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PAPER

Universal Heisenberg uncertainty relations for all massless particles

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Abstract

We derive the uncertainty relations between position and momentum (or wavector) in a uniform way for all massless particles. As expected, the lower bound $\gamma_{\rm min} = (1+\sqrt{1/4+2h})$ in the uncertainty relation grows with the particle's helicity h.

1. Introduction

The uncertainty relation for positions and momenta in nonrelativistic quantum mechanics can be stated as a mathematical theorem. It was rigorously proven by Hermann Weyl [1] right after the discovery of quantum mechanics. The uncertainty relations in relativistic quantum mechanics does not have any commonly accepted form. This is due primarily to the absence in relativistic quantum mechanics of the position operator with the full set of desired properties.

Quantum mechanics of massless particles differs so profoundly from that of massive particles that it is not obvious whether nonrelativistic properties such as the Heisenberg uncertainty relations still characterize the behavior of these ultrarelativistic objects. In our previous publications we derived such relations for photons using the second moment of the energy density [2] or the center of energy operator [3, 4]. In the present work we use a different method to define the dispersion of position which can be used for *all* massless particles. This method relies on a positive density, constructed from the particle's wave function, to quantify the spread of the wave function in position space. We previously applied this method to derive the Heisenberg uncertainty relations for massive particles [5, 6]. The application of this method to *all* massless particles completes our study of the Heisenberg uncertainty relations in relativistic quantum mechanics.

It may seem that the study of the uncertainty relation for photons is a purely theoretical activity. However, recent advances in nanotechnology in the field of photonics [7–9] may make the uncertainty relations for photons relevant in experiments. In the Supplemental Material to [7] the authors specifically refer to the role that our uncertainty relations for photons play in their experiments. We believe that in view of these experiments such statements as 'A photon was where the photodetector detected it' [10, 11] may no longer be adequate to describe all subtleties of the concept of the photon position and the corresponding uncertainty relations. In particular, one might be able to distinguish in experiments which lower bound in the photon uncertainty relation better fits the experimental data. Is it the value $4\hbar$ found in [2], the value $(1+\sqrt{5}/2)\hbar$ found in [3], or the value $5/2\hbar$ found in the present work? Of course there is no contradiction between these different values. They are different because there were obtained using 3 different measures of the uncertainty Δr in position. There is no such ambiguity in the uncertainty Δk in momentum (or wavevector). Every solution of the wave equation can be expanded into plane waves and their amplitude $f(\boldsymbol{k})$ is used to define the variance in $\boldsymbol{k}$.

2. The formulation of the uncertainty relations for massless particles

Formulating the Heisenberg uncertainty relation for relativistic particles faces a serious challenge. The position operator defined as a multiplication of the wave function by $\boldsymbol{r}$ in the nonrelativistic theory, cannot be accepted in relativistic quantum mechanics. For example, consider the electron. Its wave function is a solution of the Dirac equation with *positive* energy. The multiplication of this function by $\boldsymbol{r}$ leads to a function that no longer satisfies the positive-energy condition; it acquires a negative-energy component. One remedy is to replace the multiplication by $\boldsymbol{r}$ with the operator of the center of energy. Every relativistic theory must have such an operator because it is the generator of Lorentz boosts. We employed this operator in our previous publication [3] to derive the uncertainty relations for photons. However, one may argue that the center of the energy operator is not a good replacement for the position operator, since its components do not commute. In this work we use a different approach based on the probability densities built from the wave functions of massless particles.

We first consider the helicity values $h \neq 0$. The case of $h = 0$ is exceptional and will be discussed at the end. We trust that the use of h for helicity, will not cause confusion. The Planck constant will never appear in our calculations since we use the wave vector $\boldsymbol{k}$ instead of the momentum vector $\boldsymbol{p}$. Consequently, the right-hand-side of the uncertainty relation is a pure number.

We employ the spinorial formalism, which is particularly well suited for the description of the quantum mechanics of massless particles. In our notation we follow Corson [12]. All spinor indices $A, B, \ldots$ and the conjugate spinor indices $\dot{A}, \dot{B}, \ldots$ take on two values. In writing down the wave equations we need the spin four-vectors $g^{\mu\dot{A}B}$. These consist of the unit matrix I and the Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$, all treated as 2×2 matrices with the indices $\dot{A}B$,

$$g^{0\dot{A}B} = I,\ g^{1\dot{A}B} = \sigma_x,\ g^{2\dot{A}B} = \sigma_y,\ g^{3\dot{A}B} = \sigma_z. \tag{1}$$

According to the theory of the unitary representations of the Poincaré group [13], for every value of the helicity $h = 1/2, 1, 3/2, 2\ldots$ there are two irreducible representations. They describe massless particles with helicity $\pm h$. For all values of the helicity the wave equations have the same form [14],

$$\text{Positive helicity}\quad g^{\mu\dot{A}A}\partial_\mu \phi_{AB\ldots Z}(\boldsymbol{r}, t) = 0, \tag{2a}$$

$$\text{Negative helicity}\quad g^{\mu A\dot{A}}\partial_\mu \phi_{\dot{A}\dot{B}\ldots\dot{Z}}(\boldsymbol{r}, t) = 0, \tag{2b}$$

where $\phi_{AB\ldots Z}(\boldsymbol{r}, t)$ and $\phi_{\dot{A}\dot{B}\ldots\dot{Z}}(\boldsymbol{r}, t)$ are spinorial wave functions symmetric in all $2h$ indices. The summation convention over repeated indices μ, A and $\dot{A}$ is assumed.

In what follows, we consider only positive helicities. In the three cases realized in Nature, the wave equations are,

$$\text{Neutrino}\quad g^{\mu\dot{A}A}\partial_\mu \phi_A(\boldsymbol{r}, t) = 0, \tag{3a}$$

$$\text{Photon}\quad g^{\mu\dot{A}A}\partial_\mu \phi_{AB}(\boldsymbol{r}, t) = 0, \tag{3b}$$

$$\text{Graviton}\quad g^{\mu\dot{A}A}\partial_\mu \phi_{ABCD}(\boldsymbol{r}, t) = 0. \tag{3c}$$

So far, massless particles with $h = 3/2$ and larger have not been found.

The solutions of the wave equations (3) can be expanded into plane waves,

$$\phi_{AB\ldots Z}(\boldsymbol{r}, t) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 k}{k} \kappa_A \kappa_B \ldots \kappa_Z f(\boldsymbol{k})\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r} - \mathrm{i}\omega t}, \tag{4}$$

where κ_A is the spinor associated with the null wave vector k^μ. In spherical coordinates, we have,

$$\boldsymbol{k} = |\boldsymbol{k}| (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta), \tag{5}$$

$$\kappa_A = \sqrt{|\boldsymbol{k}|} \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\varphi/2} \cos(\theta/2) \\ \mathrm{e}^{\mathrm{i}\varphi/2} \sin(\theta/2) \end{bmatrix}, \tag{6}$$

$$\kappa_{\dot{A}} g^{\mu\dot{A}B} \kappa_B = k^\mu. \tag{7}$$

The expression (4) satisfies the wave equation (2a) because $g^{\mu\dot{A}A} k_\mu \kappa_A = 0$.

With every wave function $\phi_{AB\dots Z}(\boldsymbol{r},t)$ and its complex conjugate $\phi_{\dot{A}\dot{B}\dots\dot{Z}}(\boldsymbol{r},t)$, we can associate the conserved symmetric tensor field $P^{\mu\nu\dots\zeta}(\boldsymbol{r},t)$,

$$P^{\mu\nu\dots\zeta}(\boldsymbol{r},t)=\phi_{\dot{A}\dot{B}\dots\dot{Z}}(\boldsymbol{r},t)\,g^{\mu\dot{A}A}g^{\nu\dot{B}B}\dots g^{\zeta\dot{Z}Z}\phi_{AB\dots Z}(\boldsymbol{r},t)\,, \tag{8}$$

$$\partial_\mu P^{\mu\nu\dots\zeta}(\boldsymbol{r},t)=0. \tag{9}$$

For each value of helicity h the tensor $P^{\mu\nu\dots\zeta}(\boldsymbol{r},t)$ has a different physical interpretation. For neutrinos it is the probability current,

$$j^\mu(\boldsymbol{r},t)=\phi_{\dot{A}}(\boldsymbol{r},t)\,g^{\mu\dot{A}A}\phi_A(\boldsymbol{r},t)\,. \tag{10}$$

For photons it is the energy-momentum tensor,

$$T^{\mu\nu}(\boldsymbol{r},t)=\phi_{\dot{A}\dot{B}}(\boldsymbol{r},t)\,g^{\mu\dot{A}A}g^{\nu\dot{B}B}\phi_{AB}(\boldsymbol{r},t)\,. \tag{11}$$

For gravitons it is the linearized super-energy Bel–Robinson tensor [15],

$$T^{\mu\nu\lambda\rho}(\boldsymbol{r},t)=\phi_{\dot{A}\dot{B}\dot{C}\dot{D}}(\boldsymbol{r},t)\,g^{\mu\dot{A}A}g^{\nu\dot{B}B}g^{\mu\dot{C}C}g^{\nu\dot{D}D}\phi_{ABCD}(\boldsymbol{r},t)\,. \tag{12}$$

In all cases the time component $\rho_r(\boldsymbol{r},t)=P^{00\dots0}(\boldsymbol{r},t)$ of this tensor is positive definite because it is just the modulus squared of the spinorial wave function,

$$\rho_r(\boldsymbol{r},t)=\sum_A\sum_B\cdots\sum_Z|\phi_{AB\dots Z}(\boldsymbol{r},t)|^2. \tag{13}$$

Thus, $\rho_r(\boldsymbol{r},t)$ may serve as a natural generalization of the standard probability density; it describes the distribution of probability in space. Of course, all freely evolving wave packets expand in time. Searching for the lower bound in the uncertainty relations, we choose the origin of the time variable in such a way that at $t=0$ the extension of the wave packet is the smallest.

The probability distribution in momentum space $\rho_k(\boldsymbol{k})$, the counterpart of $\rho_r(\boldsymbol{r})$, is constructed from the Fourier transform $\tilde{\phi}_{AB\dots Z}(\boldsymbol{k})$ of the wave function (4), in full correspondence with the probability distribution in position space,

$$\tilde{\phi}_{AB\dots Z}(\boldsymbol{k})=\frac{1}{k}\kappa_A\kappa_B\dots\kappa_Z f(\boldsymbol{k})\,, \tag{14}$$

$$\rho_k(\boldsymbol{k})=\sum_A\sum_B\dots\sum_Z|\tilde{\phi}_{AB\dots Z}(\boldsymbol{k})|^2=k^{2h-2}|f(\boldsymbol{k})|^2. \tag{15}$$

The sum over indices was evaluated using $\sum_A|\kappa_A|^2=k$. The probabilistic interpretation of the $\rho's$ requires that the norm N of the wave function,

$$N=\int \mathrm{d}^3r\,\rho_r(\boldsymbol{r},t)=\int \mathrm{d}^3k\,\rho_k(\boldsymbol{k})\,, \tag{16}$$

be equal to 1. However, to avoid cumbersome Lagrange multipliers in the variational calculations, we shall not impose the normalization condition on the wave functions. Instead, we define the dispersions as follows,

$$\Delta r^2=\frac{1}{N}\int \mathrm{d}^3r\,\boldsymbol{r}^2\rho_r(\boldsymbol{r})\,, \tag{17a}$$

$$\Delta k^2=\frac{1}{N}\int \mathrm{d}^3k\,\boldsymbol{k}^2\rho_k(\boldsymbol{k})\,. \tag{17b}$$

In these definitions we assumed that the coordinate systems are chosen such that the average values of $\boldsymbol{r}$ and $\boldsymbol{k}$ vanish.

To obtain the Heisenberg uncertainty relation, one must find the lowest value $\gamma_{\min}$ of $\gamma=\Delta r\Delta k$ which will give the uncertainty relation,

$$\Delta r\Delta k\geqslant\gamma_{\min}, \tag{18}$$

where $\Delta r=\sqrt{\Delta r^2}$ and $\Delta k=\sqrt{\Delta k^2}$.

3. Variational calculation

We find the lower bound $\gamma_{\rm min}$ by applying the variational method. Using spherical coordinates, we obtain,

$$\Delta r^2 = \frac{1}{N}\int_0^\infty \mathrm{d}k\, k^{2h-2}\int_0^\pi \mathrm{d}\theta \sin\theta \int_0^{2\pi}\mathrm{d}\varphi f^*(k,\theta,\varphi)(-k^2\partial_k^2 - 2hk\partial_k - \partial_\theta^2 - \cot\theta\,\partial_\theta$$
$$+\csc^2\theta(h^2 - \partial_\varphi^2 + 2\mathrm{i}h\cos\theta\,\partial_\varphi) - 2h(h-1))f(k,\theta,\varphi), \tag{19}$$

$$\Delta k^2 = \frac{1}{N}\int_0^\infty \mathrm{d}k\, k^{2h+2}\int_0^\pi \mathrm{d}\theta \sin\theta \int_0^{2\pi}\mathrm{d}\varphi f^*(k,\theta,\varphi) f(k,\theta,\varphi), \tag{20}$$

$$N = \int_0^\infty \mathrm{d}k\, k^{2h}\int_0^\pi \mathrm{d}\theta \sin\theta \int_0^{2\pi}\mathrm{d}\varphi f^*(k,\theta,\varphi) f(k,\theta,\varphi). \tag{21}$$

The expression for Δr^2 in terms of $f(\boldsymbol{k})$ was obtained with the use of the formula,

$$\boldsymbol{r}^2\phi_{AB\ldots Z}(\boldsymbol{r},0) = -\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\times\left(\partial_k^2 + \frac{2}{k}\partial_k + \frac{1}{k^2}\left(\frac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2\right)\right)\left(\frac{\kappa_A\kappa_B\ldots\kappa_Z}{k}f(\boldsymbol{k})\right). \tag{22}$$

Varying $\Delta r^2\Delta k^2$ with respect to $f^*(k,\theta,\varphi)$ yields the following partial differential equation for $f(k,\theta,\varphi)$:

$$\left[-\partial_k^2 - \frac{2h}{k}\partial_k + \frac{(k^2\Delta r^2 - 2\gamma^2)}{\Delta k^2} - \frac{\partial_\theta^2 + \cot\theta\,\partial_\theta - \csc^2\theta\left(h^2 - \partial_\varphi^2 + 2\mathrm{i}h\cos\theta\,\partial_\varphi\right) + 2h(h-1)}{k^2}\right]f(k,\theta,\varphi)$$
$$= 0. \tag{23}$$

To determine the lowest value of γ we solve the equation (23) assuming a product form $f(k,\theta,\varphi) = \Upsilon(k)\Theta(\theta,\varphi)$ and we obtain the following two equations:

$$\left(-\partial_\theta^2 - \cot\theta\,\partial_\theta + \csc^2\theta\left(h^2 - \partial_\varphi^2 + 2\mathrm{i}h\cos\theta\,\partial_\varphi\right)\right)\Theta(\theta,\varphi) = j(j+1)\Theta(\theta,\varphi), \tag{24}$$

$$\frac{1}{2}\left(-\partial_\kappa^2 - \frac{2h}{\kappa}\partial_\kappa + \frac{j(j+1) - 2h(h-1)}{\kappa^2} + \kappa^2\right)\Upsilon(\kappa) = \gamma\Upsilon(\kappa), \tag{25}$$

where $\kappa = \sqrt{\gamma}k/\Delta k = k\sqrt{\Delta r/\Delta k}$ is the dimensionless length of the wave vector. The solutions of the first equation were found by Tamm [16] in his study of magnetic monopoles. These solutions can be expressed in terms of Jacobi polynomials $P_n^{(\alpha,\beta)}$,

$$\Theta(\theta,\varphi) = \mathrm{e}^{\mathrm{i}m\varphi}\sin^h\theta\cot^m(\theta/2)P_{j-h}^{(h-m,h+m)}(\cos\theta). \tag{26}$$

Normalizable solutions of (24) exist for $j\geqslant h$ and $j\geqslant m$.

Equation (25) is the radial Schrödinger equation for the harmonic oscillator in $2h$ dimensions. We are interested in the lowest eigenvalue γ for a given value of h. The lowest value of γ occurs for the lowest centrifugal barrier, i.e. when the value of j is the smallest possible. Therefore we set $j = h$ and we obtain the following ground state wave function for the Hamiltonian represented here by the differential equation (25),

$$\Upsilon_g(\kappa) = \kappa^{\left(1/2+\sqrt{1/4+2h}-h\right)}\mathrm{e}^{-\kappa^2/2}, \tag{27}$$

which corresponds to the following value of γ,

$$\gamma_{\rm min} = 1 + \sqrt{1/4 + 2h}. \tag{28}$$

This value for $h = 1/2$ agrees with the uncertainty relation for electrons [5] in the ultra relativistic regime when the mass term can be neglected. Note that, as in the nonrelativistic case, it is not just one wave function which saturates the inequality (18) but the whole collection of self-similar functions which differ by the values of the scaling parameter $\sqrt{\Delta r/\Delta k}$.

The wave function (27) in momentum space has the Gaussian form, as in the nonrelativistic case. However, the corresponding wave functions in position space $\phi_{gAB\ldots Z}$ have the power law decrease for large r. To illustrate this behavior, we will study the ground state wave function ϕ_{gAB} for photons in the position space ($c = 1$),

$$\phi_{gAB}(\boldsymbol{r},t) = \frac{1}{(2\pi)^{3/2}}\int\frac{\mathrm{d}^3k}{k}\,\kappa_A\kappa_B f_g(k)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}-\mathrm{i}kt}. \tag{29}$$

In this formula we will use the ground state wave function (27) as a function of κ,

$$f_g(\kappa) = \kappa\mathrm{e}^{-\kappa^2/2}. \tag{30}$$

The calculation starts from the conversion of the products of spinors κ_A to Cartesian coordinates,

$$\kappa_1\kappa_1 = \frac{1}{2\sqrt{k_x^2+k_y^2}}\left((k+k_z)\left(k_x-\mathrm{i}k_y\right)\right), \tag{31a}$$

$$\kappa_1\kappa_2 = \frac{1}{2\sqrt{k_x^2+k_y^2}}\left(k_x^2+k_y^2\right), \tag{31b}$$

$$\kappa_2\kappa_2 = \frac{1}{2\sqrt{k_x^2+k_y^2}}\left((k-k_z)\left(k_x+\mathrm{i}k_y\right)\right). \tag{31c}$$

When these formulas are inserted in the integrand of (29), we can replace the components of the wave vector by the corresponding derivatives of the integral. As a result, we have to evaluate only one integral,

$$\tilde{\Upsilon}_g(\boldsymbol{r},t) = \frac{1}{(2\pi)^{3/2}}\int\frac{\mathrm{d}^3k}{2k\sqrt{k_x^2+k_y^2}}f_g(\kappa)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}-\mathrm{i}kt}. \tag{32}$$

Second derivatives of $\tilde{\Upsilon}_g(\boldsymbol{r},t)$ with respect to x, y, z and t give the components of the photon wave function $\phi_{AB}(\boldsymbol{r},t)$ for the ground state,

$$\phi_{11}(\boldsymbol{r},t) = (\partial_t-\partial_z)\left(\partial_x-\mathrm{i}\partial_y\right)\tilde{\Upsilon}_g(\boldsymbol{r},t), \tag{33a}$$

$$\phi_{12}(\boldsymbol{r},t) = -\left(\partial_x^2+\partial_y^2\right)\tilde{\Upsilon}_g(\boldsymbol{r},t), \tag{33b}$$

$$\phi_{22}(\boldsymbol{r},t) = (\partial_t+\partial_z)\left(\partial_x+\mathrm{i}\partial_y\right)\tilde{\Upsilon}_g(\boldsymbol{r},t). \tag{33c}$$

The integral (32) cannot be expressed in terms of known functions. However, we can evaluate its approximation for large distances. To this end we write down the integral (32) for photons in spherical coordinates,

$$\tilde{\Upsilon}_g(\boldsymbol{r},t) = \int_0^{\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\int_0^{\infty}\mathrm{d}\kappa\,\kappa\,\mathrm{e}^{\mathrm{i}a\kappa}\mathrm{e}^{-\kappa^2/2}, \tag{34}$$

$$a = \Delta k((x\cos\phi+y\sin\phi)\sin\theta+z\cos\theta-t). \tag{35}$$

The integral over κ can be evaluated in a closed form,

$$\int_0^{\infty}\mathrm{d}\kappa\exp(\mathrm{i}a\kappa)\,\kappa\mathrm{e}^{-\kappa^2/2} = 1+\mathrm{i}a\mathrm{e}^{-a^2/2}\sqrt{\frac{\pi}{2}}\left(1+\mathrm{erf}\left(\mathrm{i}a/\sqrt{2}\right)\right), \tag{36}$$

where erf(z) is the error function. This function decreases as $-a^{-2}$ for large values of a. Therefore, unlike in the nonrelativistic case, the function $\tilde{\Upsilon}_{\mathrm{ph}}$ does not decrease exponentially. Of course the photon wave function (29) decreases as r^{-4} due to the second derivatives implied by the relations (33).

Even tough our spinorial formalism cannot be used for spinless particles, in this case we can still use the probability current,

$$j_\mu(\mathbf{r},t) = \mathrm{i}/2\left(\phi^*(\mathbf{r},t)\,\partial_\mu\phi(\mathbf{r},t)-\phi(\mathbf{r},t)\,\partial_\mu\phi^*(\mathbf{r},t)\right), \tag{37}$$

which can be used to formulate the uncertainty principle. The time component of this four-vector is the probability density,

$$\rho(\mathbf{r},t) = \mathrm{i}/2\left(\phi^*(\mathbf{r},t)\,\partial_t\phi(\mathbf{r},t) - \phi(\mathbf{r},t)\,\partial_t\phi^*(\mathbf{r},t)\right). \tag{38}$$

In this case, there are no complications due to helicity and the counterpart of (25) is,

$$\frac{1}{2}\left(-\partial_\kappa^2 - \frac{2h}{\kappa}\partial_\kappa + \frac{j(j+1)}{\kappa^2} + \kappa^2\right)\Upsilon(\kappa) \quad = \gamma\Upsilon(\kappa). \tag{39}$$

Therefore, our formula (28) holds also for spinless particles and it gives the same result $\gamma_{\min} = 3/2$ as in nonrelativistic quantum mechanics.

4. Conclusions

One might have expected that the lower bound $\gamma_{\min}$ in the uncertainty relation (18) for massless particles increases with the increase of the particle's helicity h. Indeed, the exact value of $\gamma_{\min} = (1 + \sqrt{1/4 + 2h})$ obtained in this work, which is valid for all massless particles, fully confirms this expectation.

The replacement of the wave vector $\boldsymbol{k}$ by momentum $\boldsymbol{p} = \hbar\boldsymbol{k}$ converts the dimensionless uncertainty relation (18) into its quantum-mechanical form,

$$\Delta r \Delta p \geqslant \hbar\gamma_{\min}. \tag{40}$$

We have avoided introducing the Planck constant in our formulation, to stress the wave nature of massless particles. The particle properties of massless particles do not play a role in the uncertainty relations. These relations are simply the consequences of the wave properties.

For photons, we confirm that the wave function does not exhibit the Gaussian fall-off in position space, unlike in nonrelativistic quantum mechanics. This was expected since it has been proven [17] that for photons the fastest possible decrease of the wave function with r is sub-exponential: $\exp(-(r/l)^\alpha)$, $\alpha < 1$.

For relativistic massive particles there is no dimensionless uncertainty relation since the conformal symmetry is broken. The Compton wavelength $\hbar/mc$ introduces a scale and the uncertainty relation depends on the particle's mass (see equation (26) in [5]).

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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