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Thermodynamics of Kerr-Newman Black Hole

Gutivan A. Syahputra and Bintoro Anang Subagyo
Deparment of Physics, Institut Teknologi Sepuluh Nopember, Indonesia

Email: gutivanaliefs@gmail.com, b_anang@physics.its.ac.id

Abstract. This paper reviews the solution of Einstein's field equations for rotating black holes coupled with Maxwell's electromagnetics. We derive the thermodynamic quantities of black hole, such as temperature, entropy, and energy with respect to mass, charge, and angular momentum, to derive the first law of thermodynamics. Also, we show that the black hole's entropy of the black hole will be directly proportional to the irreducible mass of the black hole. We present a comparison between black hole's possible maximal mass that can be extracted with it total mass.

1. Introduction

In 1969, Penrose [1] showed that it was possible to extract some energy from black holes for observers outside the black hole. It's just that Christodolou [2] shows that not all the energy of the black hole can be extracted, but there are certain restrictions. Bekenstein [3] showed the relationship between the entropy changes of black holes to changes in their surface area due to certain processes, and Hawking [4] separately obtained the same results as Bekenstein and showed the linkage of the formulation of changes in the energy magnitude of black holes to the first law and second law of thermodynamics.

In 2021, Stuchlik et al[5] introduced several variants regarding energy extraction from black holes via the Penrose process and related modifications of Kerr black holes with their naked singularities. The method he uses is to use an astrophysical variant of the Penrose process that deals with high energy. While Liu [6] tried to formulate energy extraction through the Penrose process for non-Kerr rotating black holes.

In this paper, we discuss rotating and charged black holes and the relationship between physical magnitudes and the magnitudes of thermodynamics. We also explain how mass, angular momentum, and charge affect the value of the maximum efficiency of energy a black hole can extract. This paper shows analogs between classical thermodynamics with black hole mechanics via the Penrose process. Apart from this approach, one can construct a solution to black holes thermodynamics via a semi-classical method.

2. Kerr-Newman Metric

The Kerr-Newman spacetime metric can be obtained from the Newman-Janis algorithm [7]. In general, the algorithm begins using the Reissner-Nordstrom metric in Eddington-Finklestein coordinates

$$ds_{RN}^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) du^2 + 2dudr - r^2 d\Omega^2, \quad (1)$$

with M and Q are the mass and charge of the black hole. Metric tensors can be expressed in null tetrad relating to the selected metric in the form $\{l, n, m, \bar{m}\}$

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$$g^{\mu\nu} = l^\mu n^\nu + n^\mu l^\nu - m^\mu \bar{m}^\nu - m^\nu \bar{m}^\mu, \tag{2}$$

with tetrads

$$\begin{aligned} l^\mu &= (0,1,0,0) \\ n^\mu &= \left(1, -\frac{1}{2}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right), 0,0\right) \\ m_\mu &= \frac{1}{\sqrt{2}}\left(0,0,\frac{1}{r},\frac{1}{r\sin\theta}\right). \end{aligned} \tag{3}$$

Subsequently carried out complex transformations $r \to r - ia\cos\theta$ and $u \to u + ia\cos\theta$, with a is angular momentum of black hole per it mass [8]. This transformation will obtain the Kerr-Newman metric from the Reissner-Nordstrom metric. Null tetrad on Eq. (3) after transformation will have the form

$$\begin{aligned} l^\mu &= (0,1,0,0) \\ n^\mu &= \left(1, \frac{1}{2}\left(1 - \frac{2Mr}{\rho^2} - \frac{Q^2}{\rho^2}\right), 0,0\right) \end{aligned} \tag{4}$$

$$m^\mu = \frac{1}{\sqrt{2}}\left(\frac{ia\sin\theta}{r - ia\cos\theta}, -\frac{ia\sin\theta}{r - ia\cos\theta}, \frac{1}{r - ia\cos\theta}, -\frac{i}{r\sin\theta - ia\sin\theta\cos\theta}\right)$$

and can be substituted to Eq. (2) and obtained as a new metric tensor

$$g^{\mu\nu} = \begin{pmatrix} -\frac{a^2\sin^2\theta}{\rho^2} & 1 + \frac{a^2\sin^2\theta}{\rho^2} & 0 & -\frac{a}{\rho^2} \\ \cdots & -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) & 0 & \frac{a}{\rho^2} \\ \cdots & \cdots & -\frac{1}{\rho^2} & 0 \\ \cdots & \cdots & \cdots & -\frac{1}{\rho^2\sin^2\theta} \end{pmatrix} \tag{5}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$. Furthermore, applying Eq. (5) to the transformed line element yield

$$ds^2 = \left(1 - \frac{2M - Q^2}{\rho^2}\right)(du - ia\sin\theta\ d\theta)^2 + 2\,(du - ia\sin\theta\, d\theta)(dr + ia\sin\theta) \tag{6}$$

$$+\frac{2a\sin\theta\,(2Mr - Q^2)}{\rho^2}(du - ia\sin\theta\, d\theta)d\phi - 2a\sin^2\theta\ (dr + ia\sin\theta\, d\theta)d\phi$$

$$-\rho^2 d\theta^2 - \sin^2\theta\left[(r^2 + a^2) + a^2\sin^2\theta\left(\frac{2M - Q^2}{\rho^2}\right)\right]d\phi^2.$$

Applying Giampieri transformation, $id\theta = \sin\theta\, d\phi$, we obtain

$$ds_{KN}^2 = \left(1 - \frac{2Mr - Q^2}{\rho^2}\right)du^2 + 2\,du\,dr + \frac{2a\sin^2\theta}{\rho^2}(2Mr - Q^2)du\,d\phi \tag{7}$$

$$-2\sin^2\theta\, drd\phi - \rho^2 d\theta^2 - \frac{\sin^2\theta}{\rho^2}((r^2+a^2)^2 - \Delta a\ \sin^2\theta)d\phi^2.$$

The metric Eq. (7) is the Kerr-Newman metric in Eddington-Finklestein coordinates. Metrics in Boyer-Lindquist coordinates are obtained by transformations $du \to dt - \frac{r^2+a^2}{\Delta}dr$ and $d\phi \to d\phi - \frac{a}{\Delta}dr$ [9]

$$ds^2_{KN_{BL}} = \frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\phi)^2 - \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)d\phi - adt] - \rho^2\left(\frac{dr^2}{\Delta} + d\theta^2\right). \quad (8)$$

The Kerr-Newman black hole has two event horizons when $\Delta = 0$ [10]. These two event horizons can be expressed in equation $\Delta = (r - r_{H-})(r - r_{H+})$ where

$$r_{H+} = M + \sqrt{M^2 - a^2 - Q^2}\,, \quad (9)$$

$$r_{H-} = M - \sqrt{M^2 - a^2 - Q^2}\,.$$

The black hole's surface area of its outer horizon given by

$$A_{H+} = 8\pi\left[M^2 + \sqrt{M^4 - M^2a^2 - M^2Q^2} - Q^2\right], \quad (10)$$

and the angular velocity is

$$\Omega_{H+} = \frac{a}{2M^2 + \sqrt{4M^4 - 4M^2a^2 - 4M^2Q^2} - Q^2}. \quad (11)$$

The surface gravity of the outer horizon is defined by

$$\kappa\xi^\mu = \xi^\nu \nabla_\nu \xi^\mu\ |_{\{r=r_{H+}\}}, \quad (12)$$

with ξ^μ is the Killing vector of the metric at Eddington-Finklestein coordinates. In this case, the Killing vector has the form $\xi^\mu = (1,0,0,\Omega_{H+})$, and by substituting the Killing vector into the equation, the surface gravity will be obtained on the outer horizon as

$$\kappa = \frac{\sqrt{M^2 - a^2 - Q^2}}{r_{H+}^2 + a^2}\,. \quad (13)$$

3. Penrose Process

Penrose process is extracting energy from rotating or charged or both black holes. Energy extraction is possible because the rotational and electric potential energy of a black hole are not located inside the event horizon but in the ergosphere. In this process, matter enters the black hole's ergosphere and is divided into two parts after entering. The momentum can be established in such a way that one piece of matter comes out of the black hole while the other goes into the event horizon of the black hole. By proper means, matter out of the ergosphere has greater mass-energy than originated. The piece entering the event horizon possible to possess negative mass energy. Therefore, this process results in more energy from the black hole itself. This process slightly reduces the black hole's charge and angular momentum, turning it into the extracted energy.

Changes in energy of the black holes will affect mass, angular momentum, charge, and surface area given by

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_{H+}\delta J + \Phi_{H+}\delta Q. \tag{14}$$

However, not all the black hole masses can be extracted. There is a limit to the mass that can be extracted. After reaching these limits, the energy extraction process will stop. This remaining non-extractable mass is referred to as the irreducible mass [1]. The irreducible mass for the Kerr-Newman black hole is

$$M_{irr} = \frac{1}{2}\sqrt{\left[M + \sqrt{M^2 - Q^2 - a^2}\right]^2 + a^2} \tag{15}$$

and the extracted total mass is

$$M_{ext} = M - M_{irr} = M - \frac{1}{2}\sqrt{\left[M + \sqrt{M^2 - Q^2 - a^2}\right]^2 + a^2}. \tag{16}$$

The efficiency of the Penrose process is the ratio between the extractable mass to its total mass

$$\eta = \frac{M_{ext}}{M} = 1 - \frac{1}{2M}\sqrt{\left[M + \sqrt{M^2 - Q^2 - a^2}\right]^2 + a^2} \tag{17}$$

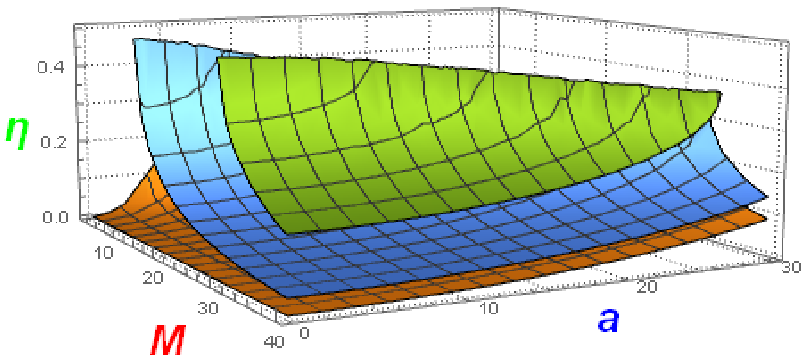

Figure 1. Efficiency of kerr-Newman black hole energy extraction as a function of M and a with $Q = 0$ (orange), $Q = 15$ (blue) and $Q = 30$ (green) units of length.

In figure 1, there are certain restrictions for black holes to extract energy. The black hole can not be extracted fully. The maximum efficiency of Penrose is 0.5. The reason is that maximum efficiency is achieved when all three parameters of a black hole are exactly almost extreme. In contrast, according to the third law of thermodynamics of a black hole, it is unlikely that a black hole can be extreme. In the Schwarzschild black hole, $a = Q = 0$, the efficiency is zero, which indicates that the entire mass from the black hole cannot be extracted. Meanwhile, the energy can still be extracted even though the Reissner-Nordstrom black hole does not have an ergosphere. This process occurs in the generalized ergosphere [9].

The efficiency of rotating and charged black holes (Kerr-Newman) is larger than Kerr black holes, with an efficiency of 0.29 with the maximum energy gained is 0,2 from its mass[11], since the electrostatic charges will store electrical energy, and this electrical energy can be extracted from black holes through the Penrose process.

4. Black Hole Thermodynamics

When the massiveness of a black hole changes, it turns out that the relationship between the massive cities of the transformed black hole has a shape similar to the relation in thermodynamics. Deep energy in thermodynamics is analogous to the mass of a black hole, temperature in thermodynamics is analog with surface gravity of a black hole, while entropy in thermodynamics is analogous to surface area.

4.1. Zeroth Law

Analogous to the zeroth law of thermodynamics, the zeroth law of thermodynamics of a black hole states that the surface gravity of the black hole will be of constant value along its event horizon. This states that the event horizon of a black hole will always be spherical, since that the gravitational value of the surface will depend on the radial distance of the singularity. Regardless of the type and parameters of the black hole, the topology of the event horizon will take the form of a sphere.

4.2. First Law

The change in energy will be related to a change in the surface area of the event horizon, a change in its angular momentum and a change in electric charge. This first law of thermodynamics of black holes is described by equations (14). The Penrose process is one of the thermodynamic processes that tries to expend energy in a black hole which causes a decrease in the angular momentum and charge of the black hole. The temperature T of the black hole and the entropy S of the black hole are successively expressed in the form of

$$T = \frac{\kappa}{2\pi} \tag{18}$$

$$S = \frac{A}{4}.$$

4.3. Second Law

The second law of thermodynamics explains the nature of entropy. The entropy of the system can only be ascending or constant. The entropy of the system will not be able to decrease. In black hole mechanics, the entropy of a black hole is described by Bekenstein's entropy whose value is proportional to the surface area of the black hole horizon. The relation between the surface area of the horizon *and* the irreducible mass is given by

$$A = 16\pi\, M_{irr}^2\ . \tag{19}$$

Reviewing the irreducible mass at (16) shows that its value is always fixed or increasing. This indicates that its surface area can also only remain or increase. This law is Hawking's law for the extent of black holes. The second legal requirement becomes the maximum limit of the Penrose process for performing energy extraction from black holes.

4.4. Third Law

The third law is based on the cosmic censorship hypothesis, where a black hole can not be extremal. The extreme conditions are achieved when the zeros of Δ have imaginary value. It states that black holes have no real horizons. This condition also limits the increase in efficiency of the Penrose process to more than 50%. Since black holes do not have a real horizon, the magnitudes associated with the horizon disappear, such as surface gravity, angular velocity, and surface area. Explicitly, this law states that it is impossible in the finite process to make a black hole reach extreme conditions.

5. Conclusion

Kerr-Newman black holes can be derived through the Newman-Janis algorithm, which uses a Reissner-Nordstrom metric as a metric seed in Eddington-Finklestein coordinates. The algorithmic process results in a Kerr-Newman metric in Eddington-Finklestein coordinates. However, a transformation can be performed to get the metric in the Boyer-Lindquist coordinates. The Kerr-Newman black hole has three parameters: mass, angular momentum, and electric charge. With some modification, the Newman-Janis algorithm can generate another rotating metric: Kerr-(A)dS, Kerr-NUT, or rotating de Sitter using a specific seed metric. All three parameters of a black hole are subject to change, whose changes are governed by the four laws of thermodynamics of a black hole.

When a black hole is extracted its energy through the Penrose process, it is seen that not all the energy from the black hole can be extracted. Certain limitations will transform the Kerr-Newman black hole into a Schwarzschild static black hole. The static approximation will lose the angular momentum and charge of the Kerr Newman black holes. However, it has been proven that the efficiency of the Penrose process in the Kerr-Newman black hole is more significant than that in the Kerr black hole.

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