Quantum motion algebra

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Abstract. A notion of Quantum Motion Algebra (QMA) allows to construct quantum state spaces for various physical systems moving under a given group of motion. The main idea of QMA is a construction of a group algebra with involution generated by a group of motion G. After defining a linear and nonnegative functional on this algebra one can construct the appropriate quantum state space by means of the Gelfand-Naimark-Segal theorem. The QMA method can be also applied in the modeling of physical systems requiring additional degrees of freedom or additional constraints.

The presented paper gives a brief description of the QMA method. As an example of the QMA application, we present a model of nuclear collective pairing where the nonnegative functional is generated by a temperature dependent quantum density operator.

1. Introduction

Every quantum system can be associated with a group of characteristic motions. In this paper, such groups are understood as Lie groups which form the configuration spaces of the corresponding physical systems. For example, for the rotational motion, the characteristic group is the SO(3) group, for the relativistic type of motion the smallest group of motions seems to be the Poincare group. Such groups not only supply required degrees of freedom but also they are equipped with some useful algebraic structures and mathematical tools. It seems to be interesting and valuable to use this natural connection among physical models and characteristic groups of motions to construct the appropriate configuration spaces. An interesting purpose of the QMA approach is to analyze such constructions.

This paper illustrates some basic principles of QMA constructions. As an example, the collective pairing model is considered. This example shows quite new features of the QMA formalism. For instance, the QMA approach allows for the construction of continuous and discrete quantum wave packets simulating both, wave and particle nature of the system under consideration. Every single element of QMA can be understood as quantum interference of a point and a field like object.

The QMA allows for the unique construction of the quantum state space of the physical system. It is generated by a positively defined function on the group of motions. This function represents some elementary amplitudes among points of the configuration space. Finally, it determines a scalar product in the state space of the quantum system. In addition, this inner product can be dependent on some additional physical parameters describing the system, e.g., temperature, shape deformation, and others. This property opens new possibilities for applications of the QMA formalism.

2. Quantum motion algebra

The Gelfand-Naimark-Segal (GNS) construction can be applied to any linear algebra with involution. In our case we deal with the Banach algebra with involution $\mathcal{A}$. The required positive linear functional $\rho : \mathcal{A} \to \mathbb{C}$ is uniquely determined by a positive function on the group of motion. It is an important ingredient of a physical model [1, 2]. It allows to use the GNS theorem to construct the Hilbert space $\mathcal{H}_\rho$ for a given quantum system. In addition, one gets a representation ϕ_ρ of the algebra $\mathcal{A}$ in a form of bounded linear operators on this Hilbert space, i.e., $\phi_\rho : \mathcal{A} \to B(\mathcal{H}_\rho)$.

The support of the constructed Hilbert space H_ρ is identified with the quotient space $\mathcal{A}/I_\rho$, where

$$I_\rho = \{a \in \mathcal{A} \,:\, \rho(a^* a) = 0\} \tag{1}$$

is a closed left ideal. This quotient space is completed by all Cauchy sequences with respect of the norm generated by the following inner product

$$\forall_{a,b\in\mathcal{A}} \quad \langle [a]_\rho | [b]_\rho \rangle = \rho(b^* a)\,. \tag{2}$$

The action of the algebra on these equivalence classes is represented by the left shift operators

$$\forall_{b\in\mathcal{A}} \quad \phi_\rho(a)[b]_\rho = [ab]_\rho\,. \tag{3}$$

The cyclic vector ξ_ρ for which $\phi_\rho(\mathcal{A})\xi_\rho$ is dense in $\mathcal{H}_\rho$ is represented by the unit element of the algebra. In the case if the algebra does not contain a unit element then ξ_ρ is equal to the approximate unit.

Let G be a locally compact Lie group and $d\mu(g)$ be a left invariant Haar measure on this group. The required convolutive algebra composes of the following pairs of functions on the group G

$$S = f(g) + \check{v}(g_n)\,, \tag{4}$$

where $f \in L^1(\mathrm{G}, d\mu(g))$ and $\check{v} \in l^1(\mathrm{G})$. In general, the function f is not a continuous function but in our terminology we call it a continuous part of the algebra, similarly the $\check{v}$ is called the discrete part, respectively. The additive operation is defined in the standard way

$$S_1 + S_2 := (f_1 + f_2) + (\check{v}_1 + \check{v}_2)\,. \tag{5}$$

The multiplication in the algebra is represented by the convolution operation over the group G

$$S_1 \circ S_2 = (f_1 + \check{v}_1) \circ (f_2 + \check{v}_2) := (f_1 \circ f_2) + (f_1 \circ \check{v}_2) + (\check{v}_1 \circ f_2) + (\check{v}_1 \circ \check{v}_2)\,, \tag{6}$$

where

$$(f_1 \circ f_2)(g) := \int_G d\mu(h)\;\; f_1(h)\; f_2(h^{-1}g)\,, \tag{7}$$

$$(\check{v}_1 \circ f_2)(g) := \sum_{h_n \in \mathrm{G}} \check{v}_1(h_n)\; f_2(h_n^{-1}g)\,, \tag{8}$$

$$(f_1 \circ \check{v}_2)(g) := \sum_{h_n \in \mathrm{G}} \Delta_G(h_n^{-1})\; f_1(g h_n^{-1})\; \check{v}_2(h_n)\,, \tag{9}$$

$$(\check{v}_1 \circ \check{v}_2)(g_n) := \sum_{h_n \in \mathrm{G}} \check{v}_1(h_n)\; \check{v}_2(h_n^{-1}g_n)\,. \tag{10}$$

$f_1 \circ f_2,\ \check{v}_1 \circ f_2,\ f_1 \circ \check{v}_2,\ \in L^1(\mathrm{G}, d\mu(g))$ and $\check{v}_1 \circ \check{v}_2 \in l^1(\mathrm{G})$. The $\Delta_G(h)$ is a modular function of the group G.

The involution operation of $S = f + \check{v} \in$ QMA is defined as follows

$$S^{\sharp} := f^{\sharp} + \check{v}^{\sharp}\,, \tag{11}$$

where

$$f^{\sharp}(g) := \Delta_G(g^{-1}) f(g^{-1})^{\star}\,, \tag{12}$$
$$\check{v}^{\,\sharp}(g) := \check{v}(g^{-1})^{\star}\,. \tag{13}$$

The positive linear functional which is needed for the GNS construction, in our case, is generated by a quantum density functional ρ acting in an auxiliary Hilbert space. The value of the functional on any group element is defined by taking the following trace

$$\rho(g) = \langle \rho; g \rangle = \mathrm{Tr}[\rho g]\,, \tag{14}$$
$$\rho(S) = \langle \rho; f + \check{v} \rangle = \int_{\mathrm{G}} d\mu(g) f(g) \langle \rho; g \rangle + \sum_{g_n \in \mathrm{G}} \check{v}(g_n) \langle \rho; g_n \rangle\;. \tag{15}$$

The inner product which comes from the GNS construction has a form

$$\langle S_1 | S_2 \rangle = \left\langle \rho; S_1^{\#} \circ S_2 \right\rangle \tag{16}$$

and the left ideal is as follows

$$I_0 = \left\{ S : \mathrm{Tr}\left[\rho S^{\#} \circ S \right] = 0 \right\}\;. \tag{17}$$

The Hilbert space obtained with the GNS construction represent the quantum state space of a physical system under consideration.

Model of nuclear collective pairing with temperature

Within the QMA formalism we can apply the generator coordinate like method to construct required equations of motion [3, 4, 5]. For simplicity, let us assume that QMA consists only of the continuous part, i.e., $S = f \in \mathrm{QMA}(G)$ ($\check{v} = 0$). Let H be a Hamiltonian of the system on an auxiliary Hilbert space $\mathcal{K}$ describing a cold nucleus with the temperature $T = 0$. Let the operators $T(g) : G \to U(\mathcal{K})$ represent an unitary representation of the group G in $\mathcal{K}$.

As it is well known, the generator coordinate method is based on the following variational principle

$$\delta\left[\langle S|H|S\rangle - E\langle S|S\rangle\right] = 0\,. \tag{18}$$

The solution of this variational principle leads to the so called Griffin-Hill-Wheeler type equation

$$\int dg' \mathrm{Tr}[\rho T(g^{-1}) H T(g')] f(g') = E \int dg' \mathrm{Tr}[\rho T(g^{-1} g')] f(g')\,, \tag{19}$$
$$\int dg' \mathcal{H}(g, g') f(g') = E \int dg' \mathcal{N}(g, g') f(g')\,. \tag{20}$$

We illustrate the above method considering a model of the nuclear collective pairing [6, 7]. Nuclear pairing is a kind of short-range nuclear attractive force that leads to the BCS type correlations in nuclei. As an effect of this pairing the fermionic quasiparticles, the so-called Cooper nucleon pairs, are produced. However, another description can be applied: the rotation about the nucleon number axis and the pairing gap can be considered as two collective variables. These collective variables can be also represented by the bosonic creation annihilation operators.

The standard pairing Hamiltonian written in terms of the creation–annihilation nucleon operators has the following form

$$H_P = -GP^+P\,, \qquad P^+ = \sum_{\nu>0} c_\nu^+ c_{\bar\nu}^+\ , \tag{21}$$

where G =const., ν defines the single particle energies numbered by integers ν, and the operators $c_{\bar\nu}^+$ create the time reversed single–particle nucleon states c_ν^+. It was shown in [8] that the collective treatment of the paring phenomena requires two collective variables (Δ, ϕ). The first one $\Delta \in (0, \infty)$ is interpreted as the pairing energy gap parameter, $\phi \in [0, 2\pi)$ is the gauge angle. By using these two variables one can construct boson creation annihilation operators

$$s_\mu^+ = \frac{1}{2}e^{i\mu\phi}\left[\alpha(\Delta) - \beta(\Delta)\frac{\partial}{\partial\Delta} + \frac{1}{\mu\alpha(\Delta)}\hat\Lambda - F(\Delta)\right]\,, \tag{22}$$

$$s_\mu = \frac{1}{2}e^{-i\mu\phi}\left[\alpha(\Delta) - \beta(\Delta)\frac{\partial}{\partial\Delta} + \frac{1}{\mu\alpha(\Delta)}\hat\Lambda + F(\Delta)\right]\,, \tag{23}$$

$$\mu = \pm 2\,, \tag{24}$$

where $\alpha(\Delta)$ and $\beta(\Delta)$ are arbitrary functions which fulfil the following condition

$$\beta(\Delta)\frac{\partial\alpha}{\partial\Delta} = 1\ . \tag{25}$$

$F(\Delta)$ is defined as follows

$$F(\Delta) = \frac{i}{2}\left[\frac{\beta(\Delta)}{\rho(\Delta)}\frac{\partial\rho}{\partial\Delta} + \frac{\partial\beta}{\partial\Delta} - \frac{1}{\alpha}\right]\,, \tag{26}$$

$$\hat\Lambda = \hat{\mathcal{N}} - \mathcal{N}_0\,, \tag{27}$$

where $\hat{\mathcal{N}}$ is a particle number operator, $\mathcal{N}_0$ is the number of nucleons in a core of the nucleus under consideration. The function $\rho(\Delta)$ is a weight function in the scalar product defined by the integral

$$\langle\Psi|\chi\rangle = \int_0^{2\pi} d\phi \int_0^\infty d\Delta\,\rho(\Delta)\,\Psi(\phi,\Delta)^*\chi(\phi,\Delta)\ . \tag{28}$$

The operators s_μ^+, s_μ can be used to construct the pairing Hamiltonian in the language of boson creation annihilation operators. The group of motion in this case is the noncompact four-dimensional symplectic group $Sp(4,R)$ generated by the bilinear forms $s_\mu^+ s_{\mu'}^+$, $s_\mu s_{\mu'}$, $s_\mu^+ s_{\mu'}$, where $\mu, \mu' = \pm 2$.

In the following we consider only one of possible quantum motions. It is the so called pairing rotation. The corresponding algebra is created by a subgroup of the full group of motion, namely the symmetry group of the Hamiltonian in the collective space. This group is generated by the excess-deficit operator $\hat\Lambda$ expressed in terms of the operators (22,23) in following form:

$$\hat\Lambda = 2\left(s_2^+ s_2 - s_{-2}^+ s_{-2}\right)\,. \tag{29}$$

These gauge transformations form the $SO(2)$ group

$$\mathcal{G}(\phi) = e^{i\phi\hat\Lambda} \quad \text{where} \quad \hat\Lambda = \hat{\mathcal{N}} - \mathcal{N}_0\ , \tag{30}$$

where $\hat{\mathcal{N}}$ is the particle number operator. This transformation can be interpreted as a rotation about the number of particle axis.

The collective pairing boson operators $s^{\pm}_{\mu}$ transform according to the same irreducible representation of the gauge group $SO(2)$ as the fermionic pairing operators P^+ and P

$$\begin{array}{c} \mathcal{G}(\phi)^\dagger s_2^+ \mathcal{G}(\phi) = e^{i2\phi} s_2^+ \\ \mathcal{G}(\phi)^\dagger s_{-2} \mathcal{G}(\phi) = e^{i2\phi} s_{-2} \end{array} \; , \; \mathcal{G}(\phi)^\dagger P^+ \mathcal{G}(\phi) = e^{i2\phi} P^+ \; , \tag{31}$$

$$\begin{array}{c} \mathcal{G}(\phi)^\dagger s_{-2}^+ \mathcal{G}(\phi) = e^{-i2\phi} s_{-2}^+ \\ \mathcal{G}(\phi)^\dagger s_2 \mathcal{G}(\phi) = e^{-i2\phi} s_2 \end{array} \; , \; \mathcal{G}(\phi)^\dagger P \mathcal{G}(\phi) = e^{-i2\phi} P \tag{32}$$

Using this property one can construct the pairing Hamiltonian. It is a two dimensional harmonic oscillator Hamiltonian with an additional term

$$\hat{H} \;=\; h_0 + \epsilon_2 s_2^+ s_2 + \epsilon_{-2} s_{-2}^+ s_{-2} + \kappa \hat{C}^2 \;=\; h_0 + \varepsilon \hat{\mathcal{N}} + \frac{1}{2}\eta \hat{\Lambda} + \kappa \hat{C}^2 \,, \tag{33}$$

where $\epsilon = \frac{1}{2}(\epsilon_2 + \epsilon_{-2})$ is the average single boson energy, $\eta = \frac{1}{2}(\epsilon_2 - \epsilon_{-2})$ is the particle asymmetry parameter and $\hat{C}$ is the Casimir operator of group $SU(1,1)$

$$\hat{C}^2 = \frac{1}{4}\left(\frac{1}{4}\hat{\Lambda}^2 - 1\right) \,. \tag{34}$$

The functional required for the GNS construction is obtained from the density operator representing the canonical ensemble where H is the Hamiltonian of the system and T is temperature of the nucleus

$$\rho = \frac{1}{Z} \exp\left(-\frac{1}{kT}\hat{H}\right) \,. \tag{35}$$

The left ideal in such a case is trivial and consists of only zero algebra element

$$I_0 = \{0\} \,. \tag{36}$$

Using the variational principle the equation of motion is of the following form

$$\int_0^{2\pi} d\phi \; \mathrm{Tr}\left[\mathcal{G}(\phi_2)^\dagger \hat{H} \mathcal{G}(\phi_1)\rho\right] f(\phi_1) = E \int_0^{2\pi} d\phi \; \mathrm{Tr}\left[\mathcal{G}(\phi_2)^\dagger \mathcal{G}(\phi_1)\rho\right] f(\phi_1) \,. \tag{37}$$

The corresponding eigensolutions are

$$f_\lambda(\phi) = e^{-i\lambda\phi}, \quad \lambda - \text{even}, \tag{38}$$

$$E_\lambda = 2\varepsilon \exp\left(-\frac{2}{kT}\varepsilon\right) + \left\{ \begin{array}{ll} h_0 - \frac{\lambda}{2}(\varepsilon - \eta) + \frac{\kappa}{4}\left(\frac{\lambda^2}{4} - 1\right) & \text{for} \quad \lambda < 0 \\ h_0 + \frac{\lambda}{2}(\varepsilon + \eta) + \frac{\kappa}{4}\left(\frac{\lambda^2}{4} - 1\right) & \text{for} \quad \lambda \geq 0 \end{array} \right. . \tag{39}$$

The quantum number λ describes the number of particles exceeding the fixed number $\mathcal{N}_0$. It means, that if lambda is lower than zero the number of particles is less than $\mathcal{N}_0$. The eigenenergies form a parabolic-like shape with respect to λ. They are dependent on temperature of a nucleus under consideration. This term controls the vertical position of eigensolution on the energy plot.

The figures Figure 1 and Figure 2 represent the contribution of pairing rotation to nuclear masses for different mass numbers. Left panels present the plots of isobar masses for the mass number A (for the Figure 1 $A = 156$, for the Figure 2 $A = 164$), N denotes the neutron number,

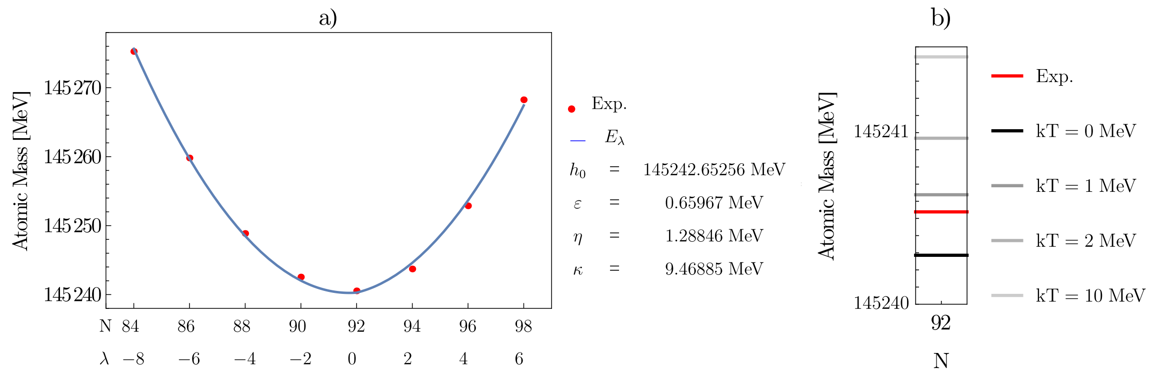

Figure 1. a) Contribution of pairing rotation to nuclear masses, N denotes the neutron number and $\lambda = N - N_0$ for $A = 156$. The red dots represent experimental masses, the blue line shows the eigenvalues E_λ for $T = 0$.
b) Temperature dependence of E_0, i.e., for $\lambda = 0$. The red line represents the experimental nuclear mass of the most stable isobar, the gray lines shows E_0 as a function of temperature kT.

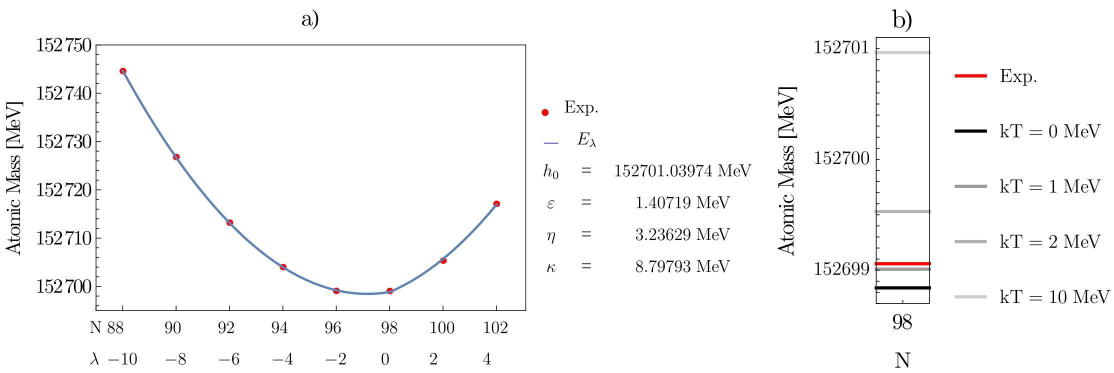

Figure 2. a) Contribution of pairing rotation to nuclear masses, N denotes the neutron number and $\lambda = N - N_0$ for $A = 164$. The red dots represent experimental masses, the blue line shows the eigenvalues E_λ for $T = 0$.
b) Temperature dependence of E_0, i.e., for $\lambda = 0$. The red line represents the experimental nuclear mass of the most stable isobar, the gray lines shows E_0 as a function of temperature kT.

and $\lambda = N - \mathcal{N}_0$ (for Figure 1 $\mathcal{N}_0 = 92$, for Figure 2 $\mathcal{N}_0 = 98$). The red dots are the experimental values. The blue curve comes from our theoretical calculations for zero temperature fitted to the experimental data, by using the least squares method. The presented values of the parameters h_0, ϵ, η, κ come from the fitting procedure.

On the right panel, we present energy for the most stable isobar ($\lambda = 0$). It is dependent on temperature of the nucleus.The e The red line corresponds to known experimental nuclear masses, the gray lines show the masses E_0 for different temperatures.

Conclusions

The QMA is a flexible tool that allows to analyze various types of quantum motions. On the one hand, this flexibility is ensured by a freedom in a choice of the positive function on the group of motions. It defines the inner product. This leads to quantum state spaces suitable for a given physical problem. One can see, that every positive functional imposes some constraints on the quantum motion. In addition, such functionals allow to implement some more realistic physical properties of the system. The presented example shows a temperature dependence of nuclear collective pairing excitation energy. More exactly, we restricted full motion only to the nuclear collective pairing rotation. This method allows for more detailed analysis of complicated quantum motions.

The QMA approach can be also used also a quantization method. This approach was discussed in the following papers [9, 10, 11]. The crucial assumption in this approach is the fact that the configuration space is formed by the parameter space of the group of motions. The quantization procedure is based on a correspondence between the points from the configuration space and a special kind of a quantum projection operators. These operators form a non-orthogonal decomposition of unity in the state space. On the other hand, these operators can be also related to some elementary quantum states interpreted as quantum points of the quantum configuration space. Using this interpretation, one can find a quantum counterpart of any classical observable as the appropriate deformation of the non-orthogonal decomposition of unity.

References

[1] Bing-Ren L 2002 *Introduction to Operator Algebras* (Singapore: World Scientific Publishing Co. Pte. Ltd.)
[2] Pędrak A et al 2019 *J. Phys.: Conf. Ser.* **1194** 012088
[3] Bogusz A and Góźdź A 1992 *J. Phys. A: Math. Gen.* **25** 4613
[4] Góźdź A and Rogatko M 1992 *J. Phys. A Math. Gen.* **25** 625
[5] Góźdź A and Rogatko M 1993 *J. Phys. A Math. Gen.* **26** L221
[6] Góźdź A Pomorski K Brack M and Werner E 1985 *Nuclear Physics* **A442** 50
[7] Góźdź A 1982 *Communications JINR Dubna* E4-82-561
[8] Bes D R et al. 1970 *Nucl. Phys.* **A143** 1
[9] Gazeau J P and Murenzi R 2016 *J. Math. Phys.* **57** 052102
[10] Góźdź A Pędrak A and Piechocki W 2022 *Class. Quantum Grav.* **39** 145005
[11] Góźdź A Pędrak A and Piechocki W 2023 *Eur. Phys. J. C* **83** 150