



# A conjectured upper bound on the Choptuik critical exponents

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Received 14 January 2021; accepted 16 February 2021

Available online 23 February 2021

Editor: Stephan Stieberger

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## Abstract

Near-critical type II gravitational collapse is characterized by the formation of arbitrarily small black holes whose horizon radii are described by the simple scaling law  $r_{\text{BH}} \propto (p - p^*)^\gamma$ , where  $\gamma$  is the matter-dependent Choptuik critical exponent and  $\Delta p \equiv p - p^*$  is the phase space distance from the exact self-similar critical evolution. We point out that all matter models studied thus far in the physics literature are characterized by the upper bound  $\gamma \leq 1$ . We conjecture that this is a generic feature of non-linear gravitational collapse scenarios.

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## 1. Introduction

Almost three decades ago, Choptuik [1] studied numerically the non-linear gravitational collapse of a real self-gravitating massless scalar field and revealed an intriguing critical phenomenon in general relativity at the threshold of black-hole formation. Characterizing the strength of the initial field configuration by a parameter  $p$  [2], Choptuik [1] has demonstrated numerically that there is a critical field parameter  $p = p^*$  for which the collapse is characterized by a self-similar behavior which leads to the formation of a central naked singularity.

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Table 1

Critical phenomena in gravitational collapse. Remarkably, one finds that all self-gravitating matter models reported in the physics literature over the past three decades [1,3–30] are characterized by the simple relation  $\gamma \leq 1$ .

Matter model	References	Critical exponent $\gamma$
Massless real scalar field	[1]	0.37
D-dimensional massless real scalar field	[4]	$0.372 \leq \gamma \leq 0.44$ for $4 \leq D \leq 11$
Charged scalar field	[5–7]	0.37
Massless Dirac field	[8]	0.26
$SU(2)$ Yang-Mills (magnetic ansatz)	[9]	0.196
General Yang-Mills	[10]	0.20
$SU(2)$ Skyrme model	[11]	0.20
$SO(3)$ Mexican hat	[12]	0.119
Yang-Mills-Higgs	[13]	first critical solution 0.118 second critical solution 0.19, 0.26
Axisymmetric massless scalar field	[14]	$0.28 \leq \gamma \leq 0.41$
Complex scalar field with angular momentum	[15]	0.11
Scalar field with angular momentum in spherical symmetry	[16]	$0.0013 \leq \gamma \leq 0.376$ for $0 \leq l \leq 6$
Axisymmetric vacuum	[17]	0.37
Vacuum gravitational collapse in $4 + 1$ dimensions	[18]	0.164
Vacuum gravitational collapse in $8 + 1$ dimensions	[19]	0.273
Axisymmetric electromagnetic waves	[20]	0.145
Einstein-Maxwell-dilaton	[21]	$0.25 < \gamma \leq 0.5$ for $0 \leq \kappa < 1$
Perfect fluid $P = k\rho$	[22,23]	$0.15 \lesssim \gamma \leq 1$ for $0.05 \leq k \leq 1$
Massless real scalar field in $(2 + 1)$ dimensions	[24]	$8/23$
2-d sigma model - complex scalar field ( $\kappa = 0$ )	[25]	0.387
2-d sigma model - axion-dilaton ( $\kappa = 1$ ) - elliptic	[26]	0.264
2-d sigma model - scalar-Brans-Dicke ( $\kappa > 0$ )	[27]	$0.205 \leq \gamma \leq 0.375$ for $0.000125 \leq \kappa \leq 2.5$
$SU(2)$ sigma model	[28]	$0.38 \leq \gamma \leq 0.5$ for $0 \leq \alpha \leq 0.42$
2-d sigma model - general $\kappa$	[29]	$0.14 \leq \gamma \leq 0.5$ for $-0.6 \leq \kappa < 13$
Axion-dilaton - hyperbolic	[30]	0.436
5-d Axion-dilaton - elliptic	[30,31]	0.372, 0.601, 0.843

Furthermore, it has been observed in [1] that field configurations in the super-critical regime  $p > p^*$  (with  $\Delta p \equiv p - p^* \ll 1$ ) are characterized by the formation of arbitrarily small black holes whose horizon radii are well fitted by the remarkably simple scaling law

$$r_{\text{BH}} \propto (p - p^*)^\gamma. \quad (1)$$

Interestingly, it has been demonstrated in [1] that the critical exponent  $\gamma$  in the Choptuik scaling relation (1) is universal in the sense that its value does not depend on the shape of the near-critical initial field configuration.

The intriguing critical phenomenon discovered in [1] has motivated many physicists and mathematicians to explore the onset of black-hole formation in self-gravitating collapse scenarios of various matter models, see [1,3–30] and references therein. These later studies have revealed that the critical scaling exponent  $\gamma$  in (1) should be regarded as a semi-universal physical parameter.

In particular, it has been demonstrated explicitly in [1,3–30] that the exact value of the critical exponent  $\gamma$  in the black-hole scaling relation (1) depends on a number of physical factors, such as: (1) the type of collapsed matter, (2) the symmetry of the initial matter configuration, and (3) the spacetime dimension.

## 2. The conjectured upper bound on the Choptuik critical exponents

Despite the fact that the critical scaling exponent  $\gamma$  in (1) is model-dependent [1,3–30], in the present compact paper we would like to raise the following physically intriguing question: Is there some underlying physical principle that *bounds* the possible numerical values of the Choptuik critical exponents?

In order to provide evidence that the answer to the above stated physically interesting question may be ‘yes’, we summarize in a compact form in Table 1 the main body of work [1,3–30] that has been mounting in the physics literature on the Choptuik critical phenomena for various self-gravitating matter models. What we find most intriguing is the fact that, for all matter models studied in the physics literature over the past three decades [1,3–30], the critical exponents are all bounded from above by the simple relation

$$\gamma \leq 1. \quad (2)$$

The observation (2) [see the numerical data presented in Table 1] may indicate that there is some yet unknown universal mechanism which, regardless of the physical characteristics of the various collapse models (the assumed matter content, symmetry, and the spacetime dimension), bounds the possible values of the Choptuik critical exponents in dynamical gravitational collapse scenarios. We hope that the present compact paper would encourage physicists and mathematicians to further explore the (in)validity of the conjectured universal relation (2) for the Choptuik critical exponents.

### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

### Acknowledgements

This research is supported by the Carmel Science Foundation. I would like to thank Matthew Choptuik, Carsten Gundlach, Piotr Bizoń, and Riccardo Antonelli for helpful correspondences. I would also like to thank Yael Oren, Arbel M. Ongo, Ayelet B. Lata, and Alona B. Tea for helpful discussions.

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