\Phi_m \nabla_\mu\,\epsilon + K^i \zeta_i\,, \tag{7.23}$$

$$\delta_\epsilon K^i = -\zeta^i \not{D}\Psi + \frac{(d-4)}{2\mathcal{R}}\zeta^i \Lambda \Psi\,, \tag{7.24}$$

where ϵ is a bosonic 16 component real chiral spinor that satisfies the conformal Killing spinor equation

$$\nabla_\mu \epsilon = \frac{1}{2\mathcal{R}}\Gamma_\mu \Lambda \epsilon\,, \tag{7.25}$$

and we have introduced seven bosonic pure spinors ζ^i defined by the following orthogonality relations [186, 207]

$$\epsilon \Gamma^M \zeta^i = 0\,, \qquad \zeta^i \Gamma^M \zeta^j = \delta^{ij} v^M\,. \tag{7.26}$$

By choosing Q to be the supercharge generated by ϵ and defining

$$V = \int \mathrm{d}^d x \sqrt{-g}\,\Psi \overline{\delta_\epsilon \Psi}\,, \tag{7.27}$$

the path integral localizes to the BPS locus defined by Q [113, 186]. Given any ϵ satisfying (7.25), the vector field $v^M = \epsilon \Gamma^M \epsilon$ automatically satisfies $v_M v^M = 0$. We can then choose ϵ such that $v^0 = 1$, $v^{8,9} = 0$, and such that it generates rotations along one particular equator of the sphere, i.e. $v_\mu v^\mu = 1$. After Wick rotating the $\Phi^0 \to i\Phi^0$, and $K^i \to iK^i$ the theory localizes onto the locus where

$$A_\mu = 0\,, \quad \Phi^I = 0 \text{ for } I \neq 0\,, \quad \nabla_\mu \Phi^0 = 0\,, \quad K_i = -\frac{(d-3)}{\mathcal{R}}\Phi^0(\zeta_i \Lambda \epsilon)\,. \tag{7.28}$$

Substituting this fixed point locus in the Lagrangian we find the classical action

$$S_0 = \frac{V_d}{2g_{\mathrm{YM}}^2}\frac{(d-1)(d-3)}{\mathcal{R}^2}\mathrm{Tr}\,(\Phi^0\Phi^0) = \frac{4\pi^{\frac{d+1}{2}}\mathcal{R}^{d-4}}{g_{\mathrm{YM}}^2 \Gamma\left(\frac{d-3}{2}\right)}\mathrm{Tr}\,\sigma^2 \tag{7.29}$$

where V_d is the volume of S^d and we have defined the dimensionless $N \times N$ Hermitian matrix $\sigma \equiv \mathcal{R}\Phi^0$. Note that this result holds for any hypermultiplet content. The full partition function for general d then reduces to [113, 182, 186]

$$Z = \int_{\text{Cartan}} [d\sigma]\ \exp\left(-\frac{4\pi^{\frac{d+1}{2}}\mathcal{R}^{d-4}}{g_{\text{YM}}^2\Gamma\left(\frac{d-3}{2}\right)}\operatorname{Tr}\sigma^2\right) Z_{1-\text{loop}}(\sigma) Z_{\text{Inst.}}\,. \tag{7.30}$$

where $Z_{\text{Inst.}}$ accounts to the non-perturbative effect of instantons and $Z_{1-\text{loop}}(\sigma)$ is the contribution of the Gaussian fluctuations around the fixed point. For maximal SYM the field content consists of only vector multiplets and when combined with the Vandermonde determinant the one-loop determinant becomes

$$Z_{1-\text{loop}}(\sigma)\prod_{\gamma>0}\langle\gamma,\sigma\rangle^2 = \prod_{\gamma>0}\prod_{n=0}^{\infty}\left(\frac{n^2+\langle\gamma,\sigma\rangle^2}{(n+d-3)^2+\langle\gamma,\sigma\rangle^2}\right)^{\frac{\Gamma(n+d-3)}{\Gamma(n+1)\Gamma(d-3)}}\,, \tag{7.31}$$

where γ are the positive roots for the gauge group. If $d<6$ then (7.31) is convergent. For $d \geq 6$ it diverges and has to be regularized. For the rest of this chapter we assume that $d<6$. The $d=6$ and $d=7$ cases will be considered separately. Note that in the matrix model defined by (7.30), the integration over σ is restricted to adjoint matrices in the Cartan of the gauge group. We can therefore fully parameterize σ by its eigenvalues σ_i.

7.4 Free energy and Wilson loop VEVs

Using localization we have reduced the full path integral to a finite dimensional matrix model. However, in most cases the instanton partition function is unknown and we cannot evaluate the matrix model exactly. Nonetheless, we can extract a lot of information by going to the large N limit where the instantons contributions are believed to be exponentially suppressed and can be ignored [54, 217]. In this limit the matrix model partition function is dominated by an eigenvalue distribution solving the following auxiliary saddle point equation

$$\frac{C_1 N}{\lambda}\sigma_i = \sum_{j\neq i} G_{16}(\sigma_{ij})\,, \qquad C_1 \equiv \frac{8\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d-3}{2}\right)}\,, \tag{7.32}$$

where we have introduced the dimensionless 't Hooft coupling

$$\lambda = g_{\text{YM}}^2 N \mathcal{R}^{4-d} \tag{7.33}$$

and $\sigma_{ij} \equiv \sigma_i - \sigma_j$. The kernel $G_{16}(\sigma)$ is given by [182],

$$\frac{iG_{16}(\sigma)}{\Gamma(4-d)} = \frac{\Gamma(-i\sigma)}{\Gamma(4-d-i\sigma)} - \frac{\Gamma(i\sigma)}{\Gamma(4-d+i\sigma)} - \frac{\Gamma(d-3-i\sigma)}{\Gamma(1-i\sigma)} + \frac{\Gamma(d-3+i\sigma)}{\Gamma(1+i\sigma)}\,. \tag{7.34}$$

Its behavior is shown in Figure 7.1 for various values of d. Notice that in this figure we are not restricting the dimension d to be an integer. Indeed, the kernel $G_{16}(\sigma)$ is a meromorphic function of d.

For small eigenvalue separations, i.e. $|\sigma_{ij}| \ll 1$, the kernel has the weak coupling behavior

$$G_{16}(\sigma_{ij}) \approx \frac{2}{\sigma_{ij}}, \tag{7.35}$$

which is independent of d. We are however more interested in strongly coupled theories where $\lambda \gg 1$. In this regime the central potential for the eigenvalues is relatively weak so the repulsive force coming from the kernel pushes the eigenvalues far apart for $d < 6$. Hence, for generic $i \neq j$ we have that the eigenvalues are widely separated, $|\sigma_{ij}| \gg 1$ when λ is large, in which case the kernel (7.34) can be approximated by

$$G_{16}(\sigma_{ij}) \approx C_2 |\sigma_{ij}|^{d-5} \mathrm{sign}(\sigma_{ij}), \tag{7.36}$$

where

$$C_2 = 2(d-3)\Gamma(5-d) \sin \tfrac{\pi(d-3)}{2}. \tag{7.37}$$

The saddle point equation then reduces to

$$\frac{C_1}{\lambda} N \sigma_i = C_2 \sum_{j \neq i} |\sigma_i - \sigma_j|^{d-5} \mathrm{sign}(\sigma_i - \sigma_j). \tag{7.38}$$

Notice that C_2 in (7.37) has a pole at $d = 6$ and a double zero at $d = 3$. This restricts our general analysis to the range $3 < d < 6$. We will return to $d = 2, 3$ in Chapter 9.

Next, we define the eigenvalue density $\rho(\sigma)$,

$$\rho(\sigma) \equiv N^{-1} \sum_{i=1}^{N} \delta(\sigma - \sigma_i), \tag{7.39}$$

which at large N becomes a continuous function. Assuming strong coupling, the saddle point equation (7.38) for $3 < d < 6$ can be written as

$$\frac{C_1}{\lambda} \sigma = C_2 \, ⨍_{-b}^{b} d\sigma' \rho(\sigma') |\sigma - \sigma'|^{d-5} \mathrm{sign}(\sigma - \sigma'), \tag{7.40}$$

where b marks the endpoints of the support of the eigenvalue distribution. Using the result in (B.1), we see that (7.40) is satisfied if the density has the form

$$\rho(\sigma) = \frac{2\pi^{\frac{d+1}{2}}}{\pi \lambda \, \Gamma(6-d) \Gamma\left(\frac{d-1}{2}\right) (b^2 - \sigma^2)^{(d-5)/2}}. \tag{7.41}$$

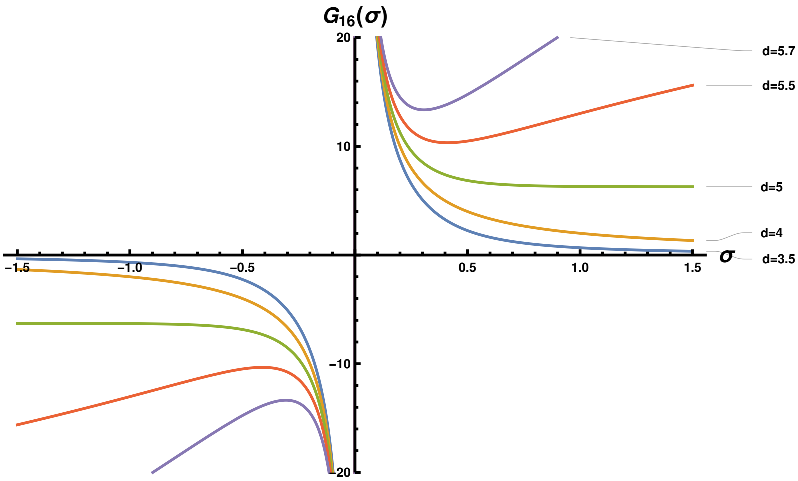


Figure 7.1: The kernel $G_{16}(\sigma)$ for various values of d. For $|\sigma| \ll 1$ the curves approach the same weak coupling behavior. For $|\sigma| > 1$ they approach different strong coupling behavior.

Using (B.2), we can determine b by properly normalizing the density,

$$b = (4\pi)^{\frac{d+1}{2(d-6)}} \left(32\lambda\, \Gamma\left(\tfrac{8-d}{2}\right) \Gamma\left(\tfrac{6-d}{2}\right) \Gamma\left(\tfrac{d-1}{2}\right)\right)^{\frac{1}{6-d}} . \tag{7.42}$$

To verify the validity of the strong coupling approximation in (7.36) we can test the solutions to the saddle point equation numerically using the full function $G_{16}(\sigma_{ij})$ defined in (7.34). As can be seen from the graphs in Figure 7.2, the numerical solutions at strong coupling are in very good agreement with the eigenvalue density (7.41) in dimensions $3 < d < 6$.

In the strong coupling regime the large N limit of the free energy, $F = -\log Z$, is given by

$$F = N^2 \left(\frac{C_1}{2\lambda} \int_{-b}^{b} d\sigma \rho(\sigma) \sigma^2 - \frac{C_2}{2(d-4)} \int_{-b}^{b} d\sigma \rho(\sigma) \int_{-b}^{b} d\sigma' \rho(\sigma') |\sigma - \sigma'|^{d-4} \right). \tag{7.43}$$

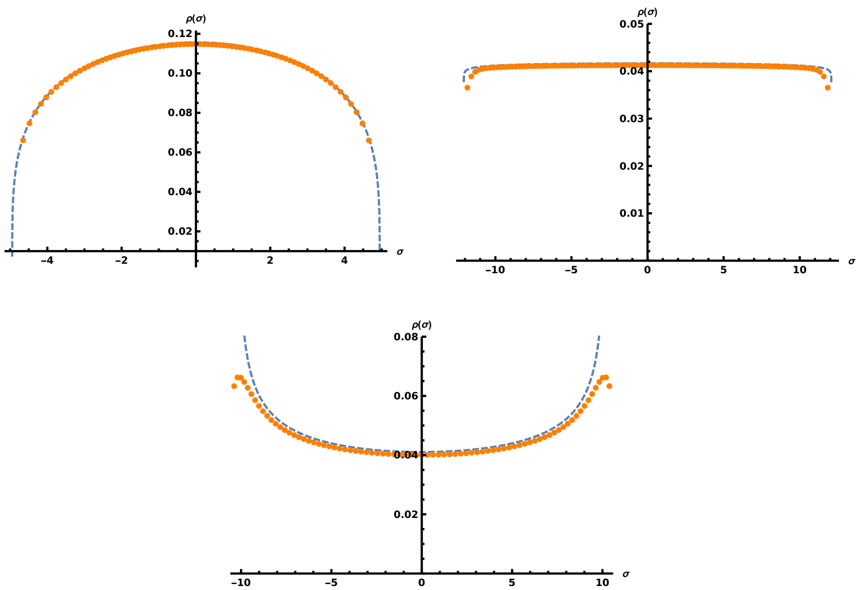


Figure 7.2: The eigenvalue density obtained from the numerical solutions of the full saddle point equations (7.32) with various choices of parameters. Top left: $d = 4.5$, $N = 80$, $\lambda = 350$. Top right: $d = 4.98$, $N = 100$, $\lambda = 500$. Bottom: $d = 5.5$, $N = 80$, $\lambda = 100$. The dashed lines represent the eigenvalue density in (7.41).

Dividing by the N^2 factor and performing the second integral over σ by parts results in

$$\frac{F}{N^2} = \frac{C_1}{2\lambda}\int_{-b}^{b} d\sigma \rho(\sigma)\sigma^2 - \frac{C_2 f(b)}{d-4}\int_{-b}^{b} d\sigma' \rho(\sigma')|b-\sigma'|^{d-4}$$
$$+\frac{C_2}{2}\int_{-b}^{b} d\sigma f(\sigma)\int_{-b}^{b} d\sigma' \rho(\sigma')|\sigma-\sigma'|^{d-5}, \qquad (7.44)$$

where $f(\sigma)$ is defined in (B.4) and we only used the fact that it is an odd function. Substituting (7.40) in the last integral and integrating once more by parts we find

$$\frac{F}{N^2} = \frac{C_1}{4\lambda}\int_{-b}^{b} d\sigma \rho(\sigma)\sigma^2 + \frac{C_1}{2\lambda} f(b) b^2 - \frac{C_2 f(b)}{d-4}\int_{-b}^{b} d\sigma' \rho(\sigma')|b-\sigma'|^{d-4}. \qquad (7.45)$$

The remaining integrals are evaluated in (B.5) and (B.6). Using these, as well as $f(b) = 1/2$ and the expression for b in (7.42), we can simplify the free energy to

$$\frac{F}{N^2} = -\frac{C_1}{2\lambda}\frac{(6-d)}{(8-d)(d-4)}\,b^2$$
$$= -\frac{16\pi^{\frac{(d+1)(4-d)}{2(6-d)}}(6-d)}{\lambda\,\Gamma\left(\frac{d-3}{2}\right)(8-d)(d-4)}\left(\frac{\lambda}{4}\Gamma\left(\tfrac{8-d}{2}\right)\Gamma\left(\tfrac{6-d}{2}\right)\Gamma\left(\tfrac{d-1}{2}\right)\right)^{\frac{2}{6-d}}.$$

This is our final result for the free energy as a function of d in the strong coupling limit.

Next, we can use our large N matrix model to compute the vacuum expectation value of a $\frac{1}{2}$-BPS Wilson loop W wrapping the equator of S^d. The VEV of this operator is given by

$$\langle W\rangle = \left\langle \mathrm{Tr}\left(\mathcal{P}\mathrm{exp}^{i\oint dx^\mu A_\mu + i\oint ds\, v_A \Phi^A}\right)\right\rangle \tag{7.46}$$

where $v_A v^A = 1$ and the direction of v_A is fixed by the choice of localizing supercharge. If the loop is chosen to be invariant with respect to the same supersymmetry used to localize the partition function then the Wilson loop can also be localized. For our choice of supersymmetry this sets $v_0 = 1$ [186, 207]. In the large N limit the Wilson loop reduces to

$$\begin{aligned}\langle W\rangle =& \left\langle \mathrm{Tr}\left(\mathcal{P}\mathrm{exp}^{i\oint ds \cdot \Phi^0}\right)\right\rangle \\ \approx& \int_{-b}^{b} d\sigma\,\rho(\sigma)\mathrm{e}^{2\pi\sigma} \\ =& (\pi b)^{\frac{d-6}{2}}\,\Gamma\left(\tfrac{8-d}{2}\right) I_{\frac{6-d}{2}}(2\pi b),\end{aligned} \tag{7.47}$$

where we used the eigenvalue density in (7.41) to evaluate the integral. The functions $I_{\frac{6-d}{2}}(2\pi b)$ are modified Bessel functions which reduce to spherical Bessel functions when d is odd.

When $d = 4$, the result (7.47) is valid for any value of λ. In Chapter 9 we will show that this is also true for $d = 3$. For all other values of d the result in (7.47) is valid only for large λ. When comparing these results to supergravity we will mainly be interested in the strong coupling limit. In this case the Wilson loop VEV is generally determined by the highest eigenvalue b, in which case we find

$$\langle W\rangle \simeq \mathrm{e}^{2\pi b}\,. \tag{7.48}$$

Chapter 8

Spherical branes

After introducing maximally supersymmetric Yang-Mills theory on the sphere we will now construct the dual supergravity backgrounds holographically describing this theory. Once these backgrounds are found, we will in the next chapter holographically compute the free energy and 1/2-BPS Wilson loop VEV.

Maximal SYM on a $(p+1)$-dimensional sphere is realized as the worldvolume theory on a stack of Dp-branes. Before embarking on the journey towards constructing supergravity solutions describing Dp-branes with a spherical worldvolume let us first review the physics of their flat counterparts.

8.1 Flat branes

Black branes are solution of ten-dimensional type II supergravity that source the metric, the dilaton, as well as $(p+2)$-form field strengths [135]. These solutions are characterized by their conserved electric charge μ_p and the ADM tension T_p. They break either all of the original 32 supercharges of type II supergravity or only half of them. We will focus on the latter case for which the branes are extremal in the sense that their tension equals their charge, i.e. $T_p = \mu_p$. Within string theory these supergravity backgrounds are interpreted as the back-reaction of a large number of fundamental Dp-branes on the ten-dimensional geometry in which they are immersed [209]. This interpretation has passed many consistency checks in string theory and ultimately led to the AdS/CFT correspondence. The low-energy physics on the flat worldvolume of the fundamental Dp-branes is described by maximal SYM in flat space. This implies that upon gravitational back-reaction the supergravity solutions describing supersymmetric black branes are holographically dual to these SYM theories as first suggested in [146].

The full ten-dimensional supergravity solution describing Dp-branes with flat worldvolume, with $p \leq 6$, in asymptotically flat space and in string frame is (see for example [49])

$$\mathrm{d}s_{10}^2 = H^{-1/2}\mathrm{d}s_{p+1}^2 + H^{1/2}\mathrm{d}s_{9-p}^2\,, \tag{8.1}$$

$$\mathrm{e}^{\Phi} = g_s H^{(3-p)/4}\,, \tag{8.2}$$

$$C_{p+1} = (g_s H)^{-1}\ \mathrm{vol}_{p+1}\,. \tag{8.3}$$

Here $\mathrm{d}s_{p+1}^2$ and $\mathrm{d}s_{9-p}^2$ denote the flat metrics on $\mathbf{R}^{1,p}$ and $\mathbf{R}^{9-p}$ respectively, vol_{p+1} is the volume form on $\mathbf{R}^{1,p}$ and H is a harmonic function on $\mathbf{R}^{9-p}$. The harmonic function has isolated singularities at the position of the branes. For a single stack of Dp-branes at the origin we have $\mathrm{d}s_{9-p}^2 = \mathrm{d}r^2 + r^2\mathrm{d}\Omega_{8-p}^2$, with $\mathrm{d}\Omega_{8-p}^2$ the unit radius metric on a round $8-p$ sphere. The harmonic function in this case is

$$H = 1 + \frac{g_s N}{\mu_{6-p} V_{6-p} r^{7-p}}\,, \tag{8.4}$$

where $V_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the volume of the unit radius n-sphere. The fundamental charge of a Dp-brane is given by[1]

$$\mu_p = \frac{2\pi}{(2\pi\ell_s)^{p+1}}\,, \tag{8.5}$$

and the Yang-Mills coupling constant of the worldvolume gauge theory is

$$g_{\mathrm{YM}}^2 = \frac{(2\pi)^2 g_s}{(2\pi\ell_s)^4 \mu_p}\,. \tag{8.6}$$

The constants in (8.4) must satisfy a Dirac quantization condition. Indeed, integrating the magnetic field strength over $\mathrm{d}\Omega_{8-p}^2$ leads to

$$\frac{1}{2\kappa_{10}^2\mu_p}\int \star\mathrm{d}C_{p+1} = \frac{N(7-p)V_{8-p}}{2\kappa_{10}^2\mu_p\mu_{6-p}V_{6-p}} = N \in \mathbf{Z}\,, \tag{8.7}$$

where the integer N is interpreted as the number of Dp-branes.

The field theory limit of Dp-branes in a holographic context was first studied in [146] (see also [60]). Introducing the dimensionless radial coordinate $U = r/(2\pi\ell_s)$, this limit is equivalent to zooming in on the near-horizon region of the branes:

$$\frac{g_s N\ U^{p-7}}{2\pi V_{6-p}} \gg 1\,. \tag{8.8}$$

[1] The ten-dimensional Newton constant is related to the string length ℓ_s through $4\pi\kappa_{10}^2 = (2\pi\ell_s)^8$, therefore $2\kappa_{10}^2\mu_p\mu_{6-p} = 2\pi$. Note that in our conventions the Newton constant does not depend on g_s.

Using (8.8), the metric and dilaton simplify to

$$\mathrm{d}s_{10}^2 = (gU)^{\frac{7-p}{2}}\, \mathrm{d}s_{p+1}^2 + (gU)^{\frac{p-7}{2}} \left(\mathrm{d}U^2 + U^2 \mathrm{d}\Omega_{8-p}^2\right), \tag{8.9}$$

$$\mathrm{e}^{\Phi} = g_s (gU)^{\frac{(p-3)(7-p)}{4}}, \tag{8.10}$$

$$C_{p+1} = g_s^{-1} (gU)^{7-p} \mathrm{vol}_{p+1}, \tag{8.11}$$

where we have introduced g as[2]

$$(2\pi \ell_s g)^{p-7} = \frac{g_s N}{2\pi V_{6-p}}. \tag{8.12}$$

For $0 < p < 3$ at high energies, $U \gg 1$, the string coupling becomes small indicating that the theory is free in the UV. As discussed in Chapter 7 this is the expected UV behavior of maximal SYM theory in $1 < d < 4$ dimensions. Conversely for $3 < p < 7$ the dilaton increases at high energies indicating that the field theory is strongly coupled. This again fits nicely with the non-renormalizability of SYM theory for $d > 4$. Clearly the case $p = 3$ is special since the string coupling is constant throughout the solution and the metric is that of $\mathrm{AdS}_5 \times S^5$. This is the well-known holographic dual description of the conformal $\mathcal{N} = 4$ SYM theory in $d = 4$. The background in (8.9)-(8.11) possesses $\mathrm{ISO}(1,\mathrm{p}) \times \mathrm{SO}(9-\mathrm{p})$ isometry for $p \neq 3$ and $\mathrm{SO}(2,4) \times \mathrm{SO}(6)$ for $p = 3$. This is the same as the global symmetry group of the SYM theories discussed in Chapter 7. It is therefore clear that this near-horizon solution nicely exhibits the physics we expect from a holographic dual to SYM on flat space. We refer to [146] and references thereof for further support of this holographic duality.

Our goal is to generalize the solutions in (8.9)-(8.11) and construct supergravity backgrounds which correspond to spherical Dp-branes and provide a holographic description of maximal SYM on S^{p+1}. This necessitates an understanding of how to construct supergravity solutions for D-branes with Euclidean worldvolume. This was addressed in several papers by Hull [137–139] where he argued that there are Euclidean branes, or E-branes, not of regular type II string theory but of the so-called type II* string theory. The existence of a low-energy supergravity limit of these type II* string theories can be deduced independently from a supergravity point of view [45]. The type II* supergravity theories admit E-brane solutions[3] for which the brane worldvolume is Euclidean and the time direction is transverse to the brane worldvolume, i.e. E-branes resemble instantons. The E-brane solutions can be obtained from the Dp-brane solutions above by analytically continuing the

[2]The real constant g will be identified with the coupling constant of the $(p+2)$-dimensional gauged supergravity theory in which the brane solutions can be effectively described.

[3]Note that in the notation of [137–139] an E$(p+1)$-brane is the Euclidean version of a Dp-brane.

time direction of the brane worldvolume into a spatial coordinate and at the same time analytically continuing the polar angle of the sphere transverse to the brane into a time-like coordinate. This analytic continuation results in changing the worldvolume of the brane from $\mathbf{R}^{1,p}$ to $\mathbf{R}^{p+1}$ and the transverse S^{8-p} sphere in (8.9) to de Sitter space, dS_{8-p}. The analytic continuation does not only affect the metric, but also changes the R-R fields. In [137–139] all R-R fields are taken to be real with "wrong sign" kinetic terms. In this paper we use an equivalent formulation in which all R-R fields are imaginary with "usual sign" kinetic terms. Finally we note that solutions of the Lorentzian type IIA* string theory should uplift to solutions of the so-called M* theory, see [137–139], which has the somewhat exotic $(2,9)$ signature of the metric, i.e. two time-like and nine spatial dimensions.

8.2 Spherical branes

To construct the near-horizon solutions for Euclidean Dp-branes wrapped on spheres we can utilize the intuition gained from the field theory discussion in Chapter 7 and make a suitable ansatz for the ten-dimensional metric. The total isometry group of the solution should be a direct product of the isometry group of the S^{p+1} which the Dp-brane is wrapping and the R-symmetry group of the Yang-Mills field theory in $p+1$ dimensions:

$$\mathrm{SO}(p+2)\times\mathrm{SU}(1,1)\times\mathrm{SO}(6-p)\,. \tag{8.13}$$

A ten-dimensional metric ansatz that implements these symmetries is given by

$$\mathrm{d}s^2_{10}=\Delta\left[\mathrm{d}r^2+\mathcal{R}^2\mathrm{e}^{2A}\mathrm{d}\Omega^2_{p+1}+\mathrm{e}^{2B}\left(\mathrm{d}\theta^2+P\cos^2\theta\,\mathrm{d}\widetilde{\Omega}^2_2+Q\sin^2\theta\,\mathrm{d}\Omega^2_{5-p}\right)\right]. \tag{8.14}$$

The functions A, B, Δ, P, and Q depend on r and θ and satisfy suitable positivity conditions such that the metric is non-degenerate and has the correct signature, while $\mathcal{R}$ is a constant that sets the radius of S^{p+1}. The metric on a unit radius round n-sphere is denoted by $\mathrm{d}\Omega^2_n$ with volume form vol_n. Clearly the $\mathrm{d}\Omega^2_{p+1}$ and $\mathrm{d}\Omega^2_{5-p}$ factors in the metric realize the $\mathrm{SO}(p+2)\times\mathrm{SO}(6-p)$ part of the isometry group in (8.13). The non-compact $\mathrm{SU}(1,1)$ factor in the R-symmetry of the SYM theory is realized as the isometry group of two-dimensional de Sitter space with metric

$$\mathrm{d}\widetilde{\Omega}^2_2=-\mathrm{d}t^2+\cosh^2 t\,\mathrm{d}\psi^2\,, \tag{8.15}$$

where ψ is 2π-periodic.

Note that for $P=Q=1$ the metric in (8.14) simplifies significantly, namely the metric transverse to the worldvolume of the branes is the round metric on dS_{8-p} which has $\mathrm{SO}(1,8-p)$ isometry group, i.e. the same as for the Euclidean Dp-branes in flat space discussed at the end of the previous section. Even before having an explicit solution for spherical branes, intuition from field theory suggests that for values of the radial coordinate much larger than the scale set by $\mathcal{R}$ the supergravity solution should reduce precisely to the Euclidean Dp-brane background with $P = Q = 1$. This is suggested by the UV limit in the field theory where the curvature of S^{p+1} should play no role in the high energy dynamics of the SYM theory and therefore should reduce to SYM in flat space.

In addition to the metric (8.14), we also have to make an appropriate ansatz for the type II NS-NS and R-R form fields and the dilaton. Dp-branes are electrically charged under C_{p+1} thus it is natural to expect there to be a non-zero component of C_{p+1} along the spherical worldvolume of the branes. We should also allow for all other form fields in the supergravity to have non-zero values as long as they preserve the isometry group in (8.13). In addition the dilaton can be an arbitrary function of r and θ. With this ansatz at hand one should analyze carefully the equations of motion and the supersymmetry variations of the ten-dimensional supergravity theory imposing that the background preserves 16 out of the 32 supercharges. This analysis should result in a system of coupled non-linear partial differential equations for the unknown functions in the ansatz. It is fair to assume that without any further insight it will be difficult to solve explicitly this system of equations. Fortunately, progress can be made by employing a well-tested strategy in top-down holography, namely reduce the ten-dimensional problem to a supergravity problem in $p+2$ dimensions. This can be achieved by employing a consistent truncation of the ten-dimensional supergravity to an appropriate gauged supergravity in $p+2$ dimensions.

8.2.1 Supergravity in $p+2$ dimensions

The gauged supergravity theories of interest are maximally supersymmetric and arise as consistent truncations of type II supergravity on S^{8-p}. The vacua of these supergravity theories are directly related to the field theory limits of Dp-branes discussed in Section 8.1 and thus for $p \neq 3$ the vacuum breaks half of the supersymmetries. In order to describe spherical branes we must analytically continue these gauged supergravity theories so as to work with an Euclidean theory. After constructing the solutions of interest we can uplift them to ten dimensions where we recover the time direction, as in (8.14), and thus obtain a fully Lorentzian solution of type II supergravity. In this section we start by briefly describing the Lorentzian gauged supergravity theories before performing the analytic continuation. Since the construction of the spherical brane solutions proceeds similarly in different dimensions we present a uniform description of the Lagrangian and BPS equations for all values of p. In Appendix D we give a more detailed discussion of the various lower-dimensional supergravity theories used in this paper and carefully analyze their analytic continuation.

The field theory discussion in Chapter 7 suggests that to construct the spherical brane solutions of interest we can restrict to an $SU(1,1) \times SO(6-p)$ invariant truncation of the maximally supersymmetric gauged supergravity theory. This ensures that the R-symmetry of the SYM theory, realized as a gauge symmetry in the supergravity theory, is preserved. In addition we are interested in supergravity solutions which preserve the $SO(p+2)$ isometry of the sphere which the branes are wrapping. This in turn implies that all fields present in the gauged supergravity theory, except the metric and scalar fields, should be set to zero. As discussed in detail in Appendix D, imposing these symmetries on the gauged supergravity leads to a consistent truncation which includes only the metric and three real scalar fields: the "dilaton" ϕ, a real scalar x and a pseudoscalar χ.[4] These scalar fields have a nice interpretation in the SYM theory on S^{p+1}. The dilaton is dual to the Yang-Mills coupling, the scalar field x is dual to the bosonic bilinear mass terms Φ^2, and the pseudoscalar is dual to the fermionic bilinear mass term $\bar{\Psi}\Lambda\Psi$ appearing in the field theory Lagrangian (7.5). It turns out that it is more convenient to work with the scalar fields η and β (discussed further in Appendix D) which are linear combinations of the scalar fields x and ϕ. In terms of these fields, the bosonic actions for the truncated gauged supergravity theories take the following uniform form for $0 < p < 6$,

$$S = \frac{1}{2\kappa_{p+2}^2} \int \star_{p+2} \left\{ R + \frac{3p}{2(p-6)} |\mathrm{d}\eta|^2 - \frac{1}{2} \left(|\mathrm{d}\beta|^2 + \mathrm{e}^{2\beta} |\mathrm{d}\chi|^2 \right) - V \right\} . \qquad (8.16)$$

V in this equation denotes the scalar potential and $\star_{p+2}$ is the $(p+2)$-dimensional Hodge star operator. It is clear from the kinetic terms that (β, χ) span an $SL(2)/SO(2)$ coset which can be conveniently parameterized by a single complex scalar

$$\tau \equiv \chi + i\mathrm{e}^{-\beta} . \qquad (8.17)$$

The kinetic term for τ can then be written in terms of the Kähler potential

$$\mathcal{K} = -\log\left(\frac{\tau - \bar{\tau}}{2} \right) = \beta , \qquad (8.18)$$

as

$$\mathcal{K}_{\tau\bar{\tau}} |\mathrm{d}\tau|^2 = \frac{1}{4} \left(|\mathrm{d}\beta|^2 + \mathrm{e}^{2\beta} |\mathrm{d}\chi|^2 \right) , \qquad (8.19)$$

where $\mathcal{K}_{\tau\bar{\tau}} = \partial_\tau \partial_{\bar{\tau}} \mathcal{K}$ is the Kähler metric. The scalar potential can be compactly expressed in terms of a superpotential which is holomorphic in τ and reads:

$$\mathcal{W} = \begin{cases} -g\ \mathrm{e}^{\frac{1}{2}\eta} \left(3\tau + (6-p) i \mathrm{e}^{-\frac{p}{6-p}\eta} \right) & \text{for } p < 3 , \\ -g\ \mathrm{e}^{\frac{3(2-p)}{2(6-p)}\eta} \left(3i\mathrm{e}^{\frac{p}{6-p}\eta} + (6-p)\tau \right) & \text{for } p > 3 . \end{cases} \qquad (8.20)$$

[4]The cases $p = 3$ and $p = 6$ are somewhat special and will be discussed separately.

Here g is the SO$(9-p)$ gauge coupling constant of the maximal gauged supergravity theory. The scalar potential is given by

$$V = \frac{1}{2}\mathrm{e}^{\mathcal{K}}\left(\frac{6-p}{3p}\left|\partial_\eta \mathcal{W}\right|^2 + \frac{1}{4}\mathcal{K}^{\tau\bar{\tau}}D_\tau \mathcal{W} D_{\bar{\tau}}\overline{\mathcal{W}} - \frac{p+1}{2p}\left|\mathcal{W}\right|^2\right), \qquad (8.21)$$

where $D_a = \partial_a + \partial_a \mathcal{K}$ is the Kähler covariant derivative.

It is clear from the kinetic term of the scalar η in (8.16) as well as the superpotential in (8.20) that $p = 6$ has to be treated separately. This is in harmony with the field theory discussion in Chapter 7 where it was shown that the R-symmetry is unbroken upon placing SYM on S^7. This in turn implies that in the supergravity theory we should retain only the complex scalar field, τ, and not include the scalar η. The eight-dimensional gravitational action for $p = 6$ then reads[5]

$$S = \frac{1}{2\kappa_8^2}\int \star_8 \left\{R - 2\mathcal{K}_{\tau\bar{\tau}}|\mathrm{d}\tau|^2 - V\right\}, \qquad (8.22)$$

where

$$V = \frac{1}{2}\mathrm{e}^{\mathcal{K}}\left(\frac{1}{4}\mathcal{K}^{\tau\bar{\tau}}D_\tau \mathcal{W} D_{\bar{\tau}}\overline{\mathcal{W}} - \frac{7}{12}\left|\mathcal{W}\right|^2\right), \qquad (8.23)$$

and

$$\mathcal{W} = -3ig\,. \qquad (8.24)$$

It is reassuring to observe that the supergravity action in (8.22) can be obtained as a formal limit of the action in (8.16) by taking $\eta/(p-6) \to 0$ and $p \to 6$.

As mentioned above, due to the runaway behavior of potential for the dilaton ϕ there is no vacuum solution of the gravitational theories in (8.16) and (8.22) that preserves all 32 supersymmetries.[6] There are however domain wall solutions which preserve 16 supercharges and are closely related to the flat brane solutions in ten dimensions discussed in Section 8.1, see for example [60]. These solutions are obtained by setting $\chi = x = 0$, or equivalently $\beta = \frac{p}{p-6}\eta$, and read

$$\mathrm{d}s^2_{p+2} = \mathrm{d}r^2 + \mathrm{e}^{\frac{2(9-p)}{(6-p)(p-3)}\eta}\mathrm{d}s^2_{p+1}\,, \qquad \mathrm{e}^{\frac{(p-3)}{6-p}\eta} = \frac{g(3-p)^2}{2p}(r-r_0)\,, \qquad (8.25)$$

where $\mathrm{d}s^2_{p+1}$ is the flat metric on Minkowski space and r_0 is an integration constant that can be set to zero by shifting appropriately the radial coordinate r. These solutions can be uplifted to solutions of type II supergravity using the uplift formulae discussed in Section 8.2.3. The end result of this uplift is given by the following

[5] Again we refer to Appendix D for more details on how to obtain this action from maximal supergravity in eight dimensions.

[6] The case $p = 3$ is an exception.

ten-dimensional background

$$\mathrm{d}s^2 = \mathrm{e}^{\eta}\left(\mathrm{d}s^2_{p+2} + \frac{1}{g^2}\mathrm{e}^{\frac{2(p-3)}{6-p}\eta}\mathrm{d}\Omega^2_{8-p}\right), \tag{8.26}$$

$$\mathrm{e}^{\Phi} = g_s \mathrm{e}^{\frac{p(7-p)}{2(6-p)}\eta}, \tag{8.27}$$

$$F_{8-p} = \frac{7-p}{g_s g^{7-p}}\mathrm{vol}_{8-p}\,. \tag{8.28}$$

This solution precisely matches the near-horizon limit of the flat Dp-brane solutions in (8.9)-(8.10) where $gU = \mathrm{e}^{\frac{2p}{(6-p)(p-3)}\eta}$ and the number of branes N is related to the supergravity coupling constant g via (8.12).

So far we have discussed only Lorentzian supergravities. However the spherical branes of interest here have a Euclidean worldvolume and thus should be described by Euclidean gauged supergravity theories. Such theories should be maximally supersymmetric with an SO$(1,8-p)$ gauge group and should be closely related to the more familiar SO$(9-p)$ maximal gauged supergravity theories in Lorentzian signature. These Euclidean supergravity theories are unfortunately not available in the literature. We resolve this impasse by performing an analytic continuation of the truncated Lorentzian supergravity theories described by the Lagrangians in (8.16) and (8.22).

At the level of the action the analytic continuation is straightforward. The metric becomes Euclidean and the only real modification to the action stems from the fact that the pseudo scalar χ becomes purely imaginary

$$\chi \to i\chi\,. \tag{8.29}$$

This results in the "wrong sign" kinetic term for χ. The scalar τ in (8.17) appears to be a purely imaginary scalar field and thus is not appropriate to describe two independent scalar fields. We must therefore consider $\tau = i(\chi + \mathrm{e}^{-\beta})$ and, what used to be its complex conjugate, $\tilde{\tau} = i(\chi - \mathrm{e}^{-\beta})$ as two independent scalar fields in the Euclidean theory.[7] Similarly we should work with two independent superpotentials, $\mathcal{W}$ as defined in (8.20) and $\widetilde{\mathcal{W}}$ obtained by complex conjugation of $\mathcal{W}$ accompanied by the replacement $\bar{\tau} \to \tilde{\tau}$,

$$\widetilde{\mathcal{W}} = \begin{cases} -g\ \mathrm{e}^{\frac{1}{2}\eta}\left(3\tilde{\tau} - (6-p)i\mathrm{e}^{-\frac{p}{6-p}\eta}\right) & \text{for } p < 3\,, \\ g\ \mathrm{e}^{\frac{3(2-p)}{2(6-p)}\eta}\left(3i\mathrm{e}^{\frac{p}{6-p}\eta} - (6-p)\tilde{\tau}\right) & \text{for } p > 3\,. \end{cases} \tag{8.30}$$

[7] This is a familiar predicament from similar constructions of Euclidean supergravity solutions in a holographic context [54, 98].

The scalar potential of the Euclidean theory is obtained by replacing $\overline{\mathcal{W}}$ by $\widetilde{\mathcal{W}}$ in (8.21).

With this Euclidean supergravity theory at hand we are now in a position to discuss how to construct the spherical branes of interest. We start by writing the following metric ansatz compatible with the spherical symmetry of the worldvolume of the brane

$$\mathrm{d}s_{p+2}^2 = \mathrm{d}r^2 + \mathcal{R}^2 \mathrm{e}^{2A} \mathrm{d}\Omega_{p+1}^2 \,. \tag{8.31}$$

In addition we assume that the scalar fields and the warp factor A only depend on the radial variable r. The constant $\mathcal{R}$ should be thought of as the radius of S^{p+1} and is auxiliary since it can be absorbed into a redefinition of metric function A.[8] To obtain the brane solutions with flat worldvolume one should take $\mathcal{R} \to \infty$. As we shall discuss below this is not a smooth limit, nevertheless it still proves useful to keep the constant $\mathcal{R}$ explicitly in the formulae below.

Equipped with this ansatz we can plug it in the supersymmetry variations of the $(p+2)$-dimensional gauged supergravity theory and look for solutions which preserve 16 supercharges. This is discussed in some detail in Appendix D. The end result is the following system of BPS equations which should be obeyed by the metric function and the three scalar fields:

$$(\eta')^2 = \mathrm{e}^{\mathcal{K}} \left(\frac{6-p}{3p} \right)^2 (\partial_\eta \mathcal{W})(\partial_\eta \widetilde{\mathcal{W}}) \,, \tag{8.32}$$

$$(\eta')(\tau') = \mathrm{e}^{\mathcal{K}} \left(\frac{6-p}{12p} \right) \big(\partial_\eta \mathcal{W}\big) \mathcal{K}^{\tau\tilde{\tau}} D_{\tilde{\tau}} \widetilde{\mathcal{W}} \,, \tag{8.33}$$

$$(\eta')(\tilde{\tau}') = \mathrm{e}^{\mathcal{K}} \left(\frac{6-p}{12p} \right) (\partial_\eta \widetilde{\mathcal{W}}) \mathcal{K}^{\tilde{\tau}\tau} D_{\tau} \mathcal{W} \,, \tag{8.34}$$

$$(\eta')(A' - \mathcal{R}^{-1}\mathrm{e}^{-A}) = -\mathrm{e}^{\mathcal{K}} \left(\frac{6-p}{6p^2} \right) (\partial_\eta \mathcal{W}) \widetilde{\mathcal{W}} \,, \tag{8.35}$$

$$(\eta')(A' + \mathcal{R}^{-1}\mathrm{e}^{-A}) = -\mathrm{e}^{\mathcal{K}} \left(\frac{6-p}{6p^2} \right) (\partial_\eta \widetilde{\mathcal{W}}) \mathcal{W} \,, \tag{8.36}$$

where $\mathcal{K}^{\tilde{\tau}\tau}$ is the inverse of the Kähler metric in (8.19). Equations (8.32), (8.33), and (8.34) arise from the spin-$\frac{1}{2}$ supersymmetry variations of the gauged supergravity theory, while (8.35) and (8.36) arise from the spin-$\frac{3}{2}$ variations.

[8] To stay in the regime of validity of supergravity we have to make sure that $\mathcal{R}\mathrm{e}^A$ is larger than the Planck and string scale throughout the solution.

Equations (8.35) and (8.36) lead to a first order differential equation together with the following algebraic relation for the metric function $A(r)$

$$\mathrm{e}^{A} = \frac{1}{\mathcal{R}g^2}\frac{2p}{6-p}\frac{\tilde{\tau}-\tau}{\tilde{\tau}+\tau}\mathrm{e}^{\frac{2(p-3)}{6-p}\eta}(\eta') \,. \tag{8.37}$$

Fortunately these two equations are compatible with each other. In addition one can explicitly check that all BPS equations in (8.32)-(8.36) are compatible with the second order equations of motion derived from the action in (8.16) after the analytic continuation in (8.29).[9]

Note that upon taking the limit $\mathcal{R} \to \infty$ in (8.32)-(8.36) accompanied with $\tau = \tilde{\tau}$, which in turn implies $\mathcal{W} = \widetilde{\mathcal{W}}$, we obtain the BPS equations for a domain wall with flat slices. These equations are then solved by the Euclidean analog of (8.25).

8.2.2 **Analysis of the BPS equations**

Let us now perform a preliminary analysis of the BPS equations (8.32)-(8.36). It proves convenient to introduce a new parameterization of the scalar fields given by

$$\begin{aligned} &\tau = i\mathrm{e}^{-\frac{p}{6-p}\eta}(X+Y)\,, \quad \tilde{\tau} = -i\mathrm{e}^{-\frac{p}{6-p}\eta}(X-Y)\,, \quad \text{for } p<3\,, \\ &\tau = i\mathrm{e}^{\frac{p}{6-p}\eta}(X+Y)\,, \quad \tilde{\tau} = -i\mathrm{e}^{\frac{p}{6-p}\eta}(X-Y)\,, \quad \text{for } p>3\,. \end{aligned} \tag{8.38}$$

When the BPS equations are solved it is important to impose appropriate boundary conditions in the IR. The physics of the SYM theory on S^{p+1} suggests that the supergravity solutions should cap off smoothly in the IR and it is thus natural to look for solutions in which close to some finite value of the radial coordinate $r \to r_{\mathrm{IR}}$ the metric looks like the metric on $(p+2)$-dimensional flat space in spherical coordinates

$$\mathrm{d}s^2_{p+2} \approx \mathrm{d}r^2 + (r-r_{\mathrm{IR}})^2\mathrm{d}\Omega^2_{p+1} \,. \tag{8.39}$$

In the UV, i.e. for large values of r, the solution should asymptotically approach the flat brane domain wall solution (8.25) as depicted in Figure 8.1. This implies that in this UV limit one should have $X = 1$ and $Y = 0$. In the IR region the scalar fields should approach a constant finite value in order to have a regular solution. These IR values for the scalars can be found as the critical points of the superpotential $\mathcal{W}$ (or equivalently, $\widetilde{\mathcal{W}}$)

$$\partial_\eta \mathcal{W} = D_\tau \mathcal{W} = 0 \,, \tag{8.40}$$

[9] The case $p = 6$ should again be treated separately and is discussed in more detail in Appendix D.1.

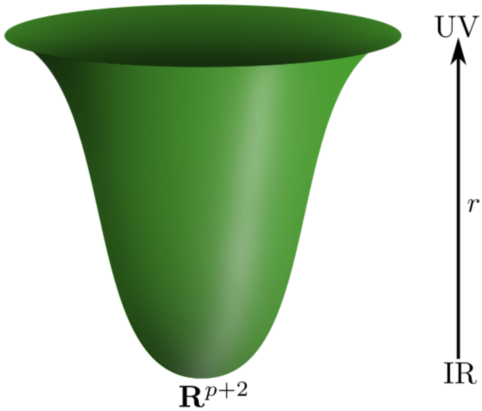


Figure 8.1: The regular geometries interpolate between flat Euclidean Dp-branes in the UV and $\mathbf{R}^{p+2}$ in the IR.

which in terms of the new variables X, Y read:

$$\begin{aligned} X_{\rm IR} &= \tfrac{p}{3} \, , & Y_{\rm IR} &= \pm\tfrac{2(p-3)}{3} \, , & &\text{for } p<3 \, , \\ X_{\rm IR} &= \tfrac{p}{(6-p)(p-2)} \, , & Y_{\rm IR} &= \pm\tfrac{2(p-3)}{(6-p)(p-2)} \, , & &\text{for } p>3 \, . \end{aligned} \tag{8.41}$$

The upper sign in the expressions above refers to a critical point of $\mathcal{W}$ whereas the lower sign refers to a critical point of $\widetilde{\mathcal{W}}$. Notice that for $p=4$ the critical value of the superpotential is at the UV point $X=1$. We will discuss this in more detail below. Even though X and Y approach fixed values in the IR, the scalar η can take any value $\eta = \eta_{\rm IR}$. As discussed in Chapter 9 below, $\eta_{\rm IR}$ is related to the effective gauge coupling constant of the dual SYM theory at the IR energy scale set by the radius of the sphere.

Finally we want to point out that when solving the BPS equations in (8.32)-(8.36) it sometimes proves useful to use the scalar X as a new radial variable. This is possible when X is a monotonic function of the original radial variable r in (8.31).

8.2.3 Uplift to ten dimensions

After this uniform treatment of the gauged supergravity theories in $p+2$ dimensions and their spherical brane solutions, we provide general uplift formulae that we use to obtain the spherical brane solutions in ten dimensions. These are distilled from the literature and brought into a universal form in Appendix D. In this section we merely quote the results. The ten-dimensional metric takes the expected form

(8.14)

$$ds_{10}^2 = \frac{e^{\eta}}{\sqrt{Q}}\left(ds_{p+2}^2 + \frac{e^{\frac{2(p-3)}{6-p}\eta}}{g^2}\left(d\theta^2 + P\cos^2\theta\ d\widetilde{\Omega}_2^2 + Q\sin^2\theta\ d\Omega_{5-p}^2\right)\right). \tag{8.42}$$

The squashing functions P and Q are determined in terms of the gauged supergravity scalars as

$$P = \begin{cases} X\big(X\sin^2\theta + (X^2-Y^2)\cos^2\theta\big)^{-1} & \text{for } p<3\,, \\ X\big(\cos^2\theta + X\sin^2\theta\big)^{-1} & \text{for } p>3\,, \end{cases} \tag{8.43}$$

$$Q = \begin{cases} X\big(\sin^2\theta + X\cos^2\theta\big)^{-1} & \text{for } p<3\,, \\ X\big(X\cos^2\theta + (X^2-Y^2)\sin^2\theta\big)^{-1} & \text{for } p>3\,. \end{cases} \tag{8.44}$$

The ten-dimensional dilaton is

$$e^{2\Phi} = g_s^2 e^{\frac{p(7-p)}{6-p}\eta}\ P\, Q^{\frac{1-p}{2}}\,, \tag{8.45}$$

and the non-vanishing type II form fields are given by

$$\begin{aligned} B_2 &= e^{\frac{p}{6-p}\eta}\frac{YP}{g^2X}\cos^3\theta\ \mathrm{vol}_2\,, \\ C_{5-p} &= i e^{-\frac{p}{6-p}\eta}\frac{YQ}{g_s g^{5-p}X}\sin^{4-p}\theta\ \mathrm{vol}_{5-p}\,, \\ C_{7-p} &= \frac{i}{g_s g^{7-p}}\left(\omega(\theta) + P\cos\theta\ \sin^{6-p}\theta\right)\mathrm{vol}_2\wedge\mathrm{vol}_{5-p}\,. \end{aligned} \tag{8.46}$$

Here vol_{5-p} and vol_2 refer to the volume forms on $d\Omega_{5-p}^2$ and $d\widetilde{\Omega}_2^2$, respectively, see (8.14) and (8.15). The function $\omega(\theta)$ is defined such that in the UV the derivative of C_{7-p} simply gives the volume form on the $(8-p)$–dimensional de Sitter space, namely

$$\frac{d}{d\theta}\left(\omega(\theta) + \cos\theta\ \sin^{6-p}\theta\right) = (7-p)\cos^2\theta\ \sin^{5-p}\theta\,. \tag{8.47}$$

8.3 Details of the solutions

In this section we perform a case-by-case study of the various spherical brane solutions. The simplest example is provided by the near-horizon geometry of Euclidean D3-branes. It is simply given by $\mathbf{H}^5 \times dS_5$. Writing the metric on $\mathbf{H}^5$ in global coordinates

$$ds_{\mathbf{H}^5}^2 = d\eta^2 + \sinh^2\eta\ d\Omega_4^2\,, \tag{8.48}$$

makes it clear that flat Euclidean D3-branes are described by the same supergravity solution as spherical D3-branes. This is of course a reflection of the fact that the worldvolume $\mathcal{N} = 4$ SYM theory is conformal and the four-sphere is a conformally flat manifold.

As discussed in Chapter 7, placing non-conformal maximal SYM theories on spheres in a supersymmetric way must be accompanied by adding particular mass terms, in addition to the standard conformal coupling term, to the Lagrangian. In the bulk supergravity solutions this is manifested by modifying the usual flat Euclidean Dp-brane solutions to genuinely new supergravity solutions which we now exhibit explicitly.

8.3.1 **D1-branes**

Let us start with the supergravity solution for spherical D1-branes. In this case we have to deviate slightly from our general approach of first finding the solution of interest in a lower-dimensional gauged supergravity and then uplifting it to ten dimensions. The reason for this is that we are not aware of an appropriate three-dimensional supergravity theory that is obtained by a consistent truncation of type IIB supergravity on S^8. Nevertheless, we are still able to make progress and find the solution directly in ten dimensions by solving a system of ordinary differential equations (ODEs), which resemble BPS equations derived from a three-dimensional supergravity theory, and are obtained by analytically continuing the equations in Section 8.2.1 to $p = 1$. We then use the solution of these effective BPS equations in a ten-dimensional background of the form presented in Section 8.2.3 with $p = 1$ and check explicitly that the equations of motion of type IIB supergravity are obeyed. It is highly non-trivial that this procedure works and we consider this sufficient evidence that the resulting solution describes the back-reaction of spherical D1-branes.

To describe the solution we use the scalar X to parameterize the radial direction. The BPS equation for Y then reduces to

$$\frac{\mathrm{d}Y^2}{\mathrm{d}X} = \frac{Y^2(7X^2 + 1 - Y^2)}{X(2(X^2 - 1) + Y^2)} . \tag{8.49}$$

A particular solution to this ODE that interpolates between the IR at $(X, Y) = (\frac{1}{3}, \pm\frac{4}{3})$ and the UV at $(X, Y) = (1, 0)$ is

$$Y^2 = \frac{(X + 1)(1 - X^2)}{X} . \tag{8.50}$$

Next, the equation for the scalar η can be can be readily integrated, resulting in

$$\eta(X) = \eta_{\mathrm{IR}} + \frac{5}{2}\log\frac{1-X}{2X} \; . \tag{8.51}$$

The three-dimensional metric in (8.42) is explicitly given in terms of the scalars Y and η by

$$\mathrm{d}s_3^2 = \frac{(1+X)^2 - Y^2}{g^2 X \mathrm{e}^{\frac{4}{5}\eta}}\left(\frac{\mathrm{d}X^2}{(2(1+X)(X-1)+Y^2)^2} + \frac{X^2}{Y^2}\mathrm{d}\Omega_2^2\right) \; . \tag{8.52}$$

One can then set $p = 1$ in the formulae in Section 8.2.3 and obtain an explicit solution of type IIB supergravity.

Regularity in the IR completely fixes the profile for Y as a function of X and the only integration constant of the solution is the one that appears in the expression for η in (8.51). A plot of the numerical solution for the scalar fields is given in Figure 8.2. We find that in the UV region the ten-dimensional background we construct is asymptotic to the flat D1-brane solution of type IIB supergravity. In the IR region the solution caps off smoothly which reflects the IR cut-off provided by the scale of the S^2 in the dual two-dimensional maximal SYM theory.

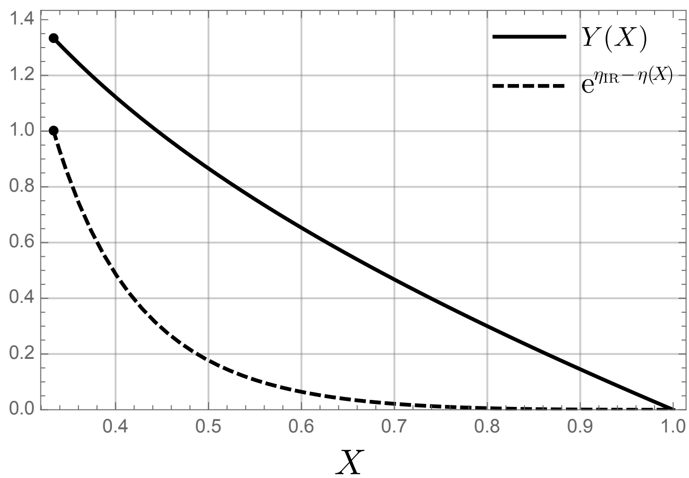


Figure 8.2: A numerical solution in the case $p = 1$. The UV region is at $X = 1$, $Y = 0$ and the IR is indicated by the solid dots.

Although the three-dimensional solution is completely regular, the ten-dimensional background appears to be singular in the IR due to the fact that there is a region in the plane spanned by X and θ for which the metric function Q becomes negative. This problem can be circumvented completely by performing a double analytic continuation,

$$\theta \to \frac{\pi}{2} + i\tilde{\theta}\,, \qquad \psi \to i\tilde{\psi}\,, \tag{8.53}$$

where θ is the coordinate appearing in the uplift formula for the metric (8.42) and ψ is a coordinate on the dS_2 in (8.15). This analytic continuation leaves the functions P and Q positive in the full range of the new coordinates, $0 \le \tilde{\theta} \le \infty$ and $\frac{2}{3} \le X \le 1$, so the metric and all other fields are now regular. Furthermore, the metric remains with signature $(1,9)$ where now $\tilde{\theta}$ parameterize the time direction. A careful study of the form fields shows that the continued solution is also a solution of type IIB* with all R-R fields purely imaginary and the NS-NS fields real. Finally, the global symmetries of the solution matches with the field theory expectation where the non-compact $SU(1,1)$ factor is now realized by the isometries of the hyperbolic plane spanned by $(t,\tilde{\psi})$. The need to perform this analytic continuation can be traced back to the the fact that in the field theory Lagrangian, (7.20), the coefficient of one of the bosonic bilinear terms changes sign as one goes from $p>3$ to $p<3$. As we will see in the next case, this interpretation is consistent with the D2-branes where we will have to perform the same analytic continuation to obtain a fully regular solution.

8.3.2 **D2-branes**

Next, we consider spherical D2-branes. These are constructed in maximal supergravity in four dimensions with ISO(7) gauge symmetry. This theory was first constructed by Hull in [136] and later argued to arise as a consistent truncation of type IIA supergravity on S^6 [141], see Appendix D.4 for more details. Again we will use the scalar X as radial coordinate to reduce the set of BPS equations (8.32)-(8.36) to a single ODE

$$\frac{dY^2}{dX} = \frac{Y^2(7X^2-4X-Y^2)}{X(2X(X-1)+Y^2)}\,. \tag{8.54}$$

This ODE is solved by

$$Y^4 = cX\Big(X(X-1)-Y^2\Big)^3\,, \tag{8.55}$$

where c is an integration constant. When setting $c=-1$ we obtain a solution connecting the UV values of the scalars $X=1$, $Y=0$ with their IR values as in (8.41) with $p=2$. This choice of integration constant still leaves us with six distinct solutions for $Y(X)$. However, two of them, $Y=\pm X$, are not physical since the metric

$$ds_4^2 = \frac{X^2-Y^2}{g^2 X e^{\eta/2}}\left(\frac{dX^2}{(2X(X-1)+Y^2)^2} + \frac{X^2}{Y^2}d\Omega_3^2\right) \tag{8.56}$$

vanishes for these flows. Of the remaining four solutions, only two flow to the (regular) IR. These are given by

$$Y^2 = \frac{(1-X)}{2X}\left((1-X)(1+2X)+\sqrt{(1-X)(1+3X)}\right) . \tag{8.57}$$

The BPS equation for η can now be readily integrated and yields

$$e^{2(\eta-\eta_{\rm IR})} = \frac{1-2X^2+X^3-XY^2}{(1-X)^2} , \tag{8.58}$$

with Y^2 given by (8.57). To illustrate this analytic solutions we plot it in Figure 8.3.

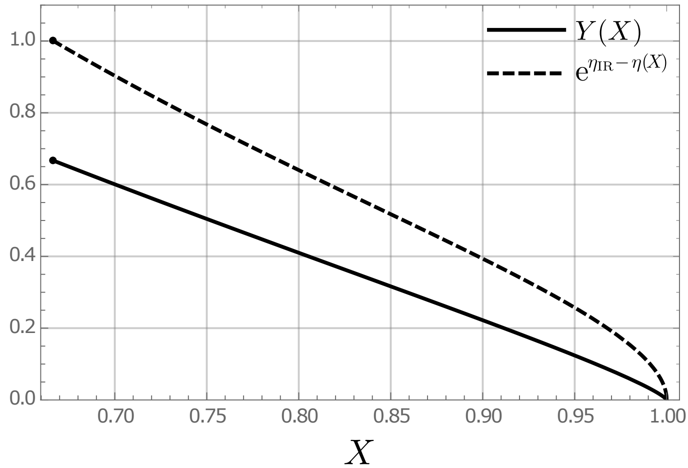


Figure 8.3: The full analytic solution for the functions $Y(X)$ and $\eta(X)$ in the case $p=2$. The UV region is at $X=1$, $Y=0$ and the IR region is indicated by the solid dots.

This four-dimensional supergravity solution can be uplifted to a ten-dimensional solution of type IIA supergravity using the uplift formulae in Section 8.2.3 with $p=2$. We have verified that all equations of motion in ten dimensions are satisfied by the above solution. In the UV one finds that $X \to 1$ and $Y \to 0$ and the ten-dimensional solution reduces to the near-horizon limit of D2-branes with flat worldvolume. Similar to the spherical D1-brane, the ten-dimensional metric becomes singular in the IR. This time however, the metric does not become negative in a region of the coordinate space but one still finds a singularity located at the IR fixed point $X = \frac{2}{3}$ and $\theta = 0$. Just as for the spherical D1-brane we can cure this problem by performing a double analytic continuation (8.53). This renders the new ten-dimensional configuration a completely regular background of type IIA* supergravity.

As discussed in Section 8.1, flat D2-branes are singular in the IR since the dilaton blows up. It is well-known that this singularity can be better interpreted in eleven-dimensional supergravity, since flat D2-branes uplift to M2-branes smeared over the M-theory circle. The IR singularity can therefore be understood as a direct consequence of the smearing and its resolution is achieved by replacing the smeared M2-branes by a point-like stack localized on the circle. In this way the singular supergravity solution is resolved by replacing it with the $AdS_4 \times S^7$ solution of eleven-dimensional supergravity. On the gauge theory side this interpretation is mirrored by the expectation that maximal SYM theory in three dimensions flows to the conformal ABJM theory in the deep IR. For our spherical D2-brane solution there is no singularity in the IR and in fact the dilaton is never so large as to warrant an uplift to eleven dimensions. In the dual field theory the interpretation is clear. Placing three-dimensional maximal SYM on a three-sphere introduces an IR cut-off and the RG flow never reaches the superconformal ABJM theory in the IR. Indeed this interpretation will be confirmed in the next chapter when we compute the free energy and Wilson loop VEV for this configuration. As a final comment we note that the spherical D2-brane supergravity solution should lie in region (b) of Figure 1 in [146], reproduced in Figure 8.4.

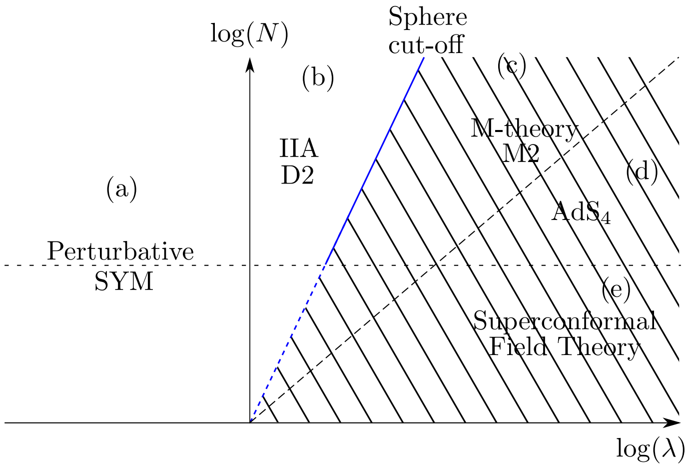


Figure 8.4: The spherical D2-brane map: The horizontal line separates small from large N. The blue line represents the cut-off provided by the sphere. Our supergravity solutions always lie in region (b). The dilaton never grows so large to warrant an uplift to eleven dimensions.

8.3.3 D4-branes

Since we have already discussed D3-branes, the next case is spherical D4-branes. these solutions are constructed in the six-dimensional gauged supergravity theory obtained by reducing the maximal SO(5) gauged supergravity in seven dimensions on a circle. As explained in more detail in Appendix D.3, we first introduce a seven-dimensional scalar x that breaks SO(5) $\to$ SU(2) $\times$ U(1) together with a U(1) gauge field $\mathcal{A}$. Reducing this theory on a circle introduces the dilaton ϕ as well as an additional scalar field arising from the component of the gauge field on the reduction circle, i.e. $\mathcal{A} = \chi \mathrm{d}\omega$ where ω is the coordinate on the circle. As a result we obtain the desired three scalar fields, x, ϕ and χ.[10] After rewriting the BPS equations using the scalar X as the radial coordinate, the system reduces to a single ODE which controls the full solution

$$\frac{\mathrm{d}Y^2}{\mathrm{d}X} = \frac{Y^2(1-12X+12X^2-4Y^2)}{2X(1-X)(1-2X)} \,. \tag{8.59}$$

This equation is solved by

$$Y^4 = cX(1-X)\Big((1-2X)^2-4Y^2\Big)^2 \,, \tag{8.60}$$

where c is an integration constant. The critical point of the superpotential determines the IR values of the scalar field as in (8.41) which for $p = 4$ yields $X_{\mathrm{IR}} = 1$ and $Y_{\mathrm{IR}} = \pm 1/2$. However, the analytic solutions (8.60) only reach the IR for diverging c, i.e. when $(1-2X)^2 - 4Y^2 = 0$. This is a solution to the BPS equation (8.59) but it is not physical since the metric

$$\mathrm{d}s_6^2 = \frac{(1-2X)^2-4Y^2}{g^2 X \mathrm{e}^{-\eta}} \left(\frac{\mathrm{d}X^2}{4(1-2X)^2(X-1)^2} + \frac{X^2}{Y^2}\mathrm{d}\Omega_5^2 \right) \,, \tag{8.61}$$

vanishes completely. All solutions in (8.60) with finite c correspond to gravitational domain walls with singular IR behavior. These singular flows still provide solutions to the ten-dimensional equations of motion via the uplift formulae in Section 8.2.3. Furthermore an uplift of these solutions to eleven-dimensional supergravity is given in Section D.3. Still, the conclusion remains that there is no smooth solution with running scalar X that connects the UV to a regular IR region. This is perhaps not surprising since the IR value of the scalar X is located at $X = 1$ which is also the UV value for X.

We are thus lead to explore solutions with constant $X = 1$. The original BPS equations (8.32)-(8.36) are solved by $2Y = \mathrm{e}^{2\eta_{\mathrm{IR}}-2\eta}$ where η_{IR} is an integration

[10] These scalars are the ones discussed in Section 8.2.1.

constant. The BPS equation for η then reduces to

$$(\eta')^2 = \frac{g^2}{16} e^{-5\eta} \left(e^{4\eta} - e^{4\eta_{\rm IR}} \right) . \tag{8.62}$$

Notice that the scalar $\chi = e^{2\eta_{\rm IR}}/2$ is constant (cf. (8.17) and (8.38)). Using η as a coordinate, the six-dimensional metric can be written as

$$ds_6^2 = \frac{8e^{3\eta-2\eta_{\rm IR}}}{g^2} \left(\sinh^{-1}(2\eta - 2\eta_{\rm IR}) d\eta^2 + \sinh(2\eta - 2\eta_{\rm IR}) d\Omega_5^2 \right) . \tag{8.63}$$

Since the six-dimensional supergravity theory used to construct this solution is obtained from a reduction of the maximal seven-dimensional SO(5) gauged supergravity it is possible to uplift the solution above to seven dimensions. Performing this uplift, see Appendix D.3, one finds that the metric and the scalar fields in seven dimensions are simply those corresponding to the maximally supersymmetric AdS_7 (or rather $\mathbf{H}_7$) vacuum of the gauged supergravity, albeit with an asymptotic $S^5 \times S^1$ metric on the boundary. There is however an additional non-vanishing gauge field $\mathcal{A} = \chi d\omega$, see (D.29), which is pure gauge since the six-dimensional scalar field χ is constant. Note that due to the topology of S^1 it requires a large gauge transformation to set the field $\mathcal{A}$ to zero.

The six-dimensional spherical D4-brane solution above can also be uplifted to ten-dimensional type IIA supergravity. The explicit form of the solution is

$$ds_{10}^2 = \frac{e^\eta}{\sqrt{Q}} \left(ds_6^2 + \frac{e^\eta}{g^2} \left(d\theta + \cos^2\theta \, d\tilde{\Omega}_2^2 + Q \sin^2\theta \, d\zeta^2 \right) \right) , \tag{8.64}$$

$$e^{2\Phi} = g_s^2 e^{6\eta} Q^{-3/2} , \tag{8.65}$$

$$B_2 = \frac{e^{2\eta_{\rm IR}}}{2g^2} \cos^3\theta \, \text{vol}_2 , \tag{8.66}$$

$$C_1 = \frac{i e^{2\eta_{\rm IR}}}{2 g_s g} e^{-4\eta} Q \, d\zeta , \tag{8.67}$$

$$C_3 = -\frac{i}{g_s g^3} \cos^3\theta \; \text{vol}_2 \wedge d\zeta . \tag{8.68}$$

This background is of the general form discussed in Section 8.2.3 with

$$P = 1 , \qquad Q = 4 \left(4 - e^{4\eta_{\rm IR} - 4\eta} \sin^2\theta \right)^{-1} . \tag{8.69}$$

In the IR region the scalar η which determines the behavior of the dilaton is finite and approaches the constant $\eta_{\rm IR}$. In the UV region however, the scalar η diverges and the the type IIA dilaton blows up. This indicates that the proper description of the solution is in eleven-dimensional supergravity. To find this eleven-dimensional background we can use the uplift formulae in Appendix A. However we should remember that we are working in the type IIA* theory of Hull which uplifts to the M*-theory in which the eleven-dimensional circle is timelike. We take this into account by using a purely imaginary x_{11}. Keeping this in mind we find the following eleven-dimensional background

$$ds_{11}^2 = \frac{1}{g_s^{2/3} g^2}\Bigg(8e^{2\tilde{\eta}}\left(\frac{d\tilde{\eta}^2}{\sinh 2\tilde{\eta}} + \sinh 2\tilde{\eta}\, d\Omega_5^2\right) - 4e^{4\tilde{\eta}} d\omega^2$$

$$+ d\theta + \cos^2\theta d\tilde{\Omega}_2^2 + \sin^2\theta (d\omega - d\zeta)^2\Bigg), \tag{8.70}$$

$$A_3 = \frac{i}{g_s g^3}\cos^3\theta\; (d\omega - d\zeta)\wedge V_2\,, \tag{8.71}$$

where we shifted η such that $e^{2\eta_{\text{IR}}-2\eta} = e^{-2\tilde{\eta}}$ and parameterized the eleventh dimension as $g g_s e^{2\eta_{\text{IR}}} x_{11}/2 = i\omega$. This eleven-dimensional solution is valid in the limit when $\tilde{\eta}$ is very large. As it turns out, the first line of (8.70) is simply the global metric on AdS_7, whereas the second line is a metric on four-dimensional de Sitter space. Indeed, the full solution in eleven dimensions is an analytic continuation of the well-known $AdS_7 \times S^4$ solution of standard eleven-dimensional supergravity. It is encouraging to find that in the far UV region of our spherical D4-brane solution we find this metric which should be associated with the near-horizon limit of Euclidean M5-branes. This is in line with the expectation discussed in Section 7 that the five-dimensional maximally symmetry SYM theory on S^5 flows to the superconformal $(2,0)$ theory on $S^5 \times S^1$ in the UV.

8.3.4 **D5/NS5-branes**

For $p = 5$ the solution is constructed in the maximal SO(4) gauged supergravity in seven dimensions, [222], which arises as a consistent truncation of type IIB supergravity on S^3, see [60]. Just like the well known SO(5) gauged supergravity theory [203], obtained by reducing eleven-dimensional supergravity on S^4, this theory has maximal supersymmetry.

Using the three scalar truncation of the SO(4) theory discussed in Appendix D.2 one can derive the BPS equations given in (8.32)-(8.36). As in the previous sections, it is convenient to use X as a coordinate and express the remaining scalar fields Y and η as functions of X. The solutions to the BPS equations are then fully determined by the following ODE

$$\frac{\mathrm{d}Y^2}{\mathrm{d}X} = \frac{Y^2}{X}\left(\frac{1-16X+15X^2-9Y^2}{2-8X+6X^2-3Y^2}\right), \tag{8.72}$$

together with the integral

$$\eta(X) = \eta_{\mathrm{IR}} + \int_{X_{\mathrm{IR}}}^{X} \frac{3\,\mathrm{d}x}{5(1-x)}\left[\left(x-\frac{1}{3}\right)\frac{\mathrm{d}\log Y(x)}{\mathrm{d}x} - 1\right]. \tag{8.73}$$

In terms of the new coordinate, the seven-dimensional metric takes the form

$$\mathrm{d}s_7^2 = \frac{(1-3X)^2-9Y^2}{g^2 X \mathrm{e}^{-4\eta}}\left(\frac{\mathrm{d}X^2}{(2-8X+6X^2-3Y^2)^2} + \frac{X^2}{Y^2}\mathrm{d}\Omega_6^2\right). \tag{8.74}$$

Unfortunately, we are not able to find an analytic solution to the equations above that connects the IR values of X and Y given in (8.41) to the UV values $X=1$, $Y=0$. To construct such solutions to (8.72), we therefore have to resort to numerical methods. A numerical plot of the solution is given in Figure 8.5. Its uplift to ten-dimensional type IIB supergravity is given by the general formulae in Section 8.2.3 with $p=5$.

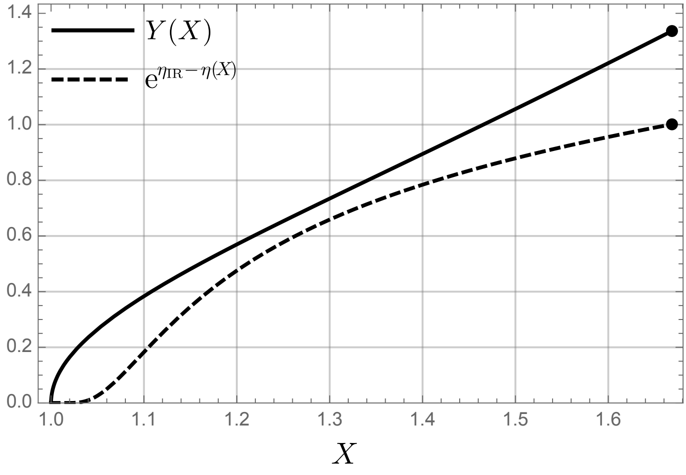


Figure 8.5: A numerical solution for the functions $Y(X)$ and $\eta(X)$ in the case $p=5$. Notice that since (8.72) is quadratic in Y, the function $-Y$ is also a solution with η unchanged. The far UV region is at $Y=0$, $X=1$ and the IR region where the S^6 smoothly caps of is indicated by the solid dots.

The numerical solution interpolates between a regular IR region where X, Y, and η approach a constant value and the metric caps off smoothly and the near-horizon geometry of D5-branes in the UV. The uplift to ten dimensions provides a full solution to the type IIB supergravity equations of motion describing D5-branes wrapped on S^6. Furthermore by performing an SL$(2,\mathbf{R})$ transformation we can obtain a solution describing NS5-branes (or more generally (p,q)-fivebranes) wrapped on S^6 and all equations of motion of course remain satisfied. The wrapped NS5-brane solution is particularly interesting since the sphere provides an IR cut-off of the linear dilaton geometry sourced by NS5-branes in flat space. As briefly discussed in Chapter 7, the UV completion of SYM theories in six dimensions is believed to be given by a non-local, non-gravitational theory called little string theory (LST). This theory can be understood as the decoupling limit of NS5-branes, where the string coupling vanishes $g_s \to 0$ [225]. A holographic model for LST was considered in [6] and studied in more detail in [110, 111]. The original construction is based on the linear dilaton vacuum which is simply the near-horizon limit of N flat NS5-branes.[11] The metric and dilaton of this type IIB solution are

$$\mathrm{d}s_{10}^2 = \mathrm{d}s_6^2 + \mathrm{d}\eta^2 + g^{-2}\mathrm{d}\Omega_3^2 \,, \qquad \Phi = \log g_s - g\eta \,, \tag{8.75}$$

where $g^2 = g_s N^{-1} \ell_s^{-2}$. Although these fields together with a Yang-Mills instanton provide an exact background of heterotic string theory [67], interpreting it in the context of holography is somewhat problematic due to the singular behavior of the dilaton for large negative η. In particular, it makes the holographic computation of LST correlation functions impossible without further information about the singular region $\eta \to -\infty$ [6]. In [110, 111] a resolution of the singularity was proposed whereby the N NS5-branes are spread out on a circle breaking the SO(4) isometry group of the space transverse to the branes to an SU(2) subgroup. In a T-dual frame the singularity corresponds to the origin of an asymptotically locally Euclidean (ALE) space

$$z_1^N + z_2^2 + z_3^2 = 0 \,, \tag{8.76}$$

and the resolution of the singularity is achieved by introducing a non-zero right-hand side in (8.76). Our type IIB supergravity solution provides an alternative way to resolve the problem. Remember that the singularity can be understood as a result of the dual SYM theory becoming weakly coupled in the IR. As we have explained, placing the SYM on S^6 introduces an effective IR cut-off set by the radius of the sphere. In the supergravity description this is manifested by the smooth cap-off of the geometry in the IR. More explicitly we find that the dilaton of our spherical NS5-brane background takes the form

$$\Phi = \log\left(\frac{g_s\sqrt{P}}{X}\right) - 5\eta \,, \tag{8.77}$$

[11] This background can be obtained from (8.9) for $p = 5$ by an SL$(2,\mathbf{R})$ transformation.

which in the IR reduces to

$$\Phi_{\rm IR} = \log\left(\frac{3g_s}{\sqrt{5(3\sin^2\theta + 5\cos^2\theta)}}\right) - 5\eta_{\rm IR}\ , \tag{8.78}$$

and therefore e^{Φ} can be made arbitrarily small throughout the full solution by suitably tuning $\eta_{\rm IR}$. In the next chapter we will study the holographic implications of our spherical NS5-brane background for the physics of little string theory.

8.3.5 **D6-branes**

Finally, let us discuss the spherical D6-branes. As discussed in Section 8.2.1, the case of spherical D6-branes is a degenerate limit of our equations since now we only have two scalar fields instead of three. This is consistent with the fact that for the maximal seven-dimensional SYM theory on S^7 the R-symmetry is unbroken. In addition we showed in (8.24) that the superpotential is a purely imaginary constant which implies that the pseudoscalar χ does not appear in the scalar potential. This in turn means that the BPS equations derived in Appendix D.1 only result in first order equations for the scalar β and the warp factor A. A first order equation for χ is obtained directly from the equations of motion. We refer to Appendix D.1 for further details on the eight-dimensional supergravity and the derivation of the BPS equations. Keeping this in mind, it is still useful to mimic the structure of the BPS equations with $p < 6$ and introduce new scalar variables

$$\tau = i(X+Y)\ , \quad \tilde{\tau} = -i(X-Y)\ . \tag{8.79}$$

In these variables the BPS equations together with the equation of motion for Y reduce to the following system of coupled first order ODEs

$$(X')^2 = \frac{9}{4}g^2X + 36\mathcal{R}^{-2}e^{-14A}X^4\ , \tag{8.80}$$

$$Y' = 6\mathcal{R}^{-1}X^2e^{-7A}\ , \tag{8.81}$$

$$(A')^2 = \frac{1}{16}g^2X^{-1} + \mathcal{R}^{-2}e^{-2A}\ , \tag{8.82}$$

where by prime we denote a derivative with respect to r. We have checked that this system of equations implies the equations of motion of the gauged supergravity theory. To solve the flow equations in (8.80) it is convenient to use the metric function A as a radial variable. One then finds

$$X = e^{6A}\ . \tag{8.83}$$

We can then proceed by defining yet another radial coordinate

$$\rho = \text{arcsinh}\left(\frac{4}{g\mathcal{R}}\mathrm{e}^{2A}\right), \tag{8.84}$$

such that the metric takes the form

$$\mathrm{d}s_8^2 = \frac{g\mathcal{R}^3}{16}\sinh\rho\left(\mathrm{d}\rho^2 + 4\mathrm{d}\Omega_7^2\right), \tag{8.85}$$

and

$$X = \left((g\mathcal{R}/4)\sinh\rho\right)^3 . \tag{8.86}$$

The full solution of the gauged supergravity theory is obtained by integrating the equation

$$\frac{\mathrm{d}Y}{\mathrm{d}\rho} = \frac{3}{16}g^3\mathcal{R}^3\sinh^3\rho\ \tanh^2\rho . \tag{8.87}$$

We will not need the lengthy analytic expression for $Y(\rho)$ in the present analysis and thus refrain from presenting it here.

Many of the interesting properties of this solution are only apparent when uplifted to ten or eleven dimensions. Unfortunately we are not able to directly use the general formulae presented in Subsection 8.2.3 since they are not valid for $p = 6$. Nevertheless, since our eight-dimensional solution is rather simple the uplift formulae of [10] can be readily applied and yield the following type IIA background

$$\mathrm{d}s_{10}^2 = \frac{\mathcal{R}^2\mathrm{e}^{2\Phi/3}}{g_s^{2/3}}\left(\frac{1}{4}\mathrm{d}\rho^2 + \mathrm{d}\Omega_7^2 + \frac{1}{16}\sinh^2\rho\ \mathrm{d}\widetilde{\Omega}_2^2\right), \tag{8.88}$$

$$H_3 = \frac{3}{g^2 g_s^2}\mathrm{e}^{2\Phi}\mathrm{d}\rho\wedge \mathrm{V}_2 , \tag{8.89}$$

$$F_2 = \frac{i}{g_s g}\mathrm{V}_2 , \tag{8.90}$$

$$\mathrm{e}^{2\Phi} = g_s^2\left(\frac{g\mathcal{R}}{4}\sinh\rho\right)^3 . \tag{8.91}$$

For $\rho \to 0$ the metric and dilaton approach that of a D6-brane in flat space:

$$\mathrm{d}s_{10}^2 \approx \frac{1}{\sqrt{H}}\mathcal{R}^2\mathrm{d}\Omega_7^2 + \sqrt{H}\left(\mathrm{d}\tilde{r}^2 + \tilde{r}^2\mathrm{d}\widetilde{\Omega}_2^2\right), \tag{8.92}$$

$$\mathrm{e}^{2\Phi} \approx (H)^{-3/2} , \tag{8.93}$$

where $16\tilde{r} = g(\mathcal{R}\rho)^2$ and $H = 1/g\tilde{r}$. The function H is precisely the harmonic function for N D6-branes in the near-horizon limit upon replacing g with N using (8.12)

$$H = \frac{g_s N \ell_s}{2\tilde{r}} . \tag{8.94}$$

We thus conclude that $\rho \to 0$ should be identified with the UV limit of the dual gauge theory. In the limit $\rho \to \infty$ we should be exploring the IR regime of the field theory where the finite size of S^7 should play a role. Indeed the metric in this limit caps off in the expected regular manner whereas the dilaton blows up. This is an indication that we must further uplift our type IIA solution to eleven dimensions and interpret it in M-theory. The uplift to eleven dimensions has to be done with some care because the two-form field strength F_2 and its one-form potential, C_1, are imaginary. The one-form potential appears in the eleven-dimensional metric as a Kaluza-Klein vector and so if it is imaginary it would render the eleven-dimensional metric complex. This is resolved by remembering that, as discussed in Section 8.1, our type II solutions can be interpreted as solutions of Hull's type II* theories [138]. Hull has argued in [137] that the type IIA* theory uplifts to an eleven-dimensional version of M-theory with two time directions called M*-theory. In our formulation this means that we can apply the standard uplift formulae presented in Appendix A with a purely imaginary M-theory circle parameterized by $i\omega$ with $\omega \in \mathbf{R}$. Performing this uplift for the solution in (8.88) we obtain, quite surprisingly, a metric on $\mathbf{H}^{2,2}/\mathbf{Z}_N \times S^7$ where $\mathbf{H}^{2,2} \equiv \mathrm{SO}(3,2)/\mathrm{SO}(2,2)$. Explicitly we find

$$\mathrm{d}s_{11}^2 = \frac{\mathcal{R}^2}{4 g_s^{2/3}} \left(\mathrm{d}s_4^2 + 4\, \mathrm{d}\Omega_7^2 \right) , \tag{8.95}$$

where

$$\mathrm{d}s_4^2 = \mathrm{d}\rho^2 - \frac{1}{4} \sinh^2 \rho \left(\mathrm{d}t^2 - \cosh^2 t\, \mathrm{d}\psi^2 + (N^{-1}\mathrm{d}\omega - \sinh t\, \mathrm{d}\psi)^2 \right) , \tag{8.96}$$

is a metric on $\mathbf{H}^{2,2}$ with three-dimensional anti-de Sitter spacetime boundary, albeit in the wrong signature. Even though the coordinate ω is timelike, it should still be treated as periodic, just like in the standard relation between type IIA and eleven-dimensional supergravity. We have parameterized the M*-theory circle such that ω has periodicity $\omega \sim \omega + 4\pi$. Notice that crucially the metric on AdS_3 is not regular unless $N = 1$. In fact the structure of the metric is precisely that of an extremal BTZ black hole. The analytic continuation of this metric to eleven spacelike dimensions yields a metric on

$$\mathbf{H}^4/\mathbf{Z}_N \times S^7 , \tag{8.97}$$

where the four-dimensional hyperbolic space has a boundary that is a Lens space $S^3/\mathbf{Z}_N$. Given that the eleven-dimensional metric above is closely related to the standard $\mathrm{AdS}_4 \times S^7$ solution of eleven-dimensional supergravity, it is not surprising to find that up to factors of N, the four-form flux is the standard one

$$G_4 = \frac{3i}{2g^2 g_s^2 \ell_s} \left(\frac{g\mathcal{R}}{4} \sinh\rho \right)^3 \mathrm{d}\rho \wedge \mathrm{vol}_2 \wedge \mathrm{d}\omega = \frac{3i}{L_4} \mathrm{vol}_{\mathbf{H}^{2,2}} \,, \tag{8.98}$$

where we have introduced

$$L_4 = \frac{\mathcal{R}}{2g_s^{1/3}} \,. \tag{8.99}$$

As expected we also find that the M2-brane flux

$$N_{\mathrm{M2}} = \frac{1}{(2\pi\ell_s)^6 i} \int G_7 = \frac{2L_4^6}{N^2\pi^2\ell_s^6} \in \mathbf{Z}\,, \tag{8.100}$$

is properly quantized. The explicit appearance of i in (8.98) and (8.100) is a result of our choice of conventions for IIA* and M* theories. We will discuss the holographic interpretation of this curious eleven-dimensional background further in the next chapter.

Chapter 9

Non-conformal precision holography

In Chapter 7, we have introduced maximal SYM on a sphere, S^{p+1}, and computed its free energy and a 1/2-Wilson loop VEV, for $1 \le p \le 6$. Next, in Chapter 8, we constructed the dual gravity solutions in ten-dimensional type II supergravity. With these solutions at hand our aim in this chapter is to holographically compute the corresponding free energies and Wilson loop VEVs.

9.1 The effective 't Hooft coupling

An important ingredient in relating the supergravity results to the field theory results obtained using supersymmetric localization is the definition of holographic 't Hooft coupling. In our conventions, the Dp-brane tension and the Yang-Mills coupling constant are given in terms of the string coupling as

$$\mu_p = \frac{2\pi}{(2\pi\ell_s)^{p+1}}\,, \qquad g_{\rm YM}^2 = \frac{(2\pi)^2 g_s}{(2\pi\ell_s)^4\mu_p} = \frac{2\pi g_s}{(2\pi\ell_s)^{3-p}}\,. \tag{9.1}$$

The dimensionless holographic 't Hooft coupling, $\lambda_{\rm hol}$, is defined by

$$\lambda_{\rm hol}(E) = g_{\rm YM}^2 N E_{\rm hol}^{p-3}\,, \tag{9.2}$$

where N is the number of Dp-branes. The quantity $E_{\rm hol}$ is a finite energy scale defined in an appropriate way through the supergravity solution. Since the supergravity backgrounds of interest here are not asymptotically locally AdS it is a priori not straightforward to define this quantity. A reasonable choice is to define it as the inverse of the effective radius $R_{\rm eff}$ of the $(p+1)$-sphere $\mathrm{d}\Omega_{p+1}^2$ in

(8.42), i.e.

$$R_{\text{eff}} = Q^{-\frac{1}{4}} \mathrm{e}^{A+\frac{\eta}{2}}\,, \tag{9.3}$$

and multiply it by the ten-dimensional dilaton e^{Φ} (8.45). This definition amounts to the following result[1]

$$E_{\text{hol}}^{p-3} = R_{\text{eff}}^{3-p} \frac{\mathrm{e}^{\Phi}}{g_s} = \mathrm{e}^{(3-p)A} \mathrm{e}^{\frac{9-p}{6-p}\eta} \frac{P^{1/2}}{Q^{1/2}}\,. \tag{9.4}$$

This energy scale is finite in the UV limit $r \to \infty$ and thus we propose to identify the holographic 't Hooft coupling in (9.2) by evaluating (9.4) in the UV where

$$A \to \frac{(p-9)}{(6-p)(3-p)}\eta + \text{const}\,. \tag{9.5}$$

The constant in this equation is fixed by regularity of the full supergravity background in the IR, it can therefore not be deduced directly by an UV analysis of the BPS equations. Using that $\lim_{r\to\infty} P(r) = \lim_{r\to\infty} Q(r) = 1$ we arrive at the following explicit result[2]

$$\lambda_{\text{hol}} \equiv \frac{2\pi g_s N}{(2\pi \ell_s)^{3-p}} \mathrm{e}^{(3-p)A} \mathrm{e}^{\frac{9-p}{6-p}\eta}\Big|_{r\to\infty}\,. \tag{9.6}$$

We will sometimes express λ_{hol} in terms of the supergravity gauge coupling g using (8.12). We note that the expression (9.6), which allows us to find a match between supergravity and field theory, does not agree with the one proposed in [150] for all values of p.

9.2 Holographic free energy

The holographic free energy of the spherical Dp-brane solutions is given by the on-shell action in $p+2$ dimensions. This action can be derived from the $(p+2)$-dimensional gauged supergravity, as was shown in the previous chapter, and takes the form

$$S = \frac{1}{2\kappa_{p+2}^2} \int \mathrm{d}^{p+2}x \sqrt{g} \left\{ R + \frac{3p}{2(p-6)} |\mathrm{d}\eta|^2 - 2\mathcal{K}_{\tau\bar{\tau}} |\mathrm{d}\tau|^2 - V \right\}, \tag{9.7}$$

[1] We divide by a factor of g_s since we have already included a factor of g_s in the definition of g_{YM}^2 in (9.1).

[2] An alternative way to obtain (9.6) is to define the *running* gauge coupling, as it appears in the probe action for Dp-branes, by $g_{\text{YM}}^2 = 2\pi \mathrm{e}^{\Phi}/(2\pi\ell_s)^{3-p}$. Then the energy scale is defined by $E = R_{\text{eff}}^{-1}$. When these two expressions are inserted into (9.2) and evaluated at $r \to \infty$ we obtain (9.6).

where the potential V is given in Appendix D in terms of a superpotential $\mathcal{W}$. The $(p+2)$-dimensional Newton constant can be expressed as[3]

$$\kappa_{p+2}^2 = \frac{(2\pi\ell_s)^8 g_s^2}{8\pi} \frac{\Gamma\left(\frac{9-p}{2}\right)}{\pi^{\frac{9-p}{2}}} g^{8-p} \,. \tag{9.8}$$

Evaluating the action in (9.7) on the spherical brane solutions leads to divergences arising from the UV region. Since for $p \neq 3$ the metric is not asymptotically locally AdS, one cannot apply the standard technology of holographic renormalization to cancel these divergences systematically. As explained in [149, 150] a useful approach to circumvent this impasse is to perform a conformal transformation of the metric to the so-called dual frame. This changes its UV asymptotics to a locally AdS form and for the solutions of interest here is achieved by the following rescaling of the metric,

$$g_{\mu\nu} = \mathrm{e}^{2a\eta}\tilde{g}_{\mu\nu} \,, \qquad \text{where} \quad a = \frac{p-3}{6-p} \,. \tag{9.9}$$

Note that the case $p=6$ needs to be treated separately. For $p=3$ the background is asymptotically AdS_5 and no rescaling is needed. In terms of this transformed metric, the action takes the form

$$S = \frac{1}{2\kappa_{p+2}^2} \int \mathrm{d}^{p+2}x \sqrt{\tilde{g}}\, \mathrm{e}^{pa\eta} \Big\{ \tilde{R} + \Big(\tfrac{3p}{2(p-6)} + a^2 p(p+1) \Big) |\mathrm{d}\eta^2| - 2\mathcal{K}_{\tau\tilde{\tau}} |\mathrm{d}\tau|^2 - \mathrm{e}^{2a\eta} V \Big\} \,. \tag{9.10}$$

In this frame the metric is asymptotically AdS and we can use the standard framework of holographic renormalization to obtain the holographic counterterm action. When transformed to the dual frame the Gibbons-Hawking boundary term is given by

$$S_{\mathrm{GH}} = \frac{1}{\kappa_{p+2}^2} \int \mathrm{d}^{p+1}x \sqrt{\tilde{h}} \mathrm{e}^{ap\eta} (p+1) \big(A' - a\eta' \big) \,. \tag{9.11}$$

The remaining divergences should be canceled by the standard curvature counterterms [91]. However, as discussed in [149], the coefficients of these counterterms should be changed with respect to the ones in [91] and are determined by the constant $\sigma = \frac{7-p}{5-p}$. These infinite counterterms are built out of the induced

[3] This expression is derived in some detail in Appendix D.5.

boundary metric in the dual frame, $\tilde{h}_{\mu\nu}$, and are given by

$$S_{\text{ct,curv}} = \frac{1}{\kappa_{p+2}^2} \int \mathrm{d}^{p+1}x \sqrt{\tilde{h}} \mathrm{e}^{ap\eta} \left[\frac{2\sigma - 1}{\sigma - 1} g + \frac{1}{4g} R_{\tilde{h}} + \frac{1}{16g^3} \frac{\sigma - 1}{\sigma - 2} \left(R_{\tilde{h}ab} R_{\tilde{h}}^{ab} - \frac{\sigma}{2(2\sigma - 1)} R_{\tilde{h}}^2 \right) \right]. \quad (9.12)$$

The counterterms in the second line of (9.12) are only needed when $p \geq 4$. Note that this infinite counterterm analysis in the "dual frame" formalism is not applicable for $p = 5$. We will treat this case separately in Section 8.3.4.

Apart from these curvature counterterms we typically need additional infinite counterterms including the scalar fields. For supersymmetric backgrounds we can take advantage of the Bogomol'nyi trick, see for example [54,98], to construct these infinite counterterms. This amounts to adding the following counterterm built out of the superpotential of the gauged supergravity theory

$$S_{\text{ct,superpot}} = \frac{1}{2\kappa_{p+2}^2} \int \mathrm{d}^{p+1}x \sqrt{\tilde{h}} \mathrm{e}^{(p+1)a\eta} \sqrt{\mathrm{e}^{\mathcal{K}} \mathcal{W} \overline{\mathcal{W}}} \bigg|_{Y \to 0}. \quad (9.13)$$

This counterterm is precisely the one that appears when regularizing the free energy of supergravity backgrounds with flat space boundary. There might be additional counterterms appearing, such as conformal couplings of the scalars or terms depending on the scalar field Y, for more general solutions such as our spherical branes. The precise form of these extra infinite counterterms terms as well as any potential finite counterterms will be determined on a case-by-case basis in Section 9.4.

9.3 Holographic Wilson loop VEV

Now, let us demonstrate how to compute supersymmetric Wilson loop vacuum expectation values. The 1/2-BPS Wilson loop captured by supersymmetric localization lies on the equator of the $(p+1)$-sphere and is invariant with respect to the localization supercharge if and only if it is aligned along the field theory scalar field Φ^0. This is realized by a fundamental string wrapping the equator of S^d in the spherical brane solutions and embedded in a specific way in the internal space. To understand this in more detail we consider the embedding of the internal space

I_{8-p} in $\mathbf{R}^{1,8-p}$,

$$X_I: \quad I_{8-p} \to \mathbf{R}^{1,8-p}: \tag{9.14}$$

$$\{\theta, t, \psi, \omega_i\} \mapsto \{\cos\theta \sinh t, \cos\theta \cosh t \sin\psi, \cos\theta \cosh t \cos\psi, \sin\theta\, Y_A\},$$

where the Y_A parameterize the standard embedding of the $(5-p)$-sphere in $\mathbf{R}^{6-p}$. This embedding provides us with an explicit map from the internal space of our supergravity solutions to the field theory scalars appearing in the Lagrangian (7.20), e.g. the scalars Φ^I can be identified with X_I. Therefore, the BPS condition requires that the corresponding holographic Wilson loop lies at constant $\theta = 0$ and $\cosh t = 0$. This implies that the holographic evaluation of the Wilson loop VEV must be performed using the analytically continued fully Euclidean background. Indeed, this is how we obtained a finite Newton constant in (9.8).

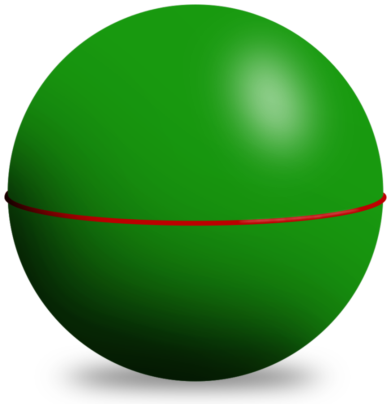


Figure 9.1: A string wrapping the equator of a $(p+1)$-sphere.

In the holographic context we are thus lead to study a probe fundamental string wrapping the equator of the spherical brane as in Figure 9.1. The expectation value of a Wilson line operator in the fundamental representation of the gauge group along a contour $\mathcal{C}$ can be calculated holographically by evaluating the regularized on-shell action of the probe string. More precisely,

$$\log\langle W(\mathcal{C})\rangle = -S^{\text{Ren.}}_{\text{string}}, \tag{9.15}$$

where $S^{\text{Ren.}}_{\text{string}}$ is the renormalized on-shell action. The probe string is governed by the Nambu-Goto action,

$$S_{\text{string}} = \frac{1}{2\pi\ell_s^2}\int_\Sigma \mathrm{d}^2\sigma \sqrt{\det P[G_{MN}]} - \frac{1}{2\pi\ell_s^2}\int_\Sigma \det P[B_2], \tag{9.16}$$

where $P[\dots]$ denotes the pull-back of the bulk fields onto the string worldsheet Σ parameterized by σ_1 and σ_2 and G_{MN} is the ten-dimensional string frame metric. In order to determine the Wilson loop expectation value we have to minimize the string action, regularize it and finally evaluate it on-shell. In order to do this, we parameterize the worldsheet by the coordinates $\sigma_1 = r$ and $\sigma_2 = \zeta \in [0, 2\pi]$, use that translations along ζ are a symmetry of the ten-dimensional solution described in Section 8.2.3, and assume that the induced fields depend only on r. Since B_2 has legs only along the internal de Sitter part of the geometry we conclude that $P[B_2] = 0$. The induced metric on the other hand takes the form

$$P[\mathrm{d}s_{10}^2] = \frac{\mathrm{e}^{\eta}}{\sqrt{Q}} \left[\left(1 + G_{mn} \frac{\partial \Theta^m}{\partial r} \frac{\partial \Theta^n}{\partial r} \right) \mathrm{d}r^2 + \mathrm{e}^{2A} \mathrm{d}\zeta^2 \right], \tag{9.17}$$

where G_{mn} is the metric on the internal space and the functions $\Theta^m(r)$ describe the profile of the string worldsheet in the internal directions. We can identify the functions Θ^m with the $8-p$ coordinates $(\theta, t, \psi, \omega_i)$ with $i = 1, \dots, 5-p$. Minimizing the string action is equivalent to minimizing

$$\det P[G_{MN}] = \frac{\mathrm{e}^{2\eta+2A}}{Q} \left(1 + G_{mn} \frac{\partial \Theta^m}{\partial r} \frac{\partial \Theta^n}{\partial r} \right). \tag{9.18}$$

Since we are performing the holographic computation for the ten-dimensional metric analytically continued to Euclidean signature, the internal metric G_{mn} is positive definite. All terms in the parentheses above are therefore manifestly positive and thus can be minimized by setting each term to zero, i.e. by taking constant Θ^m. To determine the exact position of the string in the internal space, i.e. the constant values of Θ^m, we have to minimize the function

$$\begin{aligned} \det P[G_{MN}]\big|_{\partial_r \Theta^m = 0} &= \frac{\mathrm{e}^{2\eta+2A}}{Q} \\ &= \begin{cases} \frac{\mathrm{e}^{2\eta+2A}}{X} \left(\sin^2\theta + X\cos^2\theta \right) & \text{for } p < 3\,, \\ \frac{\mathrm{e}^{2\eta+2A}}{X} \left(X\cos^2\theta + (X^2 - Y^2)\sin^2\theta \right) & \text{for } p > 3\,. \end{cases} \end{aligned} \tag{9.19}$$

The extrema of these functions are at

$$\theta = \frac{n\pi}{2} \qquad \text{for } n \in \mathbf{Z}\,. \tag{9.20}$$

Since the range of θ is $[0, \pi)$ there are only two inequivalent extrema: $\theta = 0$ and $\theta = \pi/2$. However, as explained at the beginning of this section, only $\theta = 0$ corresponds to a Wilson loop which is BPS with respect to the localizing supercharge.[4]

[4] See [57] for a similar analysis in the context of the four-dimensional $\mathcal{N} = 2^*$ theory on S^4.

We have thus arrived at the following probe string action (9.16)

$$S_{\text{string}} = \frac{1}{\ell_s^2} \int \mathrm{d}r \sqrt{\det P[G_{MN}]} = \frac{1}{\ell_s^2} \int \mathrm{d}r \mathrm{e}^{\eta + A}, \tag{9.21}$$

where we have already performed the integral over the great circle. This on-shell string action diverges close to the UV boundary of the supergravity solution and we have to renormalize it using appropriate covariant counterterms built out of the ten-dimensional supergravity fields. This can be realized by adding the following counterterm commonly used to regularize string on-shell actions [57, 89]. In terms of the gauged supergravity fields, this counterterm takes the form

$$S_{\text{string,ct}} = \frac{1}{g\ell_s^2} \mathrm{e}^{A + \frac{3}{6-p}\eta} \Big|_{r\to\infty} . \tag{9.22}$$

Note that in addition to canceling the divergences of the on-shell string action, in some cases this counterterm contains a finite contribution which will prove to be crucial for our analysis.

Before we discuss the various Dp-branes in detail it is worthwhile to study how the Wilson line VEV scales with N and λ_{hol}. Using the scaling relation (9.2), we find that

$$\log\langle W\rangle \sim N^0 \lambda_{\text{hol}}^{\frac{1}{(5-p)}} . \tag{9.23}$$

This scaling exactly matches the expectations from supersymmetric localization. In addition the same scaling of the Wilson loop vacuum expectation value was found in a holographic finite temperature setting in [61].

9.4 Case-by-case analysis

After discussing the general framework for computing the free energy and Wilson loop expectation values, both from a supergravity and field theory point of view, we proceed with a case-by-case study of the different values of p, starting at $p = 1$ and working our way up to $p = 6$. For D5- and D6-branes some aspects of the general analysis above do not apply and we treat these two cases in some more detail. To avoid confusion, in this section we will denote the QFT 't Hooft coupling in (6.1) by λ_{QFT} to explicitly distinguish it from the one used in supergravity denoted by λ_{hol}.

9.4.1 D1-branes

Field theory

In Chapter 7 we performed a general strong coupling analysis of the matrix model of [182] at large N. Strictly speaking, the matrix model is only well defined for dimensions in the interval $3 < d < 6$. To go beyond this interval let us first try to return to the general form of the kernel in (7.34). If we set $d = 2$ we find that the kernel takes the particularly simple form,

$$G_{16}(\sigma) = \frac{4}{\sigma + \sigma^3} \,. \tag{9.24}$$

A matrix model with this kernel was previously analyzed in [152] where the free energy was derived parameterically in terms of complete elliptic integrals. However, in our case the central potential has a negative sign at $d = 2$, which leads to many subtleties. In particular a straightforward analytic continuation of the results in [152] gives a complex free energy in terms of λ_{QFT}.

Instead, we propose to analytically continue the dimension to $d = 2$ in the expressions for the free energy and Wilson loop VEV in (7.46) and (7.47). Both the free energy and Wilson loop are expressible in terms of the eigenvalue endpoint which, upon substituting $d = 2$ into (7.42), becomes

$$b_2 = \left(\frac{8\lambda_{\text{QFT}}}{\pi}\right)^{1/4} , \tag{9.25}$$

which is real and positive. Having found b_2 we can read of the free energy from equation (7.46),

$$F_2 = -\frac{2\pi}{3\lambda_{\text{QFT}}} (b_2)^2 N^2 = -\frac{4(2\pi)^{1/2}}{3\lambda_{\text{QFT}}^{1/2}} N^2 \,. \tag{9.26}$$

Note that the free energy increases with increasing λ_{QFT}. The Wilson loop VEV is obtained from (7.48) by setting $b = b_2$

$$\log\langle W \rangle = 2\pi b_2 = 2^{7/4} \pi^{3/4} \lambda_{\text{QFT}}^{1/4} \,. \tag{9.27}$$

Supergravity

The supergravity solution for spherical D1-branes is most conveniently described using the scalar field X as the radial variable which in this case runs from $1/3$ in the IR to 1 in the UV. The full solution is specified in Section 8.3.1.

To compute the holographic free energy we evaluate the regularized supergravity action on the spherical D1-brane solution and subtract the counterterms (9.11), (9.12), and (9.13). In addition, due to the presence of the scalar Y we have to subtract the following infinite counterterm

$$S_{\mathrm{ct,inf}} = -\frac{1}{\kappa_3^2}\int \mathrm{d}^2x\,\sqrt{\tilde{h}}\mathrm{e}^{-\frac{2}{5}\eta}\frac{g}{4}Y^2\,. \tag{9.28}$$

Furthermore, there is a unique covariant finite counterterm that can be built out of the boundary metric and scalar fields which reads

$$S_{\mathrm{ct,fin}} = \frac{1}{\kappa_3^2}\int \mathrm{d}^2x\,\sqrt{\tilde{h}}\mathrm{e}^{-\frac{2}{5}\eta}\left(\frac{c_x}{g}\tilde{R}\log X\right)\,. \tag{9.29}$$

Evaluating the holographic 't Hooft coupling (9.6) in the UV leads to the following expression,

$$\lambda_{\mathrm{hol}} = \frac{1}{2^7 g^8 \ell_s^8 \pi^3}\mathrm{e}^{4\eta_{\mathrm{IR}}/5}\,. \tag{9.30}$$

Substituting this expression and subtracting all infinite and finite counterterms we arrive at the following result for the holographic free energy

$$F^{\mathrm{hol}} = -\frac{2(2\pi)^{1/2}N^2}{3\lambda_{\mathrm{hol}}^{1/2}}(3-4c_x)\,. \tag{9.31}$$

We do not have a rigorous argument to fix the coefficient c_x of the finite counterterm but we observe that if we set $c_x = 1/4$ the holographic result in (9.31) agrees with the field theory answer in (9.26) upon identifying λ_{hol} with λ_{QFT}. It will be most interesting to fix c_x by a first principle calculation. This can be presumably achieved by ensuring that the holographic renormalization procedure we employ is compatible with supersymmetry.

To compute the Wilson loop vacuum expectation value we start from the integral (9.21). For $p = 1$ the on-shell probe string action becomes

$$S_{\mathrm{string}} = \frac{1}{\ell_s^2}\int_{1/3}^{1}\frac{\mathrm{d}X}{X'}\mathrm{e}^{\eta(X)+A(X)} = \frac{\mathrm{e}^{\eta_{\mathrm{IR}}/5}}{\sqrt{2}g^2\ell_s^2}\int_{1/3}^{1}\frac{\mathrm{d}X}{\sqrt{1-X^2}(1-X)}\,. \tag{9.32}$$

This integral is divergent and we have to regularize it in the UV by introducing a cutoff at $X = 1-\epsilon$ and subsequently subtracting the counterterm (9.22),

$$S_{\mathrm{string,ct}} = \frac{\mathrm{e}^{\eta_{\mathrm{IR}}/5}}{g^2\ell_s^2}\frac{1}{\sqrt{\epsilon}} + \mathcal{O}(\sqrt{\epsilon})\,, \tag{9.33}$$

in order to obtain the renormalized on-shell action. Using the relation (9.30) we find the following holographic result for the Wilson loop expectation value

$$\log\langle W^{\mathrm{hol}}\rangle = 2^{7/4}\pi^{3/4}\lambda_{\mathrm{hol}}^{1/4}\,. \tag{9.34}$$

This precisely agrees with the QFT result in (9.27).

A comment on the Yang-Mills action

We close this discussion with a comment. In [39,87] (see also [75] for extensions of this analysis) it was shown that there is a Yang-Mills action for an $\mathcal{N}=(2,2)$ vector multiplet on S^2 that is Q-exact and hence the partition function is independent of the Yang-Mills coupling. In terms of the conventions used here, the $\mathcal{N}=(2,2)$ vector multiplet contains the gauge fields A_μ, the scalar fields Φ_0 and Φ_3, and the Dirac field Ψ with the projections

$$\Gamma^{6789}\Psi = \Psi\,, \quad \Gamma^{4567}\Psi = \Psi\,, \tag{9.35}$$

which reduces Ψ to four independent real components. There is also one auxiliary field K^1. All other scalar and auxiliary fields are turned off. If we restrict to four independent supersymmetry transformations where

$$\Gamma^{6789}\epsilon = \epsilon\,, \quad \Gamma^{4567}\epsilon = \epsilon\,, \tag{9.36}$$

and set $\nu^1 = \Gamma^{89}\epsilon$, the transformations on the fields in (7.22) reduce to

$$\delta_\epsilon\left(F_{12} - \frac{\Phi_3}{\mathcal{R}}\right) = -\epsilon\Gamma_{12}\not{D}\Psi \tag{9.37}$$

$$\delta_\epsilon\left(K^1 - \frac{\Phi_0}{\mathcal{R}}\right) = \epsilon\Gamma^{89}\not{D}\Psi \tag{9.38}$$

$$\delta_\epsilon\Psi = \left(F_{12} - \frac{\Phi_3}{\mathcal{R}}\right)\Gamma^{12}\epsilon + \left(K^1 - \frac{\Phi_0}{\mathcal{R}}\right)\Gamma^{89}\epsilon + D_\mu\Phi_I\Gamma^{\mu I}\epsilon - i[\Phi_0,\Phi_3]\Gamma^{03}\epsilon \tag{9.39}$$

$$\delta_\epsilon\Phi_I = \epsilon\Gamma_I\Psi\,. \tag{9.40}$$

It is then straightforward to show that the flat-space Yang-Mills Lagrangian is invariant under the transformations in (9.37) if F_{12} is replaced with $F_{12} - \frac{\Phi_3}{\mathcal{R}}$ and K^1 is replaced with $K^1 - \frac{\Phi_0}{\mathcal{R}}$. At the localization locus both terms are zero so the action is also zero.

If we were to compare this Lagrangian to the one in (7.20) at $d = 2$ and with the fields reduced as described above, then the Lagrangians differ by

$$-\frac{1}{2g_{\rm YM}^2}\mathrm{Tr}\left(\frac{2}{\mathcal{R}}F_{12}\,\Phi_3 - \frac{1}{\mathcal{R}^2}\Phi^3\Phi_3 - \frac{3}{\mathcal{R}^2}\Phi^0\Phi_0 - \frac{2}{\mathcal{R}}K^1\Phi_0 - \frac{1}{\mathcal{R}}\Psi\Lambda\Psi\right). \qquad (9.41)$$

One can show that (9.41) changes by a total derivative under the supersymmetry transformations in (9.37). Hence, both actions preserve $\mathcal{N} = (2,2)$ supersymmetry. However, only the second action can be extended to 16 supersymmetries. The extra term in (9.41) is not Q-exact so it will contribute a coupling dependent part to the partition function.

9.4.2 D2-branes

Field theory

The matrix model analysis in this case is more subtle and one has to be careful when taking the different limits to obtain the kernel. If we set $d = 3 + \epsilon$ then we can approximate $G_{16}(\sigma)$ for $\epsilon \to 0$, $(\epsilon > 0)$, as

$$G_{16}(\sigma) = \frac{2\,\epsilon^2}{\epsilon^2\sigma + \sigma^3} + \frac{\pi\sigma(\coth(\pi\sigma) + \pi\sigma\,\mathrm{csch}^2(\pi\sigma)) - 2}{\sigma^3}\,\epsilon^2 + \mathrm{O}(\epsilon^3). \qquad (9.42)$$

The first term in (9.42) comes from the $n = 0$ term in (7.31) while the second term comes from all other values of n. We can also see from (7.32) that $C_1 \approx 4\pi^2\epsilon$ in this limit, which approaches zero because the super Yang-Mills action is Q-exact in three dimensions. Aside from the first term, all other terms in (9.42) are nonsingular on the real line and of order ϵ^2 or higher. Hence they can be dropped in the saddle point equation in (7.32). Therefore, in the large N limit the saddle point equation reduces to the integral equation[5]

$$\frac{4\pi^2\epsilon}{\lambda_{\rm QFT}}\sigma = 2\,⨍_{-b}^{b}\frac{\rho(\sigma')d\sigma'}{\sigma-\sigma'} - \int_{-b}^{b}\frac{\rho(\sigma')d\sigma'}{\sigma-\sigma'+i\epsilon} - \int_{-b}^{b}\frac{\rho(\sigma')d\sigma'}{\sigma-\sigma'-i\epsilon} + \mathrm{O}(\epsilon^2). \qquad (9.43)$$

Naively it looks like the right hand side of (9.43) is even in ϵ. However, because of the poles at $\sigma \pm i\epsilon$, (9.43) reduces to

$$\frac{4\pi^2\epsilon}{\lambda_{\rm QFT}}\sigma = \pi i\Big(\rho(\sigma+i\epsilon) - \rho(\sigma-i\epsilon)\Big) + \mathrm{O}(\epsilon^2) = -2\pi\epsilon\rho'(\sigma) + \mathrm{O}(\epsilon^2). \qquad (9.44)$$

[5] After a rescaling the integral equation in (9.43) has the same form as in [152] and we could extract the the free energy by taking a limit of their results.

Hence, to leading order in ϵ we have that $\rho(\sigma) = \frac{\pi}{\lambda_{\text{QFT}}}(b^2 - \sigma^2)$. The value of b is fixed by setting $\int_{-b}^{b} \rho(\sigma) d\sigma = 1$, which gives

$$b = b_3 \equiv \left(\frac{3\lambda_{\text{QFT}}}{4\pi} \right)^{1/3} . \tag{9.45}$$

The density $\rho(\sigma)$ and value for b_3 are precisely what one finds when analytically continuing (7.41) and (7.42) to $d = 3$. We can then use (7.46) and (7.47) to find the free energy and the expectation value of the BPS Wilson loop. For the free energy we find

$$F_3 = 0 , \tag{9.46}$$

which is not surprising given the Q-exactness of the SYM action in three dimensions. However, the Wilson loop is surprisingly non-trivial. We find that

$$\langle W \rangle = \frac{3}{\xi^3} (\xi \cosh \xi - \sinh \xi) , \qquad \xi = 6^{1/3} \pi^{2/3} \lambda_{\text{QFT}}^{1/3} . \tag{9.47}$$

To compare with supergravity we note that for $\lambda_{\text{QFT}} \gg 1$ the logarithm of the Wilson loop VEV is approximately

$$\log \langle W \rangle \approx 6^{1/3} \pi^{2/3} \lambda_{\text{QFT}}^{1/3} . \tag{9.48}$$

We stress however that (9.47) is exact for any nonzero λ_{QFT}. If we expand (9.47) at small λ_{QFT} we find that

$$\langle W \rangle = 1 + \frac{1}{10} (6\pi^2 \lambda_{\text{QFT}})^{2/3} + \mathrm{O}(\lambda_{\text{QFT}}^{4/3}) , \tag{9.49}$$

hence this result cannot be reproduced in perturbation theory. Strictly speaking, the perturbative behavior is only found for $\lambda_{\text{QFT}} < \epsilon^2$ where the matrix model approaches a Gaussian model. In this sense, $d = 3$ MSYM is strongly coupled for any nonzero coupling.

One can also see that the behavior of the Wilson loop VEV is essentially an infrared effect as the only relevant contribution to $G_{16}(\sigma)$ comes from the $n = 0$ term in the partition function (7.31). The numerator of this term is the Vandermonde determinant while the denominator is the uncanceled contribution of the constant spherical harmonics about the localization locus [113].

Supergravity

The supergravity solution for spherical D2-branes was derived in 8.3.2. Like for D1-branes we use X as the radial variable which ranges from 2/3 in the IR to 1 in the UV.

In order to obtain the holographic free energy we proceed similarly to the previous case and subtract the counterterms (9.11), (9.12), (9.13) and an additional infinite counterterm

$$S_{\mathrm{ct,inf}} = -\frac{1}{\kappa_4^2}\int \mathrm{d}^3x\sqrt{\tilde{h}}\,\mathrm{e}^{-\frac{1}{2}\eta}\frac{g}{4}Y^2\,, \tag{9.50}$$

in order to obtain a finite free energy. In this case we do not find any finite counterterms. Evaluating the regularized on-shell action we find that the holographic free energy vanishes

$$F^{\mathrm{hol}} = 0\,. \tag{9.51}$$

This agrees with the supersymmetric localization result in (9.46).

In order to compute the holographic Wilson loop expectation value we have to evaluate the following integral,

$$S_{\mathrm{string}} = \frac{1}{\ell_s^2}\int_{2/3}^{1}\frac{\mathrm{d}X}{X'}\mathrm{e}^{\eta(X)+A(X)} = \frac{1}{g^2\ell_s^2}\int_{2/3}^{1}\mathrm{d}X\,\frac{\mathrm{e}^{\eta/2}(X^2-Y^2)}{Y(-2X+2X^2+Y^2)}\,. \tag{9.52}$$

By substituting the solution, (8.57)-(8.58), one can show that the integral reduces to

$$S_{\mathrm{string}} = \frac{\mathrm{e}^{\eta_{\mathrm{IR}}/2}}{g^2\ell_s^2}\int_{2/3}^{1-\epsilon}\mathrm{d}X\,\frac{1}{\sqrt{(1-X)^3(1+3X)}} = \frac{\mathrm{e}^{\eta_{\mathrm{IR}}/2}}{g^2\ell_s^2}\left(\frac{1}{\sqrt{\epsilon}}-\frac{3}{2}\right), \tag{9.53}$$

where we have introduced a cutoff $\epsilon \ll 1$. To regulate the integral we need to subtract the counterterm (9.22) given by

$$S_{\mathrm{string,ct}} = \frac{\mathrm{e}^{\eta_{\mathrm{IR}}/2}}{g^2\ell_s^2}\left(\frac{1}{\sqrt{\epsilon}}-\frac{1}{2}+\mathcal{O}(\sqrt{\epsilon})\right). \tag{9.54}$$

Note that this counterterm contains a crucial finite piece needed to match the localization result. After substituting the explicit expression (9.6) for λ_{hol},

$$\lambda_{\mathrm{hol}} = -\frac{1}{6g^6\ell_s^6\pi^2}\mathrm{e}^{3\eta_{\mathrm{IR}}/2}\,, \tag{9.55}$$

we find the following holographic result for the Wilson loop vacuum expectation value

$$\log\langle W^{\text{hol}}\rangle = -S^{\text{Ren.}}_{\text{string}} = 6^{1/3}\pi^{2/3}\lambda^{1/3}_{\text{hol}}\,. \tag{9.56}$$

This agrees with the field theory result (9.48).

9.4.3 D3-branes

The worldvolume theory on spherical D3-branes is simply the Euclidean $\mathcal{N}=4$ SYM theory on S^4. Since this is a conformal theory we can apply a conformal transformation to map S^4 to $\mathbf{R}^4$ and then analytically continue to Lorentzian signature. The supergravity dual to this the theory is the the well-known $\text{AdS}_5 \times S^5$ background of type IIB supergravity. Both the QFT and supergravity evaluations of the free energy and Wilson loop vacuum expectation value are well-known results available in the literature. Here we briefly summarize how they can be obtained from our general formalism.

Setting $d=4$ in (7.42) we find the eigenvalue endpoint

$$b_4 = \frac{\sqrt{2\lambda_{\text{QFT}}}}{2\pi}\,, \tag{9.57}$$

which is the expected result from the Wigner semi-circle distribution,

$$\rho(\sigma) = \frac{4\pi}{\lambda_{\text{QFT}}}\sqrt{b^2-\sigma^2}\,. \tag{9.58}$$

To determine the free energy, we set $d=4+\epsilon$ and take the limit $\epsilon\to 0$ since there is a singularity in (7.46) at $d=4$. We find

$$F_4 = -\frac{2\pi^2N^2}{\lambda_{\text{QFT}}\,\epsilon}\left(\frac{\lambda_{\text{QFT}}}{2\pi^2}\right)^{1+\epsilon/2} + \text{O}(\epsilon) = -\frac{N^2}{\epsilon} - \frac{N^2}{2}\log\lambda_{\text{QFT}} + \text{O}(\epsilon)\,. \tag{9.59}$$

The divergent piece proportional to ϵ^{-1} is an overall constant that can be removed, leaving the well-known result for the Gaussian matrix model. The Wilson loop VEV can be found by inserting b_4 in (7.47)

$$\log\langle W\rangle = \sqrt{2\lambda_{\text{QFT}}}\,. \tag{9.60}$$

The free energy and the Wilson line VEV for $\mathcal{N}=4$ SYM can also be computed holographically using standard results in the literature. An efficient way to obtain the end result on S^4 is to take the $m=0$ limit of the $\mathcal{N}=2^*$ calculations in [54] and [57].[6]

9.4.4 D4-branes

Field theory

Next we consider the case of spherical D4-branes which can be studied by setting $p = 4$, or equivalently $d = 5$, in the various general expressions above. From (7.42), with $d = 5$ substituted, we find that the eigenvalue endpoint is at

$$b_5 = \frac{\lambda_{\text{QFT}}}{4\pi^2}\,. \tag{9.61}$$

The free energy computed from (7.46) is then given by

$$F_5 = -\frac{\lambda_{\text{QFT}} N^2}{12\pi}\,, \tag{9.62}$$

which agrees with the results in [148, 154, 185].

To compute the VEV of the BPS Wilson loop we need to plug the expression for b_5 in (7.47) and take the large λ limit to find

$$\log\langle W\rangle = \frac{\lambda_{\text{QFT}}}{2\pi}\,. \tag{9.63}$$

Supergravity

The supergravity solution for spherical D4-branes is particularly simple as it is just a dimensional reduction of the $\text{AdS}_7 \times S^4$ solution of eleven-dimensional supergravity. In this case the AdS_7 space has an $S^5 \times S^1$ boundary. The spherical D4-brane solution is obtained by a reduction along S^1 leading to the solution derived in Section 8.3.3. Notice that in this case it is convenient to use η as the radial variable which runs from η_{IR} in the IR to infinity in the UV.

To compute the holographic free energy we follow the, by now familiar, procedure of evaluating the on-shell action and subtracting the infinite counterterms (9.11), (9.12), (9.13). No other counterterms are required in order to regularize the action. However, we do find a number of covariant counterterms which give finite

[6] Note that for $p = 3$ our convention for g^2_{YM} as given in (9.1) differs by a factor 2 from the convention used in [54, 57].

contribution to the on-shell action. These are given by[7]

$$S_{\rm ct,fin} = \frac{1}{\kappa_6^2}\int \mathrm{d}^5x\sqrt{\tilde{h}}\mathrm{e}^{2\eta}\left[c_1\left(\frac{1}{g}\tilde{R}Y^2 - 20gY^4\right) + c_2 gY^6 \right.$$
$$\left. + \frac{c_3}{g}\tilde{R}Y^4 + \frac{c_4}{g^3}\tilde{R}^2Y^2\right]. \quad (9.64)$$

Although these counterterms look innocuous in six-dimensional gauged supergravity, from the perspective of the parent SO(5) gauged seven-dimensional supergravity they are not gauge invariant. This is because the scalar field Y arises as the component of one of the SO(5) gauge fields, $\mathcal{A}_\mu$, along the S^1 direction along which we reduce the seven-dimensional theory [50]. Therefore the Y^2 term in six dimensions corresponds to terms of the form $\mathcal{A}_\mu\mathcal{A}^\mu$.

After adding all these contributions and substituting the 't Hooft coupling

$$\lambda_{\rm hol} = \frac{2\pi}{g^2\ell_s^2}\mathrm{e}^{2\eta_{\rm IR}}, \quad (9.65)$$

the renormalized holographic free energy reads

$$F^{\rm hol} = -\frac{\lambda_{\rm hol}N^2}{96\pi}(10 + 80c_1 + c_2 + 20c_3 + 400c_4). \quad (9.66)$$

Similar to the discussion of the on-shell action for spherical D1-branes in Section 8.3.1 we do not know how to fix the coefficients $c_{1,2,3,4}$ from a first principle calculation. However, we note that a convenient choice, namely

$$c_1 = c_2 = c_4 = 0, \qquad c_3 = -\frac{1}{10}, \quad (9.67)$$

makes the holographic result agree with the QFT calculation (9.62). Gauge invariance of the seven-dimensional supergravity theory would indicate that all four finite counterterms should vanish, however in this case we reproduce the result of [148] and do not find a match with the localization result. However, if we choose the counterterm coefficients as in (9.67) we obtain an agreement with the supersymmetric localization calculation at the expense of breaking the gauge invariance of the supergravity counterterms. This predicament is reminiscent of the results in [34, 35] in the context of holographic renormalization for AdS$_5$ with an $S^3 \times S^1$ boundary.

To evaluate the Wilson loop VEV we substitute the solution into the general

[7] Two more finite counterterms can be written as a product of quadratic curvature invariant times Y^2, for an S^5 boundary, these are related to the last term in (9.64).

expression (9.21) using η as a radial variable. We are then left with the following integral

$$S_{\text{string}} = \frac{1}{\ell_s^2} \int \mathrm{d}r \mathrm{e}^{\eta+A} = \frac{1}{\ell_s^2} \int \frac{\mathrm{d}\eta}{\eta'} \mathrm{e}^{\eta+A} = \frac{8}{g^2 \ell_s^2} \int_{\eta_{\mathrm{IR}}}^{\infty} \mathrm{d}\eta \, \mathrm{e}^{4\eta - 2\eta_{\mathrm{IR}}} \, . \tag{9.68}$$

Evaluating the UV regulated integral and subtracting the counterterm in (9.22) results in

$$\log\langle W^{\text{hol}} \rangle = \frac{\lambda_{\text{hol}}}{2\pi} \, , \tag{9.69}$$

which matches with the localization result in (9.63).

9.4.5 D5/NS5-branes

Field theory

Next, we discuss MSYM on S^6 which was previously investigated in [183]. This case is subtle because both (7.42) and (7.46) have an essential singularity at $d = 6$. To deal with this we set $d = 6 - \epsilon$, after which we find

$$F_6 = -\frac{32\pi^4 \epsilon N^2}{\lambda_b} \mathrm{e}^{-8/3 - \gamma_E} \left(\frac{3\lambda_b}{8\pi^3 \epsilon} \right)^{2/\epsilon} \, , \tag{9.70}$$

where γ_E is the Euler-Mascheroni constant and λ_b is bare 't Hooft coupling. Hence the free energy is negative and infinite for any value of λ_b in the limit $\epsilon \to 0_+$. However, if we substitute $d = 6 - \epsilon$ directly into (7.32) it takes the form to leading order in ϵ

$$\frac{C_1}{\lambda_b} N \sigma_i = \left(\frac{6}{\epsilon} - 6\gamma_E + 4 \right) N \sigma_i - 3 \sum_{j \neq i} (\sigma_i - \sigma_j) \log(\sigma_i - \sigma_j)^2 \, . \tag{9.71}$$

The first term on the right hand side can be absorbed into the 't Hooft coupling, hence we define the renormalized coupling λ_{QFT} in terms of λ_b as

$$\frac{1}{\lambda_{\text{QFT}}} = \frac{1}{\lambda_b} + C_\lambda \, , \tag{9.72}$$

where the constant C_λ is given by

$$C_\lambda = -\left(\frac{6}{\epsilon} - 6\gamma_E + 4 \right) C_1^{-1} = -\frac{3}{8\pi^3} \left(\frac{1}{\epsilon} + \frac{1}{2} \log(4\pi) - \frac{1}{2}\gamma_E - \frac{1}{3} \right) \tag{9.73}$$

Notice that since the r.h.s. of (9.71) contains ϵ^{-1} it is crucial to expand C_1 up to first order in ϵ to obtain C_λ to order $\mathrm{O}(\epsilon^0)$. Substituting λ_b in terms of λ_{QFT} into

(9.70) we find

$$F_6 = -12\pi e^{-8/3-\gamma_E} N^2 \left(1 - \epsilon\left(\frac{8\pi^3}{3\lambda_{\mathrm{QFT}}} - \frac{1}{3} - \frac{1}{2}\gamma_E + \frac{1}{2}\log(4\pi)\right)\right)^{2/\epsilon}$$
$$= -3N^2 \exp\left(-\frac{16\pi^3}{3\lambda_{\mathrm{QFT}}} - 2\right). \tag{9.74}$$

The −2 in the argument of the exponent could be removed by a different scheme choice for C_λ.

A similar treatment can be applied to b_6. Again using (9.72) for λ_{QFT} we find

$$b_6 = 4\sqrt{\pi} e^{-4/3-\gamma_E/2}\left(\frac{3\lambda_b}{8\pi^3\epsilon}\right)^{1/\epsilon} = 2\exp\left(-\frac{8\pi^3}{3\lambda_{\mathrm{QFT}}} - 1\right). \tag{9.75}$$

This leads to the following expectation value for the BPS Wilson loop

$$\log\langle W\rangle \approx 4\pi \exp\left(-\frac{8\pi^3}{3\lambda_{\mathrm{QFT}}} - 1\right). \tag{9.76}$$

The above results can also be directly obtained from the saddle-point equation (9.71) which we consider in detail in Appendix F. While the prefactors of the exponential functions in (9.74) and (9.76) are scheme dependent since they can be changed by a shift of the renormalized coupling λ_{QFT}, we can take the following combination of the free energy and the Wilson-loop VEV,

$$\frac{F_6}{(\log\langle W\rangle)^2} = -\frac{3N^2}{16\pi^2}, \tag{9.77}$$

which is scheme independent.

The form of (9.70) and (9.76) is also suggestive. We expect that the UV completion of 6D maximal super Yang-Mills is the (1,1) little string theory. If we now write the free energy in terms of the little string tension $T = \frac{2\pi^2}{g_{\mathrm{YM}}^2}$, we get

$$F_6 \sim N^2 \exp\left(-\frac{16\pi^3}{3}\frac{T\mathcal{R}^2}{N}\right). \tag{9.78}$$

In the large $\mathcal{R}$ limit S^6 approaches flat space and F_6 falls off to zero, consistent with the flat space free energy found in [77]. The correction away from flat space is suggestive of a non-perturbative contribution coming from the string world-sheet. It would be interesting to explore this further.

Note that (9.75) and the assumption that the eigenvalues are widely separated imply that λ_{QFT} is small and negative. However, (7.35) and (7.36) show that near $d=6$ the crossover from the weak to the strong regime happens when $|\sigma_{ij}| \sim \epsilon^{1/2}$. The approximation is then valid if $b_6 \gg \epsilon^{1/2}$ which corresponds to $\lambda_{\text{QFT}} \gg \frac{4\pi^3}{3(-\log\epsilon)}$. Therefore, in the limit $\epsilon \to 0$ the results in (9.70) and (9.76) can be trusted for any positive 't Hooft coupling.

Supergravity

As can already be seen from the localization computation, handling the divergences in this case is subtle. It is clear that the scaling relation in (9.23) breaks down for $p=5$ and there are also special features of the supergravity solution which render the evaluation of the probe string action difficult. Additionally, the dual frame formalism of [150] is not adapted to the case of five-branes.

The supergravity solution for spherical D5-branes can be obtained from the following system of equations

$$\begin{aligned} X' &= \mathrm{e}^{-2\eta}\sqrt{X}g\frac{2-8X+6X^2-3Y^2}{\sqrt{1-6X+9X^2-9Y^2}}\,, \\ (Y^2)' &= \mathrm{e}^{-2\eta}Y^2 g\frac{1-16X+15X^2-9Y^2}{\sqrt{X(1-6X+9X^2-9Y^2)}}\,, \\ \eta' &= -\frac{1}{10}\mathrm{e}^{-2\eta}g\frac{\sqrt{1-6X+9X^2-9Y^2}}{\sqrt{X}}\,, \\ \mathrm{e}^{2A} &= \frac{\mathrm{e}^{4\eta}X((1-3X)^2-9Y^2)}{g^2Y^2}\,. \end{aligned} \tag{9.79}$$

We were not able to find an analytic solution to this system of equations. However, a numerical solution that interpolates between the IR at $(X,Y^2)=(4/3,16/9)$ and the UV at $(X,Y^2)=(1,0)$ is plotted in Figure 8.5.

In order to extract holographic observables we must find asymptotic expansions for the supergravity fields. Unfortunately the BPS equations do not admit a simple UV expansion. Expressing Y^2 as a function of X results in an asymptotic series which is simultaneously an expansion in $(X-1)$ and $\mathrm{e}^{-1/(X-1)}$ of the general form

$$Y^2 = P_0(X) + \mathrm{e}^{-1/(X-1)}P_1(X) + \mathrm{e}^{-2/(X-1)}P_2(X) + \mathcal{O}(\mathrm{e}^{-3/(X-1)})\,. \tag{9.80}$$

where P_i denotes a power series (possibly with negative powers) in $(X-1)$. The first two terms P_0 and P_1 can be resummed to yield

$$
\begin{aligned}
Y^2 =& -\frac{5X-6X^2+\sqrt{X(4-3X)}}{6} \\
&+ C\mathrm{e}^{-\frac{1+\sqrt{X(4-3X)}}{X-1}} \frac{2X+3X^2+3X\sqrt{X(4-3X)}}{\sqrt{X(4-3X)}} + \mathcal{O}(\mathrm{e}^{-2/(X-1)}),
\end{aligned} \tag{9.81}
$$

where C is a constant that must be carefully chosen so that the UV expansion matches onto the IR. Note that the first line in the above expansion is in fact an exact solution of the BPS equations, However, this solution does not reach the IR since one encounters a singularity at $X = 4/3$. The corresponding UV expansion for η takes the form

$$
\begin{aligned}
\eta =& \eta_{\mathrm{UV}} + \frac{1}{20}\left(\frac{2+2\sqrt{X(4-3X)}}{X-1} + \log\frac{2-X+\sqrt{X(4-3X)}}{4X}\right) \\
&- \frac{6C\mathrm{e}^{-2/(X-1)}}{5(X-1)^2} + \mathcal{O}\left(\frac{\mathrm{e}^{-2/(X-1)}}{X-1}\right).
\end{aligned} \tag{9.82}
$$

In the IR we find an asymptotic series which follows the numerical solution to a very good approximation for a large part of the domain but deviates from the actual solution in the UV. This implies that the IR expansion will not be useful for extracting the holographic observables from the background. Instead we will use our numerical solution. As we will explain, the linear behavior of the “dilaton” η in the UV will prevent us from performing a complete holographic renormalization as we did in the previous examples.

First let us evaluate the expression (9.6) for $p = 5$ to determine the relation of the supergravity parameters to the field theory data. Surprisingly we find that λ_{hol} does not depend on η_{IR} at all. In fact we find

$$
\lambda_{\mathrm{hol}} = \lim_{X\to 1} \frac{8\pi^3 Y^2}{X((1-3X)^2-9Y^2)} = \lim_{\epsilon\to 0}\left(\frac{8\pi^3\epsilon}{3} + \mathcal{O}(\epsilon^2)\right) = 0\,, \tag{9.83}
$$

where we use $X = 1+\epsilon$ and $\epsilon \to 0^+$ in the UV. Since this vanishes in the strict $\epsilon \to 0$ limit we do not have a good definition of λ_{hol} for D5-branes. We will therefore proceed with the computation of holographic observables and extract the λ_{hol} by relating the localization and supergravity result for one of the observables, say the Wilson loop VEV. This relation can then be used to compare the supergravity result for the free energy with (9.74).

Let us therefore evaluate the Wilson loop VEV for spherical D5-branes. In order to do so we can again use X as a radial variable and evaluate the on-shell probe string action. Inserting the expressions (9.79) in the on-shell probe string action, we are left with the following integral

$$S_{\text{string}} = \frac{1}{\ell_s^2}\int \mathrm{d}r\mathrm{e}^{\eta+A} = \frac{1}{\ell_s^2}\int \frac{\mathrm{d}X}{X'}\frac{\mathrm{e}^{5\eta}}{Y}\frac{(1-3X)^2-9Y^2}{2-8X+6X^2-3Y^2}\,. \tag{9.84}$$

Notice that the integrand depends exponentially on the dilaton η, which in the UV diverges as $5\eta = 1/\epsilon + \mathcal{O}(\log\epsilon)$. This implies that the integrand diverges in the UV with a combination of polynomial and exponential powers in the cutoff $1/\epsilon$

$$\mathcal{L}_{\text{string}} = \frac{1}{g^2\ell_s^2}\left[\mathrm{e}^{1/\epsilon}\sqrt{3}\Big(\frac{1}{\epsilon}-1+\mathcal{O}(\epsilon)\Big)+\mathcal{O}(\mathrm{e}^{-1/\epsilon})\right] \tag{9.85}$$

where, as before, $X = 1+\epsilon$. Remarkably, the standard worldsheet counterterm, discussed around (9.22) cancels the entire exponential divergence and leaves a finite on-shell action. Explicitly this counterterm has the form

$$S_{\text{string,ct}} = \mathrm{e}^{5\eta}\left.\frac{\sqrt{X}\sqrt{(1-3X)^2-9Y^2}}{g^2\ell_s^2 Y}\right|_{X=1+\epsilon}\,. \tag{9.86}$$

Once the action has been made finite in the UV we can evaluate it numerically using the numerical solution to the BPS equations. The accuracy of the numerical procedure is limited due to the fact that in the implementation of holographic renormalization we have to subtract large numbers. Nevertheless, we were able to show that with 1% accuracy the following result holds

$$\log\langle W^{\text{hol}}\rangle = -S^{\text{Ren.}}_{\text{string}} \approx \frac{1}{g^2\ell_s^2}\mathrm{e}^{5\eta_{\text{IR}}}\,. \tag{9.87}$$

Comparing this expression with (9.76) suggests the relation

$$\eta_{\text{IR}} = \frac{1}{5}\left(\log(4\pi g^2\ell_s^2)-1\right)-\frac{8\pi^3}{15\lambda_{\text{QFT}}}\,. \tag{9.88}$$

Let us now return to the supergravity action with the aim to extract the holographic free energy. The UV analysis of the bulk supergravity action integrand has the following structure

$$S_{\text{on-shell}} = \frac{\pi^3}{5g^5\kappa_7^2}\left[\mathrm{e}^{2/\epsilon}\Big(\frac{576}{\epsilon^5}+\frac{1248}{\epsilon^4}+\mathcal{O}(\epsilon^{-3})\Big)+\mathcal{O}(\mathrm{e}^{1/\epsilon})\right]. \tag{9.89}$$

The polynomial divergence multiplying $\mathrm{e}^{2/\epsilon}$ can be canceled by the standard covariant counterterms. However this still leaves seemingly infinitely many finite terms multiplying an exponential divergence. But this need not be a problem, since in the case of five-branes, infinitely many counterterms are available due to the linear dilaton behavior in the UV. It therefore seems that it is required to use infinitely many counterterms to eliminate the exponential divergence in (9.89). Indeed, we have not been able to find a finite set of counterterms that renders the action finite.[8] If we nevertheless assume that (finitely or infinitely many) counterterms can be found that render the action finite, the form of the resulting expression can be deduced on general grounds. Since the bulk action is proportional to $e^{10\eta}$ we expect

$$S^{\text{Ren.}}_{\text{on-shell}} = -\frac{6\pi^3 \mathcal{I}}{g^5 \kappa_7^2} e^{10\eta_{\text{IR}}} , \tag{9.90}$$

where $\mathcal{I}$ is an undetermined constant that we are not able to evaluate without a full knowledge of the counterterms. Using (9.88) we find

$$F^{\text{hol}} = S^{\text{Ren.}}_{\text{on-shell}} = -3\,\mathcal{I} N^2 \exp\left[-\frac{16\pi^3}{3\lambda_{\text{QFT}}} - 2\right] , \tag{9.91}$$

in a nice agreement with the field theory result (9.74). As we argued above, the coefficients of the exponentials in (9.74) and (9.76) are dependent on the renormalization scheme but a scheme independent quantity can be found by combining the two as in (9.77). We observe that the same combination in holography does not rely on the map (9.88) and we find

$$\frac{S^{\text{Ren.}}_{\text{on-shell}}}{(S^{\text{Ren.}}_{\text{string}})^2} = -\frac{3N^2\mathcal{I}}{16\pi^2} , \tag{9.92}$$

which matches (9.77) if the constant $\mathcal{I}$ equals one.

9.4.6 **D6-branes**

Field theory

Finally, we turn to $d = 7$ and start by rewriting the one-loop determinant in (7.31) as

$$Z_{1-\text{loop}}(\sigma) = \exp\left(\sum_{i<j}\sum_{n=1}^{\infty} 2(n^2+1)\log\left(1+\frac{\sigma_{ij}^2}{n^2}\right)\right) . \tag{9.93}$$

[8] Such a finite set of counterterms was shown to exist in a recent study of five-branes on some curved manifolds [7].

To test the divergence we expand the log at large n, showing that the log of the determinant behaves as

$$\log Z_{1-\text{loop}}(\sigma) \sim \frac{1}{2}\sum_{i,j}\sum_{n} 2\sigma_{ij}^2\left(1+n^{-2}\right) - \sigma_{ij}^4\left(n^{-2}+n^{-4}\right) + \dots . \tag{9.94}$$

The sum over n leads to a linear divergence for the σ_{ij}^2 term while the higher terms are finite. The divergent piece can be rewritten as

$$2n_0 N \sum_i \sigma_i^2 , \tag{9.95}$$

where n_0 is a UV cutoff in n. This divergence has the form of the action in (7.30) and can be absorbed by shifting the coupling. As for the D5-brane case, we can define a bare and a renormalized 't Hooft coupling through the relation

$$\frac{1}{\lambda_{\text{QFT}}} = \frac{1}{\lambda_b} - \frac{n_0}{2\pi^4} . \tag{9.96}$$

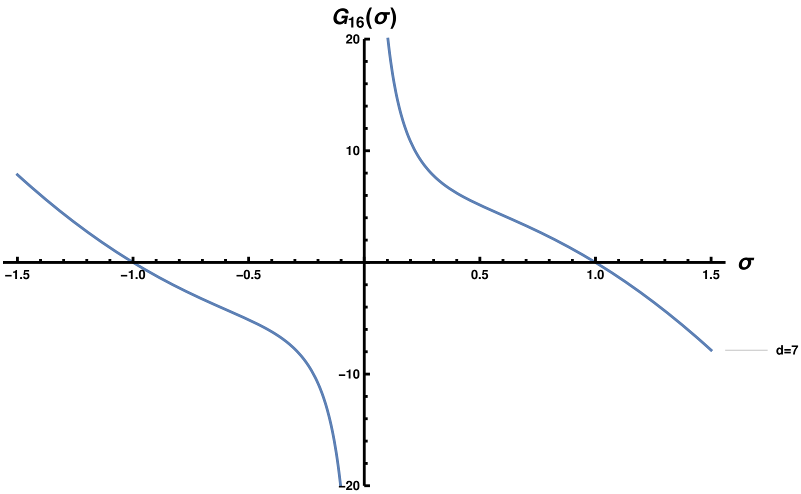


Figure 9.2: The kernel $G_{16}(\sigma)$ for $d = 7$. At $|\sigma| = 1$ the kernel crosses over from repulsive to attractive behavior.

The finite remainder from $Z_{1-\text{loop}}(\sigma)$ is what contributes to the analytic continuation of (7.34) around the singularity at $d = 7$. If we assume large separation between the eigenvalues then we can use (7.38) if we replace λ with λ_{QFT} in the left-hand side of the equation. However, C_2 in (7.37) is negative at $d = 7$. This is evident in Figure 9.2 which shows $G_{16}(\sigma)$ for $d = 7$. At short distance the kernel is repulsive, but becomes attractive for $|\sigma| > 1$. Because of this negative sign, if we analytically continue (7.42) to $d = 7$ we find that the eigenvalue endpoint is at

$$b_7 = -\frac{2\pi^3}{\lambda_{\text{QFT}}} \,. \tag{9.97}$$

The negative value for (9.97) indicates that strictly speaking this is not a solution to (7.40) assuming the eigenvalue distribution has the form in (7.39). This is obvious since (7.40) corresponds to an attractive central potential and an everywhere attractive potential between the eigenvalues. In this case the only solution has all eigenvalues at zero.

To sort this out let us consider the full $d = 7$ kernel shown in Figure 9.2,

$$G_{16}(\sigma-\sigma') = 2\pi(1-(\sigma-\sigma')^2)\coth\pi(\sigma-\sigma')\,, \tag{9.98}$$

and take the strong coupling limit so that the inverse renormalized coupling approaches $\lambda_{\text{QFT}}^{-1} \to 0_+$. While we cannot solve (7.32) analytically in this limit, we can determine the eigenvalue distribution numerically. This is shown in Figure 9.3 where we see that the short distance repulsion stabilizes the eigenvalues into a bounded two hump distribution. Hence the free energy approaches a constant multiplied by N^2 in the strong coupling limit.

Now suppose we continue $\lambda_{\text{QFT}}^{-1}$ through zero, such that $\lambda_{\text{QFT}} < 0$. The central potential is now repulsive and the eigenvalues are pushed farther away from the center, but are still stabilized by the attractive long-range force. As we let $\lambda_{\text{QFT}}^{-1}$ become more and more negative the two humps in Figure 9.3 get pushed farther apart and we can then use the large separation approximation in (7.36). In fact, in this approximation the eigenvalue density becomes two delta functions, as is shown in Appendix G.1. In Appendix G.2 we numerically show that the short range repulsion between the eigenvalues widens the delta functions to a width of order 1.

Since $\lambda_{\text{QFT}} < 0$, b_7 in (9.97) is positive. In order for the large separation assumption to be valid we require $b_7 \gg 1$, which happens when $\lambda_{\text{QFT}}^{-1} \ll -1$. Hence we are in a negative weak regime for the renormalized coupling, which is distinctly different from the usual positive weak coupling regime. Note that while $\lambda_{\text{QFT}}^{-1} \ll -1$, λ_b which appears in the original Lagrangian satisfies $\lambda_b^{-1} \gg 1$. If we now carry out the analytic continuation of (7.46) we find

$$F_7 = \frac{4\pi^4 N^2}{3\lambda_{\text{QFT}}}\left(-\frac{\lambda_r}{2\pi^3}\right)^{-2} = \frac{16\pi^{10}N^2}{3\,\lambda_{\text{QFT}}^3}\,, \tag{9.99}$$

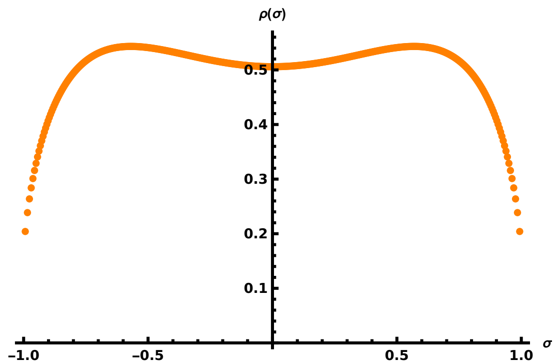


Figure 9.3: Distribution of eigenvalues obtained from (7.32) numerically for $d = 7$, $\lambda_{\mathrm{QFT}}^{-1} = 0$ and $N = 501$. The eigenvalues are clearly bounded within a finite region.

which diverges toward negative infinity as $\lambda_{\mathrm{QFT}}^{-1} \to -\infty$. Likewise, for the Wilson loop using (7.47) we find that

$$\log\langle W\rangle = \log\cosh(2\pi b_7) \approx -\frac{4\pi^4}{\lambda_{\mathrm{QFT}}}, \tag{9.100}$$

which increases as $\lambda_{\mathrm{QFT}}^{-1} \to -\infty$. Note that the cosh function is consistent with the delta function support at $d = 7$.

Since the central potential is unbounded from below, the position of the eigenvalue center of mass is unstable. However, if the gauge group is SU(N) and not U(N) then the eigenvalues satisfy the trace constraint $\sum_i \sigma_i = 0$, which keeps the center of mass of the eigenvalues at the origin. This suggests that the U(N) theory cannot be continued to negative λ_{QFT}.

Note further that the saddle point analysis is robust if λ_{QFT} is small and negative, even when N is finite. As an example, consider an SU(2) gauge group. This has two eigenvalues $\sigma_1 = -\sigma_2$, and following the analysis in Appendix G.3 the saddle point gives the same free energy as (9.99) with $N = 2$. From G.3 we also see that the fluctuations to the free energy about the saddle point are

$$\delta F = -\frac{4\pi^4}{\lambda_{\mathrm{QFT}}}(\delta\sigma_1)^2, \tag{9.101}$$

hence the fluctuations are sharply suppressed and can be ignored if $\lambda_{\mathrm{QFT}}^{-1} \ll -1$.

Supergravity

Similarly on the gravity side, the spherical D6-branes do not fit in the general framework described in Section 9.2 and 9.3. In this case the $SO(6-p)$ symmetry is trivial, hence the internal space is given simply by the two-dimensional de Sitter factor. Furthermore the eight-dimensional supergravity featured in the construction is particularly simple and contains only two scalar fields instead of the familiar three. For this reason the analysis of spherical D6-branes in supergravity will proceed differently than the cases above in several ways.

The spherical D6-brane solution is given by the following type IIA supergravity background, where we keep the radius $\mathcal{R}$ of the sphere arbitrary,

$$\begin{aligned}
ds_{10}^2 =& \frac{\mathcal{R}^2 e^{2\Phi/3}}{g_s^{2/3}} \left(\frac{1}{4} d\rho^2 + d\Omega_7^2 + \frac{1}{16} \sinh^2\rho \, d\widetilde{\Omega}_2^2 \right) , \\
H_3 =& \frac{3}{g^2 g_s^2} e^{2\Phi} d\rho \wedge \mathrm{vol}_2 , \\
F_2 =& \frac{i}{g_s g} \mathrm{vol}_2 , \\
e^{2\Phi} =& g_s^2 \left(\frac{g\mathcal{R}}{4} \sinh\rho \right)^3 .
\end{aligned} \tag{9.102}$$

The radial coordinate ρ takes values from 0 to ∞. It is convenient at this point to define the new coordinate

$$U \equiv \frac{2\pi^4 \mathcal{R}^2 \sinh^2\rho}{g_{\mathrm{YM}}^2 N} . \tag{9.103}$$

The equations in (9.102) then reduce to

$$
\begin{aligned}
ds_{10}^2 =& \ell_s^2 \Bigg(\left(\frac{g_{\rm YM}^2 N}{2(2\pi)^4 U} \right)^{1/2} \frac{dU^2}{1 + \frac{g_{\rm YM}^2 N U}{2\pi^4 \mathcal{R}^2}} + \left(\frac{2(2\pi)^4 U}{g_{\rm YM}^2 N} \right)^{1/2} \mathcal{R}^2 d\Omega_7^2 \\
& + \left(\frac{g_{\rm YM}^2 N U^3}{2(2\pi)^4} \right)^{1/2} d\widetilde{\Omega}_2^2 \Bigg), \\
H_3 =& \frac{3\ell_s^2 g_{\rm YM}^2 N U}{(2\pi)^4 \mathcal{R}} \frac{dU \wedge \mathrm{vol}_2}{\sqrt{1 + \frac{g_{\rm YM}^2 N U}{2\pi^4 \mathcal{R}^2}}}, \\
F_2 =& \frac{i N \ell_s}{2} \mathrm{vol}_2 , \\
e^{2\Phi} =& \left(\frac{g_{\rm YM}^2 U^3}{2\pi^4 N^3} \right)^{1/2} .
\end{aligned}
\tag{9.104}
$$

These equations reduce to the flat space supergravity solutions in [146] when taking $\mathcal{R} \to \infty$ while keeping U and $g_{\rm YM}^2 N$ fixed. The parameter U can be thought of as the energy of a string stretched between a probe D6-brane and the N D6-branes. For small U this is directly probing the weakly coupled 7D MSYM which has an effective coupling $g_{\rm eff}^2 = g_{\rm YM}^2 U^3$. However, in string units one sees that the curvature on the dS_2 is large for small U so supergravity can not be trusted in this regime.

Following work of Susskind and Witten [229], Peet and Polchinski observed that in fact, U is not the energy scale for a probe in supergravity [202]. Instead, this energy scale is determined by the wave equation for a field in the bulk, say a scalar ψ, which is given by

$$
\left(-\frac{\partial^2}{\partial U^2} + \frac{k^2 g_{\rm YM}^2 N}{2(2\pi)^4 U} \right) U \psi = 0 , \tag{9.105}
$$

where we have ignored the modes on dS_2. From this we see that the energy scale for supergravity is

$$
E = \left(\frac{2(2\pi)^4}{g_{\rm YM}^2 N U} \right)^{1/2} . \tag{9.106}
$$

In terms of this energy we have that effective coupling is

$$
g_{\rm eff}^2 = g_{\rm YM}^2 \left(\frac{2(2\pi)^4}{g_{\rm YM}^2 N U} \right)^{3/2} , \tag{9.107}
$$

which decreases with increasing U. At the same time, the curvature on dS_2 is small if $U^3 \gg \frac{2(2\pi)^4}{g_{\rm YM}^2 N}$, which corresponds to $g_{\rm eff}^2 N \ll 2(2\pi)^4$. Hence it seems that the supergravity is dual to a weakly coupled gauge theory, but not the standard weakly coupled gauge theory since that is found at small U where we cannot trust the supergravity.

Now let us assume that $\mathcal{R}$ is large but finite. We then see from (9.104) that we are in the flat brane regime when E as defined in (9.106) satisfies $E \gg \mathcal{R}^{-1}$. This shows that an observer starts seeing the curvature of the branes when the energy scale is on the order of the inverse radius. Furthermore, the radius of the S^7 should be small in string units, which requires that $E \ll \frac{2(2\pi)^4\mathcal{R}^3}{g_{\rm YM}^2 N}\mathcal{R}^{-1}$, hence we need weak coupling in order to trust the supergravity for distances significantly below the size of the sphere. As E approaches the scale set by the sphere its dependence on U starts to change in such a way that when $U \gg \frac{2\pi^4\mathcal{R}^2}{g_{\rm YM}^2 N}$, E scales as $(\log U)^{-1} \sim \rho^{-1}$.

As we keep increasing U the string coupling eventually becomes large and we should uplift the solution to eleven-dimensional supergravity. As demonstrated in Section 8.3.5, the uplifted metric and form fields take the form of $\mathbf{H}^{2,2}/\mathbf{Z}_N \times S^7$. Explicitly, the eleven-dimensional metric is given by

$$\mathrm{d}s_{11}^2 = \frac{L^2}{4}\left(\mathrm{d}s_4^2 + 4\mathrm{d}\Omega_7^2\right), \qquad L = \mathcal{R}/g_s^{1/3} \tag{9.108}$$

$$\mathrm{d}s_4^2 = \mathrm{d}\rho^2 - \frac{\sinh^2\rho}{4}\left(\mathrm{d}t^2 - \cosh^2 t\,\mathrm{d}\psi^2 + (N^{-1}\mathrm{d}\omega - \sinh t\,\mathrm{d}\psi)^2\right).$$

This metric has two time directions, t and ω, which is to be expected since it describes the M-theory lift of a Euclidean brane. The eleven-dimensional 4-form is given by

$$G_4 = \frac{6i}{L}\,\mathrm{vol}_{\mathbf{H}^{2,2}}, \tag{9.109}$$

where $\mathrm{vol}_{\mathbf{H}^{2,2}}$ is the volume form for the $\mathbf{H}^{2,2}/\mathbf{Z}_N$ metric. The energy scale on the sphere maintains the ρ^{-1} fall off so that for large ρ the only mode accessed is the constant one. Note that there is also a conical singularity at $\rho = 0$ if $N > 1$. This singularity is what is left of the highly curved IIA theory at small U.

Let us now use the results from the previous section to propose a dual theory to the supergravity. We saw using localization that there was a smooth transition between positive and negative $\lambda_{\rm QFT}$. We also noticed that the "strong coupling" behavior, with widely separated eigenvalues, occurs when $\lambda_{\rm QFT}^{-1} \ll -1$. If we assume that $g_{\rm YM}^2 < 0$ in the supergravity, then (8.12) and (9.1) imply that the metric and $\mathrm{e}^{2\Phi}$ in (9.102) have a negative sign. To compensate for this we can send $\rho \to -\rho$ in which case we go back to the original signs for the metric and string coupling, while the H_3 field changes sign. The eleven-dimensional supergravity metric in (9.108) is unchanged but the four-form field in (9.109) changes by a sign under these transformations. Hence, now everything looks almost the same as before, except any dictionaries we have between the supergravity and the gauge theory should have g^2_{YM} replaced with $-g^2_{\mathrm{YM}}$. For example, the condition for small curvature on the dS_2 is now $U^3 \gg -\frac{2(2\pi)^4}{g^2_{\mathrm{YM}}N}$, which translates to the relation $-g^2_{\mathrm{eff}}N \ll 2(2\pi)^4$ for the effective coupling .

We are now ready to compute the free energy and Wilson loop VEV using supergravity. One way to evaluate the free-energy of the spherical D6-brane is to use the eight-dimensional gauged supergravity used to construct the background. The eight-dimensional action is

$$S = \frac{1}{2\kappa_8^2}\int \star_8\left\{R - \frac{1}{2}\left(|\mathrm{d}\beta|^2 + \mathrm{e}^{2\beta}|\mathrm{d}\chi|^2\right) - \frac{3g^2}{2}\mathrm{e}^{\beta}\right\}\,, \tag{9.110}$$

where κ_8^2 is given by (9.8). The eight-dimensional BPS equations are written in terms of the metric

$$\mathrm{d}s_8^2 = \mathrm{d}r^2 + \mathcal{R}^2\mathrm{e}^{2A}\mathrm{d}\Omega_7^2\,, \tag{9.111}$$

where the metric function A is only a function of the radial variable r. The BPS equations read

$$\begin{aligned} \beta &= -6A\,,\\ \chi' &= 6i\mathcal{R}^{-1}\mathrm{e}^{-7A-2\beta}\,,\\ (A')^2 &= \mathcal{R}^{-2}\mathrm{e}^{-2A} + \frac{g^2\mathrm{e}^{\beta}}{16}\,. \end{aligned} \tag{9.112}$$

These equations can be easily solved by using the function A as the radial variable. It can then be related to the coordinate ρ appearing in (9.102) by the transformation

$$\mathrm{e}^{2A} = \frac{g\mathcal{R}}{4}\sinh\rho\,. \tag{9.113}$$

Evaluating the action on-shell using the above expression for the eight-dimensional fields results in

$$S = \frac{g\mathcal{R}^9}{2^{11}g_s^2\ell_s^8\pi^2}\int_0^\infty (1 + 7\cosh 2\rho)\sinh\rho\;\mathrm{d}\rho\,, \tag{9.114}$$

where we have included the Gibbons-Hawking term and performed the integral over the 7-sphere. This integral diverges as $\rho \to \infty$ which as we argued before is the IR of the geometry. The eight-dimensional metric is in fact completely regular there whereas the scalar β diverges. This statement is of course dependent on the frame we use in supergravity. It is a lucky coincidence here that the Einstein frame metric is regular whereas the string frame or any other frame which is related to the metric above via a power of the scalar field e^{β} is singular. Subtracting divergences at $\rho \to \infty$ can therefore be done as before, by changing frame and introduce curvature counterterms such that the divergences cancel. We can also perform minimal subtraction; expand out the divergent terms and remove them by hand. In both cases the result is the same, the contribution of the IR is eliminated completely. The on-shell action is completely dominated by its contribution in the UV. Using the relations in (8.12) and (9.1) with $g_{\rm YM}^2$ replaced with $-g_{\rm YM}^2$ we find

$$F^{\rm hol} = \frac{16\pi^{10}N^2}{3\lambda_{\rm hol}^3} \,, \tag{9.115}$$

where

$$\lambda_{\rm hol} \equiv N g_{\rm YM}^2 \mathcal{R}^{-3} = -\frac{2^5\pi^4\ell_s^2}{g\mathcal{R}^3} \,, \tag{9.116}$$

is defined as before but since there is no η-scalar in this case the equation (9.6) is not directly applicable. The extra minus sign is to account for the negative Yang-Mills coupling. This result is in complete agreement with the localization result in (9.99).

We can also obtain the result (9.115) from the eleven-dimensional supergravity solution. Before any Wick rotation the eleven-dimensional action is given by [137, 138]

$$S = \frac{1}{16\pi G_{11}} \int \mathrm{d}^{11}x \sqrt{g^{(11)}} \left(R^{(11)} - \tfrac{1}{2}|G_4|^2 \right) , \tag{9.117}$$

where

$$|G_4|^2 \equiv \frac{1}{4!} (G_4)_{\mu_1 \dots \mu_4} (G_4)^{\mu_1 \dots \mu_4} \,. \tag{9.118}$$

Substituting the solution (9.108)-(9.109) into (9.117) results in

$$S = \frac{1}{16\pi G_{11}} \int \mathrm{d}^{11}x \sqrt{g^{(11)}} \frac{12}{L^2} \,. \tag{9.119}$$

In order to evaluate this on-shell action we need to Wick rotate one of the time directions as $t \to -i\tau$ in (9.108). This changes the metric $\mathrm{d}s_4^2$ to

$$\mathrm{d}s_4^2 \to \mathrm{d}\rho^2 + \frac{\sinh^2\rho}{4} \left(\mathrm{d}\tau^2 + \cos^2\tau\, \mathrm{d}\psi^2 - (N^{-1}\mathrm{d}\omega + i \sin\tau\, \mathrm{d}\psi)^2 \right) , \tag{9.120}$$

Note that the M-theory circle parameterized by ω remains time-like. Even though the metric is now complex, its determinant remains real. The on-shell action then becomes[9]

$$S_{\text{on-shell}} = -\frac{1}{16\pi G_{11}} \int \mathrm{d}^{11}x\sqrt{-g^{(11)}}\frac{12}{L^2} = -\frac{1}{16\pi G_{11}}\frac{\pi^6 L^9}{2N}\int_0^{\rho_0} \mathrm{d}\rho \sinh^3\rho \,, \tag{9.121}$$

where we have introduced a UV cutoff ρ_0 to regulate the volume of $\mathbf{H}_{3,1}$ in (9.120). As we take $\rho \gg 1$ the on-shell action behaves as

$$S_{\text{on-shell}} = -\frac{1}{16\pi G_{11}}\frac{\pi^6 L^9}{2N}\left(\frac{1}{24}\mathrm{e}^{3\rho_0} - \frac{3}{8}\mathrm{e}^{\rho_0} + \frac{2}{3} + \mathcal{O}(\mathrm{e}^{-\rho_0})\right) . \tag{9.122}$$

The divergent contributions in this expression should then be removed to obtain a finite action. Using $G_{11} = 16\pi^7\ell_s^9$ and the modified AdS/CFT dictionary to account for the negative coupling, $(2\pi\ell_s)^3 g_s = -\frac{g_{\text{YM}}^2}{2\pi}$, we find[10]

$$S_{\text{on-shell}}^{\text{Ren.}} = \frac{16\pi^{10}}{3\lambda_{\text{hol}}^3}N^2 \,. \tag{9.123}$$

This again agrees nicely with the free energy in (9.99).

The BPS Wilson loop can be computed using the IIA solution in (9.102). The on-shell string action in this case is given in terms of the eight-dimensional metric function

$$S = \frac{1}{\ell_s^2}\int \mathcal{R}\mathrm{e}^{A}\,\mathrm{d}r \,. \tag{9.124}$$

Changing coordinates to the radial coordinate ρ as above we find

$$S_{\text{string}} = \frac{g\mathcal{R}^3}{8\ell_s^2}\int_0^{\infty} \mathrm{d}\rho \sinh\rho \,, \tag{9.125}$$

Just like the on-shell action, this integral diverges in the IR and can be regularized by adding a simple counterterm analogous to (9.22). This counterterm implements minimal subtraction resulting in the following expression for the Wilson line expectation value

$$\log\langle W^{\text{hol}}\rangle = -\frac{4\pi^4}{\lambda_{\text{hol}}} \,, \tag{9.126}$$

[9]The contribution to the integral over ρ in (9.121) might not be trustworthy for $\rho \lesssim |\lambda_{\text{QFT}}|^{1/3}$. However, if $|\lambda_{\text{QFT}}| \ll 1$, then this will lead to corrections of order $|\lambda_{\text{QFT}}|^{2/3}$ and the results in (9.123) can be trusted to leading order.

[10]Note that in [50] the regularized on-shell action was computed using a four-dimensional effective supergravity approach leading to a result which differs by a factor of 2 from (9.123). The eight-dimensional and eleven-dimensional approaches we use here is better justified and should be employed instead.

where we have again flipped the sign of $g_{\rm YM}^2$ in the dictionary. This precisely agrees with the localization result in (9.100).

An alternative way to compute the Wilson loop VEV is to evaluate the on-shell action of an appropriately embedded M2-brane in the eleven-dimensional solution (9.108). The M2-brane wraps the equator of S^7 and extends along ρ and the M-theory circle ω. In particular the brane is fixed along t and ψ since it should be constant along the field theory scalar ϕ_0. The holographic dual to the Wilson loop VEV is then given by

$$\log\langle W^{\rm hol}\rangle = -S_{M2}^{\rm Ren.}\,, \tag{9.127}$$

where the probe M2-brane on-shell action is given by

$$S_{\rm M2} = \mu_2 \int \mathrm{d}^3\sigma \sqrt{\det P[G_{MN}]}\,. \tag{9.128}$$

$P[G_{MN}]$ denotes the pullback of the determinant of the eleven-dimensional metric to the M2-brane worldvolume and the brane tension is given by $\mu_2 = \frac{2\pi}{(2\pi\ell_s)^2}$. Evaluating this action on our solution gives the following diverging result

$$S_{\rm M2} = \mu_2 \frac{L^3}{4}\frac{2\pi}{N}(2\pi)\int_0^\infty \mathrm{d}\rho\,\sinh\rho = \frac{gL^3}{8\ell_s^2}(\cosh\rho_0 - 1)\,. \tag{9.129}$$

where in the second step a cut-off ρ_0 was introduced to regulate the divergence. By adding a simple counterterm

$$S_{\rm W,ct} = -\frac{gL^3}{8\ell_s^2}\sinh\rho_0\,, \tag{9.130}$$

very similar in spirit to the counterterm (9.22), we obtain the following renormalized on-shell action

$$S_{\rm M2}^{\rm Ren.} = -\frac{gL^3}{8\ell_s^2}\,. \tag{9.131}$$

Inserting the expression for $\lambda_{\rm hol}$ in this equation with a sign change in the dictionary results in the following expression for the holographic Wilson loop VEV

$$\log\langle W^{\rm hol}\rangle = -\frac{4\pi^4}{\lambda_{\rm hol}}\,. \tag{9.132}$$

This agrees nicely with the type IIA calculation in (9.126) and the localization result in (9.100).

In [146] it was noted that the supergravity solution for D6-branes could be trusted even for small N. This is consistent with our results here. As we showed in the last section, the form of the free energy holds for small N, at least if N is even. In the classical supergravity we find the same free energy as a function of N so this appears to align well with the claim in [146]. There is a subtlety however for odd N. As follows from (G.18), the localization result for the free energy comes with an extra factor of $\frac{N(N-1)}{(N-1/2)^2}$. This arises because one eigenvalue has to be placed at the origin in the solution to the saddle point equation. Hence, it is essentially a quantization condition that supergravity does not directly see. In the $N = 1$ case the free energy is zero for the gauge theory, not surprisingly since the gauge group is SU(1) which is trivial. The supergravity does not look trivial although the eleven-dimensional uplift is now smooth at the origin. It would be interesting to understand this point better.

Chapter 10

Discussion and future directions

In this part we constructed explicit supergravity solutions preserving sixteen real supercharges, describing the back-reaction of spherical Dp-branes with $1 \leq p \leq 6$. We argued that these backgrounds are holographically dual to the planar limit of maximally supersymmetric Yang-Mills theory on S^{p+1}. To check this claim we computed both on the field theory and the gravity side the free energy and the vacuum expectation value of a $1/2$-BPS loop. On the field theory side, thanks to supersymmetric localization, we can compute these observables by reducing the full path integral to a finite dimensional matrix model. In the large N limit we can explicitly compute analytic expressions for both these observables. On the other hand, on the supergravity side, by employing a non-trivial application of holographic renormalization, we can compute the same observables and find an exact match at leading order in N. This constitutes a non-trivial check of holography in a non-conformal setting.

However, our work also opens up many questions and interesting directions to extend and generalize it. A first interesting path is to consider SYM theories with less supersymmetry. Both pure and matter coupled super Yang-Mills theories with eight supercharges exist on S^d with $d \leq 6$ and it is possible to study them in the large N limit using supersymmetric localization. In [113, 182] the localization computation was carried out both for pure SYM as well as SYM coupled to fundamental hypermultiplets. Furthermore for $d \leq 4$, theories preserving only four supercharges were localized as well. It would be very interesting to study the resulting matrix models in more detail and compute the free energy and various Wilson loop vacuum expectation values as well as extend the analysis to more general matter content. Constructing the dual supergravity solutions is not straightforward, since it is not a priori clear which classes of SYM theories have a weakly coupled dual. At present we do not know how to extend our construction to $d > 7$ but it is certainly an interesting question to consider. See [184] for a QFT construction that may be relevant to answer this question. Furthermore, in this work we have focused on maximal super Yang-Mills theories on the round sphere. Another way one can reduce supersymmetry is by placing supersymmetric gauge theories on other curved manifolds. A natural first example of this would be a squashed S^{2k+1} with $k \in \{1,2,3\}$. In this case, the sphere can be viewed as a $U(1)$ bundle over $\mathbf{CP}^k$ and one could squash the Einstein metric on the sphere while preserving $\mathrm{SU}(k+1) \times \mathrm{U}(1)$ invariance. This construction should preserve eight supercharges and the partition function of the gauge theory should be computable using supersymmetric localization, see for example [125, 142, 143]. Other interesting examples would be to consider products of spheres.

In the analysis of the matrix models we have mainly focused on the large N and large λ limits. It will be very interesting to extend our analysis beyond leading order to finite values of N or λ. To understand finite N we should obtain a proper understanding of the non-perturbative instanton sectors of the theories in question. This is obviously very interesting but also very hard. More feasible would be to consider finite λ, while remaining in the large N limit. This will allow us to understand whether there are any interesting phase transitions as a function of λ akin to the ones observed for $\mathcal{N} = 2^*$ in [216, 218]. For MSYM on S^3, our results appear to be valid at all values of λ. While the free energy vanishes in this case, the BPS Wilson loop has a non-trivial vacuum expectation value and does not have a form suggesting a phase transition. It would be desirable to understand this result from a perturbative analysis in weakly coupled planar limit. Alternatively, one could study the $1/\lambda$ corrections to the on-shell action of the probe string in the supergravity solution, analogous to [74].

In the holographic analysis of our spherical brane backgrounds we successfully employed a holographic renormalization procedure in non-AlAdS spacetimes. It is desirable to put this procedure on a more solid footing using the generalized conformal structures discussed in [150] and formulate a systematic procedure á la Fefferman-Graham. This would allow us to address the subtle questions on how to fix the coefficients of the finite counterterms in the on-shell actions we encountered for D2- and D4-branes and would additionally be applicable for more general non-conformal supergravity backgrounds.

Finally, we would like to stress that for D5/NS5-branes, both in the matrix model and the supergravity solution, we encounter intriguing UV divergences which we managed to regularize in a seemingly consistent way. It would be very interesting to better understand these calculations and find out what exactly they can teach us about the structure of $(1,1)$ little string theory.

Part III

Wrapped branes and punctured horizons

Chapter 11

Introduction to part III

This third and last part is adapted from [51].

Studying the low-energy physics of p-branes in string and M-theory wrapped around an n-dimensional curved manifold, $\mathcal{M}_n$, has provided a rich arena for understanding the dynamics of strongly coupled quantum field theories. This is facilitated by three distinct vantage points that provide complementing insights into the physics of these systems. One can view this setup as realizing a partial topological twist on $\mathcal{M}_n$ of the $(p+1)$-dimensional supersymmetric QFT on the world-volume of the brane. At low energies this leads to a QFT in $(p-n+1)$ dimensions which preserves part of the original supersymmetry. Alternatively, one can realize the same system more geometrically by studying the low-energy dynamics of the p-brane wrapped on a calibrated cycle $\mathcal{M}_n$ in a special holonomy manifold. Holography offers a third point of view on the same physics. When the number of p-branes is large they back-react on the geometry and this often leads to supergravity solutions dual to the QFTs of interest. In this work we will study the case $n=2$, i.e. $\mathcal{M}_n$ is a Riemann surface, and show how to construct the supergravity solutions corresponding to various p-branes in the presence of punctures on the Riemann surface.

11.1 Wrapped branes and topological twists

Studying wrapped branes on Riemann surfaces using holography was initiated in the seminal work of Maldacena and Núñez [174], see [104] for a review. In this work the large N limit of a system of branes with worldvolume $\mathbf{R}^{1,d-1}\times\Sigma_{\mathbf{g}}$ were studied, where $\Sigma_{\mathbf{g}}$ is a smooth genus $\mathbf{g}$ Riemann surface. At low energies, i.e. at energies small compared to the inverse radius of curvature of the Riemann surface, these theories become effectively d-dimensional. They constructed supergravity solutions describing the flow starting at a UV theory in $d+2$ dimensions towards a d-dimensional theory in the IR. More precisely, they studied 4d $\mathcal{N}=4$ SYM and the 6d $(2,0)$ theory wrapped on a Riemann surface. The basic technique used to find these solutions is the topological twist [47]. Apart from coupling the theory to curvature of the worldvolume, there is an additional coupling to an external SO(n) gauge field, where n is the number of transverse directions to the brane. By carefully choosing a specific form for the SO(n) background gauge field one can construct solutions preserving 1/2 or 1/4 of the original supersymmetry depending on how the holonomy group of the Riemann surface is embedded in the higher dimensional space. In the field theory limit, these distinct possibilities correspond to different normal bundles and therefore different external SO(n) gauge fields. There are various cases in which the low energy theory is a conformal field theory. In this case we find AdS_{d+1} geometries in the dual IR regime. Both the metric and the SO(n) gauge fields are modes of the maximal gauged supergravity in $d+3$ dimensions, therefore one can greatly simplify the analysis of such solutions by considering the $(d+3)$-dimensional supergravity equations.

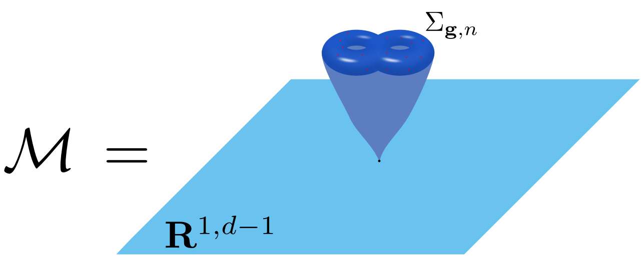


Figure 11.1: A QFT defined on a manifold of the form $\mathcal{M} = \mathbf{R}^{1,d-1} \times \Sigma_{\mathbf{g},n}$, with an appropriate topological twist, will flow in the IR to a d-dimensional SCFT. $\mathbf{g}$ is the genus of the Riemann surface, while n denotes the number of punctures. The type of puncture determines the flavor symmetry it adds to the QFT.

11.2 SCFTs of class $\mathcal{S}$

Renewed interest in the physics of these wrapped brane systems arose after understanding the four-dimensional $\mathcal{N} = 2$ quantum field theories of class $\mathcal{S}$ which emerge as the low energy limit of the worldvolume-theories of M5-branes wrapping a Riemann surface [101, 103]. The worldvolume theory living on a stack of N M5-branes is given by the six-dimensional $\mathcal{N} = (2,0)$ theory of type $\mathfrak{g}$. When compactified on a Riemann surface, the resulting four-dimensional low energy effective theory is characterized by the simply laced Lie group $\mathfrak{g}$,[1] the Riemann surface $\Sigma_{\mathbf{g},n}$, and the type of punctures on the Riemann surface. The low energy

[1]When the 6d theory is compactified on a circle, the Lie group $\mathfrak{g}$ becomes the gauge group of the resulting five-dimensional SYM theory.

theory however, only depends on the complex structure moduli of the Riemann surface which arise as exactly marginal operators in the $4d$ theory. All other UV metric fluctuations are washed away by the RG flow, they correspond to irrelevant operators. A key role in the class $\mathcal{S}$ construction is played by punctures on the Riemann surface. In the UV theory, each puncture corresponds to a codimension two defect of the $(2,0)$ theory. In the low energy theory, such punctures encode information about additional flavor symmetries and matter fields in the quantum field theory. In the brane setup these punctures arise from additional flavor M5-branes, intersecting the Riemann surface at points and sharing four dimensions with the wrapped branes. At low energies, thanks to the marginality of the complex structure moduli, we can decompose the Riemann surface in pairs of pants and cylinders. In this way we can map a decomposition of the Riemann surface to a specific four-dimensional quiver representation, see Figure 11.2. Cylinders represent simple gauge group factors, while the pairs of pants are genuinely new building blocks denoted by $\mathcal{T}^{\Lambda_1,\Lambda_2,\Lambda_3}$ or simply $\mathcal{T}_N$ when $\mathfrak{g} = A_{N-1}$ and all punctures are maximal. These new building blocks represent isolated non-Lagrangian SCFTs with no tunable couplings. Of course, there are many different pairs of pants decompositions for a given punctured Riemann surface corresponding to different weakly coupled descriptions of the same SCFT. They are all related by generalized S-dualities.

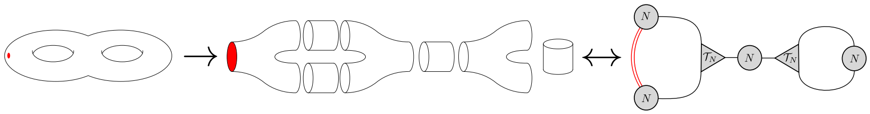


Figure 11.2: A specific pair of pants decomposition of a genus two Riemann surface with one minimal puncture ($\mathfrak{g} = A_{N-1}$). The corresponding quiver diagram is illustrated on the far right where the circles denote simple SU(N) $\mathcal{N} = 2$ vector multiplets. The red dot is a minimal puncture which corresponds to a hypermultiplet in the $4d$ theory.

Such punctures on the Riemann surface can also be incorporated in the holographic description of the class $\mathcal{S}$ setup. Indeed, it was shown in [102] that the gravitational description of this system is captured by a generalization of the class of 1/2-BPS AdS_5 solutions described in [167]. These solutions are characterized by a single function obeying the non-linear SU(∞) Toda equation which in the presence of punctures on $\Sigma_{g,n}$ is modified by including singular sources. A large number of non-trivial consistency checks of this proposal were performed in [102] and all of them lead to nice agreement with the field theory analysis of [101, 103].

Given this success it is natural to ask whether this supergravity description of wrapped branes on punctured Riemann surfaces can be generalized to D- or M-branes in other dimensions or to systems with a smaller number of supercharges. Unfortunately the approach followed in [102] is very hard to generalize to such setups. For instance, if one wants to generalize the $\mathcal{N} = 1$ class $\mathcal{S}$ construction of [18, 19, 41] to Riemann surfaces with punctures one has to study 1/4-BPS AdS_5 backgrounds of eleven-dimensional supergravity. While solutions of this type have been classified in [106] the supergravity BPS equations reduce to a complicated system of coupled nonlinear PDEs in four variables which are extremely hard to solve. Despite the progress described in [16, 17], it appears to be hard to apply the idea of [102] and introduce singular brane sources to this system of equations and find explicit solutions. This state of affairs is even more grim for other wrapped brane systems like D3- or M2-branes. In these cases one should classify 1/8-BPS AdS_3 or 1/16-BPS AdS_2 solutions of type IIB or eleven-dimensional supergravity and introduce singular sources to the corresponding BPS equations. Given this impasse it is clearly beneficial to explore alternative approaches to the construction of this type of supergravity solutions. Our goal in this part is to present one such new approach.

11.3 What about punctures?

Our strategy is based on the observation that the constructions of many wrapped branes supergravity solutions proceeds along the lines of [174]. Namely, one starts with an appropriate lower-dimensional gauged supergravity which is a consistent truncation of ten- or eleven-dimensional supergravity on a compact manifold (typically a sphere). One then studies an appropriate ansatz for the fields of the gauged supergravity theory which implements the holographic description of the partial twist of the dual SCFT on the compact Riemann surface and then constructs solutions of the supergravity BPS equations. We modify this procedure in a simple way, we allow for the metric on the Riemann surface to be general, as opposed to the constant curvature metric in [174]. We then study this setup in the maximal gauged supergravity theories in four, five, six and seven dimension relevant for the holographic description of M2-, D3-, D4-D8-, and M5-branes. We find that in all cases the supergravity BPS equations reduce to the following PDE on the Riemann surface

$$\Box\varphi + \kappa \mathrm{e}^{\varphi} = 0\,, \tag{11.1}$$

for one of the functions in the ansatz, where κ is the normalized curvature of $\Sigma_{\mathbf{g},n}$. This is the Liouville equation. Its regular solutions lead to the well-known supergravity solutions describing branes wrapped on a smooth compact Riemann surface. Our main observation is that introducing a singular source on the right hand side of the Liouville equation allows for new solutions that have not been explored before. We interpret these solutions as providing the supergravity description of branes wrapped on a punctured Riemann surface. We can then rely on a well-known supergravity uplift formulae to present the corresponding ten- or eleven-dimensional supergravity solutions. In this way we circumvent the need to classify AdS supergravity solutions and solve complicated PDEs directly in the ten or eleven-dimensional theory.

To gain confidence in the proposal described above we study in detail the case of M5-branes wrapped on a punctured Riemann surface with a general choice of topological twist preserving $\mathcal{N} = 1$ supersymmetry. We first show that for the special twist preserving $\mathcal{N} = 2$ supersymmetry our results are compatible with the ones in [102]. This comparison also exhibits a small limitation in our approach. The singular source in the Liouville equation does not capture the full information about the puncture present in the eleven-dimensional solutions of [102] and only serves as an approximate description. Nevertheless, the gauged supergravity approach provides enough information for many interesting questions, in particular in the large N approximation. We show this utility by studying more general $\mathcal{N} = 1$ setups of class $\mathcal{S}$ where we show how the results computed using the supergravity solutions agree with the M5-brane anomaly polynomial as well as explicit constructions of the dual quantum field theories using vector and hypermultiplets as well as $\mathcal{T}_N$ building blocks.

11.4 Outline

In the next chapter we present our main proposal on how to treat punctured Riemann surfaces in gauged supergravity and present some details on solutions corresponding to wrapped M2-, D3-, D4-D8-, and M5-branes. From Chapter 13 onward, we focus on M5-branes wrapped on punctured Riemann surfaces in order to accumulate evidence for our general prescription. We start by describing the different twists of the $\mathcal{N} = (2,0)$ theory and summarize how to integrate the anomaly polynomial of the M5-branes over the punctured Riemann surface to obtain the anomalies of the four-dimensional IR theories. In Chapter 14 we revisit the AdS_5 supergravity solutions of Chapter 12 and describe our treatment of singularities on the Riemann surface. We also compute the conformal anomalies of these solutions holographically and obtain an exact match with the result from the anomaly polynomial at leading order in N. Moreover, we compute the dimensions of protected operators arising from M2-branes wrapping the Riemann surface and describe the marginal deformations of our solutions. Finally, in Chapter 15 we construct the quiver gauge theories dual to a subset of our supergravity solutions. We compute the conformal anomalies, the dimensions of the M2-brane operators and the dimension of the conformal manifold and on all fronts find agreement with supergravity and the anomaly polynomial. We finish by briefly discussing various non-trivial Seiberg-like dualities. The Appendices H-K contain technical details on the supergravity constructions we employ, a review of some solutions of the Liouville equation, as well as a brief summary of our SCFT conventions.

Chapter 12

Punctured horizons

We consider d-dimensional SCFTs arising from twisted compactifications of SCFTs in $d+2$ dimensions on a punctured Riemann surface $\Sigma_{\mathbf{g},n}$. These can be realized as the theories living on the worldvolume of D- or M-branes where the worldvolume takes the form $\mathbf{R}^{1,d-1} \times \Sigma_{\mathbf{g},n}$. In general when putting a supersymmetric field theory on a curved manifold all supersymmetries are broken. However, by performing a (partial) topological twist we can preserve some supersymmetry [47, 174, 243]. The generator of supersymmetry is a spinor, ϵ, which in the presence of a background metric and R-symmetry gauge field obeys an equation of the schematic form

$$\left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} + A_\mu^I \Gamma_I\right)\epsilon = 0\,. \tag{12.1}$$

Here ω_μ^{ab} is the spin connection and A_μ is the background gauge field coupled to the R-symmetry. By identifying the structure group of $\Sigma_{\mathbf{g},n}$ with a subgroup of the R-symmetry this equation can be solved by taking a constant spinor obeying $\partial_\mu\epsilon = 0$. In order to perform a topological twist on a Riemann surface we need at least $\mathrm{U}(1)_R$ superconformal R-symmetry. In this thesis we focus on SCFTs with the maximal number of supercharges in three, four, five, and six dimensions which have a larger R-symmetry group and thus the d-dimensional SCFTs in the IR preserve some amount of supersymmetry.[1]

Following the seminal work [174], we study these twisted SCFTs using holography. To this end we consider maximally supersymmetric gauged supergravity theories in $d+3 = 4, 5, 6$ and 7 dimensions and study the most general topological twist on $\Sigma_{\mathbf{g},n}$ in every dimension. The construction follows a similar pattern for all dimensions so before we focus on each individual case we describe the general structure.

We work with a truncation of the maximal gauged supergravity which reduces the

[1] The analysis below can be extended to SCFTs and supergravity theories with smaller amount of supersymmetry, see for example [38, 53].

bosonic fields to the metric and a number of abelian gauge fields and real scalars. The supergravity solutions dual to the twisted SCFTs described above are of the following form

$$\begin{aligned} ds^2_{d+3} =& e^{2f(r,x_1,x_2)}\left(-dt^2+dz_1^2+\cdots+dz_{d-1}^2+dr^2\right)+e^{\hat{\varphi}(r,x_1,x_2)}\left(dx_1^2+dx_2^2\right), \\ A^{(i)} =& A^{(i)}_{x_1}(r,x_1,x_2)dx_1+A^{(i)}_{x_2}(r,x_1,x_2)dx_2, \\ \lambda_I =& \lambda_I(r,x_1,x_2) \qquad . \end{aligned} \tag{12.2}$$

The range of the indices $i=1,\ldots,n_A$ and $I=1,\ldots,n_\lambda$ differs on a case by case basis and will be specified for each dimension separately. In these expressions all functions – f, $\hat{\varphi}$, $A^{(i)}_{x_1,x_2}$ and λ_I – only depend on the radial coordinate r and the coordinates x_1 and x_2 of the Riemann surface. For Riemann surfaces with Gaussian curvature $\kappa=-1$, the coordinates (x_1,x_2) parameterize the hyperbolic plane $\mathbf{H}$ which we quotient by a discrete Fuchsian subgroup $\Gamma\in \mathrm{PSL}(2,\mathbf{R})$ to obtain a genus $\mathbf{g}>1$ Riemann surface. Furthermore we focus on the IR behavior of these wrapped brane solutions where the geometry becomes $\mathrm{AdS}_{d+1}\times\Sigma$. The metric functions in this IR region are fixed to

$$\begin{aligned} f =& -\log r+\log\frac{2}{g}+f_0, \\ \hat{\varphi} =& \varphi(x_1,x_2)+2\log\frac{2}{g}+\varphi_0, \end{aligned} \tag{12.3}$$

where f_0 and φ_0 are constants and the scalars take constant values. The gauge coupling g is related to the radius of the UV AdS_{d+3} solution, $R_{\mathrm{AdS}_{d+3}}=\frac{2}{g}$. It is worth pointing out that the more general BPS equations which describe the holographic RG flow from the AdS_{d+3} UV region to the AdS_{d+1} IR near-horizon region were studied in detail in [11, 56].[2] The result is that these equations admit solutions for arbitrary metric on the Riemann surface in the UV region, however in the IR the metric flows to the constant curvature metric on Σ. This behavior is known as holographic uniformization [11, 56]. From now on we concentrate solely on the IR region and investigate the resulting near-horizon geometries. As described in Appendix H one can show that the BPS equations at the IR fixed point reduce to a number of algebraic equations for the scalars together with one universal second order equation for the conformal factor φ of the metric on the Riemann surface. The gauge fields in turn are – up to a choice of twist – fully determined in terms of this function φ. The equation determining φ is given by

$$\boxed{\Box\varphi+\kappa e^{\varphi}=0} \tag{12.4}$$

[2] See [95] for a generalization of this setup to M5-branes wrapped on Kähler 4-cycles.

This is nothing but the Liouville equation for the conformal factor of the metric on the Riemann surface.[3]. When one considers smooth Riemann surfaces one finds the constant curvature metric on the covering space

$$\begin{aligned} \varphi(x_1,x_2) &= -2\log x_2 + \log 2 && \text{for} \quad \kappa=-1\,, \\ \varphi(x_1,x_2) &= 0 && \text{for} \quad \kappa=0\,, \\ \varphi(x_1,x_2) &= -2\log(1+x_1^2+x_2^2)+\log 8 && \text{for} \quad \kappa=1\,. \end{aligned} \tag{12.5}$$

The crucial observation for our work is that there are more general solutions to the Liouville equation. One can construct many more IR AdS_{d+1} solutions by allowing singular solutions to the Liouville equation where the Riemann surface includes conical defects or punctures.

Our main observation is that regular punctures on the wrapped curve correspond to conical defects of the Riemann surface in the lower dimensional supergravity description. In order to accommodate such singularities one has to add localized sources to the Liouville equation

$$\Box\varphi + \kappa \mathrm{e}^{\varphi} = \sum_i 4\pi(1-\xi_i)\delta^{(2)}(P_i)\,. \tag{12.6}$$

where i runs over all singularities and $0 \le \xi_i \le 1$ specifies the opening angle of the conical defect at the point P_i. The limiting values $\xi=0$ and $\xi=1$ correspond to respectively a true puncture and a regular point. Near a singular point the Liouville field needs to satisfy appropriate boundary conditions. Near a conical singularity with defect angle ξ, the boundary conditions are

$$\varphi = -2(1-\xi)\log r\,, \qquad \text{for} \quad r\to 0\,. \tag{12.7}$$

where $r^2 = x_1^2 + x_2^2$ and the coordinates are chosen such that the singularity lies at the origin. Once a solution to the Liouville equation on a Riemann surface with prescribed singularities is given, the gauge field strength is fully determined by the conformal factor φ, up to a choice of partial topological twist. In terms of the spin connection $\omega_\mu = \frac{1}{2}\omega_\mu^{ab}\epsilon_{ab}$, we choose the R-symmetry background gauge field A_μ such that its field strength is, up to a d-dependent prefactor, given by

$$F_{=}\sum_i F^{(i)} T_i \sim -\kappa\, \mathrm{V}_{\mathbf{g},\xi}^{-1}\, T\, \mathrm{d}\omega\,. \tag{12.8}$$

for $\kappa \neq 0$ and $F \sim -T\mathrm{vol}_{\mathbf{g},\xi}$ for $\kappa=0$. Here we defined $\mathrm{vol}_{\mathbf{g},\xi}$ to be the volume form on the singular Riemann surface of genus $\mathbf{g}$ with n conical defects with opening

[3] Some properties of the Liouville equation are summarized in Appendix I.

angles ξ_j and $\mathrm{V}_{\mathbf{g},\xi}$ is its volume

$$\mathrm{V}_{\mathbf{g},\xi} = \frac{2\pi}{\kappa}\left(2-2\mathbf{g}-\sum_{j=1}^{n}(1-\xi_j)\right). \tag{12.9}$$

This implies that in order guarantee that the volume is positive we must have $\kappa = \mathrm{sgn}[2-2\mathbf{g}-\sum_{j=1}^{n}(1-\xi_j)]$. The gauge field is taken along the generator $T = \sum a^i T_i$ where T_i are the generators of the Cartan of the R-symmetry group. The a^i are constants parameterizing the partial topological twist. In order to preserve supersymmetry a^i need to obey the constraint

$$2\pi \sum_i a^i = -\kappa\, \mathrm{V}_{\mathbf{g},\xi}\,. \tag{12.10}$$

For smooth Riemann surfaces the condition for the R-symmetry bundle to be well defined, together with the twisting condition, (12.10), imply that the a^i can only take quantized values $a^i \in \mathbf{Z}$. On the other hand, for a singular Riemann surface with conical singularities, with deficit angles $\{\xi_j\}_{j=1,\cdots,n}$, the quantization condition is slightly altered and becomes

$$\mathrm{lcm}(\{\xi_j^{-1}\})a^i \in \mathbf{Z}\,. \tag{12.11}$$

This condition corresponds to the quantization of the fluxes $F^{(i)}$ and is very similar in spirit to the quantization of electric charge in the presence of a magnetic monopole. Namely, when a monopole of magnetic charge m is present the charge quantization condition takes the form $mn \in \mathbf{Z}$.

Another useful point of view on this quantization condition is more geometric and is offered by uplifting these backgrounds to string or M-theory. There the quantization arises for imposing the normal bundle to the wrapped branes is well-defined. For an SU(N) gauge group the twisted SCFTs theories describe the low energy limit of a D- or M-brane wrapped on a singular curve Σ inside a Calabi-Yau m-fold, where $m = n_A + 1$, which is a $\bigoplus_{i=1}^{n_A}\mathcal{L}_i$ bundle over Σ.

$$\begin{array}{ccc} \mathbf{C}^{n_A} & \hookrightarrow & \bigoplus_{i=1}^{n_A}\mathcal{L}_i \\ & & \downarrow \\ & & \Sigma \end{array} \tag{12.12}$$

The degrees of the line bundles $\mathcal{L}_i$ are $\deg(\mathcal{L}_i) = -\kappa a^i$ for $\kappa \neq 0$ and $\deg(\mathcal{L}_i) = a^i$ for $\kappa = 0$. The Calabi-Yau condition of the total space reduces to the condition $\bigotimes_{i=1}^{n_A}\mathcal{L}_i = K_\Sigma$, where K_Σ is the canonical line bundle of Σ. This relation is equivalent to the twist condition (12.10). Note that due to the presence of singularities the degrees of the line bundles can take rational values [214].

It proves convenient for the subsequent analysis to split the parameters a^i into global and local parts

$$a^i = a^i_{\text{global}} + a^i_{\text{local}} \,, \tag{12.13}$$

where the global part corresponds to the background flux present for the topological twist on a smooth Σ, i.e. such that $\sum_i a^i_{\text{global}} = (2\mathbf{g}-2)$. For each puncture there is a choice to add the local contribution to one of the magnetic fields or equivalently to one of the a^i's indicating in which direction normal to the branes the puncture extends.[4] Graphically we can represent this choice by assigning a color to each conical defect, see Figure 12.1 for an illustration valid for M5-branes wrapped on Σ, as follows

$$a^1_{\text{local}} = \sum_{\text{red punctures}} (1-\xi_j)\,, \qquad a^2_{\text{local}} = \sum_{\text{blue punctures}} (1-\xi_j)\,, \qquad \cdots \tag{12.14}$$

For theories of class $\mathcal{S}$ with gauge group SU(N) regular punctures are classified

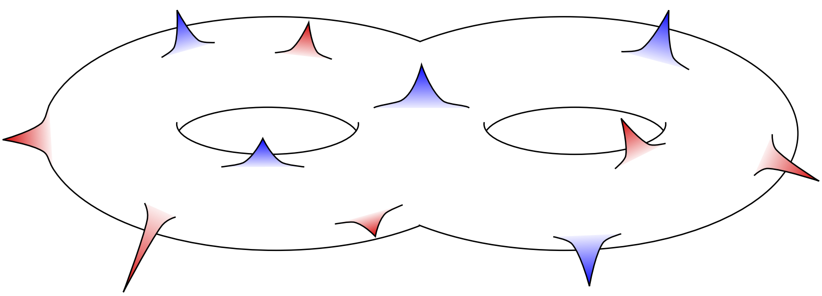


Figure 12.1: Regular punctures on the wrapped Riemann surface correspond to conical defects on the Riemann surface in the lower dimensional supergravity. The local information of a puncture is encoded in the opening angle of the corresponding conical defect. The color of the shading surrounding the singularity determines in which transverse direction the puncture extends.

by Young tableaux with N boxes [68, 101]. Similarly we conjecture that in other dimensions and with smaller number of supercharges many punctures can be classified in the same manner. Around a puncture we can uplift our supergravity

[4] One can consider the more general case where the contribution from a single singularity is split over multiple a^i. In this case the singularity still extends in a one-dimensional subspace of the transverse space the will locally still preserve half-maximal supersymmetry. By choosing a different basis for the transverse space we recover the same picture as above. However, when a singularity extends in a subspace of the transverse space with dimension $d > 1$ we have a truly different situation and the intuitive picture developed above will no longer be correct. To the best of our knowledge this situation has not been studied in the literature and we do not consider it here.

solution to ten or eleven dimensions. For $\xi^{-1} \in \mathbf{Z}$ the uplifted geometry around the puncture in 10 or 11 dimensions takes the form

$$\mathrm{d}s^2_{10(11)} \simeq \mathrm{d}s^2_{\mathrm{AdS}_{d-1}} + \mathrm{d}s^2_{\mathbf{R}^4/\mathbf{Z}_{\xi^{-1}}} + \mathrm{d}s^2_{S^{7(8)-d}} \,. \tag{12.15}$$

In class $\mathcal{S}$ theories this kind of geometry corresponds to a puncture characterized by a rectangular Young tableau with rows of equal length ξ^{-1}. The geometrical structure associated to punctures with more general Young tableaux is more complicated as the singularities are spread out along the internal space [102]. In order to analyze these singularities in full detail one should consider the full ten- or eleven-dimensional description of the solution. However, in the large N limit we can approximately describe these solutions with conical singularities with defect angles $\xi \in \{\frac{r}{N} | r = 1, \ldots, N-1\}$. A conical defect with $\xi = \frac{r}{N}$ corresponds to the set of all regular punctures described by Young tableaux with r rows; to select the specific Young tableau from this class one needs additional information encoded in the transverse geometry. Such more general punctures might violate the quantization conditions formulated above. For the moment we do not worry about this and merely consider this as an effective description in the gauged supergravity. Using this approximation we show that for $\mathcal{N}=1$ theories of class $\mathcal{S}$ we can match the conformal anomalies and dimensions of specific operators for all types of punctures in the dual field theory.

In the remainder of this chapter we construct explicit gauged supergravity solutions corresponding to M2-, D3-, D4-D8- and M5-branes wrapping the singular Riemann surface $\Sigma_{\mathbf{g},n}$.

12.1 M2-branes on singular curves

The gauge theory arising on the worldvolume of N M2-branes is given by the three-dimensional $\mathcal{N}=8$ ABJM theory [5] which at large N is dual to eleven-dimensional supergravity on $\mathrm{AdS}_4 \times S^7$.[5] A twisted compactification of this theory on a complex curve Σ is described holographically by an eleven-dimensional supergravity background which is asymptotically locally $\mathrm{AdS}_4 \times S^7$ but for which the topology at a fixed value of the radial coordinate is an S_7 fibration over Σ. An efficient way to construct these supergravity solutions is to study them in the maximal four-dimensional SO(8) gauged supergravity [84] which is a consistent truncation of the eleven-dimensional theory on S^7. For our purposes we do not need the full structure of the four-dimensional $\mathcal{N}=8$ theory and restrict to a further truncation studied in [80]. The bosonic subsector of this truncation consists of a metric, four abelian gauge fields in the Cartan of the SO(8) gauge group and three

[5] For simplicity we focus on the ABJM theory with Chern-Simons level $k=1$.

real neutral scalars. It can been shown that all solutions of this truncation can be uplifted to solutions of eleven-dimensional supergravity [80].

In [40,105] the near-horizon geometry of M2-branes wrapped on a smooth Riemann surface was analyzed using the same truncation. One can show that by inserting our ansatz in the BPS equations they indeed reduce to the Liouville equation for the conformal factor φ together with algebraic equations for the other fields [56]. In terms of φ, the gauge field strengths are given by[6]

$$F^i_{(2)} = -\kappa \frac{2\pi\, a^i}{g} \mathrm{V}^{-1}_{\mathbf{g},\xi}\, \mathrm{vol}_{\mathbf{g},\xi}\,, \tag{12.16}$$

where g is the gauge coupling constant of the supergravity theory and $i = 1,2,3,4$. The constants a^i determine the specific choice of twist and are constraint to satisfy the condition in (12.10). For generic choices of a^i the solution preserves 2 real supercharges, i.e. it is 1/16-BPS. When one of the a^i is zero the solution is 1/8-BPS, when two vanish 1/4-BPS, and when three vanish 1/2-BPS. As discussed around (12.13) all a^i consist of a global part a^i_{global} and a local part a^i_{local} accounting for the local contributions of the punctures. For each puncture there is a choice to add the puncture contribution to one of the four a^i. We can illustrate this choice by giving each puncture a color – red, green, blue or yellow. The puncture contributions now becomes

$$\begin{aligned} a^1_{\text{local}} &= \sum_{P_i=\text{green}} (1-\xi_i)\,, \quad a^2_{\text{local}} = \sum_{P_i=\text{red}} (1-\xi_i)\,, \\ a^3_{\text{local}} &= \sum_{P_i=\text{blue}} (1-\xi_i)\,, \quad a^4_{\text{local}} = \sum_{P_i=\text{yellow}} (1-\xi_i)\,. \end{aligned} \tag{12.17}$$

This construction can be phrased more geometrically in M-theory. The twisted ABJM theory describes the low-energy dynamics of N M2-branes wrapped on a holomorphic two-cycle Σ in a local Calabi-Yau five-fold X, which is constructed as four line bundles over Σ

$$\begin{array}{ccc} \mathbf{C}^4 & \hookrightarrow & \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \mathcal{L}_4 \\ & & \downarrow \\ & & \Sigma \end{array} \tag{12.18}$$

The degree of each line bundle $\mathcal{L}_i$ is a^i hence the coloring indicated in which transverse direction to Σ the puncture extends. The constraint coming from the twisting, (12.10), translates into the Calabi-Yau condition for X. The local

[6]Here and in the upcoming cases this expression for the field strengths is valid only when $\kappa = \pm 1$. When $\kappa = 0$, the field strengths are given by $F^i = \frac{a^i}{g} \mathrm{d}x_1 \wedge \mathrm{d}x_2$.

contributions a^i_{local} account for the local information encoded in the specific geometry of the punctures.

To fully specify the supergravity solution we need to solve also for the three scalar fields. They are expressed in terms of the flux parameters a^i as

$$
\begin{aligned}
e^{\lambda_1} &= \frac{2(a^2+a^3)(a^1-a^4)^2-(a^1+a^4)\left((a^2-a^3)^2+(a^1-a^4)^2\right)-8\kappa(a^4-a^1)\mathcal{T}}{2a^4(a^4-a^1+a^2-a^3)(a^4-a^1-a^2+a^3)}\,, \\
e^{\lambda_2} &= \frac{2(a^1+a^3)(a^2-a^4)^2-(a^2+a^4)\left((a^1-a^3)^2+(a^2-a^4)^2\right)-8\kappa(a^4-a^2)\mathcal{T}}{2a^4(a^4+a^1-a^2-a^3)(a^4-a^1-a^2+a^3)}\,, \\
e^{\lambda_3} &= \frac{(a^1+a^2)(a^3-a^4)^2-(a^3+a^4)\left((a^1-a^2)^2+(a^3-a^4)^2\right)-8\kappa(a^4-a^3)\mathcal{T}}{2a^4(a^4+a^1-a^2-a^3)(a^4-a^1+a^2-a^3)}\,.
\end{aligned}
\tag{12.19}
$$

Where we introduced the function

$$
\mathcal{T} = \frac{1}{2}\left(1-3\sum_i (a^i)^2\right)^{1/2} - 8\sqrt{a_1 a_2 a_3 a_4}\,. \tag{12.20}
$$

Finally the constants appearing in the metric are given by

$$
\begin{aligned}
e^{2f_0} &= \frac{2e^{\frac{1}{2}(\lambda_1+\lambda_2+\lambda_3)}}{(1+e^{\lambda_1}+e^{\lambda_2}+e^{\lambda_3})^2}\,, \\
e^{\varphi_0} &= \frac{1}{2}e^{-\frac{1}{2}(\lambda_1+\lambda_2+\lambda_3)}\left(a^1 e^{\lambda_2+\lambda_3}+a^2 e^{\lambda_1+\lambda_3}+a^3 e^{\lambda_1+\lambda_2}-a^4 e^{\lambda_1+\lambda_2+\lambda_3}\right).
\end{aligned}
\tag{12.21}
$$

We can uplift these solutions to solutions of eleven-dimensional supergravity using the uplift formulae in Appendix J.1. Locally around each of the punctures we can analyze the uplifted solution. The puncture locally preserve half of the maximal supersymmetry so we can make a local gauge and coordinate transformation such that $a^1 = \chi_{\mathbf{g},\xi}$ and $a^2 = a^3 = a^4 = 0$ and $\xi = \frac{1}{k}$. In that case we find that the eleven-dimensional solution is regular up to a $\mathbf{Z}_k$ singularity at $\alpha = 0$. Near that point the uplifted metric becomes

$$
ds^2_{11} = \Delta^{1/2}\left[ds^2_{\text{AdS}_2} + ds^2_{S^5} + ds^2_{\mathbf{R}^4/\mathbf{Z}_k}\right], \tag{12.22}
$$

which matches the expectation (12.15) from the general discussion above.

12.2 D3-branes on singular curves

The gauge theory living on the worldvolume of N D3-branes is given by four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory which at large N and large 't Hooft coupling is dual to ten-dimensional type IIB supergravity on $\mathrm{AdS}_5 \times S^5$. Compactifying the $\mathcal{N} = 4$ theory on the surface Σ is described holographically by a ten-dimensional supergravity background which is asymptotically locally $\mathrm{AdS}_5 \times S^5$ but for which the topology at a fixed value of the radial coordinate is an S_5 fibration over Σ [37, 174]. Once again the construction of these solutions is most efficient in a truncation of the maximal five-dimensional SO(6) gauged supergravity [121, 122, 204] studied in [80]. This truncation contains the metric, three abelian gauge fields in the Cartan of the SO(6) gauge group and two real scalars. All solutions of this truncated theory can be uplifted to solutions of type IIB supergravity on S^5 [80].

In [37] the near-horizon geometry of N D3-branes wrapped on a smooth Riemann surface was analyzed using this truncation. We can extend this analysis by using the more general ansatz in (12.2). The BPS equations then reduce to the Liouville equation (12.4) for the conformal factor φ. In terms of this conformal factor, the field strengths are given by

$$F^{(i)} = \kappa \frac{a^i \pi}{g} \mathrm{V}_{\mathbf{g},\xi}^{-1} \mathrm{vol}_{\mathbf{g},\xi} \,, \tag{12.23}$$

where g is the gauge coupling of the supergravity theory and $i = 1, 2, 3$. The constants a^i determine the choice of topological twist and have to satisfy (12.10). For generic choices of a^i the theory preserves $\mathcal{N} = (0, 2)$ supersymmetry, when one of the a^i is zero and the other two are equal we get $\mathcal{N} = (2, 2)$, when two a^i vanish the supersymmetry is $\mathcal{N} = (4, 4)$ and when all a^i vanish (and $\mathbf{g} = 1$) we have $\mathcal{N} = (8, 8)$ supersymmetry. As in (12.13) all a^i consist of a regular part a^i_{global} and a local part associated to the punctures a^i_{local}. For each puncture we have the choice to add the puncture contribution to one of the three a^i. We represent this choice by assigning a color to each puncture – red, green or blue. Then the puncture contribution to the a^i becomes

$$a^1_{\mathrm{local}} = \sum_{P_i = \mathrm{green}} (1 - \xi_i) \,, \qquad a^2_{\mathrm{local}} = \sum_{P_i = \mathrm{red}} (1 - \xi_i) \,, \qquad a^3_{\mathrm{local}} = \sum_{P_i = \mathrm{blue}} (1 - \xi_i) \,, \tag{12.24}$$

The geometric interpretation of this construction is by now familiar The twisted $\mathcal{N} = 4$ theory describes the low-energy dynamics of N D3-branes wrapped on a holomorphic two-cycle Σ in a local Calabi-Yau four-fold X, which is composed of three line bundles over Σ

$$\begin{array}{ccc} \mathbf{C}^3 & \hookrightarrow & \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \\ & & \downarrow \\ & & \Sigma \end{array} \tag{12.25}$$

As before, the degree of each line bundle $\mathcal{L}_1$ is a^i and the coloring describes in which part of the line bundle the puncture. Finally the twist condition (12.10) translates into the Calabi-Yau condition on X. The local information of the puncture is captured by a^i_{local}.

The solution for the two scalars, λ_1 and λ_2, is given by

$$\begin{aligned} e^{3\lambda_1+\lambda_2} &= \frac{a^3(a^1+a^2-a^3)}{a^1(-a^1+a^2+a^3)}\,, \\ e^{2\lambda_2} &= \frac{a^2(a^1-a^2+a^3)}{a^1(-a^1+a^2+a^3)}\,, \end{aligned} \tag{12.26}$$

and the constants appearing in the metric by

$$\begin{aligned} e^{3f_0} &= \frac{a^1a^2(a^1-a^2-a^3)(a^1+a^2-a^3)(a^1-a^2+a^3)}{((a^1)^2+(a^2)^2+(a^3)^2-2(a^1a^2+a^1a^3+a^2a^3))^3}\,, \\ e^{3\varphi_0} &= \frac{1}{2^6}\frac{(a^1)^2(a^2)^2(a^3)^2}{(a^1+a^2-a^3)(a^1-a^2+a^3)(-a^1+a^2+a^3)}\,. \end{aligned} \tag{12.27}$$

We can uplift these five-dimensional solutions to type IIB supergravity using the uplift formulae in Appendix J.2 and analyze the solution locally around each puncture. The punctures locally preserve $\mathcal{N}=(4,4)$ supersymmetry so we can make a local change of coordinates such that $a^1=\chi_{\mathbf{g},\xi}$ and $a^2=a^3=0$ and $\xi=\frac{1}{k}$. The result is a regular solution up to a $\mathbf{Z}_k$ singularity at $\alpha=0$. Near this point the uplifted metric becomes

$$ds^2_{10} = \Delta^{1/2}\left[ds^2_{\text{AdS}_3} + ds^2_{S^3} + ds^2_{\mathbf{R}^4/\mathbf{Z}_k}\right] \tag{12.28}$$

which matches the expectation (12.15) from the general discussion above.

12.3 D4-D8-branes on singular curves

The gauge theory living on the worldvolume of a stack of N D4-branes in a background of N_f D8-branes with an O8-plane is non-renormalizable but flows to a five-dimensional $\mathcal{N} = 1$ SCFT in the UV. At large N the SCFT is dual to ten-dimensional massive type IIA supergravity on $\mathrm{AdS}_6 \times S^4/\mathbf{Z}_2$ [62]. A twisted compactification of the SCFT on a curve Σ results in a ten-dimensional supergravity background which is asymptotically locally $\mathrm{AdS}_6 \times S^4/\mathbf{Z}_2$ but for which the topology at a fixed value of the radial coordinate is an $S^4/\mathbf{Z}_2$ fibration over Σ, see [27,79,199]. Massive type IIA supergravity admits a truncation to the Romans six-dimensional SU(2) gauged supergravity [215]. For the solutions of interest we can restrict to a further truncation containing only the metric, an abelian gauge field in the Cartan of the SU(2) gauge group and one real scalar. All solutions of this truncation can be uplifted to solutions of massive type IIA supergravity [81].[7]

As shown in [56] inserting the ansatz (12.2) in the BPS equations of the six-dimensional supergravity leads to the Liouville equation for φ. Since the supergravity truncation has only one gauge field there is only one possible twist and all fields are fully fixed in terms of φ. The field strength F is given by

$$F = -\frac{\kappa}{2g}\mathrm{vol}_{\mathbf{g},\xi}\,. \tag{12.29}$$

The scalar λ and the metric constants are given by

$$\mathrm{e}^{4\lambda} = \frac{2}{3}\,, \qquad \mathrm{e}^{f_0} = \frac{3^{1/4}}{2^{3/4}}\mathrm{e}^{-\lambda}\,, \qquad \mathrm{e}^{\varphi_0} = \frac{\sqrt{3}}{8\sqrt{2}}\,. \tag{12.30}$$

Here g is the gauge coupling of the six-dimensional supergravity which is related to the mass parameter m of massive type IIA supergravity by $m = \frac{\sqrt{2}}{3}g$.

We can uplift this solution to massive type IIA supergravity using the uplift formulae of [81] (which are summarized in Appendix J.3) and study the resulting ten-dimensional solution near a conical defect with opening angle ξ. Due to the fact that the uplift only includes half of the four-sphere, $S^4/\mathbf{Z}_2$ this case deviates slightly from the general story. The resulting geometry takes the form

$$\mathrm{d}s^2_{10} \simeq \Delta^{3/8}\left[\mathrm{d}s^2_{\mathrm{AdS}_4} + \mathrm{d}s^2_6\right]\,, \tag{12.31}$$

where the six-dimensional internal space is given by

$$\mathrm{d}s^2_6 = \frac{3}{8}\mathrm{d}\alpha^2 + \xi^2\rho^2\mathrm{d}\theta^2 + \mathrm{d}\rho^2 + (\sigma_1^2+\sigma_2^2) + (\sigma_3 + (1-\xi)\mathrm{d}\theta)^2\,, \tag{12.32}$$

[7] Solutions of the six-dimensional Romans supergravity can also be uplifted to type IIB supergravity. We do not consider this possibility here and restrict to uplifts to type IIA.

and the Riemann surface is parameterized by the coordinates ρ and θ. The $(\sigma_1^2+\sigma_2^2)$ part of the metric has the correct symmetries to account for the $\mathrm{U}(1)_R$ R-symmetry of a three-dimensional $\mathcal{N}=2$ theory however it is non-trivially fibered over the remainder of the internal space. Analyzing the global structure of this solution goes beyond the scope of this work.

12.4 M5-branes on singular curves

The gauge theory living on the worldvolume of N M5-branes is the six-dimensional $\mathcal{N}=(2,0)$ theory of type A_{N-1}. At large N this theory is dual to eleven-dimensional supergravity on $AdS_7\times S^4$. The large N dual of a twisted compactification of this theory on a complex curve Σ is an eleven-dimensional supergravity background which is asymptotically locally $AdS_7\times S^4$ but for which the topology at a fixed value of the radial coordinate is an S^4 fibration over Σ. Eleven-dimensional supergravity admits a consistent truncation to the lowest Kaluza-Klein modes given by maximal SO(5) gauged supergravity in seven dimensions [203]. Once more we can restrict to a further truncation of the theory, containing only the metric, two abelian gauge fields in the Cartan of the SO(5) gauge group, and two real scalars parameterizing the squashing of the S^4 [80, 168]. Moreover, all solutions we obtain in seven dimensions can be uplifted to eleven dimensions using the results in [80, 195, 196].

The near-horizon geometry of the M5-branes wrapping smooth curves was considered in [19, 174]. We summarize the derivation of the BPS equations for this construction in Appendix H. Yet again these equations reduce to the Liouville equation for φ and all other fields are determined in terms of this function only. The field strengths are given by

$$F^{(1)}=-\kappa\frac{a^1\pi}{4g}\mathrm{V}_{\mathbf{g},\xi}^{-1}\,\mathrm{vol}_{\mathbf{g},\xi}\,,$$
$$F^{(2)}=-\kappa\frac{a^2\pi}{4g}\mathrm{V}_{\mathbf{g},\xi}^{-1}\,\mathrm{vol}_{\mathbf{g},\xi}\,, \tag{12.33}$$

where g is the supergravity gauge coupling and we have the usual condition (12.10). As in the previous cases we can assign a color to each puncture – red or blue – indicating to which of the a^i it contributes,

$$a^1_{\mathrm{local}}=\sum_{P_i=\mathrm{red}}(1-\xi_i)\,,\qquad a^2_{\mathrm{local}}=\sum_{P_i=\mathrm{blue}}(1-\xi_i)\,. \tag{12.34}$$

This construction can again be interpreted geometrically as the low energy dynamics of N M5-branes wrapping a holomorphic two-cycle Σ in a Calabi-Yau three-fold X

with local geometry

$$\begin{array}{ccc} \mathbf{C}^2 & \hookrightarrow & \mathcal{L}_1 \oplus \mathcal{L}_2 \\ & & \downarrow \\ & & \Sigma \end{array} \tag{12.35}$$

Where $\mathcal{L}_1$ and $\mathcal{L}_2$ are two line bundles of degree a^i and the twisting condition for the a^i again translates into the Calabi-Yau condition for X.

The solution for the supergravity scalars λ_1 and λ_2 is given by

$$\begin{aligned} e^{10\lambda_1} &= \frac{1 + 7z + 7z^2 + 33z^3 + \kappa(1 + 4z + 19z^2)\sqrt{1 + 3z^2}}{4z(1 - z)^2}\,, \\ e^{2(\lambda_1 - \lambda_2)} &= \frac{2z - \kappa\sqrt{1 + 3z^2}}{1 + z}\,. \end{aligned} \tag{12.36}$$

where as in [18, 19] we have defined

$$z = \frac{a^1 - a^2}{a^1 + a^2}\,. \tag{12.37}$$

Finally the constants appearing in the metric are given by

$$\begin{aligned} e^{f_0} &= \frac{1}{2} e^{4\lambda_1 + 4\lambda_2}\,, \\ e^{\varphi_0} &= \frac{e^{2\lambda_1 + 2\lambda_2}}{64} \left((1 + z)e^{2\lambda_2} + (1 - z)e^{2\lambda_1} \right). \end{aligned} \tag{12.38}$$

Once more, we can analyze the uplifted geometry locally around a puncture with opening angle $\xi = \frac{1}{k}$ using the uplift formulae of [80], summarized in Appendix J.4. Locally around the puncture we preserve $\mathcal{N} = 2$ supersymmetry and we can thus, without loss of generality, restrict to the case $a^2 = 0$. The uplifted solution gives a regular geometry up to a single $\mathbf{Z}_k$ singularity at $\alpha = 0$. The geometry near this singularity takes the form

$$ds^2_{11} = \Delta^{1/2} \left[ds^2_{\mathrm{AdS}_5} + ds^2_{S^2} + ds^2_{\mathbf{R}^4/\mathbf{Z}_k} \right], \tag{12.39}$$

which again is in line with the general discussion above.

Chapter 13

Wrapped M5-branes and twists of the $(2,0)$ theory

From now on we focus on M5-branes and will accumulate evidence for the claim that punctures can indeed be treated in gauged supergravity as discussed in the previous chapter. We start by reviewing the six-dimensional $\mathcal{N}=(2,0)$ theory and its partial topological twists and discuss their realization as the worldvolume theory of M5-branes wrapped on complex curves. For concreteness here and in most of the following we limit ourselves to $\mathcal{N}=(2,0)$ theories of type A_{N-1}. Parts of our analysis admits generalizations to D_N or $E_{6,7,8}$ type theories.

13.1 Partial twists of the $\mathcal{N}=(2,0)$ theory

We are interested in the six-dimensional $\mathcal{N}=(2,0)$ theory of type A_{N-1} defined on a spacetime of the form

$$\mathbf{R}^{1,3}\times\Sigma \tag{13.1}$$

where Σ is a Riemann surface of genus $\mathbf{g}>1$ with prescribed singularities. To preserve supersymmetry on $\mathbf{R}^{1,3}$ we perform a partial topological twist [47, 243] by turning on a background flux for the SO(5) R-symmetry of the $(2,0)$ theory. A choice of twist corresponds to a choice of abelian subgroup $\mathrm{U}(1)'\subset\mathrm{U}(1)_\Sigma\times\mathrm{SO}(5)_R$ such that a number of supercharges are invariant under $\mathrm{U}(1)'$. Here $\mathrm{U}(1)_\Sigma$ is the structure group of the Riemann surface.

Since only an abelian factor of the structure group is being twisted it suffices to look at the Cartan of the R-symmetry group, i.e. $\mathrm{U}(1)_+\times\mathrm{U}(1)_-\subset\mathrm{SO}(5)_R$. Under the subgroup $\mathrm{SO}(1,3)\times\mathrm{U}(1)_\Sigma\times\mathrm{U}(1)_+\times\mathrm{U}(1)_-\subset\mathrm{SO}(1,5)\times\mathrm{SO}(5)_R$, the supercharges of the (2,0) theory decompose as

$$\mathbf{4}\times\mathbf{4}\to\left[(\mathbf{2},\mathbf{1})_{\frac{1}{2}}\oplus(\mathbf{1},\mathbf{2})_{-\frac{1}{2}}\right]\otimes\left[\left(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}}\right)\oplus\left(-\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}}\right)\oplus\left(\frac{\mathbf{1}}{\mathbf{2}},-\frac{\mathbf{1}}{\mathbf{2}}\right)\oplus\left(-\frac{\mathbf{1}}{\mathbf{2}},-\frac{\mathbf{1}}{\mathbf{2}}\right)\right], \quad (13.2)$$

and satisfy a reality constraint coming from the symplectic-Majorana condition. Thus under the U(1) subgroup generated by a Lie algebra element $\mathfrak{t}'=\mathfrak{t}_\Sigma+a\,\mathfrak{t}_++b\,\mathfrak{t}_-$, where the $\mathfrak{t}$'s are the generators of the respective U(1)'s, the supercharges transform with charges $\pm\frac{1}{2}\pm\frac{a}{2}\pm\frac{b}{2}$. For any choice of a and b such that $a\pm b=\pm1$ there are at least four real supercharges. Choosing $a=\frac{a^1}{a^1+a^2}$ and $b=\frac{a^2}{a^1+a^2}$ we can identify the generator of the holonomy group $U(1)_h$ with the linear combination

$$\mathfrak{t}_h=\frac{a^1}{a^1+a^2}\mathfrak{t}_++\frac{a^2}{a^1+a^2}\mathfrak{t}_-\,. \quad (13.3)$$

This twist in general preserves four supercharges and therefore leads to a $\mathcal{N}=1$ supersymmetric field theory in four dimensions. The field theory has $U(1)^2$ flavor symmetry with generators

$$R_0=\frac{1}{2}(\mathfrak{t}_++\mathfrak{t}_-)\,,\qquad \mathcal{F}=\frac{1}{2}(\mathfrak{t}_+-\mathfrak{t}_-)\,. \quad (13.4)$$

Where R_0 is the UV R-symmetry. In the IR, the superconformal R-symmetry will in general be given by a combination

$$R_{\mathcal{N}=1}=R_0+\epsilon\mathcal{F}\,. \quad (13.5)$$

of the two U(1)s. ϵ is a priori unknown but will be fixed by a-maximization [145].

When either a^1 or a^2 vanishes, the theory preserves eight supercharges and resp. $U(1)_\pm$ is enhanced to $SU(2)_\pm$. In this case, twisted compactifications of the $\mathcal{N}=(2,0)$ theory flow to four-dimensional $\mathcal{N}=2$ SCFTs of class $\mathcal{S}$ [101]. When $a^1=a^2\neq0$ the diagonal subgroup $U(1)\subset U(1)_+\times U(1)_-$ is used to perform the twist. This procedure preserves the diagonal subgroup $SU(2)_\mathcal{F}\subset SU(2)_+\times SU(2)_-$ and consequently the flavor symmetry is enhanced from $U(1)_\mathcal{F}$ to $SU(2)_\mathcal{F}$. This corresponds to the $\mathcal{N}=1$ SCFT of Maldacena and Núñez [174].

In M-theory we can construct these partially twisted theories by wrapping M5-branes on a complex curve with prescribed singularities. We can decompose the eleven-dimensional spacetime as

$$M^{1,10}\to\mathbf{R}^{1,3}\times\mathbf{R}\times CY_3\,. \quad (13.6)$$

The M5-branes extend along $\mathbf{R}^{1,3}$ and wrap a complex curve $\Sigma\subset CY_3$ inside the Calabi-Yau threefold. In general this Calabi-Yau threefold is an SU(2) bundle over the curve whose determinant line bundle equals the canonical line bundle K_Σ of the curve. When the structure group is reduced from SU(2) to U(1), in addition to the $\mathrm{U}(1)_R$ R-symmetry, the local geometry enjoys an additional $\mathrm{U}(1)_{\mathcal{F}}$ flavor symmetry under which the supercharges are invariant. Under these circumstances the local geometry takes the form presented in (12.35) where $\mathcal{L}_1$ and $\mathcal{L}_2$ are two complex line bundles subject to the condition $\mathcal{L}_1 \otimes \mathcal{L}_2 = K_\Sigma$. While the Chern class fails to be well-defined for singular Calabi-Yau manifolds, the canonical bundle and canonical class can still be defined for mild singularities.[1] The two line bundles are associated to the $\mathrm{U}(1)_\pm$ above and the Calabi-Yau condition simply reproduces the twist condition

$$a^1 + a^2 = -\chi(\Sigma,\beta) = 2\mathbf{g} - 2 + \sum_{j=1}^{n} \beta_j\,, \tag{13.7}$$

where n is the number of singularities, β_j is a puncture dependent contribution, $\chi(\Sigma,\beta)$ is the modified Euler characteristic of the singular curve and a^1 and a^2 are the degrees of the line bundles,

$$\deg(\mathcal{L}_1) = a^1\,, \qquad \deg(\mathcal{L}_2) = a^2\,. \tag{13.8}$$

For conical singularities on the Riemann surface the puncture contribution is given exactly by $\beta_j = 1 - \xi_j$ where ξ_j is defined in Chapter 12 as the defect angle at the conical singularity. For different choices of a^1 and a^2 the fields of the M5-branes transform in different representations of the flavor symmetry $\mathrm{U}(1)_{\mathcal{F}}$ and one generically ends up in different $\mathcal{N} = 1$ IR fixed points.

13.2 Central charges from the anomaly polynomial

A powerful tool to study the six-dimensional $\mathcal{N} = (2,0)$ theory and its partial topological twists is provided by anomalies. The central charges of the resulting four-dimensional theory can be computed by integrating the M5-brane anomaly polynomial over the curve Σ. This procedure was introduced in [9, 25, 70] and further explored in the present context in [21–23, 25]. Here we summarize the main ingredients of these calculations.

The a and c anomaly of a four-dimensional $\mathcal{N} = 1$ SCFT are completely determined by the linear and cubic 't Hooft anomalies of the superconformal R-symmetry [12],

$$a = \frac{3}{32}\left(3\mathrm{Tr}R^3_{\mathcal{N}=1} - \mathrm{Tr}R_{\mathcal{N}=1}\right), \qquad c = \frac{1}{32}\left(9\mathrm{Tr}R^3_{\mathcal{N}=1} - 5\mathrm{Tr}R_{\mathcal{N}=1}\right). \tag{13.9}$$

[1] The criterion for the singularities to be mild enough to still be able to define the canonical class is that all singularities have to be Gorenstein. This is the case for all singularities we consider.

These anomalies can be read off from the six-form anomaly polynomial given by

$$I_6 = \frac{\mathrm{Tr}\, R^3_{\mathcal{N}=1}}{6} c_1(F)^3 - \frac{\mathrm{Tr}\, R_{\mathcal{N}=1}}{24} c_1(F) p_1(T_4), \tag{13.10}$$

where F is the U(1) bundle which couples to the R symmetry and T_4 is the tangent bundle to the four-dimensional spacetime manifold. This anomaly six-form can in turn be obtained by integrating the anomaly eight-form of the six-dimensional theory over a (possibly) singular curve Σ. As explained in [25], the contributions to the anomaly six-form can be separated in two parts, one geometric part and a second part accommodating the local contributions of the different singularities on Σ

$$I_6 = I_6(\Sigma) + \sum_i I_6(P_i). \tag{13.11}$$

We discuss these two contributions separately below.

13.2.1 **Bulk contribution**

We start by computing the geometric part of the anomaly polynomial. In our treatment we will apply a slightly different split than the one in [25] by defining the geometric part to contain information only about the smooth Riemann surfaces. All information about the punctures is packaged in the local contributions. The anomaly eight-form for a single M5-brane is given by [127, 144, 249]

$$I_8[1] = \frac{1}{48}\left[p_2(NW) - p_2(TW) + \frac{1}{4}\left(p_1(TW) - p_1(NW)\right)^2\right], \tag{13.12}$$

where by NW and TW we denote the normal and tangent bundle to the brane world volume and p_1 and p_2 are the first and second Pontryagin classes. For a general $\mathcal{N} = (2,0)$ theory of type $\mathfrak{g} \in$ ADE, the anomaly polynomial takes the form

$$I_8[\mathfrak{g}] = r_G I_8[1] + \frac{d_G h_G}{24} p_2(NW). \tag{13.13}$$

Here r_G, d_G, and h_G stand for the rank, dimension and Coxeter number of the group G, see Table 13.1. The normal bundle can be thought of as the SO(5) bundle coupled to the R-symmetry of the six-dimensional theory. The first and second Pontryagin classes of a vector bundle E can be expressed in terms of the Chern roots e_i as

$$p_1(E) = \sum_i e_i^2, \qquad p_2(E) = \sum_{i<j} e_i^2 e_j^2. \tag{13.14}$$

$\mathfrak{g}$	$r_\mathfrak{g}$	$d_\mathfrak{g}$	$h_\mathfrak{g}$
A_{N-1}	$N-1$	N^2-1	N
D_N	N	$N(2N-1)$	$2N-2$
E_6	6	78	12
E_7	7	133	18
E_8	8	248	30

Table 13.1: Rank, dimension and Coxeter number for the simply laced Lie algebras.

To compute the anomaly six-form for a U(1) R-symmetry of the form

$$R_{\mathcal{N}=1} = R_0 + \epsilon \mathcal{F}, \tag{13.15}$$

we need to couple the symmetry to a non-trivial U(1) bundle $\mathcal{F}$ over the flat four-dimensional part of the brane worldvolume. This induces a shift in the Chern classes

$$c_1(\mathcal{L}_1) \to c_1(\mathcal{L}_1) + (1+\epsilon)c_1(\mathcal{F}), \qquad c_1(\mathcal{L}_2) \to c_1(\mathcal{L}_2) + (1-\epsilon)c_1(\mathcal{F}). \tag{13.16}$$

There are an infinite number of such decomposable bundles over the smooth Riemann surface, labeled by the Chern numbers of the line bundles

$$c_1(\mathcal{L}_1) = p, \qquad c_1(\mathcal{L}_2) = q. \tag{13.17}$$

The Calabi-Yau condition in this case reduces to $p+q = 2\mathbf{g}-2$. We can now integrate the eight-form (13.13) over the smooth curve Σ to obtain

$$I_6(\Sigma) = \int_\Sigma I_8[\mathfrak{g}] = -\frac{\chi(\Sigma)}{12}\Big[(r_G + d_G h_G)(1+\mathfrak{z}\epsilon^3) - d_G h_G(\epsilon^2 + \mathfrak{z}\epsilon)\Big]c_1(\mathcal{F})^3$$
$$+ \frac{\chi(\Sigma)}{48} r_G(1+\mathfrak{z}\epsilon)c_1(\mathcal{F})p_1(T_4), \tag{13.18}$$

where $\chi(\Sigma)$ is the Euler characteristic of the curve Σ and we have defined

$$\mathfrak{z} = \frac{p-q}{p+q}. \tag{13.19}$$

13.2.2 Punctured intermezzo

To compute the local contributions from each puncture to the anomaly polynomial we need some more information about the different types of punctures that can appear in our setup. We only consider punctures that locally preserve $\mathcal{N} = 2$

supersymmetry. Each such puncture is characterized by an embedding $\Lambda : \mathfrak{su}(2) \to \mathfrak{g}$ and a sign $\sigma_i = \pm 1$. The flavor symmetry of the puncture is determined as the commutant $\mathfrak{h} \subset \mathfrak{g}$ of the image of Λ and the sign determines whether the puncture preserves the $\mathrm{U}(1)_+ \times \mathrm{SU}(2)_-$ or $\mathrm{SU}(2)_+ \times \mathrm{U}(1)_-$ symmetry. This $\mathbf{Z}_2$ valued label determines the directions normal to the M5-branes along which the puncture extends. We represent this label by coloring each puncture as in Figure 12.1. For $\mathfrak{g} = A_{N-1}$, the choice of embedding Λ is in one-to-one correspondence with a partition of N and thus with a Young tableau Y. A Young tableau with n_h columns of length h corresponds to a puncture P with global flavor symmetry group

$$G_P = \mathrm{S}\left(\prod_h \mathrm{U}(n_h)\right). \tag{13.20}$$

A maximal puncture is represented by a Young tableau with a single row of length N, see Figure 13.1, and has the maximal amount, i.e. $\mathrm{SU}(N)$, of global symmetry associated to it, a minimal (or simple) puncture is represented by a Young tableau with one row of length 2 and $N-2$ rows of length 1 and preserves the minimum amount of global symmetry, namely $\mathrm{U}(1)$. Equivalently one can label every puncture with a set of integers, p_k, characterizing the pole structure of the degree k Seiberg-Witten differentials at the puncture.[2] These integers can be obtained from the Young tableaux as follows: Start with the first row and label the first box with $p_1 = 0$, then increase the label by one as you move to the right along the first row of the Young tableau. When this row is finished, move to the next row and label the first box of the row with the same label as the last box in the previous row. Repeat this labeling process in every row until all N boxes have a label p_k. This labeling procedure is illustrated in Figure 13.1 for some simple examples with $N = 5$. To

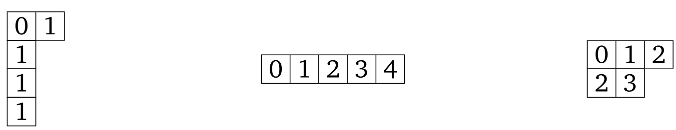


Figure 13.1: Labeling Young tableaux by their pole structure p_k. The first Young tableau corresponds to the minimal puncture with $p = (0,1,1,1,1)$. The second one is the maximal puncture with $p = (0,1,2,3,4)$ and the third diagram corresponds to an intermediate puncture with $p = (0,1,2,2,3)$.

each puncture one can associate an effective number of vector multiplets, $n_v(P_i)$,

[2] Often it is useful to rewrite the Seiberg-Witten curve $\mathcal{C}$ in terms of the Seiberg-Witten differential λ as

$$\mathcal{C}: \quad \lambda^N + \phi_2(z)\lambda^{N-2} + \cdots \phi_N(z) = 0\,, \tag{13.21}$$

where N is the rank of the gauge group. The type of puncture can then be specified by the pole structure of the degree k differentials $\phi_k(z)$.

and hypermultiplets, $n_h(P_i)$, given by [68, 101]

$$n_v(P_i) = \sum_{k=2}^{N} (2k-1) p_k \,, \qquad n_h(P_i) = n_v(P_i) + \frac{1}{2}\left(-(1+r_G) + \sum_r l_r^2 \right), \quad (13.22)$$

where l_r is the length of the rth row of the Young tableau. The numbers n_v and n_h represent the effective degrees of freedom of the specific puncture[3] and indeed when considering free theories these numbers agree with the actual number of vector and hypermultiplets. These constants can now be used to determine the local contribution of each puncture to the anomaly polynomial.

13.2.3 Contribution from a puncture

After this short intermezzo, we are ready to compute the local contribution of each puncture P_i to the anomaly polynomial. A puncture (locally) preserving flavor symmetry G with $\mathbf{Z}_2$-label σ_i contributes the following [25, 232]:

$$\begin{aligned} I_6(P_i) = \frac{1}{6}\left((1+\sigma_i \epsilon^3) n_v(P_i) - \frac{1}{4}(1+\sigma_i\epsilon)^3 n_v(P_i) \right) c_1(\mathcal{F})^3 \\ + \frac{1}{24}(1+\sigma_i\epsilon)(n_h(P_i) - n_v(P_i))\, c_1(\mathcal{F}) p_1(T_4) \\ - \frac{k_G}{3}(1+\sigma_i\epsilon)\, c_1(\mathcal{F}) c_2(F_G) \,. \quad (13.23) \end{aligned}$$

For a puncture associated to a Young tableau Y, the central charge of the flavor symmetry factor $SU(n_h)$ is given by

$$k_{\mathrm{SU}(n_h)} = 2 \sum_{i \le h} s_i \,, \qquad (13.24)$$

where s_i is the length of the i^{th} row of Y^T, the transpose of the original Young tableau.

Summing over all punctures in the theory one can easily read off the 't Hooft anomalies of the four-dimensional R-symmetry and compute the trial central charges as a function of ϵ. The correct value of the central charge is then obtained by maximizing a with respect to ϵ. When we set all $\sigma = 1$ and $q = 0$ (or $\sigma = -1$ and $p = 0$) one finds an $\mathcal{N} = 2$ theory and one can check that the anomalies reduces to

[3] The definition we use differs slightly from the ones in [68]. Our definition only accounts for the local degrees of freedom near the punctures, all global information is absorbed in $I_6(\mathcal{C})$.

the known results for $\mathcal{N}=2$ class $\mathcal{S}$ theories [102]. In the limit with no punctures the anomalies reduce to the results obtained in [19].

Chapter 14

Punctures in gauged supergravity

In this chapter we go back to the solution described in Section 12.4 and carefully study the geometry around a puncture, first in eleven and then in seven dimensions. Once we have understood how to describe punctures, we compute the holographic central charges of these AdS_5 solutions for a generic punctured Riemann surface and find an exact match at leading order in N with the results in Chapter 13. Furthermore, we compute the energy of an M2-brane wrapping the Riemann surface, which corresponds holographically to the conformal dimension of a protected baryonic operator, and discuss the exactly marginal deformations of our solutions.

14.1 Seven-dimensional supergravity solution

The seven-dimensional solution corresponding to M5-branes wrapped around a curve Σ takes the form (12.2)

$$\begin{aligned} \mathrm{d}s^2 =& \frac{4\mathrm{e}^{2f_0}}{g^2 r^2}(-\mathrm{d}t^2 + \mathrm{d}z_1^2 + \mathrm{d}z_2^2 + \mathrm{d}z_3^2 + \mathrm{d}r^2) + \frac{4\mathrm{e}^{\varphi_0+\varphi(x_1,x_2)}}{g^2}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2)\,, \\ A^{(i)} =& A^{(i)}_{x_1}(x_1,x_2)\mathrm{d}x_1 + A^{(i)}_{x_2}(x_1,x_2)\mathrm{d}x_2\,. \end{aligned} \tag{14.1}$$

where g is the coupling constant of the gauged supergravity, $i = 1,2$, $I = 1,2$ and the scalars λ_I take constant values in the IR. The Riemann surface has Gaussian curvature $\kappa = \pm 1, 0$. In this section we consider only $\kappa = \pm 1$. The analysis for $\kappa = 0$, i.e. the torus without punctures or the flat punctured sphere[1], deviates slightly from the discussion below. For the torus without punctures our solution

[1]A sphere with n punctures with puncture contributions $\sum_{i=1}^{n}(1-\xi_i) = 2$ summing exactly to 2

reduces to the one found in [19]. For the flat sphere the discussion is very similar but since the Liouville equation (12.4) reduces to the Laplace equation one has to study the solution of this equation with singular sources. We will not investigate this situation in detail here.

As discussed above the BPS equations governing this setup reduce to the Liouville equation for the conformal factor φ, (12.4), together with algebraic equations for the other fields. When the Riemann surface contains punctures or conical singularities one has to add local source terms to the right hand side of (12.4) as in (12.6). Given a solution to the Liouville equation, all other fields are fixed in terms of φ. The field strengths are given by (12.33), the scalars λ_1 and λ_2 are as in (12.36), and the constants appearing in the metric are given by (12.38). As discussed in Chapter 12, the parameters a^1 and a^2 consist of a global part and a local part accounting for the local contribution of the punctures, see (12.13). We can identify the global geometric contributions a^1_{global} and a^2_{global} with the Chern numbers of the line bundles in the smooth case and thus $a^1_{\text{global}} + a^2_{\text{global}} = 2\mathbf{g} - 2$. The parameter $\mathfrak{z}$ defined in Chapter 13 is related to the parameter z as

$$z = \frac{(2g-2)\mathfrak{z} + (a^1_{\text{local}}) - (a^2_{\text{local}})}{a^1 + a^2}\,. \tag{14.2}$$

We now explicitly uplift such a solution around a puncture and provide an interpretation of the local contribution of each puncture.

14.1.1 $\mathcal{N} = 2$ and Gaiotto-Maldacena

Locally, around a single puncture we can without loss of generality restrict ourselves to the situation $z = 1$, i.e. by performing a gauge transformation we can always locally put $a_2 = 0$. In this case supersymmetry is enhanced to $\mathcal{N} = 2$ and the solution simplifies considerably. The various fields are given by

$$\lambda_2 = -\frac{2}{3}\lambda_1 = \frac{1}{5}\log 2\,, \qquad F^{(1)}_{x_1x_2} = \frac{1}{8g}\mathrm{e}^{\varphi}\,, \qquad F^{(2)}_{x_1x_2} = 0\,. \tag{14.3}$$

where φ still solves the Liouville equation (12.4). We can uplift this solution using the uplift formulae summarized in Appendix J.4 to find the metric

$$\mathrm{d}s^2_{11} = \frac{1}{2g^2}\tilde{\Delta}^{1/3}\mathrm{d}s^2_{\mathrm{AdS}_5} + \frac{\tilde{\Delta}^{-2/3}}{4g^2}\Big(\tilde{\Delta}\mathrm{d}s^2_{\Sigma} + \tilde{\Delta}\mathrm{d}\alpha^2 + \cos^2\alpha(\mathrm{d}\beta^2 + \sin^2\beta\,\mathrm{d}\phi_2^2) + 2\sin^2\alpha(\mathrm{d}\phi_1 + 2mA^1)^2\Big)\,, \tag{14.4}$$

where $\tilde{\Delta} = 1+\cos^2\alpha$ and $ds_\Sigma^2 = e^{\varphi}(dx_1^2+dx_2^2)$ is the metric on the Riemann surface with unit Gaussian curvature. When we make the following coordinate change

$$\cos^2\alpha \to \frac{y^2}{N^2}\,, \tag{14.5}$$

we can fit this solution in the analysis of Lin-Lunin-Maldacena [167] to find the following eleven-dimensional supergravity background

$$\begin{aligned}
ds_{11}^2 &= \left(\frac{\pi\ell_p^3}{2}\right)^{2/3} e^{2\tilde{\lambda}}\left(4ds_{AdS_5}^2 + y^2 e^{-6\tilde{\lambda}} d\tilde{\Omega}_2^2 + ds_4^2\right)\\
ds_4^2 &= \frac{4}{1-y\partial_y D}(d\chi + v_i dx^i)^2 + \frac{-\partial_y D}{y}\left[dy^2 + e^D(dx_1^2+dx_2^2)\right]\\
v_i &= -\frac{1}{2}\epsilon_{ij}\partial_j D \qquad v = v_i dx_i\\
e^{-6\tilde{\lambda}} &= -\frac{\partial_y D}{y(1-y\partial_y D)}\\
G_4 &= \frac{\pi\ell_p^3}{2} F_2\wedge d\Omega_2\\
F_2 &= 2\left[(dt+v)d(y^3e^{-6\tilde{\lambda}}) + y(1-y^2e^{-6\tilde{\lambda}})dv - \frac{1}{2}\partial_y e^D dx_1\wedge dx_2\right]
\end{aligned} \tag{14.6}$$

where the function D satisfies the SU(∞) Toda equation

$$\left(\partial_{x_1}^2 + \partial_{x_2}^2\right)D + \partial_y^2 e^D = 0\,, \tag{14.7}$$

and we have used the fact that the gauge coupling constant is related to the radius of the AdS$_7$ appearing in the UV, $R_{\mathrm{AdS}_7} = \frac{2}{g} = (\pi N)^{1/3}\ell_p$. For our $\mathcal{N}=2$ solution (14.3), the function D takes the form

$$e^D = \frac{e^{\varphi}}{2}\left(N^2 - y^2\right). \tag{14.8}$$

When the Riemann surface is smooth the conformal factor φ reduces to the constant curvature metric (12.5) and the solution (14.4) reduces to the $\mathcal{N}=2$ solution of Maldacena-Núñez [174].

14.2 Punctures in eleven dimensions

Since locally all punctures we consider preserve $\mathcal{N}=2$ supersymmetry it suffices to discuss them in the $\mathcal{N}=2$ framework of [102]. After a discussion in eleven dimensions we will go back to seven dimensions and learn how to characterize the punctures there. We will then extend the analysis to include $\mathcal{N}=1$ supersymmetric solutions in seven dimensions.

Solving the Toda equation in a background with a general Riemann surface with localized punctures is hard to do in a closed form. However, locally around a puncture one can analyze the Toda equation and describe the boundary conditions for all types of regular punctures. Around an $\mathcal{N}=2$ punctures, the function D satisfies the axially symmetric SU(∞) Toda equation. By performing a Bäcklund transformation [102]

$$r^2 \mathrm{e}^D = \rho^2\,, \qquad y = \rho\partial_\rho V \equiv \dot{V}\,, \qquad \log r = \partial_\eta V \equiv V'\,, \tag{14.9}$$

we can transform the problem of solving the Toda equation to a three-dimensional axially symmetric electrostatics problem [237]. After this transformation the Toda equation becomes the cylindrically symmetric Laplace equation in three dimensions,

$$\ddot{V} + \rho^2 V'' = 0\,. \tag{14.10}$$

It was understood in [102] that solutions to this equation correspond to regular geometries, up to possibly A_{k-1} singularities, if and only if the line charge density $\dot{V}$ is given by a piece-wise linear function with decreasing integer slopes where slope changes only occur at integer values of η (or y). At points where the slope decreases by k units one finds an A_{k-1} singularity. Near a segment with constant slope $\dot{V}'$ the potential behaves as $V \sim \dot{V}(\eta)\log\rho$. This implies that $\log r = V' = \dot{V}'\log\rho$ and

$$D = 2(\log\rho - \log r) = -2\left(1 - \frac{1}{\dot{V}'}\right)\log r\,. \tag{14.11}$$

This expression is valid in the range of y where the slope is constant. The general boundary conditions for the Toda equation at a specific puncture thus become

$$\Box D = -4\pi\ell(y)\delta^{(2)}(r) \tag{14.12}$$

where $\ell(y)$ is a piece-wise constant function which only changes value at integer values of y. These constants decrease and take value $\left(1-\frac{1}{n_i}\right)$ where $n_i \in \mathbf{N}$ is the slope of the ith segment. This is illustrated in Figure 14.1.

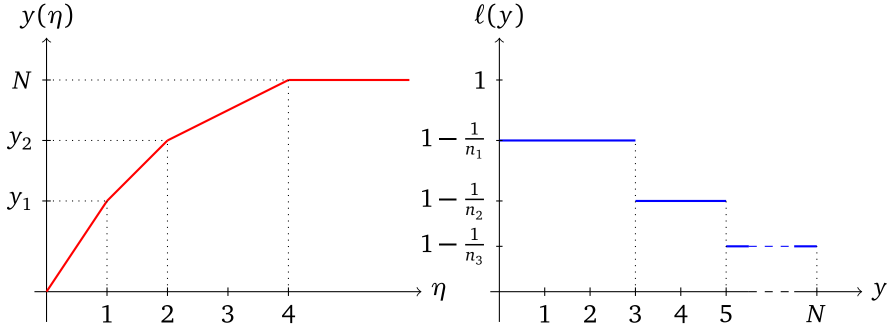


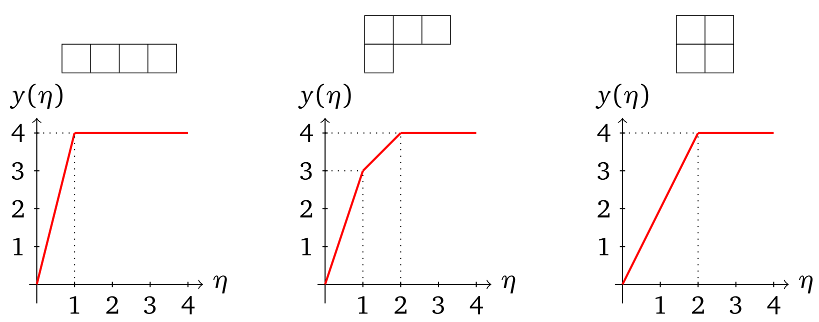
Figure 14.1: piece-wise linear function $y(\eta)$ and the accompanying step function $\ell(y)$.

If we have various segments with change in slope k_i then the total global symmetry associated to this puncture is

$$G = \frac{\prod_i U(k_i)}{\mathrm{U}(1)}, \tag{14.13}$$

The overall U(1) is related to the axial symmetry around the puncture which is not a real global symmetry of the system and hence is modded out.

Let us now relate these geometries to the punctures introduced in Chapter 13. The geometry around a puncture is fully determined by the piece-wise linear function $y(\eta)$. To every such function we can associate a set of integers n_i representing the slopes of the segments extending from $\eta = i-1$ to $\eta = i$. To the puncture with slopes n_i, we associate a Young tableau such that n_i is the length of the ith row. We illustrate this for $N = 4$ in Figure 14.2. This provides the connection between the supergravity punctures and the correct Young tableau; indeed we see that the global symmetries preserved at each puncture matches with the discussion in Chapter 13.



This analysis shows that every puncture is determined by a y dependent function

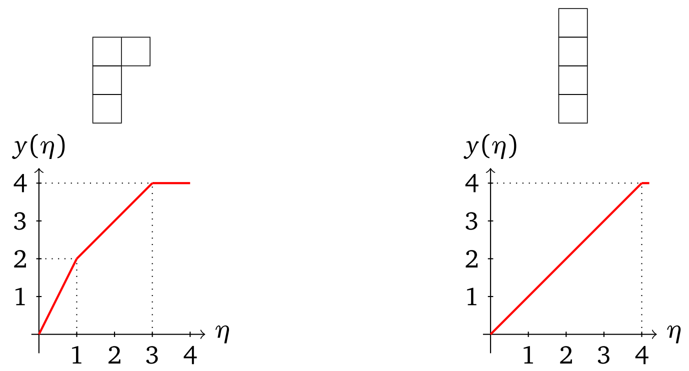


Figure 14.2: The Young diagrams and accompanying piece-wise linear functions for $N=4$. The first diagram from the left represents a maximal puncture, the third is a $\mathbf{Z}_2$ singularity, the fourth is a minimal puncture, while the rightmost diagram corresponds to a regular point.

$\eta(y)$ determining the structure of the puncture. We will not be able to capture all information contained in this function in seven dimensions but we will see that we nevertheless can extract a lot of information about the puncture solely from the seven-dimensional supergravity.

14.3 Punctures in seven dimensions

When we insert (14.8) in the SU(∞) Toda equation (14.7) we obtain the Liouville equation for φ (12.4). Similar to the Toda equation, finding global solutions to the Liouville equation in closed form on a general Riemann surface with punctures is hard. Nevertheless, we can learn a lot about the solutions of interest by analyzing them locally near a puncture. At a fixed value of y, the boundary condition of the Toda equation (14.12) reduce to the following boundary condition for the Liouville equation

$$\varphi \sim -2(1-\tfrac{1}{n_i})\log r\,, \qquad \text{as} \quad r\to 0\,. \tag{14.14}$$

This is exactly the boundary condition describing a conical defect on the Riemann surface where $0<\frac{1}{n_i}<1$ parameterizes the opening angle at the conical singularity. We thus conclude that the different punctures in eleven dimensions correspond to conical defects on the Riemann surface of our seven-dimensional solutions, where the opening angle changes as a function of y. This change of opening angle goes beyond the seven-dimensional supergravity approximation and can only be treated approximately in the seven-dimensional framework. However, for some solutions – such as $\mathbf{Z}_k$ orbifold singularities – there is only one value of the slope in eleven dimensions and exact results can be obtained purely from seven dimensions. This kind of punctures correspond to rectangular Young tableaux and for them we can identify

$$\xi = \frac{1}{n}. \tag{14.15}$$

For more general punctures, where the piece-wise linear function consists of more then one linear piece, we have to content ourselves with an approximate description of the eleven-dimensional puncture by specifying a single ξ as the inverse of the average slope of the various segments.[2]

Let us now go back to the supergravity solution in Section 14.1 and translate our results to seven dimensions. In an $\mathcal{N} = 2$ theory the field strengths $F^{(1),(2)}$ are completely fixed by the twist in terms of φ and the contribution of the punctures manifests itself only through the volume, $\mathrm{V}_{\mathbf{g},\xi_i}$, of the Riemann surface which explicitly depends on the opening angles of the conical defects,

$$\mathrm{V}_{\mathbf{g},\xi} = \frac{2\pi}{\kappa}\left(2 - 2\mathbf{g} - \sum_{P_i}(1-\xi_i)\right). \tag{14.16}$$

In an $\mathcal{N} = 1$ theory the punctures still contribute to the volume but also the field strengths will be modified. More specifically, every puncture in an $\mathcal{N} = 1$ theory is labeled by a sign σ_i indicating in which transverse direction the puncture extends. This label indicates whether the puncture P_i contributes to a^1_{local} or a^2_{local} and results in

$$\begin{aligned} a^1_{\text{local}} &= \sum_{\{P_i|\sigma_i=1\}} (1-\xi_i), \\ a^2_{\text{local}} &= \sum_{\{P_i|\sigma_i=-1\}} (1-\xi_i), \end{aligned} \tag{14.17}$$

where ξ_i is the inverse of the average slope of the ith puncture and the sums above run exclusively over punctures with $\sigma_i = \pm 1$ respectively.

14.3.1 Central charges and M2 brane energy

We can now proceed and compute the central charges a and c for our supergravity solutions using standard holographic results [129]. The holographic conformal anomalies at leading order in N are given by

$$a = c = \frac{\pi R^3_{AdS_5}}{8G^{(5)}_N} = \left(\frac{2}{g}\right)^2 \frac{2\pi^3 R^3_{AdS_5} R^4_{S^4} \mathrm{V}_{\mathbf{g},\xi} \mathrm{e}^{\varphi_0}}{3G^{(11)}_N}, \tag{14.18}$$

[2]For example in Figure 14.2 the average slopes would be resp. 4, 2, 2, 4/3, 1.

where ℓ_p is the Planck length in eleven dimensions, $G_N^{(5)}$ and $G_N^{(11)} = 16\pi^7\ell_p^9$ are the five- and eleven-dimensional Newton constants, and $R_{AdS_5} = \frac{2e^{f_0}}{g} = (\pi N)^{1/3}\ell_p e^{f_0}$ and $R_{S^4} = \frac{4}{g} = 2(\pi N)^{1/3}\ell_p$ are the radii of AdS_5 and the four-sphere respectively. Inserting this in (14.18) we obtain

$$a = c = \frac{2V_{\mathbf{g},\xi}}{3\pi} N^3 e^{3f_0+\varphi_0}\,. \tag{14.19}$$

Using the expressions (12.38) for f_0 and φ_0, we find

$$a = c = -\frac{\kappa\, V_{\mathbf{g},\xi}}{2\pi} N^3 \frac{\kappa(1-9z^2) + \left(1+3z^2\right)^{3/2}}{96z^2}\,. \tag{14.20}$$

This matches exactly the large N result obtained from the anomaly polynomial in Chapter 13. Furthermore when we remove all the punctures we recover the result for the central charges of M5-branes wrapped on a smooth Riemann surface in [19]. If all the punctures are minimal, the leading order result for the large N central charges is not affected. This is indeed as expected since minimal punctures correspond to the addition of hypermultiplets to the dual quiver gauge theory. These indeed only contribute at order N^2 and will thus not be visible at leading order in N.

For $\mathbf{g} = 0$ we find that the central charges are negative for $\sum_i(1-\xi_i) < 2$ indicating the presence of enhanced global symmetries rendering our a-maximization calculation invalid. Indeed it is a well-known fact that one can use this leftover global symmetry to fix the positions of three labeled points on the sphere. The minimal case with no extra symmetries is the sphere with three maximal punctures which corresponds to a $\mathcal{T}_N$ building block and indeed the large N anomalies (14.20) correctly describe those of a single $\mathcal{T}_N$.

Similarly for the torus without punctures the anomaly vanishes implying there is a leftover symmetry which can be used to fix the position of a single labeled point. Indeed for a torus with one or more punctures there is no extra symmetry left and we find non-vanishing positive anomalies. For a torus with one or more punctures which contribute at order N^3 we again find agreement with the anomaly polynomial. When the torus has only minimal punctures, we correctly reproduce the expected N^2 scaling of the anomalies. However, the anomaly thus obtained does not exactly match the one obtained from the anomaly polynomial due to the presence of other 'subleading' terms which also scale as N^2 which are not included in our analysis. We will discuss this setup in more detail in Section 15.3.

Our supergravity backgrounds describe the so-called wrapped brane geometries, see [104] for a review, where the curve Σ is a supersymmetric cycle to which one can associate a canonical BPS operator corresponding to an M2-brane wrapping this cycle [102]. The dimension of this operator is given by the energy of the wrapped M2-brane, which we can compute at leading order in N using the supergravity dual. The result is

$$\Delta(\mathcal{O}_{M2}) = \frac{2\mathrm{V}_{\mathbf{g},\xi_i}}{\pi} N e^{f_0+\varphi_0-2(\lambda_1+\lambda_2)} = -\kappa \frac{\mathrm{V}_{\mathbf{g},\xi_i}}{4\pi} N \left(1 - \frac{\kappa}{2}\sqrt{1+3z^2}\right), \tag{14.21}$$

which at large N and for specific values of the parameters indeed matches with the known results in [19, 41, 102]. In Section 15.4 we show how to reproduce this conformal dimension directly using the holographically dual SCFT.

14.3.2 Marginal deformations

From our supergravity solutions we can also compute the exactly marginal deformations of the dual SCFT. Doing this calculation rigorously requires computing the spectrum of KK excitations around the eleven-dimensional solutions presented above and identify the scalar modes with vanishing mass. This is a hard task which we do not know how to perform in general. Nevertheless, there is a natural set of massless modes in the solutions which we believe to exhaust the list and thus span the superconformal manifold in the dual SCFT. To illustrate this we focus on the case of Riemann surfaces with n indistinguishable minimal punctures. One in principle can straightforwardly extend this computation to compute the dimension of the moduli space of a Riemann surface with different kinds of punctures but the computations quickly become very tedious.

The first set of marginal deformations is given by the moduli space of the algebraic curve Σ denoted by $\mathcal{M}_{\mathbf{g},n}$. For $\mathbf{g} > 1$ these enter the construction in the action of the Fuchsian group Γ on the hyperbolic plane $\mathbf{H}$ spanned by (x_1, x_2). For $\mathbf{g} > 1$ the dimension of this moduli space is $\dim_{\mathbf{C}} \mathcal{M}_{\mathbf{g},n} = 3\mathbf{g} - 3 + n$.[3] These moduli correspond to the complex structure deformations of the curve and can be identified with the complex gauge couplings in the dual field theory. For $\mathcal{N} = 2$ solutions these are the only moduli present compatible with supersymmetry.

For $\mathcal{N} = 1$ solutions a second set of exactly marginal deformations arises from the freedom to shift the gauge fields by a flat connection on Σ,

$$A^{(1)} \to A^{(1)} + A_{\text{flat}}, \qquad A^{(2)} \to A^{(2)} - A_{\text{flat}}. \tag{14.22}$$

Such a shift leaves the BPS equations invariant. For generic $\mathbf{g}$ and n there are $2\mathbf{g}+n$ flat U(1) connections. Additionally every puncture contributes an additional real parameter corresponding to the $\mathbf{CP}^1$ worth of direction inside the two-dimensional fiber over Σ, see [41] for a discussion on this extra modulus. Therefore the complex

[3] For the two-sphere $\dim_{\mathbf{C}} \mathcal{M}_{0,0} = \dim_{\mathbf{C}} \mathcal{M}_{0,1} = \dim_{\mathbf{C}} \mathcal{M}_{0,2} = 0$. For $n \geq 3$ punctures $\mathcal{M}_{0,n} = n - 3$. A torus with n punctures has $\dim_{\mathbf{C}} \mathcal{M}_{1,n} = n + 1$.

dimension of the supersymmetric conformal manifold of the dual SCFT is

$$\dim_{\mathbf{C}} \mathcal{M}_C = 4\mathbf{g} - 3 + 2n\,. \tag{14.23}$$

In the special case when $z = 0$ there is an overall $SU(2)_{\mathcal{F}}$ flavor symmetry leading to additional marginal deformations related to $SU(2)_{\mathcal{F}}$ Wilson lines on Σ, see [41]. There are $6\mathbf{g} - 6 + 3n$ such flat connections and thus the dimension of the conformal manifold becomes $\dim_{\mathbf{C}} \mathcal{M}_C = 6\mathbf{g} - 6 + 3n$.

Chapter 15

Dual quiver gauge theories

In this chapter we describe the four-dimensional quiver gauge theories dual to the $\mathcal{N} = 1$ AdS_5 solutions studied in the previous chapters. As in the gravitational case we restrict ourselves to $\mathrm{SU}(N)$ quiver gauge theories originating from the A_{N-1} type $\mathcal{N} = (2,0)$ theory. It is possible to study other gauge groups and quivers originating from D_N or even E_n type theories along similar lines by using the results in [68, 69, 71–73, 101, 231]. We also restrict our analysis to minimal, or simple, punctures and compute the 't Hooft anomalies, the dimensions of protected operators and the dimension of the conformal manifold for these quiver gauge theories. Minimal punctures correspond in the quiver gauge theory to hypermultiplets connecting adjacent gauge groups. This is the simplest case to analyze but one can also study more general punctures by adding more intricate quiver tails [2, 68, 101]. We emphasize that many of the results in this chapter have appeared in the literature before or can be derived in a straightforward manner following the discussion in [2, 18–20, 24, 29, 41, 99].[1] Nevertheless, we believe that the summary below serves a useful purpose to illustrate the salient features of our construction.

15.1 Setup and symmetries

The quiver gauge theories we consider are constructed from $\mathcal{T}_N$ building blocks connected by strands of linear quivers build from vector and hyper multiplets. This type of quiver is sketched in Figure 15.1 which should be interpreted as follows:

- Nodes without ears denote $\mathcal{N} = 1$ vector multiplets with $\mathrm{SU}(N)$ gauge groups.
- Nodes with ears denote $\mathcal{N} = 2$ vector multiplets with $\mathrm{SU}(N)$ gauge groups. The ear represents the adjoint chiral.

[1] To the best of our knowledge, the relation between the L^{aba} SCFTs and theories of class $\mathcal{S}$ described in Section 15.3 has not appeared explicitly in the literature before.

- Double lines between two vector multiplets denote hypermultiplets in the bifundamental representation of the two adjacent gauge groups.
- $\mathcal{T}_N$ building blocks are denoted by triangles which are connected to three vector multiplets.
- The coloring of the matter field is associated to a $\mathbf{Z}_2$ valued label σ_i. Matter field with $\sigma_i = 1$ are colored red and those with $\sigma = -1$ are blue.

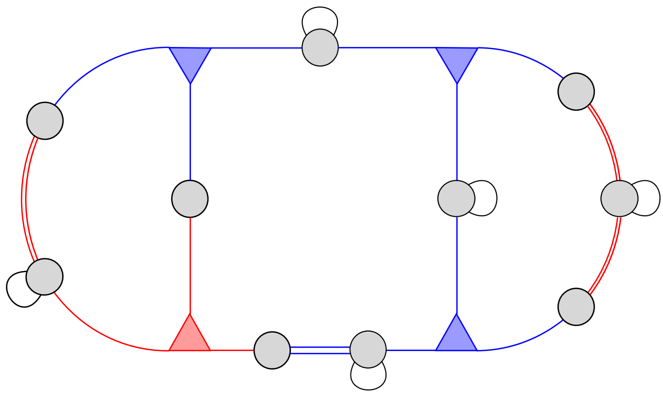


Figure 15.1: An example of a genus $\mathbf{g} = 3$ quiver.

A general quiver of genus $\mathbf{g}$ contains $2\mathbf{g} - 2$ $\mathcal{T}_N$ building blocks combined with $3\mathbf{g} - 3$ strands built out of vector and hypermultiplets. Every such linear quiver is built out of n_i hypermultiplets and $n_i + 1$ vector multiplets. Consequently, the total number of hypermultiplets is $n = \sum_i n_i$ and the number of vector multiplets is $v = 3\mathbf{g} - 3 + n$. Let us denote the number of $\mathcal{N} = 2$ vectors by v_2, analogously v_1 is the number of $\mathcal{N} = 1$ vectors.

The $\mathcal{T}_N$ theory was proposed in [101] as the low-energy theory coming from N M5-branes wrapping a trice punctured sphere, see [232] for a review. It is an $\mathcal{N} = 2$ building block with $\mathrm{SU}(N)^3 \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_r$ global symmetry. Except for the case where $N = 2$ there is no known weakly coupled Lagrangian description for these theories.[2] The spectrum of the $\mathcal{T}_N$ theory includes Higgs branch operators μ_a with $a = 1, 2, 3$ called moment maps; one triplet for each $\mathrm{SU}(N)$ flavor group. Each such operator has dimension two and transforms in the adjoint of one of the $\mathrm{SU}(N)$ factors. Additionally there are also Coulomb branch operators $u_k^{(i)}$ with $k = 3, \ldots, N$ and $i = 1, \ldots, k - 2$ associated to each $\mathcal{T}_N$ with dimension k. Finally there are dimension $N - 1$ operators Q and $\widetilde{Q}$ transforming, respectively, in the $(\mathbf{N}, \mathbf{N}, \mathbf{N})$ and $(\overline{\mathbf{N}}, \overline{\mathbf{N}}, \overline{\mathbf{N}})$ representation of $\mathrm{SU}(N)^3$. To each hypermultiplet we can also associate a

[2] In this case the $\mathcal{T}_2$ theory reduces to eight free chiral multiplets transforming in the trifundamental of the $\mathrm{SU}(2)^3$ global symmetry.

triplet of moment map operators μ transforming in the adjoint of the SU(N) flavor symmetry.

Since in general we consider $\mathcal{N}=1$ quiver gauge theories, it is convenient to think of $\mathcal{N}=2$ vector multiplets, hypermultiplets and $\mathcal{T}_N$ theories as $\mathcal{N}=1$ building blocks. An $\mathcal{N}=2$ vector multiplet can be thought of as a $\mathcal{N}=1$ vector with an additional adjoint chiral. A $\mathcal{T}_N$ on the other hand can be though of as a $\mathcal{N}=1$ building block with an additional U(1) flavor symmetry. Finally every hypermultiplet consist of two chiral multiplets in conjugate representations $H_i=\{q_i,\tilde{q}_i\}$. In terms of these constituent fields the moment map triplet of the hypermultiplet takes the form $\mu_i^+=q_i\tilde{q}_i$, $\mu_i^0=|q_i|^2-|\tilde{q}_i|^2$ and $\mu_i^-=(\mu_i^+)^*$. When expressing the theory as a $\mathcal{N}=1$ theory the generators of the $\mathcal{N}=2$ R-symmetry decompose into a generator for the $\mathcal{N}=1$ superconformal R-symmetry and a generator for an extra U(1) flavor symmetry,[3]

$$R_{\mathcal{N}=1}=\frac{1}{3}R_{\mathcal{N}=2}+\frac{4}{3}I_3\,,\quad J=R_{\mathcal{N}=2}-2I_3\,. \tag{15.1}$$

where I_3 is the Cartan of $SU(2)_R$. When the supersymmetry is broken to $\mathcal{N}=1$, $R_{\mathcal{N}=1}$ will no longer be in the same multiplet as the stress tensor and the superconformal R-symmetry at the IR fixed point may potentially mix with U(1) flavor symmetries.

Now that we have introduced all the building blocks we can use them to construct generalized quiver gauge theories by gauging the various SU(N) global symmetries. Every $\mathcal{N}=2$ gauging introduces a superpotential term of the form

$$\mathcal{W}^{\text{gauging}}_{\mathcal{N}=2}=\mathrm{Tr}\phi_i(\mu_i+\mu_{i+1})\,, \tag{15.2}$$

where the $\mu_{i,i+1}$ are the moment maps belonging to the adjacent matter building blocks. This superpotential breaks all the baryonic symmetries which are otherwise present. A general quiver (like the one in Figure 15.1) has $\mathcal{N}=1$ supersymmetry. However, in addition to $\mathcal{N}=1$ supersymmetry such quivers possess large amounts of global symmetries. We always have an overall $U(1)_R$ R-symmetry and additionally for each hyper, $\mathcal{T}_N$ and adjoint chiral there is an associated U(1). We denote the U(1) symmetries acting on the hypers and $\mathcal{T}_N$ blocks by J_i and the ones acting on the adjoint chiral by F_i. The full global symmetry is thus $U(1)^{\nu_2+2\mathbf{g}-2+n}\times U(1)_R$. However some of these symmetries are anomalous. Each gauge group, except for one global combination, provides one anomaly constraint so we end up with $2\mathbf{g}-2+\nu_2$ anomaly-free U(1) global symmetries.[4]

We can consider such gauge theories without extra superpotential terms. However, we expect these quivers to break up in smaller quivers in the IR [20]. Indeed, the

[3] See Appendix K for more detail on our SCFT conventions.

[4] This result is only valid for quivers with $\mathbf{g}>1$, for $\mathbf{g}=1$ we find ν_2+1 anomaly-free U(1)s.

one-loop beta-functions for the gauge group couplings are

$$b_0(V_i^{\mathcal{N}=2}) = 0\,, \qquad b_0(V_i^{\mathcal{N}=1}) = -N\,. \tag{15.3}$$

The gauge couplings for the $\mathcal{N}=2$ gauge groups are marginal and without superpotential, they should be marginally irrelevant [114]. As a result, the $\mathcal{N}=2$ gauge groups are non-dynamical in the IR and the quiver will break up at these sites.

We are more interested in finding situations where the IR dynamics of the gauge theory is non-trivial. We expect that for an appropriate choice of the superpotential the IR physics is governed by an $\mathcal{N}=1$ SCFT. By adding specific superpotential terms we can prevent the quivers from breaking apart in the IR. At $\mathcal{N}=1$ sites we turn on

$$\mathcal{W}_i^{\mathcal{N}=1} = \alpha_i \mathrm{Tr}\mu_{i-1}\mu_i\,, \tag{15.4}$$

where α_i are arbitrary complex numbers. At $\mathcal{N}=2$ sites on the other hand we turn on the superpotential

$$\mathcal{W}_i^{\mathcal{N}=2} = \beta_i^L \mathrm{Tr}\phi_i\mu_{i-1} + \beta_i^R \mathrm{Tr}\phi_i\mu_i\,, \tag{15.5}$$

where $\beta_i^{L,R}$ are complex numbers. The superpotentials (15.4) and (15.5) always preserve the $\mathrm{U}(1)_R$ R-symmetry R_0 generated by

$$R_0 = R_{\mathcal{N}=1} + \frac{1}{6}\sum_i J_i\,. \tag{15.6}$$

To see if these superpotentials leave another anomaly-free U(1) unbroken we first need to understand how the chiral anomaly is canceled locally at every site. At the ith node of the quiver the combination $J_{i-1} - J_i$ is always anomaly-free. When this site also contains an adjoint chiral there is a second anomaly-free U(1) with generator $J_{i-1} + J_i - 2F_i$. Note that $\mathrm{Tr}(J_{i-1} - J_i) = 0$ so this global symmetry is baryonic and will by itself be broken at $\mathcal{N}=2$ sites by the superpotential (15.5). However, by combining it with the U(1)'s coming from the adjoint chirals we are able to construct a non-baryonic anomaly-free U(1) global symmetry.

In order to construct such a global U(1) we can assign to all matter multiplets, hypermultiplets and $\mathcal{T}_N$ blocks, a sign σ_i as indicated by the coloring in Figure 15.1. When crossing an $\mathcal{N}=1$ vector the sign of two neighboring matter fields flips. When crossing an $\mathcal{N}=2$ vector multiplet the sign remains unchanged. Not every quiver configuration allows for such an assignment of $\mathbf{Z}_2$ label. If a quiver does not allow the assignment we expect that it will flow to the universal $\mathcal{N}=1$ IR fixed point discussed in [233]. If a quiver does allow for a consistent sign assignment complying to these rules we have an additional U(1) flavor symmetry with generator

$$\mathcal{F} = \frac{1}{2}\sum_{i=1}^{n}(\sigma_i J_i - (\sigma_{i-1}+\sigma_i)F_i) \,. \tag{15.7}$$

This is the only anomaly-free flavor U(1) preserved by the superpotential terms (15.4) and (15.5). This setup and the rules for finding the global flavor U(1) follow the construction in [19, 20, 29].

One might additionally want to add the superpotential term

$$\mathcal{W}_i^{\spadesuit} = \gamma_i(\mu_i)^2 \,, \tag{15.8}$$

where the γ_i are arbitrary complex numbers. However, when one of the $\gamma_i \neq 0$ the superpotential (15.8) breaks the extra flavor symmetry $\mathcal{F}$. As we will discuss in the next section, when this happens the theories always flow to the same universal IR SCFT as in [233]. In the following we tune all γ_i to zero and consider quivers gauge theories with the extra U(1) global symmetry in (15.7). These theories allow for interesting new IR SCFTs since the anomaly-free flavor symmetry can mix with the R-symmetry in (15.6). To determine the correct $\mathcal{N}=1$ superconformal R-symmetry one then has to take the linear combination

$$R_\epsilon = R_0 + \epsilon\mathcal{F} \,, \tag{15.9}$$

and fix the real number ϵ using a-maximization [145].

15.2 IR dynamics

Now that we have set the stage let us study the IR dynamics of the $\mathcal{N}=1$ quiver gauge theories introduced above. the upshot is that whenever the quiver allows for a consistent assignment of the labels σ_i, we find interacting SCFTs dual to the gravity solutions described in Chapter 14. Since the quivers with $\mathbf{g}=1$ do not contain any $\mathcal{T}_N$ building blocks we first focus on the generic situation with $\mathbf{g}>1$ and discuss $\mathbf{g}=1$ in Section 15.3. For a quiver of genus $\mathbf{g}>1$ we have $2\mathbf{g}-2$ $\mathcal{T}_N$ building blocks and $3\mathbf{g}-3$ strands of linear quiver. We have p $\mathcal{T}_N$ blocks with positive sign and q with negative sign. Similarly, we have x hypers with positive sign and y with negative sign. If the theory flows to an IR SCFT we can determine the IR superconformal R-symmetry using a-maximization [145]. We can compute the a and c anomaly and determine the dimensions of the chiral operators. The central charges a and c are given by the 't Hooft anomalies associated with the superconformal R-symmetry $R_{\mathcal{N}=1}$ as in (13.9).

Our quivers admit a one-parameter family of R-symmetries which are linear combinations of the UV R-symmetry R_0 and the global flavor symmetry $\mathcal{F}$ as in (15.9). For each R_ϵ we can compute the trial central charge $a(\epsilon)$. The superconformal R-symmetry maximizes this function $a(\epsilon)$ and in this way uniquely determines the value of ϵ. The charges of the superfields under R_ϵ are

$$R_\epsilon(q_i) = R_\epsilon(\tilde{q}_i) = \frac{1}{2}(1+\epsilon\sigma_i)\,, \quad \text{and} \quad R_\epsilon(\phi_i) = 1 - \frac{1}{2}\epsilon(\sigma_{i-1}+\sigma_i)\,. \tag{15.10}$$

Consequently, the 't Hooft anomalies for the ith hypermultiplet are

$$\mathrm{Tr}\,R_\epsilon(H_i) = N^2(\epsilon\sigma_i - 1)\,, \qquad \mathrm{Tr}\,R_\epsilon(H_i)^3 = \frac{1}{4}N^2(\epsilon\sigma_i-1)^3\,, \tag{15.11}$$

while for the ith vector multiplet we find

$$\begin{aligned} \mathrm{Tr}\,R_\epsilon(V_i) &= (N^2-1)\left[1-\frac{1}{2}\epsilon(\sigma_{i-1}+\sigma_i)\right], \\ \mathrm{Tr}\,R_\epsilon(V_i)^3 &= (N^2-1)\left[1-\frac{1}{8}\epsilon^3(\sigma_{i-1}+\sigma_i)^3\right]. \end{aligned} \tag{15.12}$$

For a single $\mathcal{T}_N$ we have

$$\begin{aligned} \mathrm{Tr}\,R_\epsilon(\mathcal{T}_{Ni}) &= \frac{1-\sigma_i\epsilon}{2}\mathrm{Tr}\,R^3_{\mathcal{N}=2}\,, \\ \mathrm{Tr}\,R_\epsilon(\mathcal{T}_{Ni})^3 &= \frac{(1-\sigma_i\epsilon)^3}{8}\mathrm{Tr}\,R^3_{\mathcal{N}=2} + \frac{3}{2}(1-\sigma_i\epsilon)(1+\sigma_i\epsilon)^2\mathrm{Tr}R_{\mathcal{N}=2}I_3^2\,, \end{aligned} \tag{15.13}$$

where

$$\begin{aligned} \mathrm{Tr}\,R_{\mathcal{N}=2}I_3^2 &= \frac{1}{12}(6-N-9N^2+4N^3)\,, \\ \mathrm{Tr}\,R^3_{\mathcal{N}=2} &= 2+N-3N^2\,. \end{aligned} \tag{15.14}$$

In the equations above the trace Tr denotes the sum over all chiral fermions in the ith hyper, vector or $\mathcal{T}_N$, together with the trace over the gauge indices.

We can now write the anomaly of one strand by summing over all fields in the linear piece to obtain[5]

$$\begin{aligned} \mathrm{Tr}R_\epsilon(H) &= N^2 l_i(\eta_i\epsilon - 1)\,, \\ \mathrm{Tr}R_\epsilon(H)^3 &= \frac{N^2 l_i}{4}\left(\eta_i\epsilon\left(3+\epsilon^2\right)-\left(1+3\epsilon^2\right)\right), \end{aligned} \tag{15.15}$$

[5]Here we sum over a linear quiver with l_i+1 vectors, l_i hypers and 2 $\mathcal{T}_N$ blocks at the endpoints. The $\mathcal{T}_N$ contribution is considered separately, only the sign coming from the $\mathcal{T}_N$ is used.

for the hypermultiplets and

$$\begin{aligned} \mathrm{Tr}R(\epsilon)(V) &= \left(N^2-1\right)\left[l_i+1-\epsilon(l_i\eta_i+\kappa_i)\right], \\ \mathrm{Tr}R(\epsilon)(V)^3 &= \left(N^2-1\right)\left[l_i+1-\epsilon^3(l_i\eta_i+\kappa_i)\right], \end{aligned} \tag{15.16}$$

for the vector multiplets. Here the trace Tr denotes the sum over all chiral fermions of the ith linear strand of the quiver together with the trace over gauge indices. In the expressions above we have introduced the new parameters

$$\eta_i = \frac{1}{l_i}\sum_{i=1}^{l_i}\sigma_i = \frac{x_i-y_i}{l_i}, \qquad \text{and} \qquad \kappa_i = \frac{\sigma_a+\sigma_b}{2}, \tag{15.17}$$

where σ_a and σ_b are the signs of the $\mathcal{T}_N$ block adjacent to the linear strand. Now summing over all linear strands and $\mathcal{T}_N$ blocks and inserting the result in (13.9) results in the trial central charge

$$\begin{aligned} a(\epsilon) =& \frac{3}{32}\left(pA(\epsilon,N)+qA(-\epsilon,N)\right)+\frac{3}{32}\frac{nN^2}{4}\left[\eta\epsilon(3\epsilon^2+5)-(9\epsilon^2-1)\right] \\ &+\frac{3}{32}(N^2-1)\left[2n+3(p+q)+n\eta(\epsilon-3\epsilon^3)+\frac{3}{2}(p-q)(\epsilon-3\epsilon^3)\right], \end{aligned} \tag{15.18}$$

where we have introduced the function

$$A(t,N) = \left[\frac{3}{8}(1-t)^3-\frac{1}{2}(1-t)\right]\mathrm{Tr}R^3_{\mathcal{N}=2}+\frac{9}{2}(1-t)(1+t)^2\mathrm{Tr}R_{\mathcal{N}=2}I_3^2, \tag{15.19}$$

and the new parameters

$$n = \sum_{i=1}^{3g-3} l_i \qquad \text{and} \qquad \eta = \frac{\sum_{i=1}^{3\mathbf{g}-3} l_i\eta_i}{\sum_{i=1}^{3\mathbf{g}-3} l_i} = \frac{x-y}{n}. \tag{15.20}$$

We also used the fact that $p+q=2\mathbf{g}-2$ and $\sum_i\kappa_i=\frac{3}{2}(p-q)$. Rewriting this using the parameter $\mathfrak{z}$ defined in (13.19) we exactly recover the result obtained from the anomaly polynomial calculation presented in Chapter 13. Note that this agreement holds even before a-maximization providing strong evidence that we have identified the correct M5-brane constructions corresponding to these four-dimensional quiver theories. When all $l_i=0$ our results reduce to the ones found in [19] for quivers corresponding to smooth Riemann surfaces. We can now maximize the function $a(\epsilon)$ to find the IR R-symmetry and the actual central charges. In general this is a rather complicated and non-illuminating function of z, η, n and N which we refrain from presenting here. However, in the large N limit, $\epsilon_{\max}$ reduces to

$$\epsilon_{\max} = \frac{1-\sqrt{1+3\mathfrak{z}^2}}{3\mathfrak{z}}\,, \tag{15.21}$$

and we find the large N central charges

$$a = c = (1-\mathbf{g})N^3\,\frac{1-9\mathfrak{z}^2-(1+3\mathfrak{z}^3)^{3/2}}{48\mathfrak{z}^2}\,. \tag{15.22}$$

This agrees with the result from the dual gravity solution presented in (14.20). To show this note that in the case of minimal punctures we have $\xi = \frac{N-1}{N}$ and at large N the supergravity parameter z in (14.20) reduces to $\mathfrak{z}$,

$$z = \frac{p+x(1-\xi)-q-y(1-\xi)}{2\mathbf{g}-2+n(1-\xi)} \quad\longrightarrow\quad \frac{p-q}{2\mathbf{g}-2} = \mathfrak{z}\,. \tag{15.23}$$

The leading order large N result remains unchanged by adding minimal punctures. From the gauge theory point of view it is easy to see that this should indeed be the case since hypermultiplets only contribute at order N^2.

15.3 $\mathbf{g}=1$ and L^{aba}

The quiver gauge theory obtained by putting the $\mathcal{N}=(2,0)$ theory on a torus with n minimal punctures, without extra flux is given by a necklace quiver composed of n vector and hypermultiplets and no $\mathcal{T}_N$ blocks, see Figure 15.2.

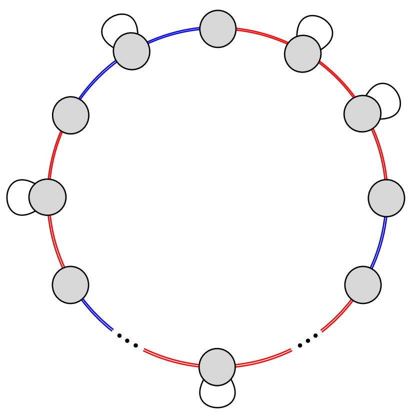


Figure 15.2: A Necklace quiver composed of $\mathcal{N}=1$ and $\mathcal{N}=2$ vector multiplets connected by hypermultiplets.

We can repeat the analysis for $\mathbf{g} > 1$ and obtain the 't Hooft anomalies by combining the R-charges of all chiral fermions in the theory. Summing over all hyper and vector multiplets in the quiver we find

$$\begin{aligned} \mathrm{Tr} R_\epsilon(H) &= N^2 n(\eta\epsilon - 1)\,, \\ \mathrm{Tr} R_\epsilon(H)^3 &= \frac{N^2 n}{4}\left(\eta\epsilon\left(3+\epsilon^2\right) - \left(1+3\epsilon^2\right)\right)\,, \end{aligned} \tag{15.24}$$

for the hypermultiplets and

$$\begin{aligned} \mathrm{Tr} R(\epsilon)(V) &= \left(N^2-1\right) n(1-\epsilon\eta)\,, \\ \mathrm{Tr} R(\epsilon)(V)^3 &= \left(N^2-1\right) n(1-\epsilon^3\eta)\,, \end{aligned} \tag{15.25}$$

for the vector multiplets. Here the trace denotes the sum over all chiral fermions in all the hypers or vectors, of the full quiver together with the trace over gauge indices. With (15.24) and (15.25) at hand we can compute the trial central charge

$$a(\epsilon) = \frac{3n}{32}\left(\frac{9}{4}N^2(1-\eta\epsilon)(1-\epsilon^2) - 2 + \eta\epsilon(1-3\epsilon^2)\right)\,. \tag{15.26}$$

This function is maximized at $\epsilon = \epsilon_{\text{max}}$ where

$$\epsilon_{\text{max}} = \frac{3N^2 - \sqrt{9N^4 + (16 - 48N^2 + 27N^4)\,\eta^2}}{3\,(4-3N^2)\,\eta}\,. \tag{15.27}$$

Inserting ϵ_{max} in (15.26) we exactly reproduce the a anomaly obtained in Chapter 13 from the anomaly polynomial. Similarly we can exactly reproduce the c anomaly by inserting ϵ_{max} in the corresponding formula for c

$$c = \frac{n}{32}\left(\frac{27}{4}N^2(1-\eta\epsilon_{\text{max}})(1-\epsilon_{\text{max}}^2) - 4 + \eta\epsilon_{\text{max}}(5-9\epsilon_{\text{max}}^2)\right)\,. \tag{15.28}$$

In [42, 66, 96] a large class of four-dimensional quiver gauge theories, known as L^{abc}, arising from D3-branes probing toric Calabi-Yau singularities were obtained. The quiver gauge theory in Figure 15.2 corresponds to the theories L^{aba} studied in Section 3.1 and 6.2.3 of [96]. The map between the parameters used in our work and the ones in [96] is

$$n = a + b\,, \qquad \xi = \frac{b-a}{a+b}\,. \tag{15.29}$$

A consistency check of this identification is provided by the agreement between the large N limit of the a-anomaly in (15.26) and the expression in Equation (6.11) of [96]. This analysis shows that the class of L^{aba} quiver gauge theories can be obtained by wrapping M5-branes on a punctured torus and therefore these theories belong to the landscape of $\mathcal{N}=1$ theories of class $\mathcal{S}$.

15.4 Consistency, duality, and the conformal manifold

Given the IR superconformal R-symmetry we can compute the scaling dimension of the various chiral primaries in our theory. These are given by

$$\begin{aligned} R_{\mathcal{N}=1}(\mu_i) &= 1+\sigma_i\epsilon_{\max}\,, \quad R_{\mathcal{N}=1}(\phi_i^2) = 2-(\sigma_{i-1}+\sigma_i)\epsilon_{\max}\,, \\ R_{\mathcal{N}=1}(u_k) &= 1-\sigma_i\epsilon_{\max}\,, \quad R_{\mathcal{N}=1}(Q_i) = \tfrac{1}{2}(N-1)(1+\sigma_i\epsilon_{\max})\,. \end{aligned} \tag{15.30}$$

The maximizing value $\epsilon_{\max}$ takes values between $-\frac{1}{3}\leq\epsilon_{\max}\leq\frac{1}{3}$ where the lower and upper bound are attained for $\eta=\mathfrak{z}=1$ and $\eta=\mathfrak{z}=-1$. Using this we find that the unitarity bound

$$\Delta=\frac{3}{2}R_{\mathcal{N}=1}\geq 1\,, \tag{15.31}$$

is always satisfied. Moreover, the Hofman-Maldacena bound [131] for $\mathcal{N}=1$ SCFTs

$$\frac{1}{2}\leq\frac{a}{c}\leq\frac{3}{2} \tag{15.32}$$

is obeyed in the case of n minimal punctures for all values of the parameters.

From now on without loss of generality we take $\epsilon_{\max}>0$. We can construct all relevant and marginal operators solely from μ and ϕ. Every $\mathcal{N}=2$ node is associated to two marginal operators, $\operatorname{Tr}\mu_i\phi_i$ and $\operatorname{Tr}\mu_{i+1}\phi_i$. If the gauge node connects two blue matter fields (i.e. $\sigma_i=-1$) it has two relevant operators, $\operatorname{Tr}\mu_i^2$ and $\operatorname{Tr}\mu_{i+1}^2$, associated to it. If it connects two red matter fields (i.e. $\sigma_i=1$) there is only one relevant operator $\operatorname{Tr}\phi_i^2$. At an $\mathcal{N}=1$ node we have a single marginal operator, $\operatorname{Tr}\mu_i\mu_{i+1}$, and a single relevant one, $\operatorname{Tr}\mu_i^2$, where the adjacent blue matter field is labeled with i. Furthermore, we can construct gauge invariant operators out of the trifundamental fields Q and $\widetilde{Q}$. These operators correspond to the wrapped M2-brane operator considered in Chapter 14 and are given by

$$\mathcal{O}_{M2}=\prod_{i=1}^{2g-2}\prod_{j=1}^{n}Q_i\mu_j^+\,,\qquad \widetilde{\mathcal{O}}_{M2}=\prod_{i=1}^{2g-2}\prod_{j=1}^{n}\widetilde{Q}_i\mu_j^-\,. \tag{15.33}$$

These operators have dimensions

$$\Delta(\mathcal{O}_{M2})=\Delta(\widetilde{\mathcal{O}}_{M2})=\frac{3}{4}(N-1)(p+q+\epsilon(p-q))+\frac{3}{2}(x+y+\epsilon(x-y))\,, \tag{15.34}$$

For large N this dimension exactly matches the energy of the wrapped brane computed in (14.21). This provides a further consistency check of our construction.

Using the results above we can also compute the dimension of the $\mathcal{N}=1$ conformal manifold using the strategy of [114, 166]. Our theory contains $3\mathbf{g}-3+n$ gauge couplings and v_1+2v_2 additional marginal operators from the vector multiplets, giving a total of $3\mathbf{g}-3+n+v_1+2v_2$ marginal deformations. However, there are constraints on the anomalous dimensions coming from each of the $2\mathbf{g}-2$ $\mathcal{T}_N$ blocks and from all $\mathcal{N}=2$ vectors with the exception of one overall combination. Thus we find the dimension of the conformal manifold,

$$\dim_{\mathbf{C}} \mathcal{M}_C = \mathbf{g}+n+v_1+v_2 = 4\mathbf{g}-3+2n\,, \tag{15.35}$$

which matches the counting in the dual gravity solutions (14.23).

Additionally we can deform our theory with relevant operators corresponding to mass terms for the adjoint chiral. In [233] it was shown that when an $\mathcal{N}=2$ SCFT with a marginal coupling is deformed by adding mass terms for all the adjoint chirals in the vector multiplet it flows to a $\mathcal{N}=1$ SCFT and the central charges of the IR theory are related to those of the UV theory by a universal linear transformation

$$a_{\mathrm{IR}} = \frac{9}{32}(4a_{\mathrm{UV}}-c_{\mathrm{UV}})\,, \qquad c_{\mathrm{IR}} = \frac{1}{32}(-12a_{\mathrm{UV}}+39c_{\mathrm{UV}})\,. \tag{15.36}$$

In the large N limit this implies that the ratio of the central charges is given by

$$\frac{a_{\mathrm{IR}}}{a_{\mathrm{UV}}} = \frac{c_{\mathrm{IR}}}{c_{\mathrm{UV}}} = \frac{27}{32}\,. \tag{15.37}$$

We observe that indeed the relations (15.36) are satisfied by our quiver gauge theories provided that the UV theory is the one with $\mathfrak{z}=1$ and the IR theory is the one with $\mathfrak{z}=0$ for fixed N and n. These are exactly the two cases where a-maximization was not needed as a result of which the central charges are rational. At the universal IR point we have $\epsilon=0$ and the superconformal R symmetry is simply R_0. At this point there are no relevant operators left solidifying its status as the inevitable universal IR fixed point. At this point the flavor symmetry is enhanced to $\mathrm{SU}(2)_{\mathcal{F}}$. Using the same arguments as in [41] we can show that the dimension of the conformal manifold in this case becomes

$$\dim_{\mathbf{C}} \mathcal{M}_C = 6g-6+3n\,. \tag{15.38}$$

To finish this chapter, we briefly consider the various Seiberg duality transformations [43, 224] of our quiver gauge theories. Our SCFTs are labeled by four parameters, $\{N, n, \mathfrak{z}, \eta\}$. For one possible set of parameters there may be multiple UV quivers realizing that particular set and we conjecture that they are all Seiberg dual. As a first example we consider the torus with $\mathfrak{z} = 0$. For $n = 6$ there are three quivers with $\eta = 0$, three quivers with $\eta = \frac{1}{3}$, one quiver with $\eta = \frac{2}{3}$ and one quiver with $\eta = 1$. All quivers with $n = 6$ and $\eta = \frac{1}{3}$ are depicted in Figure 15.3.

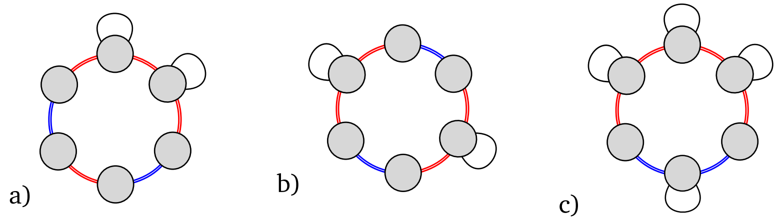


Figure 15.3: All quivers with $\mathbf{g} = 1$, $n = 6$, $\mathfrak{z} = 0$ and $\eta = \frac{1}{3}$. The three quivers are related by Seiberg dualities.

Performing a Seiberg duality on the rightmost $\mathcal{N} = 1$ node of Figure 15.3a results in Figure 15.3b. On the other hand, performing a Seiberg duality on the lower $\mathcal{N} = 1$ node of Figure 15.3a results in Figure 15.3c corroborating the conjecture that all quivers with the same parameters are Seiberg dual to each other. For higher genus Riemann surfaces there are even more intriguing Seiberg-like dualities where one can move vector multiplets across $\mathcal{T}_N$ blocks. An example is of this is presented in Figure 15.4

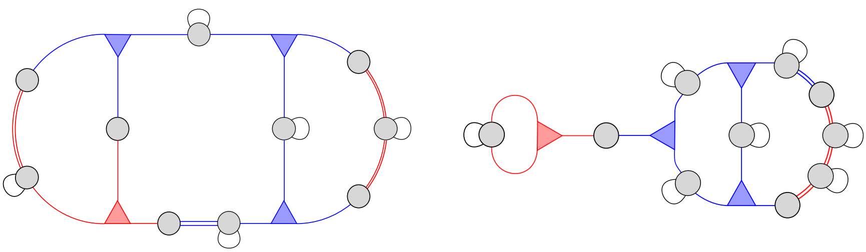


Figure 15.4: An example of two different quivers which describe the same IR SCFT. The parameters specifying the quiver are $\mathbf{g} = 3$, $z = \frac{1}{2}$, $\eta = -\frac{1}{2}$. The rank of the theory, N, is arbitrary.

As noted in [19] for $\mathbf{g} > 1$ quivers with no punctures, a naive counting of the relevant operators suggests a difference in the spectrum of chiral operators. However, when considering the $\mathcal{N} = 1$ superconformal index [32, 100, 213] of the two theories it has been shown that indeed the number of relevant operators is equal in both cases. The apparent difference in the spectrum of the chiral ring is due to non-trivial chiral ring relation that have to be taken into account properly. We thus conjecture that, in a similar vein, all quiver theories with the same discrete parameters we discussed here are dual to each other. In this way we uncover a multitude of interesting new Seiberg-like dualities similar to the ones studied in [99].

Chapter 16

Discussion and future directions

The goal of this third part of the thesis was to show that one can use gauged supergravity in 4, 5, 6, and 7 dimensions to study the IR dynamics of M2-, D3-, D4-D8- and M5-branes wrapped on a singular complex curve Σ. This goal is achieved by reducing the BPS equations of the supergravity theory to the Liouville equation on Σ. We show that singular solutions to this equation can be interpreted in supergravity as holographically dual to the topologically twisted SCFTs living on the worldvolume of the wrapped branes. To provide evidence for our construction we described the details of this picture for the case of four-dimensional $\mathcal{N} = 1$ SCFTs of class $\mathcal{S}$ arising from M5-branes wrapped on Σ. There are several natural and important directions to pursue in order to elucidate exploring these ideas and we discuss a few of them below.

Given the successful implementation of our approach to the physics of M5-branes wrapped on Σ it is natural to study in detail the physics of the other wrapped branes. The SCFTs dual to the known supergravity solutions corresponding to a smooth Σ are under much less control when compared to the class $\mathcal{S}$ theories we studied here. Indeed, this provides an instance where the supergravity analysis may inform the construction of the dual field theories. A particularly rich class of examples is offered by D3-branes wrapped on a complex curve Σ. When the Riemann surface is smooth the corresponding two-dimensional $(0,2)$ SCFTs were studied in [36, 37] and it will be most interesting to extend this to supergravity solutions with punctures on Σ. Work along these lines is currently in progress [56]. Another important question is to study higher-curvature corrections to our supergravity solutions which should capture $1/N$ effects in the dual SCFTs. A possible starting point to attack this is provided by the construction in [15]. Finally, it will be very interesting to generalize our construction to twisted compactifications of brane setups with smaller amount of supersymmetry, like the ones studied in [14, 38, 53], or to other gauged supergravity theories arising as consistent truncations of string or M-theory, see [26, 117].

Our construction provides a roundabout way to construct 1/4-BPS AdS_5 solutions of eleven-dimensional supergravity with explicit brane sources. These solutions should fall within the general classification of [106] of supersymmetric AdS_5 solutions in eleven dimensions. In fact, such backgrounds with explicit brane sources corresponding to punctured Riemann surface were studied in [16, 17] and it is desirable to make the connection between that approach and our construction more explicit. A common feature between our setup and the one in [16, 17, 102] is that the punctures/singularities on the Riemann surface are introduced "by hand" as explicit singular sources in the BPS equations of the supergravity theory. It is desirable to make this more rigorous by adding explicit sources to the supergravity Lagrangian due to the presence of branes in gauged supergravity. These arise from a dimensional reduction of the D- or M-branes in ten and eleven dimensions. The brane sources would allow us to derive more directly the BPS equations we used. A potentially useful alternative approach is to explore the connection between these constructions and SU(2) Yang-Mills theory on the Riemann surfaces [32, 94].

The gravity solutions discussed in Chapter 13 and Chapter 14 allow for the line bundles over the Riemann surface to have negative degrees. In the field theory discussion of Chapter 15, however, we restricted to positive degrees only. The quiver gauge theories dual to these negative degree line bundle setups can be constructed by introducing additional field theory building blocks $\mathcal{T}_N^{(m)}$ discussed in [3]. These building blocks can be obtained by adding adjoint chiral multiplets charged under the global symmetry of $\mathcal{T}_N$ and giving them a nilpotent vacuum expectation value, which in turn spontaneously breaks the global symmetry. In this way one can obtain more general quivers [92, 178, 192] allowing for negative degrees of the line bundles. Furthermore, using these building blocks we can construct gauge theories on the torus which include background flux. It will be interesting to study these more general setups in detail. It should also be stressed that in this work we focused on quivers built out of $SU(N)$ vector multiplets and $SU(N)$ $\mathcal{T}_N$ theories only, i.e., theories describing the infrared dynamics of the A_{N-1} type $\mathcal{N}=(2,0)$ theory on Σ. It is clear that these constructions could be generalized to quivers built out of $SO(2N)$ and $Sp(2N-2)$ vector multiplets and the $SO(2N)$ $\mathcal{T}_N$ theories to describe the infrared limit of the D_N theory compactified on Σ, see for instance [231].

In Section 15.3 we noted that the L^{aba} quiver gauge theories arising on the worldvolume of D3-branes probing a conical CY singularity can be constructed also from M5-branes wrapped on a torus with punctures. It is important to understand whether other quiver gauge theories in the L^{abc} or Y^{pq} can also be realized in terms of M5-branes in the spirit of class $\mathcal{S}$. The fact that the AdS_5 solutions associated to the Y^{pq} metrics were first constructed in [106] as (singular) solutions of eleven-dimensional supergravity suggests that such a connection may be possible.

In this work we focused on studying wrapped branes on a Riemann surface with punctures. A natural generalization is to study similar supergravity setups for branes wrapped on higher-dimensional calibrated cycles. When the cycles are smooth and compact it is well-known how to construct the corresponding supergravity solutions, see [53, 104] for a review. These constructions again typically involve studying the BPS equations of a lower-dimensional gauged supergravity. It will be interesting to apply our approach and introduce explicit singular sources to these BPS equations to study the supergravity solutions describing branes wrapped on manifolds with various defects. A particularly accessible example is offered by the case of M5-branes wrapping a four-cycle given by a product of two Riemann surfaces. The supergravity solutions corresponding to smooth surfaces were analyzed in detail in [37], see also [28], and it should be straightforward to apply our construction to this setup to construct solutions with punctures.

Given the appearance of the Liouville equation in our construction and the $SU(\infty)$ equation in the supergravity analysis of [102, 167] it is natural to wonder whether these equations are some remnant of the underlying AGT correspondence for M5-branes wrapped on Σ.

Chapter 17

Parting thoughts

That's that, I hope you enjoyed the process of reaching this page. If not, you can rejoice at the thought that the end is near.

The main goal of this thesis consisted of trying to understand better how to describe branes on curved worldvolumes from the point of view of string theory and supergravity. Through the gauge/gravity correspondence this understanding provides us with a new window into obtaining more information on the strongly coupled dual quantum field theories.

In part II, of this thesis we showed how one can construct supergravity solutions holographically corresponding to maximally supersymmetric Yang-Mills theory on a sphere. We constructed the supergravity solutions, computed various observables in the SYM theory using localization and formulated a method to match observables across the gauge/gravity duality. This provides a non-trivial precision test of holography beyond conformality.

In part III, we holographically studied IR SCFTs by wrapping various types of branes on a punctured Riemann surface. We described how one can efficiently use lower dimensional gauged supergravity solutions to compute various observables, such as anomalies or conformal dimensions of selected operators, to leading order in N. These solutions describe an RG flow across dimensions from a SCFT in d dimensions to a $(d-2)$-dimensional one. All the solutions we constructed describe flows that end in an AdS_{d-1} IR fixed point and thus describe a dual IR SCFT. Our solutions thus provide non-trivial evidence for a myriad of new IR SCFTs.

However, not all is done. Our work elucidates a small part of the road towards understanding curved branes and strongly coupled field theories but a much larger part is left unexplored. As discussed in Chapters 10 and 16, our work opens up several new interesting directions and there are many open ends left to understand. In the following years I hope to be able to continue this path and gradually clarify and understand more and more of these questions.

To be continued.

Part IV

Appendices

Appendix A

Conventions for type II supergravity

The spherical Dp-brane backgrounds constructed in this thesis solve the equations of motion of ten-dimensional type II supergravity. This theory comes in two flavors, type IIA and type IIB, depending on whether the chirality of the supersymmetry generators $\epsilon_{1,2}$ is opposite or the same. The bosonic field content consists of the NS-NS sector, common to both type IIA and type IIB, and the R-R sector. The metric G_{MN}, the dilaton Φ, and the three-form H_3 build up the NS-NS sector, while the R-R sector contains the n-form field strengths F_n, with $n = 0, 2, 4$ for type IIA and $n = 1, 3, 5$ for type IIB. In type IIA F_0, i.e. the Romans mass, does not have any propagating degrees of freedom and is set to zero throughout this paper. In type IIB F_5 has to obey a self-duality condition. The fermionic field content consists of a doublet of gravitinos, ψ_M, and a doublet of dilatinos, λ. The components of these doublets are again of opposite chirality in type IIA and of the same chirality in type IIB.

We use the democratic formalism in which the number of R-R fields is doubled such that n runs over $0, 2, 4, 6, 8, 10$ for type IIA and $1, 3, 5, 7, 9$ for type IIB [44]. This redundancy is removed by introducing duality conditions for all R-R fields

$$F_n = (-1)^{\frac{(n-1)(n-2)}{2}} \star_{10} F_{10-n} \,. \tag{A.1}$$

These duality conditions should be imposed by hand after deriving the equations of motion from the action. The bosonic part of the action written in string frame is given by[1]

$$S_{\text{bos}} = \frac{1}{2\kappa_{10}^2} \int \star_{10} \Big[e^{-2\Phi} \Big(R + 4|\mathrm{d}\Phi|^2 - \frac{1}{2}|H_3|^2 \Big) - \frac{1}{4} \sum_n |F_n|^2 \Big] , \tag{A.2}$$

[1] We mostly follow the conventions of [49].

where the ten-dimensional Newton constant κ_{10} is related to the string length through $4\pi\kappa_{10} = (2\pi l_s)^8$ and we have defined

$$\star_{10}\,|A|^2 \equiv \star_{10}\frac{1}{n!}A_{\mu_1\ldots\mu_n}A^{\mu_1\ldots\mu_n} = \star_{10}A\wedge A\,. \tag{A.3}$$

This action should be completed by its fermionic counterpart, which we do not write explicitly, and is invariant under the following supersymmetry variations[2]

$$\begin{aligned}
\delta\psi^1_M &= \Big(\nabla_M - \frac{1}{4}\not{H}_{3M}\Big)\epsilon^1 + \frac{1}{16}e^{\Phi}\sum_n \not{F}_n\Gamma_M\Gamma_{10}\epsilon^2\,,\\
\delta\psi^2_M &= \Big(\nabla_M + \frac{1}{4}\not{H}_{3M}\Big)\epsilon^2 - \frac{1}{16}e^{\Phi}\sum_n(-1)^{\frac{(n-1)(n-2)}{2}} \not{F}_n\Gamma_M\Gamma_{10}\epsilon^1\,,\\
\delta\lambda^1 &= \Big(\not{\partial}\Phi - \frac{1}{2}\not{H}_3\Big)\epsilon^1 + \frac{1}{16}e^{\Phi}\Gamma^M\sum_n \not{F}_n\Gamma_M\Gamma_{10}\epsilon^2\,,\\
\delta\lambda^2 &= \Big(\not{\partial}\Phi + \frac{1}{2}\not{H}_3\Big)\epsilon^2 - \frac{1}{16}e^{\Phi}\Gamma^M\sum_n(-1)^{\frac{(n-1)(n-2)}{2}} \not{F}_n\Gamma_M\Gamma_{10}\epsilon^1\,,
\end{aligned} \tag{A.4}$$

where Γ_M are the ten-dimensional gamma matrices and Γ_{10} is the chirality operator. The Feynman slash notation for an n-form field is defined as follows

$$(\not{A}_n)_{M_{k+1}\cdots M_n} \equiv \Gamma^{M_1\cdots M_k}(A_n)_{M_1\cdots M_k M_{k+1}\cdots M_n}\,, \tag{A.5}$$

for $k \le n$ and $\Gamma^{M_1\cdots M_k} \equiv \frac{1}{k!}\Gamma^{[M_1}\cdots\Gamma^{M_k]}$ is the totally antisymmetric product of k gamma matrices.

The Bianchi identities and equations of motion derived from the action (A.2) are

$$\mathrm{d}H_3 = 0\,, \qquad \text{and} \qquad \mathrm{d}(e^{-2\Phi}\star_{10}H_3) + \frac{1}{2}\sum_n \star_{10}F_n\wedge F_{n-2} = 0\,, \tag{A.6}$$

for the NS-NS field H_3 and

$$\mathrm{d}F_n - H_3\wedge F_{n-2} = 0\,, \tag{A.7}$$

for the R-R form fields. The NS-NS and R-R fluxes can be written in terms of potentials as

$$F_n = \mathrm{d}C_n - H_3\wedge C_{n-2}\,, \qquad H_3 = \mathrm{d}B_2\,. \tag{A.8}$$

[2]In these formulae we have implicitly chosen positive chirality spinors in type IIB supergravity.

The dilaton and the Einstein equations of motion can be written as

$$\begin{aligned} 0 &= \nabla^2\Phi - |\mathrm{d}\Phi|^2 + \frac{1}{4}R - \frac{1}{8}|H_3|^2 , \\ 0 &= R_{MN} + 2\nabla_M\nabla_N\Phi - \frac{1}{2}|H_3|^2_{MN} - \frac{1}{4}e^{2\Phi}\sum_n |F_n|^2_{MN} , \end{aligned} \tag{A.9}$$

where we have defined

$$|A_n|^2_{MN} \equiv \frac{1}{(n-1)!}(A_n)_M{}^{M_2\cdots M_n}(A_n)_{NM_2\cdots M_n} . \tag{A.10}$$

In the strong coupling limit, $g_s \gg 1$, type IIA string theory is expected to be described by M-theory. Therefore it will sometimes be useful to uplift our ten-dimensional type IIA supergravity solutions to eleven-dimensional supergravity. When compactified on a circle the eleven-dimensional theory has two parameters, the eleven-dimensional Newton constant κ_{11} and the radius of the circle R_{11}. These are related to the ten-dimensional parameters as follows

$$R_{11} = \ell_s \qquad \text{and} \qquad \kappa_{11}^2 = 2\pi R_{11}\kappa_{10}^2 . \tag{A.11}$$

The bosonic fields of eleven-dimensional supergravity are the metric and a three-form potential A_3. Their dynamics is governed by the following action

$$S = \frac{1}{2\kappa_{11}^2}\int \star_{11}\Big[R - \frac{1}{2}|\mathrm{d}A_3|^2\Big] - \frac{1}{12\kappa_{11}^2}\int A_3\wedge \mathrm{d}A_3 \wedge \mathrm{d}A_3 . \tag{A.12}$$

To reduce to ten dimensions we make the following Kaluza-Klein ansatz

$$\begin{aligned} \mathrm{d}s_{11}^2 &= \mathrm{e}^{-\frac{2}{3}\Phi}G_{MN}\mathrm{d}x^M dx^N + \mathrm{e}^{\frac{4}{3}\Phi}\left(\mathrm{d}x^{11} + C_1\right)^2 , \\ A_3 &= C_3 + B_2\wedge \mathrm{d}x^{11} . \end{aligned} \tag{A.13}$$

All fields appearing on the right hand side of (A.13) are the ten-dimensional type IIA fields in string frame.

Appendix B

Useful integrals

The following integrals are useful for the calculations in Chapter 7

$$\fint_{-b}^{b} \frac{d\sigma' |\sigma - \sigma'|^{\alpha} \mathrm{sign}(\sigma - \sigma')}{(b^2 - \sigma'^2)^{\alpha/2}} = \pi \alpha \sigma \csc\left(\tfrac{\pi\alpha}{2}\right), \tag{B.1}$$

and

$$\int_{-b}^{b} \frac{d\sigma'}{(b^2 - \sigma'^2)^{\alpha/2}} = \frac{b^{1-\alpha} \pi^{1/2} \Gamma(\frac{2-\alpha}{2})}{\Gamma(\frac{3-\alpha}{2})}, \tag{B.2}$$

where we have defined $\alpha \equiv d - 5$. Note that the result in (B.1) is independent of b. The result in (B.1) can be understood by splitting the integral into two parts,

$$\int_{-b}^{\sigma} \frac{d\sigma' (\sigma - \sigma')^{\alpha}}{(b^2 - \sigma'^2)^{\alpha/2}} - \int_{\sigma}^{b} \frac{d\sigma' (\sigma' - \sigma)^{\alpha}}{(b^2 - \sigma'^2)^{\alpha/2}}. \tag{B.3}$$

Both integrals in (B.3) are discontinuous as σ crosses the branch cuts between $-\infty < \sigma < -b$ or $b < \sigma < \infty$. However, it is straightforward to show that the discontinuities cancel between the two integrals and so the sum must be a holomorphic function of σ in the complex plane. By taking σ to a large imaginary value in (B.3) one can see that the combined integrals have a leading linear behavior in σ with the coefficient in (B.1), while the constant piece is zero because the integral in (B.1) is clearly an odd function of σ.

It proves useful to define the following function

$$f(\sigma) \equiv \frac{\Gamma(\frac{3-\alpha}{2})}{\pi^{1/2}\Gamma(\frac{2-\alpha}{2})} \frac{\sigma}{b} \, {}_2F_1\left(\frac{1}{2}, \frac{\alpha}{2}; \frac{3}{2}; \frac{\sigma^2}{b^2}\right). \tag{B.4}$$

One can show that $f'(\sigma) = \rho(\sigma)$ where $\rho(\sigma)$ is defined in (7.41), and that $f(b) = 1/2$. Note that since $\rho(\sigma)$ is an even function of σ, $f(\sigma)$ is an odd function.

Finally, we present two integrals which are useful for the calculation of the free energy

$$I_1 \equiv \int_{-b}^{b} d\sigma \rho(\sigma)\sigma^2 = \frac{b^2}{8-d}\,, \tag{B.5}$$

and

$$I_2 \equiv \int_{-b}^{b} d\sigma \rho(\sigma)(b-\sigma)^{d-4} = 2b^{d-4}\pi^{-1/2}\Gamma(\tfrac{d-1}{2})\Gamma(\tfrac{8-d}{2})\,. \tag{B.6}$$

Appendix C

Flat Euclidean branes

In this appendix we explicitly show that flat Euclidean Dp-branes are indeed supersymmetric solutions of type II* supergravity. For more details on these theories, including the type II* actions, see for example [46].

The supersymmetry variations are exactly the same as those of regular type II supergravity, see (A.4), with the only difference that the R-R fields now have to be treated as purely imaginary and the spinor obeys an unusual reality condition. For type IIA* the spinors satisfy a MW* condition

$$\epsilon^* = -\mathcal{C}A\Gamma_{(10)}\epsilon\,, \tag{C.1}$$

and similarly for type IIB*

$$\epsilon^* = \mathcal{C}A\sigma_3\epsilon\,, \tag{C.2}$$

where $\epsilon = (\epsilon_1, \epsilon_2)^T$ and A and $\mathcal{C}$ define respectively Dirac conjugation, $\bar{\chi}^D \equiv \chi^\dagger A$, and Majorana conjugation, $\bar{\chi} \equiv \chi^T \mathcal{C}$. The reality conditions for the spinors are thus equivalent to

$$\bar{\epsilon} = -\bar{\epsilon}^D \Gamma_{(10)}\,, \tag{C.3}$$

for type IIA* while for type IIB* we find

$$\bar{\epsilon} = \bar{\epsilon}^D \sigma_3\,. \tag{C.4}$$

We can now check explicitly that the flat Euclidean branes of Hull are indeed $\frac{1}{2}$-BPS solutions of type II* supergravity, i.e. they preserve 16 real supercharges. The solutions are given by

$$\mathrm{d}s_{10}^2 = H^{-1/2}\mathrm{d}s_{p+1}^2 + H^{1/2}\mathrm{d}s_{1,8-p}^2\,, \tag{C.5}$$

$$\mathrm{e}^{\phi} = g_s H^{(3-p)/4}\,, \tag{C.6}$$

$$C_{p+1} = i(g_s H)^{-1}\mathrm{vol}_{p+1} \tag{C.7}$$

In these solutions $\mathrm{d}s_{p+1}^2$ is the metric of flat $(p+1)$-dimensional Euclidean space, $\mathrm{d}s_{1,8-p}^2$ is the Minkowski metric on $\mathbf{R}^{1,8-p}$, and H is a harmonic function on this Minkowski space.

Inserting these solutions into the supersymmetry variations we see that they can indeed be solved by imposing the usual Dp-brane projector with an extra i inserted

$$\left(1 + i\Gamma^{0\ldots p}\Gamma_{(10)}\mathcal{P}\right)\epsilon = 0\,. \tag{C.8}$$

Here $\mathcal{P} = \sigma_1$ when $\frac{p(p+1)}{2}$ is even and $\mathcal{P} = i\sigma_2$ when $\frac{p(p+1)}{2}$ is odd. It is important to note that the projector above is consistent with the reality condition obeyed by the spinors in the type II* supergravity theory. We would like to stress that this subtle interplay of imaginary R-R fluxes and unusual reality conditions on the spinors is the reason why Euclidean branes preserve supersymmetry in type II* string theory and supergravity.

Appendix D

Gauged supergravity for spherical branes

In this appendix we introduce, case by case, the $(p+2)$-dimensional gauged supergravity theories used to construct the spherical Dp-brane solutions discussed in the main text. The supergravity theories available in the literature are Lorentzian and we need to analytically continue them to Euclidean signature. After presenting in some detail the construction of the gauged supergravity solutions we perform their uplifts to ten-dimensional type II and/or eleven-dimensional supergravity.

As emphasized in the main text, we start with a maximally supersymmetric gauged supergravity theory in $p+2$ dimensions and perform a consistent truncation, following the method of [238], to preserve an $SO(3) \times SO(6-p)$ subgroup of the $SO(9-p)$ gauge group, corresponding to the R-symmetry in the dual SYM theory. By analytically continuing the supergravity theory to Euclidean signature we end up with a non-compact $SO(1,2) \times SO(6-p) \simeq SU(1,1) \times SO(6-p)$ gauge group, in harmony with (7.6). We start with the case $p=6$ and work our way down to $p=2$.

D.1 Spherical D6-branes

The supergravity theory appropriate for our construction is the maximal SO(3) gauged supergravity in eight dimensions, originally constructed in [221], analytically continued to Euclidean signature and non-compact gauge group. The uplift of this theory to eleven-dimensional supergravity as well as more general gaugings are discussed in [10].

D.1.1 Maximal eight-dimensional SO(3) gauged supergravity

The maximal $\mathcal{N} = 2$ ungauged supergravity theory in eight dimensions has $E_{3(3)} \simeq SL(3,\mathbf{R}) \times SL(2,\mathbf{R})$ global symmetry under which the bosonic fields of the theory transform. In particular, the 7 scalars parameterize the five-dimensional and two-dimensional coset spaces $SL(3,\mathbf{R})/SO(3)$ and $SL(2,\mathbf{R})/SO(2)$ and are most conveniently expressed in terms of two matrices $\mathfrak{Z}$ and $\mathfrak{A}$ transforming according to

$$\begin{aligned} \mathfrak{Z} &\to G\mathfrak{Z}H\,, \qquad \text{where} \quad G \in SL(3,\mathbf{R}) \quad \text{and} \quad H \in SO(3)\,, \\ \mathfrak{A} &\to K\mathfrak{A}L\,, \qquad \text{where} \quad K \in SL(2,\mathbf{R}) \quad \text{and} \quad L \in SO(2)\,. \end{aligned} \tag{D.1}$$

The fermionic fields transform under $SO(3)\times SO(2) \simeq SU(2)\times U(1)$ which acts as the R-symmetry of the supergravity theory.[1] In total, the field content of the ungauged theory consists of the metric $g_{\mu\nu}$, two sixteen-component gravitini ψ^a_μ, six gaugini η^a_i, seven scalars $\mathfrak{Z}_M{}^i$ and $\mathfrak{A}_{IJ}$, one three-form tensor field $A_{\mu\nu\rho}$, three two-form tensor fields $A^M_{\mu\nu}$, and six one-form vector fields A^{MN}_μ. We use the following index conventions: $\mu,\nu,\rho = 0,\dots 7$ are eight-dimensional spacetime indices; $M,N = 1,2,3$ are $SL(3,\mathbf{R})$ indices in the fundamental; $I,J = 1,2$ are $SL(2,\mathbf{R})$ indices in the fundamental; $i,j = 1,2,3$ are in the $\mathbf{3}$ and $a,b = 1,2$ are in the $\mathbf{2}$ of $SU(2) \simeq SO(3)$, respectively.

To obtain a gauged supergravity theory with a non-trivial potential for the scalars a subgroup of the global symmetry group should be promoted to a local symmetry. This can be done in several inequivalent ways by gauging a subgroup of the global symmetry group. By gauging the maximal compact subgroup SO(3) in $SL(3,\mathbf{R})$ one obtains the theory studied by Salam and Sezgin in [221]. This theory can be obtained by reducing the eleven-dimensional supergravity to eight dimensions on an SU(2) group manifold. As described in [10] one can also obtain more general gaugings by reducing the eleven-dimensional supergravity on different group manifolds. One example is a reduction on an SU(1,1) group manifold resulting in the Lorentzian eight-dimensional $SO(1,2) \simeq SU(1,1)$ gauged supergravity. This case can be understood as an analytic continuation of the Salam-Sezgin theory or as a "non-compactification" of eleven-dimensional supergravity. However, this SU(1,1) gauged supergravity theory is still Lorentzian. To obtain the Euclidean action appropriate for constructing the spherical brane solutions of interest we need to combine this analytic continuation of the gauge group with an analytic continuation of the time direction in space-time.

[1] We are cavalier about the global difference between SO(3) and SU(2).

D.1.2 SO(3) invariant truncation

We begin with the SO(3) gauged supergravity of [221] and are interested in constructing solutions which preserve the SO(3) gauge symmetry and have a maximally symmetric seven-dimensional factor in the metric. These requirements eliminate all tensor fields in the supergravity theory except the metric itself. There are two scalars, a "dilaton" β and an "axion" χ, parameterizing an $SL(2,\mathbf{R})/SO(2)$ coset, which are not charged under the SO(3) gauge symmetry.

The Lagrangian for the bosonic fields in this SO(3) invariant truncation reads[2]

$$S = \frac{1}{2\kappa_8^2}\int \star_8 \left\{R - \frac{1}{2}\left(|\mathrm{d}\beta|^2 + \mathrm{e}^{2\beta}|\mathrm{d}\chi|^2\right) - V\right\}. \tag{D.2}$$

The potential is proportional to the gauge coupling constant, g, of the supergravity theory and is given by

$$V = -\frac{3}{2}g^2\mathrm{e}^{\beta}. \tag{D.3}$$

It proves convenient to introduce the complex scalar $\tau = \chi + i\mathrm{e}^{-\beta}$ as in (8.17), the Kähler potential as in (8.18), and superpotential as in (8.24) The potential in (D.3) can then be written in terms of the superpotential (8.24) using the expression (8.23).

The supersymmetry variations of this truncation of the supergravity theory can be read off from [10, 221]. They can be explicitly written as

$$\delta\psi_\mu = \nabla_\mu\epsilon + \frac{1}{8}\mathrm{e}^{\mathcal{K}}\partial_\mu(\tau+\bar{\tau})\epsilon + \frac{1}{24}\mathrm{e}^{\mathcal{K}}\mathcal{W}\gamma_9\gamma_\mu\epsilon\,, \tag{D.4}$$

$$\delta\eta_i = \left(\frac{\tau-\bar{\tau}}{2}\partial_\mu\bar{\tau}\gamma_9\gamma^\mu + \frac{1}{6}\mathrm{e}^{\mathcal{K}}D_\tau\mathcal{W}\right)\sigma_i\epsilon\,, \tag{D.5}$$

where $\gamma_9 = i\gamma^{01\ldots7}$, the spinor ϵ^a is in the $\mathbf{2}$ of SO(3), $(\sigma_i)^a_b$ are SO(3) Pauli matrices, and D_τ is the Kähler covariant derivative defined below (8.21).

As described in the main text, we are interested in an analytic continuation of this gravitational theory and its supersymmetry variations into Euclidean signature. This is achieved by changing the signature of the metric as well as replacing the pseudoscalar as follows, $\chi \to i\chi$. In addition we should treat the complex conjugate of the scalar τ as an independent scalar. We emphasize this by using the notation $\overline{\mathcal{W}} \to \widetilde{\mathcal{W}}$ and $\bar{\tau} \to \tilde{\tau}$.

[2] Our notation is different from the one in [221]. We have defined $\beta \equiv -2\phi_{\mathrm{SS}}$, $\chi \equiv -2B_{\mathrm{SS}}$, $g \equiv \frac{g_{\mathrm{SS}}}{2}$, where quantities with an SS subscript are the ones used in [221].

To find the solution of interest we impose the usual spherically symmetry domain wall ansatz for the metric as in Equation (8.31)

$$ds_8^2 = dr^2 + \mathcal{R}^2 e^{2A} d\Omega_7^2 \,, \tag{D.6}$$

and assume that the the scalar fields only depend on the radial coordinate r.

To solve the supersymmetry variations in (D.4) we use a conformal killing spinor on S^7 obeying

$$\nabla_\mu^{S^7} \epsilon = \frac{i}{2} \gamma_\mu \epsilon \,. \tag{D.7}$$

Here ∇^{S^7} is the covariant derivative on the unit radius S^7. Note that this is in harmony with the expected supersymmetry generator for the seven-dimensional SYM theory on S^7, see (7.4).

The vanishing of the gaugino and gravitino variations then leads to the following differential equations

$$\mathcal{K}_{\tilde{\tau}\tau}(\tilde{\tau}')(\tau') = \frac{1}{16} e^{\mathcal{K}} \mathcal{W}\widetilde{\mathcal{W}} \,, \tag{D.8}$$

$$(A')^2 - \mathcal{R}^{-2} e^{2A} = \frac{1}{144} e^{2\mathcal{K}} \mathcal{W}\widetilde{\mathcal{W}} \,, \tag{D.9}$$

where a prime denotes differentiation with respect to r. Notice that the equations in (D.8) correspond to a degenerate limit of (8.32)-(8.36) in which we remove the scalar η and set $p = 6$.

There is a subtlety when analyzing the BPS equations in this truncation of the eight-dimensional supergravity. There are only two independent equations in (D.8) and thus one of the two scalars in the model appears to not be constrained by a differential equation. This problem is fixed by the equations of motion which lead to the following first order differential equation for the scalar χ

$$\chi' = \frac{6}{\mathcal{R}} e^{-2\mathcal{K}} e^{-7A} \,. \tag{D.10}$$

We have a first order equation in (D.10) because the scalar χ does not appear in the potential V in (D.3) and the usual second order differential equation has an integral of motion which reduces the order of the equation. The constant coefficient on the right hand side of (D.10) is the unique value of this integral of motion which makes the BPS equations in (D.8) together with (D.10) compatible with all other equations of motion and with the integrability of the supersymmetry variations in (D.4).

The gauged supergravity solution discussed above can be uplifted to type IIA and eleven-dimensional supergravity and the explicit result is presented in Section 8.3.5.

D.2 Spherical D5/NS5-branes

To construct a supergravity solution describing spherical D5- or NS5-branes we consider the maximal seven-dimensional SO(4) gauged supergravity constructed in [222]. We then use the results of [175] to uplift this seven-dimensional solution to ten-dimensional type IIB supergravity.

D.2.1 Maximal seven-dimensional SO(4) gauged supergravity

The maximal $\mathcal{N}=4$ ungauged supergravity theory in seven dimensions has $E_{4(4)} = SL(5)$ global symmetry under which the bosonic fields transform. In particular the fourteen scalars span the coset space SL(5)/SO(5) and can be parameterized by a matrix $\mathfrak{Z}$ that transforms according to

$$\mathfrak{Z} \to G\mathfrak{Z}H\,, \quad \text{where} \quad G \in \mathrm{SL}(5) \quad \text{and} \quad H \in \mathrm{SO}(5)\,. \tag{D.11}$$

In addition to the bosonic fields, the fermions transform under SO(5) $\simeq$ USp(4) which acts as the R-symmetry group of the supergravity theory. In total the field content of the ungauged theory consists of the metric $g_{\mu\nu}$, four gravitini ψ^a_μ, five two-forms $B^M_{\mu\nu}$, ten vector fields A^{MN}_μ, sixteen gaugini χ^{abc}, and fourteen scalar fields ${\mathfrak{Z}_M}^{ab}$. We use the following index conventions: $a, b = 1, \ldots, 4$ denote USp(4) indices; $M, N = 1, \ldots, 5$ are SL(5) indices, and μ, $\nu = 0, \ldots, 6$ are seven-dimensional spacetime indices.

The global symmetries can be promoted to a local symmetry in several inequivalent ways. Gauging the maximal compact subgroup SO(5) $\subset$ SL(5) one obtains the well known gauged supergravity theory [203]. This theory has a maximally supersymmetric AdS_7 vacuum and can also be obtained by performing a consistent truncation of eleven-dimensional supergravity on S^4. Further gaugings were discovered in [205] and a complete classification was obtained in [222] using the embedding tensor formalism. We are interested in a maximal supergravity with an SO(4) gauge group which should capture domain wall solutions describing the back-reaction of NS5/D5-branes. It was anticipated in [60] that such a supergravity theory should exist and indeed it was explicitly constructed in [222].[3]

[3] A half-maximal version of the supergravity theory which can be viewed as a consistent truncation of the maximal theory was studied in [220].

In the maximal SO(4) gauged theory the SL(5) representations of the bosonic fields are decomposed into representations of the gauge group. The ten vector fields that transform in the $\overline{\mathbf{10}}$ of SL(5) transform in $\overline{\mathbf{6}}+\overline{\mathbf{4}}$ of SO(4) where the $\overline{\mathbf{6}}$ plays the role of the SO(4) gauge field. Four of the two-forms $B^M_{\mu\nu}$ become massive by combining with the $\overline{\mathbf{4}}$ other vector fields. The fifth two-form is uncharged and is present also in the $\mathcal{N}=2$ theory [220]. Finally, the scalars transform in the symmetric traceless of SL(5), i.e. the **14**, which decomposes into the $\mathbf{9}+\mathbf{4}+\mathbf{1}$ representation of SO(4).

D.2.2 SO(3) **invariant truncation**

The R-symmetry of the six-dimensional SYM theory on S^6 is SO(1,2) and thus we should find an SO(3) invariant truncation of the SO(4) gauged supergravity which we can then analytically continue. Imposing this symmetry on the theory and keeping only fields compatible with a solution preserving the isometries of S^6 leads to a consistent truncation of the SO(4) gauged supergravity consisting of the metric and three real scalar fields. This is in agreement with the field theory discussion in Chapter 7. More precisely, the three supergravity scalars should be dual to the Yang-Mills coupling and the two independent operators in the deformation Lagrangian (7.5).

The three scalars invariant under the SO(3) symmetry of interest here are the singlets in the branching rules of the breaking of SO(4) to SO(3)

$$\mathbf{9}\to\mathbf{5}\oplus\mathbf{3}\oplus\mathbf{1}\,,\qquad \mathbf{4}\to\mathbf{3}\oplus\mathbf{1}\,,\qquad \mathbf{1}\to\mathbf{1}\,. \tag{D.12}$$

In the notation of [222] the parameterization of the scalar coset element for these three scalars is

$$\mathfrak{z}=\begin{pmatrix} e^{-\phi-x} & 0 & 0 & 0 & 0\\ 0 & e^{-\phi-x} & 0 & 0 & 0\\ 0 & 0 & e^{-\phi-x} & 0 & 0\\ 0 & 0 & 0 & e^{-\phi+3x} & e^{4\phi}\chi\\ 0 & 0 & 0 & 0 & e^{4\phi}\end{pmatrix}. \tag{D.13}$$

Notice that χ is a pseudoscalar. These three scalars parameterize the following submanifold of the SL(5)/SO(5) scalar coset

$$\mathbf{R}_+\times\frac{\mathrm{SL}(2,\mathbf{R})}{\mathrm{SO}(2)}\,, \tag{D.14}$$

where $\mathbf{R}_+$ is parameterized by the combination $\eta\equiv-\phi-x$ and $\mathrm{SL}(2,\mathbf{R})/\mathrm{SO}(2)$ is parameterized by $\beta\equiv 5\phi-3x$ and χ. The bosonic action for this consistent truncation takes the familiar form (8.16) with $p = 5$

$$S = \frac{1}{2\kappa_7^2} \int \star_7 \left\{ R - \frac{15}{2} |\mathrm{d}\eta|^2 - \frac{1}{2} \left(|\mathrm{d}\beta|^2 + \mathrm{e}^{2\beta} |\mathrm{d}\chi|^2 \right) - V \right\}. \tag{D.15}$$

The potential is proportional to the gauge coupling constant g and takes the form

$$V = \frac{g^2}{2} \mathrm{e}^{\beta} (-3\mathrm{e}^{\eta} - 6\mathrm{e}^{-4\eta-\beta} + \mathrm{e}^{-9\eta-2\beta} + \mathrm{e}^{-9\eta}\chi^2). \tag{D.16}$$

After introducing the scalar $\tau = \chi + i\mathrm{e}^{-\beta}$ one can use the superpotential in (8.20) and the Kähler potential in (8.18) to write the potential in terms of the superpotential as in (8.21).

The supersymmetry variations for this consistent truncation can be obtained from the results in [222]. The vanishing of the spin-$\frac{1}{2}$ variations leads to the equations

$$\partial_\mu(\eta)\gamma^\mu\epsilon^1 = -\frac{1}{15}\mathrm{e}^{\mathcal{K}/2}\partial_\eta\mathcal{W}\epsilon^4\,, \qquad \partial_\mu(\eta)\gamma^\mu\epsilon^4 = -\frac{1}{15}\mathrm{e}^{\mathcal{K}/2}\partial_\eta\overline{\mathcal{W}}\epsilon^1\,, \tag{D.17}$$

$$\partial_\mu\bar{\tau}\gamma^\mu\epsilon^1 = \left(\mathrm{e}^{-\mathcal{K}}\mathcal{K}^{\bar{\tau}\tau}\right)^{1/2} D_\tau\mathcal{W}\epsilon^4\,, \qquad \partial_\mu\tau\gamma^\mu\epsilon^4 = \left(\mathrm{e}^{-\mathcal{K}}\mathcal{K}^{\tau\bar{\tau}}\right)^{1/2} D_{\bar{\tau}}\overline{\mathcal{W}}\epsilon^1\,. \tag{D.18}$$

From the spin-$\frac{3}{2}$ variations we find

$$\nabla_\mu\epsilon^1 + \frac{i}{8}\mathrm{e}^{\mathcal{K}}\partial_\mu(\tau+\bar{\tau})\,\epsilon^1 = -\frac{1}{20}\mathrm{e}^{\mathcal{K}/2}\mathcal{W}\gamma_\mu\epsilon^4\,, \tag{D.19}$$

$$\nabla_\mu\epsilon^4 - \frac{i}{8}\mathrm{e}^{\mathcal{K}}\partial_\mu(\tau+\bar{\tau})\,\epsilon^4 = -\frac{1}{20}\mathrm{e}^{\mathcal{K}/2}\overline{\mathcal{W}}\gamma_\mu\epsilon^1\,. \tag{D.20}$$

There are four supersymmetry generators, ϵ^a, in the maximal supergravity theory. However the equations for the pair (ϵ^2, ϵ^3) are identical to the ones presented above for (ϵ^1, ϵ^4).

As described in the main text, the analytic continuation to Euclidean signature corresponds, at this level, to the replacement $\chi \to i\chi$ accompanied by the substitutions $\overline{\mathcal{W}} \to \widetilde{\mathcal{W}}$ and $\bar{\tau} \to \tilde{\tau}$. After this analytic continuation we can look for the spherical brane solution by imposing the domain wall metric ansatz as in (8.31)

$$\mathrm{d}s_7^2 = \mathrm{d}r^2 + \mathcal{R}^2\mathrm{e}^{2A(r)}\mathrm{d}\Omega_6^2\,, \tag{D.21}$$

and assume that all scalars depend only on the radial coordinate r. Furthermore we can assume that $\epsilon_{1,4}$ are conformal Killing spinors on S^6

$$\nabla_\alpha^{S^6}\begin{pmatrix}\epsilon_1\\ \epsilon_4\end{pmatrix}=\frac{1}{2}\gamma_*\gamma_\alpha\begin{pmatrix}\epsilon_1\\ -\epsilon_4\end{pmatrix}, \tag{D.22}$$

where $\gamma_*\equiv i\gamma_{123456}$. With this at hand we can derive the system of BPS equations in (8.32)-(8.36) with $p=5$. Furthermore, combining the spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ equations we can find an algebraic equation for A as in (8.37) with $p=5$.

D.2.3 Uplift to type IIB supergravity

Any solution of the seven-dimensional SO(4) gauged supergravity can be uplifted to the ten-dimensional type IIB supergravity using the uplift formulae of [175]. When we apply these uplift formulae to the solutions of the BPS equations in (8.32)-(8.36) with $p=5$ we obtain the spherical NS5-brane solution with the following string frame metric

$$\mathrm{d}s^2=\frac{1}{\sqrt{X}}\Big(\mathrm{e}^{-4\eta}\mathrm{d}r^2+\frac{X((1-3X)^2-9Y^2)}{g^2Y^2}\mathrm{d}\Omega_6^2+\frac{1}{g^2}\mathrm{d}\theta^2+\frac{\sin^2\theta X}{g^2(\sin^2\theta+X\cos^2\theta)}\mathrm{d}\widetilde{\Omega}_2^2\Big). \tag{D.23}$$

The remaining ten-dimensional fields are given by

$$\begin{aligned}\mathrm{e}^{2\Phi}&=\frac{\mathrm{e}^{-10\eta}}{X(\sin^2\theta+X\cos^2\theta)},\\ C_0&=iY\mathrm{e}^{5\eta}\cos\theta,\\ B_2&=-\frac{1}{g^2}\Big(\theta-\frac{\sin 2\theta X}{2(\sin^2\theta+X\cos^2\theta)}\Big)\mathrm{vol}_2,\\ C_2&=-i\frac{Y\mathrm{e}^{5\eta}\sin^3\theta}{g^2(\sin^2\theta+X\cos^2\theta)}\mathrm{vol}_2,\end{aligned} \tag{D.24}$$

where vol_2 is the volume element of the $\mathrm{d}\widetilde{\Omega}_2^2$ metric in (8.15). Integrating the H and F_3 flux derived from (D.24) over the three-dimensional space spanned by θ and $\mathrm{d}\widetilde{\Omega}_2^2$ we find that the D5-brane charge is vanishing while the NS5-brane charge is not. This fits nicely with the interpretation of this background as corresponding to spherical NS5-branes.

The spherical D5-brane solution can be obtained from the spherical NS5-brane solution above by acting with the SL$(2,\mathbf{R})$ global symmetry of the type IIB supergravity. This transformation acts on the supergravity background fields as follows

$$\tau_{\rm IIB} \mapsto \frac{a\tau_{\rm IIB}+b}{c\tau_{\rm IIB}+d}\,, \qquad \begin{bmatrix} C_2 \\ B_2 \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} C_2 \\ B_2 \end{bmatrix}\,, \qquad \text{where} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbf{R})\,. \tag{D.25}$$

Here $\tau_{\rm IIB} \equiv C_0 + i e^{-\Phi}$ is the axion-dilaton field and the Einstein frame metric remains unchanged. Applying this transformation to the background in (D.24) with $a = d = 0$ and $b = -c = 1$, yields the spherical D5-brane solution in type IIB supergravity. The fluxes of this D5-brane solution are the same as the ones in (8.46) with $p = 5$. In particular the H flux integral over $\mathrm{d}\theta$ and $\mathrm{d}\widetilde{\Omega}_2^2$ vanishes while the R-R flux integral over this space does not. For other values of the $\mathrm{SL}(2,\mathbf{R})$ parameters in (D.25) we obtain more general solutions which should describe (p,q)-fivebranes wrapped on S^6.

D.3 Spherical D4-branes

If we are to follow the pattern of gauged supergravity theories used to construct spherical Dp-brane solutions we should use a maximal six-dimensional SO(5) gauged supergravity. This theory is not so well studied in the literature and the only analysis we are aware of is the one in [78] where the author constructed the six-dimensional theory through a dimensional reduction of the maximal seven-dimensional SO(5) gauged supergravity on a circle. We will thus describe the spherical D4-brane background as a solution of this maximal seven-dimensional supergravity theory. The SO(5) gauged supergravity has a maximally supersymmetric AdS_7 solution dual to the conformal vacuum of the $(2,0)$ six-dimensional SCFT which fits well with the field theory expectation, discussed in Chapter 7, that the five-dimensional maximal SYM theory on S^5 flows in the UV to the six-dimensional $(2,0)$ theory on $S^5 \times S^1$.

The maximal SO(5) gauged supergravity in seven dimensions was constructed in [203] and can be obtained as a consistent truncation of eleven-dimensional supergravity on S^4. Any solution of the seven-dimensional theory can be uplifted to eleven dimensions using the uplift formulae of [194]. The field content of the theory is the same as for the SO(4) gauged supergravity discussed in Appendix D.2.1. The difference comes from the gauging which in this case is SO(5). This gauging of course fits in the general classification of [222], whose conventions we use, and affects the details of the Lagrangian of the theory and thus the space of solutions.

D.3.1 SO(3) **invariant truncation**

The R-symmetry breaking pattern discussed around (7.6) dictates that we should look for the spherical D4-brane solutions in an SO(3) × SO(2) invariant truncation of the SO(5) gauged supergravity. This, combined with the requirement that the solution should have the isometries of $S^5 \times S^1$, leads to a consistent truncation which consists of the metric, a single real scalar field, x, and a single SO(2) gauged field, A.[4] The scalar coset matrix (D.11) for this truncation is diagonal and reads

$$\mathfrak{Z} = \mathrm{diag}(\mathrm{e}^{-x}, \mathrm{e}^{-x}, \mathrm{e}^{-x}, \mathrm{e}^{3x/2}, \mathrm{e}^{3x/2})\,. \tag{D.26}$$

The bosonic action can be obtained from [222] and reads

$$S = \frac{1}{2\kappa_7^2} \int \star_7 \left\{ R_7 - \frac{15}{2}|\mathrm{d}x|^2 - \frac{1}{2}\mathrm{e}^{6x} F_{\mu\nu}F^{\mu\nu} - V_7 \right\}, \tag{D.27}$$

where $F = \mathrm{d}\mathcal{A}$ is the gauge field strength of $\mathcal{A}$ and the potential is proportional to the gauge coupling constant g,

$$V_7 = -\frac{3}{8} g^2 \mathrm{e}^{-x}(4 + \mathrm{e}^{5x})\,. \tag{D.28}$$

We can now dimensionally reduce this theory on S^1 to a six-dimensional gravitational theory.[5] To this end we use the following metric and gauge field ansatz

$$\mathrm{d}s_7^2 = \mathrm{e}^{-\phi}\mathrm{d}s_6^2 + \mathrm{e}^{4\phi}\mathrm{d}\omega^2\,, \qquad \mathcal{A} = \chi\,\mathrm{d}\omega\,. \tag{D.29}$$

The scalar fields ϕ and χ depend only on coordinates of the six-dimensional space with metric $\mathrm{d}s_6^2$. To conform with the notation used throughout this work it is convenient to define the following combination of these two scalars

$$\beta \equiv 3x - 2\phi\,, \qquad \text{and} \qquad \eta \equiv x + \phi\,. \tag{D.30}$$

The six-dimensional Lagrangian of the dimensionally reduced theory then reads

$$S = \frac{1}{2\kappa_6^2} \int \star_6 \left\{ R - 3|\mathrm{d}\eta|^2 - \frac{1}{2}\Big(|\mathrm{d}\beta|^2 + e^{2\beta}|\mathrm{d}\chi|^2 \Big) - V \right\}, \tag{D.31}$$

[4] The SO(2) gauge field generator can be thought of as the 45 component of the 5 × 5 matrix generator of the SO(5) gauge field.

[5] It should also be possible to construct this six-dimensional theory as a consistent truncation of the six-dimensional maximal gauged supergravity studied in [78].

where R is the Ricci scalar for the metric $\mathrm{d}s_6^2$ and the six-dimensional potential is

$$V = -\frac{3}{8} g^2 \mathrm{e}^{-\eta}(4 + \mathrm{e}^{\beta+2\eta}) \,. \tag{D.32}$$

The derivation of the BPS equations now follows a familiar pattern. We work with the supersymmetry variations of the seven-dimensional maximal supergravity theory as given in [222]. To present them succinctly we define the scalar $\tau = \chi + i\mathrm{e}^{-\beta}$ and the superpotential as in (8.20) with $p = 4$. Using the Kähler potential in (8.18) one can then show that the six-dimensional potential in (D.32) can be written in the general form (8.21) with $p = 4$. Combining the gaugino and gravitino variations of [222] we find

$$\partial_\mu \eta \gamma^\mu \epsilon = \frac{1}{6} e^{\mathcal{K}/2} \partial_\eta \mathcal{W} \epsilon \,, \tag{D.33}$$

$$\partial_\mu \tau \gamma^\mu = (e^{-\mathcal{K}} \mathcal{K}^{\bar{\tau}\tau})^{1/2} D_\tau \mathcal{W} \epsilon \,, \tag{D.34}$$

$$\nabla_\mu \epsilon + \frac{i}{8} \mathrm{e}^{\mathcal{K}} \partial_\mu (\tau + \bar{\tau}) \epsilon = -\frac{1}{16} e^{\mathcal{K}/2} \mathcal{W} \gamma_\mu \epsilon \,. \tag{D.35}$$

Now we can perform the familiar analytic continuation to Euclidean signature by treating τ and $\bar{\tau} \to \tilde{\tau}$ as independent scalars and performing the substitution $\chi \to i\chi$. For the metric we use the usual spherical domain wall ansatz

$$\mathrm{d}s_6^2 = \mathrm{d}r^2 + \mathcal{R}^2 \mathrm{e}^{2A} \mathrm{d}\Omega_5^2 \,, \tag{D.36}$$

and assume that all scalar fields depend only on r. The supersymmetry parameter ϵ is a conformal Killing spinor on S^5 obeying

$$\nabla_\mu^{S^5} \epsilon = \frac{i}{2} \gamma_\mu \epsilon \,. \tag{D.37}$$

We can plug this in the supersymmetry variations (D.33) and derive the system of BPS equations in (8.32)-(8.36) and the algebraic equation in (8.37) with $p = 4$.

D.3.2 Uplift to eleven-dimensional and type IIA supergravity

The solution of the maximal seven-dimensional SO(5) gauged supergravity described above can be uplifted to eleven dimensional M* theory using the uplift formulae presented in [195, 196]. Using the functions P and Q defined in (8.43) we can write the eleven-dimensional metric as

$$\begin{aligned} ds_{11}^2 =& e^{-\eta} P^{-1/3} ds_6^2 - e^{4\eta} Q P^{-2/3} \left(d\omega + \frac{\cos^2\theta Y}{g e^{2\eta} X Q} d\xi \right)^2 \\ &+ \frac{P^{1/3}}{g^2} \left(d\theta^2 + P \sin^2\theta \, d\tilde{\Omega}_2^2 + Q \cos^2\theta \, d\xi^2 \right), \end{aligned} \tag{D.38}$$

where ω spans a timelike U(1) and ξ spans a spacelike U(1). The three-form gauge field is

$$A_{(3)} = \frac{i \sin^3\theta}{g^3(\sin^2\theta + X\cos^2\theta)} \text{vol}_2 \wedge d\xi + \frac{i e^{2\eta} Y \sin^2\theta}{g^2(\sin^2\theta + X\cos^2\theta)} \text{vol}_2 \wedge d\omega \tag{D.39}$$

where vol_2 is the volume form of the two dimensional de Sitter space in (8.15).

This eleven-dimensional solution can be dimensionally reduced to ten-dimensional type IIA* supergravity along the timelike U(1) spanned by ω using the formulae in (A.13). The result is a type IIA background of the form described in Section 8.2.3 with $p = 4$.

D.4 Spherical D2-branes

To construct spherical D2-branes we employ the four-dimensional maximal ISO(7) gauged supergravity as presented in [118]. We construct spherical brane solutions to this theory which can then be uplifted to both type IIA and eleven-dimensional supergravity.

D.4.1 ISO(7) gauged supergravity in four dimensions

The maximal ungauged supergravity in four dimensions has $E_{7(7)}$ global symmetry under which the bosonic fields transform. In particular, the scalars parameterize the 70-dimensional coset space $E_{7(7)}/SU(8)$ with coset element $\mathfrak{z}$ that transforms according to

$$\mathfrak{z} \to G\mathfrak{z}H\,, \qquad \text{where} \quad G \in E_{7(7)} \quad \text{and} \quad H \in SU(8)\,. \tag{D.40}$$

In addition to the bosonic fields, the fermions transform under SU(8) which acts as the R-symmetry of the supergravity theory. In total, the field content of the ungauged theory consists of the metric $g_{\mu\nu}$, eight gravitini ψ_μ^i, 56 gaugini χ^{ijk}, 28 gauge fields A_μ^M and 70 scalars $\mathfrak{z}_M^{ij}$. Here we use the following index conventions:

$\mu, \nu = 0, \ldots, 3$ are four-dimensional spacetime indices, $M = 1, \ldots, 56$ are $E_{7(7)}$ indices and $i, j = 1, \ldots, 8$ are SU(8) indices.

A subgroup of the global symmetry group of the supergravity theory can be promoted to a gauge group in several inequivalent ways. The well-known SO(8) gauged supergravity theory described in [84] is obtained in this way and is relevant for the low-energy dynamics of a system of coincident M2-branes since it arises as a consistent truncation of the eleven-dimensional supergravity on S^7. Here we are however interested in D2-branes and thus should use an ISO(7) gauged four-dimensional supergravity. There are two inequivalent ways to find a four-dimensional maximal supergravity theory with an ISO(7) gauge group. The "electrically gauged" theory was constructed in [136] (see also [141]) and is the one that admits an uplift to type IIA supergravity with vanishing Romans mass. This will be the theory we focus on for our analysis. The other inequivalent gauging is described in detail in [118] and is relevant for compactifications of the massive type IIA supergravity on S^6 [116].

D.4.2 SO(3) **invariant truncation**

We use the results of [118] and focus on the electric gauging with $m = 0$ relevant for type IIA supergravity with vanishing Romans mass. We want to study a solution that preserves SO(4) × SO(3) gauge symmetry and has maximally symmetric three-dimensional factor in the metric. This truncation eliminates most scalars and all tensor fields except the metric. A larger truncation of this four-dimensional supergravity which imposes only SO(4) symmetry was studied in Section 5 of [118]. The SO(4) × SO(3) truncation of interest here can be obtained from that larger truncation by setting one of the pseudoscalars in [118] to zero. To comply with the notation used in the main text we make the following change of notation with respect to [118]

$$\eta \equiv 2\varphi_{\mathrm{GV}}\,, \qquad \beta \equiv \phi_{\mathrm{GV}}\,, \qquad \chi \equiv \rho_{\mathrm{GV}}\,, \qquad \chi_{\mathrm{GV}} = 0\,, \tag{D.41}$$

where the subscript GV refers to the quantities used in Section 5 of [118].

The bosonic Lagrangian of this supergravity truncation can be read off from [118]

$$\mathcal{L} = \star_4 \left\{ R - \frac{3}{4}|\mathrm{d}\eta|^2 - \frac{1}{2}\left(|\mathrm{d}\beta|^2 + \mathrm{e}^{2\beta}|\mathrm{d}\chi|^2\right) - V \right\}\,, \tag{D.42}$$

where the potential V is given by

$$V = -\frac{1}{2}g^2 \mathrm{e}^{-\beta}\left[24\mathrm{e}^{\eta/2+\beta} + 8\mathrm{e}^{2\beta} + 3\mathrm{e}^{\eta}(1 + \chi^2\mathrm{e}^{2\beta})\right]. \tag{D.43}$$

and as usual g is the gauge coupling. Notice that this Lagrangian is of the general form in (8.16) with $p = 2$.

We are not aware of a reference in the literature where the explicit fermionic supersymmetry variations for this ISO(7) gauged supergravity were presented. However the authors of [118] write down an explicit superpotential for our truncation and, after defining $\tau = \chi + i\mathrm{e}^{-\beta}$, it can be readily checked that it coincides with the one in (8.20) with $p = 2$. Using the Kähler potential in (8.18) one can then show that the potential in (D.43) can be written in the general form (8.21) with $p = 2$. In addition one can show explicitly that the system of BPS equations in (8.32)-(8.36) and the algebraic equation (8.37) (with $p = 2$) imply the equations of motion derived from the Lagrangian in (D.42).[6] We consider these results as sufficient evidence that any solution to the system of equations in (8.32)-(8.36) describes a supersymmetric solution of the four-dimensional ISO(7) electrically gauged supergravity theory.

Finally we point out that one can use the uplift formulae provided in [116] to uplift any solution of the four-dimensional ISO(7) gauged supergravity to a solution of type IIA supergravity. For the SO(4) × SO(3) truncation described above this uplifted ten-dimensional background has the form presented in Section 8.2.3. The ten-dimensional solution has vanishing Romans mass and thus can be further uplifted to a solution of eleven-dimensional supergravity using the formulae in (A.13).

D.5 Evaluation of κ_{p+2}

Here we provide a derivation of the Newton constant in $p+2$ dimensions from the one in ten dimensions by employing dimensional reduction. Consider the metric (8.42) transformed to Einstein frame and evaluated in the UV, i.e.

$$\mathrm{d}s_{\mathrm{E}}^2 = g_s^{-1/2}\mathrm{e}^{-\frac{(8-p)(p-3)}{4(6-p)}\eta}\left(\mathrm{d}s_{p+2}^2 + \frac{\mathrm{e}^{\frac{2(p-3)}{(6-p)}\eta}}{g^2}\mathrm{d}\Omega_{8-p}^2\right). \tag{D.44}$$

The type II supergravity action is given by

$$S_{10} = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-g_{(10)}}R_{(10)} + \cdots, \tag{D.45}$$

where the dots represent other terms in the Lagrangian which are not important for the present discussion and $\kappa_{10}^2 = \frac{(2\pi\ell_s)^8}{4\pi}$. The $(p+2)$-dimensional supergravity

[6] We perform our usual analytic continuation to Euclidean signature and take a spherically symmetric ansatz for the four-dimensional metric and scalar fields.

action obtained from this action is

$$S_{p+2} = \frac{1}{2\kappa_{p+2}^2} \int \mathrm{d}^{p+2}x \sqrt{-g_{(p+2)}} R_{(p+2)} + \cdots , \tag{D.46}$$

The goal is now to obtain κ_{p+2}^2, i.e. the Newton constant on the $(p+2)$-dimensional space. To do this we insert the metric (D.44) in the ten-dimensional action and integrate over the internal $(8-p)$-dimensional space. Doing this results in a warp factor which we eliminate by performing the conformal transformation

$$\tilde{g}_{\mu\nu} = g_s^{-1/2} \mathrm{e}^{-\frac{(8-p)(p-3)}{4(6-p)}\eta} g_{\mu\nu}^{(10)} , \qquad \sqrt{-\tilde{g}}\tilde{R} = g_s^{-p/2} \mathrm{e}^{-\frac{p(8-p)(p-3)}{8(6-p)}\eta} \sqrt{-g^{(10)}} R^{(10)} . \tag{D.47}$$

This transformation exactly removes all factors of the function η from the internal space and thus we find the unambiguously defined $(p+2)$-dimensional Newton constant

$$\frac{1}{2\kappa_{p+2}^2} = \frac{1}{2\kappa_{10}^2} \frac{V_{8-p}}{g_s^2 g^{8-p}} , \tag{D.48}$$

where $V_{n-1} = 2\pi^{n/2}/\Gamma(\frac{n}{2})$ is the volume of the unit n-sphere.

Appendix E

Finite temperature

As we have emphasized numerous times in our discussion in part II, the finite size of S^{p+1} provides an IR cut-off for the low-energy dynamics of the SYM theory which is compatible with supersymmetry. A more commonly used IR cut-off is to consider SYM at finite temperature. This procedure of course breaks supersymmetry but keeps all the R-symmetry unbroken. The dual supergravity description of a $(p+1)$-dimensional maximal SYM theory at finite temperature is given by a $(p+2)$-dimensional black brane solution which we summarize below. In this appendix we consider these backgrounds in some more detail and compare with the results found for spherical branes.

The black branes of interest are most easily described as solutions to the $(p+2)$-dimensional gauged supergravity theory described in Section 8.2.1. In contrast to the spherical brane solutions, these non-supersymmetric backgrounds preserve the full gauge symmetry. This fits well with the fact that in the dual gauge theory at finite temperature the R-symmetry is preserved. The metric of the solution takes a standard black hole form

$$\mathrm{d}s^2_{p+2} = \mathrm{d}r^2 + \mathrm{e}^{2A(r)}\left(-h(r)\mathrm{d}t^2 + \mathrm{d}x_p^2\right) \,, \tag{E.1}$$

where $\mathrm{d}x_p^2$ is the usual metric on $\mathbf{R}^p$. In addition to the metric the only field with a non-trivial profile is the scalar $\eta(r)$. The equations of motion reduce to the following set of equations[1]

$$A = \frac{9-p}{(6-p)(p-3)}\eta\,, \tag{E.2}$$

$$\log(1-h) = \frac{2p(p-7)}{(6-p)(p-3)}(\eta-\eta_0)\,, \tag{E.3}$$

$$\left(\mathrm{e}^{\frac{p-3}{6-p}\eta}\right)' = -\frac{g(3-p)^2}{2p}\sqrt{h}\,, \tag{E.4}$$

where η_0 is an integration constant. When the integration constant is chosen such that $h=1$ we recover the supersymmetric flat domain wall solution in (8.25). The horizon of the black hole is located where $\eta \to \eta_0$ and the asymptotic infinity (UV) is located where $h \to 1$. Notice that for $p<3$ the UV is located at large negative η but for $p>3$ it is located at large positive η. In the near-horizon region the metric takes the form

$$\mathrm{d}s^2_{p+2} = \mathrm{d}r^2 - \frac{g^2(7-p)^2}{4}\mathrm{e}^{\frac{2(5-p)p}{(6-p)(p-3)}\eta_0}(r-r_0)^2\mathrm{d}t^2 + \mathrm{e}^{\frac{2(9-p)}{(6-p)(p-3)}\eta_0}\mathrm{d}x_p^2\,. \tag{E.5}$$

The temperature of the black hole can be determined using the standard trick of ensuring that the near-horizon metric does not have conical singularities when analytically continued to Euclidean time, $\tau = i\,t$. The result is

$$T = \frac{g(7-p)}{4\pi}\mathrm{e}^{\frac{(5-p)p}{(6-p)(p-3)}\eta_0}\,. \tag{E.6}$$

This black brane solution can be uplifted to ten dimensions using the formulae in Section 8.2.3 and the metric in string frame reads:

$$\mathrm{d}s^2_{10} = (gU)^{\frac{p-7}{2}}\left(h^{-1}\mathrm{d}U^2 + (gU)^{7-p}\left(-h\,\mathrm{d}t^2 + \mathrm{d}x_p^2\right) + U^2\mathrm{d}\Omega^2_{8-p}\right)\,, \tag{E.7}$$

where $gU = \mathrm{e}^{\frac{2p}{(6-p)(p-3)}\eta}$ such that

$$h = 1 - \frac{U_0^{7-p}}{U^{7-p}}\,, \qquad gU_0 = \mathrm{e}^{\frac{2p}{(6-p)(p-3)}\eta_0}\,. \tag{E.8}$$

The dilaton and R-R fields are the same as for the flat supersymmetric brane solution in (8.10) and (8.11).

Similar as in the supersymmetric case we should carefully analyze how to identify the holographic effective 't Hooft coupling. However, in this appendix we will only care about qualitative features so we will not compute the exact prefactors

[1] These black brane solutions are clearly well-known and studied in many references, see for example [201]. For convenience we rederive them here in our conventions and notation.

and content ourselves with the scaling behavior. Motivated by the discussion in [149, 150] and the analysis in the main text we can identify the effective 't Hooft coupling in terms of the temperature as,

$$\lambda \propto (g_s N)^{\frac{2(5-p)}{7-p}} \left(\frac{T}{g}\right)^{p-3} \propto (g_s N)(2\pi \ell_s T)^{p-3} \,. \tag{E.9}$$

An alternative way to arrive at the same scaling relation is to first identify the energy scale in the QFT, E, with the temperature of the black hole, T. The entropy of the black branes was computed in [146] and can be evaluated in terms of the Einstein frame area of the horizon which is

$$\mathcal{A}_{\text{Einst}} = \mathrm{e}^{-2\Phi(U_0)} \mathcal{A}_{\text{str}} \sim g_s^{-2} (g U_0)^{\frac{9-p}{2}} g^{p-8} V_p \,, \tag{E.10}$$

where V_p is the spatial volume of the Dp-branes. The area of the horizon determines the entropy of the black brane via $S = 2\mathcal{A}_{\text{Einst}}/\kappa_{10}^2$ which in terms of the field theory quantities takes the form

$$S \sim N^2 \lambda^{\frac{p-3}{5-p}} T^p V_p \,. \tag{E.11}$$

It is then clear that the thermal free energy $F \sim TS$ has the same scaling in terms of N and λ as the supersymmetric free energy in (7.46). This can be viewed as another consistency check of our field theory interpretation of the spherical brane backgrounds as holographically dual to the maximal SYM theory on S^{p+1}.

Appendix F

Alternative derivation for $p = 5$

In this appendix we present an alternative derivation for the free energy and Wilson loop expectation values of 6d MSYM theory that were obtained in Section 9.4.5. For this purpose we start with the saddle-point equation (9.71) in $d = 6-\epsilon$ dimensions. After renormalization (9.72) of the coupling λ_b these equations reduce to

$$\frac{16\pi^3}{\lambda_{\text{QFT}}} N\sigma_i = -3\sum_{j\neq i}(\sigma_i - \sigma_j)\log(\sigma_i - \sigma_j)^2 \,. \tag{F.1}$$

Notice that the r.h.s. of this equation describes repulsive interaction between eigenvalues at short distance of the eigenvalues and attractive at large distance. Hence in order to have stable distribution of the eigenvalues with large size of the support we should consider $\lambda_{\text{QFT}} < 0$ which is consistent with the conclusions of Section 8.3.4.

In the continuous limit we as usually introduce eigenvalue density according to (7.39) and rewrite saddle-point equation (F.1) as the following integral equation

$$-\frac{16\pi^3}{3\lambda_{\text{QFT}}}\sigma = \int_{-b}^{b} d\sigma' \rho(\sigma')(\sigma - \sigma')\log(\sigma - \sigma')^2 \,. \tag{F.2}$$

This equation has already appeared before in the context of 4d $\mathcal{N} = 2$ theories in [217, 218]. To solve it we should differentiate it twice w.r.t. σ in order to obtain

$$\int_{-b}^{b} d\sigma' \frac{\rho(\sigma')}{\sigma - \sigma'} = 0 \,, \tag{F.3}$$

which is the standard singular integral equation with Cauchy kernel. This equation has the following unbounded normalizable solution

$$\rho(\sigma)=\frac{1}{\pi\sqrt{b^2-\sigma^2}}\,, \tag{F.4}$$

In order to define position of the support endpoint b we can use the following integral:

$$\int_{-b}^{b} d\sigma'\,\frac{(\sigma-\sigma')\log(\sigma-\sigma')^2}{\pi\sqrt{b^2-\sigma'^2}}=2\sigma\log\left(\frac{be}{2}\right). \tag{F.5}$$

Comparison with (F.2) immediately gives

$$b=2\exp\left(-\frac{8\pi^3}{3\lambda_{\rm QFT}}-1\right), \tag{F.6}$$

which precisely reproduces expression (9.75) we have obtained previously considering $\epsilon\to 0$ of general expression (7.42).

It is also worth noticing that the eigenvalue density (F.4) which solves (F.2) is consistent with the $\epsilon\to 0$ limit of the general expression (7.41) provided we also use coupling renormalization (9.72). On Fig.F.1 we also compare numerical solutions of equations (F.1), (7.32) and analytical solution (F.4). As we see solutions to the equation (7.32) with full kernel agrees with the solution of (F.1) when d is close to 6. Also the solution (F.4) describes both numerical solutions very well.

Finally to find the free energy instead of substituting eigenvalue density (F.4) into free energy functional we notice the following identity

$$\frac{1}{N^2}\frac{\partial F}{\partial\lambda_r}=-\frac{8\pi^3}{\lambda_{\rm QFT}^2}\int_{-b}^{b}\rho(\sigma)\sigma^2=-\frac{16\pi^3}{\lambda_{\rm QFT}^2}{\rm e}^{-\frac{16\pi^3}{3\lambda_{\rm QFT}}-2}. \tag{F.7}$$

Integrating this identity we easily obtain

$$\frac{F}{N^2}=-3{\rm e}^{-\frac{16\pi^3}{3\lambda_{\rm QFT}}-2}, \tag{F.8}$$

which exactly reproduces expression in (9.74). We can obtain the Wilson loop from (F.4) and (F.6) which will obviously also reproduce previously obtained result (9.76).

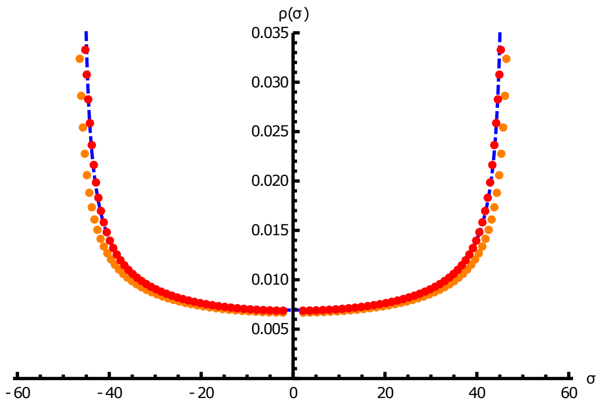


Figure F.1: The eigenvalue distribution of 6d MSYM for $N = 100$ and $\lambda_{\text{QFT}} = -20$. In particular the *orange dots* correspond to the numerical solution of (7.32) at $\lambda_b = 0.42$ and $d = 5.995$ ($\epsilon = 0.005$). The latter parameters correspond to $\lambda_{\text{QFT}} = -20$ according to (9.72). The *red dots* in turn correspond to the numerical solution of (F.1) at $N = 100$ and $\lambda_{\text{QFT}} = -20$. Finally, the dashed blue line shows the eigenvalue density (F.4) with the endpoint position b given by (F.6).

Appendix G

Alternative solution for $p = 6$

G.1 An alternative derivation for the eigenvalue density

In this part of the appendix we present an alternative way to analyze the matrix model (7.30) for the $d = 7$ case. For this purpose we have to solve the saddle point equation (7.40)

$$\frac{C_1}{\lambda_{\text{QFT}}}\,\sigma = C_2 \int_{-b}^{b} d\sigma' \rho(\sigma')|\sigma-\sigma'|^2 \text{sign}(\sigma-\sigma')\,, \tag{G.1}$$

with the regularized 't Hooft coupling λ_{QFT} which we assume is small and negative such that the eigenvalues are in general widely separated. The integral equation in (G.1) is closely related to the saddle point equation of the matrix model for five-dimensional SYM in the decompactification limit, i.e. when the radius $\mathcal{R}$ of S^5 is taken to infinity. A detailed analysis of this matrix model can be found in [197].

To solve (G.1) we differentiate both sides of the equation twice with respect to σ. This leads to the simple equation

$$\int_{-b}^{b} d\sigma' \rho(\sigma') \text{sign}(\sigma-\sigma') = \int_{-b}^{\sigma} d\sigma' \rho(\sigma') - \int_{\sigma}^{b} d\sigma' \rho(\sigma') = 0\,. \tag{G.2}$$

This equation should be satisfied for any σ on the support, but this is possible only if the eigenvalue density $\rho(\sigma)$ is zero everywhere except at the support endpoints, $\pm b$. Hence, we can assume the following form for the solution,

$$\rho(\sigma) = \frac{1}{2}\left(\delta\left(\sigma+b\right)+\delta\left(\sigma-b\right)\right), \tag{G.3}$$

where the factor of 1/2 is introduced to normalize the eigenvalue density. The endpoints of the distribution can then be found by substituting the density (G.3) back into the integral equation (G.1). This then results in the simple algebraic equation

$$C_1 \lambda_{\mathrm{QFT}}^{-1} \sigma = \frac{1}{2} C_2 \left[(\sigma + b)^2 - (\sigma - b)^2 \right], \tag{G.4}$$

which, using (7.32) and (7.37), leads to

$$b = \frac{1}{2\lambda} \frac{C_1}{C_2} = -\frac{2\pi^3}{\lambda_{\mathrm{QFT}}}. \tag{G.5}$$

The final expression agrees with (9.97) determined from the general expression. One can also easily obtain the free energy in (9.99) using the distribution in (G.3) and the value of b in (G.5).

Note that we can also derive the δ-function behavior in (G.3) directly from the expression for the density in (7.41). If we let $d = 7 - \epsilon$ then it is straightforward to see that the density approaches zero in the limit $\epsilon \to 0$, everywhere except at $\sigma = \pm b$.

G.2 Numerical solutions at weak negative coupling

In this part of the appendix we analyze numerically the solution to (7.32) for $d = 7$ and a weak negative renormalized 't Hooft constant. Here we solve a "heat equation" numerically[1], which at large "times" approaches asymptotically the solution to the saddle point equation below in (G.17). However, we also assume that the solution is symmetric around the origin, i.e. for each eigenvalue σ_i there is another eigenvalue $-\sigma_i$. As can be seen from the left panel of Figure G.1, which compares the numerical and analytical results, the solution in (G.3) indeed reproduces the behavior of the eigenvalue distribution at weak negative coupling. Notice that the graph is for $\lambda_{\mathrm{QFT}} = -1$ which is not very small. Our solution works whenever $\frac{|\lambda_{\mathrm{QFT}}|}{4\pi^3} \ll 1$, which obviously holds for $\lambda_{\mathrm{QFT}} = -1$.

For $-\lambda_{\mathrm{QFT}}^{-1} \gg \frac{1}{4\pi^3}$ the eigenvalue distribution separates into two widely separated peaks according to (G.3) with distance $\frac{4\pi^3}{|\lambda_{\mathrm{QFT}}|}$ between them. However, if we include subleading terms in the kernel we can argue that the peaks are actually humps with a width of order 1. Including the next term, $G_{16}^{(7)}(\sigma)$ in (9.98) has the expansion

$$G_{16}^{(7)}(\sigma) = 2\pi(1 - \sigma^2)\mathrm{sign}(\sigma) + \mathrm{O}(\mathrm{e}^{-\pi|\sigma|}). \tag{G.6}$$

[1] See [197] for a more detailed explanation of the numerical techniques.

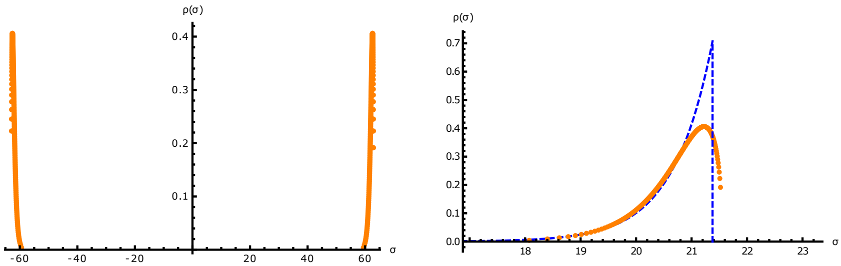


Figure G.1: Eigenvalue distribution of 7*d* MSYM for $N = 400$ and $\lambda_{\text{QFT}} = -1$ and $\lambda_{\text{QFT}} = -3$. In the *left panel* we compare numerical results for $\lambda_{\text{QFT}} = -1$ against the eigenvalue density (G.3), while on the *right panel* we compare numerics for $\lambda_{\text{QFT}} = -3$ against the analytical solution (G.13). The latter one takes into account the finite width of the eigenvalue support.

At this order in the approximation there is a repulsive force at short distance which smears the peaks into finite size humps. To estimate the size of these humps we note that the eigenvalue distribution is even about $\sigma = 0$. Hence, assuming that N is even, we can express the eigenvalues as

$$\sigma_i = -\sigma_0 - \delta\sigma_i \qquad 1 \le i \le \frac{N}{2}, \tag{G.7}$$

$$\sigma_i = \sigma_0 + \delta\sigma_{N-i} \qquad \frac{N}{2} + 1 \le i \le N.$$

Here we assume that $\sigma_0 \gg |\delta\sigma_i|$ and

$$\sum_{i=1}^{N/2} \delta\sigma_i = 0, \tag{G.8}$$

to keep the center of mass of the eigenvalues on each hump fixed at $\pm\sigma_0$. The equation of motion in (7.32) for the positive eigenvalues can then be well approximated as

$$\frac{4\pi^3 N}{\lambda_{\text{QFT}}}(\sigma_0+\delta\sigma_i) = \sum_{j\neq i}^{N/2}\left[1-\sigma_{ij}^2\right]\coth\left(\pi\sigma_{ij}\right)+\sum_{j=1}^{N/2}(1-(2\sigma_0+\delta\sigma_i+\delta\sigma_j)^2). \tag{G.9}$$

Setting $\sigma_0 = -\frac{2\pi^3}{\lambda}$, (G.9) reduces to

$$-1 = \frac{2}{N}\sum_{j\neq i}^{N/2}\left(\left[1-\sigma_{ij}^2\right]\coth\left(\pi\sigma_{ij}\right)-\sigma_{ij}^2\right), \tag{G.10}$$

where the condition on the sum in (G.8) is also imposed. This last equation has no

λ dependence and we can expect that the σ_i range over a size of order 1. This can be confirmed numerically as can be seen in the right picture of Figure G.1.

On the right hand panel of Figure G.1 we can see an exponential fall-off of the humps. We can capture this behavior using the expression (G.6) for the kernel. In this case the continuous limit of (7.32) can be written as

$$\frac{4\pi^3}{\lambda_{\mathrm{QFT}}}\sigma = \int_{-b}^{\sigma}(1-(\sigma-\sigma')^2)\rho(\sigma')d\sigma' - \int_{\sigma}^{b}(1-(\sigma-\sigma')^2)\rho(\sigma')d\sigma' . \quad \text{(G.11)}$$

Taking three derivatives with respect to σ on both sides of the equation gives

$$2\rho''(\sigma) - 4\rho(\sigma) = 0 , \quad \text{(G.12)}$$

hence

$$\rho(\sigma) = \frac{k}{\sqrt{2}}\cosh\left(\sqrt{2}\,\sigma\right) , \quad \text{(G.13)}$$

with the constraints

$$1 = \int_{-b}^{b}\frac{k}{\sqrt{2}}\cosh\left(\sqrt{2}\,\sigma\right)d\sigma = k\sinh\left(\sqrt{2}\,b\right) , \quad \text{(G.14)}$$

$$\frac{4\pi^3}{\lambda_{\mathrm{QFT}}} = \sqrt{2}k\left(\cosh\left(\sqrt{2}\,b\right) - \sqrt{2}\,b\sinh\left(\sqrt{2}\,b\right)\right) . \quad \text{(G.15)}$$

Using (G.14) we can rewrite (G.15) as

$$\sinh\left(\sqrt{1+k^2} - t\right) = \frac{1}{k} , \quad \text{(G.16)}$$

where $t \equiv 2\sqrt{2}\pi^3/\lambda_{\mathrm{QFT}}$. For a given λ_{QFT} one can solve (G.16) for k numerically, and thus determine $\rho(\sigma)$ in (G.13). The dashed line in the right picture of Figure G.1 shows this density at $\lambda_{\mathrm{QFT}} = -3$.

G.3 Solutions at weak negative coupling and finite N

In this part of the appendix we consider $d = 7$ solutions at finite N. For small negative regularized coupling we can use the approximate equations of motion in

(7.38), which for $d = 7$ are

$$-\frac{4\pi^3 N}{\lambda_{\text{QFT}}}\sigma_i = \sum_{j\neq i}\sigma_{ij}^2 \text{sign}\left(\sigma_{ij}\right) . \tag{G.17}$$

Assuming that N is even, the solution that corresponds to the large N solution in the previous section has $N/2$ eigenvalues at $\sigma_+ = b_7 = -\frac{2\pi^3}{\lambda_{\text{QFT}}}$ and $N/2$ at $\sigma_- = -\sigma_+$. However, if we put $2M$ eigenvalues at $\sigma = 0$, then we can also satisfy (G.17) if we place $N/2 - M$ eigenvalues at $\sigma_+ = -\frac{2\pi^3}{\lambda_{\text{QFT}}}\frac{N}{N-M}$ and the same number at $\sigma_- = -\sigma_+$ [2]. We have ignored short range interactions here, but as is shown in the previous section they only spread the eigenvalues an order 1 distance from the peaks.

The free energy for these more general solutions is given by

$$F = \frac{16\pi^{10}N^2}{3\lambda_{\text{QFT}}^3}\left(1 - \frac{M^2}{(N-M)^2}\right), \tag{G.18}$$

demonstrating that the free energy increases with increasing M since $\lambda_{\text{QFT}} < 0$. Assuming that λ_{QFT} is also small (G.18) shows that the solutions with nonzero M are heavily suppressed. If N is odd then M is replaced with $M + 1/2$ in (G.18).

The quadratic fluctuations about the lowest energy solution are

$$\begin{aligned}\delta F &= \frac{4\pi^4 N}{\lambda_{\text{QFT}}}\sum_i(\delta\sigma_i)^2 + 2\pi\sum_{j\neq i}|\sigma_i - \sigma_j|(\delta\sigma_i)^2 - 2\pi\sum_{j\neq i}|\sigma_i - \sigma_j|\delta\sigma_i\delta\sigma_j \\ &= \frac{4\pi^4}{\lambda_{\text{QFT}}}\left(\sum_i^{N/2}\delta\sigma_i^{(+)}\right)\left(\sum_i^{N/2}\delta\sigma_i^{(-)}\right),\end{aligned} \tag{G.19}$$

where $\delta\sigma_i^{(+)}$ are the fluctuations of the eigenvalues at σ_+ and $\delta\sigma_i^{(-)}$ are the fluctuations of the eigenvalues at σ_-. Hence, to quadratic order there is a tachyonic mode corresponding to the overall center of mass motion, which is not present for $SU(N)$, and a positive mode corresponding to the average of the left and right eigenvalues moving in the opposite direction. This latter mode has a large positive coefficient and thus is sharply suppressed. All other modes are zero modes.

The zero modes are not exact as there are nonzero cubic terms. Since the center of mass modes are either removed or suppressed, we can assume that $\sum_i \delta\sigma_i^{(+)} =$

[2] There are still other solutions, *e. g.* one can have an unequal number of eigenvalues at σ_+ and σ_-, in which case $\sigma_- \neq -\sigma_+$.

$\sum_i \delta\sigma_i^{(-)} = 0$. Then the fluctuations of the free energy are

$$\begin{aligned}\delta F &= \frac{\pi}{3}\sum_{i,j}^{N/2}\left(|\delta\sigma_i^{(+)} - \delta\sigma_j^{(+)}|^3 + |\delta\sigma_i^{(-)} - \delta\sigma_j^{(-)}|^3 + 2(\delta\sigma_i^{(+)} - \delta\sigma_j^{(-)})^3\right) \\ &= \frac{\pi}{3}\left(N\sum_i^{N/2}(\delta\sigma_i^{(+)})^3 + \sum_{i,j}^{N/2}|\delta\sigma_i^{(+)} - \delta\sigma_j^{(+)}|^3\right) \\ &\qquad + \frac{\pi}{3}\left(-N\sum_i^{N/2}(\delta\sigma_i^{(-)})^3 + \sum_{i,j}^{N/2}|\delta\sigma_i^{(-)} - \delta\sigma_j^{(-)}|^3\right),\end{aligned} \tag{G.20}$$

where we see that right and left fluctuations decouple from each other. Note that these fluctuations are of order 1 and independent of λ_{QFT}, hence their only effect is to shift the free energy by an unimportant constant and can be ignored even for small N.

Appendix H

Derivation of the BPS equations

The gauged supergravity BPS equations for the ansatz in Chapter 12 can be reduced to the Liouville equation on the Riemann surface accompanied by a set of algebraic equations. This is shown in detail in [56] for gauged supergravity theories in four, five, six and seven dimensions. Here we provide a derivation of this for the case of main interest in this paper, namely seven-dimensional maximal gauged supergravity.

As in [11, 18, 19] we work with the $U(1) \times U(1)$ invariant truncation of the seven-dimensional maximal SO(5) gauged supergravity of [203]. The supersymmetry variations for the fermionic fields were derived in [203]. For the truncation of interest here they read [168]

$$\begin{aligned}
\delta\psi_\mu &= \Big[\nabla_\mu + 2g(A^{(1)}_\mu\Gamma^{12} + A^{(2)}_\mu\Gamma^{34}) + \frac{g}{2}e^{-4(\lambda_1+\lambda_2)}\gamma_\mu + \frac{\gamma_\mu}{2}\gamma^\nu\partial_\nu(\lambda_1+\lambda_2)\Big]\varepsilon \\
&\qquad + \frac{\gamma^\nu}{2}\Big(e^{-2\lambda_1}F^{(1)}_{\mu\nu}\Gamma^{12} + e^{-2\lambda_2}F^{(2)}_{\mu\nu}\Gamma^{34}\Big)\varepsilon \ , \\
\delta\chi^{(1)} &= \Big[\frac{g}{2}(e^{2\lambda_1} - e^{-4(\lambda_1+\lambda_2)}) - \frac{\gamma^\mu}{4}\partial_\mu(3\lambda_1+2\lambda_2) - \frac{\gamma^{\mu\nu}}{8}e^{-2\lambda_1}F^{(1)}_{\mu\nu}\Gamma^{12}\Big]\varepsilon \ , \\
\delta\chi^{(2)} &= \Big[\frac{g}{2}(e^{2\lambda_2} - e^{-4(\lambda_1+\lambda_2)}) - \frac{\gamma^\mu}{4}\partial_\mu(2\lambda_1+3\lambda_2) - \frac{\gamma^{\mu\nu}}{8}e^{-2\lambda_2}F^{(2)}_{\mu\nu}\Gamma^{34}\Big]\varepsilon \ ,
\end{aligned} \tag{H.1}$$

where g is the gauge coupling of the supergravity theory which is related to the radius of the maximally symmetric AdS_7 solution. The Γ^i are SO(5) gamma matrices, the γ_μ seven-dimensional space-time gamma matrices, and we have defined $\gamma_{\mu_1\ldots\mu_p} = \gamma_{[\mu_1}\ldots\gamma_{\mu_p]}$. From now on we suppress all spinor indices and use hats to indicate tangent space indices. The partial topological twist of the boundary field theory suggests the following projectors,

$$\gamma_{\hat{x}_1\hat{x}_2}\varepsilon = \mathrm{i}\varepsilon\ , \qquad \Gamma^{12}\varepsilon = \mathrm{i}\varepsilon\ , \qquad \Gamma^{34}\varepsilon = i\varepsilon\ , \qquad \gamma_{\hat{r}}\varepsilon = \varepsilon\ . \tag{H.2}$$

Therefore, in general, our solutions preserve one quarter of the supersymmetry. Four-dimensional Poincaré invariance implies that the spinors are constant in the $\mathbf{R}^{1,3}$ directions,

$$\partial_t \varepsilon = \partial_{z_i}\varepsilon = 0\ . \tag{H.3}$$

Note however that we do not assume that the spinors are independent of the coordinates (x_1, x_2) parameterizing the Riemann surface. The conditions for the supersymmetry variations (H.1) to vanish are of two types. Vanishing of the variation of the dilatinos $\chi^{(i)}$ and the (t, z_1, z_2, z_3) components of the gravitino ψ_μ leads to differential equations for the background fields, while the integrability of this system imposes additional constraints. One can show that for the truncation of interest and with the ansatz in (12.2)-(12.3) these reduce to:

$$\begin{aligned}
&\partial_r(3\lambda_1+2\lambda_2) - 2ge^{f+2\lambda_1} + 2ge^{f-4\lambda_1-4\lambda_2} - e^{h-\hat{\varphi}-2\lambda_1}F^{(1)}_{x_1x_2} = 0\ ,\\
&\partial_r(2\lambda_1+3\lambda_2) - 2ge^{f+2\lambda_2} + 2ge^{f-4\lambda_1-4\lambda_2} - e^{f-\hat{\varphi}-2\lambda_2}F^{(2)}_{x_1x_2} = 0\ ,\\
&(\partial_{x_1} + \mathrm{i}\partial_{x_2})(3\lambda_1+2\lambda_2) - e^{-f-2\lambda_1}(F^{(1)}_{x_2r} - \mathrm{i}F^{(1)}_{x_1r}) = 0\ ,\\
&(\partial_{x_1} + \mathrm{i}\partial_{x_2})(2\lambda_1+3\lambda_2) - e^{-f-2\lambda_2}(F^{(2)}_{x_2r} - \mathrm{i}F^{(2)}_{x_1r}) = 0\ ,\\
&\partial_r\,(f+\lambda_1+\lambda_2) + ge^{f-4\lambda_1-4\lambda_2} = 0\ ,\\
&\partial_{x_1}(f+\lambda_1+\lambda_2) = \partial_{x_2}(f+\lambda_1+\lambda_2) = 0\ ,\\
&\partial_r(\hat{\varphi}/2 - 4\lambda_1 - 4\lambda_2) + 2ge^{f+2\lambda_1} + 2ge^{f+2\lambda_2} - 3ge^{f-4\lambda_1-4\lambda_2} = 0\ ,\\
&\partial_r\partial_{x_2}(\hat{\varphi}/2 - 4\lambda_1 - 4\lambda_2) + 4gF^{(1)}_{rx_1} + 4gF^{(2)}_{rx_1} = 0\ ,\\
&\partial_r\partial_{x_1}(\hat{\varphi}/2 - 4\lambda_1 - 4\lambda_2) + 4gF^{(1)}_{x_2r} + 4gF^{(2)}_{x_2r} = 0\ ,\\
&(\partial^2_{x_1} + \partial^2_{x_2})(\hat{\varphi}/2 - 4\lambda_1 - 4\lambda_2) - 4gF^{(1)}_{x_1x_2} - 4gF^{(2)}_{x_1x_2} = 0\ .
\end{aligned} \tag{H.4}$$

Imposing that the variations of the (r, x_1, x_2) components of the gravitino vanish implies the equation

$$\partial_r\epsilon - \tfrac{1}{2}(\partial_r f)\epsilon = 0, \tag{H.5}$$

which is solved by $\epsilon = \mathrm{e}^{f/2}\epsilon_0$ with ϵ_0 a constant spinor obeying the projectors in (H.2). In addition these gravitino equations imply that the two U(1) gauge fields are given by

$$A^{(1)} = \frac{a^1}{8g(a^1+a^2)}\omega\,, \qquad A^{(2)} = \frac{a^1}{8g(a^1+a^2)}\omega\,. \tag{H.6}$$

Here we show that in the IR, where the metric takes the form $\mathrm{AdS}_5 \times \mathcal{C}$ these equation reduce to a single second order equation for the conformal factor φ together with algebraic equations for the other fields. Plugging the IR fields (12.3) in (H.4) results in the following equations

$$\begin{aligned}
&2ge^{2\lambda_1} - 2ge^{-4\lambda_1-4\lambda_2} + e^{-\varphi-\varphi_0-2\lambda_1}F^{(1)}_{x_1x_2} = 0\,,\\
&2ge^{2\lambda_2} - 2ge^{-4\lambda_1-4\lambda_2} + e^{-\varphi-\varphi_0-2\lambda_2}F^{(2)}_{x_1x_2} = 0\,,\\
&1 - 2e^{f_0-4\lambda_1-4\lambda_2} = 0\,,\\
&e^{2\lambda_1} + e^{2\lambda_2} - \frac{3}{2}e^{-4\lambda_1-4\lambda_2} = 0\,,\\
&(\partial^2_{x_1} + \partial^2_{x_2})\varphi - 8g(F^{(1)}_{x_1x_2} + F^{(2)}_{x_1x_2}) = 0\,.
\end{aligned} \tag{H.7}$$

The third and fourth of these equations, together with a linear combination of the first two lead to algebraic equations for the scalars, and the metric constants. Finally, after defining,

$$\mathrm{e}^{\varphi_0} = \frac{\mathrm{e}^{4\lambda_1+4\lambda_2}}{16g^2}(\mathrm{e}^{8\lambda_1+4\lambda_2} + \mathrm{e}^{4\lambda_1+8\lambda_2} - \mathrm{e}^{-2\lambda_1} - \mathrm{e}^{2\lambda_2})^{-1} \tag{H.8}$$

and combining the second linearly independent combination, together with the last equation we find the Liouville equation

$$\Box\varphi + \kappa\mathrm{e}^{\varphi} = (\partial^2_{x_1} + \partial^2_{x_2})\varphi + \kappa\mathrm{e}^{\varphi} = 0\,, \tag{H.9}$$

which has to be satisfied by the conformal factor φ.

Appendix I

The Liouville equation

Here we give a short overview of the Liouville equation and some of its properties that are relevant in the context of this work. The Liouville equation is the non-linear partial differential equation (H.9) satisfied by the conformal factor φ of a metric $ds^2 = e^{\varphi}(dx_1^2 + dx_2^2)$ on a surface of constant Gaussian curvature κ.[1] This equation can be used to prove the uniformization theorem which states that every simply connected Riemann surface is conformally equivalent to one of three Riemann surfaces:

- the hyperbolic plane for $\kappa < 0$,
- the complex plane for $\kappa = 0$,
- the Riemann sphere for $\kappa > 0$.

In particular this implies that every Riemann surface admits a Riemannian metric of constant curvature. For compact Riemann surfaces, the hyperbolic Riemann surfaces with genus $\mathbf{g} > 1$ have the hyperbolic plane as universal cover and have a non-abelian fundamental group. The torus, $\mathbf{g} = 1$ has the complex plane as universal cover and the fundamental group is $\mathbf{Z}^2$. Finally the Riemann sphere with genus $\mathbf{g} = 0$ has a trivial fundamental group.

Apart from regular solutions to the Liouville equation we also consider Riemann surfaces with prescribed singularities. This results in the addition of localized sources on the right hand side of the Liouville equation

$$\Box\varphi + \kappa e^{\varphi} = 4\pi \sum_i (1 - \xi_i)\delta^{(2)}(p_i)\,, \tag{I.1}$$

[1]Note that it is sometimes convenient to work in complex coordinates $z = \frac{1}{2}(x_1 + i x_2)$ where $\Box = \partial_{x_1}^2 + \partial_{x_2}^2 = 4\partial_z\partial_{\bar{z}}$.

where $2\pi\xi_i$ parameterizes the opening angle of the conical defects located at the point p_i and we restrict the parameters to lie within the interval $0 < \xi_i < 1$. We can specify a Riemann surface of interest by specifying a set of parameters $\{\mathbf{g}, J, \vec{\xi}, \vec{p}, \kappa\}$, where $\mathbf{g}$ is the genus, J a complex structure, $\vec{\xi}$ and $\vec{p}$ two vectors with entries the opening angles and positions of the conical singularities and κ the Gaussian curvature. Given a metric g on the Riemann surface, we can compute the volume using the Gauss-Bonnet theorem.

$$\mathrm{V}_{\mathbf{g},\xi} = \int_{\Sigma} \kappa_g \mathrm{d}\omega_g = \frac{2\pi}{\kappa}\chi(\Sigma, \vec{\xi}) = \frac{2\pi}{\kappa}\left(\chi(\Sigma) - \sum_i (1-\xi_i)\right), \tag{I.2}$$

where $\chi(\Sigma) = 2g - 2$ is the topological Euler characteristic and we call $\chi(\Sigma, \vec{\xi})$ the conic Euler characteristic. In algebraic geometry language this represents the twisted anticanonical divisor, $-K_\Sigma - \sum_i (1-\xi_i) p_i$ on Σ, where $K_\Sigma = T^{1,0*}\Sigma$ denotes the class of the canonical divisor of the smooth Riemann surface Σ, see for example [169, 181, 235]. When a constant curvature metric exists the sign of the curvature κ_g agrees with the sign of $\chi(\Sigma, \vec{\xi})$. To extend the uniformization theorem to include Riemann surfaces with conical singularities, one looks for a metric compatible with the complex structure J, with conical singularities at points p_i and constant curvature away from the singularities. Indeed, using (I.1) it follows that the Ricci scalar, given by $R = -\mathrm{e}^{-\varphi}\Box\varphi$, is constant away from the singularities. Near a conical singularity, $\varphi \propto \log|z|$ implying that at these points $-\Box\varphi$ is given by a multiple of delta functions at $z = 0$.[2] In [169, 181, 235] it was shown that such a constant curvature metric always exists when $\chi(\Sigma, \vec{\xi}) \leq 0$ or when $\chi(\Sigma, \vec{\xi}) > 0$ and $1 - \xi_i > \sum_{j\neq i}(1-\xi_j)$ for all i. This metric is furthermore unique except when $\chi(\Sigma, \vec{\xi}) = 0$ when it is unique up to an overall constant, or when $\Sigma = S^2$ with less than three punctures when it is unique up to a Möbius transformation which fixes the position of the singularities. It is known that to each such set of parameters there exists a unique metric on Σ with constant Gaussian curvature κ and prescribed singularities at the points p_i.

Away from the singularities, the most general solution to the Liouville equation is given by

$$\varphi = \log\left(4\frac{|\partial_z u(z)|^2}{(1+\kappa|u(z)|^2)^2}\right), \tag{I.3}$$

where $u(z)$ is a meromorphic function with non-vanishing holomorphic derivative and at most simple poles. Close to the punctures, the Liouville equation implies the following asymptotic behavior:

$$\varphi = -2(1-\xi_i)\log|z - z_i|, \qquad \text{as } z \to z_i\,. \tag{I.4}$$

[2] This can be seen by excising a small circle around the origin and invoking the Stokes theorem.

The problem of finding a general solution with conical singularities is closely related to the Riemann-Hilbert problem of finding functions with prescribed monodromies in the complex plane. In order to solve this problem we introduce the Fuchsian equation, given n singularities with opening angles ξ_i this equation is given by

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \sum_{i=1}^{n} \left[\frac{(1-\xi_i)(1+\xi_i)}{4(z-z_i)^2} + \frac{c_i}{2(z-z_i)} \right] w = 0\,, \tag{I.5}$$

where the c_i are known as the accessory parameters which have to satisfy the equations

$$\begin{cases} \sum_i c_i = 0 \\ \sum_i (2c_i z_i + (1-\xi_i)(1+\xi_i)) = 0 \\ \sum_i (c_i z_i^2 + z_i(1-\xi_i)(1+\xi_i)) = 0\,. \end{cases} \tag{I.6}$$

The double poles of (I.5) fix the behavior of the solutions near the singular points to

$$w(z) \sim A(z-z_i)^{(1+\xi_i)/2} + B(z-z_i)^{(1-\xi_i)/2}\,, \tag{I.7}$$

from which one can easily read of the monodromies. Given a pair of linearly independent solutions w_1 and w_2 one can see that by plugging the function $u = w_1/w_2$ in (I.3) we find a solution φ to the Liouville equation with the prescribed singularities. In general the monodromies belong to $\mathrm{SL}(2,\mathbf{C})$, and φ will not be a single valued function. In order to find a single valued function we need to furthermore require all monodromies to lie in $\mathrm{SU}(2)$, or $\mathrm{SU}(1,1)$ for $\kappa < 1$. These conditions uniquely determine the accessory parameters c_i and consequently the function $w_{1,2}$. Since now u transforms as

$$u \to \frac{au+b}{-\bar{b}u+\bar{a}}\,, \qquad \text{where } |a|^2+|b|^2 = 1\,, \tag{I.8}$$

all monodromies leave φ invariant.

Appendix J

Uplift formulae

In this appendix we collect all the relevant uplift formulae used in part III of this thesis. The uplift formulae for the maximal four, five and seven-dimensional supergravities to string and/or M-theory were given in [80]. The uplift formulae from the maximal six-dimensional supergravity to ten-dimensional type IIA supergravity was given in [81].

J.1 S^7 reduction of eleven-dimensional supergravity

The eleven-dimensional metric is given by

$$\mathrm{d}s_{11}^2 = \Delta^{2/3}\mathrm{d}s_4^2 + \frac{\Delta^{-1/3}}{g^2}\sum_{i=1}^{4} X_i^{-1}\left(\mathrm{d}\mu_i^2 + \mu_i^2\left(\mathrm{d}\phi_i + gA^{(i)}\right)^2\right). \tag{J.1}$$

Here the four functions μ_i satisfy the constraint $\sum_i \mu_i^2 = 1$ and the X_i are defined as

$$\begin{aligned} X_1 &= \mathrm{e}^{\frac{1}{4}(3\lambda_1-\lambda_2-\lambda_3)}, & X_2 &= \mathrm{e}^{\frac{1}{4}(3\lambda_2-\lambda_1-\lambda_3)}, \\ X_3 &= \mathrm{e}^{\frac{1}{4}(3\lambda_3-\lambda_1-\lambda_2)}, & X_4 &= \mathrm{e}^{\frac{1}{4}(-\lambda_1-\lambda_2-\lambda_3)}. \end{aligned} \tag{J.2}$$

A convenient parameterization for the μ_i is given by

$$\begin{aligned} \mu_1 &= \sin\alpha\,, & \mu_2 &= \cos\alpha\sin\beta\,, \\ \mu_3 &= \cos\alpha\cos\beta\sin\gamma\,, & \mu_4 &= \cos\alpha\cos\beta\cos\gamma\,. \end{aligned} \tag{J.3}$$

The function Δ appearing in the metric is given by

$$\Delta = \sum_i X_i\mu_i^2\,. \tag{J.4}$$

There is also a four-form field strength, given by

$$F_{(4)} = 2g \sum_i \left(X_i^2 \mu_i^2 - X_i \Delta\right) \epsilon_4 - \frac{1}{2g^2} \sum_i X_i^{-2} \mathrm{d}(\mu_i^2) \wedge \left(\mathrm{d}\phi_i + gA^{(i)}\right) \wedge \star_4 F^{(i)} . \quad \text{(J.5)}$$

Here ϵ_4 is the volume form on the four-dimensional part of the metric $\mathrm{d}s_4^2$.

J.2 S^5 reduction of type IIB supergravity

The uplift to type IIB supergravity is given by

$$\mathrm{d}s_{10}^2 = \Delta^{1/2} \mathrm{d}s_5^2 + \frac{\Delta^{-1/2}}{g^2} \sum_{i=1}^{3} X_i^{-1} \left(\mathrm{d}\mu_i^2 + \mu_i^2 \left(\mathrm{d}\phi_i + gA^{(i)} \right)^2 \right) \quad \text{(J.6)}$$

where μ_i are subject to the constraint $\sum \mu_i^2 = 1$. The X_i are given by

$$X_1 = \mathrm{e}^{-\lambda_1 - \lambda_2} , \qquad X_2 = \mathrm{e}^{-\lambda_1 + \lambda_2} , \qquad X_3 = \mathrm{e}^{2\lambda_1} . \quad \text{(J.7)}$$

A convenient parameterization for the μ_i is given by

$$\mu_1 = \cos\alpha \sin\beta , \qquad \mu_2 = \cos\alpha \cos\beta , \qquad \mu_3 = \sin\alpha . \quad \text{(J.8)}$$

The function Δ is given by

$$\Delta = \sum_{i=1}^{3} X_i \mu_i^2 . \quad \text{(J.9)}$$

The self-dual five-form field strength is given by $F_5 = G_5 + \star_{10} G_5$ with

$$G_5 = 2g \sum_i \left(X_i^2 \mu_i^2 - \Delta X_i \right) \epsilon_5 - \frac{1}{2g^2} \sum_i X_i^{-2} \mathrm{d}\left(\mu_i^2\right) \wedge \left(\mathrm{d}\phi_i + gA^{(i)} \right) \wedge \star_5 F^{(i)} \quad \text{(J.10)}$$

where ϵ_5 is the volume form on $\mathrm{d}s_5^2$ and $\star_5$ is the five-dimensional Hodge dual with respect to the same metric.

J.3 $S^4/\mathbb{Z}_2$ reduction of massive type IIA supergravity

In ten dimensions, the metric is given by

$$\mathrm{d}s_{10}^2 = (\sin\alpha X^{\frac{3}{2}})^{\frac{1}{12}} \Delta^{\frac{3}{8}} \left[\mathrm{d}s_6^2 + \frac{2X^2}{g^2}\mathrm{d}\alpha^2 + \frac{1}{2g^2 X\Delta}\cos^2\alpha \left(\sigma_1^2 + \sigma_2^2 + (\sigma_3 - gA)^2\right) \right]. \tag{J.11}$$

The only non-vanishing field strength is given by

$$F_{(4)} = -\frac{\sqrt{2}\sin^{\frac{1}{3}}\alpha\cos^3\alpha}{6g^3}\Delta^{-2}U\mathrm{d}\alpha\wedge\epsilon_3 + \frac{\sin^{\frac{1}{3}}\alpha\cos\alpha}{\sqrt{2}g^2}F\wedge(\sigma_3 - gA)\wedge\mathrm{d}\alpha + \frac{\sin^{\frac{4}{3}}\alpha\cos^2\alpha}{2\sqrt{2}g^2}\Delta^{-1}X^{-3}F\wedge\mathrm{d}\sigma_3 \tag{J.12}$$

Finally the dilaton is given by

$$\mathrm{e}^{\phi} = \sin^{-\frac{5}{6}}\alpha\Delta^{\frac{1}{4}}X^{-\frac{5}{4}}. \tag{J.13}$$

In these expressions we have introduced the functions

$$\begin{aligned} \Delta =& X\cos^2\alpha + X^{-3}\sin^2\alpha, \\ U =& X^{-6}\sin^2\alpha + \left(4X^{-2} - 3X^2\right)\cos^2\alpha - 6X^{-2}. \end{aligned} \tag{J.14}$$

The σ_i are left-invariant one-forms of SU(2) which satisfy $\mathrm{d}\sigma_i = -\frac{1}{2}\epsilon_{ijk}\sigma_j\wedge\sigma_k$. The gauge coupling constant g is related to the mass parameter M of the massive type IIA theory by $M = \frac{\sqrt{2}}{3}g$.

J.4 S^4 reduction of eleven-dimensional supergravity

The eleven-dimensional metric is given by

$$\mathrm{d}s_{11}^2 = \Delta^{1/3}\mathrm{d}s_7^2 + \frac{1}{4g^2}\Delta^{-2/3}\left[X_0^{-1}\mathrm{d}\mu_0^2 + \sum_{i=1}^{2}X_i^{-1}(\mathrm{d}\mu_i^2 + \mu_i^2(\mathrm{d}\phi_i + 2gA^{(i)})^2)\right]. \tag{J.15}$$

The four-form flux takes the form

$$\star_{11} F_4 = 4g \sum_{i=0}^{2} \left(X_i^2 \mu_i^2 - \Delta X_i \right) \epsilon_7 + 2g \Delta X_0 \epsilon_7$$
$$+ \frac{1}{4g^2} \sum_{i=1}^{2} X_i^{-2} \mathrm{d}(\mu_i^2) \wedge (\mathrm{d}\phi_i + 4g A^{(i)}) \wedge \star_7 F^{(i)} \,. \quad \text{(J.16)}$$

Here we have introduced the functions $X_{0,1,2}$:

$$X_1 = \mathrm{e}^{2\lambda_1} \,, \qquad X_2 = \mathrm{e}^{2\lambda_2} \,, \qquad X_0 = (X_1 X_2)^{-2} \,. \quad \text{(J.17)}$$

and

$$\Delta = \sum_{i=0}^{2} X_i \mu_i^2 \,, \quad \text{(J.18)}$$

while the μ_i are constrained to lie on the hypersurface $\sum_{i=0}^{2} \mu_i = 1$. A convenient parameterization for the μ_i is given by

$$\mu_0 = \cos\alpha \cos\beta \,, \qquad \mu_1 = \sin\alpha \,, \qquad \mu_2 = \cos\alpha \sin\beta \,. \quad \text{(J.19)}$$

Appendix K

SCFT trivia

Here we collect some of our conventions and useful facts about SCFTs in various dimensions. The number of supercharges preserved by SCFTs in different dimensions is summarized in the table below. Additionally the supergroup and R-symmetry are given in this table.

Dimension	2 $\mathcal{Q}$'s	4 $\mathcal{Q}$'s	8 $\mathcal{Q}$'s	16 $\mathcal{Q}$'s
6	/	/	$\mathcal{N}=(1,0)$ $\mathfrak{su}(2)_R \subset \mathfrak{osp}(8\|2)$	$\mathcal{N}=(2,0)$ $\mathfrak{sp}(4)_R \subset \mathfrak{osp}(8\|4)$
5	/	/	$\mathcal{N}=1$ $\mathfrak{su}(2)_R \subset \mathfrak{f}_4$	/
4	/	$\mathcal{N}=1$ $\mathfrak{u}(1)_R \subset \mathfrak{su}(2,2\|1)$	$\mathcal{N}=2$ $\mathfrak{su}(2)_R \times \mathfrak{u}(1)_r \subset \mathfrak{su}(2,2\|2)$	$\mathcal{N}=4$ $\mathfrak{su}(4)_R \subset \mathfrak{su}(2,2\|4)$
3	$\mathcal{N}=1$ $\emptyset \subset \mathfrak{osp}(1\|4)$	$\mathcal{N}=2$ $\mathfrak{u}(1)_R \subset \mathfrak{osp}(2\|4)$	$\mathcal{N}=4$ $\mathfrak{so}(4)_R \subset \mathfrak{osp}(4\|4)$	$\mathcal{N}=8$ $\mathfrak{so}(8)_R \subset \mathfrak{osp}(8\|4)$

Table K.1: The superconformal algebra corresponding to a number of supercharges $\mathcal{Q}$ is given for dimensions $d = 3,4,5,6$, [76]. Furthermore we specify the corresponding R-symmetry.

A four-dimensional $\mathcal{N}=2$ SCFT has $\mathrm{SU}(2)_R \times \mathrm{U}(1)_r$ R-symmetry. We denote the generator of the diagonal Cartan of $\mathrm{SU}(2)_R$ by I_3 and the generator of $\mathrm{U}(1)_r$ by $R_{\mathcal{N}=2}$. The charge assignments for the components of a $\mathcal{N}=2$ vector and hypermultiplet are given as follows: Since we study both $\mathcal{N}=1$ and $\mathcal{N}=2$ theories, it is useful

$R_{\mathcal{N}=2} \setminus I_3$	$\frac{1}{2}$	0	$-\frac{1}{2}$
0		A_μ	
1	λ		λ'
2		ϕ	

$R_{\mathcal{N}=2} \setminus I_3$	$\frac{1}{2}$	0	$-\frac{1}{2}$
-1		ψ	
0	Q		$\widetilde{Q}^\dagger$
1		$\widetilde{\psi}^\dagger$	

to consider an $\mathcal{N}=1$ subalgebra of the $\mathcal{N}=2$ algebra. A choice of subalgebra corresponds to a choice of Cartan of $\mathrm{SU}(2)_R$. The unique $\mathcal{N}=1$ superconformal

R-symmetry generator in an $\mathcal{N}=2$ SCFT is given by

$$R_{\mathcal{N}=1} = \frac{1}{3} R_{\mathcal{N}=2} + \frac{4}{3} I_3 \,. \tag{K.1}$$

The linear combination

$$J = R_{\mathcal{N}=2} - 2I_3 \,, \tag{K.2}$$

commutes with the chosen $\mathcal{N}=1$ subalgebra and is thus a flavor symmetry from the $\mathcal{N}=1$ point of view.

Finally we note that the operators in a $\mathcal{T}_N$ theory have charges

	$R_{\mathcal{N}=2}$	I_3
u_k	$2k$	0
Q	0	$\frac{1}{2}(N-1)$
$\widetilde{Q}$	0	$\frac{1}{2}(N-1)$
μ	0	1

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