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The heat flow conjecture for polynomials and random matrices

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Abstract
We study the evolution of the roots of a polynomial of degree N, when the polynomial itself is evolving according to the heat flow. We propose a general conjecture for the large-N limit of this evolution. Specifically, we propose (1) that the log potential of the limiting root distribution should evolve according to a certain first-order, nonlinear PDE, and (2) that the limiting root distribution at a general time should be the push-forward of the initial distribution under a certain explicit transport map. These results should hold for sufficiently small times, that is, until singularities begin to form. We offer three lines of reasoning in support of our conjecture. First, from a random matrix perspective, the conjecture is supported by a deformation theorem for the second moment of the characteristic polynomial of certain random matrix models. Second, from a dynamical systems perspective, the conjecture is supported by the computation of the second derivative of the roots with respect to time, which is formally small before singularities form. Third, from a PDE perspective, the conjecture is supported by the exact PDE satisfied by the log potential of the empirical root distribution of the polynomial, which formally converges to the desired PDE as $N \to \infty$. We also present a "multiplicative" version of the the conjecture, supported by similar arguments. Finally, we verify rigorously that the conjectures hold at the level of the holomorphic moments.

Keywords Roots of polynomials · Heat flow · Random matrices · Calogero-Moser system · Hamilton-Jacobi PDEs

Mathematics Subject Classification 35K05 · 60B20 · 30C15

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1 Introduction

1.1 Heat flow on polynomials

Consider the standard heat equation for a function $u(x, t)$ on the real line,

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2},$$

with a polynomial initial condition,

$$u(x, 0) = p(x).$$

It is easily seen that the solution (computed, say, as the convolution of p with a Gaussian of variance t) can be expressed as a terminating power series,

$$u(x, t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{t}{2}\right)^k \left(\frac{d^2}{dx^2}\right)^k p(x), \quad x \in \mathbb{R},\ t > 0.$$

Note that $u(x, t)$ is a polynomial in x for each t, with degree equal to the degree of p. The roots of this polynomial, however, are not necessarily real, even if the roots of p are real. It is therefore natural to holomorphically extend both the initial condition and the solution in the space variable from $\mathbb{R}$ to $\mathbb{C}$ and consider the function

$$u(z,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{t}{2}\right)^k\left(\frac{d^2}{dz^2}\right)^k p(z),\quad z\in\mathbb{C},\ t>0, \tag{1.1}$$

where d/dz is the complex derivative of a holomorphic function.

Once we have this (terminating) power series representation of the solution, we note that the formula on the right-hand side of (1.1) makes sense with t replaced by an arbitrary complex number. Furthermore, in the analysis of the large-N limit, it is convenient to scale the heat equation in a way that depends on the degree N of the initial condition. We therefore define the time-τ **heat flow** operator $\exp\left(\frac{\tau}{2N}\frac{d^2}{dz^2}\right)$ on polynomials p^N of degree N by the terminating power series

$$\exp\left(\frac{\tau}{2N}\frac{d^2}{dz^2}\right)p^N(z)=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\tau}{2N}\right)^k\left(\frac{d^2}{dz^2}\right)^k p^N(z),\quad \tau\in\mathbb{C}. \tag{1.2}$$

We note the operator d^2/dz^2 appearing in (1.2) is not the Laplacian on the plane (which would be, up to a constant, $\partial^2/\partial z\partial\bar{z}$) but rather the holomorphic extension of the Laplacian d^2/dx^2 on the line. Replacing d^2/dz^2 in (1.2) by $\partial^2/\partial z\partial\bar{z}$ would not give an interesting result, since the holomorphic function $p^N(z)$ is annihilated by $\partial/\partial\bar{z}$ and we would simply get back $p^N(z)$.

If, for example, we take $\tau=-t$, with $t>0$, we obtain the backward heat flow (with a polynomial initial condition). This case is of particular interest because it preserves the class of polynomials with all real roots, by the Pólya–Benz theorem [1, Theorem 1.2]. The asymptotic behavior of real roots under the backward heat flow has a natural free probability interpretation: If the root distribution of a sequence p^N of polynomials approaches a compactly supported probability measure μ, the root distribution of $\exp\left(-\frac{t}{2N}\frac{d^2}{dz^2}\right)p^N$ approaches the free additive convolution $\mu\boxplus\mathsf{sc}_t$, where sc_t is the semicircular measure with variance t. (Free convolution with the semicircular distribution was investigated by Biane [9] as a free version of the heat equation.) See results of Kabluchko [31] and Voit and Woerner [46].

In this paper, we will investigate the behavior of *complex* roots of polynomials under the heat flow. One may think, for example, of starting with a polynomial with real roots and applying the *forward* heat flow ($\tau=t>0$), in which case, the roots will not remain real. (Terry Tao's blog post [43], for example, shows that the roots cannot remain real for all time, except in the case of polynomial of degree 1.) The behavior of complex roots under the heat flow is much more complicated than for real roots under the backward heat flow. We are, nevertheless, able to formulate some precise conjectures about the evolution of complex roots and we find, again, a connection to random matrix theory and free probability. In Sect. 6, we will also study a "multiplicative" version of the heat flow with similar conjectures and connections to random matrix theory.

We could conceivably restrict attention to the forward heat flow, taking τ to be real and positive. But once the roots are allowed to be complex, there is no meaningful differences in behavior among the forward heat flow, the backward heat flow, and the heat flow with a complex-time parameter. Indeed, it makes many of the computations simpler to work with a general complex time-parameter.

The study of roots of polynomials under the heat flow is closely connected to the study of roots of polynomials under repeated differentiation, as in (to mention just a very few of the most directly relevant examples) work of O'Rourke and Steinerberger [40], Feng and Yao [22], Hoskins and Kabluchko [30], and our own paper with Jalowy and Kabluchko [26].

1.2 Motivation from random matrix theory

In this subsection, we motivate the study of roots of polynomials under the heat flow by connecting it to random matrix theory. In recent years, connections have been developed between constructions in free probability or random matrix theory (on the one hand) and operations on polynomials (on the other hand). The first prominent example of such a connection is the work of Marcus, Spielman, and Srivastava [37] on finite free probability. In the additive case, the authors of [37] take an old construction of Walsh [48] taking two polynomials p and q of degree d and associating to it a new polynomial $p \boxplus_d q$. They give it a new interpretation as the polynomial counterpart of the free additive convolution in free probability theory. Indeed, results of Arizmendi and Perales [5, Corollary 5.5] (see also [36, Section 4]) show that as $d \to \infty$, the finite free convolution converges in an appropriate sense to the ordinary free additive convolution.

Another connection between free probability and polynomials is the relationship, discussed in Sect. 1.1, between the backward heat flow on real-rooted polynomials and the free convolution with the semicircular distribution.

Still another such connection comes from repeated differentiation of polynomials with real roots, in which we take $\lfloor Nt \rfloor$ derivatives of a polynomial of degree N, with $0 \leq t < 1$. Work of Hoskins and Kabluchko [30] and Arizmendi, Garza-Vargas, and Perales [4] establish the following result: If a sequence of real-rooted polynomials p^N of degree N have root distribution converging to μ, then the $\lfloor Nt \rfloor$-th derivative of p^N has root distribution converging to a rescaling of $\mu_t = \mu^{1/(1-t)}$, where μ^α, $\alpha \geq 1$, denotes the fractional free convolution introduced by Bercovici–Voiculescu [6] and Nica–Speicher [39].

Meanwhile, Campbell, O'Rourke, and Renfrew [14] have introduced a version of the preceding result for random polynomials with independent coefficients; such polynomials have *complex* roots with an asymptotically radial distribution. Their result also involves a version of fractional convolution, but for R-diagonal operators.

In this paper, we will consider a connection between the heat flow on polynomials with complex roots and something called the "model deformation phenomenon" in random matrix theory and free probability. We now introduce the simplest example of this phenomenon. Consider a family of "elliptic" random matrix models, having the form

$$Z_t^N = \sqrt{1-\frac{t}{2}}\, X^N + i\sqrt{\frac{t}{2}}\, Y^N, \quad 0 \le t \le 2, \tag{1.3}$$

where X^N and Y^N are independent GUE matrices. Then Z_t is a GUE matrix at $t = 0$, a Ginibre matrix at $t = 1$, and a GUE matrix multiplied by i at $t = 2$. For $0 < t < 2$, the limiting eigenvalue distribution of Z_t^N is uniform on an ellipse centered at the origin with semi-axes $2 - t$ and t. At $t = 0$ or $t = 2$, the limiting eigenvalue distribution collapses onto the real or imaginary axis, respectively. If σ_t denote the limiting eigenvalue distribution of Z_t, it is easily seen that σ_t is the push-forward of σ_1 (uniform on the unit disk) under the map

$$T_t(z) = z - (t-1)\bar{z}.$$

We now generalize the elliptic model by adding an independent Hermitian matrix:

$$A_t^N = X_0^N + Z_t^N, \tag{1.4}$$

where X_0^N is Hermitian and independent of Z_t^N. (A more general version of this model is considered in Sect. 2.1.) The distribution of X_0^N can be quite general; we assume only the eigenvalue distribution of X_0^N converges almost surely to a compactly supported probability measure on $\mathbb{R}$. If μ_t denotes the limiting eigenvalue distribution of A_t^N, results of [23] (adapted from the "multiplicative" case to the simpler "additive" case) give the following results.

- For all t with $-1 < t < 1$, the log potential $S(t, z)$ of μ_t satisfies the PDE

$$\frac{\partial S}{\partial t} = \frac{1}{4}\left(\left(\frac{\partial S}{\partial x}\right)^2 - \left(\frac{\partial S}{\partial y}\right)^2\right), \tag{1.5}$$

  except possibly on the boundary of the support of μ_t.
- For all t with $-1 < t < 1$, the measure μ_t is the push-forward of μ_0 under the map T_t given by

$$T_t(z) = z - t m_0(z),$$

  where m_0 is the Cauchy transform of the measure μ_0:

$$m_0(z) = \int_{\mathbb{C}} \frac{1}{z-w}\, d\mu_0(w).$$

The two points above are related as follows: The curves $t \mapsto T_t(z)$ are precisely the characteristic curves of the PDE (1.5). We refer to the second result as "model deformation," meaning that as we deform the random matrix model by changing t, the limiting eigenvalue distribution deforms by push-forward under a certain canonical map. As noted above, the paper [23] gives a multiplicative version of the preceding results. Furthermore, the paper [49] extends the second point to the case in which X_0^N is no longer required to be Hermitian. Finally, a version of model deformation is obtained in [26, Section 6] with connection to repeated applications of differential operators to random polynomials.

Idea 1.1 *The heat flow is the polynomial counterpart of the model deformation phenomenon.*

This idea is explained in detail in Sect. 2, but briefly it means that applying the heat flow for time $(t - t_0)$ to the characteristic polynomial of the random matrix A_{t_0} in (1.4) should give a new polynomial whose roots resemble the eigenvalues of A_t. Thus, roughly speaking, applying the heat operator has the same effect as changing the value of t in the model. This idea gives one reason to study the heat flow on polynomials—and it also gives considerable insight into how the roots of polynomials should evolve under the heat flow.

The basis for Idea 1.1 is a "deformation theorem" for the second moments of the characteristic polynomials of the matrices A_t^N. It states that varying t in the random matrix model has the same effect—at the level of the second moments—as applying the heat flow. See Theorem 2.7 in Sect. 2.3.

1.3 The main conjectures

Definition 1.2 If p^N is a polynomial of degree N, we define the **empirical root distribution** of p^N as the probability measure on $\mathbb{C}$ given by

$$\frac{1}{N}\sum_{j=1}^{N}\delta_{z_j},$$

where $z_1, \ldots, z_N$ are the roots of p^N, listed with their multiplicity. If $\{p^N\}_{N=1}^{\infty}$ is a sequence of polynomials of degree N, we say that a measure μ is the **limiting root distribution** of the sequence if the empirical root distribution of p^N converges weakly to μ as $N \to \infty$.

Definition 1.3 If μ is a compactly supported probability measure on $\mathbb{C}$, we define the **log potential** $S : \mathbb{C} \to \mathbb{R}$ and the **Cauchy transform** $m : \mathbb{C} \to \mathbb{C}$ of μ by

$$S(z) = \int_{\mathbb{C}} \log(|z - w|^2)\, d\mu(w);$$
$$m(z) = \int_{\mathbb{C}} \frac{1}{z - w}\, d\mu(w).$$

These functions are defined Lebesgue almost everywhere for any such μ and are defined everywhere if μ has a bounded density with respect to the Lebesgue measure on the plane.

We begin by considering the ordinary heat flow in (1.2). We refer to this flow as the "additive" (as opposed to "multiplicative") case, because it is connected (Sect. 2) to Gaussian random matrix models, which can be constructed as sums (as opposed to products) of large numbers of independent random matrices.

We now present a general conjecture for the behavior of high-degree polynomials under the heat flow.

Conjecture 1.4 *Suppose p_0^N is a sequence of polynomials of degree N having a compactly supported limiting root distribution μ_0 whose Cauchy transform is globally Lipschitz. Then for all $\tau \in \mathbb{C}$, the empirical root measure of*

$$p_\tau^N(z) := \exp\left(\frac{\tau}{2N}\frac{d^2}{dz^2}\right) p^N(z) \tag{1.6}$$

also converges weakly to a compactly supported probability measure μ_τ. Furthermore, there exists a constant $C > 0$ such that the following results hold.

(1) The log potential $S(z, \tau)$ of μ_τ is a $C^{1,1}$ solution of the PDE

$$\frac{\partial S}{\partial \tau} = \frac{1}{2}\left(\frac{\partial S}{\partial z}\right)^2 \tag{1.7}$$

in the set $\{z \in \mathbb{C},\ |\tau| < C\}$. Here $\partial/\partial\tau$ and $\partial/\partial z$ denote the Cauchy–Riemann operators, given by

$$\frac{\partial}{\partial \tau} = \frac{1}{2}\left(\frac{\partial}{\partial \tau_1} - i\frac{\partial}{\partial \tau_2}\right); \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$

where $\tau = \tau_1 + i\tau_2$ and $z = x + iy$.

(2) Let m_0 be the Cauchy transform of μ_0 and define a ***transport map*** *$T_\tau : \mathbb{C} \to \mathbb{C}$ by*

$$T_\tau(z) = z - \tau m_0(z). \tag{1.8}$$

Then for all τ with $|\tau| < C$, the measure μ_τ is the push-forward of μ_0 by T_τ:

$$\mu_\tau = (T_\tau)_*(\mu_0), \tag{1.9}$$

where $(f)_*(\mu)$ denotes the push-forward of a measure μ by a map f.

In Point 1, the condition $C^{1,1}$ means, more precisely, that the partial derivatives of S with respect to the real and imaginary parts of z are globally Lipschitz.

Suppose the roots of p_τ^N are distinct for τ in some domain $U \subset \mathbb{C}$, so that we can parameterize the roots holomorphically in $\tau \in U$ as $\{z_j(\tau)\}_{j=1}^N$. Then in light of (1.8) and (1.9), it is natural to expect that the roots will evolve as

$$z_j(\tau) \approx z_j(0) - \tau m_{s,\tau_0}(z_j(0)). \tag{1.10}$$

We may interpret (1.10) to mean that, with high probability when N is large, most of the trajectories $z_j(\tau)$ will be uniformly close to the right-hand side of (1.10), for sufficiently small τ. Although such a result does not follow from (1.9), the arguments in Sect. 3 will support this behavior. A rigorous result in this direction was established for the zeros of the heat-evolved Gaussian analytic function in Theorem 1.3 of our paper with Jalowy and Kabluchko [24].

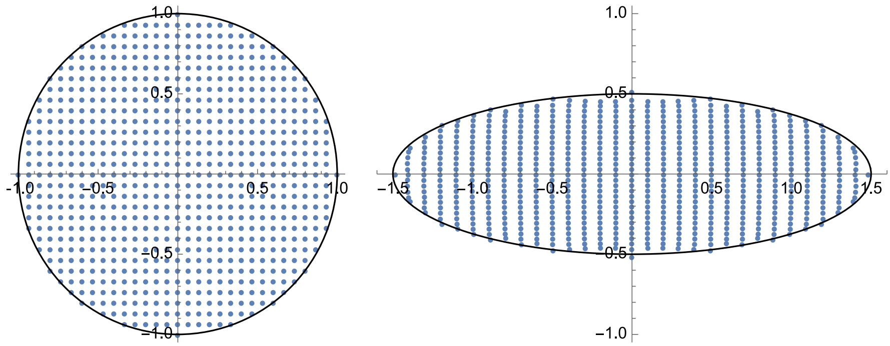

Fig. 1 Heat evolution of a polynomial whose roots form a uniform lattice inside the unit disk. Shown for $\tau = -1/2$

The simplest case of Conjecture 1.4 is the one in which μ_0 is uniform on the unit disk, in which case $m_0(z) = \bar{z}$ for all z in the disk. Then $T_\tau(z) = z - \tau\bar{z}$ and for real τ between -1 and 1, the measure μ_τ in (1.9) is uniform on an ellipse with semi-axes $1 - \tau$ and $1 + \tau$. See Fig. 1 and see also Slide 3 in the supplemental document for an animation.

Remark 1.5 If we take τ to equal a real number t in (1.7), take the real part of both sides, and then multiply by 2, we obtain the following real-variables PDE:

$$\frac{\partial S}{\partial t} = \frac{1}{4}\left(\left(\frac{\partial S}{\partial x}\right)^2 - \left(\frac{\partial S}{\partial y}\right)^2\right), \quad t \in \mathbb{R}.$$

Here $\partial/\partial t$ is the ordinary partial derivative in the real variable t.

The assumption that $m_0(z)$ is globally Lipschitz guarantees that the map T_τ is a homeomorphism of $\mathbb{C}$ to $\mathbb{C}$, when $|\tau|$ is small enough; see Lemma 4.2. This assumption will be satisfied if μ_0 is absolutely continuous with respect to Lebesgue measure on the plane, with a sufficiently smooth density. But if μ_0 is supported on the real line, m_0 will not even be continuous.

As explained in Sects. 4.1.2 and 4.1.3, if we have enough regularity in the solution to (1.7), we can hope to deduce Point 2 of Conjecture 1 from Point 1. To do this, we will take the Laplacian of both sides of (1.7) to obtain a continuity equation satisfied by the density of the measure μ_τ, from which a push-forward theorem will hold, under suitable technical assumptions. But in Sect. 3, we will give a direct argument for Point 2 of the conjecture, which does not rely on Point 1.

Remark 1.6 The polynomials in Conjecture 1.4 can be deterministic or random. In the random case, one might imagine proving the result only "in probability." That is, one might prove that both the hypothesis (convergence of the empirical root measure of p_0^N) and the conclusions (Points 1 and 2) hold in the sense of weak convergence in probability.

"Weak convergence in probability" means that with high probability, the approximating measures are close to the limit measure in the weak topology.

Remark 1.7 In a separate work with Jalowy and Kabluchko [25], we prove an "in probability" version of Conjecture 1.4 for random polynomials with independent coefficients. One needs mild regularity assumptions on the coefficients to guarantee that the constant C (denoted t_{sing} in [25]) is positive. See Theorem 5.2 (PDE) and Theorem 3.3 (push-forward theorem) in [25].

In another work with the same authors [24], we establish rigorous results in [24] for the zeros of the Gaussian analytic function under the heat flow. See also [26] for similar results on the zeros of random polynomials under repeated differentiation.

The work [25] builds on work of Kabluchko and Zaporozhets [32]. The paper [32] studies polynomials formed as sums of ordinary monomials z^j multiplied by random coefficients that are independent (but typically not identically distributed). The paper [25] applies the heat flow to a Kabluchko–Zaporozhets polynomial—which has the effect of replacing the monomial z^j by the (properly scaled) jth Hermite polynomial. The heat flow therefore leads to a major new class of random polynomials that can be analyzed by methods similar to those in [32]. This observation then opens the door to the study of even more general random polynomials by similar methods (work in progress by the present authors with Jalowy and Kabluchko).

We present three distinct lines of reasoning for Conjecture 1.4.

- An argument from the perspective of **random matrix theory** (Sect. 2). This line of reasoning is specific to the case in which p_0^N is the characteristic polynomial of a certain class of random matrix models, and we would hope for an almost sure version of the conjecture.
- An argument from the perspective of **dynamical systems** (Sect. 3). We will argue that for large N, the roots move in approximately straight lines with constant velocity given by the negative of initial Cauchy transform. This argument supports Point 2 of the conjecture.
- An argument from the perspective of **partial differential equations** (Sect. 4). We will derive an exact PDE satisfied by the log potential of the empirical root measure of a polynomial evolving by the heat equation. This PDE formally converges to the PDE (1.7) in the limit $N \to \infty$. We then show how the PDE (1.7) implies a continuity equation for the density of the measure. With enough regularity, we can then show that Point 1 of Conjecture 1.4 implies Point 2.

We now present the "multiplicative" version of Conjecture 1.4. The heat flow in (1.11) arises naturally from certain multiplicative random matrix models. See Theorem 6.5 and Lemma 6.6. This flow is almost the same as the one discussed by Terry Tao in [44].

Conjecture 1.8 *A "multiplicative" version of Conjecture 1.4 also holds, with the heat flow in (1.2) replaced by the operator*

$$\exp\left\{-\frac{(\tau-\tau_0)}{2N}\left(z^2\frac{\partial^2}{\partial z^2}-(N-2)z\frac{\partial}{\partial z}-N\right)\right\}, \tag{1.11}$$

the PDE (1.7) replaced by

$$\frac{\partial S}{\partial \tau} = -\frac{1}{2}\left(z^2\left(\frac{\partial S}{\partial z}\right)^2 - z\frac{\partial S}{\partial z}\right)$$

and the transport map T_τ in (1.8) replaced by

$$\tilde{T}_\tau(z) = z\exp\left\{\tau z m_0(z) - \frac{1}{2}\right\},$$

where m_0 is the Cauchy transform of the limiting root distribution of the initial polynomials.

Since the operator in the exponent in (1.11) maps z^j to a multiple of z^j for all j, the exponentiated operator will also have this property. Conjecture 1.8 will be supported by appropriate versions of the three lines of reasoning described above supporting Conjecture 1.4.

1.4 Rigorous results

None of the perspectives discussed above leads to a rigorous argument for Conjecture 1.4 or Conjecture 1.8. We nevertheless obtain some rigorous results, as follows. First, we establish in Theorems 2.7 and 6.5 deformation theorems for the second moment of the characteristic polynomial of certain random matrix models, in both the "additive" and "multiplicative" settings. This result provides a conceptual link between (on the one hand) the deformation of certain random matrix models with respect to a parameter and (on the other hand) the deformation of the associated characteristic polynomials by an appropriate heat flow. We expect that a version of Theorem 2.7 will hold for other families of random matrices, leading to further connections between random matrices and differential flows on polynomials. Second, we show in Sect. 5 that our two main conjectures hold at the level of the holomorphic moments. Although these moments do not uniquely determine the limiting root distributions, the results of Sect. 5 provide substantial confirmation of our conjectures. We also remind the reader of rigorous results obtained for random polynomials, in a separate paper with Jalowy and Kabluchko. See Remark 1.7.

2 A random matrix perspective

In this section, we present a conjecture related to the "model deformation phenomenon" discussed in Sect. 1.2. A multiplicative version of this conjecture is presented in Sect. 6.1.

2.1 The heat flow conjecture for random matrices

We consider now a generalization of the random matrix model denoted A_t in (1.4) in Sect. 1.2. Let X^N and Y^N be independent $N \times N$ GUE matrices and consider a matrix of the form

$$Z^N = e^{i\theta}(aX^N + ibY^N), \tag{2.1}$$

where a and b are real numbers, assumed not both zero. We call such a matrix a (rotated) **elliptic random matrix** (with the parameter θ giving the rotation). It is convenient to parameterize such matrices by a real, positive variance parameter s and a complex covariance parameter τ, given by

$$\begin{aligned} s &= \mathbb{E}\left\{\frac{1}{N}\operatorname{Trace}((Z^N)^* Z^N)\right\} \\ \tau &= \mathbb{E}\left\{\frac{1}{N}\operatorname{Trace}((Z^N)^* Z^N)\right\} - \mathbb{E}\left\{\frac{1}{N}\operatorname{Trace}((Z^N)^2)\right\}. \end{aligned} \tag{2.2}$$

This parameterization is motivated by [19], where (in the case $K_{\mathbb{C}} = \mathbb{C}^{N^2}$) the law of Z^N is denoted as $\mu_{s,\tau}$ and where the parameter τ plays the role of the complex time-variable in the Segal–Bargmann transform.

The parameters s and τ completely determine the distribution of the matrix Z^N; see [23, Section 2.1] for details. Using the Cauchy–Schwarz inequality, we can verify that

$$|\tau - s| \le s. \tag{2.3}$$

We label such a matrix as $Z^N_{s,\tau}$ and we then consider a random matrix of the form

$$X_0^N + Z^N_{s,\tau},$$

where X_0^N is independent of $Z^N_{s,\tau}$. We assume that all of the expected $*$-moments of X_0^N—that is, expectation values of normalized traces of words in X_0^N and its adjoint—are well-defined and finite. The model (1.4) considered in Sect. 1.2 corresponds to the case in which $s = 1$, τ is real, and X_0^N is Hermitian.

Suppose now that X_0^N converges almost surely in $*$-distribution to an element x_0 in a tracial von Neumann algebra. Define

$$z_{s,\tau} = e^{i\theta}(ax + iby) \tag{2.4}$$

where x and y are freely independent semicircular elements and the parameters s and τ are define by analogy to (2.2). When $|\tau - s| < s$, we may apply a result of Śniady [42] to conclude that the eigenvalue distribution of $X_0^N + X^N_{s,\tau}$ converges almost surely to the Brown measure of $x_0 + z_{s,\tau}$, where x_0 and $z_{s,\tau}$ are taken to be freely independent. (When $|\tau - s| < s$, the element Z^N in (2.1) is equal in distribution to the sum of a multiple of a Ginibre matrix and an independent elliptic matrix, so that Theorem 6 in [42] applies.)

Conjecture 2.1 (Arbitrary plus elliptic model) *Fix $s > 0$ and two complex numbers τ_0 and τ with $|\tau_0 - s| < s$ and $|\tau - s| \leq s$. Let μ_{s,τ_0} be the Brown measure of $x_0 + z_{s,\tau_0}$, where z_{s,τ_0} is defined as in (2.4) and where x_0 is freely independent of z_{s,τ_0}, and let m_{s,τ_0} be its Cauchy transform. Then m_{s,τ_0} is Lipschitz continuous and the Brown measure $\mu_{s,\tau}$ of $x_0 + z_{s,\tau}$ can be computed by the model deformation formula*

$$\mu_{s,\tau} = (\Phi_{s,\tau_0,\tau})_*(\mu_{s,\tau_0}), \tag{2.5}$$

where the map $\Phi_{s,\tau_0,\tau} : \mathbb{C} \to \mathbb{C}$ is defined as

$$\Phi_{s,\tau_0,\tau}(z) = z - (\tau - \tau_0) m_{s,\tau_0}(z).$$

Furthermore, as τ varies over the set $|\tau - s| < s$ (with x and s fixed), the log potential of $S_{s,\tau}$ is a C^1 solution of the PDE

$$\frac{\partial S_{s,\tau}}{\partial \tau} = \frac{1}{2}\left(\frac{\partial S_{s,\tau}}{\partial z}\right)^2. \tag{2.6}$$

Remark 2.2 Substantial partial results in the direction of Conjecture 2.1 have been obtained. First, when x_0 is self-adjoint, the "additive" counterpart of the results of Hall–Ho [23] apply. (The proofs from [23] —which addresses the "multiplicative" case—apply with minor modifications in the easier additive case.) In this case, the push-forward result (2.5) holds and the PDE (2.6) holds at least away from the boundary of the support of $\mu_{s,\tau}$. See Theorem 4.2, Proposition 6.3, Corollary 7.7 and Theorem 8.2 in [23]. Second, for general x_0 (assumed only to be freely independent of z_{s,τ_0}), the results of Zhong [49] apply. A push-forward result holds in this generality, at least for $\tau_0 = s$ [49, Theorem C]. But since Zhong does not use PDE methods in [49], he does not address whether the PDE (2.6) holds.

We emphasize that Conjecture 2.1 is not the main focus of this paper. We present it to explain how Conjecture 2.3 below relates to the general proposal given in Conjecture 1.4. The idea is that the change from τ_0 to τ in Conjecture 2.1 can be carried out by applying the heat flow to the characteristic polynomial of the random matrix with the first value of τ.

Conjecture 2.3 *Fix $s > 0$ and complex numbers τ_0 and τ with $|\tau_0 - s| \leq s$ and $|\tau - s| \leq s$. Let Z^N_{s,τ_0} and $Z^N_{s,\tau}$ be elliptic random matrices with parameters defined as in (2.2). Let X^N_0 be independent of $Z_{s,\tau}$ and Z_{s,τ_0} and assume X^N_0 converges almost surely in $*$-distribution to an element x_0 in a tracial von Neumann algebra. Let p_{s,τ_0} be the random characteristic polynomial of $X^N_0 + Z^N_{s,\tau_0}$ and define a new random polynomial $q_{s,\tau_0,\tau}$ by*

$$q_{s,\tau_0,\tau}(z) = \exp\left\{\frac{(\tau - \tau_0)}{2N}\frac{\partial^2}{\partial z^2}\right\} p_{s,\tau_0}(z), \quad z \in \mathbb{C}.$$

Then the empirical root measure of $q_{s,\tau_0,\tau}$ converges weakly almost surely to the limiting eigenvalue distribution of $X^N_0 + Z^N_{s,\tau}$.

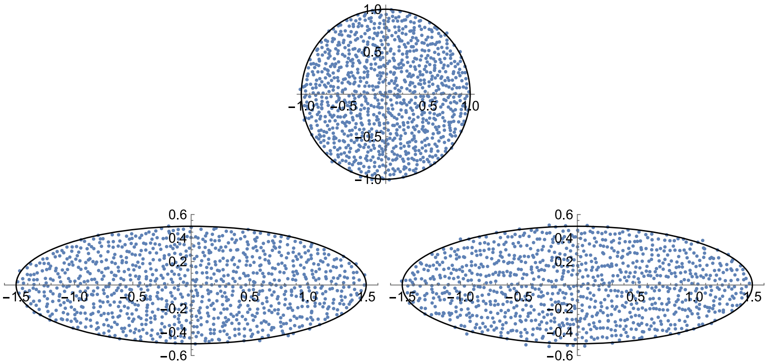

Fig. 2 Eigenvalues of a Ginibre matrix (top), eigenvalues of an elliptic matrix (bottom left), and roots of the heat evolution of the characteristic polynomial of the Ginibre matrix (bottom right)

The conjecture says that applying the heat operator for time $\tau - \tau_0$ to the polynomial p_{s,τ_0} gives a new polynomial $q_{s,\tau_0,\tau}$ whose roots resemble those of $p_{s,\tau}$—namely the eigenvalues of $X_0^N + Z_{s,\tau}^N$. Thus, the heat flow effectively changes the value of τ. We emphasize, however, that the conjecture is *not* claiming that the joint distribution of the roots of $q_{s,\tau_0,\tau}$ is the same as the joint distribution of the eigenvalues of $X_0^N + Z_{s,\tau}^N$. Rather, the conjecture merely asserts that the *limiting* eigenvalue distributions of the two collections of points are the same. The points in Fig. 5, for example, correspond to the case $X_0^N = 0$, with $s = 1$, $\tau_0 = 0$, and $\tau = 1$, so that Z_{s,τ_0} is GUE and $Z_{s,\tau}$ is Ginibre. Although the points in the figure approximate the uniform distribution on the unit disk, they are clearly distinguishable to the eye from the eigenvalues of a Ginibre matrix.

Remark 2.4 If Conjectures 2.1 and 2.3 hold, then Conjecture 1.4 also holds—that is, the expected PDE and push-forward results will hold for the limiting root distribution of $q_{s,\tau_0,\tau}$. But the argument we will present for Conjecture 2.3 in Sect. 2.3 is independent of the validity of Conjecture 2.1.

2.2 Examples

In this subsection, we present three examples of the general proposal in Conjecture 2.3. We emphasize that all the examples still have the status of a conjecture: special cases of the general conjecture. As our first example, we consider the case in which X_0^N is zero, and the parameters in (2.2) are $s = 1$ and $\tau_0 = 1$. In that case, $X_0^N + Z_{s,\tau_0}^N = Z_{1,1}^N$ is a Ginibre matrix. We then take $\tau = 1 - t$, where t is a real number between -1 and 1, so that

$$(\tau - \tau_0) = -t, \quad -1 \leq t \leq 1.$$

Example 2.5 (*Circular to elliptic and circular to semicircular*) Let Z^N be an $N \times N$ random matrix chosen from the Ginibre ensemble and let p be its random characteristic polynomial. Fix a real number t with $-1 \leq t \leq 1$ and define a new random polynomial

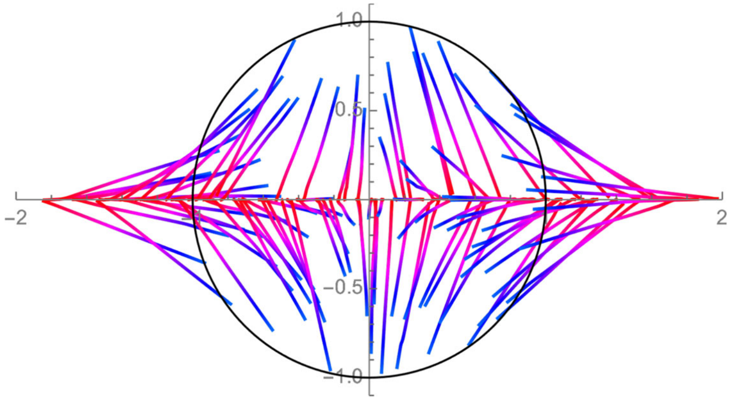

Fig. 3 Plots of 100 of the curves $z_j(t)$, $0 \leq t \leq 1$, in Example 2.5, starting from the eigenvalues of a 1000×1000 Ginibre matrix. Each curve changes color from blue to red as t increases

q_t by

$$q_t(z) = \exp\left\{-\frac{t}{2N}\frac{\partial^2}{\partial z^2}\right\} p(z), \quad z \in \mathbb{C}.$$

For $-1 < t < 1$, Conjecture 2.3 predicts that the empirical root measure of q_t will converge weakly almost surely to the uniform probability measure on the ellipse centered at the origin with semi-axes $1 + t$ and $1 - t$. For $t = 1$, Conjecture 2.3 predicts that the empirical root measure of q_1 will converge weakly almost surely to the semicircular probability measure on $[-2, 2] \subset \mathbb{R}$.

In this setting, the initial measure $\mu_{s,\tau_0} = \mu_{1,1}$ is the uniform measure on the unit disk, so that $m_{s,\tau_0}(z) = \bar{z}$ for all z in the unit disk. Thus, (1.10) predicts that, in this case, the roots $\{z_j(t)\}_{j=1}^N$ should evolve as

$$z_j(t) \approx z_j(0) + t\overline{z_j(0)}, \quad -1 \leq t \leq 1, \tag{2.7}$$

where $\{z_j(0)\}_{j=1}^N$ are the eigenvalues of Z^N. In particular, we should have

$$z_j(1) \approx 2\,\mathrm{Re}[z_j(0)].$$

Figure 2 shows the similarity between the roots of q_t and the eigenvalues of the corresponding elliptic random matrix model. See also Slide 4 in the supplemental document for an animation. Figure 3, meanwhile, illustrates the approximate straight-line motion of the roots predicted in (2.7). See also Slide 5 in the supplemental document for a dynamical version of Fig. 3. We note that a rigorous version of the first paragraph of Example 2.5 has been established in [26, Theorem 2.1]—but with the characteristic polynomial of a Ginibre matrix replaced by a Weyl polynomial. (A Weyl polynomial is a certain random polynomial whose limiting root distribution is uniform on the unit disk.) In addition, a rigorous result in the direction of (2.7) has been established for the individual zeros of the Gaussian analytic function in [24, Theorem 1.3].

As our second example of Conjecture 2.3, we consider the case in which X_0^N is zero, $s = 1$, and $\tau_0 = 0$. In that case, $X_0^N + Z_{s,\tau_0}^N$ is a GUE matrix.

Example 2.6 (*Semicircular to elliptic and semicircular to circular*) Let X^N be a GUE matrix and let p be its random characteristic polynomial. Fix a real number t with $0 \leq t \leq 2$ and define a new random polynomial q_t by

$$q_t(z) = \exp\left\{\frac{t}{2N}\frac{\partial^2}{\partial z^2}\right\} p(z), \quad z \in \mathbb{C}.$$

For $0 < t < 2$, Conjecture 2.3 predicts that the empirical root measure of q_t will converge weakly almost surely to the uniform probability measure on the ellipse centered at the origin with semi-axes $2-t$ and t. For $t = 2$, Conjecture 2.3 predicts that the empirical root measure of q_t will converge weakly almost surely to the semicircular distribution on the interval $[-2, 2]$ on the imaginary axis.

We note that the eigenvalues of a GUE matrix are distinct with probability 1. (The joint distribution of the eigenvalues is absolutely continuous on $\mathbb{R}^N$—e.g., [3, Theorem 2.5.2]—so the measure of any hyperplane where two eigenvalues are equal is zero.) Furthermore, inside the collection of polynomials of degree N with real coefficients, the set of polynomials with distinct real roots is open. Since the heat evolution is continuous, we can therefore conclude that the roots of $q_t(z)$ remain real and distinct for sufficiently small positive values of t. We believe, however, that when N is large, the roots will very quickly begin to collide and move off the real axis. (This behavior is in the spirit of how the Riemann xi function—derived from the Riemann zeta function—behaves under the heat flow; see [45].) There is therefore no contradiction in conjecturing that the $N \to \infty$ root distribution of q_t will be uniform on an ellipse in the *plane*, even for small positive t.

The top part of Fig. 4 shows the characteristic polynomial p of a GUE matrix with $N = 60$, multiplied by a suitable Gaussian to make the values of a manageable size. (Specifically, it is convenient to multiply $p(x)$ by $2^{N/2}e^{-Nx^2/4}$, which of course does not change the zeros.) The bottom part of the figure then shows the polynomial q_t with $t = 0.05$, multiplied by the same Gaussian. Already by the time $t = 0.05$, the number of real roots has dropped from 60 to 32. Figure 5 then shows the complex roots of the heat-evolved polynomial at time $t = 1$. See also Slide 6 in the supplemental document for an animation.

The anticipated trajectories of the roots in Example 2.6 are more complicated than in Example 2.5. After all, we are effectively trying to run time backward from $t = 1$ in the map (2.7), even though the $t = 1$ map $z \mapsto 2\,\mathrm{Re}\,z$ is not invertible. To put it a different way, Example 2.6 asserts that we can deform a *one*-dimensional distribution of points along the real axis into a *two*-dimensional uniform distribution on an ellipse or disk. This deformation cannot be achieved by applying a smooth map of the sort we have on the right-hand side of (2.7).

Rather, the behavior we expect is the following. If we write the roots $\{z_j(t)\}_{j=1}^N$ of q_t in the form $z_j(t) = x_j(t) + iy_j(t)$, then we expect to have the approximate

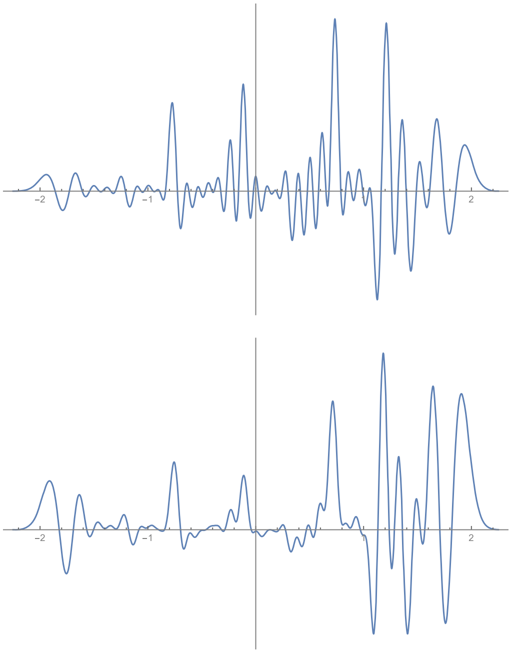

Fig. 4 The characteristic polynomial of a GUE matrix and the heat evolution of the same polynomial at $t = 0.05$, both multiplied by a Gaussian. Shown for $N = 60$. The number of real roots is 60 (top) and 32 (bottom)

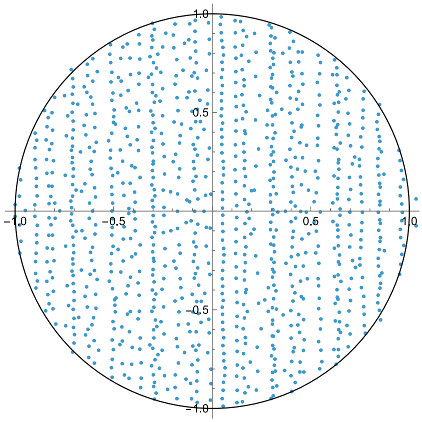

Fig. 5 The points $\{z_j(1)\}_{j=1}^N$ in Example 2.6, with $N = 1000$. The points display an obvious banding structure in the vertical direction but still approximate a uniform distribution on the unit disk

equalities

$$x_j(t) \approx x_j(0) - \frac{t}{2}x_j(0)$$
$$y_j(t) \approx c_j t\sqrt{1 - \frac{1}{4}x_j(0)^2},$$

where c_j is a *random* constant uniformly distributed between -1 and 1. This predicted behavior can be understood as the limiting case of Conjecture 2.3 with $X_0^N = 0$, $s = 1$, and τ_0 tending to zero. See Fig. 6, along with Slide 7 in the supplemental document.

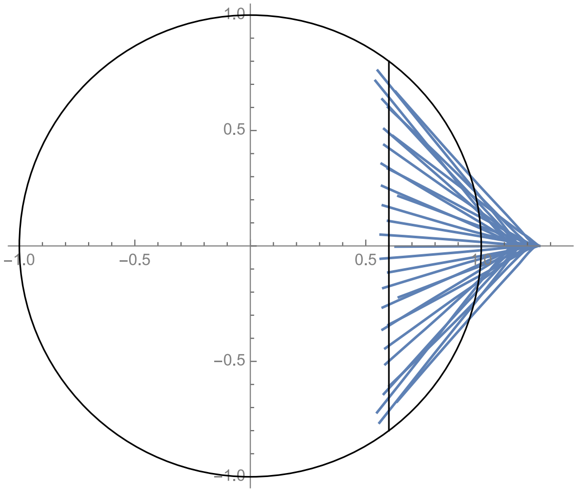

Fig. 6 Plot of 30 of the curves $z_j(t)$ in Example 2.6, with $N = 1,000$ and $z_j(0)$ close to 1.2. When $t = 1$, the distribution of points resembles a uniform distribution along the vertical segment in the unit circle with x-coordinate 0.6

As our third example of Conjecture 2.3, we consider the case in which X_0^N is Hermitian and $s = \tau_0 = 1$. In that case, the limiting eigenvalue distribution of $X_0^N + Z_{1,1}^N$ is computed in [28, Section 3] and the limiting eigenvalue distribution of $X_0^N + Z_{1,\tau}^N$ is computed in [49, Section 6] (or the additive version of [23]). We then further specialize to the case in which the limiting eigenvalue distribution of X_0^N has half of its mass at 1 and half of its mass at -1. Figure 7 shows the similarity between the eigenvalues of $X_0^N + Z_{1,\tau}^N$ and the roots of the heat evolution of the characteristic polynomial of $X_0^N + Z_{1,\tau_0}^N$. See also Slide 8 in the supplemental document for an animation.

2.3 Rigorous support for Conjecture 2.3: a deformation result for second moments

In this subsection, we present a rigorous result (Theorem 2.7) about the second moments of characteristic polynomials of random matrices. We also give a heuristic argument for why Theorem 2.7 should imply Conjecture 2.3. This argument, however, requires concentration results that we currently do not know how to establish.

Suppose p^N is a family of random polynomials of degree N. In our applications, we will take p^N to be *either* the characteristic polynomial of a random matrix *or* the heat evolution of such a characteristic polynomial. We then consider the **second moment** $D^N : \mathbb{C} \to [0, \infty)$ of p^N, defined as

$$D^N(z) = \mathbb{E}\{\left|p^N(z)\right|^2\}. \tag{2.8}$$

We then define

$$T^N(z) = \frac{1}{N}\log D^N(z) = \frac{1}{N}\log[\mathbb{E}\{\left|p^N(z)\right|^2\}]$$

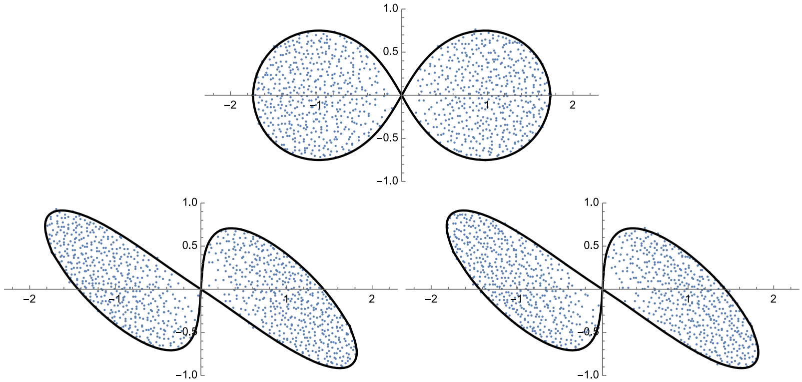

Fig. 7 The (s, τ_0) eigenvalues (top), the (s, τ) eigenvalues (bottom left), and the $(\tau - \tau_0)$ evolution of the (s, τ_0) eigenvalues (bottom right), for the case that X_0^N is Hermitian with eigenvalues equally distributed between -1 and 1. Shown for $s = 1$, $\tau_0 = 1$, and $\tau = i/2$

We expect to be able to recover the limiting root distribution of p^N from the function D^N as follows:

$$\text{limiting root distribution of } p^N = \frac{1}{4\pi}\Delta_z \left(\lim_{N\to\infty} T^N(z) \right). \tag{2.9}$$

That is to say, we expect that the large-N limit of T^N will be the log potential of the limiting root distribution of p^N. This sort of claim is taken for granted in the physics literature (e.g., [10, 11]) but it is not a rigorous result.

To understand the claim in (2.9), consider another function obtained by interchanging the expectation value with the logarithm in the formula for T^N, namely

$$\begin{aligned} S^N(z) &= \frac{1}{N}\mathbb{E}\left\{ \log \left| p^N(z) \right|^2 \right\} \\ &= \mathbb{E}\left\{ \frac{1}{N}\sum_{j=1}^{N} \log \left| z - z_j^N \right|^2 \right\}, \end{aligned}$$

where $\{z_j^N\}_{j=1}^N$ is the collection of roots of p^N, listed with their multiplicity. Then $\frac{1}{4\pi}\Delta_z S^N$ is easily seen to be the *expected empirical root distribution* of Z^N. (Put the Laplacian inside the expectation value and use that $\frac{1}{4\pi}\log|z|^2$ is the Green's function for the Laplacian on the plane.)

Suppose now that we have, as usual, a **concentration phenomenon**, in which the root distribution of p^N is approaching a deterministic limit as N goes to infinity. (In the case of characteristic polynomials of random matrices, see, for example, Sections 2.3 and 4.4 in [3].) In that case, the large-N limit of the *expected* empirical root distribution should be the almost sure limit of the eigenvalue distribution itself. In that case,

$$\text{limiting root distribution of } p^N = \frac{1}{4\pi}\Delta_z\left(\lim_{N\to\infty} S^N(z)\right).$$

But the same concentration phenomenon suggests that interchanging the expectation value with the logarithm should not have much effect, so that S^N and T^N should be almost equal. That is to say, if the empirical measure of the set $\{z_j\}_{j=1}^N$ is, with high probability, close to a deterministic measure μ, then *both* S^N and T^N should be close to the log potential of μ, and we should have

$$\begin{aligned}\text{limiting root distribution of } p^N &= \frac{1}{4\pi}\Delta_z\left(\lim_{N\to\infty} S^N(z)\right)\\ &= \frac{1}{4\pi}\Delta_z\left(\lim_{N\to\infty} T^N(z)\right),\end{aligned}$$

confirming (2.9).

We consider the second moments (as in (2.8)) of the characteristic polynomials of the random matrix models introduced in the previous section, starting from the additive case. Consider an $N\times N$ "elliptic" random matrix $Z^N_{s,\tau}$ with parameters s and τ, as in (2.1) and (2.2). Take another random matrix X^N_0 that is independent of $Z^N_{s,\tau}$ and define a function D^N by

$$D^N(z,s,\tau) = \mathbb{E}\left\{\left|\det(zI-(X^N_0+Z^N_{s,\tau}))\right|^2\right\}. \tag{2.10}$$

Here z ranges over $\mathbb{C}$, s ranges over $(0,\infty)$, and τ ranges over the set of complex numbers with $|\tau-s|<s$. Of course, this function depends also on the distribution of the random matrix X^N_0, but we suppress this dependence in the notation.

We now come to the main theorem supporting the additive heat flow conjecture (Conjecture 2.3).

Theorem 2.7 (Deformation theorem for second moment) *Suppose τ_0 and τ are complex numbers satisfying $|\tau_0-s|\le s$ and $|\tau-s|\le s$, in accordance with* (2.3). *Consider random matrices $X^N_0+Z^N_{s,\tau_0}$ and $X^N_0+Z^N_{s,\tau}$, respectively, where X^N_0 is independent of Z^N_{s,τ_0} and $Z^N_{s,\tau}$ but not necessarily Hermitian. Then the function D^N in* (2.10) *can also be computed as*

$$D^N(z,s,\tau) = \mathbb{E}\left\{\left|\exp\left(\frac{(\tau-\tau_0)}{2N}\frac{\partial^2}{\partial z^2}\right)\det(zI-(X^N_0+Z^N_{s,\tau_0}))\right|^2\right\}. \tag{2.11}$$

The proposition says that one can compute D^N in two different ways: first, according to the definition (2.10), by taking the expectation of the magnitude-squared of the characteristic polynomial of the (s,τ)-model, or, second, by applying the heat operator $\exp\left(\frac{(\tau-\tau_0)}{N}\frac{\partial^2}{\partial z^2}\right)$ to the characteristic polynomial of the (s,τ_0)-model and then taking the expectation of magnitude-square of this new polynomial.

In the notation of Conjecture 2.3, Theorem 2.7 says that the random polynomials $q_{s,\tau_0,\tau}$ and $p_{s,\tau}$ *have the same second moments*. Since we expect to recover the limiting root distributions of $q_{s,\tau_0,\tau}$ and $p_{s,\tau}$ from their second moments, as in (2.9), Theorem 2.7 provides a strong motivation for Conjecture 2.3.

Let us use the notation $\{z_j^{s,\tau}\}_{j=1}^N$ for the roots of $p_{s,\tau}$ and the notation $\{z_j^{s,\tau_0}(\tau)\}_{j=1}^N$ for the roots of $q_{s,\tau_0,\tau}$ (i.e., the roots of p_{s,τ_0}, evolved for time τ by the heat flow). Then Theorem 2.7 can be restated as

$$\mathbb{E}\left\{\left|\prod_{j=1}^N (z-z_j^{s,\tau})\right|^2\right\} = \mathbb{E}\left\{\left|\prod_{j=1}^N (z-z_j^{s,\tau_0}(\tau))\right|^2\right\}.$$

2.4 Proof of Theorem 2.7

The theorem will following easily from the following result, which is of independent interest.

Proposition 2.8 *The function D^N in (2.10) satisfies the second-order linear PDEs*

$$\frac{\partial D^N}{\partial \tau} = \frac{1}{2N}\frac{\partial^2 D^N}{\partial z^2} \tag{2.12}$$

$$\frac{\partial D^N}{\partial \bar{\tau}} = \frac{1}{2N}\frac{\partial^2 D^N}{\partial \bar{z}^2} \tag{2.13}$$

for τ in the set $|\tau - s| < s$.

In the proposition, it is understood that the distribution of the random matrix X_0^N is fixed (independent of τ) for each N.

We establish two lemmas that will be used to prove the proposition. The first gives the PDE satisfied by the probability density of the random matrix $Z_{s,\tau}^N$. The second relates derivatives of a characteristic polynomial in the matrix variable to derivatives in the complex variable.

Consider $M_N(\mathbb{C})$ as a real vector space of dimension $2N^2$, equipped with the real-valued inner product $\langle\cdot,\cdot\rangle_N$ given by the scaled Hilbert–Schmidt inner product:

$$\langle Z, W\rangle_N = N\,\mathrm{Re}[\mathrm{Trace}(Z^*W)].$$

(The density of the Ginibre ensemble is a constant times $e^{-\langle Z,Z\rangle_N}$.) We choose an orthonormal basis $\{X_j\}_{j=1}^{N^2}\cup\{Y_j\}_{j=1}^{N^2}$ such that X_j is Hermitian and $Y_j = iX_j$. We then form the translation-invariant differential operators $\tilde{X}_j$, $\tilde{Y}_j$, Z_j, and $\bar{Z}_j$ on $M_N(\mathbb{C})$

as

$$\tilde{X}_j f(A) = \frac{d}{du} f(A + uX_j)\bigg|_{u=0}; \quad \tilde{Y}_j f(A) = \frac{d}{du} f(A + uY_j)\bigg|_{u=0};$$
$$Z_j = \frac{1}{2}(\tilde{X}_j - i\tilde{Y}_j); \quad \bar{Z}_j = \frac{1}{2}(\tilde{X}_j + i\tilde{Y}_j). \tag{2.14}$$

We then introduce operators Δ, ∂^2, and $\bar{\partial}^2$ given by

$$\Delta = \sum_{j=1}^{N^2} \tilde{X}_j^2; \quad \partial^2 = \sum_{j=1}^{N^2} Z_j^2; \quad \bar{\partial}^2 = \sum_{j=1}^{N^2} \bar{Z}_j. \tag{2.15}$$

Lemma 2.9 *Let $\gamma_{s,\tau}^N$ be the density of the law of the random matrix $Z_{s,\tau}^N$ defined in Sect. 2.1. Then $\gamma_{s,\tau}^N$ satisfies the PDE*

$$\frac{\partial \gamma_{s,\tau}^N}{\partial \tau} = -\frac{1}{2}\partial^2 \gamma_{s,\tau}^N. \tag{2.16}$$

Furthermore, if f is a (not necessarily holomorphic) polynomial function on the space $M_N(\mathbb{C})$ of all $N \times N$ matrices with complex entries, we have

$$\frac{\partial}{\partial \tau}\mathbb{E}\{f(X_0^N + Z_{s,\tau}^N)\} = -\frac{1}{2}\mathbb{E}\{\partial^2 f(X_0^N + Z_{s,\tau}^N)\}. \tag{2.17}$$

Proof Let $\Gamma_{s,\tau}^N$ be the Gaussian measure on $M_N(\mathbb{C})$ describing the law of $Z_{s,\tau}^N$. It is given by

$$\gamma_{s,\tau}^N = \exp\left\{\frac{1}{2}\Delta_{s,\tau}\right\}(\delta_0)$$

where δ_0 is a δ-function at the origin and where $\Delta_{s,\tau}$ is defined as

$$\Delta_{s,\tau} = s\Delta - \tau\partial^2 - \bar{\tau}\bar{\partial}^2. \tag{2.18}$$

(The formula (2.18) is equivalent to Eq. (1.7) in [19]; see also the equation between Eqs. (1.13) and (1.14) in [19].) This operator is elliptic precisely when $|\tau - s| < s$ and semi-elliptic in the borderline case $|\tau - s| = s$. The PDE (2.16) follows formally from the commutativity of the operators on the right-hand side of (2.18). Rigorously, it is straightforward, if not terribly illuminating, to verify (2.16) from the explicit formula for $\gamma_{s,\tau}^N$ in [19, Eq. (1.15)].

Meanwhile, since X_0^N is independent of $Z_{s,\tau}^N$, we have

$$\mathbb{E}\{f(X_0^N + Z_{s,\tau}^N)\} = \int\int f(a+b)\gamma_{s,\tau}^N(b)\, db\, d\nu(a), \tag{2.19}$$

where db is the Lebesgue measure on $M_N(\mathbb{C})$ and ν is the law of X_0^N. Now, in Sect. 2.1, we assume that all of the expected $*$-moments of X_0^N are finite, which in particular guarantees that the integral on the right-hand side of (2.19) is convergent. It is then not hard to justify differentiating with respect to τ under the integral on the right-hand side of (2.19). Using (2.16), we obtain

$$\frac{\partial}{\partial\tau}\mathbb{E}\{f(X_0^N+Z_{s,\tau}^N)\}=-\frac{1}{2}\int\int f(a+b)\partial^2\gamma_{s,\tau}^N(b)\,db\,d\nu(a). \tag{2.20}$$

Then since $\gamma_{s,\tau}^N$ and its derivatives have rapid decay at infinity and f is a polynomial, we can integrate by parts in the inner integral on the right-hand side of (2.20) to obtain

$$\frac{\partial}{\partial\tau}\mathbb{E}\{f(X_0^N+Z_{s,\tau}^N)\}=-\frac{1}{2}\int\int \partial^2 f(a+b)\gamma_{s,\tau}^N(b)\,db\,d\nu(a). \tag{2.21}$$

Here, the operator ∂^2 on the right-hand side should in principle be acting in the b variable. But since ∂^2 is a translation-invariant operator, this is the same as first applying ∂^2 to the function f and then evaluating at $a+b$.

Alternatively, we may appeal to (the commutative case of) Proposition 4.7 in [19], which tells us that the inner integral in (2.19) can be computed (for polynomials functions f) as

$$\int f(a+b)\gamma_{s,\tau}^N(b)=\left(\exp\left\{\frac{1}{2}\Delta_{s,\tau}\right\}f(a+b)\right)\Big|_{b=0}, \tag{2.22}$$

where the exponential is computed as a convergent power series in powers of $\Delta_{s,\tau}$. At this point, the (commuting!) operators that make up $\Delta_{s,\tau}$ in (2.18) become operators in some *finite-dimensional* space of polynomials of degree at most k. It is therefore permissible to differentiate $\exp\left\{\frac{1}{2}\Delta_{s,\tau}\right\}$ with respect to τ in the obvious way, pulling down a factor of $-\frac{1}{2}\partial^2$ onto f. We may then apply (2.22) in the opposite direction and we will arrive at (2.21). □

Lemma 2.10 *Let A be a variable ranging over $M_N(\mathbb{C})$, let z be a variable ranging over $\mathbb{C}$, and let B be a fixed matrix. Then*

$$\partial^2\det(zI-(B+A))=-\frac{1}{N}\frac{d^2}{dz^2}\det(zI-(B+A)), \tag{2.23}$$

where ∂^2 is the operator defined in (2.15).

Proof Both sides of (2.23) are easily seen to be polynomials in z and the entries of A. It therefore suffices to establish the identity on the nonempty open set of pairs (A,z) where $zI-A$ is invertible. We will use the notation

$$Q=(zI-(B+A))$$

$$R=Q^{-1}$$

and we can verify the following basic rules for computing:

$$Z_j A = X_j,$$

where Z_j is as in (2.14). We then use Jacobi's formula [35, Section 8.3] for the derivative of the determinant (invertible case):

$$\frac{d}{du}\det(A(u)) = \det(A(u))\mathrm{Trace}\left(A^{-1}\frac{dA}{du}\right),$$

where Trace is the ordinary trace; and the formula [35, Section 8.4] for the derivative of the inverse:

$$\frac{d}{du}A(u)^{-1} = -A(u)^{-1}\frac{dA}{du}A(u)^{-1}.$$

In what follows, we use the normalized trace, $\mathrm{tr}[\cdot] = \mathrm{Trace}[\cdot]/N$. Using the tools in the previous paragraph, we can easily compute

$$\begin{aligned} Z_j \det(Q) &= -N\det(Q)\mathrm{tr}[RX_j] \\ Z_j^2 \det(Q) &= N^2\det(Q)\mathrm{tr}[RX_j]^2 - N\det(Q)\mathrm{tr}[RX_jRX_j]. \end{aligned}$$

We then use the "magic formulas" for sums involving an orthonormal basis $\{X_j\}_{j=1}^{N^2}$ as in the proof of Lemma 2.9:

$$\sum_j X_j A X_j = \mathrm{tr}[A]I \tag{2.24}$$

$$\sum_j \mathrm{tr}[X_j A]\,\mathrm{tr}[X_j B] = \frac{1}{N^2}\,\mathrm{tr}[AB]. \tag{2.25}$$

(See, for example, [18, Proposition 3.1], with a change of sign because the X_j's here are Hermitian rather than skew Hermitian.) We therefore obtain:

$$\partial^2\det(Q) = \det(Q)\mathrm{tr}[R^2] - N\det(Q)\mathrm{tr}[R]^2. \tag{2.26}$$

Meanwhile, a similar computation shows

$$\frac{d}{dz}\det(Q) = N\det(Q)\mathrm{tr}[R] \tag{2.27}$$

$$\frac{d^2}{dz^2}\det(Q) = N^2\det(Q)\mathrm{tr}[R]^2 - N\det(Q)\mathrm{tr}[R^2]. \tag{2.28}$$

Comparing (2.26) to (2.28) gives the claimed result. □

With these lemmas in hand, we are ready for the proof of Proposition 2.8.

Proof of Proposition 2.8 We apply the identity (2.17) in Lemma 2.9 to the function f given by

$$f(A) = |\det(zI - (B + A))|^2 .$$

We note that ∂^2 treats anti-holomorphic functions as constants. We thus obtain

$$\begin{aligned} \frac{\partial D^N}{\partial \tau} &= -\frac{1}{2}\mathbb{E}\{\partial^2 \left|\det(zI - (X_0^N + Z_{s,\tau}^N))\right|^2\} \\ &= -\frac{1}{2}\mathbb{E}\left\{\overline{\det(zI - (X_0^N + Z_{s,\tau}^N))}\, \partial^2 \det(zI - (X_0^N + Z_{s,\tau}^N))\right\}. \end{aligned} \tag{2.29}$$

Using Lemma 2.10, (2.29) becomes

$$\begin{aligned} \frac{\partial D^N}{\partial \tau} &= \frac{1}{2N}\mathbb{E}\left\{\overline{\det(zI - A)}\, \frac{\partial^2}{\partial z^2} \det(zI - A)\right\} \\ &= \frac{1}{2N}\frac{\partial^2}{\partial z^2} D_\tau^N, \end{aligned}$$

as claimed in (2.12). Since D^N is real-valued, (2.13) follows. □

Now for the proof of Theorem 2.7.

Proof of Theorem 2.7 Let $\tilde{D}_{\tau_0}^N(s,\tau,z)$ denote the function on the right-hand side of (2.11), so that when $\tau = \tau_0$, we have $\tilde{D}_{\tau_0}^N(s,\tau_0,z) = D^N(s,\tau_0,z)$. Our goal is to show that $\tilde{D}_{\tau_0}^N = D^N$. The function $\tilde{D}_{\tau_0}^N$ can be computed as

$$\begin{aligned} &\tilde{D}_{\tau_0}^N(s,\tau,z) \\ &= \mathbb{E}\left\{\exp\left(\frac{1}{2N}\left((\tau - \tau_0)\frac{\partial^2}{\partial z^2} + (\bar{\tau} - \bar{\tau}_0)\frac{\partial^2}{\partial \bar{z}^2}\right)\right)\left|\det(zI - (X_0^N + Z_{s,\tau_0}^N))\right|^2\right\} \\ &= \exp\left(\frac{1}{2N}\left((\tau - \tau_0)\frac{\partial^2}{\partial z^2} + (\bar{\tau} - \bar{\tau}_0)\frac{\partial^2}{\partial \bar{z}^2}\right)\right)\mathbb{E}\left\{\left|\det(zI - (X_0^N + Z_{s,\tau_0}^N))\right|^2\right\}. \end{aligned} \tag{2.30}$$

From the last expression in (2.30), we can see that $\tilde{D}_{\tau_0}^N$ satisfies the same PDEs (2.12) and (2.13) as $D^N(s,\tau,z)$, as a function of τ and z. Thus,

$$\tilde{D}_{\tau_0}^N(s,\tau_0 + t(\tau - \tau_0),\tau_0,z) \text{ and } D^N(s,\tau_0 + t(\tau - \tau_0),z) \tag{2.31}$$

will satisfy the same PDE in t and z for $0 \leq t \leq 1$, with equality at $t = 0$. Since both functions are, for all values of the other variables, polynomials in z and $\bar{z}$ of degree $2N$, the PDE in t and z is actually an ODE with values in a finite-dimensional vector space. Thus, by uniqueness of solutions of ODEs, we conclude that the two functions in (2.31) are equal for all t; setting $t = 1$ gives the claimed result. □

2.5 A "counterexample"

Example 2.6 describes what we expect to happen if we evolve the characteristic polynomial of a GUE matrix by the forward heat flow $\exp\{\frac{t}{2N}\frac{d^2}{dz^2}\}$. The roots start out at $t = 0$ with an asymptotically semicircular distribution on $[-2, 2]$. Then for $0 < t < 2$, the roots should be asymptotically uniform on an ellipse with semi-axes $2 - t$ and t.

One might then expect that if p_0^N is *any* sequence of real-rooted polynomials of degree N having an asymptotically semicircular distribution of roots on $[-2, 2]$, the heat-evolved polynomials would have the same limiting root distribution as in the previous paragraph. But this claim is false. A counterexample is provided by the (scaled) Hermite polynomials

$$H_N(z) = \exp\left\{-\frac{1}{2N}\frac{d^2}{dz^2}\right\} z^N.$$

These polynomials are known [13] to have an asymptotically semicircular distribution of roots on $[-2, 2]$. But if we apply the forward heat equation, we get

$$\exp\left\{\frac{t}{2N}\frac{d^2}{dz^2}\right\} H_N(z) = \exp\left\{-\frac{(1-t)}{2N}\frac{d^2}{dz^2}\right\} z^N.$$

For $0 < t < 1$, the above polynomials are just rescaled Hermite polynomials and have an asymptotically semicircular distribution on $[-2\sqrt{1-t}, 2\sqrt{1-t}]$.

Of course, the result in the preceding paragraph does not actually contradict the conjecture in Example 2.6, which is only about the characteristic polynomial of GUE matrices. Nor does it contradict Conjecture 1.4, since the Cauchy transform of the semicircular law is discontinuous on the interval $[-2, 2]$. It is nevertheless notable that such different behavior can arise from two sequences of polynomials with the same limiting root distribution. We attribute this difference in behavior to the very evenly spaced nature of the zeros of the Hermite polynomials, which contrasts with the random fluctuations in the spacings that occurs for the eigenvalues of a GUE matrix.

Although the (rigorous) behavior of a heat-evolved Hermite polynomial contrasts with the (conjectural) behavior of the characteristic polynomial of a GUE, there is one common aspect to the two. Consider the holomorphic moments $M_{a,N}$ of a sequence p^N of polynomials of degree N, namely

$$M_{a,N} = \frac{1}{N}\sum_{j=1}^{N}(z_j^N)^a, \quad a = 0, 1, 2, \ldots,$$

where $\{z_j^N\}_{j=1}^N$ are the roots of p^N. According to Theorem 5.1 in Sect. 5, the large-N behavior of the holomorphic moments of heat-evolved polynomials is completely determined by the large-N behavior of the holomorphic moments of the initial polynomials. Thus, heat-evolved Hermite polynomials and heat-evolved characteristic polynomials of a GUE will have the same limiting holomorphic moments. These moments, however, do not uniquely determine the limiting root distribution in the case that the roots are complex.

3 A dynamical systems perspective

In this section, we record the system of differential equations satisfied by the roots of a polynomial as the polynomial itself evolves according to the heat equation. We then use this result to argue for the push-forward result in Point 2 of Conjecture 1.4. A similar argument holds in the multiplicative case.

3.1 The dynamics of the roots

We now record a well-known (and elementary) result about the evolution of zeros of a polynomial undergoing the heat flow.

Proposition 3.1 *Let p_0^N be a polynomial of degree N and let p_τ^N be the heat evolution of p_0^N, as defined in (1.6), so that*

$$\frac{\partial p_\tau^N}{\partial \tau} = \frac{1}{2N}\frac{\partial^2 p_\tau^N}{\partial z^2}. \tag{3.1}$$

Suppose that, for some $\sigma \in \mathbb{C}$, the zeros of p_σ are distinct. Then for all τ in a neighborhood of σ, it is possible to order the zeros of p_τ^N as $z_1(\tau), \ldots, z_N(\tau)$ so that each $z_j(\tau)$ depends holomorphically on τ and so that the collection $\{z_j(\tau)\}_{j=1}^N$ satisfies the following system of holomorphic differential equations:

$$\frac{dz_j(\tau)}{d\tau} = -\frac{1}{N}\sum_{k\neq j}\frac{1}{z_j(\tau) - z_k(\tau)}. \tag{3.2}$$

The paths $z_j(\tau)$ then satisfy

$$\frac{d^2 z_j(\tau)}{d\tau^2} = -\frac{2}{N^2}\sum_{k\neq j}\frac{1}{(z_j(\tau) - z_k(\tau))^3}. \tag{3.3}$$

The sums on the right-hand side of (3.2) and (3.3) are over all k different from j, with j fixed. The result in (3.2) is discussed on Terry Tao's blog [43] and dates back at least to the work of Csordas, Smith, and Varga [17, Lemma 2.4]. The result that (3.2) implies (3.3) dates back at least to work of Choodnovsky and Choodnovsky [15, Corollary 7]. We emphasize that the polynomial p_τ^N in (1.6) is well-defined for all $\tau \in \mathbb{C}$ by (1.6), whether the roots are distinct or not.

Remark 3.2 The second-order equations in (3.3) are the equations of motion for the **rational Calogero–Moser system**. (Take $\omega = 0$ and $g^2 = -1/N$ in the notation of [12, Eq. (3)].) It follows that solutions to (3.2) are special cases of solutions to the rational Calogero–Moser system, in which the initial velocities are chosen to satisfy (3.2) at $\tau = 0$.

Remark 3.2 (together with Conjecture 2.3) indicates a novel connection between integrable systems and random matrix theory. The negative value of g^2 in the remark means that τ is real and all the z_j's are initially real, and the system is attractive and collisions can occur—allowing the points to move off the real line.

For completeness, we supply a proof of Proposition 3.1.

Proof of Proposition 3.1 The local holomorphic dependence of the roots on τ is an elementary consequence of the holomorphic version of the implicit function theorem, with the assumption that the roots of p_σ are distinct guaranteeing that dp_σ/dz is nonzero at each root.

If z_j is a simple zero of a polynomial p, an easy power series argument shows that

$$\frac{p''(z_j)}{p'(z_j)} = 2\left(\frac{p'(z)}{p(z)} - \frac{1}{z - z_j}\right)\bigg|_{z=z_j},$$

from which we easily obtain

$$\frac{p''(z_j)}{p'(z_j)} = 2\sum_{k\neq j}\frac{1}{z_j - z_k}, \tag{3.4}$$

where the sum is over all k different from j, with j fixed. We may then differentiate the identity $p_\tau^N(z_j(\tau)) = 0$ to obtain

$$\frac{\partial p_\tau^N}{\partial \tau}(z_j(\tau)) + (p_\tau^N)'(z_j(\tau))\frac{dz_j}{d\tau} = 0. \tag{3.5}$$

Using (3.1), (3.5) gives

$$\frac{dz_j}{d\tau} = -\frac{\partial p_\tau^N}{\partial \tau}(z_j(\tau))\frac{1}{(p_\tau^N)'(z_j(\tau))} = -\frac{1}{2N}\frac{(p_\tau^N)''(z_j(\tau))}{(p_\tau^N)'(z_j(\tau))}.$$

Applying (3.4) then gives (3.2).

For the second derivative, we suppress the dependence of z_j on τ and we use (3.2) to make a preliminary calculation for each pair j and k with $j \neq k$:

$$\frac{d}{d\tau}(z_j - z_k) = -\frac{1}{N}\sum_{l\neq j}\frac{1}{z_j - z_l} + \frac{1}{N}\sum_{l\neq k}\frac{1}{z_k - z_l}. \tag{3.6}$$

We then split each sum over l on the right-hand side of (3.6) into a sum over $l \notin \{j, k\}$ plus an additional term:

$$\frac{d}{d\tau}(z_j - z_k) = -\frac{1}{N}\left(\frac{1}{z_j - z_k} + \sum_{l\notin\{j,k\}}\frac{1}{z_j - z_l}\right)$$

$$+\frac{1}{N}\left(\frac{1}{z_k - z_j} + \sum_{l \notin \{j,k\}} \frac{1}{z_k - z_l}\right),$$

which simplifies to

$$\frac{d}{d\tau}(z_j - z_k) = -\frac{2}{N}\frac{1}{z_j - z_k} + \frac{1}{N}\sum_{l \notin \{j,k\}} \frac{z_j - z_k}{(z_j - z_l)(z_k - z_l)}. \tag{3.7}$$

We then differentiate (3.2) using (3.7) to get

$$\begin{aligned}
&\frac{d^2 z_j}{d\tau^2} \\
&= \frac{1}{N}\sum_{k \neq j} \frac{1}{(z_j - z_k)^2}\left(-\frac{2}{N}\frac{1}{z_j - z_k} + \frac{1}{N}\sum_{l \notin \{j,k\}} \frac{z_j - z_k}{(z_j - z_l)(z_k - z_l)}\right) \\
&= -\frac{2}{N^2}\sum_{k \neq j} \frac{1}{(z_j - z_k)^3} + \frac{1}{N^2} \sum_{\substack{k,l: \\ (j,k,l) \text{ distinct}}} \frac{1}{(z_j - z_k)(z_j - z_l)(z_k - z_l)}.
\end{aligned}$$

The last sum over k and l is zero because the range of the sum is invariant under interchange of k and l, but the summand changes sign under interchange of k and l, leaving us with the claimed result. □

We now explain how Proposition 3.1 should imply push-forward result in Conjecture 1.4.

Idea 3.3 *Suppose p_0^N is a sequence of polynomials of degree N whose limiting root distribution is a sufficiently regular probability measure μ_0 with compact support. Then we expect that the following results will hold.*

(1) The initial velocity of the roots will be approximately the Cauchy transform of μ_0 evaluated at the initial root:

$$z_j'(0) \approx -m_0(z_j(0)).$$

(2) The second derivative of the roots will be small at any time τ for which the distribution of roots approximates a smooth two-dimensional distribution in the plane:

$$z_j''(\tau) \approx 0.$$

If these results hold, then as in (1.10),

$$z_j(\tau) \approx z_j(0) - \tau m_0(z_j(0)),$$

in which case the push-forward result in Point 2 of Conjecture 1.4 will hold.

We now give a nonrigorous argument for the preceding idea.

Argument for Idea 3.3 Point 1 is obtained by approximating the sum over k in (3.2), with the factor of $1/N$, by an integral with respect to μ_0. Meanwhile, the right-hand side of (3.3) is formally of order $1/N$, since it is a sum over $N-1$ values but we are dividing by N^2. In reality, the sum is not as small as this naive calculation would suggest, because $1/z^3$ is not a locally integrable function on the plane. Thus, there will be a substantial contribution in (3.3) from the values of z_k closest to z_j. But as long as these nearest neighbors are of order $1/\sqrt{N}$—which we expect if the distribution of roots is two-dimensional—we still expect that the right-hand side of (3.3) will be of order $1/\sqrt{N}$. It is difficult to prove anything like this rigorously, in part because it is difficult to control the nearest-neighbor spacings at time τ, even if these spacings behave nicely at time zero. □

3.2 Solving the equations using the Calogero–Moser systems

In this subsection, we explain how the theory of Calogero–Moser systems can be used to give an "explicit" formula for the roots of a heat-evolved polynomial. Of course, the formula still requires a calculation: finding the eigenvalues of a matrix. Nevertheless, it is notable that we can find the roots of the heat-evolved polynomial, for any one fixed time τ, directly in a comparatively simple way.

Let p_0^N be a polynomial of degree N with roots $z_1, \ldots, z_N$, listed with their multiplicities. Let X_0 be the diagonal matrix with the z_j's on the diagonal. Let Y with diagonal entries given by

$$Y_{jj} = \frac{1}{N} \sum_{k \neq j} \frac{1}{z_j - z_k} \tag{3.8}$$

and off-diagonal entries given by

$$Y_{jk} = \frac{1}{N} \frac{1}{z_j - z_k}, \quad j \neq k. \tag{3.9}$$

(Note that (3.8) has a sum but (3.9) does not.)

Proposition 3.4 *For any polynomial p_0^N of degree N, let p_τ^N be the heat-evolved polynomial, as in (1.6). Then for all $\tau \in \mathbb{C}$, the roots of p_τ^N are the eigenvalues of the matrix*

$$X_0 - \tau Y.$$

Specifically, if p_0^N is assumed to be monic—so that p_τ^N is also monic—then the characteristic polynomial of $X_0 - \tau Y$ is equal to p_τ^N.

Proof The claimed result is a direct application of the formulas for the Calogero–Moser system (3.3), with the initial velocities given by the first-order system in (3.2). See, for example, Section 2.7, just above the remark on p. 18, of the monograph [21] by Etingof (with a different scaling). □

We now discuss how Proposition 3.4 could lead to the expected large-N behavior of the roots of p_τ^N. We would expect from (3.8) that the diagonal entries Y_{jj} of Y can be approximated for large N by the initial Cauchy transform m_0:

$$Y_{jj} \approx m_0(z_j). \tag{3.10}$$

In particular, the diagonal entries of Y—and therefore of $X_0 - \tau Y$—should be of order 1 as N tends to infinity. By contrast, the off-diagonal entries of Y—and therefore the off-diagonal entries of $X_0 - \tau Y$—should be small. Indeed, these entries are typically of order $1/N$, with some small fraction of them being larger (order $1/\sqrt{N}$ if the nearest-neighbor spacings are of order $1/\sqrt{N}$).

We then might plausibly hope that the eigenvalues of $X_0 - \tau Y$, especially if $|\tau|$ is small, might be well approximated simply by the diagonal entries of $X_0 - \tau Y$. In that case, Proposition 3.4, together with (3.10), would tell us that

$$z_j(\tau) \approx z_j - \tau m_0(z_j),$$

which is consistent with Idea 3.3 and with Point 2 of Conjecture 1.4. We emphasize, however, that we do not currently know how to make this line of reasoning rigorous. In particular, the simplest estimate for the eigenvalues of almost-diagonal matrices, namely the Gershgorin circle theorem (e.g., [29, Theorem 6.1.1]), is not sharp enough to be useful here. Specifically, the estimate in Gershgorin's theorem involves the sum of the absolute values of the off-diagonal entries of the matrix over the rows of the matrix. Such sums are of at least of order 1 in our setting.

Nevertheless, Proposition 3.4 gives an intriguing alternative way of attacking the problem. If nothing else, the proposition seems to give a numerically efficient way to compute the roots of the heat-evolved polynomial p_τ^N—for one fixed τ—at least in the regime where Conjecture 1.4 holds. Proposition 3.4 can, for example, accurately replicate Fig. 5 in the "semicircular to circular" case but with much less computational effort than numerically solving the system of ODEs in (3.2).

3.3 Comparison to first-order Coulomb systems

It is instructive to compare the system of equations in (3.2) to first-order Coulomb systems in two dimensions, such as Eqs. (1.9) or (1.10) in the ICM lecture of Serfaty [41]. These equations read as, respectively,

$$\frac{dz_j}{d\tau} = \frac{1}{N} \sum_{k \neq j} \frac{1}{\bar{z}_j - \bar{z}_k}, \tag{3.11}$$

which is the ordinary gradient flow for the Coulomb energy of charged particles in the plane, and

$$\frac{dz_j}{d\tau} = \frac{i}{N} \sum_{k \neq j} \frac{1}{\bar{z}_j - \bar{z}_k}, \tag{3.12}$$

which is the rotated gradient flow for the Coulomb energy. (One can also study the *second*-order Newtonian dynamics of charged particles in the plane, as in Eq. (1.11) in [41].) Equation (3.11) also describes the dynamics of Ginzburg–Landau vortices, as in Eqs. (2.46) and (2.47) in [47]. Equations (3.11) and (3.12) are very similar to (3.2), except for the conjugates on z_j and z_k on the right-hand sides.

One might then suppose that the methods used to rigorously analyze the large-N behavior of the systems (3.11) and (3.12)—described in [41]—could be adapted to analyze (3.2) as well. This idea, however, turns out not to be correct. In an important technical sense, (3.2) is different from (3.11) and (3.12), namely that the system (3.2) is attractive in the x-direction and repulsive in the y-direction. That is, two nearby points z_j and z_k will attract each other if they are arranged horizontally from each other and will repel each other if they are arranged vertically from each other. This behavior is to contrasted with (3.11), which is repulsive in both directions (nearby points repel each other), and with the (3.12), which is neither attractive nor repulsive (nearby points circle around each other). The mix of attractive and repulsive behavior in (3.2) means that the methods used to study (3.11) and (3.12) cannot be used without substantial modifications.

4 A PDE perspective

In this section, we offer arguments from a PDE perspective for Conjecture 1.4 and for a multiplicative analog of that conjecture.

4.1 The main argument

We consider a sequence p^N of polynomials of degree N whose empirical root distribution converges to a compactly supported probability measure μ_0. As in the statement of Conjecture 1.4, we further assume that the Cauchy transform m_0 of μ_0 (as in Definition 1.3) is globally Lipschitz. We then let L_{m_0} denote the Lipschitz constant of m_0:

$$L_{m_0} = \sup_{z_1 \neq z_2} \left| \frac{m_0(z_1) - m_0(z_2)}{z_1 - z_2} \right| < \infty. \tag{4.1}$$

We now let p_τ^N denote the associated heat-evolved polynomials, as in (1.6). We now offer arguments for Points 1 and 2 of Conjecture 1.4 from a PDE perspective.

4.1.1 Point 1 of Conjecture 1.4

Point 1 of the conjecture—that the log potential of the limiting root distribution satisfies the PDE (1.7)—is supported by the following proposition.

Proposition 4.1 *The logarithmic potential $S^N(z,\tau) = \frac{1}{N}\log|p^N(z,\tau)|^2$ satisfies the following PDE*

$$\frac{\partial}{\partial\tau} S^N(z,\tau) = \frac{1}{2}\left(\frac{\partial S^N}{\partial z}\right)^2 + \frac{1}{2N}\frac{\partial^2 S^N}{\partial z^2} \tag{4.2}$$

whenever z is not a root of $p^N(z,\tau)$. Alternatively, the second term in (4.2) can be written in terms of the roots $\{z_j(\tau)\}_{j=1}^N$ of $p^N(z,\tau)$ so that

$$\frac{\partial}{\partial\tau}S^N(z,\tau)=\frac{1}{2}\left(\frac{\partial S^N}{\partial z}\right)^2-\frac{1}{2N^2}\sum_{j=1}^{N}\frac{1}{(z-z_j(\tau))^2}.$$

Proof Notice that $p^N(z,\tau)$ is holomorphic in τ and satisfies the equation

$$\frac{\partial}{\partial\tau}p^N(z,\tau)=\frac{1}{2N}\frac{\partial^2}{\partial z^2}p^N(z,\tau).$$

Around (z,τ) where $p^N(z,\tau)\neq 0$, p^N has a branch of log, so that

$$S^N=\frac{1}{N}\log p^N+\frac{1}{N}\overline{\log p^N}.$$

Since $\log p^N$ is holomorphic in z and τ, $\overline{\log p^N}$ is anti-holomorphic in z and τ. The τ-derivative of S^N is given by

$$\begin{aligned}\frac{\partial S^N}{\partial\tau}&=\frac{1}{2N^2}\frac{(\partial^2/\partial z^2)p^N(z,\tau)}{p^N(z,\tau)}\\&=\frac{1}{2N^2}\frac{(\partial^2/\partial z^2)e^{NS^N}}{e^{NS^N}}\\&=\frac{1}{2N^2}\left[N^2\left(\frac{\partial S^N}{\partial z}\right)^2+N\frac{\partial^2 S^N}{\partial z^2}\right],\end{aligned}$$

which simplifies to (4.2).

Finally, by writing $p^N(z,\tau)$ into a product of linear factors, the first and second z-derivatives of S^N are given by

$$\frac{\partial S^N}{\partial z}=\frac{1}{N}\frac{\partial/\partial z p^N(z,\tau)}{p^N(z,\tau)}=\frac{1}{N}\sum_{j=1}^{N}\frac{1}{z-z_j(\tau)}$$

and

$$\frac{\partial^2 S^N}{\partial z^2}=-\frac{1}{N}\sum_{j=1}^{N}\frac{1}{(z-z_j(\tau))^2}.$$

This proves the last assertion. □

The PDE in (4.2) formally converges to the desired PDE (1.7) as $N\to\infty$. We now look more carefully at the question of whether the error term (the second term on the right-hand side of (4.2)) is actually small for large N.

Recall that

$$\frac{\partial^2 S^N}{\partial z^2} = -\frac{1}{N}\sum_{i=1}^{N}\frac{1}{(z - z_i(\tau))^2}.$$

We expect the estimate

$$\frac{\partial^2 S^N}{\partial z^2} = -\frac{1}{N}\sum_{i=1}^{N}\frac{1}{(z - z_i(\tau))^2} \approx -\text{P.V.}\int \frac{1}{(z - w)^2}d\mu_\tau(w).$$

Thus, the principal value integral should be finite as long as $|\tau|$ is small enough so that μ_τ remains absolutely continuous on $\mathbb{C}$. If this is true, the term $\frac{1}{2N}\frac{\partial^2 S^N}{\partial z^2} \to 0$ as $N \to \infty$, and the limit of S^N in Theorem 4.1 becomes the solution of the desired PDE (1.7).

Furthermore, the argument in the "Details of Step 1" portion of Sect. 4.1.3 (based on Theorem 1.1 and Remark 1.1 in [34]) indicates that if the solution is C^1, it will automatically by $C^{1,1}$, as claimed in Point 1 of Conjecture 1.4.

4.1.2 Brief argument for Point 2 of Conjecture 1.4

In this section, we give a brief argument in three steps why we believe the push-forward relation in Point 2 of Conjecture 1.4. Briefly, the argument is that *Point 2 should follow from Point 1*, modulo technical issues having to do with the regularity of the log potential. A separate argument for Point 2 of the conjecture was provided in Sect. 3; see Idea 3.3.

Step 1: We expect the PDE (1.7) has a unique solution for small time; that is, the logarithmic potential S of the limiting root distribution in Conjecture 1.4 is the *unique* solution of the PDE (1.7) for small $|\tau|$. The PDE (1.7) is of Hamilton–Jacobi type and (sufficiently regular) solutions can be analyzed using the method of characteristics. The global Lipschitz assumption on the Cauchy transform m_0 of μ_0 will guarantee that, for small enough τ, each point in the plane can be reached by a unique characteristic curve, giving uniqueness.

Step 2: The density W of the limiting measure μ_τ can be calculated from the logarithmic potential S by

$$W = \frac{1}{\pi}\frac{\partial^2 S}{\partial z \partial \bar{z}}.$$

If S is smooth enough, we may apply $\frac{1}{\pi}\frac{\partial}{\partial \bar{z}}$ and then $\frac{\partial}{\partial z}$ on both sides of the PDE (1.7), giving

$$\frac{\partial}{\partial \tau}\left(\frac{1}{\pi}\frac{\partial^2 S}{\partial z \partial \bar{z}}\right) = \frac{\partial}{\partial z}\left[\left(\frac{\partial S}{\partial z}\right)\cdot\frac{1}{\pi}\left(\frac{\partial^2 S}{\partial \bar{z}\partial z}\right)\right].$$

Hence, the density W satisfies a (complex-time) continuity equation

$$\frac{\partial W}{\partial \tau} + \frac{\partial}{\partial z}\left[-\frac{\partial S}{\partial z}W\right] = 0, \tag{4.3}$$

where we treat S as a known quantity—the unique solution of the PDE (1.7).

We can use general results of continuity equations by converting the complex-time continuity equation into a real-time continuity equation. Equation (4.3) can be interpreted as saying that the measure $\mu_\tau = W\,dx\,dy$ flows along the vector field $-\partial S/\partial z$. (Here we interpret the complex number $-\partial S/\partial z$ as a vector in the plane.) As long as $\partial S/\partial z$ is Lipschitz in z, general results for continuity equations (e.g., [2]) will tell us that the measure μ_τ is the push-forward of μ_0 by the map obtained by flowing along $-\partial S/\partial z$.

Step 3: Step 2 in the argument says that μ_τ is the push-forward of μ_0 under the flow of $-\partial S/\partial z$. We now show that this map can be computed in a simple, explicit form as in Point 2 of Conjecture 1.4.

The Hamilton–Jacobi equation (1.7) can be analyzed using the Hamilton's equations associated to the (complex variable) Hamiltonian

$$H(z,p) = -\frac{1}{2}p^2, \tag{4.4}$$

where p is the (complex-valued) momentum variable. The Hamiltonian is holomorphic in z and p, and independent of z. While we are not aware of a general theory on complex-time Hamilton–Jacobi equations, we can do an analysis similar to [23, Section 5] by converting the equation into a real-time Hamilton–Jacobi equation. As we will see in Sect. 4.2, the result will be Hamilton–Jacobi formulas similar to the usual real-time formulas, as in the next paragraph.

The Hamilton's equations associated to the Hamiltonian (4.4) are

$$\frac{dz}{d\tau} = \frac{\partial H}{\partial p} = -p; \quad \frac{dp}{d\tau} = -\frac{\partial H}{\partial z} = 0,$$

where we look for solutions that are holomorphic functions of τ. The initial condition z_0 for $z(\tau)$ can be chosen arbitrarily on $\mathbb{C}$, but the initial condition p_0 for $p(\tau)$ is chosen to be

$$p_0 = \frac{\partial}{\partial z_0} S(z_0, 0) = m_0(z_0). \tag{4.5}$$

The solution of the Hamilton's equations are

$$p(\tau) = p_0; \quad z(\tau) = z_0 - \tau p_0. \tag{4.6}$$

The solutions (4.6) are called the **characteristic curves** of the PDE (1.7). The first and second Hamilton–Jacobi formulas then hold:

$$S(z(\tau),\tau) = S(z_0,0) - \mathrm{Re}[\tau p_0^2] \tag{4.7}$$

$$\frac{\partial S}{\partial z}(z(\tau),\tau) = p(\tau). \tag{4.8}$$

These two formulas are derived in Sect. 4.2, Proposition 4.5.

If we combine the Hamilton's equations and the second Hamilton–Jacobi formula, we have

$$\frac{dz}{d\tau} = -p(\tau) = -\frac{\partial S}{\partial z}(z(\tau), \tau).$$

The right-hand side of the above equation is exactly the **vector field occurring in** the continuity equation (4.3). Thus, the integral curves for the vector field $-\partial S/\partial z$ are exactly the characteristic curves of the Hamilton–Jacobi equation (1.7). It is notable that the integral curves are holomorphic in τ.

The integral curves of $-\partial S/\partial z$ are the characteristic curves $z_0 - \tau p_0$, and p_0 is $m_0(z_0)$; the map T_τ defined in Point 2 of Conjecture 1.4 is just the characteristic curves evaluated at τ. Recall that the general results of the continuity equation in Step 2 tells us that the density W of the limiting root distribution flows along the characteristic curves. This gives an argument for Point 2 of Conjecture 1.4.

4.1.3 Details of Point 2 of Conjecture 1.4

In this section, we fill in the missing details in Sect. 4.1.2. We also note certain places where the argument is not rigorous, for reasons having to do with the regularity of the log potential S.

Details of Step 1: The PDE (1.7) is of Hamilton–Jacobi type with complex time and space variables. In Sect. 4.2, we transform the PDE into another Hamilton–Jacobi type PDE with a real time and two real space variables. Now, under quite general conditions (e.g., [16]), Hamilton–Jacobi equations have unique viscosity solutions. Furthermore, any C^1 solution is automatically a viscosity solution, so there can be at most one C^1 solution, giving the desired uniqueness result.

We also note that by Theorem 1.1 and Remark 1.1 in [34], one can construct a $C^{1,1}$ solution to a Hamilton–Jacobi equation by the method of characteristics, *provided* that the initial data are $C^{1,1}$ and that the flow along the characteristic curves is a bi-Lipschitz homeomorphism. Thus, under these assumptions, any C^1 solution must be given by the method of characteristics—and this C^1 solution will actually be $C^{1,1}$.

We now show that, in fact, every point in the plane is hit by a unique characteristic curve, provided that τ is small enough. We first note that, in light of (4.5) and (4.6), the characteristic curves can be computed in terms of the transport map as

$$z(\tau) = T_\tau(z_0).$$

We then establish nice behavior for T_τ, for small τ.

Lemma 4.2 *For all $|\tau| < L_{m_0}^{-1}$ (where L_{m_0} is the Lipschitz constant of m_0), T_τ is a bi-Lipschitz homeomorphism of $\mathbb{C}$ to $\mathbb{C}$. Indeed, T_τ is uniformly bi-Lipschitz for $|\tau| \le c$, for any $c < L_{m_0}$.*

Proof Suppose that $T_\tau(z_1) - T_\tau(z_2) = 0$ for some $z_1 \neq z_2$. Then after an algebraic calculation, we get

$$\frac{m_0(z_1) - m_0(z_2)}{z_1 - z_2} = -\frac{1}{\tau},$$

which is impossible if $|\tau| < L_{m_0}^{-1}$. This proves injectivity of T_τ. To prove surjectivity of T_τ, we note that $T_\tau(z) \approx z$ for large $|z|$. Thus, we can extend T_τ to a continuous map on $\mathbb{S}^2$ by defining $T_\tau(\infty) = \infty$. Suppose now that, on the contrary, there exists $z_0 \in \mathbb{C}$ such that the image $T_\tau(\mathbb{S}^2)$ does not contain z_0. Then this extended T_τ is a continuous map from $\mathbb{S}^2$ into $\mathbb{S} \setminus \{z_0\} \cong \mathbb{R}^2$. By the Borsuk–Ulam theorem, T_τ cannot be a injective map, contradicting the previously established injectivity of T_τ. (Alternatively, surjectivity may be proved using the fundamental group of the punctured plane, following a well-known proof of the fundamental theorem of algebra. See, for example, the proof of Theorem 56.1 in [38].)

Finally, we address the claimed bi-Lipschitz property of T_τ. We compute that

$$\frac{T_\tau(z_1) - T_\tau(z_2)}{z_1 - z_2} = 1 + \tau \frac{m_0(z_1) - m_0(z_2)}{z_1 - z_2}.$$

Thus, for any $c < L_{m_0}^{-1}$,

$$0 < 1 - cL_{m_0} \leq \left| \frac{T_\tau(z_1) - T_\tau(z_2)}{z_1 - z_2} \right| \leq 1 + cL_{m_0}$$

if $|\tau| \leq c$. □

Before we move to the details of Step 2, we first define the time $\tau_{\max}$ where the PDE and the transport map T_τ have nice behaviors.

Notation 4.3 *Let $\tau_{\max} = \min\left(C, L_{m_0}^{-1}\right)$, where C is the constant in Conjecture 1.4 and L_{m_0} is the Lipschitz constant of m_0, as in (4.1).*

Details of Step 2: Step 2 of the argument in Sect. 4.1.2 concerns the deformation of the measure μ_τ. Let $W(\cdot, \tau)$ be the density of μ_τ. If S is C^3, then we can take $\frac{1}{\pi}\partial^2/\partial z \partial \bar{z}$ on both sides of (1.7) to get the continuity equation. We expect that if the solution is only $C^{1,1}$, the resulting continuity equation will still hold in the weak sense.

In the following result, which is proven in Sect. 4.2, we transform the complex-time continuity equation (4.3) into a (real-time) continuity equation for W. Then in Step 3, we derive that μ_τ is the push-forward measure of μ_0 by the map T_τ.

Proposition 4.4 *Suppose that W satisfies the complex-time continuity equation* (4.3). *Let*

$$W^\tau(z, t) = W(z, t\tau).$$

Then W^τ satisfies the real-time continuity equation

$$\frac{\partial W^\tau}{\partial t} + \nabla \cdot (bW^\tau) = 0 \tag{4.9}$$

where b is the vector field

$$b(z, \tau) = b(z, \tau) = -\begin{pmatrix} \tau_1 & -\tau_2 \\ \tau_2 & \tau_1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\frac{\partial S}{\partial x} \\ -\frac{1}{2}\frac{\partial S}{\partial y} \end{pmatrix} = -\operatorname{vect}\left(\tau \frac{\partial S}{\partial z}(z, \tau)\right) \tag{4.10}$$

where the operator vect *takes a complex number $x + iy$ to the vector (x, y) in $\mathbb{R}^2$.*

By general results of the continuity equation (see, for example, Proposition 3.1 and Theorem 4.1 of [2]), if the initial condition of the continuity equation is the density of a measure, then the solution of a continuity equation is given by the push-forward of the measure by the flow of the vector field. We will show in details of Step 3 that the ODE describing the flow along the vector field (4.10) has a unique solution for every initial point on $\mathbb{C}$. Theorem 4.1 of [2] then guarantees us the solution of the continuity equation is unique.

Proof of Proposition 4.4 Using the chain rule, we compute

$$\frac{\partial W^\tau}{\partial t} = 2\,\mathrm{Re}\left[\tau \frac{\partial W}{\partial \tau}\right] = \mathrm{Re}\left[2\frac{\partial}{\partial z}\left(\tau \frac{\partial S}{\partial z} W^\tau\right)\right].$$

Expanding the right-hand side, we get

$$\frac{\partial W^\tau}{\partial t} = \frac{\partial}{\partial x}\left[\left(\frac{\tau_1}{2}\frac{\partial S}{\partial x} + \frac{\tau_2}{2}\frac{\partial S}{\partial y}\right) W^\tau\right] + \frac{\partial}{\partial y}\left[\left(\frac{\tau_2}{2}\frac{\partial S}{\partial x} - \frac{\tau_1}{2}\frac{\partial S}{\partial y}\right) W^\tau\right].$$

This means we can write the above equation into the standard form of a continuity equation

$$\frac{\partial W^\tau}{\partial t} + \nabla \cdot (bW^\tau) = 0,$$

where b is the vector field in the statement of the proposition. □

Details of Step 3: We consider the real-time continuity equation in (4.9) and we consider the integral curves of the associated vector field $-\mathrm{vect}(\tau\,\partial S/\partial z)$. That is, we look for solutions $w_\tau(t)$ to

$$\frac{dw_\tau(t)}{dt} = -\tau \frac{\partial S}{\partial z}(w_\tau(t), t\tau). \tag{4.11}$$

Now, assuming that the Hamilton–Jacobi formulas are applicable to the (real-variables version of) the equation for S, Proposition 4.6 will tell us that the right-hand side of (4.11) will be Lipschitz. Thus, by Picard–Lindelöf theorem, the ODE has a unique solutions. Then by Theorems 3.1 and 4.1 of [2], the continuity equation (4.9) has a unique solution, given by push-forward along the curves in (4.11).

We then solve (4.11) by a real-variables version of the argument we gave for Step 3 in Sect. 4.1.2. Specifically, we consider the function $S^\tau(z, t) = S(z, t\tau)$. The proof of Proposition 4.5 in Sect. 4.2 will show that the curves $z^\tau(t) = z_0 - t\tau p_0$ and $p^\tau(t) = p_0$ satisfy

$$\frac{dz^\tau}{dt} = -\tau p^\tau(t) = -\tau \frac{\partial S^\tau}{\partial z}(z^\tau(t), t) = -\tau \frac{\partial S}{\partial z}(z^\tau(t), t\tau).$$

Thus, the curves $t \mapsto z^\tau(t)$ are the unique solutions of (4.11) with the initial condition z_0. Taking $t = 1$, we conclude that the measure $W(z, \tau)\, dx\, dy$ is the push-forward of $W(z, 0)\, dx\, dy$ under the map $z \mapsto z - \tau p_0 = z - \tau m_0(z)$.

4.2 Auxiliary calculations

We will use the ordinary Hamilton–Jacobi method to prove that the solution of the PDE (1.7) satisfies the Hamilton–Jacobi formulas (4.7) and (4.8). The proof requires the solution S to be C^2, although we expect that the result will continue to hold if the solution is only $C^{1,1}$.

Proposition 4.5 *Suppose S is a C^2 solution of (1.7) for $|\tau| \leq C$. Then the values of S and $\partial S/\partial z$ can be calculated using the characteristic curves (4.6) as in (4.7) and (4.8).*

Proof We first turn the PDE (1.7) into a PDE with real time and apply the first Hamilton–Jacobi formulas. (See, for example, Proposition 5.3 of [20].) For each $|\tau| \leq C$, define

$$S^\tau(z, t) = S(z, t\tau).$$

The z-derivative of S^τ and S coincides with the same (z, τ), so that

$$\begin{aligned}\frac{\partial S^\tau}{\partial t} &= \frac{\partial S}{\partial \tau}\frac{\partial t\tau}{\partial t} + \frac{\partial S}{\partial \bar{\tau}}\frac{\partial t\bar{\tau}}{\partial t} \\ &= 2\mathrm{Re}\left[\frac{\tau}{2}\left(\frac{\partial S^\tau}{\partial z}\right)^2\right] \\ &= \frac{\mathrm{Re}\,\tau}{4}\left[\left(\frac{\partial S^\tau}{\partial x}\right)^2 - \left(\frac{\partial S^\tau}{\partial y}\right)^2\right] + \frac{\mathrm{Im}\,\tau}{2}\frac{\partial S^\tau}{\partial x}\frac{\partial S^\tau}{\partial y},\end{aligned}$$

where we have written z into real variables by $z = x + iy$. Thus, $S^\tau(z, t)$ satisfies a Hamilton–Jacobi PDE, with Hamiltonian

$$H^\tau(x, y, p_x, p_y) = -\frac{1}{4}\mathrm{Re}[\tau(p_x - ip_y)^2].$$

The Hamilton's equations for this Hamiltonian are given as

$$\dot{x} = \frac{\partial H^\tau}{\partial p_x}; \;\; \dot{y} = \frac{\partial H^\tau}{\partial p_y}; \;\; \dot{p}_x = -\frac{\partial H^\tau}{\partial x}; \;\; \dot{p}_y = -\frac{\partial H^\tau}{\partial y}.$$

Once the initial condition $z_0 = x_0 + iy_0$ is chosen, the initial conditions $p_{x,0}$ and $p_{y,0}$ of the Hamilton's equations are taken to be

$$p_{x,0} = \frac{\partial S^\tau}{\partial x}(z_0, 0); \quad p_{y,0} = \frac{\partial S^\tau}{\partial y}(z_0, 0).$$

We then easily obtain

$$\begin{aligned}p_x(t) &= p_{x,0}; \quad p_y(t) = p_{y,0} \\ x(t) &= x_0 - \frac{t}{2}\mathrm{Re}[\tau(p_{x,0} - ip_{y,0})]\end{aligned}$$

$$y(t) = y_0 - \frac{t}{2}\mathrm{Im}[\tau(p_{x,0} - ip_{y,0})].$$

We then rewrite our results using the complex-valued functions z^τ and p^τ given by

$$z^\tau(t) = x(t) + iy(t) = z_0 - t\tau p_0$$
$$p^\tau(t) = \frac{1}{2}(p_x(t) - ip_y(t)).$$

We can now apply Eqs. (5.11) and (5.12) of [20] to get the formulas

$$\begin{aligned} S^\tau(z^\tau(t), t) &= S^\tau(z_0, 0) - \mathrm{Re}\left[\tau\left(\frac{1}{2}(p_{x_0} - ip_{y,0})\right)^2\right]t \\ &= S^\tau(z_0, 0) - t\mathrm{Re}[\tau p_0^2] \end{aligned} \tag{4.12}$$

and

$$\frac{\partial S^\tau}{\partial z}(z^\tau(t), t) = p^\tau(t). \tag{4.13}$$

Evaluating (4.12) and (4.13) at $t = 1$ gives the claimed formulas. □

Proposition 4.6 *Assume that the conclusion of Proposition 4.5 holds; that is, the characteristic curves (4.6) describe the values of S and $\partial S/\partial z$ by the formulas (4.7) and (4.8). Then given any $c < L_{m_0}^{-1}$, for any $|\tau| \le c$, $\partial S/\partial z(\cdot, \tau)$ is uniformly Lipschitz.*

Proof By the second Hamilton–Jacobi formula (4.8), the global Lipschitz assumption on m_0, and Lemma 4.2, we have

$$\begin{aligned} \left|\frac{\partial S}{\partial z}(z_1, \tau) - \frac{\partial S}{\partial z}(z_1, \tau)\right| &= |m_0(T_\tau^{-1}(z_1)) - m_0(T_\tau^{-1}(z_2))| \\ &\le L_{m_0}|T_\tau^{-1}(z_1) - T_\tau^{-1}(z_2)| \\ &\le L_{m_0}(1 + cL_{m_0})\,|z_1 - z_2| \end{aligned}$$

for all $|\tau| \le c < L_{m_0}^{-1}$. □

5 Evolution of the holomorphic moments

If μ is a compactly supported probability measure on the plane, we let M_a be the a-th **holomorphic moment** of μ, given as

$$M_a = \int_{\mathbb{C}} z^a \, d\mu(z), \quad a = 0, 1, 2, \ldots,$$

so that $M_0 = 1$. We emphasize that these moments do not uniquely determine the measure. (Every rotationally invariant choice of μ, for example, gives $M_a = 0$ for all

$a \geq 1$.) Nevertheless, the holomorphic moments contain a lot of information about the measure.

Now let p_0^N be a polynomial of degree N and let p_τ^N be the heat evolution of p_0^N, as in (1.2). We then let $M_{a,N}(\tau)$ be the a-th holomorphic moment of the empirical root measure of p_τ^N, that is,

$$M_{a,N}(\tau) = \frac{1}{N}\sum_{j=1}^{N}(z_{j,N}(\tau))^a, \tag{5.1}$$

where $\{z_{j,N}(\tau)\}_{j=1}^N$ are the roots of p_τ^N.

The following theorem gives mechanism for computing these moments inductively as a function of $a = 0, 1, \ldots$, allows us to compute the large-N limit of the moments, and shows that the large-N moments are consistent with the prediction of the PDE in Point 1 in Conjecture 1.4. We may therefore say that Point 1 of Conjecture 1.4 holds "at the level of the holomorphic moments."

Theorem 5.1 *We have the following results.*

(1) *For any polynomial p_0^N of degree N, the moments $M_{a,N}(\tau)$ in (5.1) satisfy*

$$\frac{dM_{0,N}}{d\tau} = \frac{dM_{1,N}}{d\tau} = 0 \tag{5.2}$$

and for $a \geq 2$,

$$\frac{dM_{a,N}}{d\tau} = \frac{1}{2N}a(a-1)M_{a-2,N}(\tau) - \frac{a}{2}\sum_{b=0}^{a-2}M_{b,N}(\tau)M_{a-b-2,N}(\tau). \tag{5.3}$$

(2) *Suppose now that p_0^N is a sequence of polynomials of degree N such that the limit*

$$M_a(0) := \lim_{N\to\infty} M_{a,N}(0)$$

exists for all a. Then the limit

$$M_a(\tau) := \lim_{N\to\infty} M_{a,N}(\tau)$$

exists for all τ and these limiting moments satisfy the limiting version of (5.3), namely

$$\frac{dM_a}{d\tau} = -\frac{a}{2}\sum_{b=0}^{a-2}M_b(\tau)M_{a-b-2}(\tau). \tag{5.4}$$

(3) *Suppose μ_τ is a family of compactly supported probability measures whose log potential $S(z,\tau)$ satisfies the PDE (1.7) for z near infinity and τ in some open set*

$U \subset \mathbb{C}$. Let $\hat{M}_a(\tau)$ be the a-th holomorphic moment of μ_τ. Then these moments also satisfy (5.4):

$$\frac{d\hat{M}_a}{d\tau} = -\frac{a}{2}\sum_{b=0}^{a-2}\hat{M}_b(\tau)\hat{M}_{a-b-2}(\tau) \tag{5.5}$$

for $\tau \in U$.

Although we will establish (5.3) using the PDE methods from Sect. 4, it is also possible to verify the result using the ODE system (3.2) in Sect. 3.1.

Proof By (4.2), the Cauchy transform $m^N(z,\tau) = \frac{\partial S^N}{\partial z}(z,\tau)$ of the empirical root distribution of p_τ^N satisfies the PDE

$$\frac{\partial m^N}{\partial \tau} = \frac{\partial}{\partial z}\left[\frac{1}{2}(m^N)^2 + \frac{1}{2N}\frac{\partial m^N}{\partial z}\right]. \tag{5.6}$$

Meanwhile, the Cauchy transform m of any compactly supported probability measure μ can be computed near infinity as

$$\begin{aligned} m(z) &:= \int_{\mathbb{C}} \frac{1}{z-w}\,d\mu(w) \\ &= \frac{1}{z}\int_{\mathbb{C}}\left(1 + \frac{w}{z} + \frac{w^2}{z^2} + \cdots\right)d\mu(w) \\ &= \frac{1}{z}\left(1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots\right), \end{aligned}$$

where $\{a_j\}_{j=0}^\infty$ are the holomorphic moments of μ.

We then have

$$\frac{\partial}{\partial \tau}m^N(z,\tau) = \frac{1}{z}\left(\frac{a'_{1,N}(\tau)}{z} + \frac{a'_{2,N}(\tau)}{z^2} + \frac{a'_{3,N}(\tau)}{z^3} + \cdots\right) \tag{5.7}$$

and, after a short calculation,

$$\begin{aligned} &\frac{\partial}{\partial z}\left[\frac{1}{2}(m^N)^2 + \frac{1}{2N}\frac{\partial m^N}{\partial z}\right] \\ &= \sum_{j=2}^{\infty}\frac{1}{z^{j+1}}\left(\frac{1}{2N}j(j-1)a_{j-2,N}(\tau) - \frac{j}{2}\sum_{k=0}^{j-2}a_{j,N}(\tau)a_{j-k,N}(\tau)\right). \end{aligned} \tag{5.8}$$

Equating coefficients between (5.7) and (5.8) gives the $dM_{1,N}/d\tau = 0$ (for the coefficient of $1/z^2$) and gives (5.3) (for the coefficient of $1/z^{j+1}$, $j \geq 2$).

Meanwhile, we can solve for $M_{a,N}(\tau)$ inductively in a, simply by integrating (5.2) or (5.3) with respect to τ, with a fixed value of $M_{a,N}(0)$. The resulting expression will depend continuously on the parameter $C = 1/N$, up to $C = 0$. If we integrate, let $C \to 0$, and then differentiate in τ, we obtain (5.4).

Finally, if the log potential of μ_τ satisfies (1.7) near infinity, then the Cauchy transform of μ_τ satisfies the PDE (5.6), but without the $1/N$ term on the right-hand side. We may therefore repeat the above calculation without the $1/N$ term to get (5.5). □

6 The multiplicative case

In this section, we develop a "multiplicative" version of our main heat flow conjectures, in a way that closely parallels results for the standard heat flow, which we refer to as the "additive" case. The term "multiplicative" refers to the associated random matrix model, namely the Brownian motion in the general linear group, which can be approximated by multiplying together a large number of independent random matrices. (See (6.3).)

The multiplicative version of the heat flow we consider is essentially the one considered by Tao in his blog post [44]. See Remarks 6.4 and 6.8 for a precise comparison.

6.1 The multiplicative heat flow conjecture for random matrices

We now define a "multiplicative" version $B^N_{s,\tau}$ of the elliptic random matrix model $Z^N_{s,\tau}$. To do this, let X^N_r and Y^N_r be independent Brownian motions in the space of $N \times N$ Hermitian matrices, normalized so that X^N_1 and Y^N_1 are GUEs. Then, imitating (2.1), we define an **elliptic Brownian motion** by

$$Z^N_{s,\tau}(r) = e^{i\theta}(aX^N_r + ibY^N_r),$$

where the parameters a, b, and θ are chosen to give the desired values of s and τ in (2.2) at $r = 1$. Then we introduce a Brownian motion $B^N_{s,\tau}(r)$ as the solution of the following stochastic differential equation

$$dB^N_{s,\tau}(r) = B^N_{s,\tau}(r)\left(i\, dZ^N_{s,\tau}(r) - \frac{1}{2}(s-\tau)\, dr\right) \tag{6.1}$$

$$B^N_{s,\tau}(0) = I \tag{6.2}$$

Here, the dr term on the right-hand side of (6.1) is an Itô correction. The process $B^N_{s,\tau}(r)$ is a left-invariant Brownian motion living in the general linear group $GL(N;\mathbb{C})$. We typically take $r = 1$, since $B^N_{s,\tau}(r)$ has the same distribution as $B^N_{rs,r\tau}(1)$. We thus use the notation

$$B^N_{s,\tau} = B^N_{s,\tau}(r)\Big|_{r=1}.$$

The law of $B^N_{s,\tau}$ is the measure denoted as $\mu_{s,\tau}$ in [19], where it serves to define norm on the target Hilbert space in the complex-time Segal–Bargmann transform. When $\tau = 0$, the distribution of $B^N_{s,0}$ is that of a Brownian motion in the unitary group at time s. We refer to $B^N_{s,\tau}$ as a "multiplicative" random matrix model because the solution to (6.1) and (6.2) can be approximated by multiplying together independent random matrices:

$$B^N_{s,\tau} \approx \prod_{j=1}^{k} \left(I + \frac{i}{\sqrt{k}} Z_j - \frac{1}{2k}(s-\tau) I \right) \tag{6.3}$$

for some large positive integer k, where $Z_1, \ldots, Z_k$ are independent random matrices with the same distribution as $Z^N_{s,\tau}$.

We let $b_{s,\tau}(r)$ be the "free" version of $B^N_{s,\tau}(r)$, obtained by replacing X^N_r and Y^N_r by their free counterparts and then solving the free version of (6.1) and (6.2). We again take $r = 1$ and use the notation $b_{s,\tau}$ for $b_{s,\tau}(1)$. When $\tau = s$, the process $b_{s,s}$ is Biane's free multiplicative Brownian motion (denoted Λ_s in [8, Section 4.2.1]). When $\tau = 0$, the process $b_{s,0}$ is Biane's free unitary Brownian motion [7, Section 2.3]. When $\tau = t$ is real, Kemp [33] shows that the large-N limit of $B^N_{s,t}$, in the sense of $*$-distribution, is $b_{s,t}$.

In the special case $\tau = s$, Driver, Hall, and Kemp [20], building on results of Hall and Kemp [27], compute the Brown measure of $b_{s,s}$ using a novel PDE method. Ho and Zhong [28] then compute the Brown measure of $ub_{s,s}$, where u is a unitary element freely independent of $b_{s,s}$. Hall and Ho [23] then compute the Brown measure of $ub_{s,\tau}$ for general values of τ. In all cases, we believe that the Brown measure of $ub_{s,\tau}$ coincides with the large-N limit of the empirical eigenvalue distribution of $U^N_0 B^N_{s,\tau}$, where U^N_0 is independent of $B^N_{s,\tau}$ and the limiting eigenvalue distribution of U^N_0 equals the law of u.

The following conjecture describes what we believe about the Brown measure of $ab_{s,\tau}$, where a is freely independent of $b_{s,\tau}$ but not necessarily unitary.

Conjecture 6.1 *Fix $s > 0$ and two complex numbers τ_0 and τ with $|\tau_0 - s| < s$ and $|\tau - s| \leq s$. Let μ_{s,τ_0} be the Brown measure of $ab_{s,\tau}$, where a is invertible and freely independent of $b_{s,\tau}$ and let m_{s,τ_0} be its Cauchy transform. Then m_{s,τ_0} is Lipschitz continuous and the Brown measure $\mu_{s,\tau}$ of $ab_{s,\tau}$ can be computed by the model deformation formula*

$$\mu_{s,\tau} = (\Phi_{s,\tau_0,\tau})_*(\mu_{s,\tau_0}), \tag{6.4}$$

where the map $\Psi_{s,\tau_0,\tau} : \mathbb{C} \to \mathbb{C}$ is defined as

$$\Psi_{s,\tau_0,\tau}(z) = z \exp\left\{ \frac{\tau - \tau_0}{2} \left(2z m_{s,\tau_0}(z) - 1 \right) \right\}.$$

Furthermore, as τ varies over the set $|\tau - s| < s$ (with a and s fixed), the log potential of $S_{s,\tau}$ is a C^1 solution of the PDE

$$\frac{\partial S_{s,\tau}}{\partial \tau} = -\frac{1}{2} \left(z^2 \left(\frac{\partial S}{\partial z} \right)^2 - z \frac{\partial S}{\partial z} \right). \tag{6.5}$$

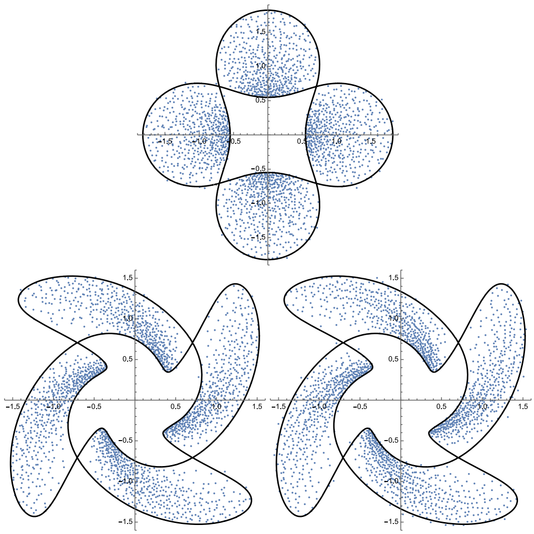

Fig. 8 The (s, τ_0) eigenvalues (top), the (s, τ) eigenvalues (bottom left), and the $(\tau - \tau_0)$ evolution of the (s, τ_0) eigenvalues (bottom right), for the case that A^N is unitary with eigenvalues equally distributed among ± 1 and $\pm i$. Shown for $s = 1$, $\tau_0 = 1$, and $\tau = 1 + i$

As in the additive case, the preceding conjecture is not the main focus of the present paper, but serves to connect Conjecture 6.3 below to the general multiplicative heat flow conjecture (Conjecture 1.8).

Remark 6.2 If a is unitary, the push-forward result in (6.4) is known from Theorem 8.2 and Proposition 8.4 of [23]. In the unitary case, the PDE (6.5) holds except possibly on the boundary of the support of $\mu_{s,\tau}$. (By Proposition 6.2 and Theorem 7.5 in [23], we can set $\varepsilon = 0$ in [23, Theorem 4.2], except possibly on the boundary of the support.) Proving the conjecture for general a is more difficult than in the corresponding additive case.

We now present the multiplicative version of Conjecture 2.3.

Conjecture 6.3 *Fix $s > 0$ and complex numbers τ_0 and τ such that $|\tau_0 - s| \leq s$ and $|\tau - s| \leq s$. Let B^N_{s,τ_0} and $B^N_{s,\tau}$ be Brownian motions as in (6.1) and (6.2). Suppose A^N a random matrix, independent of $B^N_{s,\tau}$, that is converging almost sure in the sense of $*$-distribution to an element a in a tracial von Neumann algebra. Let p_{s,τ_0} be the random characteristic polynomial of $A^N B^N_{s,\tau_0}$ and define a new random polynomial $q_{s,\tau_0,\tau}$ by*

$$q_{s,\tau_0,\tau}(z) = \exp\left\{-\frac{(\tau - \tau_0)}{2N}\left(z^2\frac{\partial^2}{\partial z^2} - (N-2)z\frac{\partial}{\partial z} - N\right)\right\} p_{s,\tau_0}(z). \tag{6.6}$$

Then the empirical root measure of $q_{s,\tau_0,\tau}$ converges weakly almost surely to the limiting eigenvalue distribution of $A^N B^N_{s,\tau}$.

See Fig. 8. See also Slide 9 in the supplemental document for an animation in the multiplicative case.

Remark 6.4 The right-hand side of (6.6) involves the heat flow

$$\exp\left\{-\frac{\tau}{2N}\left(z^2\frac{\partial^2}{\partial z^2}-(N-2)z\frac{\partial}{\partial z}-N\right)\right\}, \tag{6.7}$$

evaluated with τ replaced by $\tau-\tau_0$. The heat flow in (6.7), meanwhile, is closely related to the one in Tao's blog post [44], with g in Tao's notation identified with $N/2$ in our notation. Suppose we evaluate the flow in (6.7) on a monomial and then rescale the space variable by replacing z with $e^{\frac{\tau}{2N}}z$. After this rescaling, we find that the monomial z^j goes to

$$e^{\frac{\tau N}{2}}\exp\left\{-\frac{\tau}{2N}\left(j^2-Nj\right)\right\}z^j.$$

This agrees with the flow in [44, Equation (2)], up to a scaling of the time variable and an overall time-dependent constant (independent of j) which does not affect the roots. Note that we are scaling the space variable by the constant $e^{\frac{\tau}{2N}}$, which approaches 1 as $N\to\infty$. See also Remark 6.8 in Sect. 6.2.

We now define a multiplicative version $\tilde{D}$ of the second moment function, given by

$$\tilde{D}^N(s,\tau,z)=\mathbb{E}\{\left|\det(z-A_0^NB_{s,\tau}^N)\right|^2\}$$

Theorem 6.5 (Deformation theorem for second moment) *Suppose τ_0 and τ are complex numbers satisfying $|\tau_0-s|\le s$ and $|\tau-s|\le s$, in accordance with (2.3). Consider random matrices $A_0^NB_{s,\tau_0}^N$ and $A_0^NB_{s,\tau}^N$, respectively, where A_0^N is independent of B_{s,τ_0}^N and $B_{s,\tau}^N$. Then the function $\tilde{D}^N$ in (2.10) can also be computed as*

$$\begin{aligned}&\tilde{D}^N(s,\tau,z)\\&=\mathbb{E}\left\{\left|\exp\left\{-\frac{(\tau-\tau_0)}{2N}\left(z^2\frac{\partial^2}{\partial z^2}-(N-2)z\frac{\partial}{\partial z}-N\right)\right\}\det(z-A_0^NB_{s,\tau_0}^N)\right|^2\right\}.\end{aligned}$$

As in the additive case, we can rewrite this result as

$$\mathbb{E}\left\{\left|\prod_{j=1}^N(z-z_j^{s,\tau})\right|^2\right\}=\mathbb{E}\left\{\left|\prod_{j=1}^N(z-z_j^{s,\tau_0}(\tau))\right|^2\right\},$$

where $\{z_j^{s,\tau}\}_{j=1}^N$ are the roots of the characteristic polynomial of $A_0^NB_{s,\tau}^N$ and $\{z_j^{s,\tau_0}(\tau)\}_{j=1}^N$ are the roots of the characteristic polynomial of $A_0^NB_{s,\tau_0}^N$, after evolving for time $\tau-\tau_0$ by the multiplicative heat flow.

The proof of Theorem 6.5 is similar to the proof of Theorem 2.7 in the additive case.

We briefly indicate the differences. We construct elements X_j and Y_j as in the additive case, but with X_j taken to be skew Hermitian, so that $Y_j = iX_j$ is Hermitian. Then we define the associated left-invariant vector fields by

$$\hat{X}_j f(Z) = \left.\frac{d}{du} f(Ae^{uX_j})\right|_{u=0}; \quad \hat{Y}_j f(Z) = \left.\frac{d}{du} f(Ae^{uY_j})\right|_{u=0}.$$

Then we construct “multiplicative” operators Δ_{m}, ∂_{m}^2, and $\bar{\partial}_{\mathrm{m}}^2$ by replacing $\tilde{X}_j$ and $\tilde{Y}_j$ by $\hat{X}_j$ and $\hat{Y}_j$ in (2.15). Then the density of the law of $B_{s,\tau}^N$ is a heat kernel measure with Laplacian equal to $s\Delta_{\mathrm{m}} - \tau\partial_{\mathrm{m}}^2 - \bar{\tau}\bar{\partial}_{\mathrm{m}}^2$ [19, Lemma 5.9], and these three operators commute [19, Corollary 5.7].

The multiplicative version of the identity (2.17) in Lemma 2.9 then follows from the second proof of Lemma 2.9, using (the general version of) Proposition 4.7 in [19]. (A differential equation for the density of the law of $B_{s,\tau}^N$—as in (2.16) in the additive case—is not needed for the proof of Theorem 6.5, as long (2.17) holds.)

We then state and prove the multiplicative version of Lemma 2.10. Once this result is established, the proof of Theorem 6.5 proceeds as in the additive case.

Lemma 6.6 *Let B be a variable ranging over $M_N(\mathbb{C})$, let z be a variable ranging over $\mathbb{C}$, and let A be a fixed matrix. Then*

$$\partial_{\mathrm{m}}^2 \det(zI - AB) = \frac{1}{N}\left\{z^2\frac{\partial^2}{\partial z^2} - z(N-2)z\frac{\partial}{\partial z} - N\right\}\det(zI - AB),$$

where ∂_{m}^2 acts in the B variable.

Proof We use the notation

$$Q = zI - AB$$
$$R = Q^{-1}.$$

The multiplicative version of the operators Z_j satisfies the basic identity

$$Z_j B = BX_j.$$

Then we compute

$$\begin{aligned} Z_j \det(Q) &= -N \det(Q)\mathrm{tr}[RABX_j] \\ Z_j^2 \det(Q) &= N^2 \det(Q)\mathrm{tr}[RABX_j]\mathrm{tr}[RABX_j] \\ &\quad - N \det(Q)\mathrm{tr}[RABX_jRABX_j] \\ &\quad - N \det(Q)\mathrm{tr}[RABX_j^2] \end{aligned}$$

Using the magic formulas (2.24) and (2.25), but this time with the sign reversed because the X_j's and Y_j's are skew Hermitian, we get

$$\partial_{\mathrm{m}}^2 \det(Q) = -\det(Q)\mathrm{tr}[RABRAB] + N\det(Q)\mathrm{tr}[RAB]\mathrm{tr}[RAB] \\ + N\det(Q)\mathrm{tr}[RAB]$$

If we write

$$AB = zI - (zI - AB) = zI - Q,$$

we get

$$\mathrm{tr}[RAB] = z\mathrm{tr}[RAB] - 1$$

and

$$\mathrm{tr}[RABRAB] = 1 - 2z\mathrm{tr}[R] + z^2\mathrm{tr}[R^2]$$

and

$$\mathrm{tr}[RAB]\mathrm{tr}[RAB] = 1 - 2z\mathrm{tr}[R] + z^2\mathrm{tr}[R]^2.$$

Thus,

$$\partial_{\mathrm{m}}^2 \det(Q) = \det(Q)\{(N-1) - 2z(N-1)\mathrm{tr}[R]\} \\ + \det(Q)z^2\{N\mathrm{tr}[R]^2 - \mathrm{tr}[R^2]\} \\ + N\det(Q)(-1 + z\mathrm{tr}[R]),$$

which simplifies to

$$\partial_{\mathrm{m}}^2 \det(Q) = \det(Q)\{-1 - z(N-2)\mathrm{tr}[R]\} \\ + \det(Q)z^2\{N\mathrm{tr}[R]^2 - \mathrm{tr}[R^2]\}.$$

Using (2.27) and (2.28), we obtain the claimed result. □

6.2 The dynamics of the roots

We now describe the multiplicative version of the results of Sect. 3.1.

Proposition 6.7 *Let p_0^N be a polynomial of degree N and define p_τ^N by*

$$p_\tau^N(z) = \exp\left\{-\frac{\tau}{2N}\left(z^2\frac{\partial^2}{\partial z^2} - (N-2)z\frac{\partial}{\partial z} - N\right)\right\} p_0^N(z),$$

where the exponential, as applied to p_0^N, is defined as a convergent power series. Suppose that the zeros of p_σ are nonzero and distinct for some $\sigma \in \mathbb{C}$. Then for all τ in a neighborhood of σ, it is possible to order the zeros of p_τ as $z_1(\tau), \ldots, z_N(\tau)$ so that each $z_j(\tau)$ is nonzero and depends holomorphically on τ, and so that the collection $\{z_j(\tau)\}_{j=1}^N$ satisfies the following system of holomorphic differential equations:

$$\frac{1}{z_j(\tau)}\frac{dz_j(\tau)}{d\tau} = \frac{1}{2N}\left[1 + \sum_{k\neq j}\frac{z_j(\tau)+z_k(\tau)}{z_j(\tau)-z_k(\tau)}\right]. \tag{6.8}$$

Furthermore, if we write each $z_j(\tau)$ *as* $z_j = e^{iw_j(\tau)}$ *(but where we do not assume* $w_j(\tau)$ *is real), we have the second derivative formula*

$$\frac{d^2 w_j}{d\tau^2} = -\frac{1}{4N^2}\sum_{k\neq j}\frac{\cos((w_j - w_k)/2)}{\sin^3((w_j - w_k)/2)}. \tag{6.9}$$

Remark 6.8 The rescaled roots $\tilde{z}_j(t) = e^{-\frac{t}{2N}}z_j(t)$ satisfy a system of differential equations identical to (6.8) except with out the "1" term on the right-hand side of the equation. Now, if z and w are in the unit circle, then $(z+w)/(z-w)$ is pure imaginary. Using this observation, and interpreting the left-hand side of (6.8) as the derivative of $\log z_j(\tau)$, we can verify the following result: If $\tau_0 = 0$ and the points $\{z_j(0)\}_{j=1}^N$ are all in the unit circle, then the points $\tilde{z}_j(t)$ will be remain in the unit circle for $t \in \mathbb{R}$, for as long as they remain distinct. In the case $t > 0$, however, we are interested in going well past the time when the points collide.

The system of differential equations for the points $\tilde{z}_j(t)$ in the previous paragraph is the same as the one in Tao's blog post [44], up to a scaling of the time variable.

As for the corresponding result (3.3) in the additive case, we expect that the right-hand side of (6.9) will be small when N is large. Thus, we expect that the trajectories $w_j(\tau)$ will be approximately linear in τ, at least for small τ.

Remark 6.9 The equation (6.9) is the equation of motion for the **trigonometric Calogero–Moser** system, introduced by Sullivan. (Take $a = 1/2$ and $g^2 = -1/(2N^2)$ in Eq. (9) of [12].)

Proof of Proposition 6.7 The verification of (6.8) is very similar to the verification of (3.2) in Proposition 3.1 and is omitted. For (6.9), we first compute that

$$\frac{d}{d\tau}\frac{z_j + z_k}{z_j - z_k} = 2\frac{z_j\frac{dz_k}{d\tau} - z_k\frac{dz_j}{d\tau}}{(z_j - z_k)^2}. \tag{6.10}$$

Then since $w_j = \frac{1}{i}\log z_j$, we use (6.10) and (6.8) to compute

$$\begin{aligned}\frac{d^2 w_j}{d\tau^2} &= \frac{1}{i}\frac{d}{d\tau}\left(\frac{1}{z_j}\frac{dz_j}{d\tau}\right)\\ &= \frac{1}{2N^2}\sum_{k\neq j}\frac{1}{(z_j - z_k)^2}z_j z_k\\ &\quad\cdot\left(\sum_{l\neq k}\frac{z_k + z_l}{z_k - z_l} - \sum_{l\neq j}\frac{z_j + z_l}{z_j - z_l}\right).\end{aligned} \tag{6.11}$$

Then, as in the proof of Proposition 3.1, we write each sum over l in (6.11) as a sum over $l \notin \{j,k\}$ plus one extra term, giving

$$\frac{d^2 w_j}{d\tau^2} = \frac{1}{2iN^2}\sum_{k\neq j}\frac{1}{(z_j-z_k)^2}z_j z_k \cdot\left(-2\frac{(z_j+z_k)}{(z_k-z_k)} + \sum_{l\notin\{j,k\}}\left(\frac{z_k+z_l}{z_k-z_l}-\frac{z_j+z_l}{z_j-z_l}\right)\right).$$

This result simplifies to

$$\frac{d^2 w_j}{d\tau^2} = -\frac{1}{iN^2}\sum_{k\neq j}\frac{z_j z_k(z_j+z_k)}{(z_j-z_k)^3} + \frac{1}{2iN^2}z_j \sum_{\substack{k,l:\\ (j,k,l)\text{ distinct}}}\frac{z_k z_l}{(z_k-z_l)(z_j-z_k)(z_j-z_l)}. \tag{6.12}$$

Now, the second term on the right-hand side of (6.12) is zero because the summand changes sign under interchange of k and l. For the first term, we recall that $z_j = e^{ix_j}$ and compute that

$$\frac{z_j z_k(z_j+z_k)}{(z_j-z_k)^3} = \frac{i}{4}\frac{\cos((x_j-x_k)/2)}{\sin^3((x_j-x_k)/2)}. \tag{6.13}$$

Dropping the second term on the right-hand side of (6.12) and using (6.13) in the first term gives (6.9). □

6.3 The PDE perspective

In the multiplicative case, we again consider a sequence of polynomial p^N with distinct roots, whose empirical root distribution converges to a compactly supported probability measure ν_0 on $\mathbb{C}$ whose Cauchy transform is globally Lipschitz.

In this section, we propose the conjecture of the heat evolution of p^N using a PDE perspective. We call this the multiplicative case because the PDE is the same PDE as the one derived from multiplicative Brownian motion. The argument why we believe the conjecture holds is very similar to the additive case in Sect. 4.1.

Conjecture 6.10 *Consider the new polynomial defined by*

$$q^N(z,\tau) = \exp\left\{-\frac{\tau}{2N}\left(z^2\frac{\partial^2}{\partial z^2} - (N-2)z\frac{\partial}{\partial z} - N\right)\right\} p^N(z).$$

Let ν_τ^N be the empirical root distribution of $p^N(\cdot,\tau)$. Then the following holds.

(1) *There exists a constant $C > 0$ such that ν_τ^N converges weakly to a probability measure ν_τ on $\mathbb{C}$ whose logarithmic potential $S(z,\tau) = \int \log|z-w|^2 d\nu_\tau(w)$ is a C^1-solution of the PDE*

$$\frac{\partial S}{\partial \tau} = -\frac{1}{2}\left(z^2\left(\frac{\partial S}{\partial z}\right)^2 - z\frac{\partial S}{\partial z}\right). \tag{6.14}$$

for $|\tau| \leq C$.

(2) *There exists* $0 < \tau_{\max} < C$ *such that the probability measure* ν_τ *on* $\mathbb{C}$ *is the push-forward of* ν_0 *by*

$$T_\tau^{\mathrm{mult}}(z) = z \exp\left(\tau m_0(z) - \frac{1}{2}\right)$$

whenever $|\tau| \leq \tau_{\max}$, *where* m_0 *denotes the Cauchy transform of* ν_0.

6.4 The evolution of the holomorphic moments

We now briefly indicate how the results of Sect. 5 need to be modified in the multiplicative case.

Theorem 6.11 *The holomorphic moments in the multiplicative case satisfy Theorem 5.1, with the following changes. First,* (5.2) *is replaced by*

$$\frac{dM_{0,N}}{d\tau} = 0; \quad \frac{dM_{1,N}}{d\tau} = \frac{1}{2}M_{1,N}.$$

Second, (5.3) *is replaced by*

$$\frac{dM_{a,N}}{d\tau} = \frac{a}{2}M_{a,N} + \frac{a}{2}\sum_{b=1}^{a-1} M_{b,N}M_{a-b,N} - \frac{a}{2N}M_{a,N}. \tag{6.15}$$

Third, (5.4) *and* (5.5) *are replaced by the formal large-N limit of* (6.15) *(dropping the* $1/N$ *term). Last, in Point 3 of the theorem, the PDE* (1.7) *is replaced by* (6.14).

Note that the right-hand side of (6.15) involves the moment $M_{a,N}$. But we can still easily integrate (6.15) to obtain $M_{a,N}$ by using an exponential integrating factor. The proof of Theorem 6.11 is very similar to the proof of Theorem 5.1 and is omitted.

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Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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