Dynamical Wormholes and their Thermodynamics

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2021

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A Thesis Submitted to the Department of Mathematics,
Quaid-i-Azam University, Islamabad, in partial fulfillment of
the requirement for the degree of

Doctorate of Philosophy

in

Mathematics

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2021

To my family and my supervisor…

Acknowledgement

Foremost, I want to offer this endeavor to Allah Almighty for the wisdom he bestowed upon me, the strength, peace of my mind and blessings in order to finish this research.

I would like to express my sincere gratitude to my supervisor Prof. Khalid Saifullah for the continuous support during my Ph.D. study, for his patience, motivation and his critical way of teaching. His guidance helped me in all times not only in research work and writing of this thesis but also developing positive skills in my life. I would also like to thank Prof. Asghar Qadir for some useful discussions and suggestions that helped improve the thesis.

I express my deep gratitude to my parents for their continuous support, motivation and encouragement. Without their help and prayers I could never have reached this current level of success. Also thanks to my wife for her love, care and moral support.

I also express my profound gratitude to my research colleagues for their cooperation during this period, and finally thanks to all my well wishers who directly or indirectly helped me.

Mudassar Rehman

Abstract

This thesis deals with static and dynamical traversable wormholes. We study both charged and uncharged versions of these wormholes and analyse them in general relativity and alternate theories of gravity. We investigate thermodynamic properties of these objects including the unified first law and generalized surface gravity.

A two way traversable wormhole is a tunnel-like object comprising of trapped surfaces between horizons, defined as temporal outer trapping horizons. Usually, in a spacetime there are trapped, untrapped or marginal surfaces. On trapped surfaces both of the ingoing and outgoing light rays either converge or diverge, on untrapped surfaces one of the ingoing or outgoing light rays converges and the other diverges while on the marginal surfaces one or both of the ingoing or outgoing light rays remain constant (i.e., neither converge nor diverge but travel parallely). Trapping horizons are the hyper-surfaces foliated by marginal surfaces that may be past, future or bifurcating and further, outer, inner or degenerate. These trapping horizons coincide at the throat in static wormholes. For the purpose of studying thermodynamics, we have used a technique which was first developed in the literature for studying spherically symmetric black hole spacetimes. This technique uses a 2+2 formalism to derive the generalized surface gravity at a trapping horizon which becomes part of the first law of wormhole dynamics which is obtained from the unified first law by taking its projection along the trapping horizon. This unified first law is the rearrangement of Einstein field equations which can easily be generalized to $f(R,T)$ gravity, where R is the Ricci scalar and T is the trace of the stress-energy tensor, and non-minimal curvature-matter coupling where the equations, when written in the form of the Einstein tensor, replace the role of stress-energy tensor with an effective stress-energy tensor.

Chapter 1 is about some basic concepts that are related with the main subject of the thesis. In Chapter 2 we have reviewed the Hayward formalism and its application to the Morris-Thorne wormholes in Einstein's gravity. The generalized surface gravity,

unified first law of thermodynamics and wormhole dynamics have been studied at (bifurcating) trapping horizons. We work out the generalized surface gravity for wormholes of different shapes as well. Thermodynamic stability of Morris-Thorne wormholes has been discussed in GR. We have also investigated thermodynamics in non-minimal curvature-matter coupling which produces very complex equations. The extension of this work to $f(R,T)$ gravity has been done for Morris-Thorne wormholes.

Chapter 3 deals with thermodynamics of charged wormholes, which are static as well as spherically symmetric. The electric charge acts as additional matter to the Morris-Thorne wormhole which is already constructed by exotic matter. All the analysis (unified first law, thermodynamic stability and generalized surface gravity) done in Chapter 2 for Morris-Thorne wormholes is generalized to the charged wormholes in this chapter. In the absence of electric charge the results that have been derived in Chapter 2 can be recovered.

In Chapter 4 we study thermodynamics of dynamical traversable wormholes. We considered uncharged dynamical wormholes which are the time generalization of static Morris-Thorne wormholes. These wormholes are investigated in the background of different cosmological models, with and without the cosmological constant, and which include the power-law and exponential cosmologies also. The generalized surface gravity is evaluated at the trapping horizon and the unified first law of thermodynamics is set up. The trapping horizon in this case is not bifurcating but a past trapping horizon which does not coincide with the throat of the wormhole and it corresponds to the expanding universe. The thermodynamic stability of these wormholes has also been investigated. Some cases of asymptotically flat, de Sitter and anti-de Sitter wormholes have been considered as well. We have also extended the results from uncharged to charged dynamical wormholes. All the work done for uncharged dynamical wormholes has been generalized to charged dynamical wormholes. In the absence of charge the results derived for uncharged dynamical wormholes can be recovered.

We summarize our results and conclude the thesis in Chapter 5.

List of Publications from the Thesis

As publication is one of the requirements of the Higher Education Commission of Pakistan, we give here a list of publications from the thesis:

1. M. Rehman and K. Saifullah, Thermodynamics of dynamical wormholes, JCAP **06** (2021) 020.

2. M. Rehman and K. Saifullah, Unified first law for traversable wormholes in non-minimal coupling of curvature and matter, Ann. Phys. **407** (2019) 57.

3. M. Rehman and K. Saifullah, Thermodynamics of traversable wormholes in $f(R,T)$ gravity, Int. J. Geom. Methods Mod. Phys. **18** (2021) 2150175.

4. M. Rehman and K. Saifullah, Thermodynamics of charged wormholes, (submitted).

5. M. Rehman and K. Saifullah, Generalized surface gravity of dynamical wormholes, (submitted).

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Chapter 1

Introduction

1.1 Historical background of general relativity

Gravity is a fundamental interaction between bodies which we experience in everyday life. Gravity plays a very important role in our lives. It gives weight to objects on the Earth. The Moon's gravity produces tides in water on the Earth. Gravity keeps planets moving in their respective orbits and helps maintain the structure of a gaseous star. It is gravity which is responsible for the structure of galaxies on the large scale. Galileo Galilei studied gravity scientifically in the late 16th and early 17th century. However, gravity remained a puzzle for a long time until 1665, when Isaac Newton stated his law of universal gravitation called the "inverse square gravitational law" [1]: "*Every body attracts every other body with a force proportional to the product of their masses and inversely proportional to the square of the distance between them*". Mathematically this force of gravity is given by

$$F = G\frac{m_1 m_2}{r^2}, \tag{1.1}$$

where G is the gravitational constant, m_1, m_2 are masses of two objects and r is the distance between their centres. Newtonian gravity explained the phenomena, related to gravity, of that time successfully. But soon it was realized that this gravity needs modification to explain all the aspects of gravity. The planet Mercury does not follow the orbit prescribed for it by Newtonian physics. Classically, the perihelion shift of this planet was predicted to be $1064''$ of arc per century which is $43''$ of arc per century more than the observed value. To resolve this issue it was postulated that a new planet, which was named Vulcan, is present whose orbit lies between that of Mercury and the Sun. However it could not be detected even with best telescopes. Giving up Vulcan, it was assumed that it was a planet which lies always on the other side of the Sun relative to the Earth. This would have to lie in the same orbit as the Earth. This hypothetical planet was called anti-Earth due to its position. But this suggestion was not acceptable due to two reasons: firstly it would have perturbed Venus and Mars substantially and secondly its equilibrium position would have been unstable. Hence, no satisfactory solution to this problem could be given in Newtonian physics. This outstanding issue of that time was resolved by Albert Einstein in 1915 by introducing his famous theory of general relativity (GR) [2]. In GR, gravity is described as the curvature of spacetime which results due to the distribution of matter. Heavy and light masses produce high and low curvature in spacetime, respectively. Newtonian gravity works very well in limiting cases, where velocities of objects are small with small energies and masses are small, and it is simpler to work in this gravity. In most practical applications, Newtonian gravity is sufficient to work with. However, GR refines Newtonian gravity in a more subtle way and reduces to Newtonian gravity in the limit of small velocities and small gravitational strength of objects.

GR is the most successful and universally accepted gravitational theory which is consistent with the observations. The predictions of GR range from the existence of black holes and gravitational waves to the models of cosmology. It explains the planetary motion, physics of black holes and the deviation of light coming from the distant stars and galaxies from the straight path very well. The GR shapes our universe, it tells us that our universe contains warped regions of spacetimes (black holes).

The perihelion shift of Mercury was the first test of GR that happened to fit the observation. In 1919, bending of light in gravitational fields due to massive objects confirmed the predictions of GR. Then this theory was confirmed by many other observations and experiments which include the gravitational redshift of light and gravitational time dilation. The predictions of GR are well tested in the fields of binary pulsars, both in the weak and strong field limits. GR has proven to be a successful theory both on theoretical as well as observational fronts [2–4]. Recently this theory has successfully been confirmed by the detection of gravitational waves [5, 6]. The gravitational field equations in this theory, by adopting the gravitational Lagrangian density $\mathcal{L}_m = R$, are given by [2–4]

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{1.2}$$

Here $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $T_{\mu\nu}$ is the stress-energy tensor, G is the Newtonian constant of gravitation, c is the speed of light in vacuum and $g_{\mu\nu}$ is the metric tensor. These equations describe the gravitational phenomena of normal matter very well but the theory cannot satisfactorily explain some phenomena on large scale like the accelerated expansion of the universe, dark matter, quantum gravity and cosmic inflation etc. GR is not the ultimate theory at all. It has also faced some problems, described above, giving rise to modified theories of gravity. Since our universe is undergoing cosmic acceleration as revealed by experimental data [7–10], the late-time cosmic acceleration of the universe produces imbalance in gravitational field equations. The accelerated expansion is one of the major problems that GR could not satisfactorily explain. This accelerated expansion can be explained to some extent by adopting the Lagrangian $\mathcal{L}_m = R - 2\Lambda$, where Λ is the cosmological constant, which was included by Einstein himself in 1917. Thus Einstein field equations become

[2, 3]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{1.3}$$

These latter equations explain, to some extent, the accelerating expansion of the universe but there is still some doubt in the explanation of this phenomenon and the above mentioned problems as well [11–13]. Because of the inability of GR to account for this acceleration and other problems, several candidates have been proposed in the literature, that range from dark energy models to theories of modified gravity, among which $f(R)$ and $f(R,T)$ theories are well known, where T is the trace of the stress-energy tensor. Einstein field equations can be obtained from the Einstein-Hilbert action using an action principle in which gravitational Lagrangian density was a linear function of the scalar invariant R. However, there is no evidence that gravitational Lagrangian density must be only a linear function of R. Thus a modification of the Einstein-Hilbert action was proposed to explain this accelerated expansion and other problems too that remained unexplained by GR. For this purpose, in the modified gravitational Lagrangian density, a function $f(R)$ was introduced [14] and later it was further investigated and developed [15–17]. The first model in $f(R)$ theory was introduced in Ref. [18] and then some corrections in the gravitational action were made to explain cosmic acceleration [19]. A new term $T = g_{\mu\nu}T^{\mu\nu}$ in $f(R)$ theory was introduced and a new $f(R,T)$ theory, a generalization of the $f(R)$ theory, was also presented [20]. To obtain the modified field equations in the context of $f(R)$ gravity, the metric approach is usually used in the literature in which the action is varried with respect to metric $g_{\mu\nu}$. But there are other approaches also, which are used such as the Palatini formalism [21–26], where both the connections and metric are treated as seperate variables, and metric-affine formalism [26], where we vary the matter part of the action with respect to the connection. Thus modified gravitational field equations were obtained that not only explained the late-time accelerated expansion of the universe but also other problems mentioned above, in the context of $f(R)$ gravity. As far as the dark matter is concerned, the possibility of studying the galactic dynamics of massive test particles without taking into account the dark matter was also investigated in the framework of $f(R)$ gravity [27–30]. Exotic matter was also considered responsible for cosmic inflation at early times and accelerated expansion of the universe at late times. GR could not explain this acceleration but $f(R)$ gravity does this without the presence of exotic matter.

1.2 Wormholes and their historical background

A wormhole is a tunnel like structure that connects two distant regions of the same universe or different universes. It has two mouths connected by a throat (the minimum radius of the wormhole r_0). Wormholes are believed to be formed due to high intense gravitational fields. Wormholes are a construct of GR, which predicts wormholes mathematically, where each mouth of a wormhole is a black hole. However, a black hole which comes to existence due to natural death of a star does not make a wormhole by itself. Wormholes could create shortcuts for long journeys, but passage through them could be harmful too, due to danger of sudden collapse, high radiations and interaction of traveler with the exotic matter. However, we note that this problem can be overcome by wormholes which are supported by exotic matter. Exotic matter is characterised with a negative energy density and a huge negative pressure. Wormholes can be used to send information through them, in different regions of space, if they contain sufficient exotic matter, whether naturally occurring or added artificially.

Wormholes can also play the role of time machines if one of its mouth is moved relative to the other [31]. Wormhole spacetime structure is supported by exotic matter which violates the null energy condition (NEC) and the weak energy condition (WEC), according to Einstein field equations. This means that the matter has very strong negative pressure and even the energy density is negative according to the static observer. Some studies [9, 32–41] seem to support the idea that a major part of our universe consists of stuff that violates NEC. It was shown that phantom energy could be a form of exotic matter which has the property to violate NEC and it is the energy that supports the traversable wormhole spacetime [42, 43]. Now, as the exotic matter and ordinary matter are time-reversed versions of each other, so one may also think that wormholes and black holes are also the time-reversed versions of each other if thermodynamic behaviour of both is same. These investigations have improved the physical status of wormholes [44, 45]. However, in extended theories of relativity these violations of energy conditions can be avoided such as in $f(R)$ theory, where R is the Ricci scalar, and Gauss-Bonnet theory as these theories provide corrections to the Einstein and matter stress-energy tensors. These corrections might be ignored in the solar system regime but they play plausible role in the regime of strong gravitational field and on cosmological and galactic level [12, 46–51].

The name wormhole was firstly suggested by Misner and Wheeler [52], although it was not a new idea. In early 20th centuray, many authours including Flamm [53], Weyl [54], Einstein and Rosen [55] discussed these objects, However, Morris and Thorne worked out spherically symmetric and static wormholes in 1988 which were also traversable as they did not contain the event horizon [56]. After that many attempts were made to generalize the spacetime by introducing time-dependent factors in the metric.

The first hints pointing towards wormhole physics were made in 1916 by Ludwig Flamm [53]. In 1935 Einstein and Rosen gave the idea of "bridges" created in spacetime, which connect different points through a narrow tunnel and theoretically provide shortcuts, resulting in reducing the time of travel and distance. But passage through that bridge from one region to another was not possible due to the presence of event horizon, and such type of bridges are now referred to as Einstein-Rosen bridges [55]. The metric of Einstein-Rosen bridge can be obtained by putting $v^2 = r - 2M$ in the Schwarzschild metric

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.4}$$

so that the Schwarzschild solution in the Einstein-Rosen form is given by

$$ds^2 = -\frac{v^2}{v^2+2M}dt^2 + 4(v^2+2M)dv^2 + (v^2+2M)^2 d\Omega^2, \tag{1.5}$$

where the new coordinate v ranges in $(-\infty, \infty)$ and M represents the mass of the gravitational source. The singularities, $r = 0$ and $r = 2M$, which were appearing in Schwarzschild solution, are now avoided in these new coordinates. The region near $v = 0$ is called bridge which connects two asymptotically flat regions which are situated at $v = \infty$ and $v = -\infty$. The Einstein-Rosen bridge contains an event horizon which means we can enter from one side of the bridge but cannot reappear on the other side, just like a black hole which is one-way membrane. Also, another problem with these bridges is that their circumference fluctuates between zero to its maximum value so fast, that it becomes impossible for a traveller to cross it, even if it is moving with the speed of light. Thus Einstein-Rosen bridges cannot be used for travelling.

In 1950s, next major development in wormhole physics was by Wheeler and Misner. They created a framework which explained classical physics by Riemannian geometry of nontrivial topological manifolds. The term wormhole was first used in 1957 as: "There is a net flux of lines of force through what topologists would call a handle of the multiply-connected space and what physicists might perhaps be excused for more vividly terming a *wormhole*" [52].

In 1963, rotating black hole solutions were introduced by Kerr. From the Kerr black hole solution, in case of slow rotation, similar type of construction came into existence as the Einstein-Rosen bridge from the Schwarzschild solution, but with same problems of horizon and tidal forces. On the other hand, fast rotating black holes allow for fast transportaion but in this case the traveller does not have a choice to select the destination.

Recently a considerable interest in wormhole physics has been seen in two directions: one with the Euclidean signature metrics and the other with the Lorentzian wormholes [57–59]. Lorentzian wormholes that are traversable were first investigated by Morris and Thorne [56] in 1988. Morris and Thorne realized that wormholes could allow the traveller to go from one region to another region and that could be possible if, unlike a black hole, wormhole must be without event horizon. Their approach was to find such a wormhole spacetime that does not contain event horizon in order to be traversable from one region to the other region. They introduced a spherically symmetric and static wormhole metric. It was found that Morris-Thorne wormholes (MTWHs) violate some energy condition. However this is not surprising as energy conditions are not universal; there are number of phenomena which have been discovered to violate energy conditions. Now there are attempts to shift from static to non-static wormholes.

After the discovery of MTWHs, research advanced in various directions, concerning generalizations of MTWHs, and major ones are as follows.

Cubic and polyhedral wormholes were constructed by using thing-shell formalism, which have no constraints obeying spherical symmetry [60]. The throat was defined to be as 2-dimensional hypersurface of minimal area [61, 62]. A general class of solutions which describe spherically symmetric wormholes were also obtained [63]. The wormhole model which allows traversability to be extracted out of the quantum foam was also introduced [64]. Conformal wormholes [65, 66] and their further generalization were also described [67]. The general form of rotating axially symmetric and stationary wormhole was first described in Refs. [68, 69] and violations of energy conditions was discussed in detail [70]. Thermodynamic properties and entropy of wormholes have also been discussed in literature [44, 45, 71, 72].

Recently, macroscopic humanly traversable wormhole solutions were constructed using dark sector based on the Randall-Sundrum II model [73]. These wormholes exist in cold and flat ambient space and they permit traveler to survive the tidal forces. It takes very short proper time (less than a second) to travel through them between distant regions within our galaxy, but tens of thousands of years as seen from outside. These wormholes look like an intermediate-mass charged black holes from outside.

It was shown [74] that wormhole-like configuration can be formed between two massive objects situated in two parallel universes, modeled by two branes. The strong gravitational attraction between these objects deforms the branes, object touch and wormhole-like configuration is formed. The heavier and compact the objects are, the formation of wormhole-like configuration is more likely to occur.

1.3 Energy conditions

In discussing classical GR, there are at least seven types of energy conditions which, for the matter, are formulated in terms of its stress-energy tensor T_{ab}. These include null, weak, strong, dominant, average null, average weak and average strong energy conditions (NEC, WEC, SEC, DEC, ANEC, AWEC and ASEC) [59]. In GR these conditions are used in several theorems such as the no-hair theorem and black hole thermodynamics. For the purpose to elaborate some of these energy conditions, we consider the stress-energy tensor, given by

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{pmatrix}, \tag{1.6}$$

where ρ and p_i $(i = 1, 2, 3)$, are the energy density and three principal pressures, respectively.

1.3.1 Null energy condition

The NEC, for any null vector l^a, satisfies

$$T_{ab}l^a l^b \geq 0, \tag{1.7}$$

which in terms of energy density and pressure becomes

$$\rho + p_i \geq 0. \tag{1.8}$$

In GR one may obtain solutions by considering a metric and then solving Einstein field equations to get the matter source which is compatible with the corresponding geometry. In this way, at the wormhole throat, the flaring out condition is imposed by the tunnel like structure of a wormhole. Thus from this flaring out condition and the field equations it is revealed that the NEC is violated at or near the throat. At the throat NEC is violated or it is at the verge of violation. Thus wormhole throat is threaded by exotic matter (a matter that violates NEC). In this case energy is called phantom energy.

1.3.2 Weak energy condition

The WEC is given by

$$T_{ab}t^a t^b \geq 0, \tag{1.9}$$

for any timelike vector t^a. This condition, in addition to NEC, requires the positivity of local energy density as measured by any timelike observer. In terms of pressure it can be written as

$$\rho \geq 0 \; and \; \rho + p_i \geq 0. \tag{1.10}$$

The WEC ensures that all observers measure positive energy density, that is, normal matter is observed. However in wormholes, violation of NEC implies violation of

WEC as well.

1.3.3 Strong energy condition

The SEC asserts that for any timelike vector t^a,

$$(T_{ab} - \frac{1}{2}Tg_{ab})t^a t^b \geq 0. \tag{1.11}$$

This condition implies NEC but does not, in general, implies WEC. In terms of pressure and energy density it takes the form

$$\rho + p_i \geq 0 \, and \, \rho + \sum p_i \geq 0. \tag{1.12}$$

1.3.4 Dominant energy condition

The dominant energy condition (DEC) requires that, in addition to WEC, $T^{ab}t_a$ be null or time like. Thus it implies WEC, which further implies NEC. However SEC cannot necessarily be obtained from DEC. It says that energy density is positive and energy flux is not spacelike. For perfect fluid it yields

$$\rho \geq \mid p_i \mid . \tag{1.13}$$

In case of negative energy density, violation of NEC implies the violation of DEC.

1.4 Embedding

To capture the properties of curved spacetimes conveniently, especially with dimensions greater or equal to 3, embedding diagram can be used. A curved two dimensional surface is visualised within a flat three dimensional space using injective and structure preserving map Φ. Consider a hypersurface of dimension n, Σ_n, which is a subspace of an $n+1$ dimensional spacetime manifold, M_n. One can describe a hypersurface in terms of an embedding as

$$\Phi : \Sigma_n \rightarrow M_{n+1}. \tag{1.14}$$

If the coordinates y^a and x^α describe Σ and M, respectively, then the embedding Φ describes the points in M with coordinates x^α that corresponds to points in Σ with coordinates y^a as $\Phi : x^\alpha = x^\alpha(y^a)$. It is a good idea if some coordinates are common in the hypersurface and the spacetime into which it is embedded.

The properties of geometry can be conveniently visualized by embedding its equatorial plane in three-dimensional Euclidean spacetime, in cylindrical coordinates (Z, ρ, α). The embedding diagram is characterized by the embedding formula $Z = Z(\rho)$ determining a surface in the Euclidean space with the line element.

$$dl_E^2 = \left[1 + \Big(\frac{dz}{d\rho}\Big)^2\right] d\rho^2 + \rho^2 d\alpha^2, \tag{1.15}$$

isometric to the 2-dimensional equatorial plane of the line element [75]

$$dl^2 = h_{rr} dr^2 + h_{\phi\phi} d\phi^2. \tag{1.16}$$

The azimuthal coordinate can be identified ($\alpha \equiv \phi$) which immediately leads to

$$\Big(\frac{dZ}{d\rho}^2\Big) = h_{rr}\Big(\frac{dr}{d\rho}\Big)^2 - 1. \tag{1.17}$$

Using paremetric form, $Z(\rho) = Z(r(\rho))$, with r being the parameter, the embedding formula takes the form

$$\frac{dZ}{dr} = \pm\sqrt{h_{rr} - (\frac{d\rho}{dr})^2}. \tag{1.18}$$

If $h_{rr} - (d\rho/dr)^2 \geq 0$ then embedding diagram can be constructed.

1.5 Carter-Penrose Diagram

The Carter-Penrose diagrams are two dimensional diagrams. The purpose of these diagrams is to bring the infinities to finite distances in the diagrams. This is done by choosing the suitable coordinate transformations and using conformal rescaling of the metric. The coordinate transformations should possess two properties. Firstly, these transformations should compactify the spacetime within a finite boundary. Secondly light rays should always lie on $\pm 45^0$ angles. Under conformal rescaling, the causal nature of the vector field remains invariant, that is, a vector which was timelike or spacelike before conformal mapping remains timelike or spacelike afterward also, and light cones remain preserved ($ds^2 = 0$). Generally, timelike or spacelike geodesics do not map into each other, however null geodesics are mapped into each other.

The Carter-Penrose diagram of Einstein-Rosen bridge [76] is shown in Fig. 1.1.

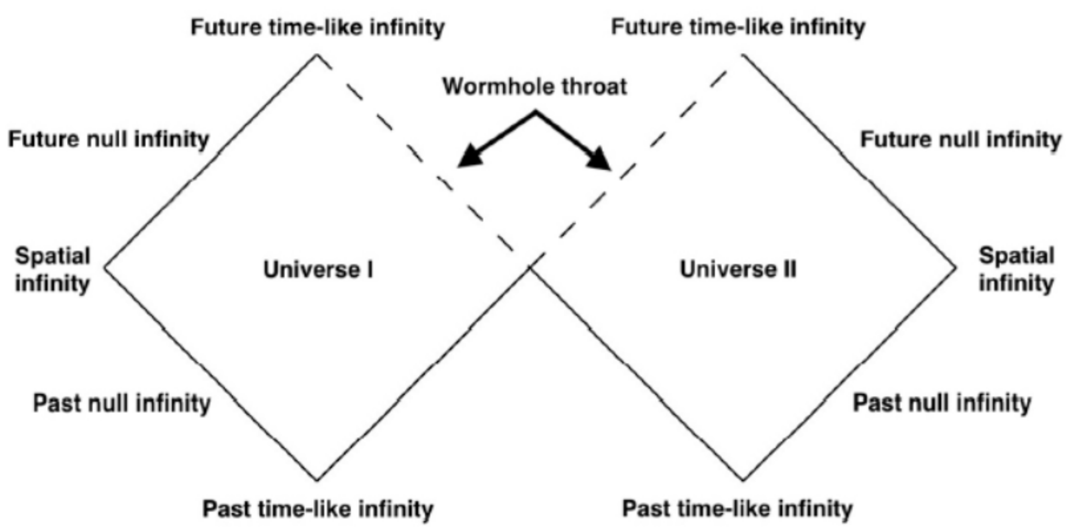

Figure 1.1. Penrose diagram of non-traversable Einstein-Rosen wormhole.

In Fig. 1.1, wormhole throat is the horizon which is shown by the two dashed lines. Every point on the wormhole throat corresponds to the two points at the same height on the two dashed lines.

Consider the metric

$$ds^2 = -\left(1 - \frac{2m}{\sqrt{r^2 + a^2}}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{\sqrt{r^2+a^2}}} + (r^2 + a^2)d\Omega^2. \tag{1.19}$$

Here t and r ranges in $(-\infty, \infty)$. This metric represents the two-way traversable wormhole for $a > 2m$ [31, 56, 60, 61]. Throat is located at $r = 0$ and the negative and positive values of radial coordinate, r, corresponds to the two universes. Its Carter-Penrose diagram [77] is shown in Fig. 1.2.

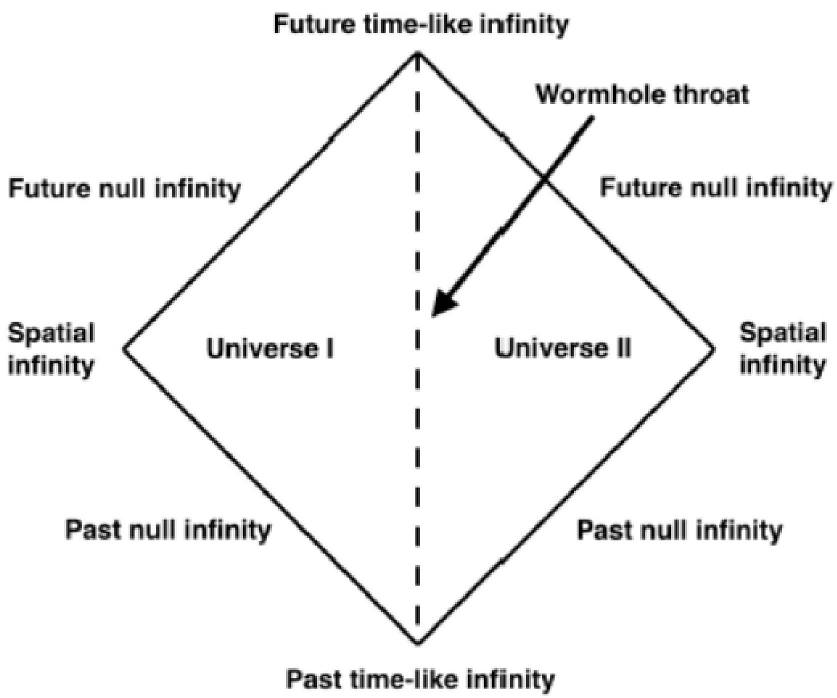

Figure 1.2. Penrose diagram of the two-way traversable wormhole.

1.6 Killing horizons

A Killing horizon is a null hypersurface in spacetime on which Killing vector becomes null. The Killing horizons define the boundaries of stationary spacetimes, black holes, white holes and cosmological regions. These are defined by the Killing vector k^a which satisfies the Killing equation

$$\nabla_a k_b + \nabla_b k_a = 0, \tag{1.20}$$

where ∇_a is the covariant derivative. Associated to Killing horizon is the geometric quantity, called surface gravity κ_{static}. If surface gravity vanishes then Killing horizon is called degenerate. In GR, the Killing horizon and stationary event horizon are same [78], for example in Schwarzschild geometry, the event horizon $r = 2M$ is also a Killing horizon, because timelike Killing vector $k^a = (\partial/\partial t)^a$ becomes null on this, which is timelike in the region $r > 2M$ and spacelike in the region $r < 2M$. Generally, in static spacetimes, any event horizon is also a Killing horizon with Killing vector $k^a = (\partial/\partial t)^a$. However, in stationary asymptotically flat spacetimes (not necessarily static), event horizon and Killing horizon match for the Killing vector which consists of two vectors, one associated with time symmetry and the other with rotational symmetry

$$k^a = (\partial/\partial t)^a + \Omega_H(\partial/\partial\phi)^a, \tag{1.21}$$

where Ω_H is the angular velocity at the horizon [79, 80]. In non-staionary spacetimes that do not admit timelike Killing vector, the concept of Killing horizon ceases to be useful. Conformal Killing horizons have been focussed in such spacetimes but this way does not seem to be very productive because this direction is *a priori* restrictive one. Hence, the Kodama vector plays its role, which resembles in some way to the Killing field, to define surface gravity and thermodynamics of spacetime horizons.

1.7 Trapping horizon

Contraction or expansion of ingoing and outgoing light rays from a surface define the surface to be trapped or untrapped. The concept of trapped surfaces was first proposed by Penrose [81], which is an important concept to describe black holes, white holes and wormholes locally in terms of trapping horizons. Consider a light flash that originates on a surface, then it forms two wave fronts, one ingoing and the other outgoing, which travel perpendicular to the surface. The expansions of these wave fronts describe the surface to be trapped or untrapped. Let Θ_+ and Θ_- be the expansions in the outgoing and ingoing directions, perpendicular to the surface, respectively. If the area of both the outgoing and ingoing wave fronts is increasing ($\Theta_\pm > 0$) or decreasing ($\Theta_\pm < 0$), or equivalently $\Theta_+\Theta_- > 0$ then surface is trapped. The former case corresponds to white holes while latter to black holes. The union of all the trapped surfaces form a trapping region and the boundary of a trapping region is called the trapping horizon, and a spacelike slice of a trapping horizon is called the apparent horizon. Outside the horizon we have untrapped surface, i.e., $\Theta_+\Theta_- < 0$, which means one of the expansions (Θ_+ or Θ_-) has changed its sign. We consider that the outgoing expansion Θ_+ changes sign across the horizon, while the ingoing expansion Θ_- keeps the same sign. Then we must have on the horizon, $\Theta_+ = 0$, which makes the product $\Theta_+\Theta_- = 0$ on the horizon. This is called a marginal surface. Thus, a trapping horizon is a hypersurface which is foliated by marginal surfaces [82–84]. Now, a spherically symmetric spacetime metric can locally be written as

$$ds^2 = -2e^{-f} d\xi^+ d\xi^- + r^2 d\Omega^2_{n-1}, \tag{1.22}$$

where the areal radius, r, and f are functions of local coordinates (ξ^+, ξ^-) and $d\Omega^2_{n-1}$ is the $(n-1)$-sphere with unit radius. We consider spacetime being time-orientable with $\partial_\pm = \partial/\partial\xi^\pm$ being future pointing. There are two null geodesics which correspond to $\xi^+ =$constant and $\xi^- =$constant, which can be obtained from considering radial null congruences, by putting $ds^2 = 0$. The expansions of these two congruences can be written as

$$\Theta_\pm = \frac{n-1}{r}\partial_\pm r. \tag{1.23}$$

Now a trapping horizon is a hypersurface on which one of the null expansions vanishes. Henceforth we take $\Theta_+ = 0$. The trapping horizon is future if $\Theta_- < 0$ or past if

$\Theta_- > 0$, it is outer if Θ_+ decreases in the ingoing direction ($\partial_-\Theta_+ < 0$) and inner if Θ_+ increases in the ingoing direction ($\partial_-\Theta_+ > 0$) [82–84].

In black holes, trapping horizon is used to study thermodynamics, and it has been claimed that it is the area of trapping horizon, instead of event horizon area, which is associated with entropy in black hole thermodynamics [85–88].

Trapping horizons and event horizons are distinct in general, and there are spacetimes where trapping horizon is present while event horizon is not [89, 90]. The difference of the area of trapping horizon from event horizon has been studied for particular spacetimes in Ref. [91].

1.8 Black holes and traversable wormholes

Wormholes gained attention from researchers and some respectability for their theoretical existence in GR, after the article published by Morris and Thorne in 1988 about traversable wormholes [56]. One motivation for studying wormholes is that they can increase our understanding of gravity where energy conditions violations take place due to Casimir effect or Hawking radiation.

Wormholes and black holes are very similar if studied using local properties. But, they are usually defined by global properties which make them quite distant from each other [56]. Also global properties do not have compatibility of event horizons with traversability. Black holes and wormholes are interconvertible objects as suggested by Hayward [92]. A mechanism was developed for such conversions and a framework was proposed for unification between black holes and wormholes. Construction of a traversable wormhole from Schwarzschild black hole has also been proposed through analytical solution [93]. Locally both these objects are defined by the presence of marginal surfaces, which are one-way traversable for black holes (or white holes) and two-way traversable for wormholes, respectively. These marginal surfaces form trapping horizon. Both black holes and wormholes are characterized by outer trapping horizons [82, 84, 92]. In the case of static black holes, examples of outer trapping horizons are event horizons or Killing horizons, while wormhole throat is an example of a double outer trapping horizon in the case of static and spherically symmetric wormholes. Standard black hole solutions and MTWHs have same spatial topology, $R \times S^2$, and spatial geometry can be identical for Schwarzschild black holes and spatially Schwarzschild wormholes [56]. In each case minimal surface joins the two asymptotically flat regions.

Now, the key difference in defining black holes and wormholes is that black hole outer trapping horizon is achronal (spacelike or null), while wormhole outer trapping horizon is temporal (timelike); this means that black holes are one-way traversable while wormholes are two-way traversable, as desired in each case [92]. The Einstein field equations imply that black holes and wormholes occur under the influence of positive and negative energy densities, respectively. This means black holes occur in natural matter or vacuum, while wormhole structure is supported by what is called exotic matter [56]. Thus black holes occur naturally while wormholes do not. However, presence of negative energy density in quantum field theory makes the possibility of constructing wormholes still open. If, theoratically, large amount of exotic matter in the universe can exist then wormholes and black holes, are equally, the prediction of GR.

The trapping horizon evolves under positive and negative energy densities, thus causal type of trapping horizon can be shifted from achronal to temporal or temporal to achronal. Addition of normal matter or dispersion of exotic matter can convert a wormhole into black hole. Geometrically, a double outer trapping horizon, which constitutes the throat of a static spherically symmetric wormhole, bifurcates under generic purturbation, which forms trapped region. If the two horizons become null and they enclose the future trapped region then it would be a black hole. Conversely, wormholes can also be formed from black holes if exotic matter is introduced. This exotic matter results in the two black hole horizons to shift from achronal to temporal, and unified as a throat of a wormhole. In the case of Schwarzschild black hole, which evaporates when considered semiclassically, the trapping horizon becomes timelike due to infalling negative energy particle inside the horizon during Hawking radiation phenomena, thus converting it into a traversable wormhole. This also suggests that the endpoint of Hawking radiation could be a wormhole [92].

1.9 Surface gravity

Surface gravity "g" of an astronomical object is the gravitational acceleration, which a hypotheticle particle experiences on its surface. Its units is that of an acceleration, which is meter per square second in SI system of units. In astrophysics, surface gravity may be expressed as $\log g$, where g is measured in cgs system [94].

Thus earth's surface gravity becomes $\log g = 2.992$. The surface gravity of a white dwarf is very high and that of neutron star is even higher. In black hole, surface gravity is measured relativistically.

Newtonian concept of gravitational acceleration is not clear cut in relativity. For a black hole, one cannot define surface gravity using Newtonian concept, because the value of surface gravity becomes infinite on the horizon. Thus a renormalized value is used which is equal to the product of Newtonian value and gravitational time dilation factor; the former becomes infinite on the horizon while the latter approaches zero on the horizon.

In relativity, surface gravity is defined in those spacetimes where event horizon is a Killing horizon. For a static Killing horizon the surface gravity 'κ_{static}' is the acceleration, as exerted at infinity, which is needed to keep an object at the horizon. In mathematical terms, if k^a is a suitably normalized Killing vector, then we define the surface gravity as [95]

$$k^a \nabla_a k^b = \kappa_{static} k^b, \tag{1.24}$$

which is evaluated at the horizon. In the case of static asymptotically flat spacetime, we choose the normalization such that $k^a k_a \to -1$ as $r \to \infty$, and so that $\kappa_{static} \geq 0$.

The surface gravity for stationary spacetimes is well defined. The reason is that all stationary spacetimes have a horizon that is Killing [95]. Recently, the surface gravity of dynamical spacetimes which do not admit a Killing vector has also been defined [96]. Various authors have suggested different definitions and, as of current, there is no agreement as to which definition, if any, is correct [97].

1.10 Kodama vector and surface gravity

Different notions of surface gravity, associated with horizons, have been introduced in the literature. For static and stationary spacetimes, timelike Killing vector field is present outside the horizon, which becomes null on it. Hence various definitions of surface gravity coincide and are well known. In dynamical situations, no timelike Killing vector field is present and notion of surface gravity is meaningless.

Kodama vector generalises the Killing vector and is used as a substitute in thermodynamics of non-static horizons. Any spherically symmetric metric can be written in the form

$$ds^2 = h_{ab} dx^a dx^b + R^2 d\Omega^2, \tag{1.25}$$

where $a, b = 0, 1$ and R is the areal radius. Now, the Kodama vector is orthogonal to 2-sphere of symmetry and it lies in 2-dimentional (t, R) space. For the metric (1.25), it is defined as [98]

$$K^a = \epsilon^{ab} \nabla_b R, \tag{1.26}$$

where ϵ^{ab} is the anti-symmetric volume form of the 2- metric h_{ab} [95]. In the case of MTWHs which we will discuss in Chapter 2, this vector becomes $\mathbf{K} = \sqrt{1 - b(r)/r}\partial_t$, which is null when $b(r) = r$ and timelike, otherwise. In the case of dynamical wormholes which we will discuss in Chapter 4, this vector becomes $\mathbf{K} = a(t)\sqrt{1 - ab/R}\partial_t + HR\sqrt{1 - ab/R}\partial_r$, which is spacelike when $HR > \sqrt{1 - b(r)/r}$, null when $HR =$

$\sqrt{1-b(r)/r}$ and timelike when $HR < \sqrt{1-b(r)/r}$. The drawback with Kodama vector is that it is defined only in spacetimes which are spherically symmetric. However there are attempts to generalise it for spacetimes which are non-spherically symmetric [99]. Kodama vector is parallel to Killing vector in static spacetimes, however they are not equal in general.

1.11 Null coordinates

Null coordinates are associated with light rays. We consider the Minkowski spacetime. On outgoing light rays, $u = t - r$ remains constant. This means that as time t increases, the radial coordinate r also increases. Different light rays relate with different values of u. Similarly on ingoing light rays, $v = t + r$ remains constant. This means that as time t increases, the value of radial coordinate r decreases. Here u changes along ingoing light ray while v remains constant. However along outgoing light ray v changes and u remains constant. Thus u and v label points on ingoing and outgoing light rays, respectively. These are called null coordinates as they label light rays.

Null coordinates can be found by integrating $ds^2 = 0$ for a radial ray ($\theta = \phi =$ constant), for instance, for the Schwarzschild metric one obtains

$$dt^2 = \frac{dr^2}{(1-\frac{2m}{r})^2} = dr_*^2, \tag{1.27}$$

where r_* is the new radial coordinate such that $u = t - r_*$ and $v = t + r_*$, where

$$r_* = r + 2m \log|\frac{r-2m}{2m}|. \tag{1.28}$$

1.12 Misner-Sharp energy

Gravitational field, produced by a massive source, contains energy. In relativity, equivalence of energy and mass means that it is only the combined energy which may be measured at a distance. Also gravitational field is non-linear in general which means the mass of a source and its kinetic and gravitational energy do not combine in a linear way to produce the active (effective) gravitational energy. In spherical symmetric spacetimes, this effective energy is the Schwarzschild energy in vacuum. Generally, there are many definitions of the so-called energy in literature, but in known physical limits those do not possess relevant physical properties, hence there is no agreed definition of gravitational energy in GR except in asymptotic flat spacetimes at infinity. At spatial infinity, one has energy called Arnowitt-Deser-Misner energy [100] and at null infinity, it is the Bondi-Sachs energy [101, 102].

In spherically symmetric spacetimes, Misner-Sharp (MS) energy (E) exists, which is given by

$$E = \frac{r}{2}(1 - \nabla^a r \nabla_a r), \tag{1.29}$$

where ∇ is the covariant derivative. This energy E possesses all the physical characteristics that the active gravitational energy has. It reduces to Newtonian energy in Newtonian limits, and behaves well in small spheres, large spheres, test particles and special relativistic limits [103].

1.13 First law of black hole statics

The first law of black hole statics concerns with stationary black holes [84]. This law involves in its expression the static definition of surface gravity. The surface gravity is obtained by solving the equation

$$k.(\nabla \wedge k_a) = \kappa_{static} k_a, \tag{1.30}$$

on the Killing horizon, generated by the Killing vector k^a. Here $\wedge$ denotes the antisymmetric tensor product and κ_{static} is the surface gravity of a black hole which determines the temperature on a black hole. Now the first law of black hole statics is given by

$$dE = \frac{\kappa_{static} dA}{8\pi} + work\ done, \tag{1.31}$$

where E is the energy on the horizon which is analogue of the internal energy in classical thermodynamics.

1.14 Wormhole thermodynamics

Thermodynamics is the branch of physics that deals with heat energy and its connection with other forms of energies. It tells us how thermal energy is converted from one form of energy to another and how matter is affected by thermal energy. The first law of thermodynamics relates the internal energy of a closed system to heat supplied or removed from a thermodynamic system, and thermodynamic work. This law is the law of conservation of energy related to thermodynamic processes, which states that "energy can neither be created nor destroyed, however, it can be transformed from one form to another form of energy".

If $\triangle U$ denotes the change in the internal energy of a closed system, Q denotes the heat energy supplied or removed from a system and W the thermodynamic work done by that closed system on its surroundings then a mathematical statement of the first law of thermodynamics can be put in the form as

$$\triangle U = Q - W, \tag{1.32}$$

where Q is positive if heat is supplied to a system and negative otherwise.

There exists a deep connection between two branches of physics, thermodynamics and gravity. Stephen Hawking suggested that black holes emit thermal radiation having temperature proportional to surface gravity and entropy proportional to the horizon area [104–106]. The Hawking temperature of accelerating and rotating black holes with electric and magnetic charges was calculated in [107, 108]. This build a relationship between Einstein field equations and thermodynamics. It was Jacobson who first derived Einstein field equations from first law of thermodynamics and a relationship between entropy and horizon area of a black hole [109]. Padmanabhan proposed that Einstein field equations when evaluated on the event horizon can be put into the form of first law of thermodynamics, $TdS = dE + PdV$, in the regime of spherically symmetric, stationary black hole spacetimes [110, 111]. Here T, S, E and P are the temperature, entropy, energy and pressure respectively. Later it was shown that the first law of thermodynamics can also be obtained from Einstein field equations at apparent horizons in various gravity theories as $TdS = dE + WdV$ [104–106, 109–112]. This relationship between thermodynamics and gravity can also be extended to braneword cosmoslogy [113, 114]. Corrections to entropy and horizon area of black holes by applying the exact differential properties to the first law of thermodynamics was discussed in Ref. [115].

Hayward developed a formalism using local quantities to describe the thermodynamic properties of spherically symmetric black holes using trapping horizons [92]. The presence of trapping horizon in wormholes suggests that this formalism can also be used to discuss the thermodynamic properties of wormholes. The thermodynamic properties of traversable wormholes have been discussed in literature [44, 45, 72, 116–118]. Here, in this thesis, we will apply this formalism to spherically symmetric traversable wormholes to discuss their thermodynamics.

Chapter 2

Thermodynamics of Morris-Thorne wormholes

In this chapter we will investigate the thermodynamic properties of MTWHs at trapping horizons using a formalism which was first used to discuss thermodynamics of spherically symmetric black holes [84]. The need and significance of characterizing black holes by using local considerations has been stressed in the literature [44, 45, 82–84, 119]. Black holes are described by the presence of event horizons, which is the global property and hence cannot be located by observers. Now, trapping horizon is a pure local concept, and in this way the thermodynamic properties of spherically symmetric dynamical black holes were studied using local considerations. We will employ the definition of surface gravity [72, 120] where we will use the trapping horizon instead of Killing horizon, and Kodama vector will play the role of Killing vector. In this chapter, we will first review Hayward formalism [84] which discusses thermodynamics of spherically symmetric spacetimes on trapping horizons and its application to MTWHs. We will also discuss thermodynamic stability of these objects and extend the formalism to non-minimal curvature-matter coupling and $f(R,T)$ gravity at the end.

2.1 Hayward formalism

Here we use a formalism [84] that defines the properties of real black holes using local quantities that are physically meaningful. This formalism recovers the thermodynamic results of black holes when we use global considerations at event horizons in the static vacuum case. Thus, this formalism generalizes the results of global considerations. In traversable wormholes, it is not possible to deduce any thermodynamic property using global considerations as there is no event horizon there. So, we use local quantities to study the thermodynamic properties of wormholes using trapping horizons. These exhibit similar properties as those of a black hole.

Now, any spherically symmetric metric can be written as

$$ds^2 = 2g_{+-}dx^+dx^- + r^2 d\Omega^2, \tag{2.1}$$

where r and g_{+-} are functions of the null coordinates (x^+, x^-), that correspond to the two preferred null normal directions for the symmetric spheres $\partial_\pm = \partial/\partial x^\pm$, and r is the so-called areal radius [84] and $d\Omega^2$ is the metric for the unit 2-sphere. We define the expansions as

$$\Theta_\pm = \frac{2}{r}\partial_\pm r. \tag{2.2}$$

These expansions tell us whether the light rays are expanding ($\Theta > 0$) or contracting ($\Theta < 0$), or equivalently, area of the sphere increases or decreases in the null directions. Since the sign of $\Theta_+\Theta_-$ is invariant, a sphere is trapped if $\Theta_+\Theta_- > 0$, untrapped if $\Theta_+\Theta_- < 0$, or marginal if $\Theta_+\Theta_- = 0$. For fixed $\Theta_+ > 0$ and $\Theta_- < 0$, ∂_+ is also fixed outgoing and ∂_- ingoing null normal vector. A surface which is foliated by marginal spheres is known as a trapping horizon. For the trapping horizon r_h, we choose

$$\Theta_+|_h = 0. \tag{2.3}$$

This trapping horizon is future if $\Theta_- < 0$, past if $\Theta_- > 0$ and bifurcating if $\Theta_- = 0$.

Further, this trapping horizon is outer if $\partial_-\Theta_+ < 0$, inner if $\partial_-\Theta_+ > 0$, or degenerate if $\partial_-\Theta_+ = 0$.

The huge bodies produce gravitational field around themselves which contain gravitational energy. This energy and the material mass produce combined effective energy in relativity, due to the equivalence principle of mass and energy. This combination of mass and energy usually takes place in a non-local and non-linear way. This is because the gravitational field is non-linear in general. In spherically symmetric cases, this is the Schwarzschild energy in vacuum. Generally, in relativity, there is no agreement on the definition of energy, except for the asymptotically flat spacetimes at infinity, where one has the Arnowitt-Deser-Misner energy and the Bondi-Sachs energy at spatial and null infinities, respectively [83]. Therefore, there should be such a definition of energy from which one can find these asymptotic energies, appropriately. Remarkably, this is the MS energy that exists in spherical symmetry, which can be written as [103]

$$E = \frac{1}{2}r(1 - \partial^a r \partial_a r) = \frac{r}{2}(1 - 2g^{+-}\partial_+ r \partial_- r), \tag{2.4}$$

which on a trapping horizon reads $E = r_h/2$.

We can formulate a unified first law (UFL) in spherically symmetric spacetimes [84]. This law describes the gradient of the active gravitational energy, using Einstein field equations, as a sum of two terms, the energy supply term and the work term. When we project this along the trapping horizon we get the first law of black hole dynamics. This expression involves the area and surface gravity and has the same form as the black hole statics if we replace the perturbations by the derivative along the trapping horizon. For the first law of black hole dynamics we need to define the generalized surface gravity (GSG) using Kodama vector and trapping horizon in the same manner as the first law of black hole statics requires the stationary definition of surface gravity using Killing vector and Killing horizon. Also, this expression involves energy at horizon rather than at infinity. This formalism can also be applied to wormholes by virtue of the presence of trapping horizon in these objects.

Using the stress-energy tensor of the background fluid we construct a function and a vector in the local coordinates as

$$\omega = -g_{+-}T^{+-}, \tag{2.5}$$

and

$$\psi = T^{++}\partial_+ r\partial_+ + T^{--}\partial_- r\partial_-. \tag{2.6}$$

Now the UFL can be written by taking gradient of the gravitational energy and using Einstein field equations as [84]

$$\partial_\pm E = A\psi_\pm + \omega\partial_\pm V, \tag{2.7}$$

where $A = 4\pi r^2$ and $V = 4\pi r^3/3$ are the area and areal volume of the spheres of symmetry and the corresponding flat space, respectively. We can interpret ω and ψ physically as the energy density and the energy flux (outward flux minus the inward flux). The right hand side of the UFL (2.7) is the sum of two terms, the first term $A\psi_\pm$, called the energy supply term, produces variation in energy of the spacetime and the second term, $\omega\partial_\pm V$, called the work term, supports the spacetime structure.

Now the Einstein field equations of interest in local coordinates are

$$\partial_\pm\Theta_\pm = -\frac{1}{2}\Theta_\pm^2 + \Theta_\pm\partial_\pm \log(-g_{+-}) - 8\pi T_{\pm\pm}, \tag{2.8}$$

$$\partial_\pm\Theta_\mp = -\Theta_+\Theta_- + \frac{1}{r^2}g_{+-} + 8\pi T_{+-}, \tag{2.9}$$

$$\partial_+\Theta_- + \partial_-\Theta_+ + \Theta_+\Theta_- = -\frac{8\pi}{r^2}T_{\theta\theta}, \tag{2.10}$$

where $T_{\mu\nu}$ is the stress energy tensor in coordinates x^+, x^-, θ, ϕ.

In non-stationary spherically symmetric spacetimes we use Kodama vector K instead of Killing vector which was introduced by Kodama [98] and which reduces to a Killing vector in stationary cases when there is vacuum. The Kodama vector in null coordinates is given by

$$K = -g^{+-}(\partial_+ r\partial_- - \partial_- r\partial_+). \tag{2.11}$$

The magnitude of $\mathbf{K}$ is

$$|\mathbf{K}|^2 = g_{ab}K^a K^b = \frac{2E}{r} - 1. \tag{2.12}$$

Note that $|\mathbf{K}|^2 = 0$ on the trapping horizon r_h.

The trapping horizon is provided by this Kodama vector which is null on a hypersurface $\partial_+ r = 0$. In a dynamical spacetime, the trapping horizon and the Kodama vector play the same role as the Killing horizon and the Killing vector play in the static case. In static spacetimes the hypersurface where the Killing vector vanishes is defined as the boundary of the spacetime but here we use Kodama vector instead. In the above, E is the Noether charge of Kodama vector. Kodama vector and Killing vector have some similar properties [83], thus allowing the definition of the GSG. The GSG κ on a trapping horizon can be expressed as [120]

$$K^a\nabla_{[b}K_{a]} = \pm\kappa K_b, \tag{2.13}$$

which, on using the Einstein field equation (2.9), can be written as

$$\kappa = \frac{E}{r_h^2} - 4\pi r_h\omega. \tag{2.14}$$

This surface gravity, from Eq. (2.13), equivalently, can also be expressed as

$$\kappa = \frac{1}{2}g^{ab}\partial_a\partial_b r, \tag{2.15}$$

on a trapping horizon. Here g^{ab} is the inverse of the metric tensor g_{ab}. It follows that $\kappa < 0$, $\kappa = 0$ and $\kappa > 0$ for inner, degenerate and outer trapping horizons, respectively. As mentioned above, in dynamical spherical spacetimes the Kodama vector is the analogue of a time-like Killing vector. We cannot define surface gravity in traversable wormholes using Killing vector because it does not vanish everywhere. But still we can use Kodama vector instead and define the GSG for static as well as dynamical traversable wormhole at a trapping horizon.

Finally, Eq. (2.7) when projected along the trapping horizon gives the first law of wormhole dynamics which can be expressed as

$$E' = \frac{\kappa A'}{8\pi} + \omega V', \tag{2.16}$$

where we have used the notation $F' = z.\nabla F$. Here $z = z^{+}\partial_{+} + z^{-}\partial_{-}$ is a tangent vector to the trapping horizon. This expression defines a relation between the surface area and geometric entropy as

$$S \propto A|_{h}. \tag{2.17}$$

2.2 Morris-Thorne wormholes

The thermodynamic properties can also be studied for a wormhole by virtue of the presence of a trapping horizon, and the results analogous to those for a black hole can be obtained [92]. We also investigate wormholes of different shapes for their thermodynamic properties. In this section we will apply the Hayward formalism to MTWHs. We consider a spherically symmetric, static and traversable wormhole given by Morris and Thorne [56]. In coordinates (t, l, θ, ϕ) this metric can be written as

$$ds^2 = -e^{2\Phi(l)}dt^2 + dl^2 + r^2(l)d\Omega^2, \tag{2.18}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and l-coordinate runs from $-\infty$ to ∞. This wormhole solution covers two asymptotically flat regions which are joined together at $l = 0$. This point $l = 0$ is the location of the wormhole throat, the minimum radius of a wormhole, $r(l) = r_0$. Thus $-\infty < l < 0$ and $0 < l < \infty$ cover the two asymptotically flat regions. Also $e^{2\Phi(l)}$ must be finite every where and when $l \to \pm\infty$ then $r(l)/|l| \to 1$ and $e^{2\Phi(l)} \to$ constant, in order to have asymptotically flat regions. In Schwarzschild coordinates metric (2.18) can be written as

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega^2, \tag{2.19}$$

where the proper radial distance is transformed as

$$l(r) = \pm \int_{r_0}^{r} \frac{dr^\star}{\sqrt{1 - b(r^\star)/r^\star}}, \tag{2.20}$$

where $\pm$ refer to the two asymptotically flat regions which are connected through the wormhole throat. In metric (2.19), $\Phi(r)$ and $b(r)$ are called the redshift and the shape functions of a wormhole, since the first corresponds to the gravitational redshift of the universe and the latter determines the shape of a wormhole. The shape of a wormhole can be seen from the embedding space in coordinates (Z, r, ϕ), where the 2-surface

$$Z(r) = \pm \int \left(\frac{r}{b(r)} - 1 \right)^{-1/2} dr \tag{2.21}$$

has the same geometry as the 2-surface $\theta = \pi/2$ and $t =$ constant, in metric (2.19). The function $Z(r)$ is called the embedding function. The graph of Eq. (2.21), when revolved around the axis of rotation, the Z-axis, gives the shape of the wormhole [64]. At the wormhole throat a coordinate singularity $b(r_0) = r_0$ occurs and $b(r) < r$ for $r > r_0$. This condition ensures the finiteness of the proper radial distance defined by Eq. (2.20). Here r, the radial coordinate, decreases from ∞ to a minimum radius r_0, the throat of the wormhole where there occurs coordinate singularity $b(r_0) = r_0$, then it increases from r_0 back to ∞. Thus both the flat regions are now represented by $r_0 < r < \infty$. In order for a wormhole to be traversable the existence of event horizon should be prohibited which are the surfaces where $e^{2\Phi(r)}$ becomes zero. Thus $\Phi(r)$ should be finite everywhere to prevent the event horizon [56].

Now for the stress-energy tensor we take the perfect fluid which is completely described by its energy density and pressure [121]. In the component form it is written as

$$T_t^t = -\rho(r),\; T_r^r = p_r(r),\; T_\theta^\theta = T_\phi^\phi = p_t(r), \tag{2.22}$$

where $\rho(r)$, $p_r(r)$ and $p_t(r)$ are, respectively, the energy density, radial pressure and tangential pressure. For isotropic pressure $p_r(r) = p_t(r)$, otherwise the pressure will be anisotropic.

For a traversable wormhole solution a flaring out condition $(b - b'r)/b^2 > 0$ at or near the throat is imposed. Further, at throat $b(r_0) = r_0$ and $b'(r_0) < 1$ is also imposed to have a wormhole solution. The violation of NEC, in fact, is because of these restrictions [56, 122, 123], since Einstien's field equations and flaring out condition imply that $\rho + p_r < 0$. Also from Einstein equations $b'(r)$ and $\rho(r)$ have same sign, therefore, it is advisable to demand $b'(r) > 0$ to minimize the exoticity.

2.3 Trapping horizons and their classification

To obtain the trapping horizon from metric (2.19), we can write it in the form of Eq. (2.1), using the null coordinates $x^+ = t + r_*$ and $x^- = t - r_*$ where r and r_* are related by the equation

$$\frac{dr}{dr_*} = \sqrt{-\frac{g_{tt}}{g_{rr}}} = e^{\Phi}\sqrt{1 - b/r}. \tag{2.23}$$

Here r and $g_{+-} = -e^{2\Phi}/2$ are functions of the null coordinates x^+ and x^-.

The stress-energy tensor from coordinates (t, r, θ, ϕ) to (x^+, x^-, θ, ϕ) is transformed through the equation

$$T^\mu_\nu = \frac{\partial x^\mu}{\partial x^a}\frac{\partial x^b}{\partial x^\nu} T^a_b \tag{2.24}$$

where μ, ν run over new coordinates (x^+, x^-, θ, ϕ) and a, b over old coordinates (t, r, θ, ϕ).

The expansions can be written as

$$\Theta_{\pm} = \pm\frac{e^{\Phi}}{r}\sqrt{1 - b/r}. \tag{2.25}$$

Now, a trapping horizon is defined as the surface foliated by spheres in which $\Theta_+\Theta_- = 0$. Here, for trapping horizon, we choose $\Theta_+|_h = 0$ which implies $\partial_+ r|_h = 0$ giving $b(r_h) = r_h$. Also, on the throat, $b(r_0) = r_0$, which implies that for metric (2.19) the trapping horizon and throat of the wormhole coincide, that is $r_h = r_0$. In our case $\Theta_+|_h = 0$ implies $\Theta_-|_h = 0$, so we have a bifurcating trapping horizon here and

$$\partial_-\Theta_+ = \frac{e^{2\Phi(r_0)}(b'(r_0) - 1)}{4r_0^2}, \tag{2.26}$$

on the trapping horizon (throat). The sign of $\partial_-\Theta_+$ depends on the value of $b'(r_0)$, and thus the trapping horizon is outer if $b'(r_0) < 1$, inner if $b'(r_0) > 1$ and degenerate if $b'(r_0) = 1$. The flaring out condition depicts that $b'(r_0) < 1$, and thus we have an outer trapping horizon which is bifurcating as well.

Since Killing vecctor is present in these spacetimes but there is no hypersurface where it is null, the Killing horizon is absent. However we have the Kodama vector which for the Morris-Thorne metric takes the form

$$K^{\pm} = e^{-\Phi(r)}\sqrt{1 - \frac{b(r)}{r}}. \tag{2.27}$$

The magnitude of $\mathbf{K}$ from Eq. (2.12) takes the form

$$|K|^2 = \frac{b(r)}{r} - 1, \tag{2.28}$$

which becomes zero on the trapping horizon r_0. Thus trapping horizon is provided by this Kodama vector which is null on a hypersurface $r = r_0$.

2.4 Thermodynamics of Morris-Thorne wormholes

2.4.1 Generalized surface gravity

Using expression (2.4) the MS energy for the metric (2.19) is given by

$$E = \frac{b(r)}{2}, \tag{2.29}$$

on a trapping horizon which reads $E = r_0/2$.

Einstein equations of interest (2.8)-(2.10) in local coordinates are

$$\partial_\pm \Theta_\pm = -\frac{1}{2}\Theta_\pm^2 + \Theta_\pm \partial_\pm \log(-g_{+-}) - 2\pi e^{2\Phi}(\rho + p_r), \tag{2.30}$$

$$\partial_\pm \Theta_\mp = -\Theta_+\Theta_- + \frac{1}{r^2} g_{+-} + 2\pi e^{2\Phi}(\rho - p_r), \tag{2.31}$$

$$\partial_+\Theta_- + \partial_-\Theta_+ + \Theta_+\Theta_- = -8\pi p_t. \tag{2.32}$$

Here, from Eq. (2.24), we have used

$$T_{++} = T_{--} = \frac{e^{2\Phi}(\rho + p_r)}{4}, \tag{2.33}$$

$$T_{+-} = T_{-+} = \frac{e^{2\Phi}(\rho - p_r)}{4}, \tag{2.34}$$

$$T_{\theta\theta} = r^2 p_t. \tag{2.35}$$

Now, because of the absence of a Killing horizon, surface gravity cannot be obtained as defined by Gibbons and Hawking [106]. However, due to the presence of trapping horizon, we can obtain it by using Kodama vector. Using Eq. (2.15), the GSG, κ, on a trapping horizon can be expressed as

$$\kappa = \frac{1 - b'(r_0)}{4r_0}, \tag{2.36}$$

which is positive since $b'(r_0) < 1$. On using Einstein field equations (2.30) and (2.31), it can also be written, respectively, as

$$\kappa = -2\pi r_0 \left(\rho + p_r\right) |_h, \tag{2.37}$$

and

$$\kappa = \frac{1}{2r_0} - 2\pi r_0 (\rho - p_r) |_h \, . \tag{2.38}$$

The Hawking temperature $T = -\kappa_h/2\pi$ [44, 45] in our case from Eq. (2.36) becomes

$$T = -\frac{1 - b'(r_0)}{8\pi r_0}, \tag{2.39}$$

which is negative for the outer trapping horizon ($b'(r_0) < 1$), since $\kappa > 0$. It means the particles coming out of a wormhole have the same properties as that of a phantom energy because this energy is linked with negative temperature as well. Or, we can say that the phantom energy is responsible for this negative temperature [124].

Now the usual surface gravity, defined by the use of Killing vector, means there is a force which acts on a test particle in a gravitational field. In our case, both Killing vector and Kodama vector are present but Kodama vector is more relevant as it vanishes on a particular hypersurface unlike Killing vector, and in the vacuum case it reduces to Killing vector as well. Thus, one could suspect that the GSG which is defined by using Kodama vector means more than just a force acting on the test particle in a gravitational field, and some extra effects on the test particle could be predicted. However, if these extra effects on a test particle vanish by some kind of symmetry then there is a possibility that such a symmetry would also give rise to a degenerate trapping horizon.

2.4.2 Some specific cases of different shape functions

Shape function $b(r) = r_0^2/r$

Here we take [123] the shape function $b(r) = r_0^2/r$. This shape function satisfies the necessary conditions, which have been discussed in the beginning, to have a traversable wormhole solution. However this corresponds to negative energy density ρ. Using this shape function Eq. (2.21) becomes

$$Z(r) = \pm r_0 \ln \frac{r + \sqrt{r^2 - r_0^2}}{r_0}. \tag{2.40}$$

This embedding function $Z(r)$ is depicted in Fig. 2.1 where we have set $r_0 = 1$.

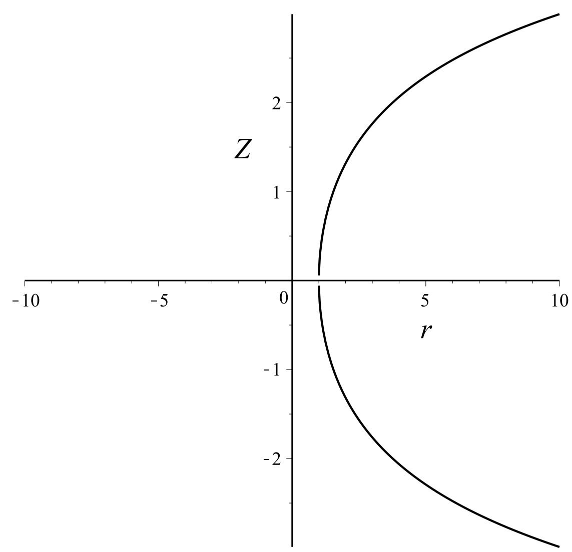

Figure 2.1. Embedding function $Z(r)$ for $b(r) = r_0^2/r$ and $r_0 = 1$.

In this case, we note that, the Kodama vector from Eq. (2.27) takes the form

$$K_{\pm} = -\frac{e^{\Phi}}{2}\sqrt{1 - \frac{r_0^2}{r^2}}. \tag{2.41}$$

Using this in Eq. (2.36) and evaluating on the trapping horizon gives the GSG

$$\kappa = \frac{1}{2r_0}. \tag{2.42}$$

This positive GSG gives the negative Hawking temperature $T = -1/4\pi r_0$.

Shape function $b(r) = \sqrt{r_0 r}$

Here we consider the shape function $b(r) = \sqrt{r_0 r}$ [123]. The necessary conditions for a traversable wormhole solution are satisfied by this shape function. The embedding function in this case from Eq. (2.21) takes the form

$$Z(r) = \pm\frac{4(r_0)^{1/4}}{3}\left[(\sqrt{r} - \sqrt{r_0})^{3/2} + 3\sqrt{r_0}(\sqrt{r} - \sqrt{r_0})^{1/2}\right]. \tag{2.43}$$

The embedding diagram for this shape function is shown in Fig. 2.2, where we have set $r_0 = 1$.

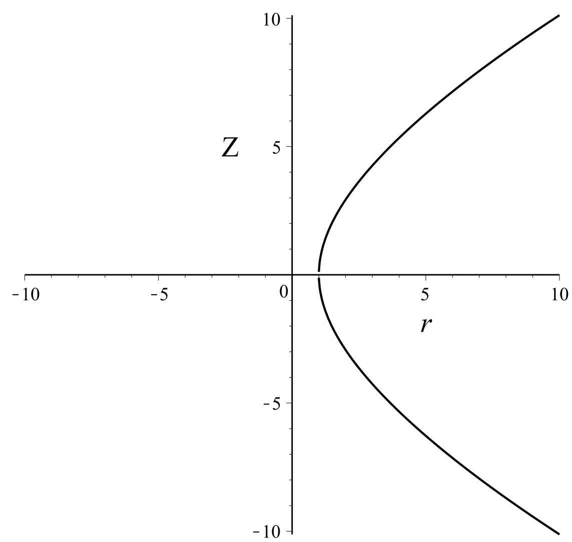

Figure 2.2. Embedding function $Z(r)$ for $b(r) = \sqrt{r_0 r}$ and $r_0 = 1$.

The Kodama vector, in this case, from Eq. (2.27) becomes

$$K_{\pm} = -\frac{e^{\Phi}}{2}\sqrt{1 - \sqrt{\frac{r_0}{r}}}. \tag{2.44}$$

Using this in Eq. (2.36) and evaluating on the trapping horizon yields the GSG

$$\kappa = \frac{1}{8r_0}, \tag{2.45}$$

which is always positive, and this gives negative Hawking temperature $T = -1/16\pi r_0$.

Shape function $b(r) = r_0(\frac{r}{r_0})^\gamma$, $0 \leq \gamma < 1$

Now, we assume the shape function $b(r) = r_0(\frac{r}{r_0})^\gamma$, $0 \leq \gamma < 1$. The embedding function in this case for $\gamma = 0$ from Eq. (2.21) is given as

$$Z(r) = \pm 2\sqrt{r_0(r - r_0)}. \tag{2.46}$$

The graph of this function is shown in Fig. 2.3, where we have taken $r_0 = 1$

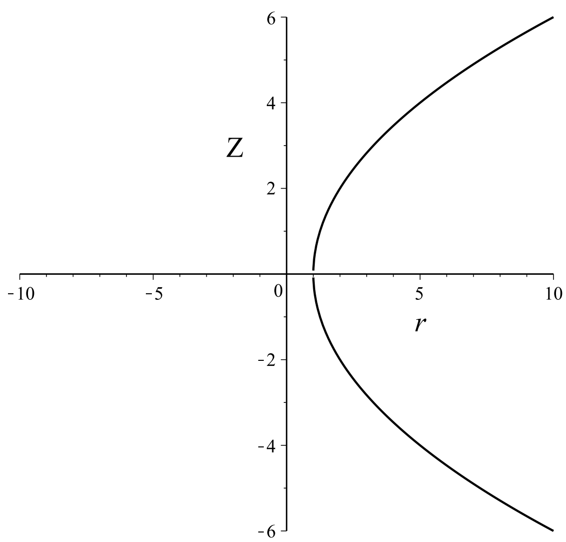

Figure 2.3. Embedding function $Z(r)$ for $b(r) = r_0$ and $r_0 = 1$.

The Kodama vector takes the form

$$K_\pm = -\frac{e^{2\Phi}}{2}\sqrt{1 - \left(\frac{r}{r_0}\right)^{\gamma-1}}. \tag{2.47}$$

In this case the GSG, from Eq. (2.36), on the trapping horizon becomes

$$\kappa = \frac{1-\gamma}{4r_0}. \tag{2.48}$$

This is positive since $0 \leq \gamma < 1$, giving negative Hawking temperature $T = -(1 - \gamma)/8\pi r_0$.

2.4.3 Unified first law

Now for deriving the UFL we first construct a fuction ω and a vector ψ using expressions (2.5) and (2.6) as

$$\omega = \frac{\rho - p_r}{2}, \tag{2.49}$$

and

$$\psi_\pm = \pm e^\Phi(r)\sqrt{1 - \frac{b}{r}}\left(\frac{\rho + p_r}{4}\right). \tag{2.50}$$

Taking the derivative of MS energy E and making use of Einstein field equations (2.30) and (2.31), we get

$$\partial_\pm E = \pm 2\pi r^2 \rho e^\Phi \sqrt{1 - \frac{b}{r}}. \tag{2.51}$$

From Eqs. (2.49)-(2.51) UFL (2.7) can be formulated. On the trapping horizon (throat) all the terms appearing in the UFL vanish, thus resulting in no evolution of the throat. However, generally, the gradient of MS energy is always positive in the outgoing direction while negative in the ingoing direction because ρ is positive. The energy density ω is positive even though energy conditions are violated. Energy flux depends on the sign of $\rho + p_r$, which is negative in our case due to exotic matter which gets energy from the spacetime, however energy removal due to energy flux term does not become so large that it could alter the sign of gradient of MS energy.

2.4.4 First law of wormhole dynamics

The first law of wormhole dynamics is obtained by projecting the UFL along the trapping horizon. This projection yields the following equation

$$E' = \frac{\kappa^{eff} A'}{8\pi} + \omega^{eff} V', \tag{2.52}$$

where, $E' = z.\nabla E$, $A' = z.\nabla A$ and $V' = z.\nabla V$ with $z = z^{+}\partial_{+} + z^{-}\partial_{-}$ being the vector tangent to the trapping horizon.

Eq. (2.52) includes in its expression the effective GSG and the area. This expression looks the same as the first law of black hole statics but here perturbations are replaced with the derivation along the trapping horizon. This first law of wormhole dynamics differs from the first law of black hole statics in the aspect that here we use the definition of the effective GSG defined at trapping horizon, instead of surface gravity defined at the Killing horizon used in the first law of black hole statics [84].

Eq. (2.52) can also be written in the form

$$E' = -TS' + \omega V', \tag{2.53}$$

on the trapping horizon with

$$S = \frac{A|_H}{4}. \tag{2.54}$$

Eq. (2.53) contains negative sign in the first term on the right hand side. It is because of the energy removed from the wormhole. Thus the first law of wormhole dynamics can be stated as "the change in the active gravitational energy is equal to the energy removed from the wormhole and the work done in the wormhole".

2.4.5 Thermodynamic stability

In this section we study the thermodynamic stability of wormholes under consideration using the variables E, T, S, P and V. We follow the usual criterion [125, 126] for thermodynamic stability, that is, $\frac{\partial \bar{P}}{\partial V} |_T \leq 0$ and $C_P \geq C_V \geq 0$, where $\bar{P} = (p_r + 2p_t)/3$ is the average pressure and C_P and C_V are specific heats at constant pressure and volume, respectively.

We subtract Eq. (2.37) from (2.38) and rearrange the terms to obtain

$$p_r = -\frac{1}{8\pi r^2}. \tag{2.55}$$

Eq. (2.32) on the trapping horizon yields

$$2p_t = \frac{\kappa}{2\pi r}. \tag{2.56}$$

From these values, using the definition of Hawking temperature ($T = -\kappa/2\pi$), we obtain the average pressure $\bar{P}$ as

$$\bar{P} = \frac{p_r + 2p_t}{3} = -\frac{1}{24\pi r^2} - \frac{T}{3r}, \tag{2.57}$$

which is the equation of state in three state parameters $T, \bar{P}$ and V. From this equation we can analyze thermodynamic stability of the wormhole.

Stable equilibrium of thermodynamic system requires that $\frac{\partial \bar{P}}{\partial V} |_T \leq 0$ where

$$\frac{\partial \bar{P}}{\partial V} |_T = \frac{(4\pi/3)^{2/3}}{36\pi V^{5/3}} + \frac{(4\pi/3)^{1/3} T}{9V^{4/3}}. \tag{2.58}$$

Now to ensure the stable equilibrium we must have

$$T \leq -\frac{1}{4\pi r}, \tag{2.59}$$

and thus the temperature assumes negative values everywhere for stable equilibrium which is attributed to the exotic matter. From Eq. (2.57) we have

$$\bar{P} \geq \frac{1}{24\pi r^2}. \tag{2.60}$$

Another condition for stable equilibrium is $C_P \geq C_V \geq 0$. Now, since constant V means constant E and S so by the definition of C_V,

$$C_V = \frac{\partial E}{\partial T} \mid_V = T\frac{\partial S}{\partial T} \mid_V = 0, \tag{2.61}$$

which means we can define heat capacity only at constant pressure as

$$C_P = T\frac{\partial S}{\partial T} \mid_P = \frac{2\pi r^2(24\pi \bar{P} r^2 + 1)}{24\pi \bar{P} r^2 - 1}, \tag{2.62}$$

where, from Eq. (2.57)

$$T = -(3r\bar{P} + \frac{1}{8\pi r}). \tag{2.63}$$

Now, from Eq. (2.60), to ensure the stable equilibrium, we can take the value of $\bar{P}$, for any non-negative ϵ, as

$$\bar{P} = \frac{1}{24\pi r^2} + \epsilon. \tag{2.64}$$

Thus Eq. (2.62) on using the above takes the form

$$C_P = \frac{1}{6\epsilon} + 2\pi r^2. \tag{2.65}$$

which is always positive. Thus the MTWHs are thermodynamically stable. This means that for stable equilibrium the average pressure is always positive, while temperature is always negative as is also depicted in Ref. [71] in which the possibility of negative temperature emerging from the exotic matter distribution was proposed.

2.5 Morris-Thorne wormholes in non-minimal curvature-matter coupling

In the regime of GR, NEC is not satisfied in wormholes, near throat which results in further violations of other energy conditions such as WEC, SEC, and DEC, in the same area [56, 59]. However these energy conditions may get some respect in modified gravitational theories which provide corrections to the Einstein tensor and stress-energy tensor. These correction terms are not very fruitful on small scale such as in solar system, however on large scale such as galactic, cosmological and on fields of strong gravitation, where there are some doubts of failure of GR, these correction terms play a significant role. Importance and applications of these extended theories of gravity has been enlightened in Refs. [12, 46–51]. In this way it may happen that wormholes be filled with no more exotic matter but ordinary natural matter which respect energy conditions. Such an approach has been used for $f(R)$ gravity [127, 128]. Recently, in 2009, It was shown that there exist a wormhole sulution which obeys the energy conditions when analyzed in f(R) gravity [123] and this solution has been generalized in Refs. [129–131].

Thus, in GR, NEC violation is necessary for a traversable wormhole. However, in higher curvature theories, such as Gauss-Bonnet theory, the matter that threads the wormhole respects NEC but it is now effective stress-energy tensor which is responsible for NEC violation. In curvature-matter coupling in $f(R, T)$ gravity solutions have been analyzed [132].

In this section we will extend the Hayward technique to the non-minimal curvature-matter coupling in the backgound of static MTWHs. Here we start with the gravitational field equations of non-minimal curvature-matter coupling in $f(R)$ gravity.

2.5.1 Gravitational field equations

In GR, while deriving gravitational field equations, the gravitational Lagrangian density, $\mathcal{L}_m = R$, is adopted in Hilbert action. This choice is not the ultimate choice. Thus in place of R, a general function $f(R)$ was introduced and thus the modified field equations were derived which explained those phenomena that GR could not account for [14]. Some models in $f(R)$ gravity, which combine dark energy and inflation, were also presented [133, 134]. The possibility to understand the galactic dynamics of huge massive test particles without considering dark matter has also been investigated [30, 135]. This modified theory was further generalized by the inclusion of explicit coupling, between matter and $f(R)$, in action [136]. This resulted in non-geodesic motion of massive particles and an extra force which is orthogonal to the four-velocity. This model was enhanced to arbitrary coupling, both in geometry and matter [137]. The coupling effects were incorporated in (effective) stress-energy tensor.

Consider the action of the non-minimal curvature-matter coupling in the context of $f(R)$ gravity given by [136]

$$S=\int\left[\frac{1}{2k}f_1(R)+\{1+\lambda_2 f_2(R)\}L_m\right]\sqrt{-g}d^4x, \tag{2.66}$$

where $f_i(R)$ $(i=1,2)$ are arbitrary functions which depend upon R, L_m is the matter Lagrangian density and $k=8\pi$, g is the determinant of the metric tensor and λ_2 is the coupling constant that characterizes the strength of interaction between curvature and matter. To obtain gravitational field equations we vary this action with respect to the metric $g_{\mu\nu}$ and get the following equations

$$\begin{aligned} F_1(R)R_{\mu\nu}-\frac{1}{2}f_1(R)g_{\mu\nu}-\nabla_\mu\nabla_\nu F_1(R)+g_{\mu\nu}\Box F_1(R) \\ =-2\lambda_2 F_2(R)L_m R_{\mu\nu}+2\lambda_2\Big(\nabla_\mu\nabla_\nu-g_{\mu\nu}\Box\Big)L_m F_2(R) \\ +\Big[1+\lambda_2 f_2(R)\Big]T^m_{\mu\nu}, \end{aligned} \tag{2.67}$$

where we have used the notation $F_i(R)=\partial f_i/\partial R$. The matter stress-energy tensor is given by

$$T^m_{\mu\nu}=-\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}. \tag{2.68}$$

Here, for simplicity, we take perfect fluid. It has also been argued [138] that $L_m=p$ is the natural choice for perfect fluid, where p is the pressure [139, 140]. This choice imposes vanishing of the extra force produced by the non-minimal curvature-matter coupling [136]. Further, $L_m = p$ does indeed reproduce the fluid equations of state but this choice is not the only one [141]. There are other choices for the matter Lagrangian density as well, such as $L_m = -\rho$ and $L_m = -na$, where ρ is the energy density, a is the physical free energy defined as $a = \rho/n - TS$, S being the entropy, T the temperature and n is the particle number density [79, 140–142]. Here, we will take $L_m = -\rho$ and consider the specific case $f_i(R) = R$, thus gravitational field equations (2.67) reduce to

$$G_{\mu\nu} = 8\pi T^{eff}_{\mu\nu}, \tag{2.69}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor and $T^{eff}_{\mu\nu}$ is called the effective stress-energy tensor given by

$$T^{eff}_{\mu\nu} = \Big(1 + \lambda_2 R\Big)T^m_{\mu\nu} + 2\lambda_2\left[\rho R_{\mu\nu} - \Big(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\Big)\rho\right]. \tag{2.70}$$

2.5.2 Generalized surface gravity

In this section we will derive the expression of the GSG in the context of $f(R)$ gravity. From the effective stress-energy tensor, on using background fluid, we can construct a function

$$\begin{aligned} \omega &= -g_{+-}T^{+-(eff)} \qquad (2.71)\\ &= \frac{\rho - p_r}{2} + \lambda_2\Bigg[\Big(1 - \frac{b}{r}\Big)\left\{(\rho + p_r)\Phi'' + (\rho + p_r)(\Phi')^2 - \frac{\rho - p_r}{r^2} + \frac{2p\Phi'}{r} - \rho'' - \rho'\Phi'\right\} \\ &+ (b - b'r)\left\{\frac{\Phi'}{2r^2}(\rho + p_r) + \frac{p}{r^3} - \frac{\rho'}{2r^2}\right\} + \frac{\rho - p_r}{r^2}\Bigg], \end{aligned}$$

and the vector

$$\psi = T^{++(eff)}\partial_+ r\partial_+ + T^{--(eff)}\partial_- r\partial_-. \tag{2.72}$$

The gravitational field equations, in the case of MTWHs, of interest are

$$-\partial_\pm\Theta_\pm - \frac{1}{2}\Theta_\pm^2 + \Theta_\pm\partial_\pm \log(-g_{+-}) = e^{2\Phi}\Bigg[\frac{\rho+p_r}{4} + \frac{\lambda_2}{2r^2}(\rho+p_r) + \lambda_2\Big(1-\frac{b}{r}\Big)\Bigg\{ -(\frac{\rho+p_r}{2})\Phi'' - (\frac{\rho+p_r}{2})(\Phi')^2 - (\frac{\rho+p_r}{2r^2}) - \frac{p\Phi'}{r} - \frac{\rho''}{2} - \frac{\rho'\Phi'}{2}\Bigg\} + \lambda_2(b-b'r)\left\{-(\rho+p_r)\frac{\Phi'}{4r^2} - \frac{(2\rho+p_r)}{2r^3} - \frac{\rho'}{4r^2}\right\}\Bigg], \tag{2.73}$$

$$\partial_\pm\Theta_\mp + \Theta_-\Theta_+ - \frac{1}{r^2}g_{\pm\mp} = \frac{e^{2\Phi}(\rho-p_r)}{4} + \frac{e^{2\Phi}\lambda_2}{2}\Bigg[\frac{\rho-p_r}{r^2} + \Big(1-\frac{b}{r}\Big)\Bigg\{(\rho+p_r)\Phi'' + (\rho+p_r)(\Phi')^2 - \frac{\rho-p_r}{r^2} + \frac{2p\Phi'}{r} - \rho'' - \rho'\Phi'\Bigg\} + (b-b'r)\left\{\frac{\Phi'}{2r^2}(\rho+p_r) + \frac{p}{r^3} - \frac{\rho'}{2r^2}\right\}\Bigg]. \tag{2.74}$$

The GSG κ satisfies

$$K^a\nabla_{[b}K_{a]} = \pm\kappa^{eff}K_b, \tag{2.75}$$

on the trapping horizon. which is equivalent to

$$\kappa^{eff} = \frac{1}{2}g^{ab}\partial_a\partial_b r. \tag{2.76}$$

For the MTWHs, on using the gravitational field equations, this gives

$$\kappa^{eff} = \frac{1}{2r_0} - 2\pi r_0\Big(\rho - p_r\Big) - 4\pi r_0\lambda_2\Bigg[\frac{\rho-p_r}{r_0^2} + \Big(1-b'(r_0)\Big)\left\{\frac{\Phi'(r_0)}{2r_0}\Big(\rho+p_r\Big) + \frac{p}{r_0^2} - \frac{\rho'(r_0)}{2r_0}\right\}\Bigg], \tag{2.77}$$

$$\begin{aligned}
\kappa^{eff} &= -2\pi\Bigg[r_0\Big(\rho+p_r\Big)-\lambda_2\Bigg\{\Big(\rho+p_r\Big)\Phi'(r_0)\Big(1-b'(r_0)\Big)\\
&+ \frac{2\Big(\rho+p_r\Big)\Big(1-b'(r_0)\Big)}{r_0}-\frac{2\Big(\rho+p_r\Big)}{r_0}\\
&+ \frac{2\rho\Big(1-b'(r_0)\Big)}{r_0}+\rho'(r_0)\Big(1-b'(r_0)\Big)\Bigg\}\Bigg]. \qquad (2.78)
\end{aligned}$$

The effective Hawking temperature can be calculated from $T^{eff} = -\kappa^{eff}/2\pi$, which is negative in our case of outer trapping horizon.

2.5.3 Unified first law

In spherical symmetric spacetimes, the UFL can also be formulated by using the gravitaional field equations for non-minimal curvature-matter coupling. According to this law the derivative of active gravitational energy, on using the gravitaional equations, is divided into two terms, the work term and the energy supply term. In components form Eq. (2.72) can be written as

$$\begin{aligned}
\psi_{\pm}^{eff} &= \pm e^{\Phi}\sqrt{1-\frac{b}{r}}\Bigg[\frac{\rho+p_r}{4}+\frac{\lambda_2}{2r^2}(\rho+p_r) \qquad (2.79)\\
&+\lambda_2\Big(1-\frac{b}{r}\Big)\Bigg\{-(\frac{\rho+p_r}{2})\Phi''-(\frac{\rho+p_r}{2})(\Phi')^2-(\frac{\rho+p_r}{2r^2})-\frac{p\Phi'}{r}-\frac{\rho''}{2}-\frac{\rho'\Phi'}{2}\Bigg\}\\
&+\lambda_2(b-b'r)\Bigg\{-(\rho+p_r)\frac{\Phi'}{4r^2}-\frac{(2\rho+p_r)}{2r^3}-\frac{\rho'}{4r^2}\Bigg\}\Bigg].
\end{aligned}$$

Now, taking gradient of the active gravitaional energy using gravitational field equations, yields the following result

$$\partial_{\pm}E = A\psi_{\pm}^{eff}+\omega^{eff}\partial_{\pm}V. \qquad (2.80)$$

This result is called the UFL, where

$$\partial_{\pm}E = \pm 4\pi r^2 e^{\Phi}\sqrt{1-\frac{b}{r}}\Bigg[\frac{\rho}{2} + \lambda_2\Big(1-\frac{b}{r}\Big)\left\{-\frac{\rho}{r^2} - \rho'' - \rho'\Phi'\right\} + \lambda_2(b-b'r)\left\{-\frac{\rho}{r^3} - \frac{\rho'}{2r^2}\right\} + \frac{\lambda_2\rho}{r^2}\Bigg], \tag{2.81}$$

and ω^{eff} is called the effective energy density while ψ^{eff}, the effective energy flux. All the terms appearing in the UFL vanish on the trapping horizon (throat) resulting in no evolution of the throat.

The right hand side of Eq. (2.80) consists of two terms: The first term, which is responsible for the variation of spacetime energy, is called the energy supply term, as due to the energy flux it produces variation in spacetime energy; and the second term, which supports the structure of spacetime, is called the work term which is carried out inside the wormhole.

2.6 Thermodynamics of Morris-Thorne wormholes in $f(R,T)$ gravity

One of the theory that gained significant attraction is the $f(R,T)$ gravity theory. This gravity was proposed by T. Harko and his collaborators and it extends the $f(R)$ gravity, by the inclusion of dependence of the Lagrangian on the trace T of the stress-energy tensor [20]. They used the metric formalism for deriving the gravitational field equations, also for test particles they obtained the equation of motion, followed from the covariant divergence of the stress-energy tensor. The models of $f(R,T)$ gravity have been used in a number of works to satisfactorily explain the cosmological problems [143–147], gravitational waves [148], thermodynamics [149–152] and accelerated expansion of the universe in the late times [153–156].

In this section we will discuss the gravitational field equations of $f(R,T)$ gravity and then using these equations we will find the GSG and formulate the UFL for the MTWHs. We will use the Kodama vector and trapping horizon of these wormholes which we have obtained earlier in this chapter. Finally we will include coupling in $f(R,T)$ gravity and extend our results for the non-minimal curvature-matter coupling.

2.6.1 $f(R,T)$ gravity

In $f(R)$ gravity, a new term T, the trace of the stress-energy tensor, was introduced in Ref. [20], thus a new modified $f(R,T)$ gravity theory was introduced. The $f(R,T)$ gravity models depend on the source term representing the variation of the matter stress-energy tensor with respect to the metric. Different choices of matter Lagrangian produce different set of field equations. Different models in $f(R,T)$ gravity, considering its explicit forms, and their properties have been discussed in [20]. In $f(R,T)$ gravity, the action can be written as

$$S = \int \sqrt{-g}\left[\frac{1}{16\pi}f(R,T) + L\right] d^4x, \tag{2.82}$$

where g is determinant of the metric tensor, $f(R,T)$ is the function of the Ricci scalar R and the trace of the stress-energy tensor T (i.e., $T = g^{\mu\nu}T_{\mu\nu}$), while L is the matter Lagrangian density. We vary this action with respect to the metric tensor $g^{\mu\nu}$, for the case $f(R,T) = R + 2\lambda_1 T$ with constant λ_1, and obtain the following field equations [20]

$$G_{\mu\nu} = 8\pi T^{eff}_{\mu\nu}, \tag{2.83}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T^{eff}_{\mu\nu}$ is the effective stress-energy tensor defined as

$$T^{eff}_{\mu\nu} = T^{(m)}_{\mu\nu} + \frac{\lambda_1}{4\pi}\left\{T^{(m)}_{\mu\nu} + \bar{P}g_{\mu\nu} + \frac{1}{2}g_{\mu\nu}T^{(m)}\right\}, \tag{2.84}$$

with $T^{(m)} = g^{\mu\nu}T^{(m)}_{\mu\nu}$ is the trace of the matter stress-energy tensor and $\bar{P} = (p_r + 2p_t)/3$ is the average pressure.

2.6.2 Generalized surface gravity and unified first law

We can also formulate the UFL in $f(R,T)$ gravity using gravitational field equations in the expression of gradient of the MS energy. The gravitational field equations of interest for MTWHs in $f(R,T)$ gravity are

$$\partial_{\pm}\Theta_{\pm} = -\frac{1}{2}\Theta_{\pm}^{2} + \Theta_{\pm}\partial_{\pm}\log(-g_{+-}) - 2\pi e^{2\Phi}(\rho + p_r)\left\{1 + \frac{\lambda_1}{4\pi}\right\}, \tag{2.85}$$

$$\partial_{\pm}\Theta_{\mp} = -\Theta_{+}\Theta_{-} + \frac{1}{r^2}g_{+-} + 2\pi e^{2\Phi}(\rho - p_r) - \lambda_1 e^{2\Phi}\left\{-\rho + \frac{4p_r}{3} + \frac{5p_t}{3}\right\}. \tag{2.86}$$

Now, from the effective stress-energy tensor (2.84) we can construct a function ω^{eff} and a vector ψ^{eff} as

$$\omega^{eff} = -g_{+-}T^{+-eff} = \frac{\rho - p_r}{2} - \frac{\lambda_1}{4\pi}\left\{-\rho + \frac{4p_r}{3} + \frac{5p_t}{3}\right\}, \tag{2.87}$$

and

$$\psi^{eff} = T^{++eff}\partial_{+}r\partial_{+} + T^{--eff}\partial_{-}r\partial_{-}. \tag{2.88}$$

On using the gravitational field equations (2.85) and (2.86), the GSG (2.36) can also be expressed as

$$\kappa^{eff} = -2\pi r_0\,(\rho + p_r)\,(1 + \frac{\lambda_1}{4\pi}), \tag{2.89}$$

and

$$\kappa^{eff} = \frac{1}{2r_0} - 2\pi r_0(\rho - p_r) + \lambda_1 r_0\left\{-\rho + \frac{4p_r}{3} + \frac{5p_t}{3}\right\}. \tag{2.90}$$

We can write Eq. (2.88) in the component form as

$$\psi_{\pm}^{eff} = \pm e^{\Phi}\sqrt{1 - \frac{b}{r}}\frac{(\rho + p_r)}{4}\left\{1 + \frac{\lambda_1}{4\pi}\right\}. \tag{2.91}$$

The UFL can also be extended from Einstein to $f(R,T)$ gravity. The gradient of the MS energy (2.4), with the use of gravitational field equations (2.85) and (2.86) can be written in the form

$$\partial_{\pm}E = A\psi_{\pm}^{eff} + \omega^{eff}\partial_{\pm}V, \tag{2.92}$$

where ω^{eff} and $\psi_{\pm}^{eff}$ are given in Eqs. (2.87) and (2.91),respectively, while

$$\partial_{\pm}E = \pm 2\pi r^2 e^{\Phi}\sqrt{1-\frac{b}{r}}\left\{\rho + \frac{\lambda_1}{4\pi}\left[\frac{3\rho}{2} - \frac{5p_r}{6} - \frac{5p_t}{3}\right]\right\}. \tag{2.93}$$

2.7 Non-minimal curvature-matter coupling in $f(R,T)$ gravity

In this section we extend the work, presented in previous section, to the non-minimal curvature-matter coupling. We consider the action for the non-minimal curvature-matter coupling, in the context of $f(R,T)$ gravity, given by

$$S = \int\left[\frac{1}{16\pi}f_1(R,T) + \{1+\lambda_2 f_2(R)\}L\right]\sqrt{-g}d^4x, \tag{2.94}$$

where $f_1(R,T)$ is an arbitrary function of R and T while $f_2(R)$ is an arbitrary function of R only. Here λ_2 is called coupling constant and it characterizes curvature-matter coupling strength. We will consider the simplest case by taking $f_1(R,T) = R + 2\lambda_1 T$ and $f_2(R) = R$. The gravitational field equations in this case, by varying the action with respect to $g^{\mu\nu}$, are obtained as

$$G_{\mu\nu} = 8\pi T_{\mu\nu}^{eff}, \tag{2.95}$$

where

$$\begin{aligned} T_{\mu\nu}^{eff} &= \Big(1+\lambda_2 R\Big)T_{\mu\nu}^{m} + 2\lambda_2\Big[\rho R_{\mu\nu} - \Big(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\Big)\rho\Big] \\ &+ \frac{\lambda_1}{4\pi}\left\{T_{\mu\nu}^{(m)} + \bar{P}g_{\mu\nu} + \frac{1}{2}g_{\mu\nu}T^{(m)}\right\}. \end{aligned} \tag{2.96}$$

is the effective stress-energy tensor. Before proceeding further, we introduce the following notation

$$\begin{aligned} u &= u(r) = 1 - \frac{b(r)}{r}, && (2.97)\\ w &= w(r) = e^{2\Phi}, && (2.98)\\ u' &= \frac{b - b'r}{r^2}, && (2.99)\\ w' &= 2e^{2\Phi}\Phi', && (2.100)\\ w'' &= 2e^{2\Phi}\Phi'' + 4e^{2\Phi}\Phi'^2. && (2.101)\end{aligned}$$

Now, the gravitational field equations of interest in this case take the form

$$\begin{aligned} \partial_\pm\Theta_\pm &= -\frac{1}{2}\Theta_\pm^2 + \Theta_\pm\partial_\pm \log(-g_{+-})\\ &- 4\pi w\left\{\frac{\rho + p_r}{2} + \frac{\lambda_1}{8\pi}(\rho + p_r) + \frac{\lambda_2 p_r l_4}{rw} + \frac{\lambda_2(\rho + p_r)}{4r^2w^2}\left[l_3 - r^2 l_1\right] - \frac{\lambda_2 l_2}{2w}\right\}, \end{aligned} \tag{2.102}$$

$$\begin{aligned} \partial_\pm\Theta_\mp &= -\Theta_+\Theta_- + \frac{1}{r^2}g_{+-}\\ &+ 4\pi w\left\{\frac{\rho - p_r}{2} - \frac{\lambda_1 l_7}{4\pi} + \frac{\lambda_2(\rho - p_r)l_6}{r^2} - \frac{\lambda_2 l_2}{2w} + \frac{\lambda_2(\rho + p_r)l_1}{4w^2} + \frac{\lambda_2 p_r l_5}{rw}\right\}. \end{aligned} \tag{2.103}$$

Here we have defined

$$l_1 = 2uww'' - uw'^2 + ww'u', \tag{2.104}$$

$$l_2 = 2wu\rho'' + wu'\rho' + \rho'w'u, \tag{2.105}$$

$$l_3 = 4w^2(1 - u - 2ru'), \tag{2.106}$$

$$l_4 = wu' - w'u, \tag{2.107}$$

$$l_5 = wu' + w'u, \tag{2.108}$$

$$l_6 = 1 - u, \tag{2.109}$$

$$l_7 = -\rho + \frac{4p_r}{3} + \frac{5p_t}{3}. \tag{2.110}$$

From the effective stress-energy tensor (2.96) we can construct a vector ψ^{eff} and a function ω^{eff} as

$$\psi^{eff} = T^{++eff}\partial_+ r\partial_+ + T^{--eff}\partial_- r\partial_-, \tag{2.111}$$

and

$$\begin{aligned} \omega^{eff} &= -g_{+-}T^{+-(eff)} = \frac{\rho - p_r}{2} - \frac{\lambda_1 l_7}{4\pi} + \frac{\lambda_2(\rho - p_r)l_6}{r^2} \\ &+ \frac{\lambda_2(\rho + p_r)l_1}{4w^2} - \frac{\lambda_2 l_2}{2w} + \frac{\lambda_2 p_r l_5}{rw}. \end{aligned} \tag{2.112}$$

Using a similar procedure as before the effective GSG, on using the gravitational field equations (2.102) and (2.103), for the non-minimal case, are obtained as

$$\begin{aligned} \kappa^{eff} &= -2\pi r_h\Bigg\{\rho + p_r + \frac{\lambda_1(\rho + p_r)}{4\pi} + \frac{2\lambda_2(\rho + p_r)}{4r_h^2 w}\Big[4w - r_h^2 w'u' - 8r_h wu'\Big] \\ &+ \frac{2\lambda_2 p_r u'}{r_h} - \lambda_2\rho'u'\Bigg\}, \end{aligned} \tag{2.113}$$

and

$$\kappa^{eff} = \frac{1}{2r_h} - 2\pi r_h(\rho - p_r) + \lambda_1 r\left[-\rho + \frac{4p_r}{3} + \frac{5p_t}{3}\right] - \frac{4\pi\lambda_2(\rho - p_r)}{r_h} - \frac{\pi\lambda_2 r_h(\rho + p_r)w'u'}{w} - 4\pi\lambda_2 p_r u' + 2\pi\lambda_2 r_h \rho' u'. \quad (2.114)$$

The effective thermal temperature, in this case, can be calculated as $T^{eff} = -\kappa^{eff}/2\pi$.

In the component form, Eq. (2.111) can be written as

$$\psi_\pm = \pm\frac{\sqrt{uw}}{2}\left\{\frac{\rho + p_r}{2} + \frac{\lambda_1}{8\pi}(\rho + p_r) - \frac{\lambda_2 l_2}{2w} + \frac{\lambda_2(\rho + p_r)}{4r^2w^2}\left[l_3 - r^2 l_1\right] + \frac{\lambda_2 p_r l_4}{rw}\right\}. \quad (2.115)$$

Now taking gradient of Eq. (2.4), and using Eqs. (2.102) and (2.103) in it, we get

$$\partial_\pm E = \pm 2\pi r^2\sqrt{uw}\left\{\rho + \frac{\lambda_1}{8\pi}\left[\rho + p_r - 2l_7\right] - \frac{\lambda_2 l_2}{w} + \frac{\lambda_2\rho}{4r^2w^2}\left[l_3 + 4w^2 l_6\right]\right\}. \quad (2.116)$$

Thus UFL in non-minimal curvature-matter coupling can also be formulated from Eqs. (2.112), (2.115) and (2.116) as

$$\partial_\pm E = A\psi_\pm^{eff} + \omega^{eff}\partial_\pm V. \quad (2.117)$$

Thus, the results of this section provide correction by replacing the stress-energy tensor of matter with the effective stress-energy tensor which includes in it further corrections due to coupling also.

Chapter 3

Thermodynamics of charged wormholes

The MTWHs can be generalized by adding extra matter to them. One way of doing this is to add charges which behave as additional matter to the static MTWHs whose structure is already maintained by the exotic matter. We will consider charged wormholes (CWHs) [157] in this chapter and study their thermodynamics, horizon mechanics and thermodynamic stability. CWHs are the charged extension of static MTWHs. Also, absence of event horizon in these objects do not disturb their traversability, however now traversability conditions are imposed on the effective shape function which will be discussed in detail in this chapter.

In this chapter, we will extend the formalism of finding GSG and UFL to CWHs. These thermodynamic workouts will be done in Einstein's gravity and then the work will be generalized to non-minimal curvature-matter coupling and further in $f(R,T)$ gravity.

3.1 Charged wormholes (CWHs)

The metric for CWHs is described by [157]

$$ds^2 = -\left[e^{2\Phi(r)} + \frac{q^2}{r^2}\right]dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r} + \frac{q^2}{r^2}} + r^2 d\Omega^2, \tag{3.1}$$

where q is the scalar electric charge. This wormhole is the combination of static MTWHs and Reissner-Nordström spacetime. If in this metric $q = 0$ then it simply becomes MTWHs. For $\Phi = 0 = b$, it represents Reissner-Nordström black hole for zero mass, and it becomes flat Minkowski metric if $b = \Phi = q = 0$. The effective shape and redshift functions of CWHs (3.1) are $b_{eff}(r) = b(r) - q^2/r$ and $\Phi_{eff}(r) = \frac{1}{2}\ln(e^{2\Phi} + q^2/r^2)$, respectively. Thus the shape of CWHs will vary with charge q by additional factor q^2/r. At the new throat $\tilde{r}_0$, a coordinate singularity $b_{eff}(\tilde{r}_0) = \tilde{r}_0$ occurs, which implies

$$\tilde{r}_0 = \frac{1}{2}(b \pm \sqrt{b^2 - 4q^2}), \tag{3.2}$$

which for $q = 0$ gives $\tilde{r}_0 = 0$ and $\tilde{r}_0 = b(\tilde{r}_0)$. The first root is meaningless, so we have only one throat corresponding to the larger root. Now the condition $b_{eff}(r) < r$ for $r > \tilde{r}_0$ in terms of $b(r)$ implies $b(r) - q^2/r < r$. The positiveness and flaring out conditions for CWHs imply $b_{eff}(r) > 0$ and $b'_{eff} < b_{eff}/r$, which can be written as

$$b > \frac{q^2}{r}, \tag{3.3}$$

and

$$b' < \frac{b}{r} - \frac{2q^2}{r^2}. \tag{3.4}$$

Thus for CWHs the flaring out condition implies that $b'(r)$ should be smaller by an additional factor of $2q^2/r^2$, as compared to MTWHs, while the value of $b(r)$ should be bigger atleast by a factor q^2/r compared to MTWHs to meet the positiveness condition of effective shape function $b_{eff}(r)$ [157].

Now for the electro-magnetic stress-energy tensor we consider a Lagrangian

$$L^e = -\frac{1}{16\pi}F_{\alpha\beta}F_{\gamma\sigma}g^{\alpha\gamma}g^{\beta\sigma}, \tag{3.5}$$

where $F_{\alpha\beta}$ is the electro-magnetic field tensor, given by

$$F_{\mu\nu} = \varepsilon(r) \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.6}$$

Now $T^{(e)}_{\mu\nu} = (1/4\pi)(F_{\mu\lambda}F^{\lambda}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma})$ is the electro-magnetic stress-energy tensor, which on using Eq. (3.6) takes the form [157]

$$T^{\mu(e)}_{\nu} = \frac{1}{8\pi}\varepsilon^2 diag(3,3,1,1)\gamma\delta, \tag{3.7}$$

where $\varepsilon = \varepsilon(r) = (q/r^2)\sqrt{|g_{00}g_{11}|}$ is the radial component of electric field, while $\gamma = (r^2e^{2\Phi(r)} + q^2)^{-1}$ and $\delta = r^2 - rb(r) + q^2$.

3.2 Trapping horizon and its classification

In this section we find trapping horizon for the CWHs (3.1). Since in CWHs event horizon is not present so we cannot use global considerations to study the thermodynamics. Instead we use the concept of trapping horizon which is defined using local considerations for spherically symmetric spacetimes. In order to obtain the expression for trapping horizon, we transform the metric (3.1) into a new form (2.1) using null coordinates. Thus metric (3.1) can be written in the form (2.1) by introducing the null coordinates $x^+ = t + r_*$ and $x^- = t - r_*$, where r and r_* satisfy the following equation

$$\frac{dr}{dr_*} = \sqrt{-\frac{g_{tt}}{g_{rr}}} = \sqrt{\left(e^{2\Phi(r)} + \frac{q^2}{r^2}\right)\left(1 - \frac{b(r)}{r} + \frac{q^2}{r^2}\right)}. \tag{3.8}$$

Here

$$g_{+-} = -\frac{1}{2}\left(e^{2\Phi(r)} + \frac{q^2}{r^2}\right), \tag{3.9}$$

and areal radius r are functions of the null coordinates, related with the outgoing and ingoing light rays normal to each symmetric spheres $\partial_\pm = \partial/\partial x^\pm$, while $d\Omega^2$ refers to the metric on the unit 2-sphere. One can define the expansions as

$$\Theta_\pm = \frac{2}{r}\partial_\pm r = \pm\frac{1}{r}\sqrt{\left(e^{2\Phi(r)} + \frac{q^2}{r^2}\right)\left(1 - \frac{b(r)}{r} + \frac{q^2}{r^2}\right)}. \tag{3.10}$$

These expansions tell us about the convergence or divergence of light rays in the null direction normal to a sphere. Also, $\Theta > 0$, means light rays are expanding while they are contracting for, $\Theta < 0$, in the null directions normal to the sphere. Or equivalently the area of a sphere is expanding or contracting in the null directions.

Now the sign of $\Theta_+\Theta_-$ is a geometrical invariant. A metric sphere is trapped if $\Theta_+\Theta_- > 0$, yielding

$$r^2 - rb(r) + q^2 < 0, \tag{3.11}$$

untrapped if $\Theta_+\Theta_- < 0$, yielding

$$r^2 - rb(r) + q^2 > 0, \tag{3.12}$$

and marginal if $\Theta_+\Theta_- = 0$, yielding

$$r^2 - rb(r) + q^2 = 0. \tag{3.13}$$

If on an untrapped sphere the orientations $\Theta_+ > 0$ and $\Theta_- < 0$ are locally assumed then ∂_+ and ∂_- are also assumed as the null normal vectors in the outgoing and ingoing directions, respectively.

For trapping horizon, a hypersurface on which, $\Theta_+\Theta_- = 0$, we choose

$$\Theta_+|_h = 0, \tag{3.14}$$

which gives

$$b(r_h) = r_h + \frac{q^2}{r_h}. \tag{3.15}$$

Thus at $r_h = \tilde{r}_0$, this trapping horizon is future if $\Theta_- < 0$, past if $\Theta_- > 0$ and bifurcating if $\Theta_- = 0$. Here $\Theta_+ = 0$ implies $\Theta_- = 0$ which makes the trapping horizon bifurcating.

This horizon may be outer, inner or degenerate depending on the sign of $\partial_-\Theta_+$. It is outer if $\partial_-\Theta_+ < 0$, which gives

$$b - b'r > \frac{2q^2}{r}, \tag{3.16}$$

inner if $\partial_-\Theta_+ > 0$, yielding

$$b - b'r < \frac{2q^2}{r}, \tag{3.17}$$

and degenerate if $\partial_-\Theta_+ = 0$, giving

$$b - b'r = \frac{2q^2}{r}. \tag{3.18}$$

3.3 Thermodynamics of charged wormholes

The MS energy for CWHs is given by

$$E = \frac{r}{2}(1 - 2g^{+-}\partial_+ r\partial_- r) = \frac{b}{2} - \frac{q^2}{2r}, \tag{3.19}$$

On the trapping horizon it becomes $E = r_h/2$.

The Kodama vector (2.11) in the covariant form, in this case, yields

$$K_\pm = -\frac{1}{2}\sqrt{\left(e^{2\Phi(r)} + \frac{q^2}{r^2}\right)\left(1 - \frac{b(r)}{r} + \frac{q^2}{r^2}\right)}. \tag{3.20}$$

This vector provides a trapping horizon which is null on hyper surface $\Theta_+|_h = 0$.

This vector is the generalization of a Killing vector that can be obtained from it in stationary vacuum cases.

Now the Einstein-Maxwell equations (2.8)-(2.10) in local coordinates are

$$\partial_\pm\Theta_\pm = -\frac{1}{2}\Theta_\pm^2 + \Theta_\pm\partial_\pm \log(-g_{+-}) - \frac{2\pi}{r^2}(\rho+p_r)(r^2e^{2\Phi}+q^2), \tag{3.21}$$

$$\partial_\pm\Theta_\mp = -\Theta_+\Theta_- + \frac{1}{r^2}g_{+-} + \frac{2\pi}{r^2}(\rho - p_r - \frac{3q^2}{4\pi r^4})(r^2e^{2\Phi}+q^2), \tag{3.22}$$

$$\partial_+\Theta_- + \partial_-\Theta_+ + \Theta_+\Theta_- = -8\pi(p_t + \frac{q^2}{8\pi r^4}). \tag{3.23}$$

where the stress-energy tensor $T_{\mu\nu}$ is the sum of the matter part $T_{\mu\nu}^{(m)}$ and electromagnetic part $T_{\mu\nu}^{(e)}$, i.e., $T_{\mu\nu} = T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(e)}$. Here From Eq. (2.24), in this case, we have used

$$\begin{aligned} T_{++} &= T_{--} = \frac{(r^2e^{2\Phi}+q^2)(\rho+p_r)}{4r^2}, & (3.24)\\ T_{+-} &= T_{-+} = \frac{(r^2e^{2\Phi}+q^2)(\rho-p_r-3q^2/4\pi r^4)}{4r^2}, & (3.25)\\ T_{\theta\theta} &= r^2(p_t + \frac{q^2}{8\pi r^4}). & (3.26)\end{aligned}$$

From the stress-energy tensor we can construct a function ω and a vector ψ as

$$\omega = -g_{+-}T^{+-} = \frac{\rho - p_r}{2} - \frac{3q^2}{8\pi r^4}, \tag{3.27}$$

and

$$\psi = T^{++}\partial_+ r\partial_+ + T^{--}\partial_- r\partial_-. \tag{3.28}$$

It is worth noticing that Killing horizon is not present in these CWHs, despite the presence of a Killing vector. Hence the GSG cannot be defined by Killing vector and instead, we make use of the Kodama vector and trapping horizon. The GSG κ

satisfies (2.13), which in the case of CWHs takes the form

$$\kappa = \frac{1 - b'(r_h) - q^2/r_h^2}{4r_h}, \tag{3.29}$$

which is positive due to the flarring out condition. The GSG (3.29) on using Einstein field equations (3.21) and (3.22), becomes

$$\kappa = -2\pi r_h \left(\rho + p_r\right), \tag{3.30}$$

and

$$\kappa = \frac{1}{2r_h} - 2\pi r_h(\rho - p_r - \frac{3q^2}{4\pi r_h^4}). \tag{3.31}$$

The thermal temperature can be calculated from the formula $T = -\kappa/2\pi$, which is negative. However one could avoid this negative temperature by making claim that this is the problem only at horizon, but ingoing radiations appearing at one mouth of the wormhole, following the classical trajectory, would reappear as outgoing radiation on the other mouth of the wormhole, unavoiding this negative temperature. This is not surprising as wormholes are argued to be constructed by phantom energy which may be characterized by negative temperature. Thus wormholes emit radiations associated with negative temperature in the same way as black holes emit radiations associated with positive temperature.

Since we are dealing with CWHs which are also spherically symmetric like MTWHs, hence we can also formulate UFL in these wormholes from the gradient of MS energy, on using the gravitational field equations.

Eq. (3.28), in the component form, can be written as

$$\psi_{\pm} = \pm\sqrt{\left(e^{2\Phi} + \frac{q^2}{r^2}\right)\left(1 - \frac{b(r)}{r} + \frac{q^2}{r^2}\right)}\frac{\rho + p_r}{4}. \tag{3.32}$$

Taking gradient of Eq. (3.19) and making use of Eqs. (3.27) and (3.32), the UFL can be formulated as

$$\partial_{\pm}E = A\psi_{\pm} + \omega\partial_{\pm}V, \tag{3.33}$$

where

$$\partial_{\pm}E = \pm 2\pi r^2\sqrt{(e^{2\phi} + \frac{q^2}{r^2})(1 - \frac{b}{r} + \frac{q^2}{r^2})}\left\{\rho - \frac{3q^2}{8\pi r^4}\right\}. \tag{3.34}$$

In Eq. (3.33), on the right hand side, there appear two terms, $A\psi_{\pm}$ and $\omega\partial_{\pm}V$. The first one is called the energy supply term which produces change in the gravitational energy due to energy flux ψ, while the second one is called the work term that is carried out in the wormhole to support its configuration.

3.3.1 Thermodynamic stability

In this section we will examine the thermodynamic stability of wormholes for a specific case ($\Phi = 0$). Thermodynamic stability of wormholes can be ensured by showing that $\frac{\partial \bar{P}}{\partial V}|_T \leq 0$ and $C_P \geq C_V \geq 0$, where $\bar{P} = (p_r + 2p_t)/3$ is the average pressure while C_P and C_V are specific heats at constant pressure and volume, respectively.

Subtracting Eq. (3.30) from (3.31), and rearranging the terms yields

$$p_r = -\frac{1}{8\pi r^2} - \frac{3q^2}{8\pi r^4}. \tag{3.35}$$

Solving Eq.(3.23) on the trapping horizon and using the definition of surface gravity we obtain

$$2p_t = -\frac{T(r^2e^{2\Phi} + q^2)}{r^3} - \frac{q^2}{4\pi r^4}. \tag{3.36}$$

Now from Eqs. (3.35) and (3.36), we can find the equation of state in three state parameters ($\bar{P}$, V, T) as

$$\bar{P} = -\left[\frac{1}{7776\pi V^2}\right]^{1/3} - 5q^2\left[\frac{\pi}{4374V^4}\right]^{1/3} - T\left[\frac{4\pi}{81V}\right]^{1/3} - \frac{4\pi Tq^2}{9V}. \tag{3.37}$$

From this equation of state, thermodynamic stability of the wormhole can be analyzed. Taking derivative with respect to V, we obtain

$$\frac{\partial \bar{P}}{\partial V}|_T = \left[\frac{1}{262244\pi V^5}\right]^{1/3} + 5q^2\left[\frac{32\pi}{59049V^7}\right]^{1/3} + T\left[\frac{4\pi}{2187V^4}\right]^{1/3} + \frac{4\pi T q^2}{9V^2}. \tag{3.38}$$

For thermodynamic stability $\frac{\partial \bar{P}}{\partial V}|_T \leq 0$, which yields

$$T \leq -\frac{r^2 + 10q^2}{4\pi r^3 + 12\pi r q^2}, \tag{3.39}$$

which shows that temperature assumes negative values everywhere, which is not surprising as it could be attributed to the presence of the exotic matter. Using this in Eq. (3.37), we get

$$\bar{P} \geq -\frac{1}{24\pi r^2} - \frac{5q^2}{24\pi r^4} + \frac{(r^2+10q^2)(r^2+q^2)}{12\pi r^6 + 36\pi r^4 q^2}. \tag{3.40}$$

For $q = 0$ the average pressure always assumes positive values, however for $q \neq 0$ it may be negative somewhere.

Stable equilibrium also requires that $C_P \geq C_V \geq 0$. Now, at constant volume S is also constant, so specific heat at constant volumes vanishes

$$C_V = \frac{\partial E}{\partial T}|_V = T\frac{\partial S}{\partial T}|_V = 0. \tag{3.41}$$

So, we can define specific heat only at constant pressure as

$$C_P = \frac{\partial E}{\partial T}|_P = T\frac{\partial S}{\partial T}|_P = \frac{2\pi r^2(r^2+q^2)}{r^2 + 3q^2 - \frac{2(r^2+q^2)(r^2+10q^2)}{r^2(24\pi r^2\bar{P}+1+5q^2/r^2)}}. \tag{3.42}$$

We substitute

$$\bar{P} = -\frac{1}{24\pi r^2} - \frac{5q^2}{24\pi r^4} + \frac{(r^2+10q^2)(r^2+q^2)}{12\pi r^6 + 36\pi r^4 q^2} + \epsilon, \tag{3.43}$$

in Eq. (3.42), which also ensures Eq. (3.40) for any $\epsilon > 0$. Thus Eq. (3.42) takes the form

$$C_P = \frac{(r^2+q^2)\Big[(r^2+q^2)(r^2+10q^2)+12\pi\epsilon r^4(r^2+3q^2)\Big]}{6\epsilon r^2(r^2+3q^2)^2}. \tag{3.44}$$

All terms involved on the right hand side are positive, thus specific heat is positive everywhere subjected to the constraint (3.40). Thus the wormholes could be thermodynamically stable.

3.4 Non-minimal curvature-matter coupling

This section deals with the extension of the work presented in this chapter before to non-minimal curvature-matter coupling. In Einstein's filed equations, Hilbert action is used in which Lagrangian density is a linear function of the Ricci scalar, R, however there is no evidence that this must be only a linear function of R, so in place of R a function $f(R)$ was proposed and modified field equations were obtained [14], and which was further investigated in Refs. [15–17]. We, in our case, consider a non-minimal curvature-matter coupling in f(R) gravity whose action is given by

$$S = \int \left[\frac{1}{16\pi}f_1(R) + \{1+\lambda_2 f_2(R)\}\left(L^m + L^e\right)\right]\sqrt{-g}d^4x, \tag{3.45}$$

where $f_1(R)$ and $f_2(R)$ are arbitrary functions of the Ricci tensor R, λ_2 is the coupling constant which characterises the strength of the curvature-matter coupling, L^m and L^e are the matter Lagrangian and the Lagrangian due to charge, respectively. The following gravitational field equations can be obtained from this action by varying with respect to $g^{\mu\nu}$ as

$$\begin{aligned} F_1(R)R_{\mu\nu} - \frac{1}{2}f_1(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F_1(R) + g_{\mu\nu}\Box F_1(R) = -2\lambda_2 F_2(R)(L^m+L^e)R_{\mu\nu} \\ + 2\lambda_2\Big(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\Big)(L^m+L^e)F_2(R) + \Big[1+\lambda_2 f_2(R)\Big](T^m_{\mu\nu}+T^e_{\mu\nu}). \end{aligned} \tag{3.46}$$

For the specific case, when $f_1(R) = f_2(R) = R$, the gravitational field equations take the form

$$G_{\mu\nu} = 8\pi T^{eff}_{\mu\nu}, \tag{3.47}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor and $T^{eff}_{\mu\nu}$ is called the effective stress-energy tensor, given by

$$T^{eff}_{\mu\nu} = \Big(1 + \lambda_2 R\Big)(T^{m}_{\mu\nu} + T^{e}_{\mu\nu}) + 2\lambda_2\left[\rho R_{\mu\nu} - \Big(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\Big)\rho\right]. \tag{3.48}$$

From now onward, we will use the notation introduced below

$$\begin{aligned} U &= U(r) = 1 - \frac{b(r)}{r} + \frac{q^2}{r^2}, && (3.49)\\ W &= W(r) = e^{2\Phi} + \frac{q^2}{r^2}, && (3.50)\\ U' &= \frac{br - b'r^2 - 2q^2}{r^3}, && (3.51)\\ W' &= \frac{2(r^3e^{2\Phi}\Phi' - q^2)}{r^3}, && (3.52)\\ W'' &= \frac{2(r^4e^{2\Phi}\Phi'' + 2r^4e^{2\Phi}\Phi'^2 + 3q^2)}{r^4}. && (3.53) \end{aligned}$$

The gravitational field equations of interest now become

$$\begin{aligned} \partial_\pm\Theta_\pm &= -\frac{1}{2}\Theta_\pm^2 + \Theta_\pm\partial_\pm\log(-g_{+-}) - 4\pi W\left\{\frac{\rho + p_r}{2} - \frac{\lambda_2}{2W}\Big[2WU\rho'' + WU'\rho'\right.\\ &+ W'U\rho'\Big] + \frac{\lambda_2(\rho + p_r)}{4r^2W^2}\Big[4W^2 + r^2W'^2U - 2r^2UWW'' - r^2WW'U'\\ &- \left. 8rW^2U' - 4W^2U\Big] + \frac{\lambda_2 p_r}{rW}\Big[WU' - W'U\Big]\right\}, \end{aligned} \tag{3.54}$$

$$\begin{aligned}
\partial_{\pm}\Theta_{\mp} &= -\Theta_{+}\Theta_{-} + \frac{1}{r^2}g_{+-} + 4\pi W\Bigg\{\frac{\rho - p_r}{2} - \frac{3q^2}{8\pi r^4} + \frac{\lambda_2(\rho + p_r)}{4W^2}\Bigg[2UWW'' \\
&- UW'^2 + WW'U'\Bigg] + \frac{3\lambda_2 q^2}{16\pi r^6 W^2}\Bigg[-r^2W'^2U + 2r^2UWW'' - 4W^2 \\
&+ r^2WW'U' + 4rWW'U + 4rW^2U' + 4W^2U\Bigg] + \frac{\lambda_2(\rho - p_r)}{r^2}\Bigg[1 - U\Bigg] \\
&+ \frac{\lambda_2 p_r}{rW}\Bigg[WU' + UW'\Bigg] + \frac{\lambda_2}{2W}\Bigg[-2WU\rho'' - WU'\rho' - \rho'W'U\Bigg]\Bigg\}.
\end{aligned} \tag{3.55}$$

From effective-stress energy tensor, we can construct a vector ψ^{eff} and a function ω^{eff} as

$$\psi^{eff} = T^{++(eff)}\partial_{+}r\partial_{+} + T^{--(eff)}\partial_{-}r\partial_{-}, \tag{3.56}$$

and

$$\begin{aligned}
\omega^{eff} &= -g_{+-}T^{+-(eff)} = \frac{\rho - p_r}{2} - \frac{3q^2}{8\pi r^4} + \frac{\lambda_2(\rho + p_r)}{4W^2}\Bigg[2UWW'' - UW'^2 + WW'U'\Bigg] \\
&+ \frac{3\lambda_2 q^2}{16\pi r^6 W^2}\Bigg[-r^2W'^2U + 2r^2UWW'' - 4W^2 + r^2WW'U' + 4rWW'U \\
&+ 4rW^2U' + 4W^2U\Bigg] + \frac{\lambda_2}{2W}\Bigg[-2WU\rho'' - WU'\rho' - \rho'W'U\Bigg] \\
&+ \frac{\lambda_2 p_r}{rW}\Bigg[WU' + UW'\Bigg] + \frac{\lambda_2(\rho - p_r)}{r^2}\Bigg[1 - U\Bigg].
\end{aligned} \tag{3.57}$$

The effective surface gravity, evaluated on the trapping horizon, in this case becomes

$$\kappa^{eff} = -2\pi r_h\Bigg\{(\rho + p_r) + \frac{2\lambda_2 p_r U'}{r_h} - \lambda_2\rho' U' + \frac{2\lambda_2(\rho + p_r)}{4r_h^2 W}\Bigg[4W - r_h^2W'U' - 8r_hWU'\Bigg]\Bigg\}, \tag{3.58}$$

and

$$\kappa^{eff} = \frac{1}{2r_h} - 2\pi r_h(\rho - p_r) + \frac{3q^2}{2r_h^3} - \frac{3\lambda_2 q^2}{4W r_h^5}\Big[r_h^2 W'U' + 4r_h WU' - 4W\Big] - 4\pi\lambda_2 p_r U' - \frac{\pi\lambda_2 r_h(\rho + p_r)W'U'}{W} - \frac{4\pi\lambda_2(\rho - p_r)}{r_h} + 2\pi\lambda_2 r_h \rho' U'. \tag{3.59}$$

The effective thermal temperature can be calculated as $T^{eff} = -\kappa^{eff}/2\pi$.

Eq. (3.56) in the component form becomes

$$\psi_\pm = \pm\frac{\sqrt{UW}}{2}\Big\{\frac{\rho + p_r}{2} - \frac{\lambda_2}{2W}\Big[2WU\rho'' + WU'\rho' + W'U\rho'\Big] + \frac{\lambda_2 p_r}{rW}\Big[WU' - W'U\Big] + \frac{\lambda_2(\rho + p_r)}{4r^2W^2}\Big[4W^2 + r^2W'^2U - 2r^2UWW'' - r^2WW'U' - 8rW^2U' - 4W^2U\Big]\Big\}. \tag{3.60}$$

The gradient of MS energy on using gravitational field equations (3.54) and (3.55) can be written as

$$\partial_\pm E = \pm 2\pi r^2\sqrt{WU}\Big\{\rho - \frac{3q^2}{8\pi r^4} + \frac{\lambda_2}{W}\Big[-2WU\rho'' - WU'\rho' - W'U\rho'\Big] + \frac{\lambda_2\rho}{r^2}\Big[2 - 2U - 2rU'\Big] + \frac{3\lambda_2 q^2}{16\pi r^6 W^2}\Big[2r^2WW''U + 4rWW'U + 4rW^2U' + 4W^2U - 4W^2 - r^2W'^2U + r^2WW'U'\Big]\Big\}. \tag{3.61}$$

Thus UFL, in non-minimal curvature-matter coupling, can be formulated from Eqs. (3.57), (3.60) and (3.61) as

$$\partial_\pm E = A\psi_\pm^{eff} + \omega^{eff}\partial_\pm V, \tag{3.62}$$

The results obtained in this section have the same form as obtained earlier in this chapter with the stress-energy tensor $T_{\mu\nu}$ being replaced by $T_{\mu\nu}^{(eff)}$.

3.5 Thermodynamics of charged wormholes in f(R,T) gravity

In this section we will consider CWHs in the context of $f(R,T)$ gravity and extend the formalism for finding GSG and UFL to $f(R,T)$ gravity. In $f(R,T)$ gravity, action can be written as [20]

$$S=\int\sqrt{-g}(\frac{1}{16\pi}f(R,T)+L^m+L^e)d^4x \tag{3.63}$$

where g is the determinant of metric tensor, $f(R,T)$ is the function of Ricci scalar R and trace of the stress-energy tensor T ($T=g^{\mu\nu}T_{\mu\nu}$), while L^m and L^e are the matter lagrangian and lagrangian due to charge, respectively. Varying this action with respect to the metric tensor $g^{\mu\nu}$, for the case $f(R,T)=R+2\lambda_1 T$ with constant λ_1, gives the following field equations

$$G_{\mu\nu}=8\pi T^{(EFF)}_{\mu\nu} \tag{3.64}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T^{(EFF)}_{\mu\nu}$ is the effective stress-energy tensor defined as

$$T^{(EFF)}_{\mu\nu}=T^{(m)}_{\mu\nu}+T^{(e)}_{\mu\nu}+\frac{\lambda_1}{4\pi}\left\{T^{(m)}_{\mu\nu}+\bar{P}g_{\mu\nu}+\frac{1}{2}g_{\mu\nu}(T^{(m)}+T^{(e)})\right\} \tag{3.65}$$

with $T^{(m)}=g^{\mu\nu}T^{(m)}_{\mu\nu}$ and $T^{(e)}=g^{\mu\nu}T^{(e)}_{\mu\nu}$, In the present case, the effective stress-energy tensor can be written as

$$T^{(EFF)}_{\mu\nu}=T^{(EFF)(m)}_{\mu\nu}+T^{(EFF)(e)}_{\mu\nu} \tag{3.66}$$

where

$$T^{(EFF)(m)}_{\mu\nu}=T^{(m)}_{\mu\nu}+(T^{(m)}_{\mu\nu}+\bar{P}g_{\mu\nu}+\frac{1}{2}T^{(m)}g_{\mu\nu}), \tag{3.67}$$

and

$$T^{(EFF)(e)}_{\mu\nu} = T^{(e)}_{\mu\nu} + \frac{\lambda_1}{8\pi} T^{(e)} g_{\mu\nu}, \tag{3.68}$$

are the effective stress-energy tensor due to matter which is threading the wormhole and effective electro-magnetic stress-energy tensor, respectively. The gravitational field equations of interest in this case become

$$\partial_\pm \Theta_\pm = -\frac{1}{2}\Theta_\pm^2 + \Theta_\pm \partial_\pm \log(-g_{+-}) - \frac{2\pi(r^2 e^{2\Phi} + q^2)}{r^2}\left\{\rho + p_r + \frac{\lambda_1}{4\pi}(\rho + p_r)\right\}, \tag{3.69}$$

$$\begin{aligned}\partial_\pm \Theta_\mp &= -\Theta_+\Theta_- + \frac{1}{r^2} g_{+-} - \frac{\lambda_1(r^2 e^{2\Phi} + q^2)}{r^2}\left\{-\rho + \frac{4p_r}{3} + \frac{5p_t}{3} + \frac{q^2}{2\pi r^4}\right\} \\ &+ \frac{2\pi(r^2 e^{2\Phi} + q^2)}{r^2}\left(\rho - p - \frac{3q^2}{4\pi r^4}\right).\end{aligned} \tag{3.70}$$

From the effective stress-energy tensor (3.65), we can construct a function ω^{eff} and a vector ψ^{eff} as

$$\omega^{eff} = -g_{+-}T^{+-eff} = \frac{\rho - p}{2} - \frac{3q^2}{8\pi r^4} - \frac{\lambda_1}{4\pi}\left\{-\rho + \frac{4p_r}{3} + \frac{5p_t}{3} + \frac{q^2}{2\pi r^4}\right\}, \tag{3.71}$$

and

$$\psi^{eff} = T^{++eff}\partial_+ r \partial_+ + T^{--eff}\partial_- r \partial_-. \tag{3.72}$$

Using Eqs. (3.69) and (3.70), the GSG (3.29) becomes

$$\kappa^{eff} = -2\pi r_h\,(\rho + p_r)\,(1 + \frac{\lambda_1}{4\pi}), \tag{3.73}$$

and

$$\kappa^{eff} = \frac{1}{2r_h} - 2\pi r_h(\rho - p_r - \frac{3q^2}{4\pi r^4}) + \lambda_1 r_h\left\{-\rho + \frac{4p_r}{3} + \frac{5p_t}{3} + \frac{q^2}{2\pi r^4}\right\}. \tag{3.74}$$

Eq. (3.72) in the component form can be written as

$$\psi_{\pm}^{eff} = \pm\frac{\sqrt{(r^2e^{2\Phi}+q^2)\,(r^2-rb(r)+q^2)}}{4r^2}\left\{\rho+p_r+\frac{\lambda_1}{4\pi}(\rho+p_r)\right\}. \tag{3.75}$$

Now, the gradient of MS energy can be written in the form of UFL on using field equations (3.69) and (3.70) as

$$\partial_{\pm}E = A\psi_{\pm}^{eff}+\omega^{eff}\partial_{\pm}V, \tag{3.76}$$

where

$$\begin{aligned}\partial_{\pm}E &= \pm 2\pi\sqrt{(r^2e^{2\Phi}+q^2)(r^2-rb(r)+q^2)}\\ &\times \left\{\rho-\frac{3q^2}{8\pi r^4}+\frac{\lambda_1}{4\pi}\left[\frac{3\rho}{2}-\frac{5p_r}{6}-\frac{5p_t}{3}-\frac{q^2}{2\pi r^4}\right]\right\}.\end{aligned} \tag{3.77}$$

3.6 Non-minimal curvature-matter coupling in $f(R,T)$ gravity

We will extend the results derived so far to the non-minimal curvature-matter coupling in $f(R,T)$ gravity for the case of CWHs. The relevant action is given by

$$S=\int\left[\frac{1}{16\pi}f_1(R,T)+\{1+\lambda_2 f_2(R)\}\,(L^m+L^e)\right]\sqrt{-g}d^4x, \tag{3.78}$$

we will consider the simple case by considering $f_1(R,T)=R+2\lambda_1T$ and $f_2(R)=R$, thus using these in above equation and varying with respect to $g^{\mu\nu}$ we get the following gravitational field equations

$$G_{\mu\nu}=8\pi T_{\mu\nu}^{eff}, \tag{3.79}$$

where

$$T^{eff}_{\mu\nu} = \Big(1+\lambda_2 R\Big)(T^m_{\mu\nu}+T^e_{\mu\nu}) + 2\lambda_2\left[\rho R_{\mu\nu} - \Big(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\Big)\rho\right] + \frac{\lambda_1}{4\pi}\left\{T^{(m)}_{\mu\nu} + \bar{P}g_{\mu\nu} + \frac{1}{2}g_{\mu\nu}(T^{(m)}+T^{(e)})\right\}, \quad (3.80)$$

is the effective stress-energy tensor of $f(R,T)$ gravity under coupling. We will use the notations defined in Eqs. (3.49) to (3.53). Thus for CWHs, in this case, the gravitational field equations of interest are

$$\partial_\pm\Theta_\pm = -\frac{1}{2}\Theta^2_\pm + \Theta_\pm\partial_\pm\log(-g_{+-}) - 4\pi W\left\{\frac{\rho+p_r}{2} + \frac{\lambda_1(\rho+p_r)}{8\pi} + \frac{\lambda_2 p_r L_4}{rW} + \frac{\lambda_2(\rho+p_r)}{4r^2W^2}\Big[-r^2L_1+L_3\Big] - \frac{\lambda_2 L_2}{2W}\right\}, \quad (3.81)$$

$$\partial_\pm\Theta_\mp = -\Theta_+\Theta_- + \frac{1}{r^2}g_{+-} + 4\pi W\left\{\frac{\rho-p_r}{2} - \frac{3q^2}{8\pi r^4} + \frac{\lambda_2(\rho-p_r)L_6}{r^2} + \frac{\lambda_2 p_r L_5}{rW} - \frac{\lambda_2 L_2}{2W} + \frac{\lambda_2(\rho+p_r)L_1}{4W^2} + \frac{3\lambda_2 q^2}{16\pi r^6 W^2}\Big[r^2L_1 - L_3 - 4rWL_4\Big] - \frac{\lambda_1 L_7}{4\pi}\right\}. \quad (3.82)$$

Here we have used

$$L_1 = 2UWW'' - UW'^2 + WW'U', \quad (3.83)$$

$$L_2 = 2WU\rho'' + WU'\rho' + \rho'W'U, \quad (3.84)$$

$$L_3 = 4W^2(1-U-2rU'), \quad (3.85)$$

$$L_4 = WU' - W'U, \quad (3.86)$$

$$L_5 = WU' + W'U, \quad (3.87)$$

$$L_6 = 1-U, \quad (3.88)$$

$$L_7 = -\rho + \frac{4p_r}{3} + \frac{5p_t}{3} + \frac{q^2}{2\pi r^4}. \quad (3.89)$$

From the effective stress-energy tensor (3.80), we can construct a vector ψ^{eff} and a function ω^{eff} as

$$\psi^{eff} = T^{++eff}\partial_{+}r\partial_{+} + T^{--eff}\partial_{-}r\partial_{-}, \tag{3.90}$$

and

$$\begin{aligned}\omega^{eff} &= -g_{+-}T^{+-(eff)} = \frac{\rho - p_r}{2} - \frac{3q^2}{8\pi r^4} - \frac{\lambda_1 L_7}{4\pi} + \frac{\lambda_2(\rho - p_r)L_6}{r^2} + \frac{\lambda_2(\rho + p_r)L_1}{4W^2} \\ &+ \frac{3\lambda_2 q^2}{16\pi r^6 W^2}\Big[r^2 L_1 - L_3 - 4rWL_4\Big] + \frac{\lambda_2 p_r L_5}{rW} - \frac{\lambda_2 L_2}{2W}.\end{aligned} \tag{3.91}$$

Eq. (3.29), on using Eqs. (3.81) and (3.82) can also be written as

$$\begin{aligned}\kappa^{eff} &= -2\pi r_h\Big\{(\rho + p_r) + \frac{\lambda_1(\rho + p_r)}{4\pi} + \frac{2\lambda_2(\rho + p_r)}{4r_h^2 W}\Big[4W - r_h^2 W'U' - 8r_h WU'\Big] \\ &+ \frac{2\lambda_2 p_r U'}{r_h} - \lambda_2\rho' U'\Big\},\end{aligned} \tag{3.92}$$

and

$$\begin{aligned}\kappa^{eff} &= \frac{1}{2r_h} - 2\pi r_h(\rho - p_r) + \frac{3q^2}{2r_h^3} + \lambda_1 r\Big[-\rho + \frac{4p_r}{3} + \frac{5p_t}{3} + \frac{q^2}{2\pi r^4}\Big] - 4\pi\lambda_2 p_r U' \\ &+ 2\pi\lambda_2 r_h \rho' U' - \frac{\pi\lambda_2 r_h(\rho + p_r)W'U'}{W} - \frac{4\pi\lambda_2(\rho - p_r)}{r_h} \\ &- \frac{3\lambda_2 q^2}{4Wr_h^5}\Big[r_h^2 W'U' + 4r_h WU' - 4W\Big].\end{aligned} \tag{3.93}$$

Eq. (3.90) in the component form becomes

$$\psi_{\pm} = \pm\frac{\sqrt{UW}}{2}\Big\{\frac{\rho + p_r}{2} + \frac{\lambda_1}{8\pi}(\rho + p_r) - \frac{\lambda_2 L_2}{2W} + \frac{\lambda_2 p_r L_4}{rW} + \frac{\lambda_2(\rho + p_r)}{4r^2W^2}\Big[-r^2L_1 + L_3\Big]\Big\}. \tag{3.94}$$

The gradient of MS energy on using the gravitational field equations (3.81) and (3.82) can be written as

$$\begin{aligned}\partial_{\pm}E &= \pm 2\pi r^2\sqrt{UW}\Bigg\{\rho-\frac{3q^2}{8\pi r^4}+\frac{\lambda_1}{8\pi}\Big[\rho+p_r-2L_7\Big]+\frac{\lambda_2\rho}{4r^2W^2}\Big[L_3+4W^2L_6\Big]\\&- \frac{\lambda_2L_2}{W}+\frac{3\lambda_2q^2}{16\pi r^6W^2}\Big[r^2L_1-L_3-4rWL_4\Big]\Bigg\}. \qquad (3.95)\end{aligned}$$

Thus UFL in non-minimal curvature-matter coupling can be formulated from Eqs. (3.91), (3.94) and (3.95) as

$$\partial_{\pm}E = A\psi_{\pm}^{eff}+\omega^{eff}\partial_{\pm}V, \qquad (3.96)$$

Chapter 4

Thermodynamics of dynamical wormholes

In GR, in static traversable wormholes, the violation of NEC is the key ingredient. However, it was shown that the NEC and WEC get some respect in some regions and for certain time durations at the throat, in the case of time dependent (dynamical) wormhole solutions [65, 66]. These wormholes are the time generalization of MTWHs. We will consider dynamical wormholes (DWHs) in this chapter with and without charge and study their thermodynamics. The UFL and GSG will be investigated in Einstein's gravity. We also work out GSG for specific uncharged DWHs for different cosmological models. Thermodynamic stability is also investigated. We will also use the areal radius coordinates to investigate these thermodynamic workouts.

4.1 Uncharged dynamical wormholes

We consider uncharged DWHs in a cosmological background, which are generalization of the MTWHs to a time dependent background [121],

$$ds^2 = -e^{2\Phi(t,r)}dt^2 + a^2(t)\left[\frac{dr^2}{1-\frac{b(r)}{r}} + r^2 d\Omega^2\right], \tag{4.1}$$

in coordinates (t, r, θ, ϕ) where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The radial coordinate r ranges in $[r_0, \infty]$. Here the minimum radius $r = r_0$ corresponds to the throat of the wormhole which connects two regions, each region is $r_0 < r < \infty$. At $r \to \infty$ this metric becomes flat, $a(t)$ is the dimensionless parameter called the scaling factor of the universe. It tells us how our universe is expanding. It is known that the expansion rate of our universe is increasing with time which implies $\ddot{a}(t) > 0$ or $\dot{a}(t)$ is an increasing function of time (here over dot represents the time derivative). $\Phi(t, r)$ is the redshift function as it corresponds to the gravitational redshift. This function should be finite everywhere in order to prevent the existence of an event horizon which is the necessary requirement for a wormhole to be traversable and when $r \to \infty$ this redshift function should vanish. At the wormhole throat a coordinate singularity $b(r_0) = r_0$ occurs and $b(r) < r$ for $r > r_0$.

The flaring out condition for wormholes requires that $b' < b(r)/r$ at or near the throat. These are the conditions on $\Phi(t, r)$ and $b(r)$ which provide a traversable wormhole solution. It is clear that when $\Phi(t, r)$ and $b(r)/r$ tend to zero then the metric (4.1) becomes the flat Friedmann-Robertson-Walker (FRW) metric, and Morris-Thorne metric is recovered when $\Phi(t, r) = \Phi(r)$ and $a(t) \to 1$. Here in this paper we take $\Phi(t, r) = 0$ so that the wormhole metric (4.1) takes the form

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega^2\right]. \tag{4.2}$$

Now for the stress-energy tensor we take the perfect fluid which is completely described by its energy density and isotropic pressure [121], with components

$$T_t^t = -\rho(t, r),\ T_r^r = p_r(t, r),\ T_\theta^\theta = T_\phi^\phi = p_t(t, r), \tag{4.3}$$

where $\rho(t, r)$, $p_r(t, r)$ and $p_t(t, r)$ are, respectively, the energy density, radial pressure and tangential pressure. For isotropic pressure $p_r(t, r) = p_t(t, r)$, otherwise the pressure will be anisotropic.

Using null coordinates (x^+, x^-), the metric (4.2) can be transformed into the form

$$ds^2 = 2g_{+-}dx^+dx^- + R^2 d\Omega^2, \tag{4.4}$$

where

$$dx^+ = \frac{dt}{a} + \frac{dr}{\sqrt{1-\frac{b}{r}}}, \tag{4.5}$$

and

$$dx^- = \frac{dt}{a} - \frac{dr}{\sqrt{1-\frac{b}{r}}}, \tag{4.6}$$

here x^+ corresponds to the outgoing radiation and x^- to the ingoing radiation. Here R and $g_{+-} = -a^2/2$ are functions of the null coordinates (x^+, x^-), that correspond to the two preferred null normal directions for the symmetric spheres $\partial_\pm = \partial/\partial x^\pm$, and $R = a(t)r$ is the so-called areal radius and $d\Omega^2$ is the metric for the unit 2-sphere. Now, we define the expansions as

$$\Theta_\pm = \frac{2}{R}\partial_\pm R = aH \pm \frac{a}{R}\sqrt{1-\frac{ab}{R}}. \tag{4.7}$$

A sphere is trapped if $\Theta_+\Theta_- > 0$, which yields

$$H^2R^2 - 1 + \frac{ab}{R} > 0, \tag{4.8}$$

untrapped if $\Theta_+\Theta_- < 0$, yielding

$$H^2R^2 - 1 + \frac{ab}{R} < 0, \tag{4.9}$$

or marginal if $\Theta_+\Theta_- = 0$, giving

$$H^2R^2 - 1 + \frac{ab}{R} = 0 \tag{4.10}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. Here, for the trapping horizon, we choose

$$\Theta_- \cong 0, \tag{4.11}$$

where the symbol ($\cong$) henceforth shows evaluation on the trapping horizon $R_h = a(t)r_h$ which gives the expression for the trapping horizon as

$$HR - \sqrt{1 - \frac{ab}{R}} \cong 0. \tag{4.12}$$

Note that the choice $\Theta_- \cong 0$ corresponds to expanding universe ($\dot{a} > 0$), on the other hand if we chose $\Theta_+ \cong 0$, then it will lead us to the contracting universe ($\dot{a} < 0$). Also note that unlike the static MTWHs, the trapping horizon and the throat of uncharged DWHs do not coincide. In the case of static MTWHs the trapping horizon is given by $b(r_0) = r_0$ which is also the value of the shape function at the throat. But in this case, because of the presence of the scaling factor $a(t)$, they do not coincide. This trapping horizon is future if $\Theta_+ < 0$ (or equivalently $\partial_+ R < 0$), giving

$$HR + \sqrt{1 - \frac{ab}{R}} < 0, \tag{4.13}$$

past if $\Theta_+ > 0$ (or equivalently $\partial_+ R > 0$), giving

$$HR + \sqrt{1 - \frac{ab}{R}} > 0, \tag{4.14}$$

and bifurcating if $\Theta_+ \cong 0$ (or equivalently $\partial_+ R \cong 0$), giving

$$HR + \sqrt{1 - \frac{ab}{R}} \cong 0. \tag{4.15}$$

Note that since we have made choice $\Theta_- \cong 0$ which corresponds to expanding universe ($\dot{a} > 0$), this makes the trapping horizon to be past as can be seen from Eq. (4.14), while Eqs. (4.13) and (4.15) are not satisfied for $\dot{a} > 0$. In our case on the trapping horizon $\Theta_+ > 0$ and $\Theta_- \cong 0$. Thus, in case of expanding universe, it is the ingoing expansion which changes sign across the trapping horizon and vanishes on it while the outgoing expansion keeps the sign same. Therefore, inside the trapping horizon we have $\Theta_\pm > 0$ and outside the trapping horizon we have $\Theta_+ > 0$ but $\Theta_- < 0$. This implies that inside the trapping horizon $HR > \sqrt{1 - ab/R}$, on the trapping horizon $HR \cong \sqrt{1 - ab/R}$ and outside the trapping horizon $HR < \sqrt{1 - ab/R}$. Further, this trapping horizon is outer if $\partial_+\Theta_- < 0$, giving

$$\frac{\dot{H}}{2} + H^2 - \frac{(ab - Rb')}{4R^3} < 0, \tag{4.16}$$

inner if $\partial_+\Theta_- > 0$, giving

$$\frac{\dot{H}}{2} + H^2 - \frac{(ab - Rb')}{4R^3} > 0, \tag{4.17}$$

or degenerate if $\partial_+\Theta_- \cong 0$, giving

$$\frac{\dot{H}}{2} + H^2 - \frac{(ab - Rb')}{4R^3} \cong 0. \tag{4.18}$$

4.1.1 Generalized surface gravity of uncharged dynamical wormholes

The MS energy for uncharged DWHs (4.2) can be expressed as

$$E = \frac{1}{2}R(1 - \partial^a R \partial_a R) = \frac{R}{2}(1 - 2g^{+-}\partial_+ R \partial_- R), \tag{4.19}$$

which gives

$$E = \frac{R}{2}\left[H^2R^2 + \frac{ab}{R}\right]. \tag{4.20}$$

On a trapping horizon this expression reads $E \cong R/2$.

Now the Einstein field equations (2.8)-(2.10) take the form

$$\partial_\pm \Theta_\pm = -\frac{1}{2}\Theta_\pm^2 + \Theta_\pm \partial_\pm \log(-g_{+-}) - 8\pi T_{\pm\pm}, \tag{4.21}$$

$$\partial_\pm \Theta_\mp = -\Theta_+ \Theta_- + \frac{1}{R^2} g_{+-} + 8\pi T_{\pm\mp}, \tag{4.22}$$

$$\Theta_+ \Theta_- = -\partial_+ \Theta_- - \partial_- \Theta_+ - \frac{8\pi}{R^2} T_{\theta\theta}, \tag{4.23}$$

where the components of stress-energy tensor, on solving Eq. (2.24), in this case take the form

$$T_{++} = T_{--} = \frac{a^2(\rho + p_r)}{4}, \tag{4.24}$$

$$T_{+-} = T_{-+} = \frac{a^2(\rho - p_r)}{4}, \tag{4.25}$$

$$T_{\theta\theta} = R^2 p_t. \tag{4.26}$$

The Kodama vector in null coordinates is given by

$$K = -g^{+-}(\partial_+ R \partial_- - \partial_- R \partial_+), \tag{4.27}$$

which for spacetime (4.2) in covariant form becomes

$$K_\pm = -\frac{a}{2}\left(\pm HR + \sqrt{1 - \frac{ab}{R}}\right). \tag{4.28}$$

The magnitude of **K** is

$$|K|^2 = \frac{2E}{R} - 1. \tag{4.29}$$

Note that $|K|^2 \cong 0$ on the trapping horizon $\Theta_- \cong 0$.

The trapping horizon is provided by this Kodama vector which is null on a hypersurface $\partial_- R \cong 0$. Now, the GSG κ on a trapping horizon can be expressed as

$$K^a \nabla_{[b} K_{a]} \cong \pm \kappa K_b. \tag{4.30}$$

For metric (4.2) the surface gravity on trapping horizon becomes

$$\kappa \cong -\frac{\dot{H}R}{2} - H^2R + \frac{1}{4R^2}\left(ab - b'R\right), \tag{4.31}$$

which on using Einstein field equations (4.21) and (4.22) can be written as

$$\kappa \cong -\dot{H}R - H^2R - 2\pi R\left(\rho + p_r\right), \tag{4.32}$$

and

$$\kappa \cong \frac{E}{R^2} - 4\pi R\omega = \frac{1}{2R} - 2\pi R(\rho - p_r). \tag{4.33}$$

This surface gravity, from Eq. (4.30), equivalently, can also be expressed as

$$\kappa \cong \frac{1}{2}g^{ab}\partial_a\partial_b R, \tag{4.34}$$

on a trapping horizon. It follows that $\kappa < 0$, $\kappa = 0$ and $\kappa > 0$ for inner, degenerate and outer trapping horizons, respectively.

The Hawking temperature is $T \cong -\kappa/2\pi$ which, in our case from Eq. (4.31), becomes

$$T \cong -\frac{\kappa}{2\pi} = -\frac{1}{2\pi}\left[-\frac{\dot{H}R}{2} - H^2R + \frac{1}{4R^2}\left(ab - b'R\right)\right], \tag{4.35}$$

which is negative for the outer trapping horizon since $\kappa > 0$.

4.1.2 Unified first law for uncharged dynamical wormholes

We can formulate the UFL for uncharged DWHs, in the same manner as for MTWHs, due to their spherical symmetric nature. Using the stress-energy tensor of the background fluid we construct a function, ω, and a vector, ψ, in the local coordinates as

$$\omega = -g_{+-}T^{+-} = \frac{\rho - p_r}{2}, \tag{4.36}$$

and

$$\psi = T^{++}\partial_+ R\partial_+ + T^{--}\partial_- R\partial_-. \tag{4.37}$$

In components form it can be written as

$$\psi_\pm = \left(\frac{\rho + p_r}{4}\right)\left(-aHR \pm a\sqrt{1 - \frac{ab}{R}}\right). \tag{4.38}$$

Now the UFL can be written by taking gradient of the gravitational energy and using Einstein field equations as

$$\partial_\pm E = A\psi_\pm + \omega\partial_\pm V, \tag{4.39}$$

with

$$\partial_\pm E = 2\pi a R^2 \left(\pm\rho\sqrt{1 - \frac{ab}{R}} - HRp_r\right), \tag{4.40}$$

where $A = 4\pi R^2$ and $V = 4\pi R^3/3$ are the area and areal volume of the spheres of symmetry and the corresponding flat space, respectively.

The variation of gravitational energy is always positive in the outgoing direction because $\rho > 0$ and $p_r < 0$, however, in the ingoing direction, it is positive inside the trapping horizon while outside the trapping horizon its sign depends on how much exotic matter is present there, for large amount of exotic matter it should be positive. The work term is also positive in the outgoing direction, as energy density ω is positive besides the fact that energy conditions are not satisfied. In the ingoing direction this is positive inside the trapping horizon and negative outside the trapping horizon. The sign of energy supply term depends on the sign of $\rho + p_r$ (in case of black holes this term corresponds to the fluid which provides energy to the spacetime and respects NEC, hence positive, while it is negative in the case of wormholes where fluid removes energy from the spacetime and violates NEC). Thus in the outgoing direction it is positive inside the trapping horizon and negative outside the trapping horizon. However, in the ingoing direction this term is always positive in our case ($\rho + p_r < 0$).

On the trapping horizon ($\partial_- R = 0$), in the outgoing direction, energy flux vanishes while both variation of gravitational energy and work term are positive. Thus change in gravitational energy equals the work done in the wormhole on the trapping horizon. In the ingoing direction, on the trapping horizon, work term vanishes while both variation of energy and energy flux are positive. Thus change in gravitational energy equals the energy supply and no work is done on the trapping horizon.

At the throat, all the terms enetering in UFL, the variation of gravitational energy, energy supply term and the work term, are always positive both in the outgoing as well as ingoing direction.

4.1.3 Thermodynamic stability of uncharged dynamical wormholes

In this section we study the thermodynamic stability of wormholes under consideration using the variables E, T, S, P and V. We follow the usual criterion for thermodynamic stability, that is $\frac{\partial \bar{P}}{\partial V} \mid_T \leq 0$ and $C_P \geq C_V \geq 0$, where $\bar{P} = (P_r + 2P_t)/3$ is the average pressure and C_P and C_V are specific heats at constant pressure and volume, respectively.

We subtract Eq. (4.32) from (4.33) and rearrange the terms to obtain

$$p_r = -\frac{1}{8\pi R^2} - \frac{\dot{H} + H^2}{4\pi}. \tag{4.41}$$

Solving Eq. (4.23) on the trapping horizon and using the definition of GSG and Hawking temperature, yields

$$2p_t = -\frac{a^2 T}{R}. \tag{4.42}$$

From Eqs. (4.41) and (4.42) we obtain the average pressure $\bar{P}$ as

$$\bar{P} = \frac{p_r + 2p_t}{3} = -\frac{1}{24\pi R^2} - \frac{\dot{H} + H^2}{12\pi} - \frac{a^2 T}{3R}, \tag{4.43}$$

which is the equation of state in three state parameters $T, \bar{P}$ and V. From this equation we can analyze the thermodynamic stability of wormhole.

Stable equilibrium of a thermodynamic system requires that $\frac{\partial \bar{P}}{\partial V} |_T \leq 0$ where

$$\frac{\partial \bar{P}}{\partial V} |_T = \frac{(4\pi/3)^{2/3}}{36\pi V^{5/3}} + \frac{(4\pi/3)^{1/3} a^2 T}{9V^{4/3}}. \tag{4.44}$$

Now, to ensure the stable equilibrium we must have

$$T \leq -\frac{1}{4\pi a^2 R}, \tag{4.45}$$

thus the temperature assumes negative values everywhere for stable equilibrium which is attributed to the exotic matter. From Eq. (4.43) we have

$$\bar{P} \geq \frac{1}{24\pi R^2} - \frac{\dot{H} + H^2}{12\pi}. \tag{4.46}$$

If the scale factor is a linear function of time then $\ddot{a} = 0$ and then $\bar{P}$ will assume the positive values everywhere, otherwise it could be negative somewhere.

Another condition for stable equilibrium is $C_P \geq C_V \geq 0$. Now since, the constant V means constant E and S so by the definition of C_V,

$$C_V = \frac{\partial E}{\partial T} |_V = T\frac{\partial S}{\partial T} |_V = 0, \tag{4.47}$$

which means we can define heat capacity only at constant pressure as

$$C_P = T\frac{\partial S}{\partial T} |_P = \frac{(24\pi \bar{P} R^2 + 2\dot{H} R^2 + 2H^2 R^2 + 1)2\pi R^2}{24\pi \bar{P} R^2 + 2\dot{H} R^2 + 2H^2 R^2 - 1}, \tag{4.48}$$

where from Eq. (4.43),

$$T = -\frac{1}{a^2}\left(3R\bar{P} + \frac{1}{8\pi R} + \frac{\dot{H}R + H^2 R}{4\pi}\right). \tag{4.49}$$

Now from Eq. (4.46), to ensure the stable equilibrium, we can take the value of $\bar{P}$, for any non-negative ϵ, as

$$\bar{P} = \frac{1}{24\pi R^2} - \frac{\dot{H} + H^2}{12\pi} + \epsilon. \tag{4.50}$$

Thus Eq. (4.48) on using Eq. (4.50) takes the form

$$C_P = \frac{1}{6\epsilon} + 2\pi R^2. \tag{4.51}$$

which is always positive. Thus the uncharged DWHs could be thermodynamically stable. This means that for stable equilibrium the average pressure is always positive for linear scale factor, however it may also have negative values for non-linear scale factor while temperature is always negative as is also depicted in Ref. [71] in which the possibility of negative temperature emerging from the exotic matter distribution was proposed.

4.1.4 Generalized surface gravity for wormholes with and without the cosmological constant

In this section we consider wormholes of different shapes in different cosmologies with and without the cosmological constant Λ. We will analyze these for anisotropic fluid where radial and tangential pressures satisfy $p_r = \omega_r \rho$ and $p_t = \omega_t \rho$. Clearly for $\omega_r = \omega_t$ pressure becomes isotropic.

Static wormholes

Here we discuss static wormholes for cosmological constant ($\Lambda = 0$). In the static case ($a(t) = 1$) we take shape function $b(r) = r_0(\frac{r}{r_0})^{-1/\omega_r}$. Here ω_r is a constant state parameter, satisfying $p_r = \omega_r \rho$ and $p_t = -\frac{1}{2}(1 + \omega_r)\rho$, where p_r and p_t are radial and tangential pressures while ρ is the energy density. For this case the metric (4.2) takes the form [158]

$$ds^2 = -dt^2 + \frac{dr^2}{1 - (r/r_0)^{-(1+\omega_r)/\omega_r}} + r^2 d\Omega^2. \tag{4.52}$$

In the range $\omega_r < -1$, we have asymptotically flat wormhole metric with positive energy density while for $\omega_r > 0$ the energy density becomes negative but still we have an asymptotically flat wormhole. This static traversable wormhole was first considered in Ref. [43]. In the static case we have a bifurcating trapping horizon on the wormhole throat location, $r_h = r_0$. The Kodama vector in this case becomes

$$K_{\pm} = -\frac{1}{2}\sqrt{1 - (r/r_0)^{-(1+\omega_r)/\omega_r}}. \tag{4.53}$$

Finally, the surface gravity from Eq. (4.30) when evaluated on the trapping horizon $r = r_h = r_0$ takes the form

$$\kappa \cong \frac{1+\omega_r}{4r_0\omega_r}. \tag{4.54}$$

Evolving wormholes with $\Lambda = 0$

We discuss a non-static wormhole with shape function

$$b(r) = r_0(\frac{r}{r_0})^{-1/\omega_r} + kr_0^3(\frac{r}{r_0})^3 - kr_0^3(\frac{r}{r_0})^{-1/\omega_r}, \tag{4.55}$$

in the background of a cosmology with the scale factor $a(t) = t\sqrt{-k} + F$, where k and F are constants and ω_r satisfies the same conditions as discussed above for the static case. This shape function also satisfies the near throat conditions discussed earlier. With these values the wormhole metric can be written as [158]

$$ds^2 = -dt^2 + (\sqrt{-k}t + F)^2 \left(\frac{dr^2}{1 - (r/r_0)^{-(1+\omega_r)/\omega_r} - kr_0^2(\frac{r}{r_0})^2 + kr_0^2(\frac{r}{r_0})^{-(1+\omega_r)/\omega_r}} + r^2 d\Omega^2 \right). \tag{4.56}$$

Here $k = -1, 0, +1$ correspond to open, flat and closed universe, respectively. In the above case must have $k \leq 0$ for preserving the Lorentzian signatures. Otherwise, for $k > 0$ the signatures changes to the Euclidean one giving rise to Euclidean wormholes. The trapping horizon for this metric is given by the expression

$$\sqrt{-k} - \sqrt{1 - (r_h/r_0)^{-(1+\omega_r)/\omega_r} - kr_0^2(\frac{r_h}{r_0})^2 + kr_0^2(\frac{r_h}{r_0})^{-(1+\omega_r)/\omega_r}} = 0, \tag{4.57}$$

whereas the Kodama vector in the component form becomes

$$K_\pm = -\frac{t\sqrt{-k} + F}{2}\left(\pm\sqrt{-k}r + \sqrt{1 - (r/r_0)^{-(1+\omega_r)/\omega_r} - kr_0^2(\frac{r}{r_0})^2 + kr_0^2(\frac{r}{r_0})^{-(1+\omega_r)/\omega_r}}\right). \tag{4.58}$$

Finally, the surface gravity from Eq. (4.30) on trapping horizon takes the form

$$\kappa \cong \frac{kr}{2(t\sqrt{-k} + F)} + \frac{1}{4r^2(\sqrt{-k}t + F)}\left[\frac{(1+\omega_r)r_0(1 - kr_0^2)}{\omega_r}(\frac{r}{r_0})^{-1/\omega_r} - 2kr_0^3(\frac{r}{r_0})^3\right]. \tag{4.59}$$

Inflating de Sitter wormholes

When we include the cosmological constant, the wormholes do not remain asymptotically flat and the expansion of the wormhole is accelerated. Here we discuss a case of exponential scale factor $a(t) = a_0 e^{\pm\sqrt{\Lambda/3}t}$ for $\Lambda > 0$. For this scale factor we take the shape function $b(r) = r_0(\frac{r}{r_0})^{-1/\omega_r}$, so that the wormhole metric takes the form

$$ds^2 = -dt^2 + a_0^2 e^{\pm 2\sqrt{\Lambda/3}t}\left[\frac{dr^2}{1 - (r/r_0)^{-(1+\omega_r)/\omega_r}} + r^2 d\Omega^2\right], \tag{4.60}$$

describing contracting and expanding wormholes. The positive sign in this scale factor represents inflation giving exponential expansion of an inflating wormhole. These wormholes were first considered in Ref. [64]. This wormhole is asymptotically de Sitter for $\omega_r < -1$ with positive energy density everywhere, while for $\omega_r > 0$ the energy density is negative everywhere and the wormhole solution is still asymptotically de Sitter universe. When Λ vanishes we obtain the static case discussed earlier. For these wormholes the trapping horizon is given by the expression

$$\pm a_0\sqrt{\Lambda/3}e^{\pm\sqrt{\Lambda/3}t}r_h - \sqrt{1-(r_h/r_0)^{-(1+\omega_r)/\omega_r}} = 0, \tag{4.61}$$

whereas the Kodama vector in the component form is given by

$$K_{\pm} = -\frac{a_0 e^{\pm\sqrt{\Lambda/3}t}}{2}\left(\pm a_0(\pm\sqrt{\Lambda/3})e^{\pm\sqrt{\Lambda/3}t}r + \sqrt{1-(r/r_0)^{-(1+\omega_r)/\omega_r}}\right), \tag{4.62}$$

yielding the surface gravity

$$\kappa \cong -\frac{a_0 r\Lambda e^{\pm\sqrt{\Lambda/3}t}}{3} + \frac{r_0(1+\omega_r)}{4a_0\omega_r r^2 e^{\pm\sqrt{\Lambda/3}t}}(\frac{r}{r_0})^{-1/\omega_r}. \tag{4.63}$$

Evolving de Sitter wormholes in closed universe

Now we discuss the more general case when $\Lambda \neq 0$, and the shape function is given by Eq. (4.55). As the cosmological constant is nonzero, the wormhole is not asymptotically flat. For different values of constant k we can have different kinds of scale factors discussed in detail in Ref. [121]. For $k = 1$ and $\Lambda > 0$, we take the scale factor given by $a(t) = \sqrt{\frac{3}{\Lambda}}\cosh(\sqrt{\frac{\Lambda}{3}}t + \phi_0)$ where ϕ_0 is a constant. With these values the de Sitter wormhole of a closed universe becomes

$$\begin{aligned} ds^2 &= -dt^2 \\ &+ \frac{3}{\Lambda}\cosh^2(\sqrt{\frac{\Lambda}{3}}t+\phi_0)\left(\frac{dr^2}{1-(r/r_0)^{-(1+\omega_r)/\omega_r} - r_0^2(\frac{r}{r_0})^2 + r_0^2(\frac{r}{r_0})^{-(1+\omega_r)/\omega_r}} + r^2 d\Omega^2\right). \end{aligned} \tag{4.64}$$

The trapping horizon for this wormhole is given by the expression

$$\sinh(\sqrt{\frac{\Lambda}{3}}t+\phi_0)r_h-\sqrt{1-(r_h/r_0)^{-(1+\omega_r)/\omega_r}-r_0^2(\frac{r_h}{r_0})^2+r_0^2(\frac{r_h}{r_0})^{-(1+\omega_r)/\omega_r}}=0, \quad (4.65)$$

and the Kodama vector in the component form is given by

$$\begin{aligned} K_{\pm} &= -\frac{\sqrt{\frac{3}{\Lambda}}\cosh(\sqrt{\frac{\Lambda}{3}}t+\phi_0)}{2}\Bigg(\pm\sinh(\sqrt{\frac{\Lambda}{3}}t+\phi_0)r \\ &+ \sqrt{1-(r/r_0)^{-(1+\omega_r)/\omega_r}-r_0^2(\frac{r}{r_0})^2+r_0^2(\frac{r}{r_0})^{-(1+\omega_r)/\omega_r}}\Bigg). \end{aligned} \quad (4.66)$$

Evaluating Eq. (4.30) on the trapping horizon gives for the surface gravity

$$\begin{aligned} \kappa &\cong -\frac{\sqrt{\Lambda}r}{2\sqrt{3}}\cosh(\sqrt{\frac{\Lambda}{3}}t+\phi_0)-\frac{\sqrt{\Lambda}r\tanh(\sqrt{\frac{\Lambda}{3}}t+\phi_0)\sinh(\sqrt{\frac{\Lambda}{3}}t+\phi_0)}{2\sqrt{3}} \\ &- \frac{\sqrt{\Lambda}}{4r^2\sqrt{3}\cosh(\sqrt{\frac{\Lambda}{3}}t+\phi_0)}\left[\frac{(1+\omega_r)r_0(1-r_0^2)}{\omega_r}(\frac{r}{r_0})^{-1/\omega_r}-2r_0^3(\frac{r}{r_0})^3\right]. \end{aligned} \quad (4.67)$$

Evolving de Sitter wormholes in open universe

If in Eq. (4.55) we take $k=-1$ then for $\Lambda>0$ the scale factor is given by $a(t)=\sqrt{\frac{3}{\Lambda}}\sinh(\sqrt{\frac{\Lambda}{3}}t+\phi_0)$ and the wormhole metric takes the form

$$\begin{aligned} ds^2 &= -dt^2 \\ &+ \frac{3}{\Lambda}\sinh^2(\sqrt{\frac{\Lambda}{3}}t+\phi_0)\left(\frac{dr^2}{1-(r/r_0)^{-(1+\omega_r)/\omega_r}+r_0^2(\frac{r}{r_0})^2-r_0^2(\frac{r}{r_0})^{-(1+\omega_r)/\omega_r}}+r^2d\Omega^2\right). \end{aligned} \quad (4.68)$$

In this case the expression for the trapping horizon is

$$\cosh(\sqrt{\frac{\Lambda}{3}}t+\phi_0)r_h-\sqrt{1-(r_h/r_0)^{-(1+\omega_r)/\omega_r}+r_0^2(\frac{r_h}{r_0})^2-r_0^2(\frac{r_h}{r_0})^{-(1+\omega_r)/\omega_r}}=0 \quad (4.69)$$

and the Kodama vector takes the form

$$K_{\pm} = -\frac{\sqrt{\frac{3}{\Lambda}}\sinh(\sqrt{\frac{\Lambda}{3}}t+\phi_0)}{2}\Bigg(\pm\cosh(\sqrt{\frac{\Lambda}{3}}t+\phi_0)r + \sqrt{1-(r/r_0)^{-(1+\omega_r)/\omega_r}+r_0^2(\frac{r}{r_0})^2-r_0^2(\frac{r}{r_0})^{-(1+\omega_r)/\omega_r}}\Bigg). \quad (4.70)$$

Thus surface gravity on trapping horizon becomes

$$\kappa \cong -\frac{\sqrt{\Lambda}r}{2\sqrt{3}}\sinh(\sqrt{\frac{\Lambda}{3}}t+\phi_0)-\frac{\sqrt{\Lambda}r\coth(\sqrt{\frac{\Lambda}{3}}t+\phi_0)\cosh(\sqrt{\frac{\Lambda}{3}}t+\phi_0)}{2\sqrt{3}} - \frac{\sqrt{\Lambda}}{4r^2\sqrt{3}\sinh(\sqrt{\frac{\Lambda}{3}}t+\phi_0)}\left[\frac{(1+\omega_r)r_0(1+r_0^2)}{\omega_r}(\frac{r}{r_0})^{-1/\omega_r}+2r_0^3(\frac{r}{r_0})^3\right]. \quad (4.71)$$

Evolving anti-de Sitter wormholes in open universe

Finally we discuss a case of negative cosmological constant ($\Lambda<0$) with $k=-1$ in Eq. (4.55). We take the scale factor as $a(t)=\sqrt{\frac{-3}{\Lambda}}sin(\sqrt{\frac{-\Lambda}{3}}t+\phi_0)$, so that the wormhole metric can be written as

$$ds^2 = -dt^2+\frac{-3}{\Lambda}\sin^2(\sqrt{\frac{-\Lambda}{3}}t+\phi_0) \times \left(\frac{dr^2}{1-(r/r_0)^{-(1+\omega_r)/\omega_r}+r_0^2(\frac{r}{r_0})^2-r_0^2(\frac{r}{r_0})^{-(1+\omega_r)/\omega_r}}+r^2d\Omega^2\right). \quad (4.72)$$

Its trapping horizon is given by the expression

$$\cos(\sqrt{\frac{-\Lambda}{3}}t+\phi_0)r_h-\sqrt{1-(r_h/r_0)^{-(1+\omega_r)/\omega_r}+r_0^2(\frac{r_h}{r_0})^2-r_0^2(\frac{r_h}{r_0})^{-(1+\omega_r)/\omega_r}}=0, \quad (4.73)$$

and the Kodama vector takes the form

$$K_{\pm} = -\frac{\sqrt{\frac{-3}{\Lambda}}sin(\sqrt{\frac{-\Lambda}{3}}t+\phi_0)}{2}\left(\pm\cos(\sqrt{\frac{-\Lambda}{3}}t+\phi_0)r + \sqrt{1-(r/r_0)^{-(1+\omega_r)/\omega_r}+r_0^2(\frac{r}{r_0})^2-r_0^2(\frac{r}{r_0})^{-(1+\omega_r)/\omega_r}}\right). \quad (4.74)$$

Using all these expressions the surface gravity becomes

$$\kappa \cong \frac{\sqrt{-\Lambda}r}{2\sqrt{3}}\sin(\sqrt{\frac{-\Lambda}{3}}t+\phi_0) - \frac{\sqrt{-\Lambda}r\cot(\sqrt{\frac{-\Lambda}{3}}t+\phi_0)\cos(\sqrt{\frac{-\Lambda}{3}}t+\phi_0)}{2\sqrt{3}} - \frac{\sqrt{-\Lambda}}{4r^2\sqrt{3}\sin(\sqrt{\frac{-\Lambda}{3}}t+\phi_0)}\left[\frac{(1+\omega_r)r_0(1+r_0^2)}{\omega_r}(\frac{r}{r_0})^{-1/\omega_r}+2r_0^3(\frac{r}{r_0})^3\right]. \quad (4.75)$$

4.1.5 Areal radius coordinates

Sometimes it is useful to employ areal radius $R \equiv a(t)r$ as a coordinate instead of r. The Schwarzschild-like coordinates are one of this kind of coordinate systems. Also, these systems provide what are called the pseudo-Painleve-Gullstrand coordinates [159]. Using the areal radius, metric (4.2) can be written in the pseudo-Painleve-Gullstrand form as

$$ds^2 = -\left[\frac{1-\frac{ab}{R}-R^2H^2}{1-\frac{ab}{R}}\right]dt^2 + \frac{dR^2}{\left(1-\frac{ab}{R}\right)} - \frac{2HR}{\left(1-\frac{ab}{R}\right)}dtdR + R^2d\Omega^2, \quad (4.76)$$

As required in the Painleve-Gullstrand coordinates the coefficient of dR^2 is not unity [160].

To obtain the Schwarzschild-like form we define a new time T by using the transformation

$$dT = \frac{1}{F}\left(dt+\beta dR\right), \quad (4.77)$$

where F is the integration factor which satisfies

$$\frac{\partial}{\partial R}(\frac{1}{F}) = \frac{\partial}{\partial t}(\frac{\beta}{F}). \quad (4.78)$$

Here $\beta(t,R)$ will be chosen later. Using Eq. (4.77) in Eq. (4.76) implies

$$ds^2 = -\left[\frac{1-\frac{ab}{R}-R^2H^2}{1-\frac{ab}{R}}\right]F^2dT^2 + \left[\frac{1+2HR\beta-\left(1-\frac{ab}{R}-R^2H^2\right)\beta^2}{1-\frac{ab}{R}}\right]dR^2 + \left[\frac{2F\beta\left(1-\frac{ab}{R}-R^2H^2\right)-2HRF}{1-\frac{ab}{R}}\right]dTdR + R^2d\Omega^2. \quad (4.79)$$

The cross term $dTdR$ is eliminated if we choose

$$\beta = \frac{HR}{1-\frac{ab}{R}-R^2H^2}. \quad (4.80)$$

Thus metric (4.79) takes the diagonal form

$$ds^2 = -\left[\frac{1-\frac{ab}{R}-R^2H^2}{1-\frac{ab}{R}}\right]F^2dT^2 + \left[\frac{1}{1-\frac{ab}{R}-R^2H^2}\right]dR^2 + R^2d\Omega^2, \quad (4.81)$$

where $F=F(T,R)$, a and H depend on T implicitly.

This metric (4.81) can be put in the form of (4.4) by using null coordinates $x^+ = T + R_*$ and $x^- = T - R_*$ where

$$dR/dR_* = \sqrt{-\frac{g_{TT}}{g_{RR}}} = \frac{\left[1-\frac{ab}{R}-R^2H^2\right]F}{\sqrt{1-\frac{ab}{R}}}. \quad (4.82)$$

The trapping horizon in this case is given by $\Theta_- \cong \frac{2}{R}\partial_- R = 0$ which gives

$$\left(1-\frac{ab}{R}\right) \cong H^2R^2. \quad (4.83)$$

Here we have bifurcating trapping horizon as $\Theta_- \cong 0$ implies $\Theta_+ \cong 0$.

The MS energy, energy flux and energy density are given, respectively, by

$$E = \frac{R}{2}\left[1-\left[1-\frac{ab}{R}-R^2H^2\right]F\right], \quad (4.84)$$

$$\psi_{\pm} = \pm(\rho + p_r)\frac{\left[1 - \frac{ab}{R} - R^2H^2\right]F}{4\sqrt{1 - \frac{ab}{R}}}, \tag{4.85}$$

$$\omega = \frac{\rho - p_r}{2}. \tag{4.86}$$

It may be noted that $E \cong R/2$ on the trapping horizon only. Now, with the quantity

$$\partial_{\pm}E = \pm\frac{2\pi R^2\rho\left[1 - \frac{ab}{R} - R^2H^2\right]F}{\sqrt{1 - \frac{ab}{R}}}, \tag{4.87}$$

the UFL is satisfied. The Kodama vector in this case takes the form

$$K^{\pm} = \frac{\sqrt{1 - \frac{ab}{R}}}{F}, \tag{4.88}$$

with $\|K\|^2 \cong 0$ on the trapping horizon. The GSG from Eq. (4.30) becomes

$$\kappa \cong -\frac{ab'}{2R} + \frac{ab}{2R^2} - H^2R, \tag{4.89}$$

which on using Einstein field equations takes the form

$$\kappa \cong -2\pi R(\rho + p_r) = \frac{E}{R^2} - 4\pi R\omega. \tag{4.90}$$

4.2 Charged dynamical wormholes

In this section we will consider DWHs which contain electric charge. Their static version has been discussed in detail in Chapter 3. Thus these wormholes generalize the wormholes which we have discussed in the previous chapters. We investigate UFL and GSG of these wormholes in Einstein's gravity and study their thermodynamic stability at the end. The charged extension of metric (4.1) can be written as

$$ds^2 = -(2e^{\Phi(t,r)} + \frac{q^2}{r^2})dt^2 + a^2(t)\left[\frac{dr^2}{1 - \frac{b(r)}{r} + \frac{q^2}{r^2}} + r^2d\Omega^2\right]. \tag{4.91}$$

If we take $a(t) = 1$ we obtain the (static) CWHs [157]. The charge act as extra matter in addition to exotic matter. Now, the effective redshift function is described as $\Phi_{eff}(t,r) = \frac{1}{2}\ln(e^{2\Phi(t,r)} + q^2/r^2)$, which should be finite everywhere for the absence of event horizon and should vanish at infinity. Here we will consider $\Phi_{eff}(t,r) = 0$ throughout this chapter. The traversability conditions, effective throat and effective shape function have already been discussed in detail in Chapter 3.

Now metric (4.91) can be transformed into the form (4.4) by introducing

$$dx^+ = \frac{dt}{a} + \frac{dr}{\sqrt{1 - \frac{ab}{R} + \frac{a^2q^2}{R^2}}}, \tag{4.92}$$

$$dx^- = \frac{dt}{a} - \frac{dr}{\sqrt{1 - \frac{ab}{R} + \frac{a^2q^2}{R^2}}}, \tag{4.93}$$

and

$$g_{+-} = -\frac{a^2}{2}. \tag{4.94}$$

The expansions in this case take the form

$$\Theta_\pm = \frac{2}{R}\partial_\pm R = \frac{a}{R}\left\{HR \pm \sqrt{1 - \frac{ab}{R} + \frac{a^2q^2}{R^2}}\right\}. \tag{4.95}$$

For trapping horizon we chose $\Theta_- \cong 0$, which gives

$$HR - \sqrt{1 - \frac{ab}{R} + \frac{a^2q^2}{R^2}} \cong 0. \tag{4.96}$$

which corresponds to the case $a(t) > 0$ and it corresponds to the expanding universe. On the trapping horizon we have $\Theta_+ \cong 2aH$ which is positive thus the trapping horizon is past. Thus, in our case on the trapping horizon $\Theta_+ > 0$ and $\Theta_- \cong 0$. Inside the trapping horizon we have $\Theta_\pm > 0$ and outside the trapping horizon we have $\Theta_+ > 0$ but $\Theta_- < 0$. Therefore, inside the trapping horizon $HR > \sqrt{1 - ab/R + a^2q^2/R^2}$, on the trapping horizon $HR \cong \sqrt{1 - ab/R + a^2q^2/R^2}$ and outside the trapping horizon

$HR < \sqrt{1 - ab/R + a^2q^2/R^2}$. However, it may be outer, inner or degenerate depending on the sign of $\partial_+\Theta_-$ as positive, negative or zero, respectively, on the trapping horizon, given by

$$\partial_+\Theta_- \cong a^2 \left\{ H^2 + \frac{\dot{H}}{2} - \frac{ab - b'R}{4R^3} + \frac{2a^2q^2}{4R^4} \right\}. \tag{4.97}$$

4.2.1 Generalised surface gravity and UFL for charged dynamical wormholes

The MS energy for charged DWHs takes the form

$$E = \frac{R}{2} \left\{ H^2R^2 + \frac{b}{r} - \frac{a^2q^2}{R^2} \right\}, \tag{4.98}$$

which is positive. This expression, on trapping horizon, becomes $E \cong R/2$. The Einstein-Maxwell field equations (4.21)-(4.23) are

$$\begin{aligned} \partial_\pm\Theta_\pm &= -\frac{1}{2}\Theta_\pm^2 + \Theta_\pm\partial_\pm \log(-g_{+-}) - 2\pi a^2(\rho + p_r), && (4.99)\\ \partial_\pm\Theta_\mp &= -\Theta_+\Theta_- + \frac{1}{R^2}g_{+-} + 2\pi a^2 \left\{ \rho - p_r - \frac{3a^4q^2}{4\pi R^4} \right\}, && (4.100)\\ \Theta_+\Theta_- &= -\partial_+\Theta_- - \partial_-\Theta_+ - 8\pi \left\{ p_t + \frac{a^4q^2}{8\pi R^4} \right\}, && (4.101) \end{aligned}$$

where the stress-energy tensor $T_{\mu\nu}$ is the sum of the matter part $T_{\mu\nu}^{(m)}$ and electromagnetic part $T_{\mu\nu}^{(e)}$, i.e., $T_{\mu\nu} = T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(e)}$. Here From Eq. (2.24), in this case, we have used

$$\begin{aligned} T_{++} &= T_{--} = \frac{a^2(\rho + p_r)}{4}, && (4.102)\\ T_{+-} &= T_{-+} = \frac{a^2(\rho - p_r - 3a^4q^2/4\pi R^4)}{4}, && (4.103)\\ T_{\theta\theta} &= R^2(p_t + \frac{a^4q^2}{8\pi R^4}). && (4.104) \end{aligned}$$

Solving Eq. (4.30) on the trapping horizon, the surface gravtiy κ, in this case, is found to be

$$\kappa \cong -\frac{\dot{H}R}{2} - H^2R + \frac{ab - b'R}{4R^2} - \frac{a^2q^2}{2R^3}. \tag{4.105}$$

This, on using Eqs. (4.99) and (4.100), can also be written as

$$\kappa \cong -\dot{H}R - H^2R - 2\pi R(\rho + p_r), \tag{4.106}$$

and

$$\kappa \cong \frac{E}{R^2} - 4\pi R\omega = \frac{1}{2R} - 2\pi R\left\{\rho - p_r - \frac{3a^4q^2}{4\pi R^4}\right\}. \tag{4.107}$$

The derivative of MS energy, on using Eqs. (4.99) and (4.100), can be written as

$$\partial_\pm E = 2\pi a R^2\left\{\pm\left(\rho - \frac{3a^4q^2}{8\pi R^4}\right)\sqrt{1 - \frac{ab}{R} + \frac{a^2q^2}{R^2}} - HR\left(p_r + \frac{3a^4q^2}{8\pi R^4}\right)\right\}. \tag{4.108}$$

From the stress-energy tensor, $T_{\mu\nu} = T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(e)}$, we can construct a function

$$\omega = -g_{+-}T^{+-} = \frac{\rho - p_r}{2} - \frac{3a^4q^2}{8\pi R^4}, \tag{4.109}$$

and a vector

$$\psi_\pm = \frac{a(\rho + p_r)}{4}\left\{\pm\sqrt{1 - \frac{ab}{R} + \frac{a^2q^2}{R^2}} - HR\right\}. \tag{4.110}$$

Thus the UFL (4.39) can be formulated using Eqs. (4.108), (4.109) and (4.110).

We must be careful about the conditions that govern the signs of the terms involved in UFL. We note that the energy supply term is always positive in the ingoing direction as $\rho + p_r < 0$. In the outgoing direction, it is positive inside the trapping horizon and negative outside the trapping horizon. The work term, in the outgoing direction, is positive if $(\rho - p_r) > 3a^4q^2/4\pi R^4$ and negative otherwise. In the ingoing direction, inside the trapping horizon it keeps the same behaviour while outside the trapping horizon its behaviour reverses. The gradient of MS energy, in the outgoing direction, is positive if $\rho > 3a^4q^2/8\pi R^4$ and $-p_r > 3a^4q^2/8\pi R^4$ while it is negative if $\rho < 3a^4q^2/8\pi R^4$ and $-p_r < 3a^4q^2/8\pi R^4$. In the ingoing direction, inside the trapping horizon it is positive when $\rho > 3a^4q^2/8\pi R^4$ and $-p_r > 3a^4q^2/8\pi R^4$ and in the range $\rho < 3a^4q^2/8\pi R^4 < -p_r$. Outside the trapping horizon it is positive when $\rho < 3a^4q^2/8\pi R^4$ and $-p_r < 3a^4q^2/8\pi R^4$, and in the range $\rho < 3a^4q^2/8\pi R^4 < -p_r$.

On the trapping horizon, in the outgoing direction, the gradient of MS energy and the work term are equal and they can be positive or negative, depending on the amount of charge in the wormhole. In the absence of charge these terms are positive, also if $\rho - p_r > 3a^4q^2/4\pi R^4$ then these terms are positive. However, they can get negative if large amount of charge is present in the wormhole such that $\rho - p_r < 3a^4q^2/4\pi R^4$. The energy supply term vanishes on the trapping horizon in the outgoing direction always. In the ingoing direction, the gradient of MS energy and the energy supply terms is same and always positive while the work term vanishes on the trapping horizon.

At the throat, in both the directions, ingoing and outgoing, the energy supply term is always positive, however, the other two terms are positive for small quantity of charge while negative for large quantity. Thus, when there is less amount of charge in the wormhole such that $(\rho - p_r) > 3a^4q^2/4\pi R^4$ then all the terms appearing in the UFL are positive. If more charge is added such that $-p_r = 3a^4q^2/8\pi R^4$ then the gradient of MS energy vanishes while the energy supply term and the work term become equal in magnitude but their signs are not same. If $(\rho - p_r) = 3a^4q^2/4\pi R^4$ then work density vanishes while gradient of MS energy and energy supply term becomes same. Here we also note that if both the conditions $-p_r = 3a^4q^2/8\pi R^4$ and $(\rho - p_r) = 3a^4q^2/4\pi R^4$ are met at the same time then this will ensure that $\rho = -p_r$, and thus it will respect the NEC. Thus exotic nature of the material supporting the wormhole may be lost. Thus both the conditions cannot be met at the same time in the case of a wormhole supported by exotic matter. If more charge is added to the wormhole such that $-p_r < 3a^4q^2/8\pi R^4$, then the gradient of MS energy and the work term become negative, however, the energy supply term is still positive. This means that increasing charge induces an increase in work density in the negative direction.

4.2.2 Thermodynamic stability of charged dynamical wormholes

We will examine thermodynamic stability of charged DWHs, in this section, using the same criterion discussed earlier for uncharged DWHs. We subtract Eq. (4.107) from (4.106), obtaining

$$p_r = -\frac{1}{8\pi R^2} - \frac{\dot{H} + H^2}{4\pi} - \frac{3a^4q^2}{8\pi R^4}. \tag{4.111}$$

Solving Eq. (4.101) on the trapping horizon, using the definition of surface gravity and Hawking temperature, we get

$$2p_t = -\frac{a^2 T}{R} - \frac{a^4 q^2}{4\pi R^4}. \tag{4.112}$$

From Eqs. (4.111) and (4.112), the average pressure $\bar{P}$ can be found as

$$\bar{P} = \frac{p_r + 2p_t}{3} = -\frac{1}{24\pi R^2} - \frac{\dot{H} + H^2}{12\pi} - \frac{a^2 T}{3R} - \frac{5a^4q^2}{24\pi R^4}. \tag{4.113}$$

The thermodynamic stability can be analyzed from equation of state (4.113). Taking derivative of this with respect to V at constant temperature, we get

$$\frac{\partial \bar{P}}{\partial V}\,|_T = \frac{(4\pi/3)^{2/3}}{36\pi V^{5/3}} + \frac{(4\pi/3)^{1/3} a^2 T}{9V^{4/3}} + 5a^4q^2 \left(\frac{32\pi}{59049 V^7}\right)^{1/3}. \tag{4.114}$$

Now, for stable thermodynamic equilibrium of a thermodynamic system, we must have $\frac{\partial \bar{P}}{\partial V}\,|_T \leq 0$ which is ensured for

$$T \leq -\frac{R^2 + 10a^4q^2}{4\pi a^2 R^3}. \tag{4.115}$$

This negative temperature can be attributed to exotic matter. Solving Eqs. (4.113) and (4.115), we find

$$\bar{P} \geq \frac{1}{24\pi R^2} - \frac{\dot{H} + H^2}{12\pi} + \frac{15a^4q^2}{24\pi R^4}. \tag{4.116}$$

If the scale factor is a linear function of time then $\ddot{a} = 0$ and then $\bar{P}$ will assume positive values everywhere, otherwise it could be negative somewhere.

Another condition for stable thermodynamic equilibrium is $C_P \geq C_V \geq 0$. Now, since constant V means constant E and S so by the definition of C_V,

$$C_V = \frac{\partial E}{\partial T} \mid_V = T\frac{\partial S}{\partial T} \mid_V = 0, \tag{4.117}$$

which means we can define heat capacity only at constant pressure as

$$C_P = T\frac{\partial S}{\partial T} \mid_P = \frac{(24\pi \bar{P} R^4 + 2\dot{H}R^4 + 2H^2R^4 + R^2 + 5a^4q^2)2\pi R^2}{24\pi \bar{P} R^4 + 2\dot{H}R^4 + 2H^2R^4 - R^2 - 15a^4q^2}, \tag{4.118}$$

where from Eq. (4.113),

$$T = -\frac{1}{a^2}(3R\bar{P} + \frac{1}{8\pi R} + \frac{\dot{H}R + H^2R}{4\pi} + \frac{5a^4q^2}{8\pi R^3}). \tag{4.119}$$

Now, from Eq. (4.116), to ensure the stable thermodynamic equilibrium, we can take the value of $\bar{P}$, for any non-negative ϵ, as

$$\bar{P} = \frac{1}{24\pi R^2} - \frac{\dot{H} + H^2}{12\pi} + \frac{15a^4q^2}{24\pi R^4} + \epsilon. \tag{4.120}$$

Thus Eq. (4.118) on using Eq. (4.120) takes the form

$$C_P = \frac{1}{6\epsilon} + 2\pi R^2 + \frac{5a^4q^2}{3R^2\epsilon}. \tag{4.121}$$

All the terms appearing in the above equation are positive which ensures $C_P \geq C_V \geq 0$. Thus charged DWHs could be thermodynamically stable.

Chapter 5

Summary and conclusion

In this thesis we have studied traversable wormholes and their thermodynamics in Einstein's gravity and alternate theories of gravity. We have studied MTWHs as well as DWHs, with and without electric charge. For our purpose we have used a $2+2$ formalism developed earlier [44, 45, 84, 92] and extended its application to DWHs. In this formalism thermodynamics of spherically symmetric spacetimes is studied using an approach which uses the local coordinates to discuss black holes and wormholes (both are characterised by the presence of outer trapping horizons). We have applied the formalism that uses the local quantities instead of global quantities to study thermodynamics of spacetimes. These local quantities allow us to find the trapping horizon, which is a generalization of the Killing horizon, on which Kodama vector, a generalization of the Killing vector, becomes null. The union of all the trapped surfaces form a trapping region and the boundary of a trapping region is called a trapping horizon which is the surface foliated by marginal spheres on which one of the null expansions becomes zero. Thus, equivalently, on trapping horizon light rays travel parallel to each other in either direction, ingoing or outgoing or both, with no increase or decrease of distance in between them which shows that the area of the sphere is constant there.

There are different types of trapping horizons depending on the sign of expansions and their derivatives. In the static case (MTWHs and CWHs) we have a bifurcating trapping horizon which results in neither expansion nor contraction of outgoing and ingoing light rays and this horizon coincides with the location of the throat. However, in non-static case (DWHs) we have a past trapping horizon which corresponds to the expanding universe.

Here, in this thesis, we have considered MTWHs, CWHs, uncharged DWHs and charged DWHs and discussed their thermodynamics in GR and $f(R,T)$ gravity and further considered non-minimal curvature-matter coupling as well for the case of MTWHs and CWHs. Due to the spherically symmetric properties of these spacetimes, the Hayward technique is applicable to discuss the thermodynamic properties of these spacetimes as well. In these wormhole spacetimes Killing horizons do not exist despite the presence of the Killing vector so that the surface gravity could not be found by using the Killing vector, hence the usual definition of finding surface gravity is not applicable here. But the Kodama vector, a generalization of the Killing vector, still exists that reduces to the Killing vector if there is vacuum. This Kodama vector allows the presence of trapping horizon which can be used to derive the GSG. Thus, in this case, the Kodama vector and trapping horizon play the role of the Killing vector and Killing horizon, respectively. Using this technique, the expression of the GSG has been derived in each case. This GSG is positive, negative or zero for outer, inner or degenerate trapping horizons, respectively. In our case of outer trapping horizon we have positive GSG and equivalently negative Hawking temperature. However, one could avoid this negative temperature by making claim that this is the problem only at horizon, but ingoing radiations appearing at one mouth of the wormhole, following the classical trajectory, would reappear as outgoing radiation on the other mouth of the wormhole, unavoiding this negative temperature. This is not surprising as wormholes are argued to be constructed by phantom energy which may be characterized by negative temperature. Thus wormholes emit radiations associated with negative temperature in the same way as black holes emit radiations associated with positive temperature. Later on this expression becomes the part of the first law of wormhole dynamics which is obtained from the UFL by projecting it along the trapping horizon.

The UFL is obtained from the MS energy by taking its gradient that results in two terms, on using the gravitational field equations, the energy removal term and the work term. In Chapters 2 and 3 we have considered MTWHs and CWHs. We observe that in GR, in the static case (MTWHs and CWHs), all the terms entering UFL, vanish on the trapping horizon (throat) of the wormhole. This results in no evolution of the throat and, in general, the variation of gravitational energy and the work term have same signs (positive in outgoing direction and negative in ingoing direction) opposite to the energy removal term. However, in the case of uncharged DWHs, discussed in Chapter 4, there is a variation of sign in different regions of the spacetime. Here, in uncharged DWHs, the situation is different; first of all the trapping horizon and throat do not coincide here and different behaviour of terms appearing in UFL is observed. We note that, on the throat, all the terms entering UFL are positive both in the ingoing and outgoing directions. Thus, the direction does not matter on the throat in the case of uncharged DWHs. However, on the trapping horizon we observe that the energy removal term and the work term vanish in the outgoing and ingoing directions, respectively. This means that the variation of gravitational energy and the work terms are equal and positive in the outgoing direction, and in the ingoing direction the variation of gravitational energy and energy supply terms are equal and positive. Inside the trapping horizon, all the terms appearing in the UFL are always positive both in the outgoing as well as ingoing direction. Outside the trapping horizon, the variation of gravitational energy and work terms are positive in the outgoing direction but the energy removal term is negative, while in the ingoing direction the variation of gravitational energy is also positive provided enough amount of exotic matter is present there, however, the energy removal and work terms are positive and negative, respectively.

We have discussed wormholes in different cosmological models, with and without the cosmological constant, in Chapter 4 for their thermodynamic properties. These include de Sitter and anti-de Sitter wormholes in open, closed and flat universes. Further, we have discussed cases of asymptotically flat and asymptotically de Sitter wormholes as well.

We have also extended our results to the charged DWHs in Chapter 4. By carefully observing the terms in the UFL we note that the behaviour of the energy supply term is the same as in the case of uncharged DWHs in each region and in both the directions. However, gradient of the MS energy and work terms are affected by different amounts of charge. When $\rho - p_r > 3a^4q^2/4\pi R^4$ then both the terms show similar behaviour as in the case of uncharged DWHs. If $\rho - p_r < 3a^4q^2/4\pi R^4$ then the sign of the work term becomes opposite to that in the case of uncharged DWHs. However, the gradient of MS energy reverses its sign always in the outgoing direction. In the ingoing direction, its sign is same on the trapping horizon and also other than the trapping horizon (provided $\rho < 3a^4q^2/8\pi R^4 < -p_r$ holds) as in uncharged DWHs, but it reverses its sign on the throat and other than the trapping horizon (provided $\rho < 3a^4q^2/8\pi R^4 < -p_r$ does not hold).

Further, a non-minimal curvature-matter coupling has been considered in MTWHs and CWHs, due to which the gravitational field equations when written in the form of the Einstein tensor replace the role of the stress-energy tensor with an effective tensor that consists of the normal matter and curvature stress-energy tensors and reduces to the normal matter stress-energy tensor if the coupling constant $\lambda_2 \to 0$. These results have been further generalized to the case of $f(R,T)$ gravity. The results of non-minimal curvature-matter coupling and those derived in $f(R,T)$ gravity reduce to the results derived for the Einstein's gravity when $\lambda_2 \to 0$ and $\lambda_1 \to 0$, respectively.

Thermodynamic stability has been discussed in GR for the wormholes. In the case of MTWHs and CWHs we have discussed thermodynamic stability for the specific case $\Phi = 0$. The MTWHs are thermodynamically stable when the temperature $T \leq -1/4\pi r$ and the average pressure $\bar{P} \geq 1/24\pi r^2$. This negative temperature could be attributed to exotic matter. It was found that the CWHs are also thermodynamically stable provided

$$
\begin{aligned}
T &\leq -\frac{r^2+10q^2}{4\pi r^3+12\pi rq^2}, \\
\bar{P} &\geq -\frac{1}{24\pi r^2}-\frac{5q^2}{24\pi r^4}+\frac{(r^2+10q^2)(r^2+q^2)}{12\pi r^6+36\pi r^4q^2},
\end{aligned}
$$

thus temperature is also negative in this case, however, the average pressure may assume negative values unlike the case of MTWHs. In the case of uncharged DWHs thermodynamic stability is ensured for

$$
\begin{aligned}
T &\leq -1/4\pi a^2R, \\
\bar{P} &\geq \frac{1}{24\pi R^2}-\frac{\dot{H}+H^2}{12\pi},
\end{aligned}
$$

here temperature is always negative but the average pressure may assume negative values for non-linear scale factor. However, for linear scale factor the average pressure is always positive. In the case of charged DWHs the thermodynamic equilibrium is maintained for

$$
\begin{aligned}
T &\leq -\frac{R^2+10a^4q^2}{4\pi a^2R^3}, \\
\bar{P} &\geq \frac{1}{24\pi R^2}-\frac{\dot{H}+H^2}{12\pi}+\frac{15a^4q^2}{24\pi R^4},
\end{aligned}
$$

thus, in this case temperature always assumes negative values, however, the average pressure is positive for linear scale factor and for non-linear scale factor it may be negative somewhere.

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