Projective State Spaces for Theories of Connections

Projektive Zustandsräume für Theorien von Zusammenhängen

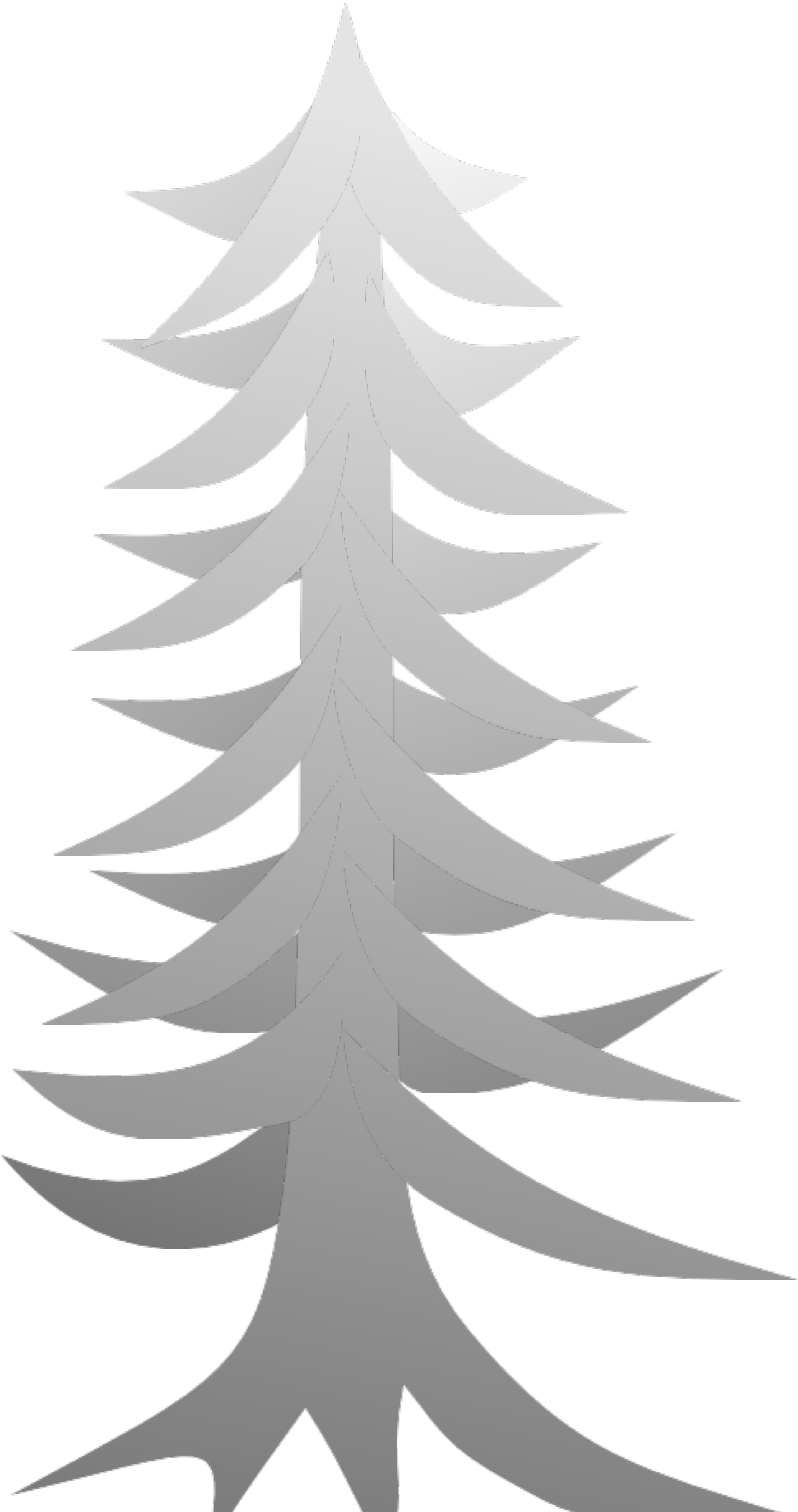
Der Naturwissenschaftlichen Fakultät
der Friedrich-Alexander-Universität
Erlangen-Nürnberg
zur
Erlangung des Doktorgrades Dr. rer. nat.
vorgelegt von
Suzanne Lanéry
aus Évreux, Frankreich

Als Dissertation genehmigt
von der Naturwissenschaftlichen Fakultät
der Friedrich-Alexander Universität Erlangen-Nürnberg

Tag der mündlichen Prüfung: Freitag, den 9. Oktober 2015

Vorsitzender des Promotionsorgans: Prof. Dr. Jörn Wilms

Gutachter: Prof. Dr. Thomas Thiemann
Dr. habil. Karim Noui
Dr. habil. Andrzej Okołów



À la mémoire de
Étienne Lanéry
mon père

Abstract: Projective State Spaces for Theories of Connections

Quantum gravity aims at combining the insights of Quantum Field Theory (QFT) and General Relativity (GR) into a consistent fundamental theory. In a *non-perturbative*, canonical approach like Loop Quantum Gravity (LQG, [4, 75, 92]), one attempts to directly quantize the spacetime geometry, while resolutely avoiding the introduction of any supporting background metric. The price for this *background independence* [77] is that the technologies commonly employed in QFT to extract physics in a computationally tractable way are not readily available. Thus, new tools need to be developed, for example to check the *semi-classical limit* of the quantum theory or to rigorously derive its *cosmological* [16] and *astrophysical implications* [83, 5, 18].

In particular, the vacuum states of Fock type used when doing QFT on Minkowski background have no equivalent when *spacetime itself* is to be quantized. While the non-standard properties of the Ashtekar-Lewandowski vacuum used in LQG [19, 8] lead to a compelling picture at Planck scale, revealing notably a fundamental *discreteness* of quantum geometry [78, 7], they also contribute to long-standing issues regarding the design of *semi-classical states* [90, 51, 36] and the implementation of the *dynamics* [88, 91]. This motivates the search for an extension of the LQG Hilbert space, that could accommodate more general quantum states while retaining the key insights of the original construction.

This is achieved in the present work using a *projective framework* [48, 66, 68], which allows to dispense altogether from the selection of a vacuum state: instead of defining states as vectors in one 'big' Hilbert space, or more generally as density matrices thereon, one constructs them as projective families of *partial density matrices* over a system of 'small' Hilbert spaces, each of which extracts specific degrees of freedom from the full quantum theory. This approach is physically motivated by interpreting each small Hilbert space as the arena to describe a *given experiment*, while the projections binding these partial descriptions together ensure the *overall consistency* of the theory.

We will set up projective state spaces of this kind for general *theories of connections*: this includes the reformulation of GR using Ashtekar variables [3, 12] that constitutes the starting point of LQG, and could have applications to other quantum gauge theories as well. To this intend, we will develop the projective framework beyond the context of linear configuration spaces in which it was originally formulated, laying down fairly generic prescriptions to turn *classical* projective system into *quantum* ones. To ascertain that the thus obtained quantum state spaces indeed *extend* existing ones, we will investigate in detail their relations with various Hilbert spaces. Finally, we will explore how this approach could help making progress on the aforementioned issues, in particular by paving the way for the development of more satisfactory semi-classical states.

Zusammenfassung: Projektive Zustandsräume für Theorien von Zusammenhängen

Ziel der Quantengravitation ist, die Erkenntnisse der Quantenfeldtheorie (QFT) mit denen der allgemeinen Relativitätstheorie (ART) in einer einheitlichen grundlegenden Theorie zu verschmelzen. Folgt man einem *nichtperturbativen*, kanonischen Ansatz, wie in der Schleifenquantengravitation (SQG, [4, 75, 92]), wird versucht gleich die Geometrie der Raumzeit zu quantisieren, wobei die Einführung jeglicher unterstützenden Hintergrundmetrik konsequent vermieden wird. Der Preis dieser *Hintergrundunabhängigkeit* [77] ist der Verzicht auf die bewährten Techniken, die es in der QFT ermöglichen, physikalische Aussage effektiv zu berechnen. Daher müssen neue Werkzeuge entwickelt werden, etwa um den *semiklassischen Limes* der Quantentheorie zu bestätigen, oder um deren *kosmologische* [16] und *astrophysikalische Bedeutung* [83, 5, 18] rigoros abzuleiten.

Insbesondere sind die Fock-artigen Vakuumzustände, die in der QFT auf dem Minkowski-Hintergrund angewendet werden, nicht mehr verfügbar, wenn die *Raumzeit selbst* quantisiert sein soll. Während die einzigartigen Eigenschaften des Ashtekar-Lewandowski Vakuums [19, 8], das der SQG zugrunde liegt, überzeugende, durch eine grundsätzliche *Diskretheit* [78, 7] auf der Planck-Skala gekennzeichnete, Einblicke in der der Quantengeometrie liefern, tragen sie auch zu andauernden Schwierigkeiten bei dem Aufstellen *semiklassischer Zustände* [90, 51, 36] sowie bei der Behandlung der *Dynamik* [88, 91] bei. Dies motiviert die Suche nach einer Erweiterung des SQG-Hilbertraumes, die die bisher fehlenden Quantenzustände einbeziehen könnte, ohne die Grundsätze der ursprünglichen Konstruktion aufzugeben.

So eine Erweiterung wird in der vorliegenden Arbeit mithilfe eines *projektiven Formalismus* [48, 66, 68] entwickelt, der uns von dem Aussuchen eines Vakuumzustandes befreit: statt Quantenzustände als Vektoren (bzw. Dichtematrizen) in einem einzigen, 'großen' Hilbertraum zu beschreiben, werden sie als projektive Familien von *partiellen Dichtematrizen* über einem System von 'kleinen' Hilberträumen zusammengesetzt. Jedem dieser kleinen Hilberträume entspricht die Auswahl bestimmter Freiheitsgrade aus der vollen Quantentheorie und wird physikalisch verstanden als die Arena, wo sich ein *gegebenes Experiment* abspielt, während die Projektionen, die diese Räume miteinander verbinden, dafür sorgen, dass aus diesen partiellen Darstellungen eine *konsistente* Theorie entsteht.

Wir werden projektive Zustandsräume dieser Art für allgemeine *Theorien von Zusammenhängen* aufbauen: dies erfasst insbesondere das Umschreiben der ART mithilfe der Ashtekar-Variablen [3, 12], das als Ausgangspunkt der SQG dient, und könnte auch bei der Beschreibung anderer Eichtheorien Anwendungen finden. Hierfür werden wir den projektiven Formalismus jenseits des Kontextes linearer Konfigurationsräume ausbauen, in dem er bislang formuliert war, und relativ systematische Vorschriften ausarbeiten, um *klassische* projektive Systeme in deren *Quantenversion* umzuwandeln. Um sicherzustellen, dass die so zusammengestellten Quantenzustandsräume die existierenden wirklich *erweitern*, werden wir detailliert deren Beziehungen zu verschiedenen Hilberträumen untersuchen. Schließlich werden wir erforschen, wie dieses Programm bei den oben genannten Schwierigkeiten helfen könnte, vor allem was die Entwicklung besseren semiklassischen Zustände betrifft.

Acknowledgements

I am deeply grateful to my advisor, Thomas Thiemann, who guided me throughout the last five years with his valuable advice, and with his example. I am also indebted to Karim Noui, without whose assistance at the critical moment, I would probably not have gone into quantum gravity, if I would have attempted a PhD at all. Further, it is a pleasure to acknowledge the solicitude of Kristina Giesel and Hanno Sahlmann, in particular their accompanying of my first teaching experiences. And I remain in great appreciation of Günter Mahler, from the University of Stuttgart, who introduced me to research years ago. Last but not least, I would like to warmly thank Antonia, Alexander, Norbert, Andreas, Christian, and all the students and postdocs of the Institute for Quantum Gravity in Erlangen, for inspiring discussions and friendship.

What I owe to my parents is beyond words. The strength of their love envelops me wherever I go, while the values and faith they gave me light my path. And the reassuring hand of my mother has been my anchor, in a world that had suddenly become unrecognizable. I also will not forget the encouragement, each in their own way, by my sisters and brothers-in-law, Odile, Hélène and Yann, Cécile and Michael, and by my longstanding friend Marie-Pierre. Nor shall I leave unmentioned the kind missives from F. Benoît, and the comforting evenings at the Offene Kirche St. Klara in Nürnberg.

This work has been financially supported by the École normale supérieure, Paris, France, by the Université François Rabelais, Tours, France (via a 3-years doctoral stipend from the French Ministry of Education – Contrat Doctoral Normalien), and by the Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany (via the Bavarian Equal Opportunities Sponsorship – Förderung von Frauen in Forschung und Lehre (FFL) – Promoting Equal Opportunities for Women in Research and Teaching).

This research project has also been supported by funds to Emerging Field Project "Quantum Geometry" from the FAU Erlangen-Nürnberg within its Emerging Fields Initiative.

Contents

Introduction

Context

Projective State Spaces in Quantum Field Theory

An important step toward the quantization of a classical theory is the choice of a home for the kinematical quantum states: typically, we look for an Hilbert space supporting a representation of an algebra of selected kinematical observables. As long as we only deal with finitely many degrees of freedom, a comprehensive survey of the available options might still be within reach [98]. But the extent and implications of this initial choice tend to get dramatically more involved in the case of a *field* theory, where the very large algebra of kinematical observables can give rise to an intricate forest of representations. Unfortunately, it is hard to concisely formalize which requirements the elected representation should satisfy to ensure that it supports the quantum states we are ultimately interested in, eg. the states that solve the dynamics.

Given these concerns, one may wonder whether we could dispense from committing ourselves to a definite representation, while still having at our disposal an *explicit and constructive* description of the state space, suitable for concrete calculations and further exploration of the quantum theory. In the present work, we will investigate such an alternative way of building the space of kinematical states, first introduced by Jerzy Kijowski in the late '70s [48] and further developed by Andrzej Okołów recently [66, 68, 69]. Instead of describing states as density matrices over a single big Hilbert space, we will construct them as projective families of density matrices over small 'building block' Hilbert spaces (subsection 5.1). The projections are given as *partial traces*, so that each small Hilbert space can be physically interpreted as selecting out specific degrees of freedom. This formalism allows us to rely more heavily on the physical interpretation of the kinematical observables, namely how they are measured in practice. This tends to give state spaces that are bigger (subsection 5.2), but nevertheless technically easier to handle. In particular, we thus start with better chances to find the states we are looking for.

Canonical Quantum Gravity

Our main motivation to study this projective framework will be its possible application to quantum gravity. More specifically, we will focus on the canonical quantization of general relativity, starting from an Hamiltonian formulation thereof. Such a formulation is provided by the ADM formalism (introduced in [70] by Arnowitt, Deser and Misner), which relies on a phase space describing 3-dimensional *spacelike slices*. The 3-geometry along the slice is parametrized by the configuration variables, while its extrinsic curvature (controlling the shape it should have when embedded in an ambient 4-dimensional spacetime) plays the role of the conjugate momenta.

On this 'kinematical' phase space, the dynamics is recovered by imposing constraints (see [105, section 1.7] and appendix A). Like in gauge theories, these constraints not only delineate a physically admissible region in phase space (aka. the constraint surface), their Hamiltonian flow also generates gauge transformations along this constraint surface. A given spatial slice belongs to the constraint surface if it can be cut out of a 4-geometry fulfilling Einstein equations, while 2 different slices are in the same gauge orbit as soon as they can be cut out of the *same* Einstein geometry. Hence, each orbit on the constraint surface can be identified with a spacetime obeying the Einstein equations, and the group of 4-diffeomorphisms acts as the gauge group running through the orbit. It is common to distinguish between the 'diffeomorphism' constraints, generating 3-diffeomorphisms along the spatial slice, and the 'Hamiltonian' ones, generating deformations and displacements of the slice (thus implementing the evolution in timelike direction).

Instead of trying to directly quantize the canonical theory in terms of the ADM variables (an approach known as geometrodynamics [103, 104]), Loop Quantum Gravity (LQG, [4, 92]) has been able to push the quantization program further through the use of the Ashtekar variables [3, 12], which provide a reformulation in a language closer to Yang-Mills theory (provided additional 'Gauss' constraints are imposed to handle the corresponding gauge invariance, see section 8 for more details).

The Ashtekar-Lewandowski Hilbert Space

The Hilbert space used in LQG is built around the Ashtekar-Lewandowski vacuum [6, 19, 8], which is an eigenstate of the spatial geometry: heuristically, the spatial slice it describes has a degenerate 3-geometry (ie. vanishing volume) and a totally indeterminate extrinsic curvature. The quantum states are then obtained by adding discrete quantum excitations on top of this vacuum: namely, *quanta of geometry* supported on of one-dimensional 'edges', so that a typical state is supported on a finite collection of edges (aka. a graph). This leads to a *discrete spectrum* for the geometrical operators measuring areas or volumes along the spatial slice [78, 7], in accordance to the physical intuition that, at Planck scale, space should no longer be smooth.

Yet, this nearly complete degeneracy of the spatial geometry, and the maximal uncertainties attached to most components of its conjugate variable (the extrinsic curvature) in any state of the Ashtekar-Lewandowski Hilbert space, make the study of the *semi-classical regime* problematic (see [51, 36] for a discussion of this problem, together with proposals to circumvent it). We would like to associate to any point in the classical phase space a corresponding quantum state, suitably peaked around that point, in both configuration and momentum variables [90]. The trouble is that an intrinsic asymmetry has been introduced in the role played by the positions versus momenta, as the vacuum was chosen as an eigenstate of the spatial geometry [11, 23, 33]: no matter how many discrete excitations are piled up on top of this vacuum, it will never be sufficient to mask this initial bias.

Similar issues arise when trying to rigorously derive symmetry reduced models [16, 30, 18] out of the full theory: an important goal here would be to obtain quantum states in the full state space that would be almost symmetric both in configuration and momentum variables, and that we could identify with the states of the reduced theory. Setting up such *quasi-symmetric states* would require techniques akin to the construction of quasi-classical ones [27].

Note that the situation here differs crucially from the Fock representation used when doing quantum field theory on a flat Minkowski background: while the Fock space also describe discrete excitations on top of a vacuum state, its vacuum is a *coherent* state with respect to a certain set of canonically conjugate variables (as provided by mode decomposition of the fields), so it does not favor half of the variables to the detriment of the others. This is however not an option for LQG, because the Ashtekar-Lewandowski vacuum is the only diffeomorphism invariant state at our disposal [55]: using a genuinely non-diffeomorphism-invariant state as vacuum would (in much the same way as just argued) lead to a quantum theory breaking diffeomorphism invariance, thus ruining its *background independence*, which is a core design principle of the LQG approach [75, 77].

Solving the Theory

Finally, one should also consider the dynamics. Following Dirac prescription [22] , the constraints of the classical theory should be promoted to operators on the so far purely kinematical Hilbert space: the quantum states that annihilate them are the physically admissible ones. The discreteness of the spatial geometry in the Ashtekar-Lewandowski state space allows for the transparent resolution of those constraints which act *along* the the spatial slice (ie. the Gauss and diffeomorphism ones). Although the solutions of the diffeomorphism constraints do not directly belong to the Ashtekar-Lewandowski Hilbert space (except for the vacuum itself), but rather to its algebraic dual [37] , the Hilbert space they span can be conveniently described in terms of *embeddingless* graphs (ie. graphs whose actual location within the spatial slice is left unspecified).

On the other hand, the Hamiltonian constraints mix non-trivially the position and momentum variables, making them hard to handle on the Ashtekar-Lewandowski Hilbert space: quantum states that were initially graph-supported are expected to spread when evolving the spatial slice. The space of solutions has therefore to be reconstructed using sophisticated techniques [91, 36] and is not yet fully understood. Moreover, this contributes to the limited usefulness of truncated semi-classical states, ie. states that would be semi-classical only with respect to selected components of the spatial geometry and extrinsic curvature: such states would span a subspace transversal to the Hamiltonian flow (due to the so-called 'graph-changing' property of the Hamiltonian constraint operator [88]) and would thus immediately lose their semi-classicality under time-directed evolution.

Outline

The projective approach to quantum field theory pointed out above was originally proposed in the context of linear configuration spaces [48, 66, 68], and could thus be applied to the quantification of general relativity rewritten using real-valued connections (a formulation known as teleparallel gravity, see [67, 69]). In the present work, we will develop this projective formalism at a fairly general level, laying the stage for a similar treatment of theories of connections on arbitrary gauge groups: this includes general relativity in the Ashtekar formulation, but could have applications to other gauge theories as well. We will illustrate on simple toy-models how this line of research could incorporate the dynamics and contribute to the study of the semi-classical limit.

Projective Limits of State Spaces

We will start by a detailed exposition of projective limits of phase spaces [89] (section 2), that build the natural *classical counterpart* of projective quantum state spaces (section 5). This will give us the opportunity to discuss the *physical motivations* for this approach. Regarding the *quantization* of classical projective structures into quantum ones, we will extend previous results by Okołów [68] to configuration spaces given by simply-connected Lie groups, and to holomorphic quantization of complex phase spaces (section 6).

Specific difficulties arise in this context when trying to implement the *dynamics* (aka. to impose the constraints, as explained above). In section 3, we will take advantage of having at our disposal a classical precursor of the formalism to analyze this question without having to deal at the same time with the inherent subtleties of the quantum dynamics. We will outline a suitable strategy, with the aim of doing justice both to the deep *physical meaning* of the issues at hand and to their practical significance for *computations*.

To support the claim that this formalism could help designing a quantum state space holding the states we need, we will also, in subsection 5.2, inspect in detail how quantum projective state spaces relate to some standard constructions (namely, inductive limits and infinite tensor products of Hilbert spaces).

Projective Structure for the Holonomy-Flux Algebra

As mentioned, the description, within this projective framework, of a theory of Abelian connections has been developed by Okołów in [68, section 5] and [69], an important insight being to use 'building blocks' labeled by combinations of edges and surfaces (instead of edges only as in the case of the Ashtekar-Lewandowski Hilbert space). We will generalize this construction to an *arbitrary gauge group* G (in particular, G is neither assumed to be Abelian nor compact). This involves refining the definition of the label set, as well as deriving explicit formulas to relate the 'small' Hilbert spaces attached to different labels (section 10 and subsection 12.1).

If the gauge group happens to be compact, we have at our disposal the well-established Ashtekar-Lewandowski Hilbert space. We will show that the quantum state space constructed using projective techniques can be thought as a natural *extension* of the space of density matrices over the Ashtekar-Lewandowski Hilbert space (subsection 12.2), while achieving a *balanced* treatment of configuration and momentum variables, as is manifest from the classical precursors of both formalisms (projective limits of phase spaces vs. configuration spaces).

However, for the possibilities offered by this approach to be fully exploited, one should be able to solve the *Gauss and diffeomorphism constraints* within the projective framework, before eventually turning to the *Hamiltonian* ones (see above). The difficulty is that the techniques developed for the standard LQG Hilbert space [8] cannot be directly transposed, as they rely strongly on having states characterized by discrete excitations of the spatial geometry. We will, in sections 13 and 20, sketch paths to explore this question.

Applications and Toy-Models

In the last part, we will review some *work in progress* built upon the formalism set up so far.

In view of handling the dynamics of LQG, we will test in two simple toy-models the strategy to implement constraints in the projective framework, that was proposed in section 3. The first one (section 15) is a very basic linear model, meant as an illustration of the *general procedure*, and we will only discuss it at the classical level. In the second one (section 16), we will reformulate the Schrödinger equation, treated as a classical field theory, within this projective framework, and proceed to its (non-relativistic) *second quantization*. We will then be able to reproduce the physical content of the usual *Fock quantization*.

Finally, we will turn to the question of identifying *semi-classical states* within the state space set up in the previous part. As we will see (section 18), treating configuration and momentum variables on equal footing, while a prerequisite for the construction of such states, is not yet sufficient:

further *obstructions* lies in the algebra of observables itself (aka. independently of our choice of *representation* for this algebra). More precisely, these difficulties are tied to the fact that we are dealing with a *continuum* of observables. This motivates the development presented in section 19, where we will outline how a *dense* but *discrete* subalgebra of observables could be extracted while preserving the *universality* and *diffeomorphism invariance* of the quantum theory.

Some very standard material is recalled in the appendices, mostly to fix notations and conventions in a form suitable for their use in the main text. Content that is non original, but presumably less well-known, has also been included for the shake of completeness, in particular:

- generalities about projective systems and the projective approach to quantum field theory in subsections 2.1 and 2.2, and subsection 5.1;
- generalities about the structures underlying Loop Quantum Gravity in section 9, subsections 10.1 and 12.2;

as well as smaller items marked as such throughout the text.

Chap. 1, chap. 2, chaps. 3 and 4, and chap. 5 have been distributed as separate articles [53] through the arXiv preprint repository (`arxiv.org`), respectively with the identifiers 1411.3589, 1411.3590, 1411.3592 and 1411.3591.

Notations and conventions

Unless otherwise stated, all manifolds are smooth manifolds [54], all maps between them are smooth, and symplectic structures on them are smooth. All submanifolds are regular (ie. embedded) submanifolds [54, chapter 5]. Where infinite dimensional manifolds are considered, these are Banach-modeled smooth manifolds, and symplectic structures on them are assumed to be strong symplectic structures [20, chap. VII].

Throughout the text, Σ will denote a finite-dimensional, analytic manifold (without boundary), d its dimension ($d \geqslant 1$), G a finite-dimensional Lie group and $\mathfrak{g}$ its Lie algebra. Where additional restrictions on Σ or G are required, they are spelled out at the beginning of the affected sections or subsections.

Index of Notations

Symbol	Description	Reference
$(\mathcal{L}, \mathcal{M}, \pi)^{\downarrow}$	projective system of phase spaces	def. 2.3
$(\mathcal{L}, X, \varphi)^{\times}$	classical factorizing system	def. 2.11
$(\mathcal{L}, \mathcal{H}, \Phi)^{\otimes}$	projective system of quantum state spaces	def. 5.1
$(\mathcal{L}, \mathcal{M}, \pi, \delta)^{\text{DYN}}$	elementary reduction of $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^{\downarrow}$	def. 3.7
$(\mathcal{L}, \mathcal{M}, \pi, \delta)^{\text{DYN},\mathcal{E}}$	regularized reduction of $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^{\downarrow}$	def. 3.16
A	smooth connection on $\boldsymbol{\mathcal{P}}$	def. 9.2
$\mathcal{A}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$	algebra of operators over $\overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$	def. 5.3
$\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$	C^{*}-algebra of operators over $\overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$	prop. 5.4
α	mapping of observables dual to σ	prop. 2.5
$B^{(d)}$	closed unit ball in $\mathbb{R}^{d}$	def. 9.7
$B(\mathcal{M})$	bounded functions on $\mathcal{M}$	def. A.2
$\boldsymbol{\mathcal{B}}$	prequantum bundle	def. B.2
$[A, B]$	commutators of two operators A and B	def. B.3
$\mathcal{C}$	configuration space	def. 2.15
$\mathcal{C}_{\Sigma}$	space of smooth connections on $\boldsymbol{\mathcal{P}}$	def. 9.2
$C^{\infty}(\mathcal{M}, \mathbb{R})$	space of smooth real-valued functions on $\mathcal{M}$	def. 2.4
$\mathcal{D}$	dense domain of a possibly unbounded operator	def. B.3
∇	covariant derivative	def. B.1
$D\nabla$	curvature 2-form	def. B.1
δ	symplectic reduction $\mathcal{M}^{\text{SHELL}} \to \mathcal{M}^{\text{DYN}}$	def. A.1
d	dimension of Σ	chap. 3
$\operatorname{div}_{\mu} X$	divergence of a vector field X with respect to the measure μ	def. B.12

Symbol	Description	Reference
E	smooth electric field	def. 9.3
$\mathcal{E}$	directed set indexing a regularization scheme	def. 3.16
$\check{e}$	encharted edge in Σ	def. 9.6
e	edge, ie. element of $\mathcal{L}_{\text{edges}}$	def. 9.6
$b(e)$	starting point of an edge e	def. 9.6
$f(e)$	ending point of an edge e	def. 9.6
$r(e)$	range of an edge e	def. 9.6
e^{-1}	reversed edge of e	prop. 10.1
$e_{[p,p']}$	subedge of an edge e	prop. 10.1
$e_2 \circ e_1$	composition of edges	prop. 10.2
F	face in a profile	prop. 10.9
$\mathcal{F}(\lambda)$	set of faces in a profile λ	prop. 10.9
$\diamond \in \left\{ \curvearrowright, \uparrow, \downarrow \right\}$	positioning of an edge with respect to a surface	prop. 10.6
$F_{\curvearrowright}(\lambda)$	set of edges indifferent to a profile λ	prop. 10.9
$\overline{F}$	free-standing face (ie. without reference to a profile)	prop. 10.21
$F^{\perp}$	set of edges indifferent to a face F	prop. 10.21
f^{DYN}	dynamical observable arising from the kinematical observable f	def. A.2
$\widehat{f}$	prequantization of an observable f	def. B.3
$\widehat{f}^{\mu}$	position quantization of an observable f	prop. B.14
G	finite-dimensional Lie group	chap. 3
$\boldsymbol{\mathcal{G}}$	$\boldsymbol{\mathcal{P}}$-associated vector bundle with fiber $\mathfrak{g}$	def. 9.2
$\mathfrak{g}$	Lie algebra of G	chap. 3
γ	graph	def. 10.3
$\mathcal{H}$	Hilbert space	def. 5.1
$\langle \cdot , \cdot \rangle_{\mathcal{H}}$	scalar product on $\mathcal{H}$	def. 5.1
$\mathcal{H}_{\text{preQ}}$	prequantum Hilbert space $L_2(\mathcal{M} \to \boldsymbol{\mathcal{B}})$	def. B.3
$\mathcal{H}_{\text{Holo}}$	Hilbert space of the holomorphic representation	prop. B.6
$\mathcal{H}^{\mu}_{\text{Pos}}$	Hilbert space of the position representation with measure μ	def. B.13
$\mathcal{H}_{\text{AL}}$	Ashtekar-Lewandowski Hilbert space	prop. 12.7
$\mathrm{h}^{(e,\boldsymbol{s},m)}$	holonomy along e as an observable on the continuum phase space $\mathcal{M}_{\Sigma}$	prop. 9.9
$\mathrm{h}^{(e,m)}$	holonomy along e as an observable on the classical projective state space $\mathcal{S}^{\downarrow}_{(\mathcal{L}_{\text{HF}},\mathcal{M},\pi)}$	prop. 10.27
$\widehat{\mathrm{h}^{(e,m)}}$	holonomy along e as an observable on the quantum projective state space $\mathcal{S}^{\otimes}_{(\mathcal{L}_{\text{HF}},\mathcal{H},\Phi)}$	prop. 12.2

Symbol	Description	Reference
$\widehat{\mathrm{h}_{\text{AL}}^{(e,m)}}$	holonomy along e as an operator on $\mathcal{H}_{\text{AL}}$	prop. 12.8
η	label, ie. element of $\mathcal{L}$	def. 2.3
J	complex structure on $\mathcal{M}$	def. B.5
$\mathcal{L}$	label set, ie. preordered, directed set	def. 2.3
$\preccurlyeq$	order on $\mathcal{L}$	def. 2.3
$\mathcal{L}_{\text{HF}}$	label set for the holonomy-flux algebra	def. 10.12
$\mathcal{L}_{\text{edges}}$	set of all analytical edges	def. 9.6
$\mathcal{L}_{\text{edges}}^{(k)}$	set of all C^k-differentiable edges	def. 12.3
$\mathcal{L}_{\text{faces}}$	set of all faces	prop. 10.21
$\mathcal{L}_{\text{graphs}}$	set of all analytical graphs	def. 10.3
$\mathcal{L}_{\text{profls}}$	set of all profiles	def. 10.10
$\mathcal{L}_{\text{surfcs}}$	set of all surfaces	def. 9.7
$\mathcal{L}_{\text{AL}}$	label set for $\mathcal{H}_{\text{AL}}$	prop. 12.4
$\mathcal{L}^{(\text{aux})}$	simplified one-dimensional version of $\mathcal{L}_{\text{HF}}$	prop. 18.10
$\mathcal{L}^{(\text{ext})}$	extended label set, *not* necessarily directed	def. 19.4
$\mathcal{L}^{[\kappa]}$	label set generated by a sequence $(\kappa_n)_{n\in\mathbb{N}}$	def. 19.7
$\mathfrak{L}$	Lie derivative	def. B.12
$[X, Y]$	Lie brackets of two vector fields X and Y	prop. 2.10
$[\,\cdot\,,\,\cdot\,]_{\mathfrak{g}}$	Lie brackets in $\mathfrak{g}$	theorem 6.2
λ	profile	def. 10.10
$\mathcal{M}, \mathcal{N}, \ldots$	symplectic manifolds	def. 2.1
$\mathcal{M}_\infty$	(possibly infinite dimensional) phase space of the continuum theory	def. 2.6
$\mathcal{M}_\Sigma$	continuum phase space for a theory of connections	def. 9.3
$\mathcal{M}^{\text{KIN}}$	kinematical phase space	def. A.1
$\mathcal{M}^{\text{SHELL}}$	constraint surface	def. A.1
$\mathcal{M}^{\text{DYN}}$	reduced phase space	def. A.1
$\mathcal{M}^{\text{FIX}}$	gauge fixing surface in $\mathcal{M}^{\text{SHELL}}$	prop. A.8
m	smooth function $G \to \mathbb{R}$ to be evaluated on holonomies	prop. 9.9
μ	smooth measure	def. B.12
$\varphi_* \mu$	push-forward measure	def. 6.1
$\varphi^* \upsilon$	pullback of the 1-form υ	def. 2.1
$\underline{\upsilon}$	vector determined from the 1-form υ via $\upsilon = \Omega_{\mathcal{M}}(\underline{\upsilon}, \cdot)$	def. 2.1
$\mathcal{O}_{\text{Holo},\mathbb{C}}$	observables compatibles with the holomorphic quantization	prop. B.7
$\mathcal{O}_{\text{Pos},\mathbb{C}}$	observables compatibles with the position quantization	prop. B.14

Symbol	Description	Reference
$\mathcal{O}_{\text{Pos}}$	real-valued observables compatibles with the position quantization	prop. B.14
$\mathcal{O}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)}$	space of observables over $\mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)}$	def. 2.4
$\mathcal{O}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$	space of observables over $\mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$	prop. 5.5
$\mathcal{P}$	momentum space	def. C.1
$\boldsymbol{\mathcal{P}}$	principal fiber bundle	subsection 9.1
$\mathsf{P}^{(S,\boldsymbol{s},u)}$	flux through S as an observable on the continuum phase space $\mathcal{M}_\Sigma$	prop. 9.9
$\mathsf{P}^{(\overline{F},u)}$	flux through $\overline{F}$ as an observable on the classical projective state space $\mathcal{S}^{\downarrow}_{(\mathcal{L}_{\text{HF}},\mathcal{M},\pi)}$	prop. 10.27
$\widehat{\mathsf{P}^{(\overline{F},u)}}$	flux through $\overline{F}$ as an observable on the quantum projective state space $\mathcal{S}^{\otimes}_{(\mathcal{L}_{\text{HF}},\mathcal{H},\Phi)}$	prop. 12.2
$\widehat{\mathsf{P}^{(\overline{F},u)}_{\text{AL}}}$	flux through $\overline{F}$ as an operator on $\mathcal{H}_{\text{AL}}$	prop. 12.9
$\boldsymbol{\pi}$	bundle projection	def. B.1
$\pi_{\eta'\to\eta}$	projection $\mathcal{M}_{\eta'} \to \mathcal{M}_\eta$ in a projective system of phase spaces	def. 2.3
$\varphi_{\eta'\to\eta}$	factorizing map $X_{\eta'} \to X_{\eta'\to\eta} \times X_\eta$ ($\eta \preccurlyeq \eta'$) in a classical factorizing system	def. 2.11
$\varphi_{\eta''\to\eta'\to\eta}$	three-spaces map $X_{\eta''\to\eta} \to X_{\eta''\to\eta'} \times X_{\eta'\to\eta}$ ($\eta \preccurlyeq \eta' \preccurlyeq \eta''$) in a classical factorizing system	def. 2.11
$\Phi_{\eta'\to\eta}$	factorizing map $\mathcal{H}_{\eta'} \to \mathcal{H}_{\eta'\to\eta} \otimes \mathcal{H}_\eta$ ($\eta \preccurlyeq \eta'$) in a projective system of quantum state spaces	def. 5.1
$\Phi_{\eta''\to\eta'\to\eta}$	three-spaces map $\mathcal{H}_{\eta''\to\eta} \to \mathcal{H}_{\eta''\to\eta'} \otimes \mathcal{H}_{\eta'\to\eta}$ ($\eta \preccurlyeq \eta' \preccurlyeq \eta''$) in a projective system of quantum state spaces	def. 5.1
ρ	density matrix, aka. (self-adjoint) positive semi-definite, traceclass operator or trace 1	def. 5.2
$\lVert\rho\rVert_1$	trace norm of ρ	lemma 5.10
Σ	finite-dimensional, analytic manifold (spatial slice)	chap. 3
$\mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)}$	classical projective state space	def. 2.3
$\mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$	quantum projective state space	def. 5.2
$\overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$	projective limit of spaces of non-negative traceclass operators	def. 5.2
$\hat{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$	space of all narrow quantum states (in the linear context)	def. 18.5
$\check{S}$	encharted surface in Σ	def. 9.7
S	surface, ie. element of $\mathcal{L}_{\text{surfcs}}$	def. 9.7
$r(S)$	range of a surface S	def. 9.7
$\boldsymbol{s}$	(local) cross-section of $\boldsymbol{\mathcal{P}}$	def. 9.5
σ	map between classical or quantum projective state spaces	prop. 2.5
$T_x(\mathcal{M})$	tangent space of $\mathcal{M}$ at x	def. 2.1

Symbol	Description	Reference
$T_x f$	tangential map (aka. derivative) of f at x	def. 2.1
$\mathcal{T}^\infty(\mathcal{M})$	space of smooth vector fields on $\mathcal{M}$	prop. B.7
$\mathrm{Tr}_{\mathcal{H}}$	trace or partial trace over $\mathcal{H}$	def. 5.2
$\mathrm{Tr}_{\eta'\to\eta}$	partial trace in a projective system of quantum state spaces	def. 5.2
Tr	evaluation of an operator in a projective quantum state	def. 5.3
T_κ	weakly continuous representation of the Weyl algebra	def. C.3
$\mathcal{T}$	group of transformations	def. 19.5
$T \triangleright \cdot$	action of a group element $T \in \mathcal{T}$ on projective quantum states and observables	def. 19.5
U_η , $\mathrm{U}_{\eta'\to\eta}$	action of a group on a projective (pre)-system of quantum state spaces	def. 19.5
$\mathcal{V}^{\downarrow}_{(\mathcal{L},\,(\mathcal{C},\mathcal{P}),\,(Q,P))}$	space of projective families of covariance matrices (in the linear context)	def. 18.6
W_ρ	Wigner characteristic function of a density matrix ρ	prop. C.7
$\mathcal{W}$	space of Wigner characteristic functions	prop. C.6
$\mathcal{W}^{\downarrow}_{(\mathcal{L},(\mathcal{C},\mathcal{P}),(Q,P))}$	space of projective families of characteristic functions	def. 18.2
X_f	Hamiltonian vector field of f	prop. 2.10
$\overline{X}_f$	Hamiltonian vector field of f projected on the configuration space	prop. B.14
Ξ	map $\mathcal{P} \to \mathcal{C}^*$ arising from Ω in the linear context	def. C.1
$\mathrm{X}_\kappa(s, t)$	generator in a weakly continuous representation of the Weyl algebra	prop. C.4
χ_η	matching of edges to their conjugate faces in a label $\eta \in \mathcal{L}_{\text{HF}}$	def. 10.12
$\Omega_{\mathcal{M}}$	symplectic structure on $\mathcal{M}$	def. 2.1
$\{\cdot,\cdot\}_{\mathcal{M}}$	Poisson brackets on $\mathcal{M}$	prop. 2.2
ω	volume form	theorem 6.2
$\zeta_{\eta'\to\eta}$	factorizing map $\boldsymbol{\mathcal{B}}_{\eta'\to\eta} \times \boldsymbol{\mathcal{B}}_\eta \to \boldsymbol{\mathcal{B}}_{\eta'}$ ($\eta \preccurlyeq \eta'$) in a factorizing system of prequantum bundles	def. 6.8
$\zeta_{\eta''\to\eta'\to\eta}$	three-spaces map $\boldsymbol{\mathcal{B}}_{\eta''\to\eta'} \times \boldsymbol{\mathcal{B}}_{\eta'\to\eta} \to \boldsymbol{\mathcal{B}}_{\eta''\to\eta}$ ($\eta \preccurlyeq \eta' \preccurlyeq \eta''$) in a factorizing system of prequantum bundles	def. 6.8

Projective Limits of State Spaces

Chapter 1 – Classical Formalism

1. Introduction

The aim of this chapter is to describe the classical structures that, while underlying the constructions considered in previous works [48, 66, 68], have not been explicitly analyzed so far. The discussion of the physical interpretation will follow closely the one that has been given in these references.

The idea of the projective framework is to assemble a complicated classical theory (typically a field theory) from a collection of easier, smaller, truncated classical theories, by appropriately sewing them together. The motivation for this is twofold.

From the physical point of view, even when considering a theory with an infinite number of degrees of freedom, any given realistic experiment will involve only a finite number of observables, since measuring an infinite number of observables would require infinite time as well as infinite memory space (in fact, this means that any experiment can only measure a finite number of *boolean* observables, but we will not be that radical here, and will satisfy us with small truncated theories that are described by finite dimensional phase spaces). We will therefore think of the small partial theories as spanned by a finite number of elementary degrees of freedom. By "elementary", we mean those that can be measured in one experimental step, hence the justification for the choice of a collection of truncations should ultimately come from a careful analysis of what concrete experiments actually measure.

From a technical point of view, the smaller and easier theories are meant to be a convenient arena to develop systematic ways of calculating physical predictions. Indeed, a theoretical model will then be optimally useful if it comes with finite algorithms prescribing how to compute, at a given precision, the outcome of any arbitrary experiment.

Note however that the intuitive understanding just sketched has some weak points. One of them is that, even if we are considering only finitely many observables, it might occur that the Poisson-algebra they are generating cannot live on a finite dimensional symplectic manifold. Our viewpoint here is that this problem should not be relevant for the *kinematical* observables (these are supposed to build an easy algebra).

Another problem is related to the formulation of deterministic predictions while considering only finitely many degrees of freedom out of a field theory. In section 3, we will therefore refine the framework to take into account the dynamics. If the equations of motion can be solved separately within each building block of the kinematical state space, it is straightforward to obtain their space of solutions, aka. the dynamical state space (subsection 3.1). Yet, this typically does not hold (we can rarely write the exact dynamics in closed form within a finite truncation of a field theory). So, we will have to *regularize* these equations over the kinematical projective structure, leading us to consider a sequence of finer and finer regularizations (subsection 3.2). The key idea will then be to *define* each dynamical state as a family of successive approximations, converging to some exact solution of the field equations. In this way, the dynamical state space will naturally acquire a projective structure, with projections mapping spaces of solutions obtained for more accurate approximations of the dynamics into the ones obtained for lesser approximations (such projections are reminiscent of the *coarse graining maps*, which play a prominent role in covariant approaches [59]).

2. Projective limits of classical phase spaces

Having a collection of partial theories is not enough, we need to say how to connect them together in a consistent way (ie. we do not want our physical predictions to depend on the particular partial theory in which we computed them). To this intend we will, in subsection 2.1, set up projective limits of classical phase spaces (aka. of symplectic manifolds), and relate them to the infinite dimensional phase spaces of classical field theories (subsection 2.2). An important observation is that such projective limits admit, at least locally, a preferred factorized description (prop. 2.10). Therefore, we will look more closely at those projective systems where the factorization holds globally: not only they are often more convenient, they also reflect the core properties of the structures we are considering, so they are well-suited to get a first hold of complex questions. This will be in particular comfortable when turning to the quantum formalism in chap. 2, but we will always try to sketch some ideas on how to strengthen those of our results that make explicit use of such a global factorization (see in particular section 7 on this point).

2.1 Projective systems of classical phase spaces

To understand how a projective system can be assembled from truncations of a classical theory, we first consider two partial theories $\mathcal{M}$ and $\mathcal{N}$, where $\mathcal{M}$ is a more detailed description of the physical system at hand, in the sense that all degrees of freedom that are retained by $\mathcal{N}$ are also retained by $\mathcal{M}$. The link between them has then two dual aspects. On the one side, we want to associate, with any state in $\mathcal{M}$, a state in $\mathcal{N}$, by forgetting the details we presently do not need. On the other side, we want to identify the observables that can be defined on $\mathcal{N}$ with a subalgebra of the ones that can be defined on $\mathcal{M}$.

Given a specific experiment, any partial theory big enough to describe that experiment (ie hosting at least all the observables involved in it) should lead to the same predictions. In other words, the two identifications mentioned above (downward identification of the states and upward identification of the observables) should intertwine the evaluation of an observable on a state.

These considerations lead to the following formulation of how some degrees of freedom, spanning a symplectic manifold $\mathcal{N}$, can be seen as being extracted out of a bigger symplectic manifold $\mathcal{M}$: what we need is a projection $\pi : \mathcal{M} \to \mathcal{N}$, and we will mount observables on $\mathcal{N}$ to observables on $\mathcal{M}$ by taking their pullback. We impose a compatibility condition between the projection π and the symplectic structures of $\mathcal{M}$ and $\mathcal{N}$ to ensure that the Poisson bracket computed between two observables in $\mathcal{N}$ is identified with the one computed between the corresponding observables mounted in $\mathcal{M}$.

Definition 2.1 A smooth, surjective map $\pi : \mathcal{M} \to \mathcal{N}$ between two smooth (possibly infinite dimensional) symplectic manifolds $\mathcal{M}, \Omega_{\mathcal{M}}$ and $\mathcal{N}, \Omega_{\mathcal{N}}$ is said to be compatible with the symplectic structures iff:

$$\forall x \in \mathcal{M}, \forall \upsilon \in T'_{\pi(x)}(\mathcal{N}), \quad \underline{\upsilon} = T_x\pi\left(\underline{\pi^*\upsilon}\right) \tag{2.1.1}$$

where $T'_{\pi(x)}(\mathcal{N})$ is the topological dual of $T_{\pi(x)}(\mathcal{N})$, $T_x\pi$ is the differential of π at x, and $\underline{\upsilon}$ (resp. $\underline{\pi^*\upsilon}$) is the unique vector in $T_{\pi(x)}(\mathcal{N})$ (resp. $T_x(\mathcal{M})$) such that $\upsilon = \Omega_{\mathcal{N},\pi(x)}(\underline{\upsilon}, \cdot)$ (resp. $\pi^*\upsilon = \Omega_{\mathcal{M},x}(\underline{\pi^*\upsilon}, \cdot)$).

Proposition 2.2 If $\pi : \mathcal{M} \to \mathcal{N}$ satisfies def. 2.1 and $f, g : \mathcal{N} \to \mathbb{R}$ are smooth maps on $\mathcal{N}$, then $\{f, g\}_{\mathcal{N}} \circ \pi = \{f \circ \pi, g \circ \pi\}_{\mathcal{M}}$ where $\{\cdot, \cdot\}_{\mathcal{N}}$ (resp. $\{\cdot, \cdot\}_{\mathcal{M}}$) denotes the Poisson brackets on $\mathcal{N}$ (resp. $\mathcal{M}$).

Proof Eq. (2.1.1) is equivalent to:

$$\forall x \in \mathcal{M}, \forall \mu, \upsilon \in T^*_{\pi(x)}(\mathcal{N}),\ \mu\left(\underline{\upsilon}\right) = \pi^*\mu\left(\underline{\pi^*\upsilon}\right).$$

Using the definition of the Poisson brackets, we therefore have:

$$\forall x \in \mathcal{M},\ \{f, g\}_{\mathcal{N}} \circ \pi(x) = dg_{\pi(x)}\left(\underline{df_{\pi(x)}}\right) = d(g \circ \pi)_x\left(\underline{d(f \circ \pi)_x}\right) = \{f \circ \pi, g \circ \pi\}_{\mathcal{M}}(x).$$

□

Next, the collection of partial theories, together with the projections between them, can be arranged into a structure of projective limit. Such a construction has been considered for example in [89].

That the label set $\mathcal{L}$ indexing the partial theories should be directed is manifest if we go back to the interpretation of these small theories as the arenas to describe specific experiments: if we want to describe an elaborate experimental protocol, combining two sub-experiments, that can be described respectively in $\mathcal{M}_\eta$ and $\mathcal{M}_{\eta'}$, we need a symplectic manifold $\mathcal{M}_{\eta''}$, containing the degrees

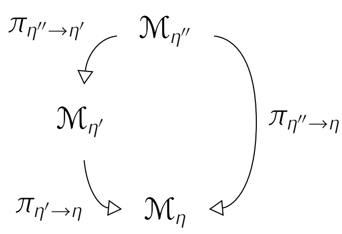


Figure 2.1 – Three-spaces consistency for projective systems of phase spaces

of freedom in $\mathcal{M}_\eta$ as well as the ones in $\mathcal{M}_{\eta''}$, in order to model the full experiment. And the three-spaces consistency condition (fig. 2.1) ensures that the connection between a bigger partial theory $\mathcal{M}_{\eta''}$ and a smaller one $\mathcal{M}_\eta$ is unambiguous, namely that it coincides with the identification we get if we perform the truncation in two successive steps, going first from $\mathcal{M}_{\eta''}$ to an intermediary $\mathcal{M}_{\eta'}$ and then from $\mathcal{M}_{\eta'}$ to $\mathcal{M}_\eta$.

With this structure for the state space, the observables naturally build an inductive limit, which is consistent with the discussion above regarding the mounting of observables and indeed corresponds to the standard construction when looking at functions on a projective limit.

Definition 2.3 A projective system of phase spaces is a triple $\left(\mathcal{L}, \left(\mathcal{M}_\eta\right)_{\eta\in\mathcal{L}}, \left(\pi_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'}\right)$ where:

1. $\mathcal{L}$ is a preordered, directed set (we denote the pre-order, ie. a reflexive and transitive binary relation, by $\preccurlyeq$, its inverse by $\succcurlyeq$);
2. $\left(\mathcal{M}_\eta\right)_{\eta\in\mathcal{L}}$ is a family of symplectic manifolds indexed by $\mathcal{L}$;
3. $\left(\pi_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'}$ is a family of surjective maps $\pi_{\eta'\to\eta} : \mathcal{M}_{\eta'} \to \mathcal{M}_\eta$ indexed by $\{\eta, \eta' \in \mathcal{L} \mid \eta \preccurlyeq \eta'\}$ such that $\pi_{\eta'\to\eta}$ is compatible with the symplectic structures, $\pi_{\eta\to\eta} = \mathrm{id}_{\mathcal{M}_\eta}$ and $\forall \eta, \eta', \eta'' \in \mathcal{L}$, $\eta \preccurlyeq \eta' \preccurlyeq \eta'' \Rightarrow \pi_{\eta''\to\eta} = \pi_{\eta'\to\eta} \circ \pi_{\eta''\to\eta'}$.

Whenever possible, we will use the shortened notation $(\mathcal{L}, \mathcal{M}, \pi)^\downarrow$ instead of $\left(\mathcal{L}, \left(\mathcal{M}_\eta\right)_{\eta\in\mathcal{L}}, \left(\pi_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'}\right)$.

The projective limit of $(\mathcal{L}, \mathcal{M}, \pi)^\downarrow$, denoted by $\mathcal{S}^\downarrow_{(\mathcal{L},\mathcal{M},\pi)}$, is the space:

$$\mathcal{S}^\downarrow_{(\mathcal{L},\mathcal{M},\pi)} := \left\{ (x_\eta)_{\eta\in\mathcal{L}} \in \prod_{\eta\in\mathcal{L}} \mathcal{M}_\eta \;\middle|\; \forall \eta \preccurlyeq \eta',\ \pi_{\eta'\to\eta}(x_{\eta'}) = x_\eta \right\}.$$

On $\mathcal{S}^\downarrow_{(\mathcal{L},\mathcal{M},\pi)}$ we put the initial topology with respect to the family of projections $\left(\pi_\eta\right)_{\eta\in\mathcal{L}}$ where:

$$\begin{aligned} \pi_\eta :\ & \mathcal{S}^\downarrow_{(\mathcal{L},\mathcal{M},\pi)} \to \mathcal{M}_\eta \\ & (x_{\eta'})_{\eta'\in\mathcal{L}} \mapsto x_\eta =: \left[(x_{\eta'})_{\eta'\in\mathcal{L}}\right]_\eta \end{aligned}.$$

Definition 2.4 An observable over a projective limit of phase spaces $\mathcal{S}^\downarrow_{(\mathcal{L},\mathcal{M},\pi)}$ is an equivalence class in $\bigcup_{\eta\in\mathcal{L}} C^\infty(\mathcal{M}_\eta, \mathbb{R})$ for the equivalence relation defined by:

$$\forall \eta, \eta' \in \mathcal{L},\ \forall f_\eta \in C^\infty(\mathcal{M}_\eta, \mathbb{R}),\ \forall f_{\eta'} \in C^\infty(\mathcal{M}_{\eta'}, \mathbb{R}),$$

$$f_\eta \sim f_{\eta'} \Leftrightarrow \left(\exists \eta'' \in \mathcal{L} \,/\, \eta \preccurlyeq \eta'', \eta' \preccurlyeq \eta'' \quad \& \quad f_\eta \circ \pi_{\eta'' \to \eta} = f_{\eta'} \circ \pi_{\eta'' \to \eta'}\right) \tag{2.4.1}$$

The space of observables over $\mathcal{S}^\downarrow_{(\mathcal{L},\mathcal{M},\pi)}$ will be denoted by $\mathcal{O}^\downarrow_{(\mathcal{L},\mathcal{M},\pi)}$. The definition of the equivalence relation ensures that the evaluation $f(x) = f_\eta(x_\eta)$ of an element of $f = [f_\eta]_\sim$ of $\mathcal{O}^\downarrow_{(\mathcal{L},\mathcal{M},\pi)}$ on a point $x = (x_\eta)_{\eta \in \mathcal{L}}$ in $\mathcal{S}^\downarrow_{(\mathcal{L},\mathcal{M},\pi)}$ is well-defined. From prop. 2.2 the Poisson bracket of two elements of $\mathcal{O}^\downarrow_{(\mathcal{L},\mathcal{M},\pi)}$ is well-defined as an element of $\mathcal{O}^\downarrow_{(\mathcal{L},\mathcal{M},\pi)}$ ($\forall \eta' \succcurlyeq \eta$, $f_\eta \circ \pi_{\eta' \to \eta} \in [f_\eta]_\sim$, hence, $\mathcal{L}$ being directed, we can find a common label to compute the Poisson bracket).

2.2 Maps between classical state spaces

A question that occurs frequently when working with the structure introduced above, is to ask what happens if we restrict ourselves to a directed subset $\mathcal{L}'$ of the label set $\mathcal{L}$. It is immediate that a state $\left(x_\eta\right)_{\eta \in \mathcal{L}}$ in the projective structure based on $\mathcal{L}$ defines a state $\left(x_\eta\right)_{\eta \in \mathcal{L}'}$ in the one based on $\mathcal{L}'$, simply by throwing away all the x_η for $\eta \in \mathcal{L} \setminus \mathcal{L}'$. But this map from $\mathcal{S}^\downarrow_{(\mathcal{L},\mathcal{M},\pi)}$ into $\mathcal{S}^\downarrow_{(\mathcal{L}',\mathcal{M},\pi)}$ will in general neither be injective nor surjective.

The injectivity might fail because the structure based on $\mathcal{L}'$ retains less observables than the structure based on $\mathcal{L}$, and states that can, thanks to these additional observables, be distinguished in the latter may be indistinguishable in the former. That also the surjectivity might fails is more subtle: it can occur if $\mathcal{L}$ has a label η that is above an infinite number of labels in $\mathcal{L}'$. Then, given a state $\left(x_\eta\right)_{\eta \in \mathcal{L}'}$ in $\mathcal{S}^\downarrow_{(\mathcal{L}',\mathcal{M},\pi)}$, it may indeed not be possible to find an x_η that will project correctly on all the $x_{\eta'}$ for $\eta' \in \mathcal{L}$ with $\eta' \preccurlyeq \eta$.

In the particular case of $\mathcal{L}'$ being cofinal in $\mathcal{L}$, we can however completely identify the two projective structure, since we can reconstruct any thrown away x_η for $\eta \in \mathcal{L} \setminus \mathcal{L}'$ by projecting down from some $\eta' \in \mathcal{L}'$ above η.

Proposition 2.5 Let $(\mathcal{L}, \mathcal{M}, \pi)^\downarrow$ be a projective system of phase spaces and let $\mathcal{L}'$ be a directed subset of $\mathcal{L}$. We define the map:

$$\begin{aligned} \sigma : \mathcal{S}^\downarrow_{(\mathcal{L},\mathcal{M},\pi)} &\to \mathcal{S}^\downarrow_{(\mathcal{L}',\mathcal{M},\pi)} \\ \left(x_\eta\right)_{\eta \in \mathcal{L}} &\mapsto \left(x_\eta\right)_{\eta \in \mathcal{L}'} \end{aligned}.$$

Then, we have a map $\alpha : \mathcal{O}^\downarrow_{(\mathcal{L}',\mathcal{M},\pi)} \to \mathcal{O}^\downarrow_{(\mathcal{L},\mathcal{M},\pi)}$ such that:

$$\forall x \in \mathcal{S}^\downarrow_{(\mathcal{L},\mathcal{M},\pi)}, \forall f \in \mathcal{O}^\downarrow_{(\mathcal{L}',\mathcal{M},\pi)},\ \alpha(f)(x) = f(\sigma(x)), \tag{2.5.1}$$

and:

$$\forall f,\, g \in \mathcal{O}^\downarrow_{(\mathcal{L}',\mathcal{M},\pi)},\ \{\alpha(f),\, \alpha(g)\} = \alpha\left(\,\{f,\, g\}\,\right). \tag{2.5.2}$$

If $\mathcal{L}'$ is cofinal in $\mathcal{L}$, we have in addition that σ and α are bijective maps.

Proof It's an immediate check that σ is indeed valued in $\mathcal{S}^{\downarrow}_{(\mathcal{L}',\mathcal{M},\pi)}$.

Then, for $f = [f_\eta]_{\sim,\mathcal{L}'} \in \mathcal{O}^{\downarrow}_{(\mathcal{L}',\mathcal{M},\pi)}$, we define:

$\alpha(f) = [f_\eta]_{\sim,\mathcal{L}}$ for f_η a representative of f.

We have:

$$\forall \eta, \eta' \in \mathcal{L}',\ \forall f_\eta \in C^\infty(\mathcal{M}_\eta\,,\ \mathbb{R}), \forall f_{\eta'} \in C^\infty(\mathcal{M}_{\eta'}\,,\ \mathbb{R}),\ f_\eta \sim_{\mathcal{L}'} f_{\eta'} \quad \Rightarrow \quad f_\eta \sim_{\mathcal{L}} f_{\eta'}\,,$$

hence α is well-defined as a map $\mathcal{O}^{\downarrow}_{(\mathcal{L}',\mathcal{M},\pi)} \rightarrow \mathcal{O}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)}$.

Eq. (2.5.1) and eq. (2.5.2) hold because we can choose any representative we want to carry out the evaluation or to compute the Poisson brackets.

We now suppose that $\mathcal{L}$ is cofinal. Then, we can define:

$$\begin{array}{rl}\widetilde{\sigma} : \mathcal{S}^{\downarrow}_{(\mathcal{L}',\mathcal{M},\pi)} & \rightarrow \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)} \\ \left(x_\eta\right)_{\eta\in\mathcal{L}'} & \mapsto \left(\widetilde{x}_\eta\right)_{\eta\in\mathcal{L}}\end{array},$$

where for $\eta \in \mathcal{L}$, $\widetilde{x}_\eta = \pi_{\eta'\to\eta}\, x_{\eta'}$, with $\eta' \in \mathcal{L}'$ and $\eta' \succcurlyeq \eta$. If η'' is an other element of $\mathcal{L}'$ such that $\eta'' \succcurlyeq \eta$, there exists $\eta''' \in \mathcal{L}' \,/\, \eta' \preccurlyeq \eta''' \quad \& \quad \eta'' \preccurlyeq \eta'''$ ($\mathcal{L}'$ is directed by hypothesis), hence:

$$\pi_{\eta'\to\eta}\, x_{\eta'} = \pi_{\eta'\to\eta}\, \pi_{\eta'''\to\eta'}\, x_{\eta'''} = \pi_{\eta'''\to\eta}\, x_{\eta'''} = \pi_{\eta''\to\eta}\, \pi_{\eta'''\to\eta''}\, x_{\eta'''} = \pi_{\eta''\to\eta}\, x_{\eta''}\,.$$

If $\eta \in \mathcal{L}'$, we can choose $\eta = \eta'$, so that $\widetilde{x}_\eta = x_\eta$, therefore $\sigma \circ \widetilde{\sigma} = \mathrm{id}_{\mathcal{S}^{\downarrow}_{(\mathcal{L}',\mathcal{M},\pi)}}$. On the other hand, if there exists an element $\left(\overline{x}_\eta\right)_{\eta\in\mathcal{L}} \in \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)}$, such that $\forall \eta' \in \mathcal{L}'$, $x_{\eta'} = \overline{x}_{\eta'}$, then $\widetilde{x}_\eta = \pi_{\eta'\to\eta}\, \overline{x}_{\eta'} = \overline{x}_\eta$, therefore $\widetilde{\sigma} \circ \sigma = \mathrm{id}_{\mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)}}$.

Then, for $f = [f_\eta]_{\sim,\mathcal{L}} \in \mathcal{O}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)}$, we define:

$$\widetilde{\alpha}(f) = [f_\eta \circ \pi_{\eta'\to\eta}]_{\sim,\mathcal{L}'}\,,$$

for f_η a representative of f and $\eta' \in \mathcal{L}'$ such that $\eta' \succcurlyeq \eta$. If η'' is an other element of $\mathcal{L}'$ such that $\eta'' \succcurlyeq \eta$, there exists $\eta''' \in \mathcal{L}' \,/\, \eta' \preccurlyeq \eta''' \quad \& \quad \eta'' \preccurlyeq \eta'''$ ($\mathcal{L}'$ is directed by hypothesis), hence:

$$(f_\eta \circ \pi_{\eta'\to\eta}) \circ \pi_{\eta'''\to\eta'} = f_\eta \circ \pi_{\eta'''\to\eta} = (f_\eta \circ \pi_{\eta''\to\eta}) \circ \pi_{\eta'''\to\eta''}\,,$$

so that $f_\eta \circ \pi_{\eta'\to\eta} \sim_{\mathcal{L}'} f_\eta \circ \pi_{\eta''\to\eta}$.

If f_κ is an other representative of f, there exists $\mu \in \mathcal{L} \,/\, \mu \succcurlyeq \eta \quad \& \quad \mu \succcurlyeq \kappa$ such that $f_\eta \circ \pi_{\mu\to\eta} = f_\kappa \circ \pi_{\mu\to\kappa}$. Since $\mathcal{L}'$ is cofinal in $\mathcal{L}$, we can choose $\mu' \in \mathcal{L}$ such that $\mu' \succcurlyeq \mu$, and we have:

$$f_\eta \circ \pi_{\mu'\to\eta} = f_\eta \circ \pi_{\mu\to\eta} \circ \pi_{\mu'\to\mu} = f_\kappa \circ \pi_{\mu\to\kappa} \circ \pi_{\mu'\to\mu} = f_\kappa \circ \pi_{\mu'\to\kappa}\,,$$

hence $\widetilde{\alpha}$ is well-defined as a map $\mathcal{O}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)} \rightarrow \mathcal{O}^{\downarrow}_{(\mathcal{L}',\mathcal{M},\pi)}$.

If f_η is a representative of f with $\eta \in \mathcal{L}'$, we can choose $\eta' = \eta$, so that $\widetilde{\alpha}(f) = [f_\eta]_{\sim,\mathcal{L}'}$, therefore $\widetilde{\alpha} \circ \alpha = \mathrm{id}_{\mathcal{O}^{\downarrow}_{(\mathcal{L}',\mathcal{M},\pi)}}$. On the other hand, we have for all $\eta \in \mathcal{L}$ and all $\eta' \in \mathcal{L}$ with $\eta' \succcurlyeq \eta$, $[f_\eta \circ \pi_{\eta'\to\eta}]_{\sim,\mathcal{L}} = [f_\eta]_{\sim,\mathcal{L}}$, therefore $\alpha \circ \widetilde{\alpha} = \mathrm{id}_{\mathcal{O}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)}}$. □

We can now rewrite in terms of the concepts we have introduced the program that has been followed in [48, 66, 68]. When considering a field theory constructed on an infinite dimensional manifold $\mathcal{M}_\infty$, we will first, relying on our understanding of how physical effects are measured in practice, undertake to identify what the elementary observables should be, and try to construct a corresponding collection of interconnected partial theories, where each partial theory $\mathcal{M}_\eta$ will be associated to a finite subset of elementary observables (those that can be defined on $\mathcal{M}_\eta$, ie. that only depend on the degrees of freedom retained by $\mathcal{M}_\eta$). Since we have naturally a projection from $\mathcal{M}_\infty$ into each partial theory $\mathcal{M}_\eta$, we also immediately have a map from $\mathcal{M}_\infty$ into $\mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)}$.

If the set of all elementary observables separate the points of $\mathcal{M}_\infty$, then this map will be injective. Moreover, the projective limit of phase spaces will provide an extension of $\mathcal{M}_\infty$ in the sense that $\mathcal{M}_\infty$ can be identified with a dense subspace of $\mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)}$.

Note that choosing a collection of elementary observables is not the same as choosing preferred coordinates on $\mathcal{M}_\infty$, for the construction here does not require the elementary observables to be *independent*, they can form an overdetermined system. This can make physically a crucial difference: to illustrate this point, one can think at the set of elementary observables as analogous to the set of all the linear forms on a vector space, while preferred coordinates would correspond to the choice of a basis (compare the two examples given in chap. 5 of projective structures implemented along these lines: the model in section 15 relies on a choice of basis, while the one in section 16 does not). The set of all linear forms encodes nothing less but nothing more than the linear structure of the vector space, and this structure might indeed have a deep physical relevance, while we probably want to avoid relying on a preferred basis, in order not to break the invariance under isomorphisms.

Definition 2.6 We say that a (possibly infinite dimensional) symplectic manifold $\mathcal{M}_\infty$ is rendered by a projective system of phase spaces $(\mathcal{L},\mathcal{M},\pi)^{\downarrow}$ if for all $\eta \in \mathcal{L}$ there exists an application $\pi_{\infty\to\eta} : \mathcal{M}_\infty \to \mathcal{M}_\eta$ such that:

1. $\forall \eta \in \mathcal{L}$, $\pi_{\infty\to\eta}$ is surjective and compatible with the symplectic structures;
2. $\forall \eta \preccurlyeq \eta' \in \mathcal{L}$, $\pi_{\infty\to\eta} = \pi_{\eta'\to\eta} \circ \pi_{\infty\to\eta'}$.

Hence, we have a projective system of phase spaces $(\mathcal{L} \sqcup \{\infty\}, \mathcal{M}, \pi)^{\downarrow}$, where we extend the preorder of $\mathcal{L}$ to $\mathcal{L} \sqcup \{\infty\}$ by requiring $\forall \eta \in \mathcal{L},\ \infty \succ \eta$. From prop. 2.5, we have maps $\sigma_{\searrow}$: $\mathcal{S}^{\downarrow}_{(\mathcal{L}\sqcup\{\infty\},\mathcal{M},\pi)} \to \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)}$ and $\sigma_\infty^{-1} : \mathcal{S}^{\downarrow}_{(\{\infty\},\mathcal{M},\pi)} \to \mathcal{S}^{\downarrow}_{(\mathcal{L}\sqcup\{\infty\},\mathcal{M},\pi)}$ (since $\{\infty\}$ is cofinal in $\mathcal{L} \sqcup \{\infty\}$), so by identifying $\mathcal{S}^{\downarrow}_{(\{\infty\},\mathcal{M},\pi)}$ with $\mathcal{M}_\infty$, we define:

$$\sigma_{\downarrow} := \sigma_{\searrow} \circ \sigma_\infty^{-1} : \mathcal{M}_\infty \to \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)}\,.$$

Similarly, we have $\alpha_{\nwarrow} : \mathcal{O}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)} \to \mathcal{O}^{\downarrow}_{(\mathcal{L}\sqcup\{\infty\},\mathcal{M},\pi)}$ and $\alpha_\infty^{-1} : \mathcal{O}^{\downarrow}_{(\mathcal{L}\sqcup\{\infty\},\mathcal{M},\pi)} \to \mathcal{O}^{\downarrow}_{(\{\infty\},\mathcal{M},\pi)}$, so by identifying $\mathcal{O}^{\downarrow}_{(\{\infty\},\mathcal{M},\pi)}$ with $C^\infty(\mathcal{M}_\infty, \mathbb{R})$, we define:

$$\alpha_{\uparrow} := \alpha_\infty^{-1} \circ \alpha_{\nwarrow} : \mathcal{O}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)} \to C^\infty(\mathcal{M}_\infty, \mathbb{R})\,.$$

Proposition 2.7 With the notations of def. 2.6, $\sigma_{\downarrow}\langle\mathcal{M}_\infty\rangle$ is dense in $\mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)}$.

Proof Let $\left(x_\eta\right)_{\eta\in\mathcal{L}} \in \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)}$. For $\eta \in \mathcal{L}$, we choose $y^\eta \in \mathcal{M}_\infty$ such that $\pi_{\infty\to\eta}(y^\eta) = x_\eta$ (this is possible, since $\pi_{\infty\to\eta}$ is surjective). We have:

$$\forall \eta \in \mathcal{L},\ \forall \eta' \succcurlyeq \eta,\ \left[\sigma_{\downarrow}\left(y^{\eta'}\right)\right]_{\eta} = \pi_{\infty\to\eta}\left(y^{\eta'}\right) = \pi_{\eta'\to\eta} \circ \pi_{\infty\to\eta'}\left(y^{\eta'}\right) = \pi_{\eta'\to\eta}\left(x_{\eta'}\right) = x_{\eta}\,.$$

Hence, the net $\left(\sigma_{\downarrow}\left(y^{\eta'}\right)\right)_{\eta'\in\mathcal{L}}$ converges in $\mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)}$ to $\left(x_{\eta}\right)_{\eta\in\mathcal{L}}$, therefore $\left(x_{\eta}\right)_{\eta\in\mathcal{L}} \in \overline{\operatorname{Im}\sigma_{\downarrow}}$. □

We close this subsection by mentioning the construction of a different kind of maps between projective systems of phase spaces, that will be of interest when dealing with concrete examples. Indeed, we will often encounter the situation of having a projective system that has been originally constructed over a very large and complicated label set (in particular this can be a side-effect of the way we will handle constraints, as exhibited in section 3), but whose structure happens to be considerably simpler, because we can group the labels into classes partitioning $\mathcal{L}$, in such a way that the projective consistency conditions force a symplectomorphic identification between the manifolds $\mathcal{M}_{\eta}$ for all η belonging to the same class. Then, we probably want to define a label set $\mathcal{L}^{\bullet}$ by quotienting $\mathcal{L}$ according to those classes, and to identify the original projective system on $\mathcal{L}$ with an easier one built on $\mathcal{L}^{\bullet}$.

For example, suppose that the elements of $\mathcal{L}$ are pairs (ε, θ), ordered in the product order (aka. $(\varepsilon,\theta) \preccurlyeq (\varepsilon',\theta') \Leftrightarrow \varepsilon \preccurlyeq \varepsilon' \quad \& \quad \theta \preccurlyeq \theta$). Now, if it turns out that $\mathcal{M}_{(\varepsilon,\theta)}$ only depends on ε and $\pi_{(\varepsilon',\theta')\to(\varepsilon,\theta)}$ only on ε and ε', then the projective condition on the states will actually impose $x_{(\varepsilon,\theta_1)} = x_{(\varepsilon,\theta_2)}$. Thus this projective limit is in reality just a projective limit on the set of all ε.

This is a tool that we will use repeatedly in chap. 5 (and whose equivalent at the quantum level will be instrumental in the result of theorem 12.11).

Proposition 2.8 Let $\mathcal{L}$ and $\mathcal{L}^{\bullet}$ be directed preordered sets and assume that we are given:

1. a surjective map $\ell : \mathcal{L} \to \mathcal{L}^{\bullet}$ such that $\forall \eta \preccurlyeq \eta' \in \mathcal{L},\ \ell(\eta) \preccurlyeq \ell(\eta')$;
2. a projective system of phase spaces $(\mathcal{L}^{\bullet}, \mathcal{M}^{\bullet}, \pi^{\bullet})^{\downarrow}$ on $\mathcal{L}^{\bullet}$;
3. and for all $\eta \in \mathcal{L}$, a symplectic manifold $\mathcal{M}_{\eta}$ together with a symplectomorphism $\mu_{\eta} : \mathcal{M}_{\eta} \to \mathcal{M}^{\bullet}_{\ell(\eta)}$.

Then, defining for all $\eta \preccurlyeq \eta' \in \mathcal{L}$ the projection:

$$\pi_{\eta'\to\eta} := \mu_{\eta}^{-1} \circ \pi^{\bullet}_{\ell(\eta')\to\ell(\eta)} \circ \mu_{\eta'}\,, \tag{2.8.1}$$

$(\mathcal{L}, \mathcal{M}, \pi)^{\downarrow}$ is a projective system of phase spaces and the map:

$$\begin{array}{rccc} \kappa & : & \mathcal{S}^{\downarrow}_{(\mathcal{L}^{\bullet},\mathcal{M}^{\bullet},\pi^{\bullet})} & \to & \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)} \\ & & \left(x^{\bullet}_{\eta^{\bullet}}\right)_{\eta^{\bullet}\in\mathcal{L}^{\bullet}} & \mapsto & \left(\mu_{\eta}^{-1}\left(x^{\bullet}_{\ell(\eta)}\right)\right)_{\eta\in\mathcal{L}} \end{array}, \tag{2.8.2}$$

is bijective. Moreover, there exists a bijective map $\lambda : \mathcal{O}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)} \to \mathcal{O}^{\downarrow}_{(\mathcal{L}^{\bullet},\mathcal{M}^{\bullet},\pi^{\bullet})}$ such that:

$$\forall x^{\bullet} \in \mathcal{S}^{\downarrow}_{(\mathcal{L}^{\bullet},\mathcal{M}^{\bullet},\pi^{\bullet})},\ \forall f \in \mathcal{O}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)},\ \lambda(f)(x^{\bullet}) = f(\kappa(x^{\bullet}))\,. \tag{2.8.3}$$

Proof First, we check that κ is well-defined. Let $\left(x^{\bullet}_{\eta^{\bullet}}\right)_{\eta^{\bullet}\in\mathcal{L}^{\bullet}} \in \mathcal{S}^{\downarrow}_{(\mathcal{L}^{\bullet},\mathcal{M}^{\bullet},\pi^{\bullet})}$ and let $\eta \preccurlyeq \eta' \in \mathcal{L}$. We have $\ell(\eta) \preccurlyeq \ell(\eta')$ and from eq. (2.8.1):

$$\pi_{\eta'\to\eta}\left(\mu_{\eta'}^{-1}\left(x^\bullet_{\ell(\eta')}\right)\right) = \mu_\eta^{-1}\circ\pi^\bullet_{\ell(\eta')\to\ell(\eta)}\left(x^\bullet_{\ell(\eta')}\right) = \mu_\eta^{-1}\left(x^\bullet_{\ell(\eta)}\right),$$

hence $\left(\mu_\eta^{-1}\left(x^\bullet_{\ell(\eta)}\right)\right)_{\eta\in\mathcal{L}} \in \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)}$.

To prove that κ is bijective, we define:

$$\begin{array}{rcrcl} \widetilde{\kappa} & : & \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)} & \to & \mathcal{S}^{\downarrow}_{(\mathcal{L}^\bullet,\mathcal{M}^\bullet,\pi^\bullet)} \\ & & \left(x_\eta\right)_{\eta\in\mathcal{L}} & \mapsto & \left(x^\bullet_{\eta^\bullet}\right)_{\eta^\bullet\in\mathcal{L}^\bullet} \end{array},$$

where $\forall\eta^\bullet\in\mathcal{L}^\bullet$, $x^\bullet_{\eta^\bullet} := \mu_\eta\left(x_\eta\right)$ for any η such that $\ell(\eta)=\eta^\bullet$ (making use of the surjectivity of ℓ). $x^\bullet_{\eta^\bullet}$ does not depend on the choice of $\eta\in\ell^{-1}\left\langle\eta^\bullet\right\rangle$; indeed, if $\ell(\eta)=\ell(\eta')$, there exists $\eta''\in\mathcal{L}$ such that $\eta''\succcurlyeq\eta,\eta'$, hence:

$$\mu_\eta\left(x_\eta\right) = \mu_\eta\circ\pi_{\eta''\to\eta}\left(x_{\eta''}\right) = \pi^\bullet_{\ell(\eta'')\to\ell(\eta)}\circ\mu_{\eta''}\left(x_{\eta''}\right)$$

$$= \pi^\bullet_{\ell(\eta'')\to\ell(\eta')}\circ\mu_{\eta''}\left(x_{\eta''}\right) = \mu_{\eta'}\circ\pi_{\eta''\to\eta'}\left(x_{\eta''}\right) = \mu_{\eta'}\left(x_{\eta'}\right).$$

And by construction of $\widetilde{\kappa}$, we have $\kappa\circ\widetilde{\kappa} = \mathrm{id}_{\mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)}}$ as well as $\widetilde{\kappa}\circ\kappa = \mathrm{id}_{\mathcal{S}^{\downarrow}_{(\mathcal{L}^\bullet,\mathcal{M}^\bullet,\pi^\bullet)}}$.

Now, we define λ by:

$$\begin{array}{rcrcl} \lambda & : & \mathcal{O}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)} & \to & \mathcal{O}^{\downarrow}_{(\mathcal{L}^\bullet,\mathcal{M}^\bullet,\pi^\bullet)} \\ & & \left[f_\eta\right]_\sim & \mapsto & \left[f_\eta\circ\mu_\eta^{-1}\right]_\sim \end{array}.$$

λ is well-defined, for we have:

$$\begin{aligned} \forall\eta,\eta'\in\mathcal{L},\quad f_\eta\sim f_{\eta'} \quad &\Leftrightarrow \quad \left(\exists\eta''\succcurlyeq\eta',\eta \,/\, f_\eta\circ\pi_{\eta''\to\eta} = f_{\eta'}\circ\pi_{\eta''\to\eta'}\right) \\ &\Leftrightarrow \quad \left(\exists\eta''\succcurlyeq\eta',\eta \,/\, f_\eta\circ\mu_\eta^{-1}\circ\pi^\bullet_{\ell(\eta'')\to\ell(\eta)} = f_{\eta'}\circ\mu_{\eta'}^{-1}\circ\pi^\bullet_{\ell(\eta'')\to\ell(\eta')}\right) \\ &\Rightarrow \quad \left(f_\eta\circ\mu_\eta^{-1}\sim f_{\eta'}\circ\mu_{\eta'}^{-1}\right). \end{aligned}$$

And by construction of λ, eq. (2.8.3) is fulfilled.

Finally, to prove that λ is bijective, we construct a map $\widetilde{\lambda}$ by:

$$\begin{array}{rcrcl} \widetilde{\lambda} & : & \mathcal{O}^{\downarrow}_{(\mathcal{L}^\bullet,\mathcal{M}^\bullet,\pi^\bullet)} & \to & \mathcal{O}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)} \\ & & \left[f^\bullet_{\eta^\bullet}\right]_\sim & \mapsto & \left[f_\eta\right]_\sim \end{array},$$

where f_η is defined for any η such that $\ell(\eta)=\eta^\bullet$ by $f_\eta = f^\bullet_{\eta^\bullet}\circ\mu_\eta$. To check that $\widetilde{\lambda}$ is well-defined, let $\eta,\eta'\in\mathcal{L}$ such that there exist $f^\bullet_{\ell(\eta)}, f^\bullet_{\ell(\eta')}\in\left[f^\bullet_{\eta^\bullet}\right]_\sim$ (note that this also covers the case $\ell(\eta)=\ell(\eta')=\eta^\bullet$). Then, there exists $\eta^{\bullet\prime\prime}$ such that:

$$f^\bullet_{\ell(\eta)}\circ\pi^\bullet_{\eta^{\bullet\prime\prime}\to\ell(\eta)} = f^\bullet_{\ell(\eta')}\circ\pi^\bullet_{\eta^{\bullet\prime\prime}\to\ell(\eta')},$$

and, since ℓ is surjective, there exists $\eta''\in\mathcal{L}$ such that $\ell(\eta'')=\eta^{\bullet\prime\prime}$. Next, using that $\mathcal{L}$ is a directed set, there exists $\eta'''\in\mathcal{L}$ with $\eta'''\succcurlyeq\eta,\eta',\eta''$. Therefore, we have:

$$f^\bullet_{\ell(\eta)}\circ\pi^\bullet_{\ell(\eta''')\to\ell(\eta)} = f^\bullet_{\ell(\eta')}\circ\pi^\bullet_{\ell(\eta''')\to\ell(\eta')}$$

$$f^\bullet_{\ell(\eta)}\circ\mu_\eta\circ\pi_{\eta'''\to\eta} = f^\bullet_{\ell(\eta')}\circ\mu_{\eta'}\circ\pi_{\eta'''\to\eta'}$$

$$f^\bullet_{\ell(\eta)}\circ\mu_\eta\sim f^\bullet_{\ell(\eta')}\circ\mu_{\eta'}.$$

And by construction of $\widetilde{\lambda}$, we have $\widetilde{\lambda} \circ \lambda = \mathrm{id}_{\mathcal{O}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)}}$ as well as $\lambda \circ \widetilde{\lambda} = \mathrm{id}_{\mathcal{O}^{\downarrow}_{(\mathcal{L}^\bullet,\mathcal{M}^\bullet,\pi^\bullet)}}$. □

Proposition 2.9 The previous result still holds if, instead of requiring ℓ to be surjective, we simply require $\ell \langle \mathcal{L} \rangle$ to be a cofinal part of $\mathcal{L}^\bullet$.

Proof This follows by combining prop. 2.8 with prop. 2.5. □

2.3 Factorizing systems

For technical convenience, we will often specialize to a particular class of projective systems of phase phases, namely the situation where for any $\eta \preccurlyeq \eta'$, the symplectic manifold $\mathcal{M}_{\eta'}$ can be identified with the Cartesian product of $\mathcal{M}_\eta$ with a symplectic manifold $\mathcal{M}_{\eta' \to \eta}$ (in other words the discarded degrees of freedom can be collected into a phase space $\mathcal{M}_{\eta' \to \eta}$).

This restriction is in fact not as radical as one could first think, for given a projection $\pi : \mathcal{M} \to \mathcal{N}$ as in def. 2.1, $\mathcal{M}$ can always be *locally* written as a Cartesian product of symplectic manifolds in such a way that π correspond to the projection map on one factor of the product. Moreover, there is only one (local) decomposition having this property.

At the level of observables, writing $\mathcal{M}$ as a Cartesian product $\mathcal{N}^\perp \times \mathcal{N}$ implies that the algebra $\mathcal{O}'$ of all observables over $\mathcal{M}$ is generated by $\mathcal{O} \cup \mathcal{O}^\perp$, with $\mathcal{O}$ the subalgebra of $\mathcal{O}'$ defined by the observables over $\mathcal{N}$ and $\mathcal{O}^\perp$ by the ones over $\mathcal{N}^\perp$. And asking the symplectic structure on $\mathcal{M}$ to agree with the symplectic structure on the Cartesian product moreover requires that any observable in $\mathcal{O}$ Poisson-commutes with any observable in $\mathcal{O}^\perp$. This is the reason why, at least locally, the symplectic structure on $\mathcal{M}$ prescribes how to choose a subalgebra $\mathcal{O}^\perp$ completing $\mathcal{O}$: $\mathcal{O}^\perp$ has to be the set of all observables having vanishing Poisson brackets with any observable in $\mathcal{O}$.

To understand better why this factorization of $\mathcal{M}$ will not always hold globally, we can examine how the proof of prop. 2.10 below is done: what we have is a foliation of $\mathcal{M}$, of which each leaf is locally diffeomorphic to $\mathcal{N}$ via π. It is precisely when this local diffeomorphic identification fails to be a global one, that we will not get a global factorization. This can happen at two different levels. First, the restriction of π to a given leaf is not necessarily a covering map, although it is locally diffeomorphic: there can be 'completeness' issues, as exemplified by the ad hoc situation where $\mathcal{M} = \big\{(x_1, x_2; p_1, p_2) \,\big|\, |x_2| < \exp(x_1)\big\} \subset T^*(\mathbb{R}^2)$ and $\mathcal{N} = \big\{(x_1; p_1)\big\} \subset T^*(\mathbb{R})$. Second, a covering map need not be bijective, unless $\mathcal{N}$ is simply-connected: for example, if $\mathcal{M}$ is a symplectomorphic multiple cover of $\mathcal{N}$, it provides a projection that is compatible with the symplectic structures in the sense of def. 2.1, but there is no corresponding factorization.

Unless otherwise stated, all manifolds considered in the present subsection will be *finite* dimensional manifolds.

Proposition 2.10 Let $\mathcal{M}$, $\mathcal{N}$ be finite dimensional symplectic manifolds and suppose that there exists $\pi : \mathcal{M} \to \mathcal{N}$ satisfying def. 2.1.

Then, for $x \in \mathcal{M}$, there exist an open neighborhood $\mathcal{U}$ of x in $\mathcal{M}$, an open neighborhood $\mathcal{V}$ of $\pi(x)$ in $\mathcal{N}$, a manifold $\mathcal{W}$ and a symplectic structure $\Omega_\mathcal{W}$ on $\mathcal{W}$ such that there exists a diffeomorphism

$\Phi : \mathcal{V} \times \mathcal{W} \to \mathcal{U}$ satisfying $\forall y \in \mathcal{V}, \forall w \in \mathcal{W}$, $\pi \circ \Phi(y, w) = y$ and $\Phi^*\Omega_{\mathcal{M}} = \Omega_{\mathcal{N}} \times \Omega_{\mathcal{W}}$.

Moreover, Φ is unique in the following sense: if $\mathcal{U}'$ is an open subset of $\mathcal{U}$, $\mathcal{V}'$ is a connected open subspace of $\mathcal{V}$, $\mathcal{W}'$ is a symplectic manifold and $\Phi' : \mathcal{V}' \times \mathcal{W}' \to \mathcal{U}'$ is a symplectomorphism such that $\forall y \in \mathcal{V}', \forall z \in \mathcal{W}'$, $\pi \circ \Phi'(y, z) = y$, then there exists a symplectomorphism $\psi : \mathcal{W}' \to \mathcal{W}''$ (with $\mathcal{W}''$ an open subset of $\mathcal{W}$) such that $\forall y \in \mathcal{V}', \forall z \in \mathcal{W}'$, $\Phi'(y, z) = \Phi(y, \psi(z))$.

Proof *Existence.* We call $D = \dim(\mathcal{M})$, $d = \dim(\mathcal{N})$ and $n = D - d$. For all $x \in \mathcal{M}$, we define:

$$W_x = \{w \in T_x(\mathcal{M}) \mid T_x\pi(w) = 0\} \text{ and } V_x = \{v \in T_x(\mathcal{M}) \mid \forall w \in W_x,\ \Omega_{\mathcal{M},x}(v, w) = 0\}.$$

We have $\forall \upsilon \in T^*_{\pi(x)}(\mathcal{N}), \forall w \in W_x$, $\Omega_{\mathcal{M},x}(\underline{\pi^*\upsilon}, w) = \upsilon \circ T_x\pi(w) = 0$, hence $\forall \upsilon \in T^*_{\pi(x)}(\mathcal{N})$, $\underline{\pi^*\upsilon} \in V_x$. For $u \in T_x(\mathcal{M})$, we define $\upsilon_u = \Omega_{\mathcal{N},\pi(x)}(T_x\pi(u), \cdot)$. Using eq. (2.1.1), we get $\forall u \in T_x(\mathcal{M})$, $T_x\pi\left(\underline{\pi^*\upsilon_u}\right)$ $= \underline{\upsilon_u} = T_x\pi(u)$. So we can write $u \in T_x(\mathcal{M})$ as $u = (u - \underline{\pi^*\upsilon_u}) + \underline{\pi^*\upsilon_u}$ with $u - \underline{\pi^*\upsilon_u} \in W_x$ and $\underline{\pi^*\upsilon_u} \in V_x$.

Hence, we have $W_x + V_x = T_x(\mathcal{M})$, and therefore $W_x \oplus V_x = T_x(\mathcal{M})$, since $\dim(V_x) = \dim(\mathcal{M}) - \dim(W_x)$. Moreover, since eq. (2.1.1) implies that $T_x\pi$ is surjective, we have $\dim(W_x) = D - d$ and $\dim(V_x) = D - (D - d) = d$.

Now we choose $x \in \mathcal{M}$ and we consider a coordinate patch $\mathcal{V}_1$ on $\mathcal{N}$ containing $\pi(x)$, with coordinates $y_1, \ldots, y_d$. We define:

$$X_{i,x'} := \underline{\pi^* dy_{i,\pi(x')}} = \underline{d(y_i \circ \pi)_{x'}} \text{ for all } x' \in \mathcal{U}_1 := \pi^{-1}\langle V_1\rangle.$$

$X_1, \ldots, X_d$ are vector fields on $\mathcal{U}_1$ such that $\forall x' \in \mathcal{U}_1$, $(X_{1,x'}, \ldots, X_{d,x'})$ is a basis of $V_{x'}$. We calculate the Lie brackets between two of these vector fields:

$$\left[X_i, X_j\right] = \left[\underline{dy_i \circ \pi}, \underline{dy_j \circ \pi}\right] = \underline{d\left(\{y_i \circ \pi, y_j \circ \pi\}\right)} = \underline{d\left(\{y_i, y_j\} \circ \pi\right)} = \underline{\pi^* d\left(\{y_i, y_j\}\right)}$$

where the second equality expresses the Lie brackets of two Hamiltonian vector fields and the third equality comes from prop. 2.2.

Therefore, we have $\forall x' \in \mathcal{U}_1$, $\left[X_i, X_j\right]_{x'} \in V_{x'}$. From Frobenius theorem [54, theorem 14.5], there exist an open neighborhood $\mathcal{U}_2$ of x in $\mathcal{U}_1$ and coordinates $x_1, \ldots, x_d, x_{d+1}, \ldots, x_D$ over $\mathcal{U}_2$ such that $\forall x' \in \mathcal{U}_2$, $\partial_{x_1,x'}, \ldots, \partial_{x_d,x'}$ is a basis of $V_{x'}$. We define:

$$\begin{aligned}\widetilde{\Phi} : \mathcal{U}_2 &\to \mathcal{N} \times \mathbb{R}^n \\ x' &\mapsto \pi(x'), (x_{d+1}(x'), \ldots, x_D(x'))\end{aligned}.$$

We can now show that $T_x\widetilde{\Phi} : T_x(\mathcal{M}) \to T_{\pi(x)}(\mathcal{N}) \times \mathbb{R}^n$ is bijective. Indeed, let $u \in T_x(\mathcal{M})$ such that $T_x\widetilde{\Phi}(u) = 0$. Then, in particular, we have $T_x\pi(u) = 0$, so $u \in W_x$. On the other hand, we have $dx_{k,x}(u) = 0$ for $k = d + 1, \ldots, D$, so u is a linear combination of $\partial_{x_1,x}, \ldots, \partial_{x_d,x}$, hence $u \in V_x$. From $W_x \oplus V_x = T_x(\mathcal{M})$, $u = 0$. Therefore $T_x\widetilde{\Phi}$ is injective, thus bijective, for $\dim\left(T_x(\mathcal{M})\right) = \dim\left(T_{\pi(x)}(\mathcal{N}) \times \mathbb{R}^n\right)$.

From the inverse function theorem [54, theorem 5.11], there exists an open neighborhood $\mathcal{U}_3$ of x in $\mathcal{U}_2$ such that $\widetilde{\Phi}\Big|_{\mathcal{U}_3} : \mathcal{U}_3 \to \widetilde{\Phi}\langle\mathcal{U}_3\rangle$ is a diffeomorphism. Hence there exist an open connected neighborhood $\mathcal{V}$ of $\pi(x)$ in $\mathcal{N}$, an open subset $\mathcal{W}$ of $\mathbb{R}^n$, an open neighborhood $\mathcal{U}$ of x in $\mathcal{U}_3$ and a diffeomorphism $\Phi : \mathcal{V} \times \mathcal{W} \to \mathcal{U}$ such that $\forall y \in \mathcal{V}, \forall z \in \mathcal{W}$, $\widetilde{\Phi}(\Phi(y, z)) = (y, z)$. In particular,

$\forall y \in \mathcal{V}, \forall z \in \mathcal{W},\ \pi \circ \Phi(y, z) = y$.

At every point $x' \in \mathcal{M}$ and for every vector $v, w \in T_{\Phi^{-1}(x')}(\mathcal{V} \times \mathcal{W})$, $T_{\Phi^{-1}(x')}\Phi(v, 0) \in V_{x'}$ and $T_{\Phi^{-1}(x')}\Phi(0, w) \in W_{x'}$. In particular, we have $\Omega_{\mathcal{M},x'}\left(T_{\Phi^{-1}(x')}\Phi(v, 0),\ T_{\Phi^{-1}(x')}\Phi(0, w)\right) = 0$.

We now consider $z \in \mathcal{W}$, $w, w' \in T_z(\mathcal{W})$ and we define for all $y \in \mathcal{V}$,

$$\Omega^y_{\mathcal{W},z}(w, w') = \Omega_{\mathcal{M},\Phi(y,z)}(T_{(y,z)}\Phi(0, w), T_{(y,z)}\Phi(0, w')).$$

Let $\widetilde{Y}$ be a vector field on $\mathcal{V}$, let $\widetilde{Z}, \widetilde{Z}'$ be vector fields on $\mathcal{W}$ such that $\widetilde{Z}_z = w$, $\widetilde{Z}'_z = w'$. We define the vector fields $Y = \Phi_*\left(\widetilde{Y}, 0\right)$, $Z = \Phi_*\left(0, \widetilde{Z}\right)$ and $Z' = \Phi_*\left(0, \widetilde{Z}'\right)$ on $\mathcal{M}$. From $[Y,\ Z] = [Y,\ Z'] = 0$ and $d\Omega_{\mathcal{M}} = 0$, we have:

$$Y\left(\Omega_{\mathcal{M}}\left(Z, Z'\right)\right) = Z\left(\Omega_{\mathcal{M}}\left(Y, Z'\right)\right) - Z'\left(\Omega_{\mathcal{M}}\left(Y, Z\right)\right) + \Omega_{\mathcal{M}}\left([Z,\ Z'],\ Y\right)$$

since $Y_{x'} \in V_{x'}$ and $Z_{x'}, Z'_{x'}, [Z,\ Z']_{x'} \in W_{x'}$, we have $Y\left(\Omega_{\mathcal{M}}\left(Z, Z'\right)\right) = 0$. Therefore the differential of $y \mapsto \Omega^y_{\mathcal{W},z}(w, w')$ is zero at every point $y \in \mathcal{V}$, and $\mathcal{V}$ being connected, $\Omega^y_{\mathcal{W},z}(w, w')$ does not depend on y. So, we define $\Omega_{\mathcal{W},z}(w, w') = \Omega^y_{\mathcal{W},z}(w, w')$.

We now can check using eq. (2.1.1) and the definition of $\Omega_{\mathcal{W}}$ that $\Phi^*\Omega_{\mathcal{M}} = \Omega_{\mathcal{N}} \times \Omega_{\mathcal{W}}$. Therefore $\Omega_{\mathcal{W}}$ is a symplectic structure on $\mathcal{W}$ and $\Phi : \mathcal{V} \times \mathcal{W} \to \mathcal{U}$ is a symplectomorphism.

Uniqueness. We consider symplectic manifolds $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{W}'$, a connected open subset $\mathcal{V}'$ of $\mathcal{V}$, and an application $\widetilde{\psi} : \mathcal{V}' \times \mathcal{W}' \to \mathcal{W}$ such that:

$$\begin{aligned} \Psi : \mathcal{V}' \times \mathcal{W}' &\to \mathcal{V} \times \mathcal{W} \\ y, z &\mapsto y, \widetilde{\psi}(y, z) \end{aligned}$$

induces a symplectomorphism $\mathcal{V}' \times \mathcal{W}' \to \Psi\left\langle \mathcal{V}' \times \mathcal{W}' \right\rangle$.

For $y \in \mathcal{V}', z \in \mathcal{W}', v \in T_y(\mathcal{V}'), w \in T_z(\mathcal{W}')$, we then have:

$$0 = \Omega_{\mathcal{V}\times\mathcal{W},\Psi(y,z)}\left(T_{(y,z)}\Psi(v, 0),\ T_{(y,z)}\Psi(0, w)\right) = \Omega_{\mathcal{W},\psi(y,z)}\left(T_{(y,z)}\widetilde{\psi}(v, 0),\ T_{(y,z)}\widetilde{\psi}(0, w)\right).$$

However, for Ψ to be a diffeomorphism, $T_{(y,z)}\widetilde{\psi}(0, w)$ should run through $T_{\psi(y,z)}(\mathcal{W})$ when w runs through $T_z(\mathcal{W})$. Therefore, we should have $T_{(y,z)}\widetilde{\psi}(v, 0) = 0$. Hence, $\mathcal{V}'$ being connected, $\widetilde{\psi}(y, z)$ cannot depend upon y. Accordingly, we define $\psi(z) := \widetilde{\psi}(y, z)$, and $\Psi|_{\mathcal{V}'\times\mathcal{W}'\to\Psi\langle\mathcal{V}'\times\mathcal{W}'\rangle}$ being a symplectomorphism requires that $\psi|_{\mathcal{W}'\to\psi\langle\mathcal{W}'\rangle}$ should be a symplectomorphism.

Note. A more concise (albeit less instructive) proof of this result can be achieved by considering the closed 2-form $\sigma := \Omega_{\mathcal{M}} - \pi^*\,\Omega_{\mathcal{N}}$ and applying a standard result of symplectic geometry [84, § 5.24], telling us that the kernel of σ is an involutive distribution, and that σ defines a symplectic form on the quotient. □

In order to build a structure describing a collection of interconnected partial theories, where the relation between a more detailed partial theory $\mathcal{M}_{\eta'}$ and a less detailed one $\mathcal{M}_{\eta}$ is given by a factorization of $\mathcal{M}_{\eta'}$ as $\mathcal{M}_{\eta'\to\eta} \times \mathcal{M}_{\eta}$, we also need to reformulate the three-spaces consistency condition that we had for a projective system (fig. 2.1) in terms of a factorization requirement. For this, we ask for the symplectic manifold $\mathcal{M}_{\eta''\to\eta}$, that holds the degrees of freedom discarded when

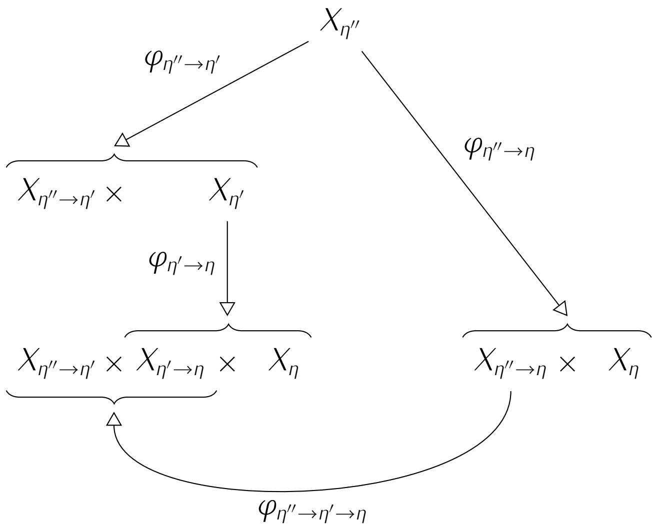


Figure 2.2 – Three-spaces consistency for factorizing systems

going directly from $\mathcal{M}_{\eta''}$ to $\mathcal{M}_\eta$, to decompose as the Cartesian product of $\mathcal{M}_{\eta''\to\eta'}$ with $\mathcal{M}_{\eta'\to\eta}$, where $\mathcal{M}_{\eta''\to\eta'}$ holds the degrees of freedom discarded when going as a first step from $\mathcal{M}_{\eta''}$ to $\mathcal{M}_{\eta'}$, and $\mathcal{M}_{\eta'\to\eta}$ holds the ones discarded when going as a second step from $\mathcal{M}_{\eta'}$ to $\mathcal{M}_\eta$ (fig. 2.2).

Having a factorizing system defined this way then provides us immediately with a projective system as above. Reciprocally, if we give us a projective system of phase spaces in which any projection $\pi_{\eta'\to\eta}$ can be understood as projecting on a factor of a Cartesian product (that is, if the result of prop. 2.10 happens to hold globally and not just locally) and if moreover all the $\mathcal{M}_\eta$ are connected (which sounds physically sensible when speaking of phases spaces), we can construct a corresponding factorizing system of phase spaces.

Definition 2.11 A factorizing system is a quintuple:

$$\left(\mathcal{L}, \left(X_\eta\right)_{\eta\in\mathcal{L}}, \left(X_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'}, \left(\varphi_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'}, \left(\varphi_{\eta''\to\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'\preccurlyeq\eta''}\right)$$

where:

1. $\mathcal{L}$ is a preordered, directed set;
2. $\left(X_\eta\right)_{\eta\in\mathcal{L}}$ is a family of spaces indexed by $\mathcal{L}$;
3. $\left(X_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'}$ is a family of spaces indexed by $\{\eta, \eta' \in \mathcal{L} \mid \eta \preccurlyeq \eta'\}$, such that, for all $\eta \in \mathcal{L}$, $X_{\eta\to\eta}$ has only one element;
4. $\left(\varphi_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'}$ is a family of bijective maps $\varphi_{\eta'\to\eta} : X_{\eta'} \to X_{\eta'\to\eta} \times X_\eta$ indexed by $\{\eta, \eta' \in \mathcal{L} \mid \eta \preccurlyeq \eta'\}$ such that $\varphi_{\eta\to\eta}$ is trivial;
5. $\left(\varphi_{\eta''\to\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'\preccurlyeq\eta''}$ is a family of bijective maps $\varphi_{\eta''\to\eta'\to\eta} : X_{\eta''\to\eta} \to X_{\eta''\to\eta'} \times X_{\eta'\to\eta}$ indexed by $\{\eta, \eta', \eta'' \in \mathcal{L} \mid \eta \preccurlyeq \eta' \preccurlyeq \eta''\}$, such that $\varphi_{\eta''\to\eta'\to\eta}$ is trivial whenever $\eta = \eta'$ or $\eta' = \eta''$, and:

$$\forall \eta, \eta', \eta'' \in \mathcal{L} \,/\, \eta \preccurlyeq \eta' \preccurlyeq \eta'', \quad (\varphi_{\eta''\to\eta'\to\eta} \times \mathrm{id}_{X_\eta}) \circ \varphi_{\eta''\to\eta} = (\mathrm{id}_{X_{\eta''\to\eta'}} \times \varphi_{\eta'\to\eta}) \circ \varphi_{\eta''\to\eta'} \,. \quad (2.11.1)$$

Whenever possible, we will use the shortened notation $(\mathcal{L}, X, \varphi)^\times$ instead of $\Big(\mathcal{L},\ \big(X_\eta\big)_{\eta\in\mathcal{L}}\,,\ \big(X_{\eta'\to\eta}\big)_{\eta\preccurlyeq\eta'}\,,$ $\big(\varphi_{\eta'\to\eta}\big)_{\eta\preccurlyeq\eta'}\,,\ \big(\varphi_{\eta''\to\eta'\to\eta}\big)_{\eta\preccurlyeq\eta'\preccurlyeq\eta''}\Big)$.

Definition 2.12 A factorizing system of phase spaces is a factorizing system $(\mathcal{L}, \mathcal{M}, \varphi)^\times$ where:

1. for all $\eta \in \mathcal{L}$, $\mathcal{M}_\eta$ is a symplectic manifold, and for all $\eta \preccurlyeq \eta' \in \mathcal{L}$, $\mathcal{M}_{\eta'\to\eta}$ is a symplectic manifold, except if $\eta' = \eta$ in which case $\mathcal{M}_{\eta\to\eta}$ is a set with just one element;
2. for all $\eta \preccurlyeq \eta' \in \mathcal{L}$, $\varphi_{\eta'\to\eta}$ is a symplectomorphism, and for all $\eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}$, $\varphi_{\eta''\to\eta'\to\eta}$ is a symplectomorphism.

Proposition 2.13 If $(\mathcal{L}, \mathcal{M}, \varphi)^\times$ fulfills def. 2.12 and if, for $\eta \preccurlyeq \eta' \in \mathcal{L}$, we define:

$$\begin{array}{rl} s_{\eta'\to\eta} : \mathcal{M}_{\eta'\to\eta} \times \mathcal{M}_\eta & \to \mathcal{M}_\eta \\ (y, x) & \mapsto x \end{array} \quad \text{and} \quad \pi_{\eta'\to\eta} = s_{\eta'\to\eta} \circ \varphi_{\eta'\to\eta} \qquad (2.13.1)$$

then $(\mathcal{L}, \mathcal{M}, \pi)^\downarrow$ is a projective system of phase spaces.

Accordingly, we define the space of states by $\mathcal{S}^\times_{(\mathcal{L},\mathcal{M},\varphi)} := \mathcal{S}^\downarrow_{(\mathcal{L},\mathcal{M},\pi)}$ (def. 2.3) and the space of observables by $\mathcal{O}^\times_{(\mathcal{L},\mathcal{M},\varphi)} := \mathcal{O}^\downarrow_{(\mathcal{L},\mathcal{M},\pi)}$ (def. 2.4).

Proof We need to prove that $\forall \eta \preccurlyeq \eta' \in \mathcal{L}$, $\pi_{\eta'\to\eta}$ is a surjective map compatible with the symplectic structures, that $\forall \eta \in \mathcal{L}$, $\pi_{\eta\to\eta} = \mathrm{id}_{\mathcal{M}_\eta}$, and that $\forall \eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}$, $\pi_{\eta''\to\eta} = \pi_{\eta'\to\eta} \circ \pi_{\eta''\to\eta'}$.

For $\eta \in \mathcal{L}$, we have $\pi_{\eta\to\eta} = \mathrm{id}_{\mathcal{M}_\eta}$ (identifying $\mathcal{M}_\eta$ and its trivial Cartesian product with a one-element set), so in particular it is a surjective map compatible with the symplectic structures.

Let $\eta \prec \eta' \in \mathcal{L}$. $\mathcal{M}_{\eta'\to\eta} \neq \varnothing$ (as a manifold), hence $s_{\eta'\to\eta}$ is surjective, therefore $\pi_{\eta'\to\eta}$ is a surjective map.

Let $(y, x) \in \mathcal{M}_{\eta'\to\eta} \times \mathcal{M}_\eta$ and let $\upsilon \in T^*_x(\mathcal{M}_\eta)$. We have:

$$\forall w, v \in T_{(y,x)}(\mathcal{M}_{\eta'\to\eta} \times \mathcal{M}_\eta),$$

$$\upsilon \circ \left[T_{(y,x)} s_{\eta'\to\eta}\right](w,\ v) = \upsilon(v) = \Omega_{\mathcal{M}_\eta, x}\left(\underline{\upsilon},\ v\right) = \Omega_{\mathcal{M}_{\eta'\to\eta}\times\mathcal{M}_\eta, (y,x)}\left((0, \underline{\upsilon}),\ (w, v)\right),$$

so that $\underline{s^*_{\eta'\to\eta}\upsilon} = (0, \underline{\upsilon})$, hence $\left[T_{(y,x)} s_{\eta'\to\eta}\right](\underline{s^*_{\eta'\to\eta}\upsilon}) = \underline{\upsilon}$. Therefore $s_{\eta'\to\eta}$ is compatible with the symplectic structures, and since $\varphi_{\eta'\to\eta}$ is a symplectomorphism, $\pi_{\eta'\to\eta}$ is compatible with the symplectic structures.

Let $\eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}$ and define:

$$\begin{array}{rl} s_{\eta''\to\eta'\to\eta} : \mathcal{M}_{\eta''\to\eta'} \times \mathcal{M}_{\eta'\to\eta} \times \mathcal{M}_\eta & \to \mathcal{M}_\eta \\ (z, y, x) & \mapsto x \end{array}.$$

We have:

$$s_{\eta''\to\eta'\to\eta} \circ \left(\mathrm{id}_{\mathcal{M}_{\eta''\to\eta'}} \times \varphi_{\eta'\to\eta}\right) = s_{\eta'\to\eta} \circ \varphi_{\eta'\to\eta} \circ s_{\eta''\to\eta'}\,,$$

$$\text{and} \quad s_{\eta''\to\eta'\to\eta} \circ \left(\varphi_{\eta''\to\eta'\to\eta} \times \mathrm{id}_{\mathcal{M}_\eta}\right) = s_{\eta''\to\eta}\,.$$

Hence, composing eq. (2.11.1) to the right with $s_{\eta''\to\eta'\to\eta}$ gives:

$$s_{\eta''\to\eta} \circ \varphi_{\eta''\to\eta} = s_{\eta'\to\eta} \circ \varphi_{\eta'\to\eta} \circ s_{\eta''\to\eta'} \circ \varphi_{\eta''\to\eta'} \,,$$

so that we have $\pi_{\eta''\to\eta} = \pi_{\eta'\to\eta} \circ \pi_{\eta''\to\eta'}$. □

Proposition 2.14 Let $(\mathcal{L}, \mathcal{M}, \pi)^{\downarrow}$ be a projective system of phase spaces and suppose that:

1. for all $\eta \in \mathcal{L}$, $\mathcal{M}_\eta$ is connected;
2. for all $\eta \prec \eta' \in \mathcal{L}$, there exist a symplectic manifold $\mathcal{M}_{\eta'\to\eta}$ and a symplectomorphism $\varphi_{\eta'\to\eta}$: $\mathcal{M}_{\eta'} \to \mathcal{M}_{\eta'\to\eta} \times \mathcal{M}_\eta$ such that $\pi_{\eta'\to\eta} = s_{\eta'\to\eta} \circ \varphi_{\eta'\to\eta}$.

Then, we can complete this input into a factorizing system $(\mathcal{L}, \mathcal{M}, \varphi)^{\times}$.

Proof For $\eta \in \mathcal{L}$, we define $\mathcal{M}_{\eta\to\eta}$ to be a space with one element and $\varphi_{\eta\to\eta}$ to be the trivial identification.

Let $\eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}$. What we need to show is that there exists a symplectomorphism $\varphi_{\eta''\to\eta'\to\eta} : \mathcal{M}_{\eta''\to\eta} \to \mathcal{M}_{\eta''\to\eta'} \times \mathcal{M}_{\eta'\to\eta}$ such that eq. (2.11.1) is fulfilled. If $\eta = \eta'$ or $\eta' = \eta''$, we can choose $\varphi_{\eta''\to\eta'\to\eta}$ to be the trivial identification, so we now consider the case $\eta \prec \eta' \prec \eta''$.

We define:

$$\Psi := \varphi_{\eta''\to\eta} \circ \varphi^{-1}_{\eta''\to\eta'} \circ \left(\mathrm{id}_{\eta''\to\eta'} \times \varphi_{\eta'\to\eta}\right)^{-1} : \mathcal{M}_{\eta''\to\eta'} \times \mathcal{M}_{\eta'\to\eta} \times \mathcal{M}_\eta \to \mathcal{M}_{\eta''\to\eta} \times \mathcal{M}_\eta \,.$$

Ψ is a symplectomorphism and satisfies:

$$\forall (z, y, x) \in \mathcal{M}_{\eta''\to\eta'} \times \mathcal{M}_{\eta'\to\eta} \times \mathcal{M}_\eta, \quad s_{\eta''\to\eta} \circ \Psi(z, y, x) = x.$$

Hence, applying the uniqueness part from the proof of prop. 2.10 (with $\mathcal{V} = \mathcal{V}' = \mathcal{M}_\eta$, $\mathcal{W} = \mathcal{M}_{\eta''\to\eta}$ and $\mathcal{W}' = \mathcal{M}_{\eta''\to\eta'} \times \mathcal{M}_{\eta'\to\eta}$, using that $\mathcal{M}_\eta$ is connected, as $\mathcal{V}'$ must be), there exists a symplectomorphism $\psi : \mathcal{M}_{\eta''\to\eta'} \times \mathcal{M}_{\eta'\to\eta} \to \mathcal{M}_{\eta''\to\eta}$ such that $\Psi = \psi \times \mathrm{id}_{\mathcal{M}_\eta}$. Thus we define $\varphi_{\eta''\to\eta'\to\eta} = \psi^{-1}$. □

If we have a family of finite dimensional symplectic manifolds, where each $\mathcal{M}_\eta$ modeling a partial theory can be written as a cotangent bundle on a configuration space $\mathcal{C}_\eta$, then a factorizing system built over the family $\left(\mathcal{C}_\eta\right)_{\eta\in\mathcal{L}}$ can automatically be lifted as a factorizing system over the family $\left(\mathcal{M}_\eta\right)_{\eta\in\mathcal{L}}$. Reciprocally, if we build a projective system of symplectic manifolds over this family, such that each projection can be understood as arising from a factorization of the underlying configuration spaces, and if additionally all the configuration spaces are connected, then not only can the projective system of symplectic manifolds be put into a factorizing form (as follows from prop. 2.14), but this factorizing form goes down to a factorizing system of the configuration spaces.

It is important to note that, at the level of configuration spaces, a factorizing system contains much more input than a projective system does. The situation here is different than what we have at the level of phase spaces, where projective and factorizing systems can, let aside global considerations, be matched unambiguously. The reason for this disparity is that the symplectic structure on the phase spaces played a crucial role in the proof of prop. 2.10: when looking at a projection between configuration spaces, that retains only a subset of the configuration variables, we have no additional structure that would allows us to select a preferred complementary set of discarded variables.

Definition 2.15 A factorizing system of smooth manifolds is a factorizing system $(\mathcal{L}, \mathcal{C}, \varphi)^{\times}$ (def. 2.11; in particular eq. (2.11.1) holds) where:

1. for all $\eta \in \mathcal{L}$, $\mathcal{C}_{\eta}$ is a smooth manifold, and for all $\eta \preccurlyeq \eta' \in \mathcal{L}$, $\mathcal{C}_{\eta' \to \eta}$ is a smooth manifold, except if $\eta' = \eta$ in which case $\mathcal{C}_{\eta \to \eta}$ is a set with just one element;
2. for all $\eta \preccurlyeq \eta' \in \mathcal{L}$, $\varphi_{\eta' \to \eta}$ is a diffeomorphism, and for all $\eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}$, $\varphi_{\eta'' \to \eta' \to \eta}$ is a diffeomorphism.

Proposition 2.16 If $(\mathcal{L}, \mathcal{C}, \varphi)^{\times}$ fulfills def. 2.15 and if:

1. for all $\eta \in \mathcal{L}$ (resp. all $\eta, \eta' \in \mathcal{L}$ with $\eta \prec \eta'$), we define $\mathcal{M}_{\eta} := T^*(\mathcal{C}_{\eta})$ (resp. $\mathcal{M}_{\eta' \to \eta} := T^*(\mathcal{C}_{\eta' \to \eta})$), equipped with the canonical symplectic structure on a cotangent bundle;
2. for all $\eta, \eta' \in \mathcal{L}$ with $\eta \prec \eta'$ (resp. all $\eta, \eta', \eta'' \in \mathcal{L}$ with $\eta \prec \eta' \prec \eta''$), we naturally lift $\varphi_{\eta' \to \eta} : \mathcal{C}_{\eta'} \to \mathcal{C}_{\eta' \to \eta} \times \mathcal{C}_{\eta}$ (resp. $\varphi_{\eta'' \to \eta' \to \eta} : \mathcal{C}_{\eta'' \to \eta} \to \mathcal{C}_{\eta'' \to \eta'} \times \mathcal{C}_{\eta' \to \eta}$) to a map $\widetilde{\varphi}_{\eta' \to \eta} : \mathcal{M}_{\eta'} \to \mathcal{M}_{\eta' \to \eta} \times \mathcal{M}_{\eta}$ (resp. $\widetilde{\varphi}_{\eta'' \to \eta' \to \eta} : \mathcal{M}_{\eta'' \to \eta} \to \mathcal{M}_{\eta'' \to \eta'} \times \mathcal{M}_{\eta' \to \eta}$) between the cotangent bundles;
3. for all $\eta \in \mathcal{L}$, we define $\mathcal{M}_{\eta \to \eta}$ to be a set with one element, and for all $\eta \in \mathcal{L}$ (resp. all $\eta, \eta', \eta'' \in \mathcal{L}$ with $\eta \preccurlyeq \eta' \preccurlyeq \eta''$, such that $\eta = \eta'$ or $\eta' = \eta''$) we define $\widetilde{\varphi}_{\eta \to \eta}$ (resp. $\widetilde{\varphi}_{\eta'' \to \eta' \to \eta}$) to be the trivial identification;

then $(\mathcal{L}, \mathcal{M}, \widetilde{\varphi})^{\times}$ is a factorizing system of phase spaces.

Proof We need to prove that $\forall \eta \prec \eta'$, $\widetilde{\varphi}_{\eta' \to \eta}$ is a symplectomorphism, that $\forall \eta \prec \eta' \prec \eta''$, $\widetilde{\varphi}_{\eta'' \to \eta' \to \eta}$ is a symplectomorphism and that eq. (2.11.1) for the maps φ is lifted up to the corresponding equation for the maps $\widetilde{\varphi}$.

For $\eta \in \mathcal{L}$, the symplectic structure on $\mathcal{M}_{\eta} = T^*(\mathcal{C}_{\eta})$ is defined by:

$$\forall (x, p) \in \mathcal{M}_{\eta},\ \forall w,\ w' \in T_{(x,p)}(\mathcal{M}_{\eta}),\ \ \Omega_{\mathcal{M}_{\eta},(x,p)}\left(w,\ w'\right) := w'_{\text{VER}}\left(w_{\text{HOR}}\right) - w_{\text{VER}}\left(w'_{\text{HOR}}\right), \tag{2.16.1}$$

where we define for $w \in T_{(x,p)}(\mathcal{M}_{\eta})$, $w_{\text{HOR}} \in T_x(\mathcal{C}_{\eta})$ to be the horizontal projection of w, and $w_{\text{VER}} \in T^*_x(\mathcal{C}_{\eta})$ to be the vertical part of w defined using some local coordinate system around x (the map $w \mapsto w_{\text{VER}}$ depends on this choice of local coordinates, however the anti-symmetrization in eq. (2.16.1) ensures that the definition of $\Omega_{\mathcal{M}_{\eta},(x,p)}$ is independent of this choice).

For $\eta \prec \eta' \in \mathcal{L}$, the map $\widetilde{\varphi}_{\eta' \to \eta} : \mathcal{M}_{\eta'} \to \mathcal{M}_{\eta' \to \eta} \times \mathcal{M}_{\eta}$ is defined by:

$$\forall (x', p') \in \mathcal{M}_{\eta'},\ \widetilde{\varphi}_{\eta' \to \eta}(x', p') := \Bigg(\left(f_{\eta' \to \eta} \circ \varphi_{\eta' \to \eta}(x'),\ p' \circ \left[T_{\varphi_{\eta' \to \eta}(x')}\varphi^{-1}_{\eta' \to \eta}\right](\,\cdot\,,\ 0)\right),$$
$$\left(s_{\eta' \to \eta} \circ \varphi_{\eta' \to \eta}(x'),\ p' \circ \left[T_{\varphi_{\eta' \to \eta}(x')}\varphi^{-1}_{\eta' \to \eta}\right](0,\ \cdot\,)\right) \Bigg),$$

where $f_{\eta' \to \eta} : \mathcal{C}_{\eta' \to \eta} \times \mathcal{C}_{\eta} \to \mathcal{C}_{\eta' \to \eta}$ and $s_{\eta' \to \eta} : \mathcal{C}_{\eta' \to \eta} \times \mathcal{C}_{\eta} \to \mathcal{C}_{\eta}$ are the projection maps of the Cartesian product. This map is bijective, because $\varphi_{\eta' \to \eta}$ and $\left[T_{\varphi_{\eta' \to \eta}(x')}\varphi^{-1}_{\eta' \to \eta}\right]$ are.

Let $(x,\ p) \in \mathcal{M}_{\eta}$, $(y,\ q) \in \mathcal{M}_{\eta' \to \eta}$ and $(x',\ p') = \widetilde{\varphi}_{\eta' \to \eta}\left((y,\ q),\ (x,\ p)\right)$. From the definition of $\widetilde{\varphi}_{\eta' \to \eta}$, we have for all $w \in T_{(x',p')}(\mathcal{M}_{\eta'})$:

$$\left([T_{x',p'}\,\widetilde{\varphi}_{\eta' \to \eta}](w)\right)_{\text{HOR}} = \left([T_{x'}\,f_{\eta' \to \eta} \circ \varphi_{\eta' \to \eta}](w_{\text{HOR}}),\ [T_{x'}\,s_{\eta' \to \eta} \circ \varphi_{\eta' \to \eta}](w_{\text{HOR}})\right).$$

Now, we choose local coordinates around x in $\mathcal{C}_\eta$ and around y in $\mathcal{C}_{\eta'\to\eta}$, so we have local coordinates around in (y, x) in $\mathcal{C}_{\eta'\to\eta} \times \mathcal{C}_\eta$ that we can transport through $\varphi^{-1}_{\eta'\to\eta}$ as local coordinates around $x' = \varphi^{-1}_{\eta'\to\eta}(y, x)$ in $\mathcal{C}_{\eta'}$. Using these to define $(\,\cdot\,)_{\text{VER}}$ in $T_{(x,p)}(\mathcal{M}_\eta)$, $T_{(y,q)}(\mathcal{M}_{\eta'\to\eta})$ and $T_{(x',p')}(\mathcal{M}_{\eta'})$, we have for all $w \in T_{(x',p')}(\mathcal{M}_{\eta'})$:

$$\left([T_{x',p'}\,\widetilde{\varphi}_{\eta'\to\eta}](w)\right)_{\text{VER}} = \left(w_{\text{VER}} \circ \left[T_{y,x}\varphi^{-1}_{\eta'\to\eta}\right](\,\cdot\,,0),\; w_{\text{VER}} \circ \left[T_{y,x}\varphi^{-1}_{\eta'\to\eta}\right](0,\,\cdot\,)\right).$$

Therefore:

$$\forall w,\, w' \in T_{(x',p')}(\mathcal{M}_{\eta'}),\quad \Omega_{\mathcal{M}_{\eta'\to\eta}\times\mathcal{M}_\eta,((y,x),(q,p))}\left([T_{x',p'}\,\widetilde{\varphi}_{\eta'\to\eta}](w),\, [T_{x',p'}\,\widetilde{\varphi}_{\eta'\to\eta}](w')\right) =$$

$$= w'_{\text{VER}} \circ \left[T_{y,x}\varphi^{-1}_{\eta'\to\eta}\right]\left([T_{x'}\,f_{\eta'\to\eta} \circ \varphi_{\eta'\to\eta}]\,w_{\text{HOR}}),\, 0\right) +$$

$$+\, w'_{\text{VER}} \circ \left[T_{y,x}\varphi^{-1}_{\eta'\to\eta}\right]\left(0,\, [T_{x'}\,s_{\eta'\to\eta} \circ \varphi_{\eta'\to\eta}](w_{\text{HOR}})\right) - \left(w \leftrightarrow w'\right)$$

$$= w'_{\text{VER}} \circ \left[T_{y,x}\varphi^{-1}_{\eta'\to\eta}\right] \circ [T_{x'}\varphi_{\eta'\to\eta}](w_{\text{HOR}}) - \left(w \leftrightarrow w'\right)$$

$$= \Omega_{\mathcal{M}_{\eta'},(x',p')}\left(w,\, w'\right).$$

So $\widetilde{\varphi}_{\eta'\to\eta}$ is a symplectomorphism, and in the same way we prove that for all $\eta \prec \eta' \prec \eta''$, $\widetilde{\varphi}_{\eta''\to\eta'\to\eta}$ is a symplectomorphism.

Let $\eta \prec \eta' \prec \eta'' \in \mathcal{L}$, eq. (2.11.1) for the maps φ implies:

$$f_{\eta''\to\eta'\to\eta} \circ \varphi_{\eta''\to\eta'\to\eta} \circ f_{\eta''\to\eta} \circ \varphi_{\eta''\to\eta} = f_{\eta''\to\eta'} \circ \varphi_{\eta''\to\eta'}\,,$$

$$s_{\eta''\to\eta'\to\eta} \circ \varphi_{\eta''\to\eta'\to\eta} \circ f_{\eta''\to\eta} \circ \varphi_{\eta''\to\eta} = f_{\eta'\to\eta} \circ \varphi_{\eta'\to\eta} \circ s_{\eta''\to\eta'} \circ \varphi_{\eta''\to\eta'}\,,$$

$$\&\quad s_{\eta''\to\eta} \circ \varphi_{\eta''\to\eta} = s_{\eta'\to\eta} \circ \varphi_{\eta'\to\eta} \circ s_{\eta''\to\eta'} \circ \varphi_{\eta''\to\eta'}\,,$$

(where $f_{\eta''\to\eta'\to\eta} : \mathcal{C}_{\eta''\to\eta'} \times \mathcal{C}_{\eta'\to\eta} \to \mathcal{C}_{\eta''\to\eta}$ and $s_{\eta''\to\eta'\to\eta} : \mathcal{C}_{\eta''\to\eta'} \times \mathcal{C}_{\eta'\to\eta} \to \mathcal{C}_{\eta''\to\eta}$ are the projection maps of the Cartesian product), and, for all $z, y, x \in \mathcal{C}_{\eta''\to\eta'} \times \mathcal{C}_{\eta'\to\eta} \times \mathcal{C}_\eta$:

$$\left[T_{\varphi^{-1}_{\eta''\to\eta'\to\eta}(z,y),x}\,\varphi^{-1}_{\eta''\to\eta}\right](\,\cdot\,,0) \circ \left[T_{z,y}\,\varphi^{-1}_{\eta''\to\eta'\to\eta}\right](\,\cdot\,,0) = \left[T_{z,\varphi^{-1}_{\eta'\to\eta}(y,x)}\,\varphi^{-1}_{\eta''\to\eta'}\right](\,\cdot\,,0)\,,$$

$$\left[T_{\varphi^{-1}_{\eta''\to\eta'\to\eta}(z,y),x}\,\varphi^{-1}_{\eta''\to\eta}\right](\,\cdot\,,0) \circ \left[T_{z,y}\,\varphi^{-1}_{\eta''\to\eta'\to\eta}\right](0,\,\cdot\,) = \left[T_{z,\varphi^{-1}_{\eta'\to\eta}(y,x)}\,\varphi^{-1}_{\eta''\to\eta'}\right](0,\,\cdot\,) \circ \left[T_{y,x}\,\varphi^{-1}_{\eta'\to\eta}\right](\,\cdot\,,0)\,,$$

$$\&\quad \left[T_{\varphi^{-1}_{\eta''\to\eta'\to\eta}(z,y),x}\,\varphi^{-1}_{\eta''\to\eta}\right](0,\,\cdot\,) = \left[T_{z,\varphi^{-1}_{\eta'\to\eta}(y,x)}\,\varphi^{-1}_{\eta''\to\eta'}\right](0,\,\cdot\,) \circ \left[T_{y,x}\,\varphi^{-1}_{\eta'\to\eta}\right](0,\,\cdot\,)\,,$$

therefore eq. (2.11.1) is fulfilled for the maps $\widetilde{\varphi}$. □

Proposition 2.17 Let $(\mathcal{L},\, \mathcal{M},\, \pi)^{\downarrow}$ be a projective system of phase spaces and suppose that:

1. $\forall \eta \in \mathcal{L}$, $\mathcal{M}_\eta = T^*(\mathcal{C}_\eta)$ where $\mathcal{C}_\eta$ is a smooth connected manifold;
2. $\forall \eta \prec \eta' \in \mathcal{L}$, there exist a smooth manifold $\mathcal{C}_{\eta'\to\eta}$ and a diffeomorphism $\varphi_{\eta'\to\eta} : \mathcal{C}_{\eta'} \to \mathcal{C}_{\eta'\to\eta} \times \mathcal{C}_\eta$ such that $\pi_{\eta'\to\eta} = \widetilde{s}_{\eta'\to\eta} \circ \widetilde{\varphi}_{\eta'\to\eta}$, where $\widetilde{s}_{\eta'\to\eta} : T^*\left(\mathcal{C}_{\eta'\to\eta} \times \mathcal{C}_\eta\right) \simeq T^*\left(\mathcal{C}_{\eta'\to\eta}\right) \times T^*\left(\mathcal{C}_\eta\right) \to T^*\left(\mathcal{C}_\eta\right)$ is the projection on the second Cartesian factor and $\widetilde{\varphi}_{\eta'\to\eta} : T^*\left(\mathcal{C}_{\eta'}\right) \to T^*\left(\mathcal{C}_{\eta'\to\eta} \times \mathcal{C}_\eta\right)$ is the cotangent lift of $\varphi_{\eta'\to\eta}$.

Then, we can complete this input into a factorizing system $(\mathcal{L}, \mathcal{C}, \varphi)^\times$.

Proof For $\eta \in \mathcal{L}$, we define $\mathcal{C}_{\eta\to\eta}$ to be a space with one element and $\varphi_{\eta\to\eta}$ to be the trivial identification.

Let $\eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}$. What we need to show is that there exists a diffeomorphism $\varphi_{\eta''\to\eta'\to\eta} : \mathcal{C}_{\eta''\to\eta} \to \mathcal{C}_{\eta''\to\eta'} \times \mathcal{C}_{\eta'\to\eta}$ such that eq. (2.11.1) is fulfilled. If $\eta = \eta'$ or $\eta' = \eta''$, we can choose $\varphi_{\eta''\to\eta'\to\eta}$ to be the trivial identification, so we now consider the case $\eta \prec \eta' \prec \eta''$.

Let $x'', p'' \in T^*(\mathcal{C}_{\eta'})$ and define:

$$(y', q'; x', p') = \widetilde{\varphi}_{\eta''\to\eta'}(x'', p'')\,,$$

$$(y, q; x, p) = \widetilde{\varphi}_{\eta'\to\eta}(x', p')\,,$$

$$\text{and}\quad (z, r; x^\bullet, p^\bullet) = \widetilde{\varphi}_{\eta''\to\eta}(x'', p'')\,.$$

Now, from $\pi_{\eta''\to\eta} = \pi_{\eta'\to\eta} \circ \pi_{\eta''\to\eta'}$, we have $x = x^\bullet$ and $p = p^\bullet$, hence:

$$s_{\eta'\to\eta} \circ \varphi_{\eta'\to\eta} \circ s_{\eta''\to\eta'} \circ \varphi_{\eta''\to\eta'}(x'') = s_{\eta''\to\eta} \circ \varphi_{\eta''\to\eta}(x'')\,,$$

$$\text{and}\quad p'' \circ \left[T_{y',x'}\, \varphi^{-1}_{\eta''\to\eta'}\right](0,\,\cdot\,) \circ \left[T_{y,x}\, \varphi^{-1}_{\eta'\to\eta}\right](0,\,\cdot\,) = p'' \circ \left[T_{z,x}\, \varphi^{-1}_{\eta''\to\eta}\right](0,\,\cdot\,)\,,$$

thus, we get:

$$t_{\eta''\to\eta'\to\eta} \circ \Psi = s_{\eta''\to\eta}\,,$$

$$\text{and}\quad (0,\,0,\,\cdot\,) = [T\Psi](0,\,\cdot\,)\,,$$

where $\Psi := \left(\mathrm{id}_{\mathcal{C}_{\eta''\to\eta'}} \times \varphi_{\eta'\to\eta}\right) \circ \varphi_{\eta''\to\eta'} \circ \varphi^{-1}_{\eta''\to\eta}$ and $t_{\eta''\to\eta'\to\eta} : \mathcal{C}_{\eta''\to\eta'} \times \mathcal{C}_{\eta'\to\eta} \times \mathcal{C}_\eta \to \mathcal{C}_\eta$ is the projection on the third Cartesian factor.

Finally, since $\mathcal{C}_\eta$ is connected, there exists a diffeomorphism $\varphi_{\eta''\to\eta'\to\eta} : \mathcal{C}_{\eta''\to\eta} \to \mathcal{C}_{\eta''\to\eta'} \times \mathcal{C}_{\eta'\to\eta}$ such that $\Psi = \varphi_{\eta''\to\eta'\to\eta} \times \mathrm{id}_{\mathcal{C}_\eta}$. □

3. Constraints and regularization

When we try to incorporate the dynamics in the formalism described in the previous section, we quickly realize that the intuitive picture we were relying on was quite oversimplified. For, although it should be true that we only need a finite dimensional truncation of the kinematical theory to hold the elementary kinematical observables associated to any given real experiment, in general we cannot write the dynamics in a closed form within such a truncation.

We will work in the context of constrained classical theories reviewed in appendix A (initially unconstrained ones can always be rewritten in this language [105, section 1.8]: we will briefly illustrate in subsection 16.1 how this is done). As developed in the discussion preceding def. A.2, we take the point of view that from each kinematical observable arises a corresponding dynamical observable and, considering a family of functionally independent kinematical observables, it might be possible to write functional relations connecting the associated dynamical observables: here lies the predictive contents of the theory. However, such a functional relation can involve an infinite number of observables, and thus get silently dropped, if we never look at more than a finite number of observables at a time. When looking at a typical field theory, the interesting content of the dynamics lies precisely in those functional relations that can only be written over an infinite number of observables, and do not emerge from simpler relations within finite set of observables (a partial differential equation is mostly useless if we only dispose of a discrete, finite set of initial values).

On the other hand, if the the theory is to have any physically relevant predictivity, namely if it is to be usable to formulate predictions for the output of some real experiments, it should at least be possible to *approximate* the dynamics with relations over finite sets of elementary dynamical observables (we do nothing else when elaborating numerical techniques to deal with partial differential equations). In other words, although we may not be able to state *exact* predictions for any specific realistic experiment, we can restore predictivity in a weaker sense, by describing how to refine an experiment and the associated approximate predictions to make them better and better.

This concept of convergence is physically useful, notwithstanding the fact that we will *not* perform the infinite chain of experiments (that would again be a case of measuring an infinite number of observables, and we already mentioned that this is excluded in practice), because we can convert it into a notion of *plausibility*, by stating how to design an experimental protocol such that it will be highly unlikely that the output lies outside some confidence domain.

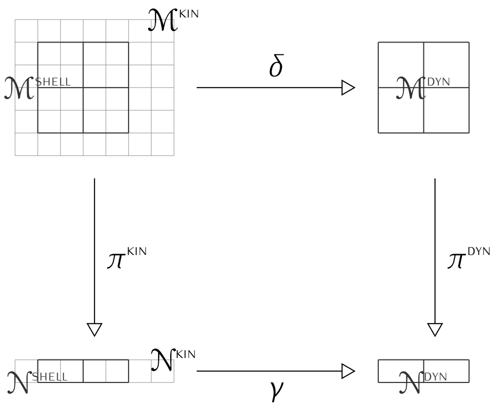
The object of this section is to formulate this raw idea more precisely, in order to develop a procedure to solve constraints in a projective system of phase spaces.

3.1 Elementary reductions

We begin by studying in detail under which conditions the dynamics actually *can* be formulated straightforwardly within a projective system of phase spaces, for this will be our building block when addressing the generic case.

Our aim here is the following: we want to write in each partial kinematical theory $\mathcal{M}_\eta^{\text{KIN}}$ a constraint surface $\mathcal{M}_\eta^{\text{SHELL}}$, and to reassemble the resulting reduced phase spaces $\mathcal{M}_\eta^{\text{DYN}}$ (see appendix A) into a new projective system of phase spaces. And we want to accomplish this in such a way that we can glue together the maps that, for each η, associate to the kinematical observables on $\mathcal{M}_\eta^{\text{KIN}}$ the corresponding dynamical observables on $\mathcal{M}_\eta^{\text{DYN}}$, thus building a map from the set of all observables on the kinematical projective system into the set of all observables on the dynamical projective system. For this map to accurately reproduce a given dynamics, it should give rise to functional relations between the dynamical observables that catch the full predictive power of the theory and it should account for the correct dynamical Poisson commutation relations.

We start by looking at a symplectic manifold $\mathcal{N}^{\text{KIN}}$, that extracts, via a projection π^{KIN}, specific degrees of freedom out of a bigger symplectic manifold $\mathcal{M}^{\text{KIN}}$ (as were introduced in def. 2.1). Given a phase space reduction on $\mathcal{M}^{\text{KIN}}$, with reduced phase space $\mathcal{M}^{\text{DYN}}$, we ask whether it is possible to



Here, we sketch a symplectic manifold by a grid, each square of which is to be thought as a point in the manifold (and is emblematic for infinitely many other points).

Figure 3.1 – Phase space reductions on $\mathcal{M}^{\text{KIN}}$ and $\mathcal{N}^{\text{KIN}}$, related by a projection π^{KIN}

write closed equations, involving only the degrees of freedom retained in $\mathcal{N}^{\text{KIN}}$, and capturing all what the dynamics on $\mathcal{M}^{\text{KIN}}$ has to say concerning these degrees of freedom.

More precisely, we are looking for a phase space reduction on $\mathcal{N}^{\text{KIN}}$, but also for a projection π^{DYN} allowing to understand the reduced phase space $\mathcal{N}^{\text{DYN}}$ as a selection of dynamical degrees of freedom out of $\mathcal{M}^{\text{DYN}}$ (fig. 3.1). Indeed, if we consider an observable O^{KIN} on $\mathcal{N}^{\text{KIN}}$, we can pull it back by π^{KIN} into an observable $O^{\text{KIN}\prime}$ on $\mathcal{M}^{\text{KIN}}$. So using the dynamics on $\mathcal{M}^{\text{KIN}}$, we can obtain a corresponding dynamical observable $O^{\text{DYN}\prime}$ on $\mathcal{M}^{\text{DYN}}$. Now, if we can write the dynamics in closed form on $\mathcal{N}^{\text{KIN}}$, we can also map directly O^{KIN} to a dynamical observable O^{DYN} on $\mathcal{N}^{\text{DYN}}$. The role of the projection π^{DYN} is then to ensure that the dynamics we have on $\mathcal{N}$ is actually consistent with the one on $\mathcal{M}$, by requiring $O^{\text{DYN}\prime}$ to be precisely the pullback of O^{DYN} by π^{DYN}.

If this is at all possible, both the reduction on $\mathcal{N}^{\text{KIN}}$ and the projection π^{DYN} are uniquely determined by the dynamics we choose on $\mathcal{M}^{\text{KIN}}$. Indeed, the constraint surface in $\mathcal{N}^{\text{KIN}}$ has to be the projection by π^{KIN} of the one in $\mathcal{M}^{\text{KIN}}$ (for the constraint surface can be reconstructed if we know which kinematical observables are mapped to a vanishing dynamical observables), and we have from prop. A.6 (at least in the finite dimensional case) that a reduction is completely determined by its constraint surface. Then, the uniqueness of π^{DYN} is enforced by requiring that it correctly makes the connection between the dynamics on $\mathcal{N}^{\text{KIN}}$ and the aforementioned map $O^{\text{KIN}} \mapsto O^{\text{DYN}\prime}$ (that only depends of π^{KIN} and of the reduction on $\mathcal{M}^{\text{KIN}}$).

Definition 3.1 Let $\mathcal{M}^{\text{KIN}}$ and $\mathcal{N}^{\text{KIN}}$ be two symplectic manifolds and $\pi^{\text{KIN}} : \mathcal{M}^{\text{KIN}} \to \mathcal{N}^{\text{KIN}}$ a surjective map compatible with the symplectic structures (def. 2.1). Let $(\mathcal{M}^{\text{DYN}}, \mathcal{M}^{\text{SHELL}}, \delta)$, resp. $(\mathcal{N}^{\text{DYN}}, \mathcal{N}^{\text{SHELL}}, \gamma)$, be phase space reductions of $\mathcal{M}^{\text{KIN}}$, resp. $\mathcal{N}^{\text{KIN}}$ (def. A.1). We say that these reductions are related by π^{KIN} if:

1. $\pi^{\text{KIN}} \left\langle \mathcal{M}^{\text{SHELL}} \right\rangle = \mathcal{N}^{\text{SHELL}}$;
2. there exists a surjective map $\pi^{\text{DYN}} : \mathcal{M}^{\text{DYN}} \to \mathcal{N}^{\text{DYN}}$, compatible with the symplectic structures, such that:

$$\forall x \in \mathcal{N}^{\text{SHELL}}, \forall y' \in \mathcal{M}^{\text{DYN}}, \quad \left(\exists x' \in \mathcal{M}^{\text{SHELL}} \,/\, \delta(x') = y' \quad \& \quad \pi^{\text{KIN}}(x') = x\right) \Leftrightarrow \left(\gamma(x) = \pi^{\text{DYN}}(y')\right). \tag{3.1.1}$$

Proposition 3.2 With the notations of def. 3.1, if $\pi^{\text{DYN},1}$ and $\pi^{\text{DYN},2}$ are two surjective maps satisfying eq. (3.1.1), then $\pi^{\text{DYN},1} = \pi^{\text{DYN},2}$.

Proof Let $y' \in \mathcal{M}^{\text{DYN}}$. Since δ is surjective, there exists $x' \in \mathcal{M}^{\text{SHELL}}$ such that $\delta(x') = y'$. Hence, $\pi^{\text{DYN},1}(y') = \gamma \circ \pi^{\text{KIN}}(x') = \pi^{\text{DYN},2}(y')$. □

Proposition 3.3 We consider the same objects as in def. 3.1 and use the notations introduced in def. A.2. For $f \in B(\mathcal{N}^{\text{KIN}})$, we have $f \circ \pi^{\text{KIN}} \in B(\mathcal{M}^{\text{KIN}})$ and:

$$(f \circ \pi^{\text{KIN}})^{\text{DYN}} = f^{\text{DYN}} \circ \pi^{\text{DYN}}.$$

Proof Let $y' \in \mathcal{M}^{\text{DYN}}$. Using eq. (3.1.1) into eq. (A.2.1), we have:

$$\begin{aligned}(f \circ \pi^{\text{KIN}})^{\text{DYN}}(y') &= \sup\left\{ f \circ \pi^{\text{KIN}}(x') \;\middle|\; x' \in \delta^{-1}\langle y' \rangle \right\} \\ &= \sup\left\{ f(x) \;\middle|\; x \in \gamma^{-1}\langle \pi^{\text{DYN}}(y') \rangle \right\} = f^{\text{DYN}} \circ \pi^{\text{DYN}}(y').\end{aligned}$$

□

Why do we need to require eq. (3.1.1) for π^{DYN} instead of the seemingly more natural condition $\gamma \circ \pi^{\text{KIN}} = \pi^{\text{DYN}} \circ \delta$? The physical reason behind eq. (3.1.1) is that we shall not look at the map δ but rather at $\delta^{-1}\langle \cdot \rangle$, that sends a point in $\mathcal{M}^{\text{DYN}}$ to an orbit in $\mathcal{M}^{\text{SHELL}}$ (and similarly at $\gamma^{-1}\langle \cdot \rangle$ instead of γ), for this is the map that is dual to the application associating a kinematical observable to a dynamical one (in a way similar to π^{KIN} being dual to the application that sends an observable on $\mathcal{N}^{\text{KIN}}$ into an observable on $\mathcal{M}^{\text{KIN}}$). And, indeed, we can rewrite eq. (3.1.1) as $\pi^{\text{KIN}}\langle \cdot \rangle \circ \delta^{-1}\langle \cdot \rangle = \gamma^{-1}\langle \cdot \rangle \circ \pi^{\text{DYN}}$.

That eq. (3.1.1) could fail in situations where $\gamma \circ \pi^{\text{KIN}} = \pi^{\text{DYN}} \circ \delta$ does hold, can have local as well as global causes, as illustrated by the examples below. It happens when the projection of an orbit in $\mathcal{M}^{\text{SHELL}}$, though included in an orbit of $\mathcal{N}^{\text{SHELL}}$, does not *fill* it.

Proposition 3.4 If we replace in def. 3.1 the condition given by eq. (3.1.1) by the weaker assumption:

$$\gamma \circ \pi^{\text{KIN}} = \pi^{\text{DYN}} \circ \delta, \tag{3.4.1}$$

then the previous result (prop. 3.3) does not hold.

Proof As a counter example, we consider the following situation:

1. $\mathcal{M}^{\text{KIN}} = \left(\mathbb{R}^2\right)^3$, $\mathcal{M}^{\text{DYN}} = \left(\mathbb{R}^2\right)^2$, $\mathcal{N}^{\text{KIN}} = \left(\mathbb{R}^2\right)^2$, $\mathcal{N}^{\text{DYN}} = \mathbb{R}^2$ (with the standard symplectic structure on $\mathbb{R}^2$: $\Omega_{\mathbb{R}^2}(x, p;\, x', p') = x\,p' - x'\,p$);
2. $\forall (x_i, p_i)_{i\in\{0,\dots,2\}} \in \mathcal{M}^{\text{KIN}}$, $\pi^{\text{KIN}}\left((x_i, p_i)_{i\in\{0,\dots,2\}}\right) = (x_i, p_i)_{i\in\{0,1\}}$;
3. $\mathcal{M}^{\text{SHELL}} = \left\{(x_i, p_i)_{i\in\{0,\dots,2\}} \;\middle|\; p_1 = 0 \;\;\&\;\; x_1 = x_2\right\}$ and $\forall (x_i, p_i)_{i\in\{0,\dots,2\}} \in \mathcal{M}^{\text{SHELL}}$, $\delta\left((x_i, p_i)_{i\in\{0,\dots,2\}}\right) = (x_i, p_i)_{i\in\{0,2\}}$;
4. $\mathcal{N}^{\text{SHELL}} = \left\{(x_i, p_i)_{i\in\{0,1\}} \;\middle|\; p_1 = 0\right\}$ and $\forall (x_i, p_i)_{i\in\{0,1\}} \in \mathcal{N}^{\text{SHELL}}$, $\gamma\left((x_i, p_i)_{i\in\{0,1\}}\right) = (x_0, p_0)$;
5. $\forall (x_i, p_i)_{i\in\{0,2\}} \in \mathcal{M}^{\text{DYN}}$, $\pi^{\text{DYN}}\left((x_i, p_i)_{i\in\{0,2\}}\right) = (x_0, p_0)$.

We can check that $(\mathcal{M}^{\text{DYN}}, \mathcal{M}^{\text{SHELL}}, \delta)$ is a phase space reduction of $\mathcal{M}^{\text{KIN}}$ and $(\mathcal{N}^{\text{DYN}}, \mathcal{N}^{\text{SHELL}}, \gamma)$ is a phase space reduction of $\mathcal{N}^{\text{KIN}}$. π^{KIN} and π^{DYN} are surjective maps compatible with the symplectic structures, satisfying $\pi^{\text{KIN}}\langle \mathcal{M}^{\text{SHELL}} \rangle = \mathcal{N}^{\text{SHELL}}$ and $\gamma \circ \pi^{\text{KIN}} = \pi^{\text{DYN}} \circ \delta$.

However, if we consider $f \in B(\mathcal{N}^{\text{KIN}})$ defined by:

$$\forall (x_i, p_i)_{i\in\{0,1\}} \in \mathcal{N}^{\text{KIN}},\ f\left((x_i, p_i)_{i\in\{0,1\}}\right) = \begin{cases} 1 \text{ if } x_1 \leqslant 0 \\ 0 \text{ else} \end{cases},$$

we have $f^{\text{DYN}} \circ \pi^{\text{DYN}} \equiv 1$, but:

$$\forall (x_i, p_i)_{i\in\{0,2\}} \in \mathcal{M}^{\text{DYN}},\ (f \circ \pi^{\text{KIN}})^{\text{DYN}}\left((x_i, p_i)_{i\in\{0,2\}}\right) = \begin{cases} 1 \text{ if } x_2 \leqslant 0 \\ 0 \text{ else} \end{cases}.$$

Requiring local conditions in addition to eq. (3.4.1) would not help either, since even if everything works well locally, it may still goes wrong globally, as the following example shows:

6. $\mathcal{M}^{\text{KIN}} = \left(\mathbb{R}^2\right)^4$, $\mathcal{M}^{\text{DYN}} = \left(\mathbb{R}^2\right)^2$, $\mathcal{N}^{\text{KIN}} = \left(\mathbb{R}^2\right)^2$, $\mathcal{N}^{\text{DYN}} = \mathbb{R}^2$;
7. $\forall (x_i, p_i)_{i\in\{0,\dots,3\}} \in \mathcal{M}^{\text{KIN}},\ \pi^{\text{KIN}}\left((x_i, p_i)_{i\in\{0,\dots,3\}}\right) = (x_i, p_i)_{i\in\{0,1\}}$;
8. $\mathcal{M}^{\text{SHELL}} = \left\{(x_i, p_i)_{i\in\{0,\dots,3\}} \mid p_1 = 0,\ p_2 = 0 \ \ \&\ \ x_1 = x_3 + \exp(x_2)\right\}$ and $\forall (x_i, p_i)_{i\in\{0,\dots,3\}} \in \mathcal{M}^{\text{SHELL}}$, $\delta\left((x_i, p_i)_{i\in\{0,\dots,3\}}\right) = (x_i, p_i)_{i\in\{0,3\}}$;
9. $\mathcal{N}^{\text{SHELL}} = \left\{(x_i, p_i)_{i\in\{0,1\}} \mid p_1 = 0\right\}$ and $\forall (x_i, p_i)_{i\in\{0,1\}} \in \mathcal{N}^{\text{SHELL}},\ \gamma\left((x_i, p_i)_{i\in\{0,1\}}\right) = (x_0, p_0)$;
10. $\forall (x_i, p_i)_{i\in\{0,3\}} \in \mathcal{M}^{\text{DYN}},\ \pi^{\text{DYN}}\left((x_i, p_i)_{i\in\{0,3\}}\right) = (x_0, p_0)$.

We can check that $(\mathcal{M}^{\text{DYN}}, \mathcal{M}^{\text{SHELL}}, \delta)$ is a phase space reduction of $\mathcal{M}^{\text{KIN}}$ and $(\mathcal{N}^{\text{DYN}}, \mathcal{N}^{\text{SHELL}}, \gamma)$ is a phase space reduction of $\mathcal{N}^{\text{KIN}}$. π^{KIN} and π^{DYN} are surjective maps compatible with the symplectic structures, satisfying $\pi^{\text{KIN}}\left\langle\mathcal{M}^{\text{SHELL}}\right\rangle = \mathcal{N}^{\text{SHELL}}$ and $\gamma \circ \pi^{\text{KIN}} = \pi^{\text{DYN}} \circ \delta$.

Moreover, eq. (3.1.1) holds at the linear level, namely:

$$\forall x' \in \mathcal{M}^{\text{SHELL}},\quad \forall v \in T_{\pi^{\text{KIN}}(x')}(\mathcal{N}^{\text{SHELL}}),\ \forall w' \in T_{\delta(x')}(\mathcal{M}^{\text{DYN}}),$$

$$\left(\exists\, v' \in T_{x'}(\mathcal{M}^{\text{SHELL}}) \ / \ T_{x'}\delta(v') = w' \quad \& \quad T_{x'}\pi^{\text{KIN}}(v') = v\right) \Leftrightarrow \left(T_{\pi^{\text{KIN}}(x')}\gamma(v) = T_{\delta(x')}\pi^{\text{DYN}}(w')\right),$$

for this reduces in the present example to:

$$\forall x_2 \in \mathbb{R},\quad \forall x_1^{v} \in \mathbb{R},\ \forall x_3^{w'} \in \mathbb{R},\quad \left(\exists\, x_2^{v'},\ x_3^{v'} \in \mathbb{R}^2 \ / \ x_3^{v'} = x_3^{w'} \quad \& \quad x_3^{v'} + \exp(x_2)\, x_2^{v'} = x_1^{v}\right).$$

However, if we consider the same $f \in B(\mathcal{N}^{\text{KIN}})$ as before, we have $f^{\text{DYN}} \circ \pi^{\text{DYN}} \equiv 1$, but:

$$\forall (x_i, p_i)_{i\in\{0,3\}} \in \mathcal{M}^{\text{DYN}},\ (f \circ \pi^{\text{KIN}})^{\text{DYN}}\left((x_i, p_i)_{i\in\{0,3\}}\right) = \begin{cases} 1 \text{ if } x_3 \leqslant 0 \\ 0 \text{ else} \end{cases}.$$

□

Asking for the dynamics on $\mathcal{M}^{\text{KIN}}$ to define a dynamics on $\mathcal{N}^{\text{KIN}}$ in the sense above actually puts strong restrictions (local as well as global ones) on what the constraint surface in $\mathcal{M}^{\text{KIN}}$ can be.

If we consider the special case where $\mathcal{M}^{\text{KIN}}$ and $\mathcal{N}^{\text{KIN}}$ are symplectic vector spaces, and π^{KIN} is a linear map, the symplectic structure provides a natural decomposition of $\mathcal{M}^{\text{KIN}}$ as $\mathcal{P}^{\text{KIN}} \oplus (\mathcal{P}^{\text{KIN}})^{\perp}$, with $\mathcal{P}^{\text{KIN}} = \operatorname{Ker}\pi^{\text{KIN}}$ and $(\mathcal{P}^{\text{KIN}})^{\perp} \approx \mathcal{N}^{\text{KIN}}$ (where the orthogonal subspace is defined with respect to the symplectic structure; this is the linear version of prop. 2.10). What are the conditions for a vector subspace $\mathcal{M}^{\text{SHELL}}$ of $\mathcal{M}^{\text{KIN}}$ to define a (linear) dynamics that will descend well through π^{KIN}? An obvious way of fulfilling this wish is to have a constraint surface $\mathcal{M}^{\text{SHELL}}$ that decomposes as $\mathcal{M}^{\text{SHELL}} = W \oplus V$

where W and V are vector subsets of $\mathcal{P}^{\text{KIN}}$ and $(\mathcal{P}^{\text{KIN}})^{\perp}$ respectively: this would be a dynamics with no interaction between the degrees of freedom in $\mathcal{N}^{\text{KIN}}$ and the ones in $\mathcal{P}^{\text{KIN}}$, so clearly we can write separately the dynamics on $\mathcal{N}^{\text{KIN}}$. However, a closer study of what is really needed shows that we have an additional freedom to construct admissible constraint surfaces $\mathcal{M}^{\text{SHELL}}$: instead of choosing V as a vector subset of $(\mathcal{P}^{\text{KIN}})^{\perp}$, it is enough for V to be included in $W^{\perp}$, provided π^{KIN} identifies the restriction to V of the symplectic structure $\Omega_{\mathcal{M}^{\text{KIN}}}$ with the restriction to $\mathcal{N}^{\text{SHELL}} = \pi^{\text{KIN}}\langle V\rangle$ of $\Omega_{\mathcal{N}^{\text{KIN}}}$.

This study of the linear case essentially translates to local necessary conditions in the generic case. However, this holds only at the points in $\mathcal{M}^{\text{SHELL}}$ where the derivative of π^{KIN} maps the tangent space of the orbit of $\mathcal{M}^{\text{SHELL}}$ going through that point into the tangent space of an orbit of $\mathcal{N}^{\text{SHELL}}$: for, although eq. (3.1.1) implies that π^{KIN} should map an orbit into an orbit, this does not need to hold at the linear level (the derivative of a surjective map does not need to be surjective; nonetheless Sard's theorem [81] tells us, in a specific sense, that this 'rarely' fails).

Proposition 3.5 Let $\mathcal{M}^{\text{KIN}}$ and $\mathcal{N}^{\text{KIN}}$ be two finite dimensional symplectic manifolds and $\pi^{\text{KIN}} : \mathcal{M}^{\text{KIN}} \to \mathcal{N}^{\text{KIN}}$ a surjective map compatible with the symplectic structures. Let $(\mathcal{M}^{\text{DYN}}, \mathcal{M}^{\text{SHELL}}, \delta)$, resp. $(\mathcal{N}^{\text{DYN}}, \mathcal{N}^{\text{SHELL}}, \gamma)$, be phase space reductions of $\mathcal{M}^{\text{KIN}}$, resp. $\mathcal{N}^{\text{KIN}}$. Assume these reductions are related by π^{KIN}, and let $x' \in \mathcal{M}^{\text{SHELL}}$, $y' := \delta(x') \in \mathcal{M}^{\text{DYN}}$, $x := \pi^{\text{KIN}}(x') \in \mathcal{N}^{\text{SHELL}}$ and $y := \gamma(x) \in \mathcal{N}^{\text{DYN}}$. Then, π^{KIN} induces a surjective map $\delta^{-1}\langle y'\rangle \to \gamma^{-1}\langle y\rangle$.

If moreover $T_{x'}\pi^{\text{KIN}}\left\langle T_{x'}\left(\delta^{-1}\langle y'\rangle\right)\right\rangle = T_x\left(\gamma^{-1}\langle y\rangle\right)$, then there exist $V_{x'}, W_{x'}$ vector subspaces of $T_{x'}(\mathcal{M}^{\text{SHELL}})$ such that:

1. $T_{x'}(\mathcal{M}^{\text{SHELL}}) = V_{x'} \oplus W_{x'}$ & $\Omega_{\mathcal{M}^{\text{KIN}},x'}(V_{x'},\, W_{x'}) = \{0\}$;
2. $V_{x'} \cap \operatorname{Ker} T_{x'}\pi^{\text{KIN}} = \{0\}$ & $W_{x'} \subset \operatorname{Ker} T_{x'}\pi^{\text{KIN}}$;
3. $\pi^{\text{KIN},*}_{x'}\, \Omega_{\mathcal{N}^{\text{KIN}},x}\big|_{V_{x'}} = \Omega_{\mathcal{M}^{\text{KIN}},x'}\big|_{V_{x'}}$.

Proof Let π^{DYN} be as in def. 3.1.2. From eq. (3.1.1), we have $\gamma \circ \pi^{\text{KIN}} = \pi^{\text{DYN}} \circ \delta$, hence $y = \pi^{\text{DYN}}(y')$, and:

$$\forall z \in \mathcal{N}^{\text{SHELL}},\ \left(z \in \gamma^{-1}\langle y\rangle\right) \Leftrightarrow \left(\gamma(z) = \pi^{\text{DYN}}(y')\right) \Leftrightarrow \left(\exists\, z' \in \delta^{-1}\langle y'\rangle \,/\, \pi^{\text{KIN}}(z') = z\right)$$

$$\Leftrightarrow \left(z \in \pi^{\text{KIN}}\left\langle \delta^{-1}\langle y'\rangle\right\rangle\right),$$

therefore $\pi^{\text{KIN}}\left\langle \delta^{-1}\langle y'\rangle\right\rangle = \gamma^{-1}\langle y\rangle$.

We now moreover assume that $T_{x'}\pi^{\text{KIN}}$ induces a surjective linear map from $\operatorname{Ker} T_{x'}\delta = T_{x'}\left(\delta^{-1}\langle y'\rangle\right)$ into $\operatorname{Ker} T_x\gamma = T_x\left(\gamma^{-1}\langle y\rangle\right)$. Then, there exist vector subspaces $V^o_{x'}$ and $W^o_{x'}$ of $\operatorname{Ker} T_{x'}\delta$ such that:

$$W^o_{x'} = \operatorname{Ker} T_{x'}\pi^{\text{KIN}} \cap \operatorname{Ker} T_{x'}\delta \quad \& \quad \operatorname{Ker} T_{x'}\delta = V^o_{x'} \oplus W^o_{x'},$$

and $T_{x'}\pi^{\text{KIN}}$ induces a bijection $V^o_{x'} \to \operatorname{Ker} T_x\gamma$.

Next, we define the vector subspaces $V^1_{y'}$ and $W^1_{y'}$ of $T_{y'}(\mathcal{M}^{\text{DYN}})$ by:

$$W^1_{y'} := \operatorname{Ker} T_{y'}\pi^{\text{DYN}} \quad \& \quad V^1_{y'} := \left(W^1_{y'}\right)^{\perp} = \left\{v \in T_{y'}(\mathcal{M}^{\text{DYN}}) \,\middle|\, \Omega_{\text{DYN},y'}\left(v,\, W^1_{y'}\right) = \{0\}\right\},$$

and since π^{DYN} is compatible with the symplectic structures, we have $T_{y'}(\mathcal{M}^{\text{DYN}}) = V^1_{y'} \oplus W^1_{y'}$ and $T_{y'}\pi^{\text{DYN}}$ being surjective, it induces a bijection $V^1_{y'} \to T_y(\mathcal{N}^{\text{DYN}})$, such that:

$$\pi^{\text{DYN},*}_{y'} \Omega_{\mathcal{N}^{\text{DYN}},y}\Big|_{V^1_{y'}} = \Omega_{\mathcal{M}^{\text{DYN}},y'}\big|_{V^1_{y'}} .$$

Let $\widetilde{W}^2_{x'} := [T_{x'}\delta]^{-1} \left\langle W^1_{y'} \right\rangle$. We have $\operatorname{Ker} T_{x'}\delta \subset \widetilde{W}^2_{x'}$, and from $T_x\gamma \circ T_{x'}\pi^{\text{KIN}} \left\langle \widetilde{W}^2_{x'} \right\rangle = T_{y'}\pi^{\text{DYN}} \circ T_{x'}\delta \left\langle \widetilde{W}^2_{x'} \right\rangle = \{0\}$ and $\operatorname{Ker} T_x\gamma = T_{x'}\pi^{\text{KIN}} \left\langle \operatorname{Ker} T_{x'}\delta \right\rangle \subset T_{x'}\pi^{\text{KIN}} \left\langle \widetilde{W}^2_{x'} \right\rangle$, we also have $T_{x'}\pi^{\text{KIN}} \left\langle \widetilde{W}^2_{x'} \right\rangle = \operatorname{Ker} T_x\gamma$, hence there exists a vector subspace $W^2_{x'}$ of $\widetilde{W}^2_{x'}$ such that:

$$W^o_{x'} \oplus W^2_{x'} = \operatorname{Ker} T_{x'}\pi^{\text{KIN}} \cap \widetilde{W}^2_{x'} \quad \& \quad \widetilde{W}^2_{x'} = V^o_{x'} \oplus W^o_{x'} \oplus W^2_{x'} ,$$

and $T_{x'}\delta \left\langle W^2_{x'} \right\rangle = T_{x'}\delta \left\langle \widetilde{W}^2_{x'} \right\rangle = W^1_{y'}$ for $T_{x'}\delta$ is surjective. Additionally, since $T_{x'}\delta$ is surjective, there exists a vector subspace $V^2_{x'}$ of $T_{x'}(\mathcal{M}^{\text{SHELL}})$ such that:

$$T_{x'}(\mathcal{M}^{\text{SHELL}}) = V^o_{x'} \oplus W^o_{x'} \oplus V^2_{x'} \oplus W^2_{x'} ,$$

with $T_{x'}\delta$ inducing a bijective map $V^2_{x'} \to V^1_{y'}$. So $T_x\gamma \circ T_{x'}\pi^{\text{KIN}} = T_{y'}\pi^{\text{DYN}} \circ T_{x'}\delta$ induce a bijective map $V^2_{x'} \to T_y\left(\mathcal{N}^{\text{DYN}}\right) = T_x\gamma \left\langle T_x(\mathcal{N}^{\text{SHELL}}) \right\rangle$, therefore $T_{x'}\pi^{\text{KIN}}$ induce a bijective map $V^o_{x'} \oplus V^2_{x'} \to T_x\left(\mathcal{N}^{\text{SHELL}}\right)$, such that, for all $u,\, v \in V^o_{x'} \oplus V^2_{x'}$:

$$\begin{aligned}
\Omega_{\mathcal{N}^{\text{KIN}},x}\left(T_{x'}\pi^{\text{KIN}}(u),\, T_{x'}\pi^{\text{KIN}}(v)\right) &= \Omega_{\mathcal{N}^{\text{DYN}},y}\left(T_x\gamma \circ T_{x'}\pi^{\text{KIN}}(u),\, T_x\gamma \circ T_{x'}\pi^{\text{KIN}}(v)\right) \\
&= \Omega_{\mathcal{N}^{\text{DYN}},y}\left(T_{y'}\pi^{\text{DYN}} \circ T_{x'}\delta(u),\, T_{y'}\pi^{\text{DYN}} \circ T_{x'}\delta(v)\right) \\
&= \Omega_{\mathcal{M}^{\text{DYN}},y'}\left(T_{x'}\delta(u),\, T_{x'}\delta(v)\right) \\
&= \Omega_{\mathcal{M}^{\text{KIN}},x'}\left(u,\, v\right) .
\end{aligned}$$

Finally, defining $V_{x'} := V^o_{x'} \oplus V^2_{x'}$ and $W_{x'} := W^o_{x'} \oplus W^2_{x'}$, we have:

$$\begin{aligned}
\Omega_{\text{KIN},x'}\left(V_{x'},\, W_{x'}\right) &= \Omega_{\text{KIN},x'}\left(V^2_{x'},\, W^2_{x'}\right) \text{ (for } \Omega_{\text{KIN},x'}\left(T_{x'}(\mathcal{M}^{\text{SHELL}}),\, \operatorname{Ker} T_{x'}\delta\right) = \{0\}) \\
&= \Omega_{\text{DYN},y'}\left(V^1_{y'},\, W^1_{y'}\right) \text{ (for } \Omega_{\text{KIN},x'}\big|_{T_{x'}(\mathcal{M}^{\text{SHELL}})} = \delta^*_{x'}\,\Omega_{\text{DYN},y'}) \\
&= \{0\} \text{ (for } V^1_{y'} = \left(W^1_{y'}\right)^{\perp}),
\end{aligned}$$

and $W_{x'} \subset \operatorname{Ker} T_{x'}\pi^{\text{KIN}}$, while $\operatorname{Ker} T_{x'}\pi^{\text{KIN}} \cap V_{x'} = \{0\}$. □

Returning to the linear case previously mentioned, we can reformulate in terms of constraints the condition we had for $\mathcal{M}^{\text{SHELL}}$ to define a closed dynamics on $\mathcal{N}^{\text{KIN}}$ (through the straightforward duality between the description of $\mathcal{M}^{\text{SHELL}}$ as a vector subspace and its description by linear constraints). This provides a specification of $\mathcal{M}^{\text{SHELL}}$ as characterized by three sets of constraints $C^{\mathcal{P}}_i$, $C^{\mathcal{N}}_j$, and C^{mix}_k, where the $C^{\mathcal{P}}_i$ only depend on the variables from $\mathcal{P}^{\text{KIN}}$ and characterize in $\mathcal{P}^{\text{KIN}}$ the projection $\mathcal{P}^{\text{SHELL}}$ of $\mathcal{M}^{\text{SHELL}}$, similarly the $C^{\mathcal{N}}_j$ only depend on the variables from $\mathcal{N}^{\text{KIN}}$ and characterize $\mathcal{N}^{\text{SHELL}}$ in $\mathcal{N}^{\text{KIN}}$, while the C^{mix}_k account for possible interactions. These interactions cannot be arbitrary: the requirements on V discussed above prescribe that the constraints $C^{\text{mix},x}_k$, obtained on $\mathcal{P}^{\text{KIN}}$ from the C^{mix}_k by fixing some $x \in \mathcal{N}^{\text{SHELL}}$, should perform a partial gauge fixing of $\mathcal{P}^{\text{SHELL}}$ (prop. A.8).

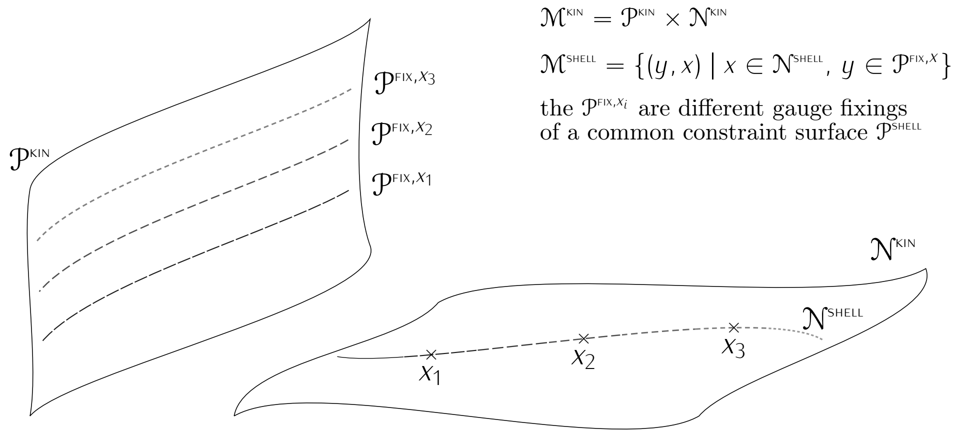


Figure 3.2 – A (rather broad but not exhaustive) way to construct an admissible dynamics on $\mathcal{M}^{\text{KIN}}$ in the factorizing case

In the generic case of a symplectic manifold $\mathcal{M}^{\text{KIN}}$ factorizing as $\mathcal{M}^{\text{KIN}} = \mathcal{P}^{\text{KIN}} \times \mathcal{N}^{\text{KIN}}$ (such as considered in subsection 2.3), the insight we gain from the linear case suggests a possibility, depicted in fig. 3.2, to design dynamics on $\mathcal{M}^{\text{SHELL}}$ that will project well on $\mathcal{N}^{\text{KIN}}$. This provides a much broader class of admissible dynamics than the trivial ones splitting into independent dynamics on $\mathcal{P}^{\text{KIN}}$ and $\mathcal{N}^{\text{KIN}}$.

Nevertheless, this procedure only corresponds to a *sufficient* condition for def. 3.1 to be fulfilled. Note that the gap between the necessary condition at the linear level supplied by prop. 3.5 and the characterization of $\mathcal{M}^{\text{SHELL}}$ considered here does not solely arise from global considerations: for $\mathcal{M}^{\text{SHELL}}$ to be of this form, some additional integrability conditions (ie. requirements at the second order) need to hold, so that we can combine the prescriptions in the tangent space of each point into prescriptions in small open patches.

Proposition 3.6 Let $\mathcal{M}^{\text{KIN}} = \mathcal{P}^{\text{KIN}} \times \mathcal{N}^{\text{KIN}}$, where $\mathcal{M}^{\text{KIN}}$, $\mathcal{P}^{\text{KIN}}$ and $\mathcal{N}^{\text{KIN}}$ are finite dimensional symplectic manifolds, and define:

$$\begin{aligned} \pi^{\text{KIN}} : \mathcal{M}^{\text{KIN}} &\to \mathcal{N}^{\text{KIN}} \\ y, x &\mapsto x \end{aligned}.$$

Let $(\mathcal{M}^{\text{DYN}}, \mathcal{M}^{\text{SHELL}}, \delta)$, $(\mathcal{P}^{\text{DYN}}, \mathcal{P}^{\text{SHELL}}, \theta)$, resp. $(\mathcal{N}^{\text{DYN}}, \mathcal{N}^{\text{SHELL}}, \gamma)$, be phase space reductions of $\mathcal{M}^{\text{KIN}}$, $\mathcal{P}^{\text{KIN}}$, resp. $\mathcal{N}^{\text{KIN}}$. Assume that there exist a submanifold $\mathcal{P}^{\text{FIX}}$ of $\mathcal{P}^{\text{SHELL}}$ and a smooth map:

$$\begin{aligned} \Psi : \mathcal{P}^{\text{FIX}} \times \mathcal{N}^{\text{SHELL}} &\to \mathcal{P}^{\text{SHELL}} \times \mathcal{N}^{\text{SHELL}} \\ y, x &\mapsto \psi(y, x),\; x \end{aligned},$$

such that:

1. $\text{Im}\Psi = \mathcal{M}^{\text{SHELL}}$ and $\Psi|_{\mathcal{P}^{\text{FIX}} \times \mathcal{N}^{\text{SHELL}} \to \mathcal{M}^{\text{SHELL}}}$ is a diffeomorphism;
2. $\Psi^* \left(\Omega_{\mathcal{M}^{\text{KIN}}}|_{T(\mathcal{M}^{\text{SHELL}})} \right) = \Omega_{\mathcal{P}^{\text{KIN}}}|_{T(\mathcal{P}^{\text{FIX}})} \times \Omega_{\mathcal{N}^{\text{KIN}}}|_{T(\mathcal{N}^{\text{SHELL}})}$;
3. $\forall x \in \mathcal{N}^{\text{SHELL}}$, $\mathcal{P}^{\text{FIX},x} := \psi \left\langle \mathcal{P}^{\text{FIX}} \times \{x\} \right\rangle$ defines a partial gauge fixing of $(\mathcal{P}^{\text{DYN}}, \mathcal{P}^{\text{SHELL}}, \theta)$ (prop. A.8).

Then, $(\mathcal{M}^{\text{DYN}}, \mathcal{M}^{\text{SHELL}}, \delta)$ and $(\mathcal{N}^{\text{DYN}}, \mathcal{N}^{\text{SHELL}}, \gamma)$ are related by π^{KIN}.

Proof From the definition of Ψ, we have $\pi^{\text{KIN}}\left\langle \mathcal{M}^{\text{SHELL}}\right\rangle = \operatorname{Im}\pi^{\text{KIN}}\circ\Psi = \mathcal{N}^{\text{SHELL}}$.

Let $x\in\mathcal{N}^{\text{SHELL}}$. Using assumption 3.6.1 together with the definition of $\mathcal{P}^{\text{FIX},x}$, the map:

$$\begin{array}{rcccl} \psi^x & : & \mathcal{P}^{\text{FIX}} & \to & \mathcal{P}^{\text{FIX},x} \\ & & y & \mapsto & \psi(y,x) \end{array}$$

is a diffeomorphism and, by 3.6.2, it satisfies $\psi^{x,*}\left(\Omega_{\mathcal{P}^{\text{KIN}}}\big|_{T(\mathcal{P}^{\text{FIX},x})}\right) = \Omega_{\mathcal{P}^{\text{KIN}}}\big|_{T(\mathcal{P}^{\text{FIX}})}$.

Now, from prop. A.8, $\left(\mathcal{P}^{\text{DYN}},\,\mathcal{P}^{\text{FIX},x},\,\theta|_{\mathcal{P}^{\text{FIX},x}}\right)$ is a phase space reduction of $\mathcal{P}^{\text{KIN}}$. Hence, defining $\theta^{\text{FIX},x} := \theta|_{\mathcal{P}^{\text{FIX},x}}\circ\psi^x$, $(\mathcal{P}^{\text{DYN}},\,\mathcal{P}^{\text{FIX}},\,\theta^{\text{FIX},x})$ is a phase space reduction of $\mathcal{P}^{\text{KIN}}$.

Using 3.6.2, we have for all $x\in\mathcal{N}^{\text{SHELL}}$, $y\in\mathcal{P}^{\text{FIX}}$ and for all $v\in T_x(\mathcal{N}^{\text{SHELL}})$, $w\in T_y(\mathcal{P}^{\text{FIX}})$:

$$0 = \Omega_{\mathcal{M}^{\text{KIN}}}\left(\left(\left[T_{(y,x)}\psi\right](0,v),\,v\right),\,\left(\left[T_{(y,x)}\psi\right](w,0),\,0\right)\right) = \Omega_{\mathcal{P}^{\text{KIN}}}\left(\left[T_{(y,x)}\psi\right](0,v),\,\left[T_{(y,x)}\psi\right](w,0)\right).$$

However, since ψ^x is a diffeomorphism, $\left[T_{(y,x)}\psi\right](w,0)$ runs through $T_{\psi(y,x)}\left(\mathcal{P}^{\text{FIX},x}\right)$ when w runs through $T_y(\mathcal{P}^{\text{FIX}})$, so $\left[T_{(y,x)}\psi\right](0,v)\in\left(T_{\psi(y,x)}\left(\mathcal{P}^{\text{FIX},x}\right)\right)^{\perp}\cap T_{\psi(y,x)}\left(\mathcal{P}^{\text{SHELL}}\right)$. As $\mathcal{P}^{\text{FIX},x}$ defines a partial gauge fixing of $(\mathcal{P}^{\text{DYN}},\mathcal{P}^{\text{SHELL}},\theta)$, we have $T_{\psi(y,x)}\left(\mathcal{P}^{\text{SHELL}}\right) = T_{\psi(y,x)}\left(\mathcal{P}^{\text{FIX},x}\right) + K_{\psi(y,x)}\left(\mathcal{P}^{\text{SHELL}}\right)$, hence $\left[T_{(y,x)}\psi\right](0,v)\in K_{\psi(y,x)}\left(\mathcal{P}^{\text{SHELL}}\right)$. Therefore, $\partial_x\theta^{\text{FIX},x} = 0$.

Without loss of generality, we can assume that $\mathcal{N}^{\text{SHELL}}$ is connected (otherwise $\mathcal{M}^{\text{SHELL}}$ is not connected either and we can consider each connected part of $\mathcal{N}^{\text{SHELL}}$ separately). Then, we can define $\theta^{\text{FIX}} := \theta^{\text{FIX},x}$.

Using θ^{FIX}, we define:

$$\begin{array}{rcccl} \widetilde{\delta} & : & \mathcal{M}^{\text{SHELL}} & \to & \mathcal{P}^{\text{DYN}}\times\mathcal{N}^{\text{DYN}} \\ & & y,x & \mapsto & (\theta^{\text{FIX}}\times\gamma)\circ\left(\Psi|_{\mathcal{P}^{\text{FIX}}\times\mathcal{N}^{\text{SHELL}}\to\mathcal{M}^{\text{SHELL}}}\right)^{-1}(y,x) = \left(\theta(y),\,\gamma(x)\right) \end{array}.$$

We want to prove that $\left(\mathcal{P}^{\text{DYN}}\times\mathcal{N}^{\text{DYN}},\,\mathcal{M}^{\text{SHELL}},\,\widetilde{\delta}\right)$ is a phase space reduction of $\mathcal{M}^{\text{KIN}}$.

First, we need to show that $\widetilde{\delta}$ is surjective, that its derivative is surjective at each point, and transports correctly the restriction to $\mathcal{M}^{\text{SHELL}}$ of the symplectic structure. Since $\left(\Psi|_{\mathcal{P}^{\text{SHELL}}\times\mathcal{N}^{\text{FIX}}\to\mathcal{M}^{\text{SHELL}}}\right)^{-1}$ is a diffeomorphism and transports the symplectic structure, we need only to check the corresponding properties of $\theta^{\text{FIX}}\times\gamma$. Now, since θ^{FIX} and γ corresponds to phase space reductions, they indeed have the required properties, and so does $\theta^{\text{FIX}}\times\gamma$.

Let $(y,x)\in\mathcal{P}^{\text{FIX}}\times\mathcal{N}^{\text{SHELL}}$. We choose a basis $(e_i)_{i\leqslant k}$ of $K_x(\mathcal{N}^{\text{SHELL}})$ (with $k := \dim K_x(\mathcal{N}^{\text{SHELL}})$) and we complete it into a basis $(e_i)_{i\leqslant n}$ of $T_x(\mathcal{N}^{\text{SHELL}})$ (with $n := \dim T_x(\mathcal{N}^{\text{SHELL}})$). We also choose a basis $(f_j)_{j\leqslant l}$ of $K_y(\mathcal{P}^{\text{FIX}})$ (with $l := \dim K_y(\mathcal{P}^{\text{FIX}})$) and complete it into a basis $(f_j)_{j\leqslant p}$ of $T_y(\mathcal{P}^{\text{FIX}})$ (with $p := \dim T_y(\mathcal{P}^{\text{FIX}})$). Then, we have:

$$T_{\Psi(y,x)}(\mathcal{M}^{\text{SHELL}}) = \operatorname{Vect}\left\{\left(\left[T_{(y,x)}\psi\right](0,e_i),\,e_i\right)\;\middle|\;i\leqslant n\right\} + \operatorname{Vect}\left\{\left(\left[T_{(y,x)}\psi\right](f_j,0),\,0\right)\;\middle|\;j\leqslant p\right\}.$$

As proved above, we have $\forall v\in T_x(\mathcal{N}^{\text{SHELL}})$, $\left[T_{(y,x)}\psi\right](0,v)\in K_{\psi(y,x)}\left(\mathcal{P}^{\text{SHELL}}\right)$. Since ψ^x is a diffeomorphism $\mathcal{P}^{\text{FIX}}\to\mathcal{P}^{\text{FIX},x}$, the $\left[T_{(y,x)}\psi\right](f_j,0)$ for $j\leqslant p$ span $T_{\psi(y,x)}\left(\mathcal{P}^{\text{FIX},x}\right)$. And since ψ^x transports the symplectic structure, we also have $\forall w\in T_y(\mathcal{P}^{\text{FIX}})$, $\left[T_{(y,x)}\psi\right](w,0)\in K_{\psi(y,x)}\left(\mathcal{P}^{\text{FIX},x}\right)\Leftrightarrow w\in K_y\left(\mathcal{P}^{\text{FIX}}\right)$. Therefore, we have:

$$K_{\Psi(y,x)}(\mathcal{M}^{\text{SHELL}}) = \operatorname{Vect}\left\{\left(\left[T_{(y,x)}\psi\right](0,e_i),\,e_i\right)\;\middle|\;i\leqslant k\right\} + \operatorname{Vect}\left\{\left(\left[T_{(y,x)}\psi\right](f_j,0),\,0\right)\;\middle|\;j\leqslant l\right\}.$$

Using $K_{\psi(y,x)}\left(\mathcal{P}^{\text{FIX},x}\right) \subset K_{\psi(y,x)}\left(\mathcal{P}^{\text{SHELL}}\right)$, we can now check that $T_{\Psi(y,x)}\,\widetilde{\delta}\left\langle K_{\Psi(y,x)}(\mathcal{M}^{\text{SHELL}})\right\rangle = \{0\}$. Therefore, the leaf of the foliation $K(\mathcal{M}^{\text{SHELL}})$ that goes through $\Psi(y,x)$ is included in $\widetilde{\delta}^{-1}\left\langle\{\theta^{\text{FIX}}(y),\,\gamma(x)\}\right\rangle$ (as leaves of foliation are by definition connected).

On the other hand, $\widetilde{\delta}^{-1}\left\langle\{\theta^{\text{FIX}}(y),\,\gamma(x)\}\right\rangle = \Psi\left\langle\theta^{\text{FIX},-1}\left\langle\{\theta^{\text{FIX}}(y)\}\right\rangle \times \gamma^{-1}\left\langle\{\gamma(x)\}\right\rangle\right\rangle$ is connected as image by a continuous map of a Cartesian product of connected spaces. And its tangent space at $\Psi(y,x)$ is given by:

$$T_{(y,x)}\Psi\left\langle K_y\left(\mathcal{P}^{\text{FIX}}\right) \times K_x\left(\mathcal{N}^{\text{SHELL}}\right)\right\rangle = K_{\Psi(y,x)}(\mathcal{M}^{\text{SHELL}}) \text{ (using 3.6.2).}$$

Therefore, $\widetilde{\delta}^{-1}\left\langle\{\theta^{\text{FIX}}(y),\,\gamma(x)\}\right\rangle$ is included in the leaf of the foliation $K(\mathcal{M}^{\text{SHELL}})$ that goes through $\Psi(y,x)$.

This concludes the proof that $\left(\mathcal{P}^{\text{DYN}} \times \mathcal{N}^{\text{DYN}},\, \mathcal{M}^{\text{SHELL}},\, \widetilde{\delta}\right)$ is a phase space reduction of $\mathcal{M}^{\text{KIN}}$. Now, using prop. A.6, there exists a symplectomorphism $\Phi : \mathcal{M}^{\text{DYN}} \to \mathcal{P}^{\text{DYN}} \times \mathcal{N}^{\text{DYN}}$ such that $\Phi \circ \delta = \widetilde{\delta}$. π^{DYN} is then the projection corresponding to this factorization of $\mathcal{M}^{\text{DYN}}$. □

We are now ready to consider a projective system of phase spaces $\mathcal{M}^{\text{KIN}}_\eta$, with a phase space reduction of $\mathcal{M}^{\text{KIN}}_\eta$ for each η. As announced at the beginning of this subsection, we want to examine the situation where the reduced phase spaces $\mathcal{M}^{\text{DYN}}_\eta$ can be arranged into a new projective system of phase spaces, in such a way that the maps, that translate the kinematical observables into dynamical ones for each η, are intertwined by the projections on both sides. Thus, we can associate to an observable on the projective limit of the $\mathcal{M}^{\text{KIN}}_\eta$ an observable on the projective limit of the $\mathcal{M}^{\text{DYN}}_\eta$. In a dual way, to each state on this dynamical projective system of phase spaces corresponds a projective family of orbits in the constraint surfaces $\mathcal{M}^{\text{SHELL}}_\eta$ (another option here would be to consider projective family of probability measures, aka. statistical states, in which case we would map dynamical statistical states to on-shell supported, gauge invariant, kinematical statistical states).

The previous study, examining a projection that relates the phase space reductions on two symplectic manifolds, is the key element for this construction. Indeed the requirement that the dynamical phase spaces should readily assemble into a new projective system can actually be enforced by asking, for each pair of index $\eta \preccurlyeq \eta'$, that the reductions on $\mathcal{M}^{\text{KIN}}_\eta$ and $\mathcal{M}^{\text{KIN}}_{\eta'}$ should be related by $\pi^{\text{KIN}}_{\eta'\to\eta}$.

Definition 3.7 Let $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^\downarrow$ be a projective system of phase spaces. An elementary reduction of $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^\downarrow$ is a quadruple $\left(\left(\mathcal{M}^{\text{DYN}}_\eta\right)_{\eta\in\mathcal{L}}, \left(\mathcal{M}^{\text{SHELL}}_\eta\right)_{\eta\in\mathcal{L}}, \left(\pi^{\text{DYN}}_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'}, \left(\delta_\eta\right)_{\eta\in\mathcal{L}}\right)$ such that:

1. $(\mathcal{L}, \mathcal{M}^{\text{DYN}}, \pi^{\text{DYN}})^\downarrow$ is a projective system of phase spaces;
2. $\forall \eta \in \mathcal{L}$, $(\mathcal{M}^{\text{DYN}}_\eta, \mathcal{M}^{\text{SHELL}}_\eta, \delta_\eta)$ is a phase space reduction of $\mathcal{M}^{\text{KIN}}_\eta$;
3. $\forall \eta \preccurlyeq \eta' \in \mathcal{L}$, $\pi^{\text{KIN}}_{\eta'\to\eta}\left\langle\mathcal{M}^{\text{SHELL}}_{\eta'}\right\rangle = \mathcal{M}^{\text{SHELL}}_\eta$ and:

   $\forall x_\eta \in \mathcal{M}^{\text{SHELL}}_\eta$, $\forall y_{\eta'} \in \mathcal{M}^{\text{DYN}}_{\eta'}$,

$$\left(\exists x_{\eta'} \in \mathcal{M}^{\text{SHELL}}_{\eta'} \;\; / \;\; \delta_{\eta'}(x_{\eta'}) = y_{\eta'} \quad \& \quad \pi^{\text{KIN}}_{\eta'\to\eta}(x_{\eta'}) = x_\eta\right) \Leftrightarrow \left(\delta_\eta(x_\eta) = \pi^{\text{DYN}}_{\eta'\to\eta}(y_{\eta'})\right).$$

Whenever possible, we will use the shortened notation $(\mathcal{L}, \mathcal{M}, \pi, \delta)^{\text{DYN}}$ instead of $\left(\left(\mathcal{M}^{\text{DYN}}_\eta\right)_{\eta\in\mathcal{L}},\right.$ $\left.\left(\mathcal{M}^{\text{SHELL}}_\eta\right)_{\eta\in\mathcal{L}}, \left(\pi^{\text{DYN}}_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'}, \left(\delta_\eta\right)_{\eta\in\mathcal{L}}\right)$.

Definition 3.8 We consider the same objects as in def. 3.7 and we define (in analogy to the definition of $\mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M},\pi)}$ in def. 2.3) $\widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ as:

$$\widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})} := \left\{ (D_\eta)_{\eta\in\mathcal{L}} \in \prod_{\eta\in\mathcal{L}} \mathcal{P}(\mathcal{M}^{\text{KIN}}_\eta) \;\middle|\; \forall \eta \preccurlyeq \eta',\, \pi^{\text{KIN}}_{\eta'\to\eta}\left\langle D_{\eta'}\right\rangle = D_\eta \right\},$$

where, for $\eta \in \mathcal{L}$, $\mathcal{P}(\mathcal{M}^{\text{KIN}}_\eta)$ is the set of subsets of $\mathcal{M}^{\text{KIN}}_\eta$.

Then, we define:

$$\begin{array}{rcl} \Delta \;:\; \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})} & \to & \widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})} \\ (y_\eta)_{\eta\in\mathcal{L}} & \mapsto & \left(\delta^{-1}_\eta \left\langle \{y_\eta\}\right\rangle\right)_{\eta\in\mathcal{L}} \end{array},$$

which is well-defined as a map $\mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})} \to \widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$, for we have $\forall \eta \preccurlyeq \eta' \in \mathcal{L}$, $\forall y_{\eta'} \in \mathcal{M}^{\text{DYN}}_{\eta'}$, $\delta^{-1}_\eta \left\langle \left\{\pi^{\text{DYN}}_{\eta'\to\eta}(y_{\eta'})\right\}\right\rangle = \pi^{\text{KIN}}_{\eta'\to\eta}\left\langle \delta^{-1}_{\eta'}\left\langle y_{\eta'}\right\rangle\right\rangle$.

Proposition 3.9 Let $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^{\downarrow}$ be a projective system of phase spaces and let $(\mathcal{L}, \mathcal{M}, \pi, \delta)^{\text{DYN}}$ be an elementary reduction of $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^{\downarrow}$. We define (in analogy to def. 2.4) $\mathcal{A}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ as the set of equivalence classes in $\bigcup_{\eta\in\mathcal{L}} B(\mathcal{M}_\eta)$ for the equivalence relation defined by:

$$\forall \eta, \eta' \in \mathcal{L},\, \forall f_\eta \in B(\mathcal{M}^{\text{KIN}}_\eta),\, \forall f_{\eta'} \in B(\mathcal{M}^{\text{KIN}}_{\eta'}),$$

$$f_\eta \sim^{\text{KIN}} f_{\eta'} \quad\Leftrightarrow\quad \left(\exists\, \eta'' \in \mathcal{L} \;\;/\;\; \eta \preccurlyeq \eta'', \eta' \preccurlyeq \eta'' \quad \& \quad f_\eta \circ \pi^{\text{KIN}}_{\eta''\to\eta} = f_{\eta'} \circ \pi^{\text{KIN}}_{\eta''\to\eta'}\right),$$

and similarly $\mathcal{A}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})}$ with the equivalence relation $\sim^{\text{DYN}}$.

Then, the map:

$$\begin{array}{rcl} (\,\cdot\,)^{\text{DYN}} \;:\; \mathcal{A}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})} & \to & \mathcal{A}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})} \\ \left[f_\eta\right]_{\sim^{\text{KIN}}} & \mapsto & \left[f^{\text{DYN}}_\eta\right]_{\sim^{\text{DYN}}} \end{array}$$

is well-defined.

For $(D_\eta)_{\eta\in\mathcal{L}} \in \widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ and $f = [f_\eta]_{\sim^{\text{KIN}}} \in \mathcal{A}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$, we define:

$$[f_\eta]_{\sim^{\text{KIN}}}\left((D_\eta)_{\eta\in\mathcal{L}}\right) := \sup\left\{f_\eta(x) \mid x \in D_\eta\right\},$$

(the definition of the equivalence relation $\sim^{\text{KIN}}$ ensures that this is well-defined)

Then, we have for all $y \in \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})}$ and all $f \in \mathcal{A}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$:

$$f^{\text{DYN}}(y) = f\left(\Delta(y)\right). \tag{3.9.1}$$

Proof What we need to show is that for $\eta, \eta' \in \mathcal{L}$, $f_\eta \in B(\mathcal{M}_\eta)$ and $f_{\eta'} \in B(\mathcal{M}_{\eta'})$, $\left(f_\eta \sim^{\text{KIN}} f_{\eta'}\right) \Rightarrow \left(f^{\text{DYN}}_\eta \sim^{\text{DYN}} f^{\text{DYN}}_{\eta'}\right)$. Indeed if there exist $\eta'' \in \mathcal{L}$, with $\eta'' \succcurlyeq \eta$, $\eta'' \succcurlyeq \eta'$, and $f_{\eta''} \in B(\mathcal{M}_{\eta''})$ such that

$f_\eta \circ \pi^{\text{KIN}}_{\eta''\to\eta} = f_{\eta'} \circ \pi^{\text{KIN}}_{\eta''\to\eta'}$, then, from prop. 3.3:

$$f^{\text{DYN}}_\eta \circ \pi^{\text{DYN}}_{\eta''\to\eta} = \left(f_\eta \circ \pi^{\text{KIN}}_{\eta''\to\eta}\right)^{\text{DYN}} = \left(f_{\eta'} \circ \pi^{\text{KIN}}_{\eta''\to\eta'}\right)^{\text{DYN}} = f^{\text{DYN}}_{\eta'} \circ \pi^{\text{DYN}}_{\eta''\to\eta'}.$$

Then, we only need to check eq. (3.9.1) for a particular representative f_η of f:

$$f^{\text{DYN}}(y) = f^{\text{DYN}}_\eta(y_\eta) = \sup_{x\in\delta_\eta^{-1}\left\langle\{y_\eta\}\right\rangle} f_\eta(x) = \sup_{x\in(\delta^{\text{DYN}}(y))_\eta} f_\eta(x) = f(\Delta(y)).$$

□

Proposition 3.10 Let $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^\downarrow$ be a projective system of phase spaces. For all $\eta \in \mathcal{L}$, we give ourselves a phase space reduction $(\mathcal{M}^{\text{DYN}}_\eta, \mathcal{M}^{\text{SHELL}}_\eta, \delta_\eta)$ of $\mathcal{M}_\eta$. The following statements are equivalent:

1. there exists a family of surjective maps $\left(\pi^{\text{DYN}}_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'}$ such that $(\mathcal{L}, \mathcal{M}, \pi, \delta)^{\text{DYN}}$ is an elementary reduction of $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^\downarrow$;
2. $\forall \eta \preccurlyeq \eta'$, $(\mathcal{M}^{\text{DYN}}_{\eta'}, \mathcal{M}^{\text{SHELL}}_{\eta'}, \delta_{\eta'})$ and $(\mathcal{M}^{\text{DYN}}_\eta, \mathcal{M}^{\text{SHELL}}_\eta, \delta_\eta)$ are related by $\pi^{\text{KIN}}_{\eta'\to\eta}$.

Proof By definition of an elementary reduction of $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^\downarrow$, we have 3.10.1 $\Rightarrow$ 3.10.2.

To prove the other direction, we need to show that the $\pi^{\text{DYN}}_{\eta'\to\eta}$ induced by the $\pi^{\text{KIN}}_{\eta'\to\eta}$ satisfy the three-spaces consistency condition:

$$\forall \eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L},\ \pi^{\text{DYN}}_{\eta''\to\eta} = \pi^{\text{DYN}}_{\eta'\to\eta} \circ \pi^{\text{DYN}}_{\eta''\to\eta'}.$$

For $x_\eta \in \mathcal{M}^{\text{SHELL}}_\eta$, $y_{\eta''} \in \mathcal{M}^{\text{DYN}}_{\eta''}$, we have (using def. 3.7.3 for $\eta \preccurlyeq \eta'$ and $\eta' \preccurlyeq \eta''$):

$$\left(\delta_\eta(x_\eta) = \pi^{\text{DYN}}_{\eta'\to\eta}\left(\pi^{\text{DYN}}_{\eta''\to\eta'}(y_{\eta''})\right)\right) \Leftrightarrow \left(\exists x_{\eta'} \in \mathcal{M}^{\text{SHELL}}_{\eta'} \,/\, \delta_{\eta'}(x_{\eta'}) = \pi^{\text{DYN}}_{\eta''\to\eta'}(y_{\eta''}) \quad \& \quad \pi^{\text{KIN}}_{\eta'\to\eta}(x_{\eta'}) = x_\eta\right)$$

$$\Leftrightarrow \left(\exists x_{\eta'} \in \mathcal{M}^{\text{SHELL}}_{\eta'},\ \exists x_{\eta''} \in \mathcal{M}^{\text{SHELL}}_{\eta''} \,/\, \delta_{\eta''}(x_{\eta''}) = y_{\eta''} \quad \& \quad \pi^{\text{KIN}}_{\eta''\to\eta'}(x_{\eta''}) = x_{\eta'} \quad \& \quad \pi^{\text{KIN}}_{\eta'\to\eta}(x_{\eta'}) = x_\eta\right)$$

$$\Leftrightarrow \left(\exists x_{\eta''} \in \mathcal{M}^{\text{SHELL}}_{\eta''} \,/\, \delta_{\eta''}(x_{\eta''}) = y_{\eta''} \quad \& \quad \pi^{\text{KIN}}_{\eta'\to\eta}\left(\pi^{\text{KIN}}_{\eta''\to\eta'}(x_{\eta''})\right) = x_\eta\right).$$

Hence, using $\pi^{\text{KIN}}_{\eta'\to\eta} \circ \pi^{\text{KIN}}_{\eta''\to\eta'} = \pi^{\text{KIN}}_{\eta''\to\eta'}$, and applying prop. 3.2 with $\pi^{\text{DYN}}_{\eta''\to\eta}$ and $\pi^{\text{DYN}}_{\eta'\to\eta} \circ \pi^{\text{DYN}}_{\eta''\to\eta'}$, we have $\pi^{\text{DYN}}_{\eta''\to\eta} = \pi^{\text{DYN}}_{\eta'\to\eta} \circ \pi^{\text{DYN}}_{\eta''\to\eta'}$. □

Recalling the discussion of subsection 2.2, regarding restrictions and extensions of the label set, we would like to understand how elementary reductions pass through these operations. It is quite straightforward that everything will go smoothly if we *restrict* the label set.

The interesting question occurs when we have an elementary reduction on a subset $\mathcal{L}'$ of $\mathcal{L}$. In particular, if $\mathcal{L}'$ is cofinal in $\mathcal{L}$, we can identify the kinematical spaces of states and observables over the projective system restricted to $\mathcal{L}'$ with the ones over the original projective system on $\mathcal{L}$ (prop. 2.5), thus the transport of observables (from the kinematical to the dynamical theory) and states (from the dynamical to the kinematical theory) arising from an elementary reduction on $\mathcal{L}'$ immediately defines corresponding transport maps between the kinematical projective structure on $\mathcal{L}$ and the dynamical projective structure (which is then only defined for the label set $\mathcal{L}'$). In other words, we are still able to glue together the dynamical phase spaces $\mathcal{M}^{\text{DYN}}_\eta$ ($\eta \in \mathcal{L}'$) into a dynamical projective structure, to inherit observables on this structure and to project back its states. However, in general, there will *not* exist an elementary reduction on $\mathcal{L}$, that would reproduce the same transport maps (modulo the identification of the thus obtained dynamical structure on $\mathcal{L}$ with its restriction to $\mathcal{L}'$). This point will play a key role when moving to the regularization of a dynamics that does not break down well on the projective structure (subsection 3.2).

What is lacking, when trying to extend to $\mathcal{L}$ an elementary reduction on $\mathcal{L}'$, is the assurance that there will exist phase space reductions of the $\mathcal{M}^{\text{KIN}}_\eta$ for $\eta \in \mathcal{L} \setminus \mathcal{L}'$, and that these reductions will be compatible with each other, as well as with the given reductions on $\mathcal{L}'$: specifically, for any pair of labels $\eta \preccurlyeq \eta'$ (one or both being in $\mathcal{L} \setminus \mathcal{L}'$) the reductions should be related by $\pi^{\text{KIN}}_{\eta'\to\eta}$ (the elementary reduction on $\mathcal{L}'$ already accounts for the compatibility when both labels are in $\mathcal{L}'$). Prop. 3.12 shows slightly weaker hypotheses under which the extension *is* possible, provided the $\mathcal{M}^{\text{KIN}}_\eta$ for $\eta \in \mathcal{L} \setminus \mathcal{L}'$ are finite dimensional.

Proposition 3.11 Let $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^\downarrow$ be a projective system of phase spaces and $(\mathcal{L}, \mathcal{M}, \pi, \delta)^{\text{DYN}}$ be an elementary reduction of $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^\downarrow$. If $\mathcal{L}'$ is a directed subset of $\mathcal{L}$, $\left(\mathcal{L}', \mathcal{M}, \pi, \delta\right)^{\text{DYN}}$ is an elementary reduction of $\left(\mathcal{L}', \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}}\right)^\downarrow$ and we have:

$$\widehat{\sigma}^{\text{KIN}}_{\mathcal{L}\to\mathcal{L}'} \circ \Delta = \Delta' \circ \sigma^{\text{DYN}}_{\mathcal{L}\to\mathcal{L}'} ,$$

where $\widehat{\sigma}^{\text{KIN}}_{\mathcal{L}\to\mathcal{L}'} : \widehat{\mathcal{S}}^\downarrow_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})} \to \widehat{\mathcal{S}}^\downarrow_{(\mathcal{L}',\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$, $\sigma^{\text{DYN}}_{\mathcal{L}\to\mathcal{L}'} : \mathcal{S}^\downarrow_{(\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})} \to \mathcal{S}^\downarrow_{(\mathcal{L}',\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})}$ are defined in analogy to prop. 2.5, while $\Delta : \mathcal{S}^\downarrow_{(\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})} \to \widehat{\mathcal{S}}^\downarrow_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ and $\Delta' : \mathcal{S}^\downarrow_{(\mathcal{L}',\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})} \to \widehat{\mathcal{S}}^\downarrow_{(\mathcal{L}',\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ are defined as in def. 3.8.

In addition, for any $f \in \mathcal{A}^\downarrow_{(\mathcal{L}',\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$, we have:

$$\left(\beta^{\text{KIN}}_{\mathcal{L}\leftarrow\mathcal{L}'}(f)\right)^{\text{DYN}} = \beta^{\text{DYN}}_{\mathcal{L}\leftarrow\mathcal{L}'}\left(f^{\text{DYN}}\right),$$

where $\beta^{\text{KIN/DYN}}_{\mathcal{L}\leftarrow\mathcal{L}'} : \mathcal{A}^\downarrow_{\left(\mathcal{L}',\mathcal{M}^{\text{KIN/DYN}},\pi^{\text{KIN/DYN}}\right)} \to \mathcal{A}^\downarrow_{\left(\mathcal{L},\mathcal{M}^{\text{KIN/DYN}},\pi^{\text{KIN/DYN}}\right)}$ are defined in analogy to prop. 2.5, while $(\,\cdot\,)^{\text{DYN}} : \mathcal{A}^\downarrow_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})} \to \mathcal{A}^\downarrow_{(\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})}$ and $(\,\cdot\,)^{\text{DYN}} : \mathcal{A}^\downarrow_{(\mathcal{L}',\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})} \to \mathcal{A}^\downarrow_{(\mathcal{L}',\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})}$ are defined as in prop. 3.9.

Proof That $\left(\mathcal{L}', \mathcal{M}, \pi, \delta\right)^{\text{DYN}}$ is an elementary reduction of $\left(\mathcal{L}', \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}}\right)^\downarrow$ can be immediately checked from def. 3.7.

Let $\left(y_\eta\right)_{\eta\in\mathcal{L}} \in \mathcal{S}^\downarrow_{(\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})}$. We have:

$$\widehat{\sigma}^{\text{KIN}}_{\mathcal{L}\to\mathcal{L}'} \circ \Delta\left(\left(y_\eta\right)_{\eta\in\mathcal{L}}\right) = \widehat{\sigma}^{\text{KIN}}_{\mathcal{L}\to\mathcal{L}'}\left(\left(\delta_\eta^{-1}\left\langle y_\eta\right\rangle\right)_{\eta\in\mathcal{L}}\right) = \left(\delta_\eta^{-1}\left\langle y_\eta\right\rangle\right)_{\eta\in\mathcal{L}'}$$

$$= \Delta'\left(\left(y_\eta\right)_{\eta\in\mathcal{L}'}\right) = \Delta' \circ \sigma^{\text{DYN}}_{\mathcal{L}\to\mathcal{L}'}\left(\left(y_\eta\right)_{\eta\in\mathcal{L}}\right).$$

Let $f = [f_\eta]_{\sim^{\text{KIN}}} \in \mathcal{A}^\downarrow_{(\mathcal{L}',\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$. We have:

$$\left(\beta^{\text{KIN}}_{\mathcal{L}\leftarrow\mathcal{L}'}(f)\right)^{\text{DYN}} = \left[f^{\text{DYN}}_\eta\right]_{\sim^{\text{DYN}}} = \beta^{\text{DYN}}_{\mathcal{L}\leftarrow\mathcal{L}'}\left(f^{\text{DYN}}_{\prime}\right).$$

□

Proposition 3.12 Let $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^\downarrow$ be a projective system of phase spaces and let $\mathcal{L}'$ be a cofinal subset of $\mathcal{L}$. We assume:

1. that we are given an elementary reduction $\left(\mathcal{L}', \mathcal{M}, \pi, \delta\right)^{\text{DYN}}$ of $\left(\mathcal{L}', \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}}\right)^{\downarrow}$;
2. that for any $\eta \in \mathcal{L} \setminus \mathcal{L}'$, $\mathcal{M}^{\text{KIN}}_{\eta}$ is finite dimensional and we are given a phase space reduction $\left(\mathcal{M}^{\text{DYN}}_{\eta}, \mathcal{M}^{\text{SHELL}}_{\eta}, \delta_{\eta}\right)$ of $\mathcal{M}^{\text{KIN}}_{\eta}$;
3. that for any $\eta \in \mathcal{L}$, and for any $\eta' \in \mathcal{L}'$ with $\eta' \succcurlyeq \eta$, the reductions on $\mathcal{M}^{\text{KIN}}_{\eta'}$ and $\mathcal{M}^{\text{KIN}}_{\eta}$ are related by $\pi^{\text{KIN}}_{\eta'\to\eta}$.

Then, $\left(\mathcal{L}', \mathcal{M}, \pi, \delta\right)^{\text{DYN}}$ can be completed into an elementary reduction $(\mathcal{L}, \mathcal{M}, \pi, \delta)^{\text{DYN}}$ of $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^{\downarrow}$.

Lemma 3.13 Let $\mathcal{M}^{\text{KIN}}$, $\mathcal{N}^{\text{KIN}}$ and $\mathcal{P}^{\text{KIN}}$ be symplectic manifolds and assume that $\mathcal{N}^{\text{KIN}}$ and $\mathcal{P}^{\text{KIN}}$ are finite dimensional. Let $\pi^{\text{KIN}}_1 : \mathcal{M}^{\text{KIN}} \to \mathcal{N}^{\text{KIN}}$, $\pi^{\text{KIN}}_2 : \mathcal{N}^{\text{KIN}} \to \mathcal{P}^{\text{KIN}}$, and $\pi^{\text{KIN}}_3 : \mathcal{M}^{\text{KIN}} \to \mathcal{P}^{\text{KIN}}$ be projections compatible with the symplectic structures, satisfying $\pi^{\text{KIN}}_3 = \pi^{\text{KIN}}_2 \circ \pi^{\text{KIN}}_1$. Let $(\mathcal{M}^{\text{DYN}}, \mathcal{M}^{\text{SHELL}}, \delta)$, $(\mathcal{N}^{\text{DYN}}, \mathcal{N}^{\text{SHELL}}, \gamma)$ and $(\mathcal{P}^{\text{DYN}}, \mathcal{P}^{\text{SHELL}}, \eta)$ be phase space reductions of $\mathcal{M}^{\text{KIN}}$, $\mathcal{N}^{\text{KIN}}$ and $\mathcal{P}^{\text{KIN}}$ respectively.

If the reductions on $\mathcal{M}^{\text{KIN}}$ and $\mathcal{N}^{\text{KIN}}$ are related by π^{KIN}_1 and the reductions on $\mathcal{M}^{\text{KIN}}$ and $\mathcal{P}^{\text{KIN}}$ are related by π^{KIN}_3, then the reductions on $\mathcal{N}^{\text{KIN}}$ and $\mathcal{P}^{\text{KIN}}$ are related by π^{KIN}_2.

Proof Applying def. 3.1.1 for π^{KIN}_1 and π^{KIN}_3, we have:

$$\pi^{\text{KIN}}_2 \left\langle \mathcal{N}^{\text{SHELL}} \right\rangle = \pi^{\text{KIN}}_2 \left\langle \pi^{\text{KIN}}_1 \left\langle \mathcal{M}^{\text{SHELL}} \right\rangle\right\rangle = \pi^{\text{KIN}}_3 \left\langle \mathcal{M}^{\text{SHELL}} \right\rangle = \mathcal{P}^{\text{SHELL}}.$$

Let $\pi^{\text{DYN}}_1 : \mathcal{M}^{\text{DYN}} \to \mathcal{N}^{\text{DYN}}$ and $\pi^{\text{DYN}}_3 : \mathcal{M}^{\text{DYN}} \to \mathcal{P}^{\text{DYN}}$ be as in def. 3.1.2. For any $y, y' \in \mathcal{M}^{\text{DYN}}$ such that $\pi^{\text{DYN}}_1(y) = \pi^{\text{DYN}}_1(y')$, there exists $z \in \gamma^{-1}\left\langle \pi^{\text{DYN}}_1(y)\right\rangle \subset \mathcal{N}^{\text{SHELL}}$ (for γ is surjective from def. A.1.2) and, using eq. (3.1.1), there exist $x, x' \in \mathcal{M}^{\text{SHELL}}$ such that:

$$\delta(x) = y,\ \delta(x') = y' \text{ and } \pi^{\text{KIN}}_1(x) = z = \pi^{\text{KIN}}_1(x').$$

Therefore, $\pi^{\text{KIN}}_3(x) = \pi^{\text{KIN}}_2(z) = \pi^{\text{KIN}}_3(x')$, so using again eq. (3.1.1), we have $\pi^{\text{DYN}}_3(y) = \eta \circ \pi^{\text{KIN}}_2(z) = \pi^{\text{DYN}}_3(y')$.

Hence, π^{DYN}_3 is constant on the level sets of π^{DYN}_1, so there exists a map $\pi^{\text{DYN}}_2 : \mathcal{N}^{\text{DYN}} \to \mathcal{P}^{\text{DYN}}$ such that $\pi^{\text{DYN}}_3 = \pi^{\text{DYN}}_2 \circ \pi^{\text{DYN}}_1$.

Now, for $z' \in \mathcal{P}^{\text{SHELL}}$ and $w \in \mathcal{N}^{\text{DYN}}$, there exists $y \in \mathcal{M}^{\text{DYN}}$ such that $\pi^{\text{DYN}}_1(y) = w$ (for π^{DYN}_1 is surjective) and we have:

$$\left(\eta(z') = \pi^{\text{DYN}}_2(w)\right) \Leftrightarrow \left(\eta(z') = \pi^{\text{DYN}}_3(y)\right) \Leftrightarrow \left(\exists\, x \in \mathcal{M}^{\text{SHELL}} \,/\, \delta(x) = y \;\;\&\;\; \pi^{\text{KIN}}_3(x) = z'\right)$$

$$\Leftrightarrow \left(\exists\, z \in \mathcal{N}^{\text{SHELL}} \,/\, \gamma(z) = w \;\;\&\;\; \pi^{\text{KIN}}_2(z) = z'\right),$$

where the last equivalence comes from setting $z = \pi^{\text{KIN}}_1(x)$ (for proving '⇒') and using eq. (3.1.1) with $\gamma(z) = \pi^{\text{DYN}}_1(y)$ (for '⇐'). Hence, π^{DYN}_2 fulfills eq. (3.1.1).

In particular, we then have $\eta \circ \pi^{\text{KIN}}_2 = \pi^{\text{DYN}}_2 \circ \gamma$. Thus, since $\mathcal{N}^{\text{KIN}}$, $\mathcal{N}^{\text{DYN}}$ and $\mathcal{P}^{\text{DYN}}$ are smooth finite dimensional manifolds, $\eta \circ \pi^{\text{KIN}}_2$ is smooth and γ is surjective with surjective derivative at any point (def. A.1.3), the rank theorem implies [54, prop. 5.19] that π^{DYN}_2 is smooth.

Finally, we need to show that π^{DYN}_2 is a surjective map compatible with the symplectic structures. We have $\pi^{\text{DYN}}_2 \left\langle \mathcal{N}^{\text{DYN}}\right\rangle = \pi^{\text{DYN}}_2 \left\langle \pi^{\text{DYN}}_1 \left\langle \mathcal{M}^{\text{DYN}}\right\rangle\right\rangle = \pi^{\text{DYN}}_3 \left\langle \mathcal{M}^{\text{DYN}}\right\rangle = \mathcal{P}^{\text{DYN}}$. And, for any $w \in \mathcal{N}^{\text{DYN}}$, there exists $y \in \mathcal{M}^{\text{DYN}}$ with $\pi^{\text{DYN}}_1(y) = w$, so that for any $\upsilon \in T^*_{\pi^{\text{DYN}}_2(w)}(\mathcal{P}^{\text{DYN}})$:

$$[T_w \pi_2^{\text{DYN}}] \left(\underline{\upsilon \circ [T_w \pi_2^{\text{DYN}}]} \right) = [T_w \pi_2^{\text{DYN}}] \circ \left[T_y \pi_1^{\text{DYN}}\right] \left(\underline{\upsilon \circ [T_w \pi_2^{\text{DYN}}] \circ \left[T_y \pi_1^{\text{DYN}}\right]} \right)$$

$$= \left[T_y \pi_3^{\text{DYN}}\right] \left(\underline{\upsilon \circ \left[T_y \pi_3^{\text{DYN}}\right]} \right) = \underline{\upsilon}\,,$$

therefore π_2^{DYN} fulfills eq. (2.1.1). □

Proof of prop. 3.12 Let $\eta \in \mathcal{L}$ and $\eta' \in \mathcal{L} \setminus \mathcal{L}'$, with $\eta' \succcurlyeq \eta$. Since $\mathcal{L}'$ is a cofinal part of $\mathcal{L}$, there exists $\eta'' \in \mathcal{L}'$ such that $\eta'' \succcurlyeq \eta' \succcurlyeq \eta$. Using lemma 3.13 ($\mathcal{M}_{\eta'}^{\text{KIN}}$ is finite dimensional, for $\eta' \in \mathcal{L} \setminus \mathcal{L}'$, so $\mathcal{M}_{\eta}^{\text{KIN}}$ is finite dimensional, for $\eta \preccurlyeq \eta'$), the reductions on $\mathcal{M}_{\eta'}^{\text{KIN}}$ and $\mathcal{M}_{\eta}^{\text{KIN}}$ are related by $\pi_{\eta' \to \eta}^{\text{KIN}}$.

Hence, using prop. 3.10, there exists an elementary reduction $(\mathcal{L}, \mathcal{M}, \widetilde{\pi}, \delta)^{\text{DYN}}$ of $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})$, where $\forall \eta \preccurlyeq \eta' \in \mathcal{L}$, $\widetilde{\pi}_{\eta' \to \eta}^{\text{KIN}} = \pi_{\eta' \to \eta}^{\text{KIN}}$. And by prop. 3.2, $\forall \eta \preccurlyeq \eta' \in \mathcal{L}'$, $\widetilde{\pi}_{\eta' \to \eta}^{\text{DYN}} = \pi_{\eta' \to \eta}^{\text{DYN}}$, which supplies the desired result. □

In practice, we will be interested in a kinematical projective structure that is a rendering, by a system of finite dimensional manifolds $\mathcal{M}_{\eta}^{\text{KIN}}$, of an infinite dimensional symplectic manifold $\mathcal{M}_{\infty}^{\text{KIN}}$ (def. 2.6). If the phase space reduction on $\mathcal{M}_{\infty}^{\text{KIN}}$ satisfies the (admittedly very restrictive) requirement that it projects as a closed dynamics on $\mathcal{M}_{\eta}^{\text{KIN}}$ for all η, we will get an elementary reduction of the kinematical projective structure, and the thus obtained dynamical projective system will automatically be a rendering of the physical phase space $\mathcal{M}_{\infty}^{\text{DYN}}$ (fig. 3.3).

Moreover, the map turning observables on the kinematical projective structure into observables on the dynamical structure coincides with the one that can be defined directly from the phase space reduction of $\mathcal{M}_{\infty}^{\text{KIN}}$ (identifying the observables on the projective structures with functions on $\mathcal{M}_{\infty}^{\text{KIN}}$ or $\mathcal{M}_{\infty}^{\text{DYN}}$, as described in def. 2.6). It cannot be too much emphasized that this is a crucial point, for a physical theory is more than just a space of states: it is also a *labeling* of the observables over this state space, that associates to the elementary observables a particular physical meaning. This labeling is the interface that allows us to make the connection between a given concrete measure protocol and an observable of the theory, between the experimental world and the mathematical formalism. Hence, from a physical point of view, a rendering of the physical phase space would be useless if we do not tell at the same time how the elementary observables of our theory are constructed in this rendering.

As already mentioned above, we have, dual to the translation of observables, the possibility of transporting dynamical states back to the kinematical theory (as projective families of orbits), and again this transport reflects the map $\delta_{\infty}^{-1}\langle \cdot \rangle$ that sends a point in $\mathcal{M}_{\infty}^{\text{DYN}}$ to an orbit in $\mathcal{M}_{\infty}^{\text{SHELL}}$. This is probably not needed when the constraints are there to implement dynamics, since, as soon as we have obtained the physical state space (and observables thereon!), the kinematical theory has played its role and can be discarded. However, the same mathematical formalism of imposing constraints can also describe the symmetry restriction of a theory. It has in this case an entirely different physical interpretation, and we are then not only interested in the symmetry restricted theory itself, but we also want to understand its states as special, symmetric states in the full theory (note that the constraints describing symmetry restriction being second class, we map a state on the restricted state to a state on the unrestricted side: orbits are in this case just single points).

Proposition 3.14 Let $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^{\downarrow}$ be a rendering of a (possibly infinite dimensional) symplectic

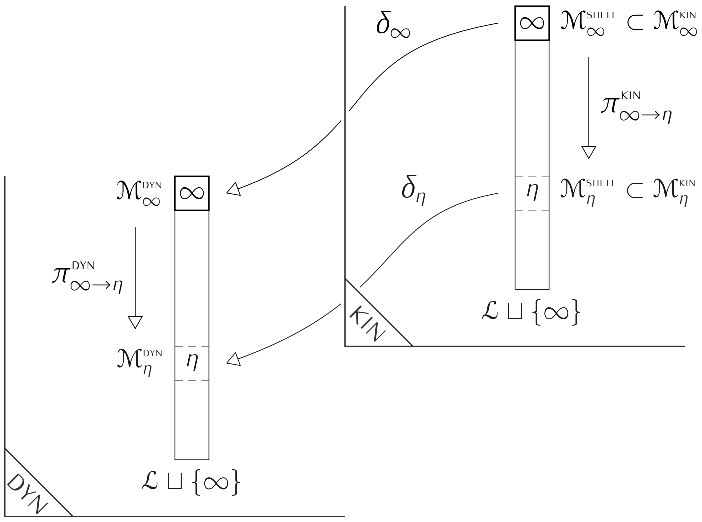


Figure 3.3 – Elementary reduction and rendering

manifold $\mathcal{M}^{\text{KIN}}_\infty$ (def. 2.6) and let $(\mathcal{M}^{\text{DYN}}_\infty, \mathcal{M}^{\text{SHELL}}_\infty, \delta_\infty)$ be a phase space reduction of $\mathcal{M}^{\text{KIN}}_\infty$. We suppose that, for all $\eta \in \mathcal{L}$:

1. $\mathcal{M}_\eta$ is finite dimensional;
2. we are given a phase space reduction $\left(\mathcal{M}^{\text{DYN}}_\eta, \mathcal{M}^{\text{SHELL}}_\eta, \delta_\eta\right)$ of $\mathcal{M}^{\text{KIN}}_\eta$ that is related by $\pi^{\text{KIN}}_{\infty\to\eta}$ to the reduction of $\mathcal{M}^{\text{KIN}}_\infty$.

Then, we have an elementary reduction $(\mathcal{L}, \mathcal{M}, \pi, \delta)^{\text{DYN}}$ of $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^\downarrow$ and a rendering of $\mathcal{M}^{\text{DYN}}_\infty$ by $(\mathcal{L}, \mathcal{M}^{\text{DYN}}, \pi^{\text{DYN}})^\downarrow$ such that, for any $y_\infty \in \mathcal{M}^{\text{DYN}}_\infty$:

$$\widehat{\sigma}^{\text{KIN}}_\downarrow \left(\delta^{-1}_\infty \langle y_\infty \rangle\right) = \Delta \circ \sigma^{\text{DYN}}_\downarrow (y_\infty), \tag{3.14.1}$$

where $\widehat{\sigma}^{\text{KIN}}_\downarrow : \mathcal{P}(\mathcal{M}^{\text{KIN}}_\infty) \to \widehat{\mathcal{S}}^\downarrow_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$, $\sigma^{\text{DYN}}_\downarrow : \mathcal{M}^{\text{DYN}}_\infty \to \mathcal{S}^\downarrow_{(\mathcal{L}',\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})}$ are defined in analogy to def. 2.6. Moreover, for any $f \in \mathcal{A}^\downarrow_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$, we have:

$$\left(\beta^{\text{KIN}}_\uparrow (f)\right)^{\text{DYN}} = \beta^{\text{DYN}}_\uparrow (f^{\text{DYN}}), \tag{3.14.2}$$

where $\beta^{\text{KIN/DYN}}_\uparrow : \mathcal{A}^\downarrow_{\left(\mathcal{L},\mathcal{M}^{\text{KIN/DYN}},\pi^{\text{KIN/DYN}}\right)} \to B(\mathcal{M}^{\text{KIN/DYN}}_\infty)$ are defined in the same way as $\alpha^{\text{KIN/DYN}}_\uparrow : \mathcal{O}^\downarrow_{\left(\mathcal{L},\mathcal{M}^{\text{KIN/DYN}},\pi^{\text{KIN/DYN}}\right)} \to C^\infty(\mathcal{M}^{\text{KIN/DYN}}_\infty, \mathbb{R})$ (def. 2.6).

Proof From prop. 3.12, we can complete the phase space reduction $(\mathcal{M}^{\text{DYN}}_\infty, \mathcal{M}^{\text{SHELL}}_\infty, \delta_\infty)$ of $\mathcal{M}^{\text{KIN}}_\infty$ into an elementary reduction $(\mathcal{L} \sqcup \{\infty\}, \mathcal{M}, \pi, \delta)^{\text{DYN}}$ of $(\mathcal{L} \sqcup \{\infty\}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^\downarrow$. In particular, $(\mathcal{L} \sqcup \{\infty\}, \mathcal{M}^{\text{DYN}}, \pi^{\text{DYN}})^\downarrow$ is a projective system of phase spaces; in other words, $(\mathcal{L}, \mathcal{M}^{\text{DYN}}, \pi^{\text{DYN}})^\downarrow$ is a rendering of $\mathcal{M}^{\text{DYN}}_\infty$.

Eq. (3.14.1) and eq. (3.14.2) then follow by applying twice the corresponding results from prop. 3.11 (to go down from $\mathcal{L} \sqcup \{\infty\}$ to both $\mathcal{L}$ and $\{\infty\}$), together with:

$$\widehat{\sigma}^{\text{KIN}}_\downarrow = \widehat{\sigma}^{\text{KIN}}_{\mathcal{L}\sqcup\{\infty\}\to\mathcal{L}} \circ \widehat{\sigma}^{\text{KIN},-1}_{\mathcal{L}\sqcup\{\infty\}\to\{\infty\}} \quad \& \quad \sigma^{\text{DYN}}_\downarrow = \sigma^{\text{DYN}}_{\mathcal{L}\sqcup\{\infty\}\to\mathcal{L}} \circ \sigma^{\text{DYN},-1}_{\mathcal{L}\sqcup\{\infty\}\to\{\infty\}},$$

and:

$$\beta^{\text{KIN}}_{\uparrow} = \beta^{\text{KIN},-1}_{\mathcal{L}\sqcup\{\infty\}\leftarrow\{\infty\}} \circ \beta^{\text{KIN}}_{\mathcal{L}\sqcup\{\infty\}\leftarrow\mathcal{L}} \quad \& \quad \beta^{\text{DYN}}_{\uparrow} = \beta^{\text{DYN},-1}_{\mathcal{L}\sqcup\{\infty\}\leftarrow\{\infty\}} \circ \beta^{\text{DYN}}_{\mathcal{L}\sqcup\{\infty\}\leftarrow\mathcal{L}} .$$

□

3.2 Regularized reductions

We now turn to the general case, where, typically, the prerequisites of the previous section will *not* be satisfied.

Recall that, as underlined above (prop. 3.12), these prerequisites will become milder and milder if we look for elementary reductions only defined on smaller and smaller cofinal subsets of the label set $\mathcal{L}$: the argument is that it's easier to write closed dynamics over truncations of the theory if we consent to give up the coarsest truncations for lost and to only try to formulate such truncated dynamics in partial theories retaining enough elementary observables (and being thus able to exhibit finer properties of the states). On the other hand, for what we are interested in (namely, defining a projective structure for the dynamical theory, constructing on it the observables inherited from the kinematical theory, and, if need be, embedding its states in the initial projective structure), such an elementary restriction restricted to a cofinal part of $\mathcal{L}$ is all we need.

This observation motivates the following strategy: we will try to design an approximating scheme, indexed by a directed set $\mathcal{E}$, that approaches the exact constraints (unadapted to the projective structure) by approximate constraints, projecting well on the $\mathcal{M}^{\text{KIN}}_{\eta}$ at least for all $\eta \in \mathcal{L}^{\varepsilon}$ (where $\varepsilon \in \mathcal{E}$ parametrizes the level of approximation and $\mathcal{L}^{\varepsilon}$ is a cofinal part of $\mathcal{L}$ that depends on ε). We expect that the label subset $\mathcal{L}^{\varepsilon}$ will get smaller and smaller (yet remaining cofinal), since formulating more accurate approximations of the dynamics will require deeper and deeper knowledge of the properties of the states (such knowledge that is only accessible in partial theories with labels at the high end of $\mathcal{L}$).

As an illustration of this idea, suppose that $\mathcal{L}$ consists of all possible finite subsets of points on the real line (ordered by inclusion), take $\mathcal{E}$ to be the set of positive reals ε and define $\mathcal{L}^{\varepsilon} \subset \mathcal{L}$ to select those subsets in which next neighbor points have a distance of at most ε. Thus a label $\eta \in \mathcal{L}$ will only qualify for belonging to $\mathcal{L}^{\varepsilon}$ if, given a real function f, it can provide an approximation $f\big|_{\eta}$ with at least a resolution of ε (over the convex hull of η). As ε gets smaller and smaller, we retain less and less labels η, yet $\mathcal{L}^{\varepsilon}$ will keep cofinal. Notice that in this example we would use on $\mathcal{E}$ the reverse order $\varepsilon \preccurlyeq \varepsilon' \Leftrightarrow \varepsilon \geqslant \varepsilon'$, because we think of the partial order on $\mathcal{E}$ in terms of coarser lattices being included in finer ones, rather than as an ordering of the lattice parameters (thus we will sometimes refer to the continuum limit as $\varepsilon = \infty$, in the sense of having an infinitely fine lattice, although in the present case $\varepsilon = 0$ would have been more intuitive).

To make it more precise what we mean by approaching the exact constraints, we want the approximation scheme to come with an additional input, namely a family of projections, going from the space of exact solutions of the dynamics $\mathcal{M}^{\text{DYN}}$ into each space of approximate solutions $\mathcal{M}^{\text{DYN},\varepsilon}$: it will tell us, for each level of approximation $\varepsilon \in \mathcal{E}$, how to map the exacts orbits in $\mathcal{M}^{\text{SHELL}}$ to their approximate versions in $\mathcal{M}^{\text{SHELL},\varepsilon}$. In other words, we will associate to each orbit in the exact constraint surface a family (indexed by $\mathcal{E}$) of orbits intended to approach it, thus setting the stage to formulate a notion of convergence (this point will be examined more closely in the second half of the present subsection).

Besides, it is sensible that the map from $\mathcal{M}^{\text{DYN}}$ to $\mathcal{M}^{\text{DYN},\mathcal{E}}$ does not retain all degrees of freedom, so that it only depends on the most distinctive properties of the dynamical states in $\mathcal{M}^{\text{DYN}}$: the approximation of an exact solution should drop those finest details, that can anyway not be handled correctly by the coarse dynamics underlying $\mathcal{M}^{\text{DYN},\mathcal{E}}$. More precisely, we will require that the family of approximated theories build on their own a projective system of phase spaces (with label set $\mathcal{E}$), and, in addition, we would like the approximating maps, bringing us from a finer approximated dynamical theory to a coarser one, to be expressible at the level of the truncated theories $\mathcal{M}_\eta^{\text{DYN},\varepsilon}$, so that we can assemble all $\mathcal{M}_\eta^{\text{DYN},\varepsilon}$ into a big projective system of phase spaces (whose label set will be a part of $\mathcal{E} \times \mathcal{L}$). The return of these quite restrictive requirements is that it supplies immediately a dynamical projective structure, where we can represent the dynamical states and start doing calculations with them, even before we have settled the question of convergence.

Clearly, we are assuming here that we are provided with some non-trivial input, that will have to come from a precise understanding of the system under study. The examples in chap. 5, besides demonstrating that the procedure described here can indeed be put into practice in simple systems, also give some insights on how the needed input can be obtained, but it will require more extensive investigations to develop systematic ways of constructing suitable approximating schemes in the sense above.

Proposition 3.15 Let $\mathcal{E}, \preccurlyeq$ and $\mathcal{L}, \preccurlyeq$ be preordered, directed sets and suppose there exists for all $\varepsilon \in \mathcal{E}$ a cofinal part $\mathcal{L}^\varepsilon$ of $\mathcal{L}$ such that:

$$\forall \varepsilon \preccurlyeq \varepsilon', \mathcal{L}^\varepsilon \supset \mathcal{L}^{\varepsilon'}.$$

We define $\mathcal{E}\mathcal{L} := \left\{(\varepsilon, \eta) \mid \varepsilon \in \mathcal{E}, \eta \in \mathcal{L}^\varepsilon\right\}$ and equip it with the preorder:

$$\forall (\varepsilon, \eta), (\varepsilon', \eta') \in \mathcal{E}\mathcal{L}, \quad (\varepsilon, \eta) \preccurlyeq (\varepsilon', \eta') \Leftrightarrow \left(\varepsilon \preccurlyeq \varepsilon' \quad \& \quad \eta \preccurlyeq \eta'\right).$$

Then $\mathcal{E}\mathcal{L}, \preccurlyeq$ is directed.

Proof Let $(\varepsilon, \eta), (\varepsilon', \eta') \in \mathcal{E}\mathcal{L}$. Since $\mathcal{E}$ and $\mathcal{L}$ are directed, there exist $\varepsilon'' \in \mathcal{E}$ and $\widetilde{\eta} \in \mathcal{L}$ such that $\varepsilon, \varepsilon' \preccurlyeq \varepsilon''$ and $\eta, \eta' \preccurlyeq \widetilde{\eta}$. $\mathcal{L}^{\varepsilon''}$ being cofinal in $\mathcal{L}$, there exists $\eta'' \in \mathcal{L}^{\varepsilon''} / \widetilde{\eta} \preccurlyeq \eta''$. □

Definition 3.16 Let $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^\downarrow$ be a projective system of phase spaces. A regularized reduction of $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^\downarrow$ is a sextuple:

$$\left(\mathcal{E}, (\mathcal{L}^\varepsilon)_{\varepsilon \in \mathcal{E}}, \left(\mathcal{M}_\eta^{\text{DYN},\varepsilon}\right)_{(\varepsilon,\eta) \in \mathcal{E}\mathcal{L}}, \left(\mathcal{M}_\eta^{\text{SHELL},\varepsilon}\right)_{(\varepsilon,\eta) \in \mathcal{E}\mathcal{L}}, \left(\pi_{\eta' \to \eta}^{\text{DYN},\varepsilon' \to \varepsilon}\right)_{(\varepsilon,\eta) \preccurlyeq (\varepsilon',\eta')}, \left(\delta_\eta^\varepsilon\right)_{(\varepsilon,\eta) \in \mathcal{E}\mathcal{L}}\right)$$

such that:

1. $\mathcal{E}$ is a directed set indexing a family $(\mathcal{L}^\varepsilon)_{\varepsilon \in \mathcal{E}}$ of decreasing ($\forall \varepsilon \preccurlyeq \varepsilon', \mathcal{L}^\varepsilon \supset \mathcal{L}^{\varepsilon'}$), cofinal parts of $\mathcal{L}$ as in prop. 3.15;

2. $\forall \varepsilon \in \mathcal{E}$, $\left(\left(\mathcal{M}_\eta^{\text{DYN},\varepsilon}\right)_{\eta \in \mathcal{L}^\varepsilon}, \left(\mathcal{M}_\eta^{\text{SHELL},\varepsilon}\right)_{\eta \in \mathcal{L}^\varepsilon}, \left(\pi_{\eta' \to \eta}^{\text{DYN},\varepsilon \to \varepsilon}\right)_{\eta \preccurlyeq \eta'}, \left(\delta_\eta^\varepsilon\right)_{\eta \in \mathcal{L}^\varepsilon}\right)$ is an elementary reduction of $(\mathcal{L}^\varepsilon, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^\downarrow$;

3. $\left(\mathcal{E}\mathcal{L}, \left(\mathcal{M}_\eta^{\text{DYN},\varepsilon}\right)_{(\varepsilon,\eta) \in \mathcal{E}\mathcal{L}}, \left(\pi_{\eta' \to \eta}^{\text{DYN},\varepsilon' \to \varepsilon}\right)_{(\varepsilon,\eta) \preccurlyeq (\varepsilon',\eta')}\right)$ is a projective system of phase spaces.

Whenever possible, we will use the shortened notation $(\mathcal{L}, \mathcal{M}, \pi, \delta)^{\text{DYN},\mathcal{E}}$ instead of $\Big(\mathcal{E},\, (\mathcal{L}^{\varepsilon})_{\varepsilon\in\mathcal{E}}\,,$ $\big(\mathcal{M}^{\text{DYN},\varepsilon}_{\eta}\big)_{(\varepsilon,\eta)\in\mathcal{E}\mathcal{L}}\,,\ \big(\mathcal{M}^{\text{SHELL},\varepsilon}_{\eta}\big)_{(\varepsilon,\eta)\in\mathcal{E}\mathcal{L}}\,,\ \Big(\pi^{\text{DYN},\varepsilon\to\varepsilon}_{\eta'\to\eta}\Big)_{(\varepsilon,\eta)\preccurlyeq(\varepsilon',\eta')}\,,\ \big(\delta^{\varepsilon}_{\eta}\big)_{(\varepsilon,\eta)\in\mathcal{E}\mathcal{L}}\Big)$.

At that point we have written a projective structure for the dynamical theory, but as we emphasized in the previous subsection, constructing the space of physical states is of little use if we do not prescribe how to define on it the observables inherited from the kinematical theory. As a first step in this direction, we will construct maps that transport kinematical observables into the dynamical theory *at some level of approximation* ε: given a particular kinematical observable, the dynamical observables constructed this way, for all possible ε, should be thought of as successive approximations of the exact dynamical version of this kinematical observable.

Moreover, we can check that these maps transform well under restriction of the label sets $\mathcal{L}$ and $\mathcal{E}$ (provided the label subsets $\mathcal{L}'$ and $\mathcal{E}'$ considered are such that we still have a regularized reduction after restricting ourselves to $\mathcal{E}'\mathcal{L}'$). We will make use of this result at the end of the present subsection, when we will consider how regularized reductions interact with renderings.

Definition 3.17 We consider the same objects as in def. 3.16. For $\varepsilon \in \mathcal{E}$, we define $\Delta^{\varepsilon} : \mathcal{S}^{\downarrow}_{(\mathcal{E}\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})} \to \widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ as:

$$\Delta^{\varepsilon} := \widehat{\sigma}^{\text{KIN},-1}_{\mathcal{L}\to\mathcal{L}^{\varepsilon}} \circ \Delta^{\varepsilon}_{\mathcal{L}^{\varepsilon}} \circ \sigma^{\text{DYN}}_{\mathcal{E}\mathcal{L}\to\mathcal{L}^{\varepsilon}}\,,$$

where $\sigma^{\text{DYN}}_{\mathcal{E}\mathcal{L}\to\mathcal{L}^{\varepsilon}} : \mathcal{S}^{\downarrow}_{(\mathcal{E}\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})} \to \mathcal{S}^{\downarrow}_{(\mathcal{L}^{\varepsilon},\mathcal{M}^{\text{DYN},\varepsilon},\pi^{\text{DYN},\varepsilon\to\varepsilon})}$ is defined as in prop. 2.5 (for the directed part $\{(\varepsilon,\eta) \mid \eta \in \mathcal{L}^{\varepsilon}\}$ of $\mathcal{E}\mathcal{L}$), $\Delta^{\varepsilon}_{\mathcal{L}^{\varepsilon}} : \mathcal{S}^{\downarrow}_{(\mathcal{L}^{\varepsilon},\mathcal{M}^{\text{DYN},\varepsilon},\pi^{\text{DYN},\varepsilon\to\varepsilon})} \to \widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L}^{\varepsilon},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ is defined as in def. 3.8 (for the elementary reduction $(\mathcal{L}^{\varepsilon}, \mathcal{M}^{\varepsilon}, \pi^{\varepsilon\to\varepsilon}, \delta^{\varepsilon})^{\downarrow}$ of $(\mathcal{L}^{\varepsilon}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^{\downarrow}$), and $\widehat{\sigma}^{\text{KIN}}_{\mathcal{L}\to\mathcal{L}^{\varepsilon}} : \widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})} \to \widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L}^{\varepsilon},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ is defined in analogy to prop. 2.5 (for the cofinal part $\mathcal{L}^{\varepsilon}$ of $\mathcal{L}$).

Similarly, we define $(\,\cdot\,)^{\varepsilon} : \mathcal{A}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})} \to \mathcal{A}^{\downarrow}_{(\mathcal{E}\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})}$ as:

$$(\,\cdot\,)^{\varepsilon} := \beta^{\text{DYN}}_{\mathcal{E}\mathcal{L}\leftarrow\mathcal{L}^{\varepsilon}} \circ (\,\cdot\,)^{\text{DYN},\varepsilon} \circ \beta^{\text{KIN},-1}_{\mathcal{L}\leftarrow\mathcal{L}^{\varepsilon}}\,,$$

where $\beta^{\text{DYN}}_{\mathcal{E}\mathcal{L}\leftarrow\mathcal{L}^{\varepsilon}} : \mathcal{A}^{\downarrow}_{(\mathcal{L}^{\varepsilon},\mathcal{M}^{\text{DYN},\varepsilon},\pi^{\text{DYN},\varepsilon\to\varepsilon})} \to \mathcal{A}^{\downarrow}_{(\mathcal{E}\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})}$ and $\beta^{\text{KIN}}_{\mathcal{L}\leftarrow\mathcal{L}^{\varepsilon}} : \mathcal{A}^{\downarrow}_{(\mathcal{L}^{\varepsilon},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})} \to \mathcal{A}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ are defined as in analogy to prop. 2.5 (for the directed part $\{(\varepsilon,\eta) \mid \eta \in \mathcal{L}^{\varepsilon}\}$ of $\mathcal{E}\mathcal{L}$ and the cofinal part $\mathcal{L}^{\varepsilon}$ of $\mathcal{L}$) and $(\,\cdot\,)^{\text{DYN},\varepsilon} : \mathcal{A}^{\downarrow}_{(\mathcal{L}^{\varepsilon},\mathcal{M}^{\text{DYN},\varepsilon},\pi^{\text{DYN},\varepsilon\to\varepsilon})} \to \mathcal{A}^{\downarrow}_{(\mathcal{L}^{\varepsilon},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ is defined as in prop. 3.9 (for the elementary reduction $(\mathcal{L}^{\varepsilon}, \mathcal{M}^{\varepsilon}, \pi^{\varepsilon\to\varepsilon}, \delta^{\varepsilon})^{\downarrow}$ of $(\mathcal{L}^{\varepsilon}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^{\downarrow}$).

We have for all $y \in \mathcal{S}^{\downarrow}_{(\mathcal{E}\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})}$ and all $f \in \mathcal{A}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$:

$$f^{\varepsilon}(y) = f(\Delta^{\varepsilon}(y))\,.$$

Proposition 3.18 Let $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^{\downarrow}$ be a projective system of phase spaces and let $(\mathcal{L}, \mathcal{M}, \pi, \delta)^{\text{DYN},\mathcal{E}}$ be a regularized reduction of $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^{\downarrow}$. Let $\mathcal{L}'$ and $\mathcal{E}'$ be directed subsets of $\mathcal{L}$ and $\mathcal{E}$ respectively, such that, for all $\varepsilon \in \mathcal{E}'$, $\mathcal{L}'^{\varepsilon} := \mathcal{L}^{\varepsilon} \cap \mathcal{L}'$ is a cofinal part of $\mathcal{L}'$.

Then, $\big(\mathcal{L}', \mathcal{M}, \pi, \delta\big)^{\text{DYN},\mathcal{E}'}$ is a regularized reduction of $\big(\mathcal{L}', \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}}\big)^{\downarrow}$ and, for any $\varepsilon \in \mathcal{E}'$, we have:

$$\widehat{\sigma}^{\text{KIN}}_{\mathcal{L}\to\mathcal{L}'} \circ \Delta^{\varepsilon} = \Delta'^{\varepsilon} \circ \sigma^{\text{DYN}}_{\mathcal{E}\mathcal{L}\to\mathcal{E}'\mathcal{L}'}\,, \tag{3.18.1}$$

where $\widehat{\sigma}^{\text{KIN}}_{\mathcal{L}\to\mathcal{L}'} : \widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})} \to \widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L}',\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$, $\sigma^{\text{DYN}}_{\mathcal{E}\mathcal{L}\to\mathcal{E}'\mathcal{L}'} : \mathcal{S}^{\downarrow}_{(\mathcal{E}\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})} \to \mathcal{S}^{\downarrow}_{(\mathcal{E}'\mathcal{L}',\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})}$ are defined in analogy to prop. 2.5, while $\Delta^{\varepsilon} : \mathcal{S}^{\downarrow}_{(\mathcal{E}\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})} \to \widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ and $\Delta'^{\varepsilon} : \mathcal{S}^{\downarrow}_{(\mathcal{E}'\mathcal{L}',\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})} \to \widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L}',\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ are defined as in def. 3.17.

In addition, for any $f \in \mathcal{A}^{\downarrow}_{(\mathcal{L}',\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ and any $\varepsilon \in \mathcal{E}'$, we have:

$$\left(\beta^{\text{KIN}}_{\mathcal{L}\leftarrow\mathcal{L}'}(f)\right)^{\varepsilon} = \beta^{\text{DYN}}_{\mathcal{E}\mathcal{L}\leftarrow\mathcal{E}'\mathcal{L}'}\left(f^{\varepsilon}\right), \tag{3.18.2}$$

where $\beta^{\text{KIN}}_{\mathcal{L}\leftarrow\mathcal{L}'} : \mathcal{A}^{\downarrow}_{(\mathcal{L}',\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})} \to \mathcal{A}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ and $\beta^{\text{DYN}}_{\mathcal{L}\leftarrow\mathcal{L}'} : \mathcal{A}^{\downarrow}_{(\mathcal{E}'\mathcal{L}',\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})} \to \mathcal{A}^{\downarrow}_{(\mathcal{E}\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})}$ are defined in analogy to prop. 2.5, while $(\,\cdot\,)^{\varepsilon} : \mathcal{A}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})} \to \mathcal{A}^{\downarrow}_{(\mathcal{E}\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})}$ and $(\,\cdot\,)^{\varepsilon} : \mathcal{A}^{\downarrow}_{(\mathcal{L}',\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})} \to \mathcal{A}^{\downarrow}_{(\mathcal{E}'\mathcal{L}',\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})}$ are defined as in def. 3.17.

Proof $\mathcal{L}'^{\varepsilon}$ being a cofinal part of $\mathcal{L}'$ for all $\varepsilon \in \mathcal{E}'$ ensures that def. 3.16.1 is fullfiled. Moreover, $\mathcal{E}'\mathcal{L}'$ is then a directed subset of $\mathcal{E}\mathcal{L}$, hence def. 3.16.3 holds. Lastly, def. 3.16.2 follows from prop. 3.11, since, for any $\varepsilon \in \mathcal{E}'$, $\mathcal{L}'^{\varepsilon}$ is a directed part of $\mathcal{L}^{\varepsilon}$ (as a cofinal part of the directed set $\mathcal{L}'$).

Let $\varepsilon \in \mathcal{E}'$. We have $\mathcal{L} \supset \mathcal{L}', \mathcal{L}^{\varepsilon} \supset \mathcal{L}'^{\varepsilon}$, hence:

$$\widehat{\sigma}^{\text{KIN}}_{\mathcal{L}'\to\mathcal{L}'^{\varepsilon}} \circ \widehat{\sigma}^{\text{KIN}}_{\mathcal{L}\to\mathcal{L}'} = \widehat{\sigma}^{\text{KIN}}_{\mathcal{L}^{\varepsilon}\to\mathcal{L}'^{\varepsilon}} \circ \widehat{\sigma}^{\text{KIN}}_{\mathcal{L}\to\mathcal{L}^{\varepsilon}}\,.$$

And, from $\mathcal{E}\mathcal{L} \supset \mathcal{E}'\mathcal{L}', \mathcal{L}^{\varepsilon} \supset \mathcal{L}'^{\varepsilon}$ (identifying $\mathcal{L}^{\varepsilon}$ with the subset $\{(\varepsilon,\eta) \mid \eta \in \mathcal{L}^{\varepsilon}\}$ of $\mathcal{E}\mathcal{L}$ and $\mathcal{L}'^{\varepsilon}$ with the subset $\left\{(\varepsilon,\eta) \,\middle|\, \eta \in \mathcal{L}'^{\varepsilon}\right\}$ of $\mathcal{E}'\mathcal{L}'$ as in def. 3.17), we also have:

$$\sigma^{\text{DYN}}_{\mathcal{L}^{\varepsilon}\to\mathcal{L}'^{\varepsilon}} \circ \sigma^{\text{DYN}}_{\mathcal{E}\mathcal{L}\to\mathcal{L}^{\varepsilon}} = \sigma^{\text{DYN}}_{\mathcal{E}'\mathcal{L}'\to\mathcal{L}'^{\varepsilon}} \circ \sigma^{\text{DYN}}_{\mathcal{E}\mathcal{L}\to\mathcal{E}'\mathcal{L}'}\,.$$

So, using the definition of Δ^{ε} and Δ'^{ε} from def. 3.17:

$$\begin{aligned}
\widehat{\sigma}^{\text{KIN}}_{\mathcal{L}\to\mathcal{L}'} \circ \Delta^{\varepsilon} &= \widehat{\sigma}^{\text{KIN}}_{\mathcal{L}\to\mathcal{L}'} \circ \widehat{\sigma}^{\text{KIN},-1}_{\mathcal{L}\to\mathcal{L}^{\varepsilon}} \circ \Delta^{\varepsilon}_{\mathcal{L}^{\varepsilon}} \circ \sigma^{\text{DYN}}_{\mathcal{E}\mathcal{L}\to\mathcal{L}^{\varepsilon}} \\
&= \widehat{\sigma}^{\text{KIN},-1}_{\mathcal{L}'\to\mathcal{L}'^{\varepsilon}} \circ \widehat{\sigma}^{\text{KIN}}_{\mathcal{L}^{\varepsilon}\to\mathcal{L}'^{\varepsilon}} \circ \Delta^{\varepsilon}_{\mathcal{L}^{\varepsilon}} \circ \sigma^{\text{DYN}}_{\mathcal{E}\mathcal{L}\to\mathcal{L}^{\varepsilon}} \\
&= \widehat{\sigma}^{\text{KIN},-1}_{\mathcal{L}'\to\mathcal{L}'^{\varepsilon}} \circ \Delta'^{\varepsilon}_{\mathcal{L}'^{\varepsilon}} \circ \sigma^{\text{DYN}}_{\mathcal{L}^{\varepsilon}\to\mathcal{L}'^{\varepsilon}} \circ \sigma^{\text{DYN}}_{\mathcal{E}\mathcal{L}\to\mathcal{L}^{\varepsilon}} \text{ (using prop. 3.11)} \\
&= \widehat{\sigma}^{\text{KIN},-1}_{\mathcal{L}'\to\mathcal{L}'^{\varepsilon}} \circ \Delta'^{\varepsilon}_{\mathcal{L}'^{\varepsilon}} \circ \sigma^{\text{DYN}}_{\mathcal{E}'\mathcal{L}'\to\mathcal{L}'^{\varepsilon}} \circ \sigma^{\text{DYN}}_{\mathcal{E}\mathcal{L}\to\mathcal{E}'\mathcal{L}'} \\
&= \Delta'^{\varepsilon} \circ \sigma^{\text{DYN}}_{\mathcal{E}\mathcal{L}\to\mathcal{E}'\mathcal{L}'}\,.
\end{aligned}$$

Similarly, we have:

$$(\,\cdot\,)^{\varepsilon} \circ \beta^{\text{KIN}}_{\mathcal{L}\leftarrow\mathcal{L}'} = \beta^{\text{DYN}}_{\mathcal{E}\mathcal{L}\leftarrow\mathcal{E}'\mathcal{L}'} \circ (\,\cdot\,)^{\varepsilon}\,.$$

□

Now, we would like to give a precise definition of the convergence we have been hinting at repeatedly above. To ascertain convergence is crucial for ensuring that we will get consistent predictions when refining the level of approximation at which we are conducting the calculations.

Our unchanged goal is to make it possible to transport kinematical observables over to the dynamical theory, not only in an approximated fashion, but in such a way that we faithfully realize the transport prescribed by the exact dynamics we are trying to implement (and, if the constraints

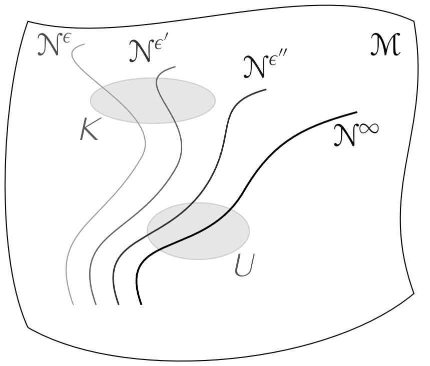


Figure 3.4 – Convergence of a net of orbits

we are considering describe a symmetry restriction, we are also interested in the correct embedding of the symmetric states in the full theory; as we already underlined in the previous subsection, the map between observables and the one between states are the two dual aspects of the bond between the initial or kinematical theory and the restricted or dynamical one). Additionally, we would like to be able to investigate the properties of this correct dynamics (at a certain level of precision) by making use of the approximated dynamics, where calculations will probably be more tractable.

The straightforward course is to obtain the correct transport map as the limit of the net of approximated maps introduced previously. As illustrated in fig. 3.4, we begin by defining, given a family of orbits in a manifold, a notion of convergence, which is adjusted to our method for transposing kinematical observables into dynamical ones (as explained in appendix A, this method is itself motivated by taking the indicator functions as model observables).

Definition 3.19 Let $\mathcal{M}$ be a finite dimensional manifold and let $(\mathcal{N}^{\varepsilon})_{\varepsilon\in\mathcal{E}}$ be a net of subsets of $\mathcal{M}$. We say that the net $(\mathcal{N}^{\varepsilon})_{\varepsilon\in\mathcal{E}}$ converges to the subset $\mathcal{N}^{\infty}$ if:

1. $\forall U$ open set $\subset \mathcal{M}$ such that $\mathcal{N}^{\infty} \cap U \neq \varnothing$, $\exists \varepsilon \in \mathcal{E} \,/\, \forall \varepsilon' \succcurlyeq \varepsilon$, $\mathcal{N}^{\varepsilon'} \cap U \neq \varnothing$;
2. and $\forall K$ compact set $\subset \mathcal{M}$ such that $\mathcal{N}^{\infty} \cap K = \varnothing$, $\exists \varepsilon \in \mathcal{E} \,/\, \forall \varepsilon' \succcurlyeq \varepsilon$, $\mathcal{N}^{\varepsilon'} \cap K = \varnothing$.

Proposition 3.20 Let $\mathcal{M}$ and $(\mathcal{N}^{\varepsilon})_{\varepsilon\in\mathcal{E}}$ be as in def. 3.19 and let $f \in C_o^{\infty}(\mathcal{M}, \mathbb{R})$ (the space of compactly supported, smooth, real-valued functions on $\mathcal{M}$). We define:

$$f^{\varepsilon} := \sup\left\{f(x) \mid x \in \mathcal{N}^{\varepsilon}\right\} \quad \& \quad f^{\infty} := \sup\left\{f(x) \mid x \in \mathcal{N}^{\infty}\right\}.$$

Then, $\lim\limits_{\varepsilon\in\mathcal{E},\preccurlyeq} f^{\varepsilon} = f^{\infty}$.

Proof Let $\delta > 0$. We choose $x \in \mathcal{N}^{\infty}$ such that $f(x) > f^{\infty} - \delta/2$. f is smooth, so there exists an open neighborhood U of x such that $\forall x' \in U$, $f(x') > f^{\infty} - \delta$. From def. 3.19.1, there exists ε_1 such that $\forall \varepsilon' \succcurlyeq \varepsilon_1$, $\mathcal{N}^{\varepsilon'} \cap U \neq \varnothing$. Hence, $\forall \varepsilon' \succcurlyeq \varepsilon_1$, $f^{\varepsilon'} > f^{\infty} - \delta$.

Let $K := \{x \in \mathcal{M} \mid f(x) \geqslant f^{\infty} + \delta\}$. Since f is compactly supported, K is compact. We have $K \cap \mathcal{N}^{\infty} = \varnothing$, so, from def. 3.19.2, there exists ε_2 such that $\forall \varepsilon' \succcurlyeq \varepsilon_2$, $\mathcal{N}^{\varepsilon'} \cap K = \varnothing$. Hence, $\forall \varepsilon' \succcurlyeq \varepsilon_2$, $f^{\varepsilon'} \leqslant f^{\infty} + \delta$.

$\mathcal{E}$ being directed, there exists $\varepsilon \succcurlyeq \varepsilon_1, \varepsilon_2$. Then, $\forall \varepsilon' \succcurlyeq \varepsilon$, $\left|f^\infty - f^{\varepsilon'}\right| \leqslant \delta$. □

For any state over the dynamical projective system, and any $\eta \in \mathcal{L}$, the framework laid at the beginning of the present subsection allows to construct a net of approximated orbits representing this dynamical state in $\mathcal{M}^{\text{KIN}}_\eta$ (def. 3.17). Thus we can define the space $\mathcal{R}$ of all dynamical states such that, for all η, the net of their approached projections on $\mathcal{M}^{\text{KIN}}_\eta$ converges in the sense above.

Hopefully, $\mathcal{R}$ will be dense in the space of all dynamical states, but we do not require both spaces to coincide. It is in fact not really surprising that formulating the exact dynamics may require to consider only states that are well-behaved enough (we can view this prescription on the same footing as, for example, the routine requirement for fields to be smooth so that we can describe their dynamics by partial differential equations).

Reciprocally, we can also associate to any (sufficiently regular) kinematical observable its corresponding exact dynamical version, as an observable defined on $\mathcal{R}$. At this point we can comment on the issue raised in section 1, namely that even the Poisson-algebra generated by finitely many observables could be too complicated to be represented on a finite dimensional symplectic manifold. We had argued that this problem should not arise when looking at kinematical observables, yet it might (and generically will) occur for the dynamical observables. By defining a dynamical observable as the limit of a family of imperfect estimations, we escape this difficulty. On the one hand, each such estimation can be expressed over a sufficiently big partial theory, while keeping the partial theories finite dimensional, because the regularization allows us to keep under control the algebra generated by finitely many of these approximated versions of the observables. On the other hand, an exact dynamical observable, being a limit, is allowed to depend on the full projective state $\left(y^\varepsilon_\eta\right)_{(\varepsilon,\eta)\in\mathcal{E}\mathcal{L}} \in \mathcal{R}$.

Definition 3.21 Let $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^\downarrow$ be a projective system of finite dimensional phase spaces and let $(Y^\varepsilon)_{\varepsilon\in\mathcal{E}}$ be a net in $\widehat{\mathcal{S}}^\downarrow_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$. We say that the net $(Y^\varepsilon)_{\varepsilon\in\mathcal{E}}$ converges to the element $Y^\infty \in \widehat{\mathcal{S}}^\downarrow_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ iff:

1. $\forall \eta \in \mathcal{L}$, the net $(Y^\varepsilon_\eta)_{\varepsilon\in\mathcal{E}}$ converges to the subset Y^∞_η of $\mathcal{M}^{\text{KIN}}_\eta$ in the sense of def. 3.19.

If $(\mathcal{L}, \mathcal{M}, \pi, \delta)^{\text{DYN},\mathcal{E}}$ is a regularized reduction of $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^\downarrow$, we say that the regularization converges on a subset $\mathcal{R}$ of $\mathcal{S}^\downarrow_{(\mathcal{E}\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})}$ iff:

2. $\forall y \in \mathcal{R}$, the net $(\Delta^\varepsilon(y))_{\varepsilon\in\mathcal{E}}$ converges in $\widehat{\mathcal{S}}^\downarrow_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$.

Proposition 3.22 Let $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^\downarrow$ be a projective system of finite dimensional phase spaces. We define:

$$\mathcal{A}^{o,\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})} = \left\{ f \in \mathcal{A}^\downarrow_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})} \,\middle|\, \exists \eta \in \mathcal{L},\, \exists f_\eta \in f \,/\, f_\eta \in C^\infty_o(\mathcal{M}^{\text{KIN}}_\eta, \mathbb{R}) \right\}.$$

If the net $(Y^\varepsilon)_{\varepsilon\in\mathcal{E}}$ in $\widehat{\mathcal{S}}^\downarrow_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ converges to the element $Y^\infty \in \widehat{\mathcal{S}}^\downarrow_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$, then, for all $f \in \mathcal{A}^{o,\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$, the net $(f(Y^\varepsilon))_{\varepsilon\in\mathcal{E}}$ converges to $f(Y^\infty)$.

If $(\mathcal{L}, \mathcal{M}, \pi, \delta)^{\text{DYN},\mathcal{E}}$ is a regularized reduction of $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^\downarrow$ such that the regularization converges on $\mathcal{R} \subset \mathcal{S}^{\downarrow}_{(\mathcal{E}\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})}$, then, for all $f \in \mathcal{A}^{o,\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$, we can define an application f^{DYN} on $\mathcal{R}$ by:

$$\forall y \in \mathcal{R}, f^{\text{DYN}}(y) := \lim_{\varepsilon \in \mathcal{E},\preccurlyeq} f^{\varepsilon}(y) .$$

Proof Let $f \in \mathcal{A}^{o,\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ and let $\eta \in \mathcal{L}$, $f_\eta \in f$ such that $f_\eta \in C^{\infty}_{o}(\mathcal{M}^{\text{KIN}}_{\eta}, \mathbb{R})$. For all $\varepsilon \in \mathcal{E}$, we have $f(Y^{\varepsilon}) = \sup_{Y^{\varepsilon}_{\eta}} f_\eta$ (for we can choose any representative of f to evaluate f on Y^{ε}). Now, the net $(Y^{\varepsilon}_{\eta})_{\varepsilon \in \mathcal{E}}$ converges to the subset Y^{∞}_{η} of $\mathcal{M}^{\text{KIN}}_{\eta}$, hence, using prop. 3.20:

$$\lim_{\varepsilon \in \mathcal{E},\preccurlyeq} f(Y^{\varepsilon}) = \sup_{Y^{\infty}_{\eta}} f_\eta = f(Y^{\infty}) .$$

Now, if $y \in \mathcal{R}$, we have from def. 3.21 that the net $(\Delta^{\varepsilon}(y))_{\varepsilon \in \mathcal{E}}$ converges in $\widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$. Hence, the net $(f(\Delta^{\varepsilon}(y)))_{\varepsilon \in \mathcal{E}}$ converges, but we have $\forall \varepsilon \in \mathcal{E}$, $f(\Delta^{\varepsilon}(y)) = f^{\varepsilon}(y)$ (def. 3.17). □

Finally, we want to discuss how renderings (def. 2.6) can be incorporated in this procedure, and more specifically, how regularized reductions can be used to mirror phase space reductions in infinite dimensional symplectic manifolds. Given a rendering of some infinite dimensional symplectic manifold $\mathcal{M}^{\text{KIN}}_{\infty}$, and a phase space reduction thereof, we will aim at constructing a regularized reduction whose dynamical projective system renders the dynamical phase space $\mathcal{M}^{\text{DYN}}_{\infty}$. Additionally, we will require that the regularization converges (at least) on the dynamical states that are identified, through this rendering, with points in $\mathcal{M}^{\text{DYN}}_{\infty}$, and that, for any such state, the family of orbits reflecting it in the kinematical structure can be identified with the corresponding orbit in $\mathcal{M}^{\text{KIN}}_{\infty}$.

Then, besides being provided with a rendering of the dynamical theory, this last point will ensure that the maps linking the kinematical side and the dynamical one are appropriately intertwined by the identifications arising from the renderings on both sides.

In prop. 3.24, we formulate more concise assumptions that are sufficient to bring forth this optimal setting. As illustrated in fig. 3.5, it relies on the successive approximations of the dynamics being formulated as phase space reductions of $\mathcal{M}^{\text{KIN}}_{\infty}$, and the thus defined dynamical phase spaces $\mathcal{M}^{\text{DYN},\varepsilon}_{\infty}$ building a rendering of the exact dynamical theory (denoted by $\mathcal{M}^{\text{DYN},\infty}_{\infty}$).

Proposition 3.23 Let $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^{\downarrow}$ be a projective system of finite dimensional phase spaces and let $(\mathcal{L}, \mathcal{M}, \pi, \delta)^{\text{DYN},\mathcal{E}}$ be a regularized reduction of $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^{\downarrow}$. Assume that we have a symplectic manifold $\mathcal{M}^{\text{KIN}}_{\infty}$ and a phase space reduction $(\mathcal{M}^{\text{DYN}}_{\infty}, \mathcal{M}^{\text{SHELL}}_{\infty}, \delta_\infty)$ of $\mathcal{M}^{\text{KIN}}_{\infty}$ such that:

1. we have a rendering of $\mathcal{M}^{\text{KIN}}_{\infty}$ by $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^{\downarrow}$ and of $\mathcal{M}^{\text{DYN}}_{\infty}$ by $(\mathcal{E}\mathcal{L}, \mathcal{M}^{\text{DYN}}, \pi^{\text{DYN}})^{\downarrow}$;
2. for all y in $\mathcal{M}^{\text{DYN}}_{\infty}$, the net $\left(\Delta^{\varepsilon} \circ \sigma^{\text{DYN}}_{\downarrow}(y)\right)_{\varepsilon \in \mathcal{E}}$ converges in $\widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ to $\widehat{\sigma}^{\text{KIN}}_{\downarrow}\left(\delta^{-1}_{\infty}\left\langle \{y\} \right\rangle\right)$, where $\sigma^{\text{DYN}}_{\downarrow} : \mathcal{M}^{\text{DYN}}_{\infty} \to \mathcal{S}^{\downarrow}_{(\mathcal{E}\mathcal{L},\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})}$ is defined as in def. 2.6 and $\widehat{\sigma}^{\text{KIN}}_{\downarrow} : \mathcal{P}(\mathcal{M}^{\text{KIN}}_{\infty}) \to \widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ is defined in a similar way.

Then, the regularization converges on $\mathcal{R} := \operatorname{Im} \sigma^{\text{DYN}}_{\downarrow}$ and for all $y \in \mathcal{M}^{\text{DYN}}_{\infty}$, for all $f \in \mathcal{A}^{o,\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$, we have:

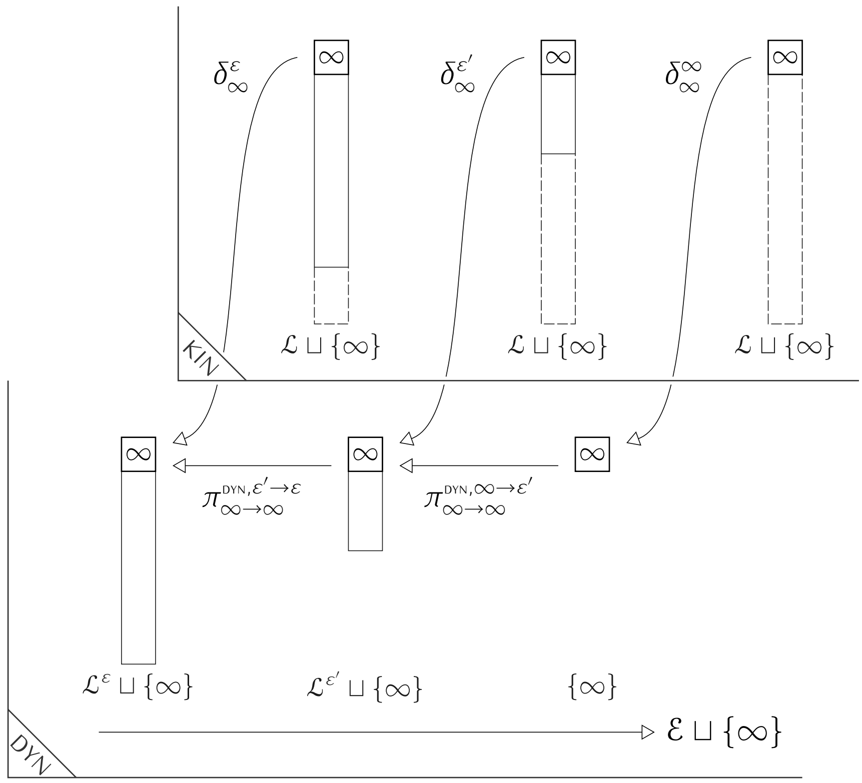


Figure 3.5 – Regularized reduction and rendering

$$f^{\text{DYN}} \circ \sigma^{\text{DYN}}_{\downarrow}(y) = \left(\beta^{\text{KIN}}_{\uparrow}(f)\right)^{\text{DYN}}(y)\,.$$

Proof Let $y \in \mathcal{M}^{\text{DYN}}_{\infty}$ and $f \in \mathcal{A}^{o,\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$. We have, using prop. 3.22:

$$f^{\text{DYN}} \circ \sigma^{\text{DYN}}_{\downarrow}(y) = \lim_{\varepsilon\in\mathcal{E},\preccurlyeq} f^{\varepsilon} \circ \sigma^{\text{DYN}}_{\downarrow}(y) = \lim_{\varepsilon\in\mathcal{E},\preccurlyeq} f \circ \Delta^{\varepsilon} \circ \sigma^{\text{DYN}}_{\downarrow}(y) = f \circ \widehat{\sigma}^{\text{KIN}}_{\downarrow}\left(\delta^{-1}_{\infty}\left\langle\{y\}\right\rangle\right)$$

$$= \beta^{\text{KIN}}_{\uparrow}(f)\left(\delta^{-1}_{\infty}\left\langle\{y\}\right\rangle\right) = \sup_{\delta^{-1}_{\infty}\langle\{y\}\rangle} \beta^{\text{KIN}}_{\uparrow}(f) = \left(\beta^{\text{KIN}}_{\uparrow}(f)\right)^{\text{DYN}}(y)\,.$$

□

Proposition 3.24 Let $(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})^{\downarrow}$ be a projective system of finite dimensional phase spaces yielding a rendering of a symplectic manifold $\mathcal{M}^{\text{KIN}}_{\infty}$. Let $\mathcal{E}$ be a directed preordered set and assume that:

1. for any $\varepsilon \in \mathcal{E} \sqcup \{\infty\}$, we have a phase space reduction $(\mathcal{M}^{\text{DYN},\varepsilon}_{\infty}, \mathcal{M}^{\text{SHELL},\varepsilon}_{\infty}, \delta^{\varepsilon}_{\infty})$ of $\mathcal{M}^{\text{KIN}}_{\infty}$;
2. for any $\varepsilon \in \mathcal{E}$, we have a cofinal subset $\mathcal{L}^{\varepsilon}$ of $\mathcal{L}$ and an elementary reduction $\left(\left(\mathcal{M}^{\text{DYN},\varepsilon}_{\eta}\right)_{\eta\in\mathcal{L}^{\varepsilon}\sqcup\{\infty\}}, \left(\mathcal{M}^{\text{SHELL},\varepsilon}_{\eta}\right)_{\eta\in\mathcal{L}^{\varepsilon}\sqcup\{\infty\}}, \left(\pi^{\text{DYN},\varepsilon\to\varepsilon}_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'}, \left(\delta^{\varepsilon}_{\eta}\right)_{\eta\in\mathcal{L}^{\varepsilon}\sqcup\{\infty\}}\right)$ of $(\mathcal{L}^{\varepsilon} \sqcup \{\infty\}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^{\downarrow}$, arising from $(\mathcal{M}^{\text{DYN},\varepsilon}_{\infty}, \mathcal{M}^{\text{SHELL},\varepsilon}_{\infty}, \delta^{\varepsilon}_{\infty})$;
3. we have a rendering of $\mathcal{M}^{\text{DYN},\infty}_{\infty}$ by $(\mathcal{E}, \mathcal{M}^{\text{DYN}}_{\infty}, \pi^{\text{DYN}}_{\infty\to\infty})^{\downarrow}$;
4. for any $\varepsilon \preccurlyeq \varepsilon'$, $\mathcal{L}^{\varepsilon'} \subset \mathcal{L}^{\varepsilon}$ and, for any $\eta \in \mathcal{L}^{\varepsilon'}$, we have a projection $\pi^{\text{DYN},\varepsilon'\to\varepsilon}_{\eta\to\eta} : \mathcal{M}^{\text{DYN},\varepsilon'}_{\eta} \to \mathcal{M}^{\text{DYN},\varepsilon}_{\eta}$, compatible with the symplectic structures, and such that $\pi^{\text{DYN},\varepsilon\to\varepsilon}_{\infty\to\eta} \circ \pi^{\text{DYN},\varepsilon'\to\varepsilon}_{\infty\to\infty} = \pi^{\text{DYN},\varepsilon'\to\varepsilon}_{\eta\to\eta} \circ \pi^{\text{DYN},\varepsilon'\to\varepsilon'}_{\infty\to\eta}$.

Then, defining $\widetilde{\mathcal{L}} := \mathcal{L} \sqcup \{\infty\}$ and $\widetilde{\mathcal{E}} := \mathcal{E} \sqcup \{\infty\}$ (extending the preorders in such a way that ∞

is a greatest element), we can complete this input to build a regularized reduction $\left(\widetilde{\mathcal{L}}, \mathcal{M}, \pi, \delta\right)^{\text{DYN},\widetilde{\mathcal{E}}}$ of $\left(\widetilde{\mathcal{L}}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}}\right)^{\downarrow}$.

If we moreover have:

5. for any $y \in \mathcal{M}^{\text{DYN},\infty}_{\infty}$, the net $\left(\widehat{\sigma}^{\text{KIN}}_{\downarrow}\left(\delta^{\varepsilon,-1}_{\infty}\left\langle\pi^{\text{DYN},\infty\to\varepsilon}_{\infty\to\infty}(y)\right\rangle\right)\right)_{\varepsilon\in\mathcal{E}}$ converges in $\widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ to $\widehat{\sigma}^{\text{KIN}}_{\downarrow}\left(\delta^{\infty,-1}_{\infty}\left\langle y\right\rangle\right)$;

then the hypotheses of prop. 3.23 are fulfilled.

Proof For any $\varepsilon \in \mathcal{E}$ we define $\widetilde{\mathcal{L}}^{\varepsilon} := \mathcal{L}^{\varepsilon} \sqcup \{\infty\}$ and we additionally define $\widetilde{\mathcal{L}}^{\infty} := \{\infty\}$. With this def. 3.16.1 is fulfilled.

For any $\varepsilon \in \mathcal{E}$, def. 3.16.2 comes from assumption 3.24.2, and for $\infty \in \widetilde{\mathcal{E}}$, it reduces to $(\mathcal{M}^{\text{DYN},\infty}_{\infty}, \mathcal{M}^{\text{SHELL},\infty}_{\infty}, \delta^{\infty}_{\infty})$ being a phase space reduction of $\mathcal{M}^{\text{KIN}}_{\infty}$, which has been assumed in 3.24.1.

For any $\varepsilon \preccurlyeq \varepsilon' \in \widetilde{\mathcal{E}}$ and any $\eta \in \widetilde{\mathcal{L}}^{\varepsilon}$, $\eta' \in \widetilde{\mathcal{L}}^{\varepsilon'}$ with $\eta \preccurlyeq \eta'$, we define $\pi^{\text{DYN},\varepsilon'\to\varepsilon}_{\eta'\to\eta} := \pi^{\text{DYN},\varepsilon\to\varepsilon}_{\eta'\to\eta} \circ \pi^{\text{DYN},\varepsilon'\to\varepsilon}_{\eta'\to\eta'}$. From assumption 3.24.2, $\pi^{\text{DYN},\varepsilon\to\varepsilon}_{\eta'\to\eta} : \mathcal{M}^{\text{DYN},\varepsilon}_{\eta'} \to \mathcal{M}^{\text{DYN},\varepsilon}_{\eta}$ is a projection compatible with the symplectic structures, and from assumption 3.24.4 (or 3.24.3 if $\eta' = \infty$), $\pi^{\text{DYN},\varepsilon'\to\varepsilon}_{\eta'\to\eta'} : \mathcal{M}^{\text{DYN},\varepsilon'}_{\eta'} \to \mathcal{M}^{\text{DYN},\varepsilon}_{\eta'}$ is a projection compatible with the symplectic structures. Hence, $\pi^{\text{DYN},\varepsilon'\to\varepsilon}_{\eta'\to\eta} : \mathcal{M}^{\text{DYN},\varepsilon'}_{\eta'} \to \mathcal{M}^{\text{DYN},\varepsilon}_{\eta}$ is a projection compatible with the symplectic structures.

Let $(\varepsilon, \eta) \preccurlyeq (\varepsilon', \eta') \preccurlyeq (\varepsilon'', \eta'') \in \widetilde{\mathcal{E}}\widetilde{\mathcal{L}}$. We have $\eta', \eta'' \in \mathcal{L}^{\varepsilon'}$, hence, using points 3.24.2 and 3.24.4:

$$\pi^{\text{DYN},\varepsilon\to\varepsilon}_{\eta''\to\eta'} \circ \pi^{\text{DYN},\varepsilon'\to\varepsilon}_{\eta''\to\eta''} \circ \pi^{\text{DYN},\varepsilon'\to\varepsilon'}_{\infty\to\eta''} = \pi^{\text{DYN},\varepsilon\to\varepsilon}_{\infty\to\eta'} \circ \pi^{\text{DYN},\varepsilon'\to\varepsilon}_{\infty\to\infty} = \pi^{\text{DYN},\varepsilon'\to\varepsilon}_{\eta'\to\eta'} \circ \pi^{\text{DYN},\varepsilon'\to\varepsilon'}_{\eta''\to\eta'} \circ \pi^{\text{DYN},\varepsilon'\to\varepsilon'}_{\infty\to\eta''}.$$

Since $\pi^{\text{DYN},\varepsilon'\to\varepsilon'}_{\infty\to\eta''}$ is surjective, we then have:

$$\pi^{\text{DYN},\varepsilon\to\varepsilon}_{\eta''\to\eta'} \circ \pi^{\text{DYN},\varepsilon'\to\varepsilon}_{\eta''\to\eta''} = \pi^{\text{DYN},\varepsilon'\to\varepsilon}_{\eta'\to\eta'} \circ \pi^{\text{DYN},\varepsilon'\to\varepsilon'}_{\eta''\to\eta'},$$

hence, using once more from 3.24.2:

$$\begin{aligned}\pi^{\text{DYN},\varepsilon'\to\varepsilon}_{\eta'\to\eta} \circ \pi^{\text{DYN},\varepsilon''\to\varepsilon'}_{\eta''\to\eta'} &= \pi^{\text{DYN},\varepsilon\to\varepsilon}_{\eta'\to\eta} \circ \pi^{\text{DYN},\varepsilon\to\varepsilon}_{\eta''\to\eta'} \circ \pi^{\text{DYN},\varepsilon'\to\varepsilon}_{\eta''\to\eta''} \circ \pi^{\text{DYN},\varepsilon''\to\varepsilon'}_{\eta''\to\eta''} \\ &= \pi^{\text{DYN},\varepsilon\to\varepsilon}_{\eta''\to\eta} \circ \pi^{\text{DYN},\varepsilon'\to\varepsilon}_{\eta''\to\eta''} \circ \pi^{\text{DYN},\varepsilon''\to\varepsilon'}_{\eta''\to\eta''}. \end{aligned} \tag{3.24.1}$$

Now, using repeatedly 3.24.4, together with 3.24.3, we have:

$$\begin{aligned}\pi^{\text{DYN},\varepsilon'\to\varepsilon}_{\eta''\to\eta''} \circ \pi^{\text{DYN},\varepsilon''\to\varepsilon'}_{\eta''\to\eta''} \circ \pi^{\text{DYN},\varepsilon''\to\varepsilon''}_{\infty\to\eta''} &= \pi^{\text{DYN},\varepsilon\to\varepsilon}_{\infty\to\eta''} \circ \pi^{\text{DYN},\varepsilon'\to\varepsilon}_{\infty\to\infty} \circ \pi^{\text{DYN},\varepsilon''\to\varepsilon'}_{\infty\to\infty} \\ &= \pi^{\text{DYN},\varepsilon''\to\varepsilon}_{\eta''\to\eta''} \circ \pi^{\text{DYN},\varepsilon''\to\varepsilon''}_{\infty\to\eta''}\end{aligned}$$

and, since $\pi^{\text{DYN},\varepsilon''\to\varepsilon''}_{\infty\to\eta''}$ is surjective:

$$\pi^{\text{DYN},\varepsilon'\to\varepsilon}_{\eta''\to\eta''} \circ \pi^{\text{DYN},\varepsilon''\to\varepsilon'}_{\eta''\to\eta''} = \pi^{\text{DYN},\varepsilon''\to\varepsilon}_{\eta''\to\eta''}. \tag{3.24.2}$$

Combining eq. (3.24.1) and eq. (3.24.2), we get:

$$\pi^{\text{DYN},\varepsilon'\to\varepsilon}_{\eta'\to\eta} \circ \pi^{\text{DYN},\varepsilon''\to\varepsilon'}_{\eta''\to\eta'} = \pi^{\text{DYN},\varepsilon\to\varepsilon}_{\eta''\to\eta} \circ \pi^{\text{DYN},\varepsilon''\to\varepsilon}_{\eta''\to\eta''} = \pi^{\text{DYN},\varepsilon''\to\varepsilon}_{\eta''\to\eta},$$

therefore $\left(\widetilde{\mathcal{E}}\widetilde{\mathcal{L}}, \mathcal{M}^{\text{DYN}}, \pi^{\text{DYN}}\right)^{\downarrow}$ is a projective system of phase spaces, so def. 3.16.3 holds.

Thus, using prop. 3.18 with assumption 3.24.2, $(\mathcal{L}, \mathcal{M}, \pi, \delta)^{\text{DYN},\mathcal{E}}$ is a regularized reduction of $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^{\downarrow}$, while $(\mathcal{EL}, \mathcal{M}^{\text{DYN}}, \pi^{\text{DYN}})^{\downarrow}$ is a rendering of $\mathcal{M}^{\text{DYN},\infty}_{\infty}$.

We now assume that assumption 3.24.5 holds. Using eq. (3.18.1) for $\mathcal{EL} \subset \widetilde{\mathcal{E}}\widetilde{\mathcal{L}}$, we have:

$$\widehat{\sigma}^{\text{KIN}}_{\widetilde{\mathcal{L}}\to\mathcal{L}} \circ \widetilde{\Delta}^{\varepsilon} = \Delta^{\varepsilon} \circ \sigma^{\text{DYN}}_{\widetilde{\mathcal{E}}\widetilde{\mathcal{L}}\to\mathcal{EL}} ,$$

and using it for $\widetilde{\mathcal{E}}\,\{\infty\} \subset \widetilde{\mathcal{E}}\widetilde{\mathcal{L}}$:

$$\begin{aligned}\widehat{\sigma}^{\text{KIN}}_{\widetilde{\mathcal{L}}\to\{\infty\}} \circ \widetilde{\Delta}^{\varepsilon} &= \delta^{\varepsilon,-1}_{\infty} \left\langle \,\cdot\, \right\rangle \circ \sigma^{\text{DYN}}_{\widetilde{\mathcal{E}}\{\infty\}\to\{(\varepsilon,\infty)\}} \circ \sigma^{\text{DYN}}_{\widetilde{\mathcal{E}}\widetilde{\mathcal{L}}\to\widetilde{\mathcal{E}}\{\infty\}} \\ &= \delta^{\varepsilon,-1}_{\infty} \left\langle \,\cdot\, \right\rangle \circ \sigma^{\text{DYN}}_{\widetilde{\mathcal{E}}\widetilde{\mathcal{L}}\to\{(\varepsilon,\infty)\}} ,\end{aligned}$$

therefore:

$$\begin{aligned}\Delta^{\varepsilon} \circ \sigma^{\text{DYN}}_{\downarrow} &= \Delta^{\varepsilon} \circ \sigma^{\text{DYN}}_{\widetilde{\mathcal{E}}\widetilde{\mathcal{L}}\to\mathcal{EL}} \circ \sigma^{\text{DYN},-1}_{\widetilde{\mathcal{E}}\widetilde{\mathcal{L}}\to\{(\infty,\infty)\}} \\ &= \widehat{\sigma}^{\text{KIN}}_{\widetilde{\mathcal{L}}\to\mathcal{L}} \circ \widehat{\sigma}^{\text{KIN},-1}_{\widetilde{\mathcal{L}}\to\{\infty\}} \circ \delta^{\varepsilon,-1}_{\infty} \left\langle \,\cdot\, \right\rangle \circ \sigma^{\text{DYN}}_{\widetilde{\mathcal{E}}\widetilde{\mathcal{L}}\to\{(\varepsilon,\infty)\}} \circ \sigma^{\text{DYN},-1}_{\widetilde{\mathcal{E}}\widetilde{\mathcal{L}}\to\{(\infty,\infty)\}} \\ &= \widehat{\sigma}^{\text{KIN}}_{\downarrow} \circ \delta^{\varepsilon,-1}_{\infty} \left\langle \,\cdot\, \right\rangle \circ \pi^{\text{DYN},\infty\to\varepsilon}_{\infty\to\infty} .\end{aligned}$$

Hence, for all $y \in \mathcal{M}^{\text{DYN},\infty}_{\infty}$, the net $\left(\Delta^{\varepsilon} \circ \sigma^{\text{DYN}}_{\downarrow}(y)\right)_{\varepsilon\in\mathcal{E}}$ converges in $\widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ to $\widehat{\sigma}^{\text{KIN}}_{\downarrow}\left(\delta^{\infty,-1}_{\infty}\left\langle y\right\rangle\right)$. □

At this point, the question that remains open is how to construct a rendering of $\mathcal{M}^{\text{DYN},\infty}$ by a net of reduced phase spaces $\mathcal{M}^{\text{DYN},\epsilon}$, arising from constraint surfaces approaching $\mathcal{M}^{\text{SHELL},\infty}$. In other words, we are lacking systematic recipes for setting up regularization schemes in the sense of the procedure just described.

Among the tools that are at our disposal is the gauge fixing/unfixing trick (taken from [14], where it was however used in a completely different context), that would consist in first partially gauge fixing (prop. A.8) the original phase space reduction, and then gauge unfixing it in a slightly different direction: thus we would deform the orbits (in the view of improving their projectability), and get an approximation of the dynamics that should be satisfactory in some neighborhood of the common gauge fixing surface (this technique is the one used in section 16). Another option, that might be in particular relevant when the gauge orbits are infinite dimensional, could be to drastically gauge fix them, before progressively lifting the gauge fixing conditions, thus approaching a given orbit by an increasing net of submanifolds inside it. In both cases, we get a natural symplectomorphism between $\mathcal{M}^{\text{DYN},\infty}$ and each $\mathcal{M}^{\text{DYN},\epsilon}$, so we probably want to combine such methods with projections from $\mathcal{M}^{\text{DYN},\infty}$ into symplectic submanifolds of it, to drop the degrees of freedom that are disproportionately accurate at a given level of approximation.

Also, there is presumably some link between the regularization procedure we are considering and various concepts developed eg. in the context of Loop Quantum Gravity (often within a Lagrangian setting), exploring the interplay between discretization, coarse graining, diffeomorphism invariance, and the continuum limit [59]. Studying more precisely how these approaches are related to the strategy proposed here could in particular help incorporate renormalization group ideas into the picture.

Chapter 2 – Quantum Formalism

4. Introduction

While finite dimensional symplectic manifolds are comparatively easy to quantize, allowing the formulation of rather systematic procedures (such as geometric quantization [105], of which a brief account is given in appendix B), quantizing infinite dimensional ones (aka. field theories) is substantially more involved: typically, we have to rely on some additional insight, telling us how to break them down into a stack of finite dimensional truncations and how to afterwards reassemble the pieces into a consistent quantum theory. The projective approach to quantum field theory introduced in [48, 68], and reviewed in subsection 5.1 below, makes this concept central to the formalism, by representing each truncation on a 'small' Hilbert space, before sewing the thus obtained partial quantum theories together into a projective structure. Like in the classical formalism described in the previous chapter, the labels indexing these partial theories are to be thought as selecting finitely many degrees of freedom out of the considered infinite dimensional theory. Whenever the degrees of freedom retained by some label η are also covered by a finer label η', the usual rules of quantum mechanics suggest to identify the small Hilbert space $\mathcal{H}_\eta$ associated to η as a tensor product factor in $\mathcal{H}_{\eta'}$: this allows to compute partial traces of density matrices on $\mathcal{H}_{\eta'}$, which is how the relation between the corresponding partial quantum theories will be formalized in def. 5.2.

We pointed out in the main introduction that this approach has the potential to deliver bigger state spaces, as it sidesteps the need to specialize into a single representation of the algebra of observables. To demonstrate this property, we will, in subsection 5.2, inspect how the quantum state spaces built this way compare to those provided by other quantization methods, that are also assembled from 'small' Hilbert spaces, yet sewed in a different manner (this analysis will then be applied to the relation with the standard LQG Hilbert space in subsection 12.2, and with the Fock space of QFT in subsection 16.2).

As stressed in section 1, there is a correspondence between projective limits of symplectic manifolds on the classical side and projective limits of state spaces on the quantum side. Accordingly, one can try to formulate a quantization program to turn a classical projective system into a quantum one. In [68] Andrzej Okołów established such a quantization prescription: by identifying appropriate assumptions, he was able to set up what we would call, in the terminology of def. 2.15, a factorizing system of linear configuration spaces, which could then be quantized in a projective form. He subsequently used this construction to obtain the kinematical state space of a certain theory of quantum gravity [69]. In subsection 6.1, we will extend this result to configuration spaces given as simply-connected Lie groups. This is meant as a preparation for a corresponding treatment of Loop Quantum Gravity, but could probably have applications to other gauge field theories as well (viz. chaps. 3 and 4). Additionally, holomorphic quantization will be discussed in subsection 6.2, following the lines of geometric quantization (note that the quantum projective structure used in subsection 16.2 could be seen as arising from such an holomorphic quantization, although we will arrive at it from a different perspective).

Note that the heuristic picture that was presented at the beginning of subsection 2.1 as a justification for the projective formalism becomes a bit more involved when we go over to the quantum theory. In particular, we had justified the directedness of the label set $\mathcal{L}$ by arguing that, given any two experiments, involving observables included respectively in the labels η and η', we should be able to describe the simultaneous realization of both experiments, hence the need for a label $\eta'' \succcurlyeq \eta, \eta'$. However, this argument obviously does not hold any more in the quantum theory, where complementarity forbids the simultaneous measurement of non-commuting observables. As a way out, we could simply decide to restrict the elementary observables (the ones that are accounted for in $\mathcal{L}$) to a set of mutually compatible observables, but that would force us to hard-code in the theory which observables we intend to actually measure (and to drop those that could *a priori* be measured but will not). This in turn creates severe difficulties, because we need to prescribe how to select in advance the set of those truly measured observables without spoiling predictivity (see eg. the concerns raised in [24]).

In the following, we bypass this discussion by assuming that we get the *kinematical* quantum state space via the quantization of a nice classical projective limit, so that it will not be a problem, for any finite set of kinematical observables to represent them on a 'simple' Hilbert space, whether these kinematical observables can be simultaneously measured or not. On the other hand it is to be expected that the algebra generated by a finite set of *dynamical* quantum observables will not be easily represented: already on the classical side we had underlined that a finite set of dynamical classical observables may generate an intricate Poisson-algebra (recall the comment before def. 3.21). Like in the classical formalism, we must therefore expect that the exact dynamical observables will have to be approximated by approached ones, which in particular build a tractable algebra.

5. Projective limits of quantum state spaces

The crucial insight of the projective formalism, that goes back to Jerzy Kijowski [48], is that quantum states will be realized as projective families of *density matrices*, and *not* as families of vector states. This is actually a repercussion of the specific viewpoint of this formalism, namely that labels in $\mathcal{L}$ stand for a selection of observables (and not eg. for a selection of states). Indeed, in order to project a state from a more detailed partial quantum theory, represented on an Hilbert space $\mathcal{H}_{\eta'}$, to a coarser one, with Hilbert space $\mathcal{H}_{\eta}$, we need a map that will retain from a state only the features needed to compute expectation values of the observables on $\mathcal{H}_{\eta}$: this is what the partial trace on a tensor product factor accomplishes but it can only be defined as a map between density matrices (the partial trace of a pure state can be a mixed state and conversely).

While this has been previously rather seen as a weakness of the construction (see the discussion in [68, section 6.2]), we argue that such a generalized framework will be, in practice, indistinguishable from a more traditional theory on a Hilbert space: given a particular experiment, we should be able to select an η such that everything takes place within $\mathcal{H}_\eta$. Moreover it seems advantageous to start with a very large kinematical state space, thus avoiding the inherent arbitrariness of restricting it *a priori* to some particular subspace, and to wait until there is a real need for such a restriction, together with clear requirements on how to perform it (in the light of section 3, this could be because we are forced to consider only the states for which the regularization scheme, needed to implement the dynamics, converges; this intuition is supported by the toy-model studied in section 16).

In order to clarify the claim made above, that quantum state spaces described as projective limits tend to be 'bigger', we will examine precisely how they compare to those built on inductive limits of Hilbert spaces (theorem 5.9) or on infinite tensor products (theorem 5.11): these are constructions that incorporate ingredients somewhat similar to the projective approach, but differ notably by seeking a global Hilbert space for the quantum theory.

5.1 Projective systems of quantum state spaces

In this subsection, we present the projective approach to quantum field theory [48, 68], formulating it in a form as close as possible to the classical formalism set up in section 2. Indeed, the projective systems of quantum state spaces described here can be seen as the direct equivalents of the factorizing systems we had on the classical side (subsection 2.3), replacing Cartesian products of classical phase spaces by tensor products of Hilbert spaces, in accordance to the basic principles of quantum mechanics (in particular, the three-spaces consistency from fig. 2.1 is straightforwardly transformed into its quantum version illustrated in fig. 5.1).

While we had convinced ourselves that the factorizing systems are quite generic among the projective systems of classical phase spaces (see the argument laid in prop. 2.10), their quantum counterparts seem to be even more broadly applicable (viz. section 7).

Definition 5.1 A projective system of quantum state spaces is a quintuple:

$$\left(\mathcal{L}, \left(\mathcal{H}_\eta\right)_{\eta\in\mathcal{L}}, \left(\mathcal{H}_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'}, \left(\Phi_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'}, \left(\Phi_{\eta''\to\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'\preccurlyeq\eta''}\right)$$

where:

1. $\mathcal{L}$ is a preordered, directed set (we denote the pre-order by $\preccurlyeq$, its inverse by $\succcurlyeq$);
2. $\left(\mathcal{H}_\eta\right)_{\eta\in\mathcal{L}}$ is a family of Hilbert spaces indexed by $\mathcal{L}$;
3. $\left(\mathcal{H}_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'}$ is a family of Hilbert spaces indexed by $\{\eta, \eta' \in \mathcal{L} \mid \eta \preccurlyeq \eta'\}$, such that $\dim(\mathcal{H}_{\eta\to\eta}) = 1$ for all $\eta \in \mathcal{L}$;
4. $\left(\Phi_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'}$ is a family of isomorphisms of Hilbert spaces $\Phi_{\eta'\to\eta} : \mathcal{H}_{\eta'} \to \mathcal{H}_{\eta'\to\eta} \otimes \mathcal{H}_\eta$ indexed by $\{\eta, \eta' \in \mathcal{L} \mid \eta \preccurlyeq \eta'\}$ such that $\Phi_{\eta\to\eta}$ is trivial (by isomorphism of Hilbert spaces we mean a

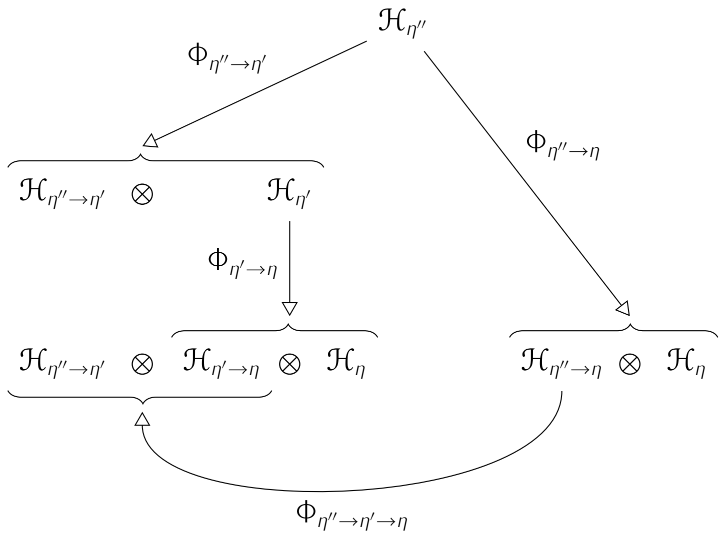


Figure 5.1 – Three-spaces consistency for projective systems of quantum state spaces

(bijective) unitary map between Hilbert spaces);

5. $\left(\Phi_{\eta''\to\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'\preccurlyeq\eta''}$ is a family of isomorphisms of Hilbert spaces $\Phi_{\eta''\to\eta'\to\eta} : \mathcal{H}_{\eta''\to\eta} \to \mathcal{H}_{\eta''\to\eta'} \otimes \mathcal{H}_{\eta'\to\eta}$ indexed by $\{\eta, \eta', \eta'' \in \mathcal{L} \mid \eta \preccurlyeq \eta' \preccurlyeq \eta''\}$, such that $\Phi_{\eta''\to\eta'\to\eta}$ is trivial whenever $\eta = \eta'$ or $\eta' = \eta''$, and:

$$\forall \eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L},\ \ (\Phi_{\eta''\to\eta'\to\eta} \otimes \mathrm{id}_{\mathcal{H}_\eta}) \circ \Phi_{\eta''\to\eta} = (\mathrm{id}_{\mathcal{H}_{\eta''\to\eta'}} \otimes \Phi_{\eta'\to\eta}) \circ \Phi_{\eta''\to\eta'}. \tag{5.1.1}$$

Whenever possible, we will use the shortened notation $(\mathcal{L}, \mathcal{H}, \Phi)^\otimes$ instead of $\Big(\mathcal{L},\ \left(\mathcal{H}_\eta\right)_{\eta\in\mathcal{L}},\ \left(\mathcal{H}_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'},\ \left(\Phi_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'},\ \left(\Phi_{\eta''\to\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'\preccurlyeq\eta''}\Big)$.

Definition 5.2 Let $(\mathcal{L}, \mathcal{H}, \Phi)^\otimes$ be a projective system of quantum state spaces. For $\eta \in \mathcal{L}$, we define $\overline{\mathcal{S}}_\eta$ the space of (self-adjoint) positive semi-definite, traceclass operators on $\mathcal{H}_\eta$ and $\mathcal{S}_\eta$ the space of density matrices:

$$\mathcal{S}_\eta = \left\{\rho_\eta \in \overline{\mathcal{S}}_\eta \;\middle|\; \mathrm{Tr}_{\mathcal{H}_\eta}\, \rho_\eta = 1\right\}.$$

For $\eta \preccurlyeq \eta' \in \mathcal{L}$, we define:

$$\begin{array}{rccl} \mathrm{Tr}_{\eta'\to\eta} & : & \overline{\mathcal{S}}_{\eta'} & \to \overline{\mathcal{S}}_\eta \\ & & \rho_{\eta'} & \mapsto \mathrm{Tr}_{\mathcal{H}_{\eta'\to\eta}}\left(\Phi_{\eta'\to\eta} \circ \rho_{\eta'} \circ \Phi^{-1}_{\eta'\to\eta}\right) \end{array}.$$

From eq. (5.1.1), we have:

$$\forall \eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L},\ \mathrm{Tr}_{\eta'\to\eta} \circ \mathrm{Tr}_{\eta''\to\eta'} = \mathrm{Tr}_{\eta''\to\eta}.$$

Hence, $\left(\mathcal{L},\ \left(\overline{\mathcal{S}}_\eta\right)_{\eta\in\mathcal{L}},\ \left(\mathrm{Tr}_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'}\right)$ forms a projective system and we denote its projective limit by $\overline{\mathcal{S}}^\otimes_{(\mathcal{L},\mathcal{H},\Phi)}$. The maps $\mathrm{Tr}_{\eta'\to\eta}$ being linear under conical combinations (ie. under addition and multiplication by positive reals), $\overline{\mathcal{S}}^\otimes_{(\mathcal{L},\mathcal{H},\Phi)}$ forms a cone (ie. we can equip it with a notion of addition and positive multiplication).

Now, for all $\eta \preccurlyeq \eta' \in \mathcal{L}$, $\mathrm{Tr}_{\eta' \to \eta} \langle \mathcal{S}_{\eta'} \rangle = \mathcal{S}_\eta$, hence $\left(\mathcal{L}, \left(\mathcal{S}_\eta\right)_{\eta \in \mathcal{L}}, \left(\mathrm{Tr}_{\eta' \to \eta}\right)_{\eta \preccurlyeq \eta'}\right)$ also forms a projective system. Accordingly, a state on $(\mathcal{L}, \mathcal{H}, \Phi)^\otimes$ is a family $\left(\rho_\eta\right)_{\eta \in \mathcal{L}}$ such that $\forall \eta \in \mathcal{L}$, ρ_η is a density matrix over $\mathcal{H}_\eta$ and $\forall \eta \preccurlyeq \eta'$, $\mathrm{Tr}_{\eta' \to \eta} \rho_{\eta'} = \rho_\eta$. We denote the space of states by $\mathcal{S}^\otimes_{(\mathcal{L}, \mathcal{H}, \Phi)}$.

The projective structure conferred to the space of states comes with a natural inductive structure for the observables (see [48, section 6] or [68, section 6.2] as well as [86, section 1]). Thus, we can define a C^*-algebra $\overline{\mathcal{A}}^\otimes_{(\mathcal{L}, \mathcal{H}, \Phi)}$ of bounded operators, as the inductive limit of the algebras $\mathcal{A}_\eta$ that live over each $\mathcal{H}_\eta$. Then, the states defined above can be seen as states on $\overline{\mathcal{A}}^\otimes_{(\mathcal{L}, \mathcal{H}, \Phi)}$ in the sense of [41, part III, def. 2.2.8]. Looking at states over a C^*-algebra is in a sense more fundamental than looking at density matrices over a specific representation of this algebra. Indeed, any such representation can be split into cyclic components, and each cyclic component arises from a state over the algebra via the GNS construction [34]. Reciprocally, any state ρ over the algebra defines a corresponding GNS representation, and the irreducible representations are precisely the ones that arise from pure states (states that cannot be written as a non trivial convex superposition of other states) [41, part III, theorem 2.2.17].

The states that can be represented as density matrices over the GNS representation of a state ρ form the folium of ρ (see also prop. 5.8 on this point), and Fell's theorem [29] tells us that, whenever ρ yields a faithful GNS representation, its folium will be, in a definite sense, dense in the set of all states over the algebra. While this mitigates the universality concerns posed by the (more or less arbitrary) choice of a representation (that we raised in the general introduction), it does not completely eliminate them: although we could, in principle, approximate any desired state by a family of states that belong to the chosen representation, we would still have to carefully check the convergence properties of this family (to ensure that it provides a consistent physical picture). On the other hand, by working at the level of the full space of states over the algebra, we would lack any constructive description of the states. The framework presented here can thus be seen as a middle ground: we are trying not to restrict the space of states too much (see prop. 5.12 and the discussion at the end of section 18), while providing a fairly explicit characterization for it (especially if the label set can be made countable, as we will investigate in details in section 19).

Note that this framework is somewhat reminiscent of other approaches where one is looking at states over an inductive limit C^*-algebra, like Algebraic Quantum Field Theory [41, part III], or the General Boundary Formulation of quantum field theory (in its positive version [63]). There are, however, substantial differences. Notably, the building blocks of our algebra of observables will be in practice very *small* algebras: each $\mathcal{A}_\eta$, instead of being meant to include all the operators needed to interpret any arbitrary experiment taking place in some given region of spacetime, should be thought as only containing the operators needed for the description of *finitely* many experiments. More deeply, the purpose of giving the algebra of observables an inductive limit structure is in our case not so much to encode additional physically essential information (eg. the localization of the operators and associated causal structure) but rather to arrive at a description of the space of states as concrete and as convenient as possible (by building it from small representations that are well under control and suitable for calculation). Of course, we could combine both aspects, by decorating the projective structure with this extra information: for example, we could map a region in spacetime to the set of all η that can be seen as contained in this region.

Definition 5.3 We consider the same objects as in def. 5.2. For $\eta \in \mathcal{L}$, we denote by $\mathcal{A}_\eta$ the algebra of bounded operators on $\mathcal{H}_\eta$ and, for $\eta \preccurlyeq \eta' \in \mathcal{L}$, we define:

$$\begin{array}{rccl} \iota_{\eta'\leftarrow\eta} : & \mathcal{A}_\eta & \to & \mathcal{A}_{\eta'} \\ & A_\eta & \mapsto & \Phi^{-1}_{\eta'\to\eta} \circ \left(\mathrm{id}_{\mathcal{H}_{\eta'\to\eta}} \otimes A_\eta\right) \circ \Phi_{\eta'\to\eta} \end{array} .$$

By definition $\iota_{\eta'\leftarrow\eta}$ is injective and, from eq. (5.1.1), we have:

$$\forall \eta \preccurlyeq \eta' \preccurlyeq \eta'',\ \iota_{\eta''\leftarrow\eta'} \circ \iota_{\eta'\leftarrow\eta} = \iota_{\eta''\leftarrow\eta}\,. \tag{5.3.1}$$

Accordingly, an operator over $\overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ is an equivalence class in $\bigcup_{\eta\in\mathcal{L}} \mathcal{A}_\eta$ for the equivalence relation defined by:

$$\begin{array}{l} \forall \eta, \eta' \in \mathcal{L}, \forall A_\eta \in \mathcal{A}_\eta, \forall A_{\eta'} \in \mathcal{A}_{\eta'}, \\ \qquad A_\eta \sim A_{\eta'} \Leftrightarrow \left(\exists \eta'' \in \mathcal{L} \,/\, \eta \preccurlyeq \eta'', \eta' \preccurlyeq \eta'' \ \ \&\ \ \iota_{\eta''\leftarrow\eta}\left(A_\eta\right) = \iota_{\eta''\leftarrow\eta'}\left(A_{\eta'}\right)\right) \end{array} \tag{5.3.2}$$

The space of operators over $\overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ will be denoted by $\mathcal{A}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$. For $A = [A_\eta]_\sim \in \mathcal{A}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ and $\rho = (\rho_\eta)_{\eta\in\mathcal{L}} \in \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$, we can define $\mathrm{Tr}\rho A := \mathrm{Tr}_{\mathcal{H}_\eta}\rho_\eta A_\eta$. The definition of the equivalence relation ensures that this is well-defined.

Proposition 5.4 For $\eta \preccurlyeq \eta' \in \mathcal{L}$, the map $\iota_{\eta'\leftarrow\eta}$ is an injective C^*-algebra morphism (ie. an injective, isometric $*$-algebra morphism). Hence, $\mathcal{A}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ can be equipped with a normed $*$-algebra structure as an inductive limit of C^*-algebras. And denoting by $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ the completion of $\mathcal{A}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ with respect to the operator norm, $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ is a C^*-algebra.

Then, for all $\rho \in \mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$, $\mathrm{Tr}\left(\rho\ \cdot\right)$ can be extended by continuity as a state over $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$.

Proof That $\iota_{\eta'\leftarrow\eta}$ is a C^*-algebra morphism for any $\eta \preccurlyeq \eta'$ is ensured by $\Phi_{\eta'\to\eta}$ being a Hilbert space isomorphism and by the properties of the tensor product of operators.

Next, let $\rho \in \mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$, and let $[A_\eta]_\sim, [B_{\eta'}]_\sim \in \mathcal{A}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$. For any $\eta'' \succcurlyeq \eta', \eta$ and any $a, b \in \mathbb{C}$, we have:

1. $\mathrm{Tr}\,\rho\,(a\,A + b\,B) = \mathrm{Tr}_{\mathcal{H}_{\eta''}}\,\rho_{\eta''}\,\left(a\,\iota_{\eta''\leftarrow\eta}(A_\eta) + b\,\iota_{\eta''\leftarrow\eta'}(B_{\eta'})\right) = a\,\mathrm{Tr}\,(\rho\,A) + b\,\mathrm{Tr}\,(\rho\,B)\,;$
2. $\mathrm{Tr}\,(\rho\,A) = \mathrm{Tr}_{\mathcal{H}_\eta}\,(\rho_\eta\,A_\eta) \leqslant \|A_\eta\| = \|A\|\,;$
3. $\mathrm{Tr}\,(\rho\,\mathbb{1}) = \mathrm{Tr}_{\mathcal{H}_\eta}\,(\rho_\eta\,\mathrm{id}_{\mathcal{H}_\eta}) = 1\,;$
4. $\mathrm{Tr}\,(\rho\,A^+\,A) = \mathrm{Tr}_{\mathcal{H}_\eta}\,(\rho_\eta\,A^+_\eta\,A_\eta) \geqslant 0\,.$

□

Proposition 5.5 For $\eta \in \mathcal{L}$, we denote by $\mathcal{O}_\eta$ the algebra of densely defined (possibly unbounded) self-adjoint operators on $\mathcal{H}_\eta$. For $\eta \preccurlyeq \eta' \in \mathcal{L}$, the map $\iota_{\eta'\leftarrow\eta}$ (def. 5.3) can be extended to $\mathcal{O}_\eta \to \mathcal{O}_{\eta'}$. Indeed, if $O_\eta \in \mathcal{O}_\eta$ is a self-adjoint operator on $\mathcal{H}_\eta$, with dense domain $\mathcal{D}_\eta \subset \mathcal{H}_\eta$, then the operator $\Phi^{-1}_{\eta'\to\eta} \circ \left(\mathrm{id}_{\mathcal{H}_{\eta'\to\eta}} \otimes O_\eta\right) \circ \Phi_{\eta'\to\eta}$, defined on the dense subset $\Phi^{-1}_{\eta'\to\eta}\left\langle \mathcal{H}_{\eta'\to\eta} \otimes \mathcal{D}_\eta \right\rangle$ of $\mathcal{H}_{\eta'}$ (where the $\otimes$ is understood as a tensor product of vector spaces, ie. *without* any completion), is essentially self-adjoint and we define $\iota_{\eta'\leftarrow\eta}(O_\eta)$ as its unique self-adjoint extension [73, section VIII.10]. An observable over a projective limit of quantum state spaces $\mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ is then defined as an equivalence class in $\bigcup_{\eta\in\mathcal{L}} \mathcal{O}_\eta$ in analogy to eq. (5.3.2).

The space of observables over $\mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ will be denoted by $\mathcal{O}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$. For $O = [O_\eta]_\sim \in \mathcal{O}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$, we can define the spectrum $\varsigma(O)$ of O as the spectrum $\varsigma(O_\eta)$ of any representative O_η of O and, for W a measurable subset of $\varsigma(O)$, we can define $\mathbb{I}_W(O)$ as the equivalence class $[\mathbb{I}_W(O_\eta)]_\sim \in \mathcal{A}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ of the spectral projector $\mathbb{I}_W(O_\eta)$.

Hence, for a state $\rho = (\rho_\eta)_{\eta\in\mathcal{L}} \in \mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$, we can define the probability of measuring O in W as $\rho[O \in W] := \operatorname{Tr} \rho\, \mathbb{I}_W(O)$.

Proof $\varsigma(O_\eta)$ being independent of the choice of a representative O_η comes from:

$$\forall \eta' \succcurlyeq \eta,\ \varsigma\left(\Phi^{-1}_{\eta'\to\eta} \circ \left(\mathrm{id}_{\mathcal{H}_{\eta'\to\eta}} \otimes O_\eta\right) \circ \Phi_{\eta'\to\eta}\right) = \varsigma\left(O_\eta\right),$$

where we used [73, theorem VIII.33] together with the fact that $\Phi_{\eta'\to\eta}$ is an Hilbert space isomorphism.

That $O_\eta \sim O_{\eta'}$ implies $\mathbb{I}_W(O_\eta) \sim \mathbb{I}_W(O_{\eta'})$ comes from:

$$\forall \eta' \succcurlyeq \eta,\ \mathbb{I}_W\left(\Phi^{-1}_{\eta'\to\eta} \circ \left(\mathrm{id}_{\mathcal{H}_{\eta'\to\eta}} \otimes O_\eta\right) \circ \Phi_{\eta'\to\eta}\right) = \Phi^{-1}_{\eta'\to\eta} \circ \left(\mathrm{id}_{\mathcal{H}_{\eta'\to\eta}} \otimes \mathbb{I}_W(O_\eta)\right) \circ \Phi_{\eta'\to\eta},$$

as can be ascertained using the spectral theorem in its multiplication operator form [73, theorem VIII.4]. □

5.2 Maps between quantum state spaces

In order to investigate the relations between the spaces of quantum states assembled this way and more standard constructions, we will use the same tool as we used to make the connection between classical projective structures and infinite dimensional symplectic manifolds, namely extensions and restrictions of the label set. As in subsection 2.2, the strategy will be to extend the label set by adding to it a greatest element (associated to the 'big' Hilbert space to which we want to make contact), before restricting ourselves to this greatest element alone (thus ending with a trivial projective system that can be identified with the target Hilbert space). A bit unintuitively, the non-trivial switch of state space occurs during the first step: this is because a greatest element forms by definition a cofinal part, which makes the second step innocuous.

Proposition 5.6 Let $(\mathcal{L},\mathcal{H},\Phi)^{\otimes}$ be a projective system of quantum state spaces and let $\mathcal{L}'$ be a directed subset of $\mathcal{L}$. We define the map:

$$\sigma : \begin{array}{rcl} \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} & \to & \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}',\mathcal{H},\Phi)} \\ \left(\rho_\eta\right)_{\eta\in\mathcal{L}} & \mapsto & \left(\rho_\eta\right)_{\eta\in\mathcal{L}'} \end{array}.$$

σ is conically linear (ie. compatible with addition and multiplication by positive scalars).

Moreover, we have a map $\alpha : \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}',\mathcal{H},\Phi)} \to \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ such that:

$$\forall \rho \in \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}, \forall A \in \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}',\mathcal{H},\Phi)},\ \mathrm{Tr}\,(\rho\,\alpha(A)) = \mathrm{Tr}\,(\sigma(\rho)\,A)\,, \tag{5.6.1}$$

and α is a C^*-algebra morphism.

If $\mathcal{L}'$ is cofinal in $\mathcal{L}$, we have in addition that σ and α are bijective maps.

Proof The proof works in the same way as in the classical case (prop. 2.5). The conical linearity of σ and morphism property of α comes from their definition (with α being defined in analogy to the classical case). Eq. (5.6.1) can be first checked for $\mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ and $\mathcal{A}_{(\mathcal{L}',\mathcal{H},\Phi)}$ and expanded by continuity and conical linearity. □

Proposition 5.7 In particular, if $\mathcal{L}$ admits a greatest element Λ, there exist bijective maps $\sigma : \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \to \overline{\mathcal{S}}_{\Lambda}$ ($\overline{\mathcal{S}}_{\Lambda}$ being the space of (self-adjoint) positive semi-definite, traceclass operators over $\mathcal{H}_{\Lambda}$) and $\alpha : \mathcal{A}_{\Lambda} \to \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ such that $\forall \rho \in \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}, \forall A \in \mathcal{A}_{\Lambda}$, $\mathrm{Tr}\,(\rho\,\alpha(A)) = \mathrm{Tr}_{\Lambda}\,(\sigma(\rho)\,A)$.

Proof This is an application of prop. 5.6 for the cofinal part $\mathcal{L}' = \{\Lambda\}$ of $\mathcal{L}$, using the obvious identification of $\overline{\mathcal{S}}^{\otimes}_{(\{\Lambda\},\mathcal{H},\Phi)}$ with $\overline{\mathcal{S}}_{\Lambda}$ and $\overline{\mathcal{A}}^{\otimes}_{(\{\Lambda\},\mathcal{H},\Phi)}$ with $\mathcal{A}_{\Lambda}$. □

If we choose a particular state in a projective limit of quantum state spaces, we can use it as a vacuum to construct a corresponding GNS representation of the inductive limit algebra of observables [68, section 6.2], and this representation will naturally inherit a structure of inductive limit Hilbert space (like the Hilbert space used as starting point in LQG, see [6, 60, 8]). We will specialize in the case where the vacuum state we are using projects as a pure state on every $\mathcal{H}_{\eta}$. Whether there exists such a pure projective state of course depends on the specific projective structure under consideration, however we will be interested in situations where the natural vacuum turns out to be of this type (notably in prop. 6.5 and prop. 16.17). In this case, the inductive limit Hilbert space is obtained from a collection of 'reference states' $\zeta_{\eta'\to\eta} \in \mathcal{H}_{\eta'\to\eta}$, that allows us to see the tensor product factor $\mathcal{H}_{\eta}$ as a vector subspace in $\mathcal{H}_{\eta'}$. Any density matrix over such an inductive limit $\mathcal{H}_{\zeta}$ can then be unambiguously mapped to a state in the projective limit, but the converse typically does not hold, and in fact, we can formulate a handy condition to check if a state on the projective structure has its counterpart as a density matrix on $\mathcal{H}_{\zeta}$.

Proposition 5.8 Let $(\mathcal{L}, \mathcal{H}, \Phi)^{\otimes}$ be a projective system of quantum state spaces and let $\rho = \left(\rho_{\eta}\right)_{\eta\in\mathcal{L}} \in \mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$. For all $\eta \in \mathcal{L}$, ρ_{η} is a state over $\mathcal{A}_{\eta}$ and we denote by $\mathcal{H}^{\text{GNS}}_{\rho_{\eta}}$, $(\,\cdot\,)^{\text{GNS}}$ the GNS representation constructed from this state [41, section III.2.2].

Then, for all $\eta \preccurlyeq \eta' \in \mathcal{L}$, there exists an injective linear map $\tau_{\eta'\leftarrow\eta} : \mathcal{H}^{\text{GNS}}_{\rho_{\eta}} \to \mathcal{H}^{\text{GNS}}_{\rho_{\eta'}}$. $\tau_{\eta'\leftarrow\eta}$ is isometric onto its image and satisfies:

$$\forall A_{\eta} \in \mathcal{A}_{\eta},\ \tau_{\eta'\leftarrow\eta} \circ A^{\text{GNS}}_{\eta} = \left(\iota_{\eta'\leftarrow\eta}(A_{\eta})\right)^{\text{GNS}} \circ \tau_{\eta'\leftarrow\eta}\,. \tag{5.8.1}$$

Moreover, for all $\eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}$, we have $\tau_{\eta''\leftarrow\eta'} \circ \tau_{\eta'\leftarrow\eta} = \tau_{\eta''\leftarrow\eta}$, hence we can define an Hilbert space $\mathcal{H}^{\text{GNS}}_{\rho}$ as (the completion of) the inductive limit of $\left(\mathcal{L},\ \left(\mathcal{H}^{\text{GNS}}_{\rho_{\eta}}\right)_{\eta\in\mathcal{L}},\ \left(\tau_{\eta'\leftarrow\eta}\right)_{\eta\preccurlyeq\eta'}\right)$ and $\mathcal{H}^{\text{GNS}}_{\rho}$ can be identified with the GNS representation of $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ arising from the state ρ (prop. 5.4).

If in addition there exists, for all $\eta \in \mathcal{L}$, a vector $\zeta_{\eta} \in \mathcal{H}_{\eta}$ such that $\rho_{\eta} = |\zeta_{\eta}\,\rangle\langle\,\zeta_{\eta}|$, then

$\mathcal{H}_{\rho_\eta}^{\text{GNS}} \approx \mathcal{H}_\eta$ and for all $\eta \preccurlyeq \eta'$ there exists a vector $\zeta_{\eta'\to\eta} \in \mathcal{H}_{\eta'\to\eta}$ such that the map $\tau_{\eta'\leftarrow\eta}$ is given by:

$$\forall \psi \in \mathcal{H}_\eta,\ \tau_{\eta'\leftarrow\eta}(\psi) = \Phi_{\eta'\to\eta}^{-1}\left(\zeta_{\eta'\to\eta} \otimes \psi\right). \tag{5.8.2}$$

Proof *Explicit description of* $\mathcal{H}_{\rho_\eta}^{\text{GNS}}$. Let $\eta \in \mathcal{L}$. By applying the spectral theorem to the (self-adjoint) positive semi-definite, traceclass, normalized operator ρ_η, there exist $N_\eta \in \mathbb{N} \sqcup \{\infty\}$, an orthonormal family $\left(\zeta_\eta^{(k)}\right)_{k\leqslant N_\eta}$ in $\mathcal{H}_\eta$ and a family of *strictly* positive reals $\left(p_\eta^{(k)}\right)_{k\leqslant N_\eta}$ such that:

$$\rho_\eta = \sum_{k\leqslant N_\eta} p_\eta^{(k)} \left|\zeta_\eta^{(k)}\right\rangle\left\langle\zeta_\eta^{(k)}\right|,$$

with $\sum_{k\leqslant N_\eta} p_\eta^{(k)} = 1$.

We define the map Ψ_{ρ_η} by:

$$\begin{array}{rcl} \Psi_{\rho_\eta} : \mathcal{A}_\eta & \to & \mathcal{H}_\eta^{+} \otimes \mathcal{H}_\eta \\ A & \mapsto & \sum_{k\leqslant N_\eta} \sqrt{p_\eta^{(k)}}\, \left\langle \zeta_\eta^{(k)}\right| \otimes \left|A\, \zeta_\eta^{(k)}\right\rangle \end{array},$$

where $\mathcal{H}_\eta^{+}$ is the topological dual of $\mathcal{H}_\eta$ equipped with its natural Hilbert space structure. The map Ψ_{ρ_η} is well-defined, for we have:

$$\sum_{k\leqslant N_\eta} \left\| \sqrt{p_\eta^{(k)}}\, \left\langle \zeta_\eta^{(k)}\right| \otimes \left|A\, \zeta_\eta^{(k)}\right\rangle \right\|^2 = \sum_{k\leqslant N_\eta} p_\eta^{(k)}\, \left\|\left|A\, \zeta_\eta^{(k)}\right\rangle\right\|^2 \leqslant \|A\|^2. \tag{5.8.3}$$

Moreover, it has following properties:

1. Ψ_{ρ_η} is $\mathbb{C}$-linear;
2. $\forall A, B \in \mathcal{A}_\eta,\ \Psi_{\rho_\eta}(A\,B) = \left(\text{id}_{\mathcal{H}_\eta^{+}} \otimes A\right)\, \Psi_{\rho_\eta}(B)$;
3. $\forall A, B \in \mathcal{A}_\eta,\ \left\langle \Psi_{\rho_\eta}(A),\ \Psi_{\rho_\eta}(B)\right\rangle_{\mathcal{H}_\eta^{+}\otimes\mathcal{H}_\eta} =$

$$= \sum_{k,k'\leqslant N_\eta} \sqrt{p_\eta^{(k)}}\sqrt{p_\eta^{(k')}}\, \left\langle \zeta_\eta^{(k')} \,\middle|\, \zeta_\eta^{(k)}\right\rangle \otimes \left\langle A\, \zeta_\eta^{(k)} \,\middle|\, B\, \zeta_\eta^{(k')}\right\rangle$$

$$= \sum_{k\leqslant N_\eta} p_\eta^{(k)} \otimes \left\langle \zeta_\eta^{(k)} \,\middle|\, A^{+} B\, \zeta_\eta^{(k)}\right\rangle = \text{Tr}_{\mathcal{H}_\eta}\, \rho_\eta\, A^{+} B\,;$$

4. $\overline{\Psi_{\rho_\eta}\left\langle \mathcal{A}_\eta\right\rangle} = \overline{\text{Vect}\left\{\sqrt{p_\eta^{(k)}}\, \left\langle \zeta_\eta^{(k)}\right| \otimes |\psi\rangle \mid k \leqslant N_\eta,\ \psi \in \mathcal{H}_\eta\right\}}$

   ('$\subset$': by definition of Ψ_{ρ_η}; '$\supset$': by considering operators of the form $\left|\psi\right\rangle\left\langle \zeta_\eta^{(k)}\right|$)

$$= \overline{\text{Vect}\left\{\sqrt{p_\eta^{(k)}}\, \left\langle \zeta_\eta^{(k)}\right| \mid k \leqslant N_\eta\right\}} \otimes \mathcal{H}_\eta =: \mathcal{K}_{\rho_\eta} \otimes \mathcal{H}_\eta\,.$$

Therefore, we can identify $\mathcal{H}_{\rho_\eta}^{\text{GNS}}$ with $\mathcal{K}_{\rho_\eta} \otimes \mathcal{H}_\eta$ and we have:

$$\forall A \in \mathcal{A}_\eta,\, A^{\text{GNS}} = \mathrm{id}_{\mathcal{K}_{\rho_\eta}} \otimes A\,.$$

Definition of the injections $\tau_{\eta'\leftarrow\eta}$. Let $\eta \preccurlyeq \eta' \in \mathcal{L}$. We define a $\mathbb{C}$-linear map:

$$\tau_{\eta'\leftarrow\eta} : \mathrm{Vect}\left\{\sqrt{p_\eta^{(k)}}\,\left\langle \zeta_\eta^{(k)}\right| \,\middle|\, k \leqslant N_\eta\right\} \otimes \mathcal{H}_\eta \to \mathrm{Vect}\left\{\sqrt{p_{\eta'}^{(k')}}\,\left\langle \zeta_{\eta'}^{(k')}\right| \,\middle|\, k' \leqslant N_{\eta'}\right\} \otimes \mathcal{H}_{\eta'}$$

by:

$$\forall k \leqslant N_\eta,\, \forall \psi \in \mathcal{H}_\eta,\ \tau_{\eta'\leftarrow\eta}\left(\sqrt{p_\eta^{(k)}}\,\left\langle \zeta_\eta^{(k)}\right| \otimes \left|\psi\right\rangle\right) =$$

$$= \sum_{k' \leqslant N_{\eta'},e} \left\langle \Phi^{-1}_{\eta'\to\eta}\left(e \otimes \zeta_\eta^{(k)}\right) \,\middle|\, \zeta_{\eta'}^{(k')}\right\rangle \sqrt{p_{\eta'}^{(k')}}\,\left\langle \zeta_{\eta'}^{(k')}\right| \otimes \left|\Phi^{-1}_{\eta'\to\eta}\left(e \otimes \psi\right)\right\rangle$$

$$= \sum_e \left\langle \sqrt{\rho_{\eta'}} \circ \Phi^{-1}_{\eta'\to\eta}\left(e \otimes \zeta_\eta^{(k)}\right)\right| \otimes \left|\Phi^{-1}_{\eta'\to\eta}\left(e \otimes \psi\right)\right\rangle,$$

where (e) is an orthonormal basis of $\mathcal{H}_{\eta'\to\eta}$ and $\sqrt{\rho_{\eta'}}$ is defined via spectral resolution. We have:

$$\forall k \leqslant N_\eta,\, \forall \psi \in \mathcal{H}_\eta,\ \left\|\tau_{\eta'\leftarrow\eta}\left(\sqrt{p_\eta^{(k)}}\,\left\langle \zeta_\eta^{(k)}\right| \otimes \left|\psi\right\rangle\right)\right\|^2 =$$

$$= \sum_{e,e'} \left\langle \sqrt{\rho_{\eta'}} \circ \Phi^{-1}_{\eta'\to\eta}\left(e \otimes \zeta_\eta^{(k)}\right) \,\middle|\, \sqrt{\rho_{\eta'}} \circ \Phi^{-1}_{\eta'\to\eta}\left(e' \otimes \zeta_\eta^{(k)}\right)\right\rangle \left\langle \Phi^{-1}_{\eta'\to\eta}\left(e' \otimes \psi\right) \,\middle|\, \Phi^{-1}_{\eta'\to\eta}\left(e \otimes \psi\right)\right\rangle$$

$$= \sum_e \left\langle \Phi^{-1}_{\eta'\to\eta}\left(e \otimes \zeta_\eta^{(k)}\right) \,\middle|\, \rho_{\eta'} \circ \Phi^{-1}_{\eta'\to\eta}\left(e \otimes \zeta_\eta^{(k)}\right)\right\rangle \left\|\psi\right\|^2$$

$$= \left\langle \zeta_\eta^{(k)} \,\middle|\, \left(\mathrm{Tr}_{\eta'\to\eta}\,\rho_{\eta'}\right)\, \zeta_\eta^{(k)}\right\rangle \left\|\psi\right\|^2$$

$$= p_\eta^{(k)}\,\left\|\psi\right\|^2 = \left\|\sqrt{p_\eta^{(k)}}\,\left\langle \zeta_\eta^{(k)}\right| \otimes \left|\psi\right\rangle\right\|^2.$$

Therefore $\tau_{\eta'\leftarrow\eta}$ is well-defined and can be extended as an injection $\mathcal{H}^{\text{GNS}}_{\rho_\eta} \to \mathcal{H}^{\text{GNS}}_{\rho_{\eta'}}$, which is isometric onto its image. For $A_\eta \in \mathcal{A}_\eta$ we can check directly that eq. (5.8.1) is satisfied, using $A_\eta^{\text{GNS}} = \mathrm{id}_{\mathcal{K}_{\rho_\eta}} \otimes A_\eta$.

Next, we have:

$$\sum_{k \leqslant N_\eta} \mathrm{Tr}_{\mathcal{H}_{\eta'}}\,\rho_{\eta'}\,\iota_{\eta'\leftarrow\eta}\left(\left|\zeta_\eta^{(k)}\right\rangle\left\langle \zeta_\eta^{(k)}\right|\right) = \mathrm{Tr}_{\mathcal{H}_\eta}\,\rho_\eta = 1$$

$$= \sum_{k \leqslant N_\eta, k' \leqslant N_{\eta'},e} p_{\eta'}^{(k')}\,\left|\left\langle \zeta_{\eta'}^{(k')} \,\middle|\, \Phi^{-1}_{\eta'\to\eta}\left(e \otimes \zeta_\eta^{(k)}\right)\right\rangle\right|^2,$$

but since all $p_{\eta'}^{(k')}$ are strictly positive and $\sum_{k' \leqslant N_{\eta'}} p_{\eta'}^{(k')} = 1$, this implies:

$$\forall k' \leqslant N_{\eta'},\ \sum_{k \leqslant N_\eta, e} \left| \left\langle \zeta_{\eta'}^{(k')} \,\middle|\, \Phi_{\eta' \to \eta}^{-1} \left(e \otimes \zeta_\eta^{(k)} \right) \right\rangle \right|^2 = 1 = \left\| \zeta_{\eta'}^{(k')} \right\|^2,$$

and therefore:

$$\forall k' \leqslant N_{\eta'},\ \zeta_{\eta'}^{(k')} = \sum_{k \leqslant N_\eta, e} \left\langle \Phi_{\eta' \to \eta}^{-1} \left(e \otimes \zeta_\eta^{(k)} \right) \,\middle|\, \zeta_{\eta'}^{(k')} \right\rangle \left| \Phi_{\eta' \to \eta}^{-1} \left(e \otimes \zeta_\eta^{(k)} \right) \right\rangle. \tag{5.8.4}$$

With this we can now prove:

$$\forall A_\eta \in \mathcal{A}_\eta,\ \tau_{\eta' \leftarrow \eta} \circ \Psi_{\rho_\eta}(A_\eta) =$$

$$= \sum_{k \leqslant N_\eta, k' \leqslant N_{\eta'}, e} \left\langle \Phi_{\eta' \to \eta}^{-1} \left(e \otimes \zeta_\eta^{(k)} \right) \,\middle|\, \zeta_{\eta'}^{(k')} \right\rangle \sqrt{p_{\eta'}^{(k')}} \left\langle \zeta_{\eta'}^{(k')} \right| \otimes \left| \Phi_{\eta' \to \eta}^{-1} \left(e \otimes A_\eta\, \zeta_\eta^{(k)} \right) \right\rangle$$

$$= \sum_{k' \leqslant N_{\eta'}} \sqrt{p_{\eta'}^{(k')}} \left\langle \zeta_{\eta'}^{(k')} \right| \otimes \left| \iota_{\eta' \leftarrow \eta} \left(A_\eta \right)\ \zeta_{\eta'}^{(k')} \right\rangle$$

$$= \Psi_{\rho_{\eta'}} \circ \iota_{\eta' \leftarrow \eta} \left(A_\eta \right). \tag{5.8.5}$$

Thus, for $\eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}$, $\tau_{\eta'' \leftarrow \eta'} \circ \tau_{\eta' \leftarrow \eta} = \tau_{\eta'' \leftarrow \eta}$ follows from eq. (5.3.1) together with point 5.8.4 above.

Inductive limit Hilbert space as GNS representation of $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$. Let $\mathcal{H}_\rho^{\text{GNS}}$ be (the completion of) the inductive limit of $\left(\mathcal{L},\ \left(\mathcal{H}_{\rho_\eta}^{\text{GNS}} \right)_{\eta \in \mathcal{L}},\ \left(\tau_{\eta' \leftarrow \eta} \right)_{\eta \preccurlyeq \eta'} \right)$. Eq. (5.8.1) provides a representation of $\mathcal{A}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$, hence of $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$, on $\mathcal{H}_\rho^{\text{GNS}}$, which we will denote by $A \mapsto A_\rho^{\text{GNS}}$. In addition, eq. (5.8.5) ensures that we can consistently assemble the family of maps $\left(\Psi_{\rho_\eta} \right)_{\eta \in \mathcal{L}}$ into a map $\Psi_\rho : \mathcal{A}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \to \mathcal{H}_\rho^{\text{GNS}}$, and, by eq. (5.8.3), we can extend this map to $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$. Now, the properties of the individual Ψ_{ρ_η} ensure that Ψ_ρ has the following properties:

5. Ψ_ρ is $\mathbb{C}$-linear;
6. $\forall \eta, \eta' \in \mathcal{L}$, $\forall A = [A_\eta]_\sim$, $B = [B_{\eta'}]_\sim \in \mathcal{A}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$, $\forall \eta'' \succcurlyeq \eta, \eta'$,
$$\Psi_\rho (A\, B) = \Psi_\rho \left([A_{\eta''}\, B_{\eta''}]_\sim \right) = \left[A_{\eta''}^{\text{GNS}}\, \Psi_{\rho_{\eta''}}(B_{\eta''}) \right]_\sim = A_\rho^{\text{GNS}}\, \Psi_\rho (B);$$
7. $\forall A,\ B \in \mathcal{A}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$, $\langle \Psi_\rho (A)\,,\ \Psi_\rho (B) \rangle_{\mathcal{H}_\rho^{\text{GNS}}} = \operatorname{Tr} \rho\, A^+ B$;
8. $\overline{\Psi_\rho \left\langle \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \right\rangle} = \mathcal{H}_\rho^{\text{GNS}}$.

Therefore, we can identify $\mathcal{H}_\rho^{\text{GNS}}$ with the GNS representation of $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ arising from the state ρ.

Note. This result could be proved at a more abstract level, by directly using eq. (5.8.5) to *define* $\tau_{\eta' \leftarrow \eta}$. Here we gave the explicit expressions as a bonus.

Pure projective state. We now assume that for all $\eta \in \mathcal{L}$ $N_\eta = 0$ and we define $\forall \eta \in \mathcal{L}$, $\zeta_\eta := \zeta_\eta^{(0)}$. Thus $\forall \eta \in \mathcal{L}$, $\mathcal{K}_{\rho_\eta} \approx \mathbb{C}$ and therefore $\mathcal{H}_{\rho_\eta}^{\text{GNS}} \approx \mathcal{H}_\eta$. Then, for $\eta \preccurlyeq \eta'$, eq. (5.8.4) becomes:

$$\zeta_{\eta'} = \sum_{e \text{ BON of } \mathcal{H}_{\eta'\to\eta}} \left\langle \Phi^{-1}_{\eta'\to\eta}\left(e \otimes \zeta_\eta\right) \mid \zeta_{\eta'} \right\rangle \left|\Phi^{-1}_{\eta'\to\eta}\left(e \otimes \zeta_\eta\right)\right\rangle,$$

hence, defining $\zeta_{\eta'\to\eta} := \sum_{e \text{ BON of } \mathcal{H}_{\eta'\to\eta}} \left\langle \Phi^{-1}_{\eta'\to\eta}\left(e \otimes \zeta_\eta\right) \mid \zeta_{\eta'} \right\rangle |e\rangle$, we get $\zeta_{\eta'} = \Phi^{-1}_{\eta'\to\eta}\left(\zeta_{\eta'\to\eta} \otimes \zeta_\eta\right)$.

Inserting into the definition of $\tau_{\eta'\leftarrow\eta}$ and applying the identification $\mathcal{K}_{\rho_\eta} \approx \mathbb{C}$ provides eq. (5.8.2).□

Theorem 5.9 Let $(\mathcal{L}, \mathcal{H}, \Phi)^\otimes$ be a projective system of quantum state spaces and suppose there exists a family of vectors $\left(\zeta_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'}$ such that:

1. $\forall \eta \preccurlyeq \eta'$, $\zeta_{\eta'\to\eta} \in \mathcal{H}_{\eta'\to\eta}$ & $\|\zeta_{\eta'\to\eta}\| = 1$;
2. $\forall \eta \preccurlyeq \eta' \preccurlyeq \eta''$, $\Phi_{\eta''\to\eta'\to\eta}(\zeta_{\eta''\to\eta}) = \zeta_{\eta''\to\eta'} \otimes \zeta_{\eta'\to\eta}$.

We define an Hilbert space $\mathcal{H}_\zeta$ as (the completion of) the inductive limit of $\left(\mathcal{L},\ \left(\mathcal{H}_\eta\right)_{\eta\in\mathcal{L}},\ \left(\tau_{\eta'\leftarrow\eta}\right)_{\eta\preccurlyeq\eta'}\right)$, where the injective maps $\tau_{\eta'\leftarrow\eta}$ are defined as:

$$\forall \eta \preccurlyeq \eta' \in \mathcal{L},\ \begin{array}{rrcl} \tau_{\eta'\leftarrow\eta} : & \mathcal{H}_\eta & \to & \mathcal{H}_{\eta'} \\ & \psi & \mapsto & \Phi^{-1}_{\eta'\to\eta}\left(\zeta_{\eta'\to\eta} \otimes \psi\right) \end{array}.$$

Then, there exist maps $\sigma : \overline{\mathcal{S}}_\zeta \to \overline{\mathcal{S}}^\otimes_{(\mathcal{L},\mathcal{H},\Phi)}$ and $\alpha : \overline{\mathcal{A}}^\otimes_{(\mathcal{L},\mathcal{H},\Phi)} \to \mathcal{A}_\zeta$ ($\overline{\mathcal{S}}_\zeta$ being the space of (self-adjoint) positive semi-definite, traceclass operators over $\mathcal{H}_\zeta$ and $\mathcal{A}_\zeta$ the algebra of bounded operators on $\mathcal{H}_\zeta$) such that:

3. $\forall \rho \in \overline{\mathcal{S}}_\zeta, \forall A \in \overline{\mathcal{A}}^\otimes_{(\mathcal{L},\mathcal{H},\Phi)}$, $\mathrm{Tr}_{\mathcal{H}_\zeta}\left(\rho\, \alpha(A)\right) = \mathrm{Tr}\left(\sigma(\rho)\, A\right)$;
4. σ is injective;
5. $\sigma\left\langle \mathcal{S}_\zeta \right\rangle = \left\{ \left(\rho_\eta\right)_{\eta\in\mathcal{L}} \,\middle|\, \sup_{\eta\in\mathcal{L}} \left(\inf_{\eta'\succcurlyeq\eta} \mathrm{Tr}_{\mathcal{H}_{\eta'}} \left(\rho_{\eta'}\, \Theta_{\eta'|\eta}\right)\right) = \mathrm{Tr}\,\rho = 1 \right\}$,

   where $\mathcal{S}_\zeta$ is the space of density matrices over $\mathcal{H}_\zeta$ and:

   $$\Theta_{\eta'|\eta} := \Phi^{-1}_{\eta'\to\eta} \circ \left(|\zeta_{\eta'\to\eta}\rangle\langle\zeta_{\eta'\to\eta}| \otimes \mathrm{id}_{\mathcal{H}_\eta}\right) \circ \Phi_{\eta'\to\eta}.$$

We will, in the proof below, rely heavily on the so-called trace norm, which, for a positive traceclass operator is just its trace. The reason why this is the appropriate norm for our purpose is twofold. First, it plays nicely with the partial traces, since the trace norm of a partial trace of ρ is always bounded by the trace norm of ρ itself (it is obviously equal in the case of a positive ρ, and the bound follows by decomposing a general ρ into positive and negative parts, or by invoking the next point). Second, it supports the physical interpretation of quantum states, revolving around the evaluation of observable expectation values, since the trace norm of ρ is precisely the norm of the continuous linear functional $A \mapsto \mathrm{Tr}\,\rho A$ defined on the algebra of bounded operators (this can be proven using the polar decomposition of ρ [73, theorem VI.10]). An additional advantage is that the traceclass operators form a Banach space with respect to this norm [82].

Lemma 5.10 Let $\mathcal{H}$ be an Hilbert space. For any traceclass operator ρ on $\mathcal{H}$ we define its trace norm $\|\rho\|_1$ (aka. Schatten-norm with $p = 1$ [82]) by:

$$\|\rho\|_1 := \mathrm{Tr}\sqrt{\rho^+ \rho}\,.$$

Let $(\mathcal{J}_\alpha)_\alpha$ be a family of closed vector subspaces of $\mathcal{H}$, forming a directed preordered set under inclusion, and such that:

$$\mathcal{H} = \overline{\bigcup_\alpha \mathcal{J}_\alpha}\,.$$

Define Θ_α to be the orthogonal projection on $\mathcal{J}_\alpha$.

The following statements hold:

1. for any (self-adjoint) positive semi-definite, traceclass operator ρ on $\mathcal{H}$, the net $(\Theta_\alpha\,\rho\,\Theta_\alpha)_\alpha$ converges in trace norm to ρ;
2. if $(\rho_\alpha)_\alpha$ is a net of (self-adjoint) positive semi-definite, traceclass operators on $\mathcal{H}$ such that:

$$\forall \alpha,\ \alpha' \,/\, \mathcal{J}_\alpha \subset \mathcal{J}_{\alpha'},\ \rho_\alpha = \Theta_\alpha\,\rho_{\alpha'}\,\Theta_\alpha\,,$$

   and if $\sup_\alpha \mathrm{Tr}\,\rho_\alpha = l < \infty$, then there exists a (self-adjoint) positive semi-definite, traceclass operator ρ on $\mathcal{H}$ such that $\rho_\alpha = \Theta_\alpha\,\rho\,\Theta_\alpha$ and $\mathrm{Tr}\,\rho = l$.

Proof The trace norm is well-defined, since for any traceclass operator ρ on $\mathcal{H}$, $\rho^+\,\rho$ is a self-adjoint positive semi-definite operator on $\mathcal{H}$, so its square-root can be defined by spectral resolution, and this square-root is traceclass (by definition of ρ being traceclass).

Statement 5.10.1. Let ρ be a (self-adjoint) positive semi-definite, traceclass operator on $\mathcal{H}$. There exist real numbers $p_k \geqslant 0$ ($k \in \mathbb{N}$) and vectors ψ_k in $\mathcal{H}$ with $\|\psi_k\| = 1$ such that:

$$\rho = \sum_k p_k\, |\psi_k\,\rangle\langle\,\psi_k| \qquad \& \qquad \sum_k p_k = \mathrm{Tr}\,\rho \geqslant 0\,.$$

Hence, we have for any α:

$$\|\rho - \Theta_\alpha\,\rho\,\Theta_\alpha\|_1 \leqslant \sum_k p_k\, \big\|\, |\psi_k\,\rangle\langle\,\psi_k| - |\Theta_\alpha\,\psi_k\,\rangle\langle\,\Theta_\alpha\,\psi_k|\,\big\|_1\,.$$

Let $\epsilon > 0$ and let $N \in \mathbb{N}$ such that:

$$\sum_{k>N} 2\,p_k \leqslant \frac{\epsilon}{2}\,.$$

Since $\mathcal{H}$ is the completion of the union of the $\mathcal{J}_\alpha$ (which are directed with respect to inclusion subordinate to the labels α), there exists, for every $k \leqslant N$, an α_k such that $\|\psi_k - \Theta_{\alpha_k}\,\psi_k\| \leqslant \frac{\epsilon}{6}$. And since the family $(\mathcal{J}_\alpha)_\alpha$ is directed under inclusion, there exists α such that $\bigcup_{k\leqslant N} J_{\alpha_k} \subset \mathcal{J}_\alpha$. Let α' such that $\mathcal{J}_{\alpha'} \supset \mathcal{J}_\alpha$. Then, we have:

$$\forall k \leqslant N,\ \|\psi_k - \Theta_{\alpha'}\,\psi_k\| \leqslant \frac{\epsilon}{6}\,.$$

On the other hand, for any $k \leqslant N$, the non-zero eigenvalues of $|\psi_k\,\rangle\langle\,\psi_k| - |\Theta_{\alpha'}\,\psi_k\,\rangle\langle\,\Theta_{\alpha'}\,\psi_k|$ are:

$$\lambda_\pm = \frac{\mu^2}{2} \pm \mu\sqrt{1 - \frac{3\mu^2}{4}} \text{ with } \mu := \|\psi_k - \Theta_{\alpha'}\,\psi_k\|\,,$$

each one with multiplicity 1. So from $\mu \leqslant \frac{\epsilon}{6}$ and $\mu \leqslant 1$ ($\|\psi_k\| = 1$), we have:

$$\big\| \, |\psi_k\rangle\langle\psi_k| - |\Theta_{\alpha'}\psi_k\rangle\langle\Theta_{\alpha'}\psi_k| \, \big\|_1 = |\lambda_+| + |\lambda_-| \leqslant 3\,\mu \leqslant \frac{\epsilon}{2}\,.$$

Therefore,

$$\|\rho - \Theta_{\alpha'}\,\rho\,\Theta_{\alpha'}\|_1 \leqslant \left(\sum_{k \leqslant N} p_k\,\frac{\epsilon}{2}\right) + \frac{\epsilon}{2} \leqslant \epsilon\,.$$

Statement 5.10.2. Since the family $(\mathcal{J}_\alpha)_\alpha$ is directed and each $\mathcal{J}_\alpha$ is a vector subspace of $\mathcal{H}$, $\mathcal{J} := \bigcup_\alpha \mathcal{J}_\alpha$ is a vector subspace of $\mathcal{H}$ and, by hypothesis, $\mathcal{J}$ is dense in $\mathcal{H}$.

For any $\psi,\, \psi' \in \mathcal{J}$, we define:

$$\rho_{\psi,\psi'} := \langle\psi,\, \rho_\alpha\,\psi'\rangle \text{ for } \alpha \text{ such that } \psi, \psi' \in \mathcal{J}_\alpha\,.$$

$\rho_{\psi,\psi'}$ is well-defined, since there exists α_ψ, resp. $\alpha_{\psi'}$, such that $\psi \in \mathcal{J}_{\alpha_\psi}$, resp. $\psi \in \mathcal{J}_{\alpha_\psi}$, hence there exists α such that $\psi, \psi' \in \mathcal{J}_\alpha$; and if α' is an other index such that $\psi, \psi' \in \mathcal{J}_{\alpha'}$, then there exists α'' with $\mathcal{J}_\alpha,\, \mathcal{J}_{\alpha'} \subset \mathcal{J}_{\alpha''}$, so we have:

$$\langle\psi,\, \rho_\alpha\,\psi'\rangle = \langle\Theta_\alpha\,\psi,\, \rho_{\alpha''}\,\Theta_\alpha\,\psi'\rangle = \langle\psi,\, \rho_{\alpha''}\,\psi'\rangle = \langle\Theta_{\alpha'}\,\psi,\, \rho_{\alpha''}\,\Theta_{\alpha'}\,\psi'\rangle = \langle\psi,\, \rho_{\alpha'}\,\psi'\rangle\,.$$

Moreover, $(\psi, \psi') \mapsto \rho_{\psi,\psi'}$ is a positive semi-definite, sesquilinear form on $\mathcal{J}$ and:

$$\forall\psi,\, \psi' \in \mathcal{J},\, |\rho_{\psi,\psi'}| \leqslant l\, \|\psi\|\, \|\psi'\|\,,$$

hence, there exists a positive semi-definite, self-adjoint, bounded operator ρ on $\mathcal{H}$, such that:

$$\forall\psi,\, \psi' \in \mathcal{J},\, \rho_{\psi,\psi'} := \langle\psi,\, \rho\,\psi'\rangle\,.$$

So, for any α and any $\psi, \psi' \in \mathcal{H}$, we have:

$$\langle\psi,\, \Theta_\alpha\,\rho\,\Theta_\alpha\,\psi'\rangle = \langle\Theta_\alpha\,\psi,\, \rho\,\Theta_\alpha\,\psi'\rangle = \langle\Theta_\alpha\,\psi,\, \rho_\alpha\,\Theta_\alpha\,\psi'\rangle = \langle\psi,\, \rho_\alpha\,\psi'\rangle\,,$$

therefore $\rho_\alpha = \Theta_\alpha\,\rho\,\Theta_\alpha$.

Now, suppose ρ would not be traceclass. Then, there would exist a finite orthonormal family ψ_k, $k \in \{1, \ldots, N\}$ such that:

$$\sum_{k=1}^{N} \langle\psi_k,\, \rho\,\psi_k\rangle > l + 1\,,$$

Next, like in the proof of statement 5.10.1, we can find α satisfying:

$$\forall k \leqslant N,\, \forall\alpha' \,/\, \mathcal{J}_{\alpha'} \supset \mathcal{J}_\alpha,\quad \|\psi_k - \Theta_{\alpha'}\,\psi_k\| \leqslant \frac{1}{N\,(2l+1)}\,.$$

Hence, using $\|\rho\| \leqslant l$ (where $\|\cdot\|$ is the operator norm):

$$\sum_{k=1}^{N} \langle\psi_k,\, \rho\,\psi_k\rangle \leqslant \sum_{k=1}^{N} \left(\langle\Theta_{\alpha'}\,\psi_k,\, \rho\,\Theta_{\alpha'}\,\psi_k\rangle + \frac{2l+1}{N\,(2l+1)}\right)$$

$$\leqslant \operatorname{Tr}\rho_{\alpha'} + 1 \leqslant l + 1\,,$$

which would then be contradictory.

Lastly, ρ being a (self-adjoint) positive semi-definite, traceclass operator on $\mathcal{H}$, the first statement implies that the net $(\rho_\alpha)_\alpha$ converges in trace norm to ρ, hence $\lim_\alpha \operatorname{Tr} \rho_\alpha = \operatorname{Tr} \rho$. So $\operatorname{Tr} \rho \leqslant l$ and, for any $\epsilon > 0$, there exists α_ϵ such that:

$$\forall \alpha' \,/\, \mathcal{J}_{\alpha'} \supset \mathcal{J}_{\alpha_\epsilon},\ \operatorname{Tr} \rho_{\alpha'} \leqslant \operatorname{Tr} \rho + \epsilon\,.$$

Therefore, for any α, choosing α' such that $\mathcal{J}_\alpha \cup \mathcal{J}_{\alpha_\epsilon} \subset \mathcal{J}_{\alpha'}$, we have:

$$\operatorname{Tr} \rho_\alpha \leqslant \operatorname{Tr} \rho_{\alpha'} \leqslant \operatorname{Tr} \rho + \epsilon\,.$$

Thus, $l \leqslant \operatorname{Tr} \rho$, hence $\operatorname{Tr} \rho = l$. □

Proof of theorem 5.9 *Existence of σ and α satisfying 5.9.3.* The inductive limit defining $\mathcal{H}_\zeta$ is consistent since for all $\eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}$ and for all $\psi \in \mathcal{H}_\eta$:

$$\begin{aligned}
\tau_{\eta'' \leftarrow \eta'} \circ \tau_{\eta' \leftarrow \eta}(\psi) &= \Phi^{-1}_{\eta'' \to \eta'} \circ \left(\mathrm{id}_{\mathcal{H}_{\eta'' \to \eta'}} \otimes \Phi_{\eta' \to \eta}\right)^{-1} \left(\zeta_{\eta'' \to \eta'} \otimes \left(\zeta_{\eta' \to \eta} \otimes \psi\right)\right) \\
&= \Phi^{-1}_{\eta'' \to \eta} \circ \left(\Phi_{\eta'' \to \eta' \to \eta} \otimes \mathrm{id}_{\mathcal{H}_\eta}\right)^{-1} \left(\left(\zeta_{\eta'' \to \eta'} \otimes \zeta_{\eta' \to \eta}\right) \otimes \psi\right) \\
&= \Phi^{-1}_{\eta'' \to \eta} \left(\zeta_{\eta'' \to \eta} \otimes \psi\right) = \tau_{\eta'' \leftarrow \eta}(\psi)\,.
\end{aligned}$$

Additionally, for all $\eta \in \mathcal{L}$, we call $\tau_{\zeta \leftarrow \eta}$ the injective map $\mathcal{H}_\eta \to \mathcal{H}_\zeta$.

Let $\eta \in \mathcal{L}$. We define an Hilbert space $\mathcal{H}_{\zeta \to \eta}$ as (the completion of) the inductive limit of $\left(\{\kappa \in \mathcal{L} \mid \kappa \succcurlyeq \eta\}, \left(\mathcal{H}_{\kappa \to \eta}\right)_{\kappa \succcurlyeq \eta}, \left(\tau_{\kappa' \leftarrow \kappa \to \eta}\right)_{\kappa' \succcurlyeq \kappa \succcurlyeq \eta}\right)$, where the injective maps $\tau_{\kappa' \leftarrow \kappa \to \eta}$ are defined as:

$$\forall \kappa' \succcurlyeq \kappa \succcurlyeq \eta,\ \begin{array}{rccl} \tau_{\kappa' \leftarrow \kappa \to \eta} : & \mathcal{H}_{\kappa \to \eta} & \to & \mathcal{H}_{\kappa' \to \eta} \\ & \psi & \mapsto & \Phi^{-1}_{\kappa' \to \kappa \to \eta}\left(\zeta_{\kappa' \to \kappa} \otimes \psi\right) \end{array}.$$

We can prove that $\forall \kappa'' \succcurlyeq \kappa' \succcurlyeq \kappa \succcurlyeq \eta$, $\tau_{\kappa'' \leftarrow \kappa' \to \eta} \circ \tau_{\kappa' \leftarrow \kappa \to \eta} = \tau_{\kappa'' \leftarrow \kappa \to \eta}$ in a way similar to above, using:

$$\left(\mathrm{id}_{\mathcal{H}_{\kappa'' \to \kappa'}} \otimes \Phi_{\kappa' \to \kappa \to \eta}\right) \circ \Phi_{\kappa'' \to \kappa' \to \eta} = \left(\Phi_{\kappa'' \to \kappa' \to \kappa} \otimes \mathrm{id}_{\mathcal{H}_{\kappa \to \eta}}\right) \circ \Phi_{\kappa'' \to \kappa \to \eta}\,, \tag{5.9.1}$$

which can be proved by acting on both sides with $\left(\,\cdot\, \otimes \mathrm{id}_{\mathcal{H}_\eta}\right) \circ \Phi_{\kappa'' \to \eta}$ and using repeatedly eq. (5.1.1). Additionally, for all $\kappa \succcurlyeq \eta$, we call $\tau_{\zeta \leftarrow \kappa \to \eta}$ the injective map $\mathcal{H}_{\kappa \to \eta} \to \mathcal{H}_{\zeta \to \eta}$.

Then, we can combine the isomorphisms $\Phi_{\kappa \to \eta} : \mathcal{H}_\kappa \to \mathcal{H}_{\kappa \to \eta} \otimes \mathcal{H}_\eta$ defined for $\kappa \succcurlyeq \eta$ into an isomorphism $\Phi_{\zeta \to \eta} : \mathcal{H}_\zeta \to \mathcal{H}_{\zeta \to \eta} \otimes \mathcal{H}_\eta$, for we have, for all $\kappa' \succcurlyeq \kappa \succcurlyeq \eta$:

$$\Phi_{\kappa' \to \eta} \circ \tau_{\kappa' \leftarrow \kappa} = \left(\tau_{\kappa' \leftarrow \kappa \to \eta} \otimes \mathrm{id}_{\mathcal{H}_\eta}\right) \circ \Phi_{\kappa \to \eta}\,,$$

as can be shown using eq. (5.1.1).

Similarly, we can combine the isomorphisms $\Phi_{\kappa \to \eta' \to \eta} : \mathcal{H}_{\kappa \to \eta} \to \mathcal{H}_{\kappa \to \eta'} \otimes \mathcal{H}_{\eta' \to \eta}$ defined for $\kappa \succcurlyeq \eta' \succcurlyeq \eta$ into an isomorphism $\Phi_{\zeta \to \eta' \to \eta} : \mathcal{H}_{\zeta \to \eta} \to \mathcal{H}_{\zeta \to \eta'} \otimes \mathcal{H}_{\eta' \to \eta}$, for we have, for all $\kappa' \succcurlyeq \kappa \succcurlyeq \eta' \succcurlyeq \eta$:

$$\Phi_{\kappa' \to \eta' \to \eta} \circ \tau_{\kappa' \leftarrow \kappa \to \eta} = \left(\tau_{\kappa' \leftarrow \kappa \to \eta'} \otimes \mathrm{id}_{\mathcal{H}_{\eta' \to \eta}}\right) \circ \Phi_{\kappa \to \eta' \to \eta}\,,$$

as can be shown using eq. (5.9.1).

Moreover, we have (again from eq. (5.1.1)):

$$\left(\Phi_{\zeta\to\eta'\to\eta}\otimes \mathrm{id}_{\mathcal{H}_\eta}\right)\circ\Phi_{\zeta\to\eta}=\left(\mathrm{id}_{\mathcal{H}_{\zeta\to\eta'}}\otimes\Phi_{\eta'\to\eta}\right)\circ\Phi_{\zeta\to\eta'}\,.$$

Now, if we define $\mathcal{L}_\zeta := \mathcal{L}\sqcup\{\zeta\}$ and extend the preorder on $\mathcal{L}$ to $\mathcal{L}_\zeta$ by requiring $\forall\eta\in\mathcal{L},\ \eta\prec\zeta$, we can therefore assemble these objects into a projective system of quantum state spaces $\left(\mathcal{L}_\zeta,\mathcal{H},\Phi\right)^\otimes$.

Using prop. 5.6, we then have maps $\sigma_\searrow : \overline{\mathcal{S}}^\otimes_{(\mathcal{L}\sqcup\{\zeta\},\mathcal{H},\Phi)}\to\overline{\mathcal{S}}^\otimes_{(\mathcal{L},\mathcal{H},\Phi)}$ and $\alpha_\nwarrow : \overline{\mathcal{A}}^\otimes_{(\mathcal{L},\mathcal{H},\Phi)}\to\overline{\mathcal{A}}^\otimes_{(\mathcal{L}\sqcup\{\zeta\},\mathcal{H},\Phi)}$, and, using prop. 5.7, we have maps $\sigma_\zeta^{-1} : \overline{\mathcal{S}}_\zeta\to\overline{\mathcal{S}}^\otimes_{(\mathcal{L}\sqcup\{\zeta\},\mathcal{H},\Phi)}$ and $\alpha_\zeta^{-1} : \overline{\mathcal{A}}^\otimes_{(\mathcal{L}\sqcup\{\zeta\},\mathcal{H},\Phi)}\to\mathcal{A}_\zeta$. Hence, we define:

$$\sigma := \sigma_\searrow\circ\sigma_\zeta^{-1}\quad\&\quad\alpha := \alpha_\zeta^{-1}\circ\alpha_\nwarrow\,.$$

Properties of σ (5.9.4 and 5.9.5). For all $\eta\preccurlyeq\eta'$, we define:

$$\Theta_{\eta'|\eta} := \Phi^{-1}_{\eta'\to\eta}\circ\left(|\zeta_{\eta'\to\eta}\rangle\langle\zeta_{\eta'\to\eta}|\otimes\mathrm{id}_{\mathcal{H}_\eta}\right)\circ\Phi_{\eta'\to\eta}=\tau_{\eta'\leftarrow\eta}\,\tau^+_{\eta'\leftarrow\eta}\,,$$

which is the orthogonal projection on the image of $\tau_{\eta'\leftarrow\eta}$ in $\mathcal{H}_{\eta'}$, and, for all $\eta\in\mathcal{L}$, $\Theta_{\zeta|\eta}$, which is the orthogonal projection on the image of $\tau_{\zeta\leftarrow\eta}$ in $\mathcal{H}_\zeta$ and satisfy:

$$\Theta_{\zeta|\eta}\circ\tau_{\zeta\leftarrow\eta}=\tau_{\zeta\leftarrow\eta}\qquad\&\qquad\forall\eta'\succcurlyeq\eta,\ \Theta_{\zeta|\eta}\circ\tau_{\zeta\leftarrow\eta'}=\tau_{\zeta\leftarrow\eta'}\circ\Theta_{\eta'|\eta}\,.$$

We start by deriving a useful identity, for $\eta\preccurlyeq\eta'\in\mathcal{L}$ and A a self-adjoint, traceclass operator on $\mathcal{H}_{\eta'}$:

$$\begin{aligned}\tau^+_{\eta'\leftarrow\eta}\,A\,\tau_{\eta'\leftarrow\eta} &= \mathrm{Tr}_{\mathcal{H}_{\eta'\to\eta}}\left[\left(|\zeta_{\eta'\to\eta}\rangle\langle\zeta_{\eta'\to\eta}|\otimes\mathrm{id}_{\mathcal{H}_\eta}\right)\left(\Phi_{\eta'\to\eta}\,A\,\Phi^{-1}_{\eta'\to\eta}\right)\left(|\zeta_{\eta'\to\eta}\rangle\langle\zeta_{\eta'\to\eta}|\otimes\mathrm{id}_{\mathcal{H}_\eta}\right)\right]\\ &= \mathrm{Tr}_{\eta'\to\eta}\left(\Theta_{\eta'|\eta}\,A\,\Theta_{\eta'|\eta}\right).\end{aligned}\tag{5.9.2}$$

Let $\rho_\zeta\in\mathcal{S}_\zeta$. For $\kappa\in\mathcal{L}$, we define $\widetilde{\rho}_\kappa := \Theta_{\zeta|\kappa}\,\rho_\zeta\,\Theta_{\zeta|\kappa}$, and $\delta_\kappa := \rho_\zeta-\widetilde{\rho}_\kappa$. $\widetilde{\rho}_\kappa$ is a positive semi-definite, traceclass operator, and, from lemma 5.10.1, δ_κ converges in trace norm to 0.

Moreover, for any $\kappa\in\mathcal{L}$, δ_κ is self-adjoint, so we can write $\delta_\kappa=\delta^+_\kappa-\delta^-_\kappa$ where $\delta^\pm_\kappa$ are the positive and negative parts of δ_κ (defined by spectral resolution), hence $\left(\sigma(\delta^\pm_\kappa)\right)_\eta$ are positive semi-definite, self-adjoint operators on $\mathcal{H}_\eta$, and from the conical linearity of σ:

$$\forall\kappa\in\mathcal{L},\ \sigma(\rho_\zeta)=\sigma(\widetilde{\rho}_\kappa)+\sigma(\delta^+_\kappa)-\sigma(\delta^-_\kappa)=:\sigma(\widetilde{\rho}_\kappa)+\sigma(\delta_\kappa)\,,$$

hence:

$$\forall\kappa\in\mathcal{L},\ \forall\eta\in\mathcal{L},\ \left(\sigma(\rho_\zeta)\right)_\eta=\left(\sigma(\widetilde{\rho}_\kappa)\right)_\eta+\left(\sigma(\delta_\kappa)\right)_\eta.$$

Additionally, we have:

$$\mathrm{Tr}_{\mathcal{H}_\eta}\left(\sigma(\delta^\pm_\kappa)\right)_\eta=\mathrm{Tr}\left(\sigma(\delta^\pm_\kappa)\,\mathbb{1}\right)=\mathrm{Tr}_{\mathcal{H}_\zeta}\left(\delta^\pm_\kappa\,\alpha(\mathbb{1})\right)=\mathrm{Tr}_{\mathcal{H}_\zeta}\left(\delta^\pm_\kappa\,\mathrm{id}_{\mathcal{H}_\zeta}\right)=\mathrm{Tr}_{\mathcal{H}_\zeta}\,\delta^\pm_\kappa\,,$$

where $\mathbb{1}\in\overline{\mathcal{A}}^\otimes_{(\mathcal{L},\mathcal{H},\Phi)}$ is the equivalence class of $\mathrm{id}_{\mathcal{H}_\eta}$. So, we get:

$$\forall\kappa\in\mathcal{L},\ \left\|\left(\sigma(\delta_\kappa)\right)_\eta\right\|_1\leqslant\left\|\left(\sigma(\delta^+_\kappa)\right)_\eta\right\|_1+\left\|\left(\sigma(\delta^-_\kappa)\right)_\eta\right\|_1=\mathrm{Tr}_{\mathcal{H}_\zeta}\,\delta^+_\kappa+\mathrm{Tr}_{\mathcal{H}_\zeta}\,\delta^-_\kappa=\|\delta_\kappa\|_1\,,$$

therefore the net $\left(\left(\sigma(\widetilde{\rho}_\kappa)\right)_\eta\right)_{\kappa\in\mathcal{L}}$ converges in trace norm to $\left(\sigma(\rho_\zeta)\right)_\eta$.

Now, for $\eta'\succcurlyeq\kappa\in\mathcal{L}$, we have:

$$
\begin{aligned}
(\sigma(\widetilde{\rho}_\kappa))_{\eta'} &= \mathrm{Tr}_{\zeta\to\eta'}\, \widetilde{\rho}_\kappa = \mathrm{Tr}_{\zeta\to\eta'}\ \Theta_{\zeta|\eta'}\, \widetilde{\rho}_\kappa\, \Theta_{\zeta|\eta'} \\
&= \tau^+_{\zeta\leftarrow\eta'}\ \ \widetilde{\rho}_\kappa\ \ \tau_{\zeta\leftarrow\eta'} \\
&= \tau_{\eta'\leftarrow\kappa}\, \tau^+_{\zeta\leftarrow\kappa}\ \ \rho_\zeta\ \ \tau_{\zeta\leftarrow\kappa}\, \tau^+_{\eta'\leftarrow\kappa} \\
&= \Phi^{-1}_{\eta'\to\kappa}\, \left(|\zeta_{\eta'\to\kappa}\rangle\langle\zeta_{\eta'\to\kappa}| \otimes \left(\tau^+_{\zeta\leftarrow\kappa}\ \ \rho_\zeta\ \ \tau_{\zeta\leftarrow\kappa} \right) \right)\, \Phi_{\eta'\to\kappa}\,,
\end{aligned}
\tag{5.9.3}
$$

and, for $\eta' \succcurlyeq \kappa' \succcurlyeq \kappa \in \mathcal{L}$:

$$
\begin{aligned}
\Theta_{\eta'|\kappa}\ (\sigma(\widetilde{\rho}_{\kappa'}))_{\eta'}\ \Theta_{\eta'|\kappa} &= \tau_{\eta'\leftarrow\kappa}\, \tau^+_{\kappa'\leftarrow\kappa}\, \tau^+_{\zeta\leftarrow\kappa'}\ \ \rho_\zeta\ \ \tau_{\zeta\leftarrow\kappa'}\, \tau_{\kappa'\leftarrow\kappa}\, \tau^+_{\eta'\leftarrow\kappa} \\
&= (\sigma(\widetilde{\rho}_\kappa))_{\eta'}\,.
\end{aligned}
$$

Hence, for all $\eta, \kappa \in \mathcal{L}$, and for all $\eta' \in \mathcal{L}$ such that $\eta' \succcurlyeq \eta$ and $\eta' \succcurlyeq \kappa$, we have:

$$
\mathrm{Tr}_{\eta'\to\eta}\, \Theta_{\eta'|\kappa}\, \left(\sigma(\widetilde{\rho}_{\eta'})\right)_{\eta'}\, \Theta_{\eta'|\kappa} = (\sigma(\widetilde{\rho}_\kappa))_\eta\,. \tag{5.9.4}
$$

On the other hand:

$$
\begin{aligned}
&\left\| \left[\mathrm{Tr}_{\eta'\to\eta}\, \Theta_{\eta'|\kappa}\, \left(\sigma(\rho_\zeta)\right)_{\eta'}\, \Theta_{\eta'|\kappa} \right] - (\sigma(\widetilde{\rho}_\kappa))_\eta \right\|_1 \\
&\quad = \left\| \left[\mathrm{Tr}_{\eta'\to\eta}\, \Theta_{\eta'|\kappa}\, \left(\sigma(\rho_\zeta)\right)_{\eta'}\, \Theta_{\eta'|\kappa} \right] - \left[\mathrm{Tr}_{\eta'\to\eta}\, \Theta_{\eta'|\kappa}\, \left(\sigma(\widetilde{\rho}_{\eta'})\right)_{\eta'}\, \Theta_{\eta'|\kappa} \right] \right\|_1 \\
&\quad \leqslant \left\| \Theta_{\eta'|\kappa}\, \left(\sigma(\rho_\zeta) - \sigma(\widetilde{\rho}_{\eta'})\right)_{\eta'}\, \Theta_{\eta'|\kappa} \right\|_1 \\
&\quad = \left\| \Theta_{\eta'|\kappa}\, \left(\sigma(\delta_{\eta'})\right)_{\eta'}\, \Theta_{\eta'|\kappa} \right\|_1 \leqslant \|\delta_{\eta'}\|_1\,,
\end{aligned}
$$

as can be shown by decomposing the self-adjoint operator $\delta_{\eta'}$ in positive and negative parts. Therefore, we have:

$$
\lim_{\eta'\succcurlyeq\kappa,\eta} \mathrm{Tr}_{\eta'\to\eta}\, \left(\Theta_{\eta'|\kappa}\, \left(\sigma(\rho_\zeta)\right)_{\eta'}\, \Theta_{\eta'|\kappa} \right) = (\sigma(\widetilde{\rho}_\kappa))_\eta\,, \tag{5.9.5}
$$

where the limit is taken in the trace norm. And we can now take the net limit on κ:

$$
\lim_{\kappa\in\mathcal{L}}\, \lim_{\eta'\succcurlyeq\kappa,\eta} \mathrm{Tr}_{\eta'\to\eta}\, \left(\Theta_{\eta'|\kappa}\, \left(\sigma(\rho_\zeta)\right)_{\eta'}\, \Theta_{\eta'|\kappa} \right) = \left(\sigma(\rho_\zeta)\right)_\eta\,. \tag{5.9.6}
$$

Now, for $\rho_\zeta \neq \rho'_\zeta \in \mathcal{S}_\zeta$, there should exist $\kappa \in \mathcal{L}$ such that:

$$
\left(\tau^+_{\zeta\leftarrow\kappa} \circ \rho_\zeta \circ \tau_{\zeta\leftarrow\kappa} \right) \neq \left(\tau^+_{\zeta\leftarrow\kappa} \circ \rho'_\zeta \circ \tau_{\zeta\leftarrow\kappa} \right),
$$

which, from eq. (5.9.3), implies:

$$
\forall \eta \succcurlyeq \kappa,\ \ (\sigma(\widetilde{\rho}_\kappa))_\eta \neq \left(\sigma(\widetilde{\rho}'_\kappa)\right)_\eta,
$$

but, using eq. (5.9.5), $(\sigma(\widetilde{\rho}_\kappa))_\eta$ can be computed from $\sigma(\rho_\zeta)$, hence $\sigma(\rho_\zeta) \neq \sigma(\rho'_\zeta)$. Therefore, $\sigma|_{\mathcal{S}_\zeta}$ is injective, so, from the conical linearity of σ, σ is injective.

Then, for $\rho_\zeta \in \mathcal{S}_\zeta$, we have (from eq. (5.9.6)):

$$\lim_{\eta \in \mathcal{L}} \lim_{\eta' \succcurlyeq \eta} \mathrm{Tr}_{\mathcal{H}_{\eta'}} \left(\left(\sigma(\rho_\zeta)\right)_{\eta'} \Theta_{\eta'|\eta} \right) = \mathrm{Tr}_{\mathcal{H}_\zeta} \rho_\zeta = 1\,,$$

and, since the net $\left(\mathrm{Tr}_{\mathcal{H}_{\eta'}} \left(\left(\sigma(\rho_\zeta)\right)_{\eta'} \Theta_{\eta'|\eta} \right) \right)_{\eta' \succcurlyeq \eta}$ is decreasing, while the net:

$$\left(\lim_{\eta' \succcurlyeq \eta} \mathrm{Tr}_{\mathcal{H}_{\eta'}} \left(\left(\sigma(\rho_\zeta)\right)_{\eta'} \Theta_{\eta'|\eta} \right) \right)_{\eta \in \mathcal{L}} = \left(\mathrm{Tr}_{\mathcal{H}_\zeta} \left(\widetilde{\rho}_\eta\right) \right)_{\eta \in \mathcal{L}}\,,$$

is increasing, the limits are given respectively by the infimum and by the supremum, so:

$$\sigma(\rho_\zeta) \in \left\{ \left(\rho_\eta\right)_{\eta \in \mathcal{L}} \in \mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \;\middle|\; \sup_{\eta \in \mathcal{L}} \left(\inf_{\eta' \succcurlyeq \eta} \mathrm{Tr}_{\mathcal{H}_{\eta'}} \left(\rho_{\eta'}\, \Theta_{\eta'|\eta}\right) \right) = 1 \right\}.$$

To prove that this condition indeed characterizes $\sigma\left\langle \mathcal{S}_\zeta \right\rangle$, we now consider $\left(\rho_\eta\right)_{\eta \in \mathcal{L}} \in \mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ such that:

$$\sup_{\eta \in \mathcal{L}} \left(\inf_{\eta' \succcurlyeq \eta} \mathrm{Tr}_{\mathcal{H}_{\eta'}} \left(\rho_{\eta'}\, \Theta_{\eta'|\eta}\right) \right) = 1\,.$$

Let $\eta \in \mathcal{L}$. We have $0 \leqslant \inf_{\eta' \succcurlyeq \eta} \mathrm{Tr}_{\mathcal{H}_{\eta'}} \left(\rho_{\eta'}\, \Theta_{\eta'|\eta}\right) = \mu_\eta \leqslant 1$. We consider the net $\left(\check{\rho}_{\eta'|\eta}\right)_{\eta' \succcurlyeq \eta'}$, where $\check{\rho}_{\eta'|\eta}$ is a positive semi-definite, traceclass operator on $\mathcal{H}_\eta$ defined by:

$$\check{\rho}_{\eta'|\eta} := \mathrm{Tr}_{\eta' \to \eta} \left(\Theta_{\eta'|\eta}\, \rho_{\eta'}\, \Theta_{\eta'|\eta}\right).$$

For $\eta'' \succcurlyeq \eta' \succcurlyeq \eta \in \mathcal{L}$, we have:

$$\check{\rho}_{\eta'|\eta} - \check{\rho}_{\eta''|\eta} = \mathrm{Tr}_{\eta'' \to \eta} \left(\left(\iota_{\eta'' \leftarrow \eta'}(\Theta_{\eta'|\eta}) - \Theta_{\eta''|\eta}\right) \rho_{\eta''} \left(\iota_{\eta'' \leftarrow \eta'}(\Theta_{\eta'|\eta}) - \Theta_{\eta''|\eta}\right) \right),$$

$$\text{and } \iota_{\eta'' \leftarrow \eta'}(\Theta_{\eta'|\eta}) - \Theta_{\eta''|\eta} = \Phi^{-1}_{\eta'' \to \eta'} \circ \left[\left(\mathrm{id}_{\mathcal{H}_{\eta'' \to \eta'}} - |\zeta_{\eta'' \to \eta'}\rangle\langle\zeta_{\eta'' \to \eta'}| \right) \otimes \Theta_{\eta'|\eta} \right] \circ \Phi_{\eta'' \to \eta'}\,,$$

hence $\check{\rho}_{\eta'|\eta} - \check{\rho}_{\eta''|\eta}$ is also a positive semi-definite, traceclass operator on $\mathcal{H}_\eta$. Its trace is $\mathrm{Tr}_{\mathcal{H}_{\eta'}} \left(\rho_{\eta'}\, \Theta_{\eta'|\eta}\right) - \mathrm{Tr}_{\mathcal{H}_{\eta''}} \left(\rho_{\eta''}\, \Theta_{\eta''|\eta}\right) = \mathrm{Tr}_{\mathcal{H}_\eta} \check{\rho}_{\eta'|\eta} - \mathrm{Tr}_{\mathcal{H}_\eta} \check{\rho}_{\eta''|\eta}$, therefore $\left(\mathrm{Tr}_{\mathcal{H}_\eta} \check{\rho}_{\eta'|\eta}\right)_{\eta' \succcurlyeq \eta}$ is decreasing and converges to μ_η.

Thus, $\left(\check{\rho}_{\eta'|\eta}\right)_{\eta' \succcurlyeq \eta}$ is a Cauchy net and, since the traceclass operators form a Banach space with respect to the trace norm [82], it converges in trace norm to a positive semi-definite, traceclass operator $\check{\rho}_\eta$ on $\mathcal{H}_\eta$, with $\mathrm{Tr}_{\mathcal{H}_\eta} \check{\rho}_\eta = \mu_\eta$.

Moreover, for $\kappa \preccurlyeq \kappa' \in \mathcal{L}$, we have:

$$\begin{aligned} \tau^{+}_{\kappa' \leftarrow \kappa}\, \check{\rho}_{\kappa'}\, \tau_{\kappa' \leftarrow \kappa} &= \lim_{\eta \succcurlyeq \kappa'} \tau^{+}_{\kappa' \leftarrow \kappa}\, \check{\rho}_{\eta|\kappa'}\, \tau_{\kappa' \leftarrow \kappa} \\ &= \lim_{\eta \succcurlyeq \kappa'} \check{\rho}_{\eta|\kappa} = \check{\rho}_\kappa \quad \text{(using eq. (5.9.2))}. \end{aligned}$$

Hence, since $\sup_{\kappa \in \mathcal{L}} \mathrm{Tr}_{\mathcal{H}_\kappa} \check{\rho}_\kappa = 1$, there exists, from lemma 5.10.2, an operator $\check{\rho}_\zeta \in \mathcal{S}_\zeta$ satisfying:

$$\forall \kappa \in \mathcal{L},\; \tau^{+}_{\zeta \leftarrow \kappa}\, \check{\rho}_\zeta\, \tau_{\zeta \leftarrow \kappa} = \check{\rho}_\kappa\,.$$

Therefore, we have:

$$\forall \eta, \kappa \in \mathcal{L},\ \forall \eta' \succcurlyeq \eta, \kappa,\ \left(\sigma\left(\widetilde{\left(\check{\rho}_\zeta\right)_\kappa}\right)\right)_\eta = \mathrm{Tr}_{\eta' \to \eta}\, \Theta_{\eta'|\kappa}\, \check{\rho}_{\eta'}\, \Theta_{\eta'|\kappa} \text{ (using eqs. (5.9.3) and (5.9.4))},$$

hence, applying for $\eta' = \kappa \succcurlyeq \eta \in \mathcal{L}$:

$$\left(\sigma\left(\check{\rho}_\zeta\right)\right)_\eta = \lim_{\kappa \succcurlyeq \eta} \mathrm{Tr}_{\kappa \to \eta}\, \check{\rho}_\kappa$$

$$= \lim_{\kappa \succcurlyeq \eta} \lim_{\kappa' \succcurlyeq \kappa} \mathrm{Tr}_{\kappa \to \eta}\, \check{\rho}_{\kappa'|\kappa}$$

On the other hand, we can show as above that for any $\kappa \preccurlyeq \kappa'$, $\rho_\kappa - \check{\rho}_{\kappa'|\kappa}$ is a positive semi-definite, traceclass operator on $\mathcal{H}_\kappa$, with trace smaller than $1 - \mu_\kappa$. Thus, $\rho = \sigma\left(\check{\rho}_\zeta\right)$. □

As a corollary of the previous result, we also consider the case of infinite tensor products [99, 93]. Given a family of Hilbert spaces $(\mathcal{J}_\lambda)_{\lambda \in \mathcal{F}}$, we can build its infinite tensor product (ITP) $\mathcal{H}_\mathcal{F}$, which will in general be a non-separable Hilbert space. On the other hand, we can also build a projective system of quantum state spaces where the 'small' Hilbert spaces are given by tensor products of finitely many $\mathcal{J}_\lambda$. In this case, we still can map density matrices on $\mathcal{H}_\mathcal{F}$ to states in the projective limit, but this mapping will no longer be injective, because we can define considerably more observables over the infinite tensor product, and we can use them to distinguish between states that are indistinguishable if we solely use the algebra of observables defined over the projective system. However, if we believe that these latter observables (which can be sensible only to correlations between finitely many $\mathcal{J}_\lambda$) are the only *experimentally measurable* ones, additional distinctions between states might be objectionable.

Interestingly, the ITP $\mathcal{H}_\mathcal{F}$, while being a really huge Hilbert space, still fails (except in absolutely degenerate cases) to reproduce the full state space of the projective system: this can be traced back to the fact that the latter allows to model states that are patently more 'statistical' than any state realizable on $\mathcal{H}_\mathcal{F}$. Also, grouping the tensor product factors $\mathcal{J}_\lambda$ into finite tensor products *before* performing the ITP construction generically gives rise to inequivalent Hilbert spaces (ie. ITP's are not associative, see [99, section 4.2]), while such a grouping does not affect the projective state space (as a consequence of prop. 5.6). We will further investigate the comparison between projective state spaces and ITP Hilbert spaces in props. 19.2 and 19.13.

Theorem 5.11 Let $(\mathcal{J}_\lambda)_{\lambda \in \mathcal{F}}$ be a family of Hilbert spaces (with $\dim \mathcal{J}_\lambda \neq 0$ for each $\lambda \in \mathcal{F}$) and define:

1. $\mathcal{L} := \{\Lambda \subset \mathcal{F} \mid \#\Lambda < \infty\}$ equipped with the preorder $\subset$;
2. $\forall \Lambda \in \mathcal{L}$, $\mathcal{H}_\Lambda := \bigotimes_{\lambda \in \Lambda} \mathcal{J}_\lambda$;
3. $\forall \Lambda \subset \Lambda' \in \mathcal{L}$, $\mathcal{H}_{\Lambda' \to \Lambda} := \mathcal{H}_{\Lambda' \setminus \Lambda}$ with $\Phi_{\Lambda' \to \Lambda}$ the natural identification $\mathcal{H}_{\Lambda'} \to \mathcal{H}_{\Lambda' \setminus \Lambda} \otimes \mathcal{H}_\Lambda$.

Then, we can complete these elements into a projective system of quantum state spaces $(\mathcal{L}, \mathcal{H}, \Phi)^\otimes$.

Let $\mathcal{H}_\mathcal{F}$ be infinite tensor product of $(\mathcal{J}_\lambda)_{\lambda \in \mathcal{F}}$. There exist maps $\sigma : \overline{\mathcal{S}}_\mathcal{F} \to \overline{\mathcal{S}}^\otimes_{(\mathcal{L}, \mathcal{H}, \Phi)}$ and $\alpha : \overline{\mathcal{A}}^\otimes_{(\mathcal{L}, \mathcal{H}, \Phi)} \to \mathcal{A}_\mathcal{F}$ such that:

4. $\forall \rho \in \overline{\mathcal{S}}_\mathcal{F}, \forall A \in \overline{\mathcal{A}}^\otimes_{(\mathcal{L}, \mathcal{H}, \Phi)}$, $\mathrm{Tr}_{\mathcal{H}_\mathcal{F}}\left(\rho\, \alpha(A)\right) = \mathrm{Tr}\left(\sigma(\rho)\, A\right)$;
5. if $\{\lambda \in \mathcal{F} \mid \dim \mathcal{J}_\lambda > 1\}$ is infinite, σ is neither injective nor surjective.

Proof Clearly, $(\mathcal{L}, \subset)$ is a directed set and, defining, for any $\Lambda \subset \Lambda' \subset \Lambda'' \in \mathcal{L}$, $\Phi_{\Lambda''\to\Lambda'\to\Lambda}$ as the natural identification $\mathcal{H}_{\Lambda''\setminus\Lambda} \to \mathcal{H}_{\Lambda''\setminus\Lambda'} \otimes \mathcal{H}_{\Lambda'\setminus\Lambda}$, we obtain a projective system of quantum state spaces $(\mathcal{L}, \mathcal{H}, \Phi)^{\otimes}$.

Existance of σ and α satisfying 5.11.4. The ITP $\mathcal{H}_{\mathcal{F}}$ arising from $(\mathcal{J}_\lambda)_{\lambda\in\mathcal{F}}$ can be written as [99, chapter 4]:

$$\mathcal{H}_{\mathcal{F}} = \overline{\bigoplus_{[f]} \mathcal{H}_{[f]}},$$

where the $[f]$ are equivalence classes in $\left\{ (f_\lambda)_{\lambda\in\mathcal{F}} \in (\mathcal{J}_\lambda)_{\lambda\in\mathcal{F}} \;\middle|\; \sum_{\lambda\in\mathcal{F}} \left| \|f_\lambda\|_{\mathcal{J}_\lambda} - 1 \right| \text{ converges } \right\}$ for the equivalence relation:

$$(f_\lambda)_{\lambda\in\mathcal{F}} \simeq (g_\lambda)_{\lambda\in\mathcal{F}} \quad\Leftrightarrow\quad \sum_{\lambda\in\mathcal{F}} \left| \langle f_\lambda, g_\lambda\rangle_{\mathcal{J}_\lambda} - 1 \right| \text{ converges },$$

and the Hilbert space $\mathcal{H}_{[f]}$ is (the completion of) the inductive limit of $\left(\mathcal{L}, \left(\bigotimes_{\lambda\in\Lambda} \mathcal{J}_\lambda\right)_{\Lambda\in\mathcal{L}}, \left(\tau^f_{\Lambda'\leftarrow\Lambda}\right)_{\Lambda\subset\Lambda'}\right)$, the inductive maps $\tau^f_{\Lambda'\leftarrow\Lambda}$ being defined as:

$$\forall \Lambda \subset \Lambda', \quad \begin{array}{rccc} \tau^f_{\Lambda'\leftarrow\Lambda} : & \bigotimes_{\lambda\in\Lambda} \mathcal{J}_\lambda & \to & \bigotimes_{\lambda\in\Lambda'} \mathcal{J}_\lambda \\ & \psi & \mapsto & \left(\bigotimes_{\lambda\in\Lambda'\setminus\Lambda} f_\lambda\right) \otimes \psi \end{array},$$

for f some representative of $[f]$ such that $\forall \lambda \in \mathcal{F}$, $\|f_\lambda\|_{\mathcal{J}_\lambda} = 1$ (note that every equivalence class admits normalized representatives: given a representative $(g_\lambda)_{\lambda\in\mathcal{F}}$, there can be at most finitely many λ such that $g_\lambda = 0$, so, defining f_λ to be some normalized vector of $\mathcal{J}_\lambda$, if $g_\lambda = 0$, and $f_\lambda := {g_\lambda}/{\|g_\lambda\|_{\mathcal{J}_\lambda}}$ otherwise, we have $f_\lambda \simeq g_\lambda$).

Now, *choosing* such a normalized representative f for each equivalence class $[f]$, we can identify $\mathcal{H}_{[f]}$ with the Hilbert space $\mathcal{H}_{\zeta^f}$ constructed as in theorem 5.9 for the family $\left(\zeta^f_{\Lambda'\to\Lambda}\right)_{\Lambda\subset\Lambda'}$ given by:

$$\forall \Lambda, \zeta^f_{\Lambda\to\Lambda} = 1 \quad \& \quad \forall \Lambda \subsetneq \Lambda', \zeta^f_{\Lambda'\to\Lambda} := \bigotimes_{\lambda\in\Lambda'\setminus\Lambda} f_\lambda .$$

Hence, as in the proof of theorem 5.9, we can construct for all $\Lambda \in \mathcal{L}$ an Hilbert space $\mathcal{H}_{[f]\to\Lambda}$ and an Hilbert space isomorphism $\Phi_{[f]\to\Lambda} : \mathcal{H}_{[f]} \to \mathcal{H}_{[f]\to\Lambda} \otimes \mathcal{H}_\Lambda$, and for all $\Lambda \subset \Lambda'$, we can construct an Hilbert space isomorphism $\Phi_{[f]\to\Lambda'\to\Lambda} : \mathcal{H}_{[f]\to\Lambda} \to \mathcal{H}_{[f]\to\Lambda'} \otimes \mathcal{H}_{\Lambda'\to\Lambda}$, satisfying:

$$\left(\Phi_{[f]\to\Lambda'\to\Lambda} \otimes \mathrm{id}_{\mathcal{H}_\Lambda}\right) \circ \Phi_{[f]\to\Lambda} = \left(\mathrm{id}_{\mathcal{H}_{[f]\to\Lambda'}} \otimes \Phi_{\Lambda'\to\Lambda}\right) \circ \Phi_{[f]\to\Lambda'} .$$

We define:

$$\forall \Lambda \in \mathcal{L}, \quad \mathcal{H}_{\mathcal{F}\to\Lambda} = \overline{\bigoplus_{[f]} \mathcal{H}_{[f]\to\Lambda}}, \; \Phi_{\mathcal{F}\to\Lambda} = \overline{\bigoplus_{[f]} \Phi_{[f]\to\Lambda}} \quad \& \quad \forall \Lambda \subset \Lambda', \quad \Phi_{\mathcal{F}\to\Lambda'\to\Lambda} = \overline{\bigoplus_{[f]} \Phi_{[f]\to\Lambda'\to\Lambda}} .$$

Note that $\mathcal{H}_{\mathcal{F}\to\Lambda}$ can also be identified as the ITP of $(\mathcal{J}_\lambda)_{\lambda\in\mathcal{F}\setminus\Lambda}$.

Defining $\overline{\mathcal{L}} := \mathcal{L} \cup \{\mathcal{F}\}$ and extending the preorder on $\mathcal{L}$ to $\overline{\mathcal{L}}$ by $\forall \Lambda \in \mathcal{L}$, $\Lambda \subset \mathcal{F}$, we thus have a projective system of quantum state spaces $\left(\overline{\mathcal{L}}, \mathcal{H}, \Phi\right)^{\otimes}$. As in the proof of theorem 5.9, we can then define σ and α by first using prop. 5.7 to go from $\mathcal{H}_{\mathcal{F}}$ to $\left(\overline{\mathcal{L}}, \mathcal{H}, \Phi\right)^{\otimes}$ and then using prop. 5.6

to go from $\left(\overline{\mathcal{L}}, \mathcal{H}, \Phi\right)^{\otimes}$ to $(\mathcal{L}, \mathcal{H}, \Phi)^{\otimes}$.

Properties of σ (5.11.5). Let $\rho_{\mathcal{F}} \in \overline{\mathcal{S}}_{\mathcal{F}}$. For $\Lambda \in \mathcal{L}$, we have:

$$\left(\sigma(\rho_{\mathcal{F}})\right)_\Lambda = \mathrm{Tr}_{\mathcal{F}\to\Lambda}\, \rho_{\mathcal{F}} = \mathrm{Tr}_{\mathcal{H}_{\mathcal{F}\to\Lambda}} \left(\Phi_{\mathcal{F}\to\Lambda} \circ \rho_{\mathcal{F}} \circ \Phi^{-1}_{\mathcal{F}\to\Lambda}\right)$$

$$= \sum_{[f]} \mathrm{Tr}_{\mathcal{H}_{[f]\to\Lambda}} \left(\Phi_{[f]\to\Lambda} \circ \Pi_{[f]} \circ \rho_{\mathcal{F}} \circ \Pi^+_{[f]} \circ \Phi^{-1}_{[f]\to\Lambda}\right),$$

where $\Pi_{[f]} : \mathcal{H}_{\mathcal{F}} \to \mathcal{H}_{[f]}$ is the orthogonal projection on $\mathcal{H}_{[f]}$.

Thus, σ does not see the correlations between different $\mathcal{H}_{[f]}$ that might be contained in $\rho_{\mathcal{F}}$, and therefore σ cannot be injective if there exist more than one equivalence class (as will be the case if $\{\lambda \in \mathcal{F} \mid \dim \mathcal{J}_\lambda > 1\}$ is infinite, see below). Indeed, if ψ, ψ' are two normalized states in $\mathcal{H}_{\mathcal{F}}$ with $\psi \in \mathcal{H}_{[f]}$, $\psi' \in \mathcal{H}_{[g]}$ and $[f] \neq [g]$, then:

$$\sigma\left(\frac{1}{2}\left|\psi\middle\rangle\middle\langle\psi\right| + \frac{1}{2}\left|\psi'\middle\rangle\middle\langle\psi'\right|\right) = \sigma\left(\left|\frac{\psi+\psi'}{\sqrt{2}}\middle\rangle\middle\langle\frac{\psi+\psi'}{\sqrt{2}}\right|\right).$$

On the other hand, let $\rho = (\rho_\Lambda)_{\Lambda\in\mathcal{L}} \in \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ and suppose there exists $\rho_{\mathcal{F}} \in \overline{\mathcal{S}}_{\mathcal{F}}$ such that $\rho = \sigma(\rho_{\mathcal{F}})$. If $\mathrm{Tr}_{\mathcal{H}_{\mathcal{F}}}\, \rho_{\mathcal{F}} = 0$, then $\rho_{\mathcal{F}} = 0$, hence $\rho = 0$. Therefore, if $\rho \neq 0$, we have $\mathrm{Tr}_{\mathcal{H}_{\mathcal{F}}}\, \rho_{\mathcal{F}} > 0$, and therefore there should exist at least one $[f]$ such that $\mathrm{Tr}_{\mathcal{H}_{[f]}} \left(\Pi_{[f]}\, \rho_{\mathcal{F}}\, \Pi^+_{[f]}\right) > 0$. Using theorem 5.9, we then have:

$$\sup_{\Lambda\in\mathcal{L}} \left(\inf_{\Lambda'\supset\Lambda} \mathrm{Tr}_{\mathcal{H}_{\Lambda'}} \left(\rho_{\Lambda'}\Theta^f_{\Lambda'|\Lambda}\right)\right) = \sum_{[g]} \sup_{\Lambda\in\mathcal{L}} \inf_{\Lambda'\supset\Lambda} \mathrm{Tr}_{\mathcal{H}_{\Lambda'}} \left(\mathrm{Tr}_{\mathcal{H}_{[g]\to\Lambda'}} \left(\Phi_{[g]\to\Lambda'} \circ \Pi_{[g]}\, \rho_{\mathcal{F}}\, \Pi^+_{[g]} \circ \Phi^{-1}_{[g]\to\Lambda'}\right) \Theta^f_{\Lambda'|\Lambda}\right)$$

$$\geqslant \sup_{\Lambda\in\mathcal{L}} \inf_{\Lambda'\supset\Lambda} \mathrm{Tr}_{\mathcal{H}_{\Lambda'}} \left(\mathrm{Tr}_{\mathcal{H}_{[f]\to\Lambda'}} \left(\Phi_{[f]\to\Lambda'} \circ \Pi_{[f]}\, \rho_{\mathcal{F}}\, \Pi^+_{[f]} \circ \Phi^{-1}_{[f]\to\Lambda'}\right) \Theta^f_{\Lambda'|\Lambda}\right)$$

$$= \mathrm{Tr}_{\mathcal{H}_{[f]}} \left(\Pi_{[f]}\, \rho_{\mathcal{F}}\, \Pi^+_{[f]}\right) > 0,$$

with $\Theta^f_{\Lambda'|\Lambda} = \Phi^{-1}_{\Lambda'\to\Lambda} \circ \left(\left|\zeta^f_{\Lambda'\to\Lambda}\middle\rangle\middle\langle\zeta^f_{\Lambda'\to\Lambda}\right| \otimes \mathrm{id}_{\mathcal{H}_\Lambda}\right) \circ \Phi_{\Lambda'\to\Lambda}$ (we have used in the first line that the sum $\sum_{[g]}$ is absolutely convergent in trace norm, and that the argument of $\inf_{\Lambda'\supset\Lambda}$, resp. of $\sup_{\Lambda\in\mathcal{L}}$, is positive and decreasing with Λ', resp. increasing with Λ, see the proof of theorem 5.9).

Now, we suppose that we have a infinite part $\Gamma \subset \mathcal{F}$ such that $\forall \gamma \in \Gamma$, $\dim \mathcal{J}_\gamma > 1$, and, for all $\gamma \in \Gamma$, we choose g^1_γ and g^2_γ two normalized vectors in $\mathcal{J}_\gamma$ that are orthogonal with each other. We define:

$$\forall \Lambda \in \mathcal{L} \,/\, \Lambda \subset \Gamma,\ \forall \left(\epsilon_\gamma\right)_{\gamma\in\Lambda} \in \{0,1\}^\Lambda,\ g^{(\epsilon)}_\Lambda := \bigotimes_{\gamma\in\Lambda} g^{\epsilon_\gamma}_\gamma \in \mathcal{H}_\Lambda.$$

Then, we choose some $(g_\lambda)_{\lambda\in\mathcal{F}\setminus\Gamma}$, with $\forall \lambda \in \mathcal{F} \setminus \Gamma$, $g_\lambda \in \mathcal{J}_\lambda$ and $\|g_\lambda\|_{\mathcal{J}_\lambda} = 1$. We define, for any $\Lambda \in \mathcal{L}$:

$$\forall \left(\epsilon_\gamma\right)_{\gamma\in\Lambda\cap\Gamma} \in \{0,1\}^{\Lambda\cap\Gamma},\ g^{(\epsilon)}_\Lambda := \left(\bigotimes_{\lambda\in\Lambda\setminus\Gamma} g_\lambda\right) \otimes g^{(\epsilon_\gamma)}_{\Lambda\cap\Gamma}$$

$$\text{and } \rho_\Lambda := \frac{1}{2^{\#(\Lambda\cap\Gamma)}} \sum_{(\epsilon)} \left|g^{(\epsilon)}_\Lambda\middle\rangle\middle\langle g^{(\epsilon)}_\Lambda\right|.$$

We can check that $(\rho_\Lambda)_{\Lambda\in\mathcal{L}} \in \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ and we have, for any f and any $\Lambda \subset \Lambda'$:

$$\mathrm{Tr}_{\mathcal{H}_{\Lambda'}}\left(\rho_{\Lambda'}\Theta^f_{\Lambda'|\Lambda}\right) = \prod_{\gamma\in(\Lambda'\setminus\Lambda)\cap\Gamma} \mathrm{Tr}_{\mathcal{J}_\gamma}\left(\tfrac{1}{2}\left(|g^1_\gamma\rangle\langle g^1_\gamma| + |g^2_\gamma\rangle\langle g^2_\gamma|\right)\,|f_\gamma\rangle\langle f_\gamma|\right) \prod_{\lambda\in(\Lambda'\setminus\Lambda)\setminus\Gamma} \mathrm{Tr}_{\mathcal{J}_\lambda}\left(|g_\lambda\rangle\langle g_\lambda|\,|f_\lambda\rangle\langle f_\lambda|\right)$$

$$= \prod_{\gamma\in(\Lambda'\setminus\Lambda)\cap\Gamma} \tfrac{1}{2}\left(\left|\langle g^1_\gamma, f_\gamma\rangle\right|^2 + \left|\langle g^2_\gamma, f_\gamma\rangle\right|^2\right) \prod_{\lambda\in(\Lambda'\setminus\Lambda)\setminus\Gamma} \mathrm{Tr}_{\mathcal{J}_\lambda}\left(\left|\langle g_\lambda, f_\lambda\rangle\right|^2\right).$$

Thus, we get for any f and any $\Lambda \subset \Lambda'$ the bound:

$$\mathrm{Tr}_{\mathcal{H}_{\Lambda'}}\left(\rho_{\Lambda'}\Theta^f_{\Lambda'|\Lambda}\right) \leqslant \frac{2^{\#(\Lambda\cap\Gamma)}}{2^{\#(\Lambda'\cap\Gamma)}}.$$

Hence,

$$\forall\Lambda\in\mathcal{L},\ \inf_{\Lambda'\supset\Lambda} \mathrm{Tr}_{\mathcal{H}_{\Lambda'}}\left(\rho_{\Lambda'}\Theta^f_{\Lambda'|\Lambda}\right) = 0 \quad\text{and}\quad \sup_{\Lambda\in\mathcal{L}}\left(\inf_{\Lambda'\supset\Lambda} \mathrm{Tr}_{\mathcal{H}_{\Lambda'}}\left(\rho_{\Lambda'}\Theta^f_{\Lambda'|\Lambda}\right)\right) = 0,$$

so $(\rho_\Lambda)_{\Lambda\in\mathcal{L}} \notin \mathrm{Im}\,\sigma$, and therefore σ is not surjective. □

As expressed by theorems 5.9.3 or 5.11.4, the mappings considered above rely on matching different explicit representations of certain abstract states on the algebra $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ (def. 5.3 and prop. 5.4). Finally, we can also characterize the space of projective states within the space of all states over the algebra: such a characterization is easily adapted from the characterization of normal states over a W^*-algebra [87, corollary III.3.11].

While the space of states on an algebra is never empty [1] (in fact, these states even separate the points of the algebra, as follows from [87, theorem I.9.23]), it is not quite clear whether the space of projective states $\mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ could turn out to be empty. Indeed, even a seemingly healthy projective limit (namely, an inverse limit limit defined on a directed label set, such that all involved spaces are non empty and all gluing maps are surjective, aka. are projection maps) can actually be empty. An instructive example is given in [101], and we will come back to this potential issue with regard to a concrete projective system at the end of section 18. Note that this issue cannot affect *countable* label sets, which have the additional advantage of allowing for a constructive description of the projective states. Hence, the development of section 19, aiming at restricting an uncountable label set to a suitably selected countable subset thereof, could provide a way out if necessary.

Proposition 5.12 We consider the same objects as in prop. 5.4. Let $\mathcal{Q}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ denote the space of states over $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ (ie. the positive linear forms of norm 1 on $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$). Then the map:

$$\begin{array}{rccc} \chi\ : & \mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} & \to & \mathcal{Q}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \\ & \rho & \mapsto & \mathrm{Tr}\,(\rho\,\cdot\,) \end{array},$$

is injective and its image is characterized by:

$$\upsilon \in \chi\left\langle \mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}\right\rangle$$

1. Thanks to Alexander Stottmeister for pointing out this result to me.

$\Leftrightarrow \forall\, \eta \in \mathcal{L}$, $\forall\, (J_i)_{i\in I}$ familly of closed, mutually orthogonal vector subspaces of $\mathcal{H}_\eta$,

$$\upsilon\left(\iota_{\leftarrow\eta}\Big(\sum_{i\in I}\Pi_i\Big)\right) = \sum_{i\in I}\upsilon\left(\iota_{\leftarrow\eta}(\Pi_i)\right),$$

where $\iota_{\leftarrow\eta}$ denotes the canonical injection of $\mathcal{A}_\eta$ in $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ and, for $i \in I$, Π_i denotes the orthgonal projection on the closed vector subspace J_i of $\mathcal{H}_\eta$.

Proof That χ is well-defined as a map $\mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \to \mathcal{Q}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ has been discussed in prop. 5.4. Let $\rho \in \mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$, $\eta \in \mathcal{L}$ and $(J_i)_{i\in I}$ be a family of closed, mutually orthogonal vector subspaces of $\mathcal{H}_\eta$. Let, for any finite subset $\alpha \subset I$, $\mathcal{J}_\alpha := \bigoplus_{i\in\alpha} J_i$, and $\mathcal{J} := \overline{\bigoplus_{i\in I} J_i}$. Let, for any finite subset $\alpha \subset I$, $\Theta_\alpha := \sum_{i\in\alpha}\Pi_i$ be the orthogonal projection on $\mathcal{J}_\alpha$, and $\Theta := \sum_{i\in I}\Pi_i$ be the orthogonal projection on $\mathcal{J}$. From lemma 5.10.1 (applied to the non-negative, traceclass operator $\Theta\,\rho_\eta\,\Theta$ on $\mathcal{J} \subset \mathcal{H}_\eta$), the net $\left(\Theta_\alpha\,\rho_\eta\,\Theta_\alpha\right)_\alpha$ converges in trace norm to $\Theta\,\rho_\eta\,\Theta$, hence:

$$\sum_{i\in I}\operatorname{Tr}\left(\rho\,\iota \leftarrow \eta(\Pi_i)\right) = \lim_\alpha \sum_{i\in\alpha}\operatorname{Tr}_{\mathcal{H}_\eta}\left(\rho_\eta\,\Pi_i\right) = \lim_\alpha \operatorname{Tr}_{\mathcal{H}_\eta}\left(\Theta_\alpha\,\rho_\eta\,\Theta_\alpha\right)$$

$$= \operatorname{Tr}_{\mathcal{H}_\eta}\left(\Theta\,\rho_\eta\,\Theta\right) = \operatorname{Tr}\left(\rho\,\iota_{\leftarrow\eta}\Big(\sum_{i\in I}\Pi_i\Big)\right).$$

Next, let $\rho,\ \rho' \in \mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ such that $\chi(\rho) = \chi(\rho')$ and $\eta \in \mathcal{H}_\eta$. For any $\psi,\ \psi' \in \mathcal{H}_\eta$, we have:

$$\langle\psi',\,\rho_\eta\,\psi\rangle = \operatorname{Tr}_{\mathcal{H}_\eta}\rho_\eta\,|\psi\,\rangle\langle\,\psi'| = \operatorname{Tr}\left(\rho\,\iota_{\leftarrow\eta}\left(|\psi\,\rangle\langle\,\psi'|\right)\right) = \operatorname{Tr}\left(\rho'\,\iota_{\leftarrow\eta}\left(|\psi\,\rangle\langle\,\psi'|\right)\right) = \langle\psi',\,\rho'_\eta\,\psi\rangle,$$

hence $\rho_\eta = \rho'_\eta$. Since this holds for any $\eta \in \mathcal{L}$ and for any $\rho, \rho' \in \mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ such that $\chi(\rho) = \chi(\rho')$, χ is injective.

Reciprocally, let $\upsilon \in \mathcal{Q}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ such that, for any $\eta \in \mathcal{L}$ and any familly $(J_i)_{i\in I}$ of closed, mutually orthogonal vector subspaces of $\mathcal{H}_\eta$:

$$\upsilon\left(\iota_{\leftarrow\eta}\Big(\sum_{i\in I}\Pi_i\Big)\right) = \sum_{i\in I}\upsilon\left(\iota_{\leftarrow\eta}(\Pi_i)\right). \tag{5.12.1}$$

Let $\eta \in \mathcal{L}$ and let $(e_i)_{i\in I}$ be an orthonormal basis of $\mathcal{H}_\eta$. For any finite subset $\alpha \subset I$, we define the operator $\rho^{(\alpha)}_\eta$ on the finite dimensional vector subspace $\mathcal{J}_\alpha := \operatorname{Vect}\{e_i \mid i \in \alpha\}$ of $\mathcal{H}_\eta$ by:

$$\forall i, j \in \alpha,\ \left\langle e_j \,\middle|\, \rho^{(\alpha)}_\eta\, e_i\right\rangle := \upsilon\left(\iota_{\leftarrow\eta}\left(\left|e_i\,\right\rangle\left\langle\,e_j\right|\right)\right).$$

$\upsilon \circ \iota_{\leftarrow\eta}$ being a positive linear form on $\mathcal{A}_\eta$, $\rho^{(\alpha)}_\eta$ is a positive (semi-definite) operator, and, $\mathcal{J}_\alpha$ being finite dimensional, it is traceclass. Moreover, for any finite subsets $\alpha \subset \alpha' \subset I$, $\rho^{(\alpha)}_\eta = \Theta_{\alpha'\to\alpha}\,\rho^{(\alpha')}_\eta\,\Theta_{\alpha'\to\alpha}$, with $\Theta_{\alpha'\to\alpha}$ the orthogonal projection on the vector subspace $\mathcal{J}_\alpha$ in $\mathcal{J}_{\alpha'}$. And, using eq. (5.12.1), we have:

$$\sup_\alpha \operatorname{Tr}_{\mathcal{J}_\alpha}\rho^{(\alpha)}_\eta = \sup_\alpha \sum_{i\in\alpha}\upsilon\left(\iota_{\leftarrow\eta}\left(\left|e_i\,\right\rangle\left\langle\,e_i\right|\right)\right) = \upsilon\left(\iota_{\leftarrow\eta}\left(\mathrm{id}_{\mathcal{H}_\eta}\right)\right) = 1.$$

So, from lemma 5.10.2, there exists a positive (semi-definite) traceclass operator ρ_η on $\mathcal{H}_\eta$, with $\mathrm{Tr}_{\mathcal{H}_\eta} \rho_\eta = 1$ and:

$$\forall i, j \in I,\ \left\langle e_j \,\middle|\, \rho_\eta\, e_i \right\rangle = \upsilon\left(\iota_{\leftarrow\eta}\big(\,|e_i\,\rangle\langle\, e_j|\,\big)\right).$$

Let $\psi \in \mathcal{H}_\eta$ and, for any finite subset $\alpha \subset I$, let ψ_α be the orthogonal projection of ψ on $\mathcal{J}_\alpha$. The net $(\psi_\alpha)_\alpha$ converges in norm to ψ, hence:

$$\langle \psi \mid \rho_\eta\, \psi \rangle = \lim_\alpha \langle \psi_\alpha \mid \rho_\eta\, \psi_\alpha \rangle = \lim_\alpha \upsilon\left(\iota_{\leftarrow\eta}\big(\,|\psi_\alpha\,\rangle\langle\,\psi_\alpha|\,\big)\right) = \upsilon\left(\iota_{\leftarrow\eta}\big(\,|\psi\,\rangle\langle\,\psi|\,\big)\right),$$

where we have used that υ is linear and bounded. Thus, for any orthogonal projection Π on a finite dimensional vector subspace of $\mathcal{H}_\eta$, we have, by linearity:

$$\mathrm{Tr}_{\mathcal{H}_\eta} \rho_\eta\, \Pi = \upsilon \circ \iota_{\leftarrow\eta}\big(\Pi\big). \tag{5.12.2}$$

Now, eq. (5.12.1) holds not only for $\upsilon \circ \iota_{\leftarrow\eta}$ but also for $\mathrm{Tr}_{\mathcal{H}_\eta}\left(\rho_\eta\ \cdot\ \right)$, as was shown in the first part of the present proof, so eq. (5.12.2) can be extended to any orthogonal projection Π on a closed vector subspace of $\mathcal{H}_\eta$. By linearity and norm-continuity, it can therefore be extended to any bounded self-adjoint operator on $\mathcal{H}_\eta$ (since the linear span of the orthogonal projections is norm-dense in the space of bounded self-adjoint operators, as follows from the spectral theorem). But any bounded operator A_η on $\mathcal{H}_\eta$ can be written as $A_\eta^{(+)} + i\, A_\eta^{(-)}$ with $A_\eta^{(+)}$, $A_\eta^{(-)}$ bounded self-adjoint operators on $\mathcal{H}_\eta$, so eq. (5.12.2) holds for any such A_η as well. In other words, we have:

$$\mathrm{Tr}_{\mathcal{H}_\eta}\left(\rho_\eta\ \cdot\ \right) = \upsilon \circ \iota_{\leftarrow\eta}, \tag{5.12.3}$$

as linear forms on $\mathcal{A}_\eta$.

Finally, let $\eta \preccurlyeq \eta' \in \mathcal{L}$, and let ρ_η, resp. $\rho_{\eta'}$, be the positive traceclass operator thus constructed on $\mathcal{H}_\eta$, resp. $\mathcal{H}_{\eta'}$. Let $\rho_\eta^{(\eta')} := \mathrm{Tr}_{\eta'\to\eta}\, \rho_{\eta'}$. For any $\psi, \psi' \in \mathcal{H}_\eta$, we have:

$$\begin{aligned}\langle \psi' \mid \rho_\eta\, \psi \rangle &= \upsilon \circ \iota_{\leftarrow\eta}\big(\,|\psi\,\rangle\langle\,\psi'|\,\big) = \upsilon \circ \iota_{\leftarrow\eta'}\Big(\iota_{\eta'\leftarrow\eta}\big(\,|\psi\,\rangle\langle\,\psi'|\,\big)\Big)\\ &= \mathrm{Tr}_{\mathcal{H}_{\eta'}}\Big(\rho_{\eta'}\, \iota_{\eta'\leftarrow\eta}\big(\,|\psi\,\rangle\langle\,\psi'|\,\big)\Big) = \left\langle \psi' \,\middle|\, \rho_\eta^{(\eta')}\, \psi \right\rangle.\end{aligned}$$

Thus, $\forall \eta \preccurlyeq \eta'$, $\mathrm{Tr}_{\eta'\to\eta}\, \rho_{\eta'} = \rho_\eta$, so $\rho := \left(\rho_\eta\right)_{\eta\in\mathcal{L}} \in \mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$, and, by norm-continuity, eq. (5.12.3) extends to all $\overline{A}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ as $\chi(\rho) = \upsilon$. □

6. Quantization in special cases

The motivation of section 2 was to pave the way for a better understanding of how a quantum projective structure as described in the previous section can be constructed starting from a classical field theory. The procedure we have in mind here, is, given an infinite dimensional symplectic manifold, to first build its rendering by a system of finite dimensional manifolds (the partial theories, that encapsulate insights from a careful analysis of how measurements are done experimentally), and then quantize this projective system (with the aim of getting a quantum theory assembled from 'small' Hilbert spaces, on which calculations should be workable).

In this section, we will consider two basic, yet fairly generic cases, namely position and holomorphic representations, assuming that we have a factorizing system on the classical side (see subsection 2.3). In both cases, the key prerequisite is that the polarizations, that endow each symplectic manifold $\mathcal{M}_\eta$ with the additional structure needed for quantization (the choice of configuration variables or the complex structure, respectively), should be compatible, in an appropriate sense, with the projections defining the projective system.

In this section, all manifolds are assumed to be smooth *finite dimensional* manifolds.

6.1 Position representation

The starting point for position quantization will be a projective limit of classical phase spaces arising from a factorizing system of configuration spaces as described in prop. 2.16. Then, there is only one additional ingredient required, namely we need to find a family of measures on these configuration spaces that are intertwined by the factorization maps. With this, constructing the projective system of quantum state spaces is a straightforward generalization of [68, subsections 3.4.3 to 3.4.5], since an L_2-space over a Cartesian product of measure spaces has a natural tensor product factorization.

Surely, given an arbitrary projective system of phase spaces, it will in general not be possible to rewrite it as arising from a factorizing system of configuration spaces. However, we consider in theorem 6.2 an important case where we indeed get such a factorizing form automatically, namely when each individual phase space can be identified with the cotangent bundle on a simply-connected Lie group (assuming some appropriate compatibility conditions between the projections and these identifications). The idea is that the group structure, together with the favorable topology, fills exactly the gap between the local result from prop. 2.10 and the global factorization we want to have. Also, using Haar measures, we can easily build a family of measures for this factorizing system.

Note that this result in particular covers the situation considered in [68] (looking at $\mathbb{R}^n$ as an additive Lie group), the proof we will give below being in fact nothing but the non-linear version of the procedure described in [68, section 3.4]. At the same time, it lays the ground to address the question raised in this reference, as to whether the construction can be generalized to non-Abelian gauge theories (this will be the endeavor of chaps. 3 and 4, but for the limitation acknowledged at the end of section 9 regarding gauge invariance). To make the relation clearer between the objects in [68] and the ones we are using here, let us look in more detail at the assumptions of theorem 6.2. That each 'small' phase space $\mathcal{M}_\eta$ is a cotangent bundle on a simply-connected Lie group, equipped with its canonical symplectic structure, is a weaker version of assumptions 2, 3b and 4 in [68]. The most crucial assumption is that we start from a projective system of phase spaces: on the one hand, the compatibility of the projections with the symplectic structures provides the seeds of the desired factorizations, on the other hand its three-spaces consistency condition will turn into the corresponding condition for the quantum projective system (eq. (5.1.1)). This is ensured in [68] by assumptions 3a and 6. Finally, the condition 6.2.2, corresponding to the rest of assumption 6 in [68], ensures the compatibility of the projection maps both with the configuration polarizations (so that the factorization of the phase spaces will descend to a factorization of the configuration spaces) and with the group structure (otherwise we would not be able to really make use of this structure). Note that, thanks to the compatibility of the projections with the symplectic structures, simply assuming that the map, besides acting independently on the position and momentum variables, is linear in the momentum variables is sufficient to ensure a full compatibility with the group structure, as expressed by eqs. (6.2.1) and (6.2.2).

Definition 6.1 A factorizing system of measured manifolds is a factorizing system of smooth, finite dimensional, manifolds $(\mathcal{L}, \mathcal{C}, \varphi)^\times$ (def. 2.15) such that:

1. for all $\eta \in \mathcal{L}$, $\mathcal{C}_\eta$ is equipped with a smooth measure μ_η (def. B.12);
2. for all $\eta \prec \eta' \in \mathcal{L}$, $\mathcal{C}_{\eta'\to\eta}$ is equipped with a smooth measure $\mu_{\eta'\to\eta}$ (on $\mathcal{C}_{\eta\to\eta}$ we will use the counting measure);
3. for all $\eta \prec \eta' \in \mathcal{L}$, $\varphi_{\eta'\to\eta}$ is volume-preserving, and for all $\eta \prec \eta' \prec \eta'' \in \mathcal{L}$, $\varphi_{\eta''\to\eta'\to\eta}$ is volume-preserving; in other words, we require:
$$\forall \eta \prec \eta',\ \ \varphi_{\eta'\to\eta,*}\, \mu_{\eta'} = \mu_{\eta'\to\eta} \times \mu_\eta\,,$$
$$\text{and}\quad \forall \eta \prec \eta' \prec \eta'',\ \ \varphi_{\eta''\to\eta'\to\eta,*}\, \mu_{\eta''\to\eta} = \mu_{\eta''\to\eta'} \times \mu_{\eta'\to\eta}\,.$$

Theorem 6.2 Let $(\mathcal{L},\, \mathcal{M},\, \pi)^\downarrow$ be a projective system of phase spaces such that:

1. $\forall \eta \in \mathcal{L}$, $\mathcal{M}_\eta = T^*(\mathcal{C}_\eta)$ where $\mathcal{C}_\eta$ is a *simply-connected* Lie group; by relying on left translations, we thus have an identification $L_\eta : \mathcal{M}_\eta \to \mathcal{C}_\eta \times \mathrm{Lie}^*(\mathcal{C}_\eta)$;
2. $\forall \eta \preccurlyeq \eta'$, $\pi_{\eta'\to\eta} = L_\eta^{-1} \circ (\rho_{\eta'\to\eta} \times \lambda_{\eta'\to\eta}) \circ L_{\eta'}$ where $\rho_{\eta'\to\eta}$ is a map $\mathcal{C}_{\eta'} \to \mathcal{C}_\eta$ and $\lambda_{\eta'\to\eta}$ is a *linear* map $\mathrm{Lie}^*(\mathcal{C}_{\eta'}) \to \mathrm{Lie}^*(\mathcal{C}_\eta)$.

Then, there exists a factorizing system of measured manifolds $(\mathcal{L},\, (\mathcal{C}, \mu),\, \varphi)^\times$ such that $(\mathcal{L},\, \mathcal{M},\, \pi)^\downarrow$ arises from $(\mathcal{L},\, \mathcal{C},\, \varphi)^\times$ (in the sense of prop. 2.16).

Proof *Conditions on $\rho_{\eta'\to\eta}$ and $\lambda_{\eta'\to\eta}$.* Let $\eta \in \mathcal{L}$ and $x \in \mathcal{C}_\eta$. There exists an open neighborhood U of 0 in $\mathrm{Lie}(\mathcal{C}_\eta)$ such that the map:
$$\begin{array}{rcccl} \Psi_x & : & U & \to & \mathcal{C}_\eta \\ & & X & \mapsto & x\,.\,\exp(X) \end{array}\,,$$
is a diffeomorphism onto its image, hence it provides a local coordinate system around x in $\mathcal{C}_\eta$. We can lift it to a local trivialization of the cotangent bundle $\mathcal{M}_\eta = T^*(\mathcal{C}_\eta)$:
$$\begin{array}{rcccl} \widetilde{\Psi}_x & : & U \times \mathrm{Lie}^*(\mathcal{C}_\eta) & \to & \mathcal{M}_\eta \\ & & X,\, \ell & \mapsto & x\,.\,\exp(X),\, \ell \circ \left[T_X \Psi_x\right]^{-1} \end{array}\,.$$
Using eq. (2.16.1), we then get, for all $\ell \in \mathrm{Lie}^*(\mathcal{C}_\eta)$ and for all $(u, v), (u', v') \in \mathrm{Lie}(\mathcal{C}_\eta) \times \mathrm{Lie}^*(\mathcal{C}_\eta)$:
$$\Omega_{\mathcal{M}_\eta,\, \widetilde{\Psi}_x(0,\ell)} \left(\left[T_{(0,\ell)} \widetilde{\Psi}_x\right](u, v),\, \left[T_{(0,\ell)} \widetilde{\Psi}_x\right](u', v') \right) = v'(u) - v(u')\,.$$
Next, we have:

$$L_\eta \circ \widetilde{\Psi}_x \ : \ U \times \mathrm{Lie}^*(\mathcal{C}_\eta) \ \rightarrow \ \mathcal{C}_\eta \times \mathrm{Lie}^*(\mathcal{C}_\eta)$$
$$X, \ell \ \mapsto \ x\,.\,\exp(X),\ \ell \circ \left[T_X \Psi_x\right]^{-1} \circ \left[T_1 \left(x\,.\,\exp(X)\,.\,\cdot\right)\right] = \ell \circ \frac{\mathrm{ad}_X}{\mathrm{id}_{\mathrm{Lie}(\mathcal{C}_\eta)} - e^{-\mathrm{ad}_X}} \ ,$$

where we have used:

$$\left[T_{\exp(X)} \left(\exp(-X)\,.\,\cdot\right)\right] \circ \left[T_X \exp\right] = \frac{\mathrm{id}_{\mathrm{Lie}(\mathcal{C}_\eta)} - e^{-\mathrm{ad}_X}}{\mathrm{ad}_X} \,,$$

as follows from the Baker–Campbell–Hausdorff formula.

Therefore, for all $x, \ell \in \mathcal{C}_\eta \times \mathrm{Lie}^*(\mathcal{C}_\eta)$ and for all $(u, v), (u', v') \in T_x(\mathcal{C}_\eta) \times \mathrm{Lie}^*(\mathcal{C}_\eta)$, the symplectic form on $\mathcal{M}_\eta$ is given by:

$$\left(L_\eta^{-1,*}\,\Omega_{\mathcal{M}_\eta}\right)_{(x,\ell)} \left((u, v),\,(u', v')\right) = v' \circ L_{\eta,x}^{-1}(u) - v \circ L_{\eta,x}^{-1}(u') + \ell\left(\left[L_{\eta,x}^{-1}(u),\ L_{\eta,x}^{-1}(u')\right]_{\mathrm{Lie}(\mathcal{C}_\eta)}\right),$$

where $L_{\eta,x} := \left[T_1 \left(x\,.\,\cdot\right)\right] : \mathrm{Lie}(\mathcal{C}_\eta) \rightarrow T_x(\mathcal{C}_\eta)$. This allows us to express the map $\underline{\cdot} : T^*(\mathcal{M}_\eta) \rightarrow T(\mathcal{M}_\eta)$ (defined from the symplectic structure as in def. 2.1) and we have for any $p, k \in T_x^*(\mathcal{C}_\eta) \times \mathrm{Lie}(\mathcal{C}_\eta)$:

$$\underline{(p, k) \circ \left[T_{L_\eta^{-1}(x,\ell)}\, L_\eta\right]} = \left[T_{(x,\ell)} L_\eta^{-1}\right] \left(L_{\eta,x}(k),\ \ell\left(\left[k,\ \cdot\right]_{\mathrm{Lie}(\mathcal{C}_\eta)}\right) - p \circ L_{\eta,x}\right).$$

Let $\eta \preccurlyeq \eta'$. We can now formulate the conditions on $\rho_{\eta'\rightarrow\eta}$ and $\lambda_{\eta'\rightarrow\eta}$ for $\pi_{\eta'\rightarrow\eta}$ to be compatible with the symplectic structures as:

$$\left[T_{x'}\rho_{\eta'\rightarrow\eta}\right] \circ L_{\eta',x'} \circ \lambda^*_{\eta'\rightarrow\eta} = L_{\eta,\rho_{\eta'\rightarrow\eta}(x')}\,, \tag{6.2.1}$$

$$\text{and} \quad \left[\lambda^*_{\eta'\rightarrow\eta}(\,\cdot\,),\ \lambda^*_{\eta'\rightarrow\eta}(\,\cdot\,)\right]_{\mathrm{Lie}(\mathcal{C}_{\eta'})} = \lambda^*_{\eta'\rightarrow\eta}\left(\left[\,\cdot\,,\ \cdot\,\right]_{\mathrm{Lie}(\mathcal{C}_\eta)}\right). \tag{6.2.2}$$

$\mathcal{C}_\eta$ as a Lie subgroup of $\mathcal{C}_{\eta'}$. $\pi_{\eta'\rightarrow\eta}$ being surjective, so is $\lambda_{\eta'\rightarrow\eta}$, thus $\lambda^*_{\eta'\rightarrow\eta} : \mathrm{Lie}(\mathcal{C}_\eta) \rightarrow \mathrm{Lie}(\mathcal{C}_{\eta'})$ is injective, and, from eq. (6.2.2), it is a Lie algebra morphism. Therefore, $\lambda^*_{\eta'\rightarrow\eta}\left\langle \mathrm{Lie}(\mathcal{C}_\eta)\right\rangle$ is a Lie subalgebra of $\mathrm{Lie}(\mathcal{C}_{\eta'})$ so there exists a unique connected Lie subgroup $\widetilde{\mathcal{C}}_\eta$ of $\mathcal{C}_{\eta'}$ such that $T_1\left(\widetilde{\mathcal{C}}_\eta\right) = \lambda^*_{\eta'\rightarrow\eta}\left\langle \mathrm{Lie}(\mathcal{C}_\eta)\right\rangle$ [100, theorem 3.19]. $\widetilde{\mathcal{C}}_\eta$ is an immersed submanifold in $\mathcal{C}_{\eta'}$ and its tangent space at $x' \in \widetilde{\mathcal{C}}_\eta$ is given by:

$$T_{x'}\left(\widetilde{\mathcal{C}}_\eta\right) = L_{\eta',x'} \circ \lambda^*_{\eta'\rightarrow\eta}\left\langle \mathrm{Lie}(\mathcal{C}_\eta)\right\rangle.$$

Let $x_1', x_2' \in \mathcal{C}_{\eta'}$ and define:

$$\kappa_{x_1',x_2'} \ : \ \mathcal{C}_{\eta'} \ \rightarrow \ \mathcal{C}_\eta$$
$$x' \ \mapsto \ \rho_{\eta'\rightarrow\eta}(x_1'\,.\,x')\,.\,\rho_{\eta'\rightarrow\eta}(x_2'\,.\,x')^{-1} \ .$$

For any $k \in \mathrm{Lie}(\mathcal{C}_\eta)$, we have:

$$\left[T_1\,\kappa_{x_1',x_2'}\right] \circ \lambda^*_{\eta'\rightarrow\eta}(k) =$$

$$= \left[T_{\rho_{\eta'\rightarrow\eta}(x_1')}\left(\,\cdot\,.\,\rho_{\eta'\rightarrow\eta}(x_2')^{-1}\right)\right] \circ \left[T_{x_1'}\,\rho_{\eta'\rightarrow\eta}\right] \circ L_{\eta',x_1'} \circ \lambda^*_{\eta'\rightarrow\eta}(k) +$$
$$+ \left[T_{\rho_{\eta'\rightarrow\eta}(x_2')}\left(\rho_{\eta'\rightarrow\eta}(x_1')\,.\,(\,\cdot\,)^{-1}\right)\right] \circ \left[T_{x_2'}\,\rho_{\eta'\rightarrow\eta}\right] \circ L_{\eta',x_2'} \circ \lambda^*_{\eta'\rightarrow\eta}(k)$$

$$= \left[T_{\rho_{\eta'\to\eta}(x_1')}\left(\,\cdot\,.\,\rho_{\eta'\to\eta}(x_2')^{-1}\right)\right] \circ L_{\eta,\rho_{\eta'\to\eta}(x_1')}(k) + \left[T_{\rho_{\eta'\to\eta}(x_2')}\left(\rho_{\eta'\to\eta}(x_1')\,.\,(\,\cdot\,)^{-1}\right)\right] \circ L_{\eta,\rho_{\eta'\to\eta}(x_2')}(k)$$

(using eq. (6.2.1))

$$= \left[T_{\mathbf{1}}\left(x \mapsto \rho_{\eta'\to\eta}(x_1')\,.\,x\,.\,x^{-1}\,.\,\rho_{\eta'\to\eta}(x_2')^{-1}\right)\right](k) = 0\,.$$

With this, we get, for any $x' \in \widetilde{\mathfrak{C}}_\eta$:

$$\left[T_{x'}\,\kappa_{x_1',x_2'}\right]\left\langle T_{x'}\left(\widetilde{\mathfrak{C}}_\eta\right)\right\rangle = \left[T_{x'}\,\kappa_{x_1',x_2'}\right] \circ L_{\eta',x'} \circ \lambda^*_{\eta'\to\eta}\left\langle \mathrm{Lie}\left(\mathfrak{C}_\eta\right)\right\rangle$$

$$= \left[T_{\mathbf{1}}\,\kappa_{x_1'.x',\,x_2'.x'}\right] \circ \lambda^*_{\eta'\to\eta}\left\langle \mathrm{Lie}\left(\mathfrak{C}_\eta\right)\right\rangle = \{0\}\,,$$

thus, $\widetilde{\mathfrak{C}}_\eta$ being connected by definition, $\kappa_{x_1',x_2'}$ is constant on $\widetilde{\mathfrak{C}}_\eta$ for any x_1', $x_2' \in \mathfrak{C}_{\eta'}$. In particular, applying with $x_2' = \mathbf{1}$ gives:

$$\forall x_o' \in \mathfrak{C}_{\eta'},\ \forall x' \in \widetilde{\mathfrak{C}}_\eta,\ \widetilde{\rho}_{\eta'\to\eta}(x_o'\,.\,x') = \widetilde{\rho}_{\eta'\to\eta}(x_o')\,.\,\widetilde{\rho}_{\eta'\to\eta}(x')\,, \tag{6.2.3}$$

where $\forall x' \in \mathfrak{C}_{\eta'}, \widetilde{\rho}_{\eta'\to\eta}(x') := \rho_{\eta'\to\eta}(\mathbf{1})^{-1}\,.\,\rho_{\eta'\to\eta}(x')\,.$

Therefore, $\widetilde{\rho}_{\eta'\to\eta}|_{\widetilde{\mathfrak{C}}_\eta\to\mathfrak{C}_\eta}$ is a smooth group homomorphism, and, moreover, its derivative at $\mathbf{1}$ is a Lie algebra isomorphism, for we have using eq. (6.2.1):

$$[T_{\mathbf{1}}\,\widetilde{\rho}_{\eta'\to\eta}] \circ \lambda^*_{\eta'\to\eta} = L^{-1}_{\eta,\rho_{\eta'\to\eta}(\mathbf{1})} \circ [T_{\mathbf{1}}\,\rho_{\eta'\to\eta}] \circ \lambda^*_{\eta'\to\eta} = \mathrm{id}_{\mathrm{Lie}(\mathfrak{C}_\eta)}\,.$$

Hence, $\widetilde{\rho}_{\eta'\to\eta}|_{\widetilde{\mathfrak{C}}_\eta\to\mathfrak{C}_\eta}$ is a Lie group isomorphism, for $\widetilde{\mathfrak{C}}_\eta$ is connected and $\mathfrak{C}_\eta$ is simply-connected [100, prop. 3.26]. We will denote by $\Lambda_{\eta'\leftarrow\eta} : \mathfrak{C}_\eta \to \widetilde{\mathfrak{C}}_\eta$ its inverse.

Factorizing system. We define $\mathfrak{C}_{\eta'\to\eta} := \widetilde{\rho}^{-1}_{\eta'\to\eta}\langle\mathbf{1}\rangle$. $\widetilde{\rho}_{\eta'\to\eta}$ has surjective derivative at each point, so $\mathfrak{C}_{\eta'\to\eta}$ is a smooth manifold as level set of a constant rank map [54, theorem 5.22]. Next, we define the map $\varphi_{\eta'\to\eta}$ by:

$$\begin{array}{rccl} \varphi_{\eta'\to\eta} & : & \mathfrak{C}_{\eta'} & \to & \mathfrak{C}_{\eta'\to\eta} \times \mathfrak{C}_\eta \\ & & x' & \mapsto & x'\,.\,\left(\Lambda_{\eta'\leftarrow\eta} \circ \widetilde{\rho}_{\eta'\to\eta}(x')\right)^{-1},\ \rho_{\eta'\to\eta}(x') \end{array}.$$

$\varphi_{\eta'\to\eta}$ is well-defined for, using eq. (6.2.3), we have for all $x' \in \mathfrak{C}_{\eta'}$:

$$\widetilde{\rho}_{\eta'\to\eta}\left(x'\,.\,\Lambda_{\eta'\leftarrow\eta}\left(\widetilde{\rho}_{\eta'\to\eta}(x')^{-1}\right)\right) = \widetilde{\rho}_{\eta'\to\eta}(x')\,.\,\widetilde{\rho}_{\eta'\to\eta}(x')^{-1} = \mathbf{1}\,.$$

To prove that $\varphi_{\eta'\to\eta}$ is a bijective map, we define a map $\widetilde{\varphi}_{\eta'\to\eta}$ by:

$$\begin{array}{rccl} \widetilde{\varphi}_{\eta'\to\eta} & : & \mathfrak{C}_{\eta'\to\eta} \times \mathfrak{C}_\eta & \to & \mathfrak{C}_{\eta'} \\ & & y,\ x & \mapsto & y\,.\,\sigma^{-1}\,.\,\Lambda_{\eta'\leftarrow\eta}(x) \end{array},$$

where $\sigma := \Lambda_{\eta'\leftarrow\eta}\left(\rho_{\eta'\to\eta}(\mathbf{1})\right)$. Using again eq. (6.2.3), we can check that $\varphi_{\eta'\to\eta} \circ \widetilde{\varphi}_{\eta'\to\eta} = \mathrm{id}_{\mathfrak{C}_{\eta'\to\eta}\times\mathfrak{C}_\eta}$ and $\widetilde{\varphi}_{\eta'\to\eta} \circ \varphi_{\eta'\to\eta} = \mathrm{id}_{\mathfrak{C}_{\eta'}}$. Since both $\varphi_{\eta'\to\eta}$ and $\widetilde{\varphi}_{\eta'\to\eta}$ are smooth, $\varphi_{\eta'\to\eta}$ is a diffeomorphism.

From eq. (6.2.1), we have:

$$\forall x \in \mathfrak{C}_\eta,\ [T_x\Lambda_{\eta'\leftarrow\eta}] = L_{\eta',\Lambda_{\eta'\leftarrow\eta}(x)} \circ \lambda^*_{\eta'\to\eta} \circ L^{-1}_{\eta,x}\,.$$

Thus, for any $y, x \in \mathfrak{C}_{\eta'\to\eta} \times \mathfrak{C}_\eta$, the derivative of $\varphi^{-1}_{\eta'\to\eta}$ satisfies:

$$\forall u \in T_x(\mathcal{C}_\eta),\quad \left[T_{y,x}\,\varphi^{-1}_{\eta'\to\eta}\right](0,u) = L_{\eta',\,\varphi^{-1}_{\eta'\to\eta}(y,x)} \circ \lambda^*_{\eta'\to\eta} \circ L^{-1}_{\eta,x}(u)\,. \tag{6.2.4}$$

So we get, for any $x', p' \in T^*(\mathcal{C}_{\eta'})$:

$$\pi_{\eta'\to\eta}(x',\, p') = \left(\rho_{\eta'\to\eta}(x'),\, p' \circ L_{\eta',x'} \circ \lambda^*_{\eta'\to\eta} \circ L^{-1}_{\eta,\rho_{\eta'\to\eta}(x')}\right)$$

$$= \left(s_{\eta'\to\eta} \circ \varphi_{\eta'\to\eta}(x'),\, p' \circ \left[T_{x'}\,\varphi_{\eta'\to\eta}\right]^{-1}(0,\,\cdot\,)\right),$$

where $s_{\eta'\to\eta} : \mathcal{C}_{\eta'\to\eta} \times \mathcal{C}_\eta \to \mathcal{C}_\eta$ is the projection on the second Cartesian factor. We can now use prop. 2.17 to build from these objects a factorizing system $(\mathcal{L},\,\mathcal{C},\,\varphi)^\times$ that gives rise to $(\mathcal{L},\,\mathcal{M},\,\pi)^\downarrow$.

Volume forms. For any $\eta \in \mathcal{L}$, we choose a non-zero d_η-form $\widetilde{\omega}_\eta$ on $\mathrm{Lie}(\mathcal{C}_\eta)$ (with $d_\eta := \dim \mathcal{C}_\eta$) and we define a right-invariant volume form ω_η on $\mathcal{C}_\eta$ by:

$$\forall x \in \mathcal{C}_\eta,\ \forall u_1, \dots, u_{d_\eta} \in T_x(\mathcal{C}_\eta),\ \omega_\eta(u_1, \dots, u_{d_\eta}) := \widetilde{\omega}_\eta\left(R^{-1}_{\eta,x}\,u_1,\ \dots\ ,\ R^{-1}_{\eta,x}\,u_{d_\eta}\right),$$

where $R_{\eta,x} := T_{\mathbf{1}}\left(\,\cdot\,.\,x\right)$. We call μ_η the smooth measure arising from the volume form ω_η.

Let $\eta \preccurlyeq \eta'$, $y \in \mathcal{C}_{\eta'\to\eta}$ and $w_1, \dots, w_{d_{\eta'\to\eta}} \in T_y(\mathcal{C}_{\eta'\to\eta}) = \mathrm{Ker}\left[T_y\,\rho_{\eta'\to\eta}\right]$ (with $d_{\eta'\to\eta} := \dim \mathcal{C}_{\eta'\to\eta} = d_{\eta'} - d_\eta$). The map:

$$\begin{array}{rccl} \alpha\ : & \mathrm{Lie}(\mathcal{C}_\eta)^{d_\eta} & \to & \mathbb{R} \\ & u_1, \dots, u_{d_\eta} & \mapsto & \widetilde{\omega}_{\eta'}\left(R^{-1}_{\eta',y}(w_1),\ \dots\ ,\ R^{-1}_{\eta',y}(w_{d_{\eta'\to\eta}}),\ A_{\eta'\leftarrow\eta,y}(u_1),\ \dots\ ,\ A_{\eta'\leftarrow\eta,y}(u_{d_\eta})\right) \end{array},$$

where $A_{\eta'\leftarrow\eta,y} := \mathrm{Ad}_y \circ \lambda^*_{\eta'\to\eta} \circ \mathrm{Ad}^{-1}_{\rho_{\eta'\to\eta}(\mathbf{1})}$, is a d_η-form on $\mathrm{Lie}(\mathcal{C}_\eta)$, so there exists $\omega_{\eta'\to\eta,y}(w_1,\ \dots\ ,\ w_{d_{\eta'\to\eta}})$ $\in \mathbb{R}$ such that:

$$\alpha(u_1,\ \dots\ ,\ u_{d_\eta}) = \omega_{\eta'\to\eta,y}(w_1,\ \dots\ ,\ w_{d_{\eta'\to\eta}})\,\widetilde{\omega}_\eta(u_1,\ \dots\ ,\ u_{d_\eta})\,.$$

Now, using the expression for $\varphi^{-1}_{\eta'\to\eta}$ given above, we have, for any $y, x \in \mathcal{C}_{\eta'\to\eta} \times \mathcal{C}_\eta$:

$$\forall w \in T_y(\mathcal{C}_{\eta'\to\eta}),\ R^{-1}_{\eta',\varphi^{-1}_{\eta'\to\eta}(y,x)} \circ \left[T_{y,x}\,\varphi^{-1}_{\eta'\to\eta}\right](w,0) = R^{-1}_{\eta',y}(w)\,,$$

and, from eq. (6.2.4), we also have:

$$\forall u \in T_x(\mathcal{C}_\eta),\quad R^{-1}_{\eta',\varphi^{-1}_{\eta'\to\eta}(y,x)} \circ \left[T_{y,x}\,\varphi^{-1}_{\eta'\to\eta}\right](0,u) = A_{\eta'\leftarrow\eta,y} \circ R^{-1}_{\eta,x}(u)\,,$$

where we have used that $\lambda^*_{\eta'\to\eta} = T_{\mathbf{1}}\Lambda_{\eta'\leftarrow\eta}$ with $\Lambda_{\eta'\leftarrow\eta}$ a group homomorphism. With this, we can check that $\varphi^{-1,*}_{\eta'\to\eta}\,\omega_{\eta'} = \omega_{\eta'\to\eta} \wedge \omega_\eta$. In particular, this implies that $\omega_{\eta'\to\eta}$ is a smooth volume form on $\mathcal{C}_{\eta'\to\eta}$. Thus, defining $\mu_{\eta'\to\eta}$ to be the corresponding smooth measure, we get $\varphi_{\eta'\to\eta,*}\,\mu_{\eta'} = \mu_{\eta'\to\eta} \times \mu_\eta$.

Finally, for any $\eta \preccurlyeq \eta' \preccurlyeq \eta''$, $\varphi_{\eta''\to\eta'}$, $\varphi_{\eta'\to\eta}$ and $\varphi_{\eta''\to\eta}$ are volume-preserving, hence so is $\varphi_{\eta''\to\eta'\to\eta} \times \mathrm{id}_{\mathcal{C}_\eta}$ (using eq. (2.11.1)) and therefore $\varphi_{\eta''\to\eta'\to\eta}$ itself. □

Proposition 6.3 Let $(\mathcal{L}, \mathcal{C}, \varphi)^\times$ be a factorizing system of measured manifolds. We define:

1. for $\eta \in \mathcal{L}$, $\mathcal{H}_\eta := L_2(\mathcal{C}_\eta,\, d\mu_\eta)$;
2. for $\eta \prec \eta' \in \mathcal{L}$, $\mathcal{H}_{\eta'\to\eta} := L_2(\mathcal{C}_{\eta'\to\eta},\, d\mu_{\eta'\to\eta})$, and:

$$\begin{array}{rccl} \Phi_{\eta'\to\eta}\ : & \mathcal{H}_{\eta'} & \to & \mathcal{H}_{\eta'\to\eta} \otimes \mathcal{H}_\eta \\ & \psi & \mapsto & \psi \circ \varphi^{-1}_{\eta'\to\eta} \end{array},$$

with the natural identification of $L_2(\mathcal{C}_{\eta'\to\eta},\, d\mu_{\eta'\to\eta}) \otimes L_2(\mathcal{C}_\eta,\, d\mu_\eta)$ with $L_2(\mathcal{C}_{\eta'\to\eta} \times \mathcal{C}_\eta,\, d\mu_{\eta'\to\eta} \times d\mu_\eta)$.

Then, we can complete these objects to build a projective system of quantum state spaces $(\mathcal{L}, \mathcal{H}, \Phi)^\otimes$.

Proof We define:

3. for $\eta \prec \eta' \prec \eta'' \in \mathcal{L}$,

$$\begin{array}{rccl} \Phi_{\eta''\to\eta'\to\eta} & : \mathcal{H}_{\eta''\to\eta} & \to & \mathcal{H}_{\eta''\to\eta'} \otimes \mathcal{H}_{\eta'\to\eta} \\ & \psi & \mapsto & \psi \circ \varphi^{-1}_{\eta''\to\eta'\to\eta} \end{array},$$

   with the natural identification of $L_2(\mathcal{C}_{\eta''\to\eta'},\, d\mu_{\eta''\to\eta'}) \otimes L_2(\mathcal{C}_{\eta'\to\eta},\, d\mu_{\eta'\to\eta})$ with $L_2(\mathcal{C}_{\eta''\to\eta'} \times \mathcal{C}_{\eta'\to\eta},\, d\mu_{\eta''\to\eta'} \times d\mu_{\eta'\to\eta})$;

4. for $\eta \in \mathcal{L}$, $\mathcal{H}_{\eta\to\eta} := \mathbb{C}$ and we define $\Phi_{\eta\to\eta}$ to be the natural isomorphic identification between $\mathcal{H}_\eta$ and $\mathbb{C} \otimes \mathcal{H}_\eta$;

5. for $\eta \preccurlyeq \eta' \in \mathcal{L}$, we define $\Phi_{\eta'\to\eta\to\eta}$ (resp. $\Phi_{\eta'\to\eta'\to\eta}$) to be the natural isomorphic identification between $\mathcal{H}_{\eta'\to\eta}$ and $\mathcal{H}_{\eta'\to\eta} \otimes \mathbb{C}$ (resp. $\mathbb{C} \otimes \mathcal{H}_{\eta'\to\eta}$).

That $\Phi_{\eta'\to\eta}$ for $\eta \prec \eta'$ defines an Hilbert space isomorphism comes from the volume-preserving property of $\varphi_{\eta'\to\eta}$ and from the fact that $L_2(\mathcal{C},\, d\mu) \otimes L_2(\mathcal{C}',\, d\mu')$ can be unitarily identified with $L_2(\mathcal{C} \times \mathcal{C}',\, d\mu \times d\mu')$ (thanks to Fubini's theorem). Similarly, $\Phi_{\eta''\to\eta'\to\eta}$ for $\eta \prec \eta' \prec \eta''$ is an Hilbert space isomorphism.

We now just need to check the three-spaces consistency condition eq. (5.1.1). We consider $\eta \prec \eta' \prec \eta''$ (since the condition is trivially satisfied whenever $\eta = \eta'$ or $\eta' = \eta''$):

$$\begin{aligned} \forall \psi \in \mathcal{H}_\eta,\ &\left(\Phi_{\eta''\to\eta'\to\eta} \otimes \mathrm{id}_{\mathcal{H}_\eta}\right) \circ \Phi_{\eta''\to\eta}(\psi) = \\ &= \left(\psi \circ \varphi^{-1}_{\eta''\to\eta}\right) \circ \left(\varphi^{-1}_{\eta''\to\eta'\to\eta} \otimes \mathrm{id}_{\mathcal{C}_\eta}\right) \\ &= \left(\psi \circ \varphi^{-1}_{\eta''\to\eta'}\right) \circ \left(\mathrm{id}_{\mathcal{C}_{\eta''\to\eta'}} \otimes \varphi^{-1}_{\eta'\to\eta}\right) \text{ (using eq. (2.11.1))} \\ &= \left(\mathrm{id}_{\mathcal{H}_{\eta''\to\eta'}} \otimes \Phi_{\eta'\to\eta}\right) \circ \Phi_{\eta''\to\eta'}(\psi)\,. \end{aligned}$$

□

To argue that the quantum projective system composed above actually provides a quantization of the classical one (as specified by the factorizing system of configuration spaces we started from), we need to say how classical observables on the latter are turned to quantum observables on the former. For this, we import the prescriptions of geometric quantization (summarized in appendix B.3, especially in prop. B.14, and rewritten here more explicitly using the benefit of working in a phase space given as a cotangent bundle). Thus, for each η, we can formulate the quantization condition (the choice of preferred configuration variables is tied to a selection of which observables can be directly quantized) as well as the definition of the quantized observables. The key statement is that the compatibility conditions imposed on the family of measures is sufficient to ensure that these prescriptions, supplied separately for each η, will fit readily into a coherent picture.

Proposition 6.4 We consider the same objects as in prop. 6.3. Let $(\mathcal{L}, \mathcal{M}, \widetilde{\varphi})^\times$ be the factorizing system of phase spaces constructed from $(\mathcal{L}, \mathcal{C}, \varphi)^\times$ as in prop. 2.16 and let $f = [f_\eta]_\sim \in \mathcal{O}^\times_{(\mathcal{L},\mathcal{M},\widetilde{\varphi})}$ (prop. 2.13).

If there exists a representative f_η of f such that:

$$\exists\, \overline{X}_{f_\eta} \textit{ complete } \text{vector field on } \mathcal{C}_\eta \,/\;\; \forall (x,p) \in \mathcal{M}_\eta,\, \left[T_{(x,p)}\, \gamma_\eta\right]\left(X_{f_\eta,(x,p)}\right) = \overline{X}_{f_\eta,x}\,, \tag{6.4.1}$$

where $\gamma_\eta : \mathcal{M}_\eta = T^*(\mathcal{C}_\eta) \to \mathcal{C}_\eta$ is the bundle projection, then this condition is satisfied by all representatives of f. Accordingly, we define:

$$\mathcal{O}^{\times,\text{Pos}}_{(\mathcal{L},\mathcal{C},\varphi)} := \left\{ f \in \mathcal{O}^\times_{(\mathcal{L},\mathcal{M},\widetilde{\varphi})} \;\middle|\; \exists\, f_\eta \in f \text{ satisfying eq. (6.4.1)} \right\}.$$

For $f_\eta \in f \in \mathcal{O}^{\times,\text{Pos}}_{(\mathcal{L},\mathcal{C},\varphi)}$, we can define $\widehat{f_\eta}^{\mu_\eta}$ as a densely defined, essentially self-adjoint operator on $\mathcal{H}_\eta$ (with dense domain $\mathcal{D}_\eta \subset \mathcal{H}_\eta$) by:

$$\forall \psi \in \mathcal{D}_\eta,\, \forall x \in \mathcal{C}_\eta,\;\; \widehat{f_\eta}^{\mu_\eta}(\psi)(x) := \psi(x)\, f_\eta(x,0) + i\, [d\psi]_x \left(\overline{X}_{f_\eta,x}\right) + \frac{i}{2}\, \psi(x) \left(\text{div}_{\mu_\eta} \overline{X}_{f_\eta}\right)(x)\,,$$

where $\text{div}_{\mu_\eta} \overline{X}_{f_\eta}$ is defined by $\mathfrak{L}_{\overline{X}_{f_\eta}}\, \mu_\eta = \left(\text{div}_{\mu_\eta} \overline{X}_{f_\eta}\right)\, \mu_\eta$ (def. B.12).

Then, the application:

$$\begin{array}{rccc} \widehat{\cdot}^{\mu} : & \mathcal{O}^{\times,\text{Pos}}_{(\mathcal{L},\mathcal{C},\varphi)} & \to & \mathcal{O}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \\ & [f_\eta]_\sim & \to & \left[\widehat{f_\eta}^{\mu_\eta}\right]_\sim \end{array},$$

is well-defined ($\mathcal{O}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ has been defined in prop. 5.5).

Proof *Quantization condition.* For $\eta \preccurlyeq \eta' \in \mathcal{L}$, we define $\pi_{\eta'\to\eta} : \mathcal{C}_{\eta'} \to \mathcal{C}_\eta$, $\lambda_{\eta'\to\eta} : \mathcal{C}_{\eta'} \to \mathcal{C}_{\eta'\to\eta}$, and $\widetilde{\pi}_{\eta'\to\eta} : \mathcal{M}_{\eta'} \to \mathcal{M}_\eta$, $\widetilde{\lambda}_{\eta'\to\eta} : \mathcal{M}_{\eta'} \to \mathcal{M}_{\eta'\to\eta}$, such that:

$$\forall x' \in \mathcal{C}_{\eta'},\, \varphi_{\eta'\to\eta}(x') = \left(\lambda_{\eta'\to\eta}(x'),\, \pi_{\eta'\to\eta}(x')\right)$$

$$\&\quad \forall (x',p') \in \mathcal{M}_{\eta'},\, \widetilde{\varphi}_{\eta'\to\eta}(x',p') = \left(\widetilde{\lambda}_{\eta'\to\eta}(x',p'),\, \widetilde{\pi}_{\eta'\to\eta}(x',p')\right).$$

From prop. 2.16, we then have:

$$\forall \eta \preccurlyeq \eta' \in \mathcal{L},\, \gamma_\eta \circ \widetilde{\pi}_{\eta'\to\eta} = \pi_{\eta'\to\eta} \circ \gamma_{\eta'} \quad \& \quad \gamma_{\eta'\to\eta} \circ \widetilde{\lambda}_{\eta'\to\eta} = \lambda_{\eta'\to\eta} \circ \gamma_{\eta'}\,.$$

Let $f_\eta \in C^\infty(\mathcal{M}_\eta, \mathbb{R})$ satisfying eq. (6.4.1) and let $\eta' \succcurlyeq \eta$. Using the previous identity, we have:

$$\forall (x',p') \in \mathcal{M}_{\eta'},\, \left[T_{x'}\, \pi_{\eta'\to\eta}\right] \circ \left[T_{(x',p')}\, \gamma_{\eta'}\right]\left(X_{f_\eta \circ \widetilde{\pi}_{\eta'\to\eta},(x',p')}\right) =$$

$$= \left[T_{\widetilde{\pi}_{\eta'\to\eta}(x',p')}\, \gamma_\eta\right] \circ \left[T_{(x',p')}\, \widetilde{\pi}_{\eta'\to\eta}\right]\left(\underline{\left[\widetilde{\pi}^*_{\eta'\to\eta}\, df_\eta\right]_{x',p'}}\right)$$

$$= \left[T_{\widetilde{\pi}_{\eta'\to\eta}(x',p')}\, \gamma_\eta\right]\left(\underline{[df_\eta]_{\widetilde{\pi}_{\eta'\to\eta}(x',p')}}\right) \text{ (using eq. (2.1.1))}$$

$$= \left[T_{\widetilde{\pi}_{\eta'\to\eta}(x',p')}\, \gamma_\eta\right]\left(X_{f_\eta,\widetilde{\pi}_{\eta'\to\eta}(x',p')}\right) = \overline{X}_{f_\eta,\pi_{\eta'\to\eta}(x')}\,,$$

and:

$$\forall (x', p') \in \mathcal{M}_{\eta'},\ [T_{x'}\, \lambda_{\eta'\to\eta}] \circ [T_{(x',p')}\, \gamma_{\eta'}] \left(X_{f_\eta \circ \widetilde{\pi}_{\eta'\to\eta}, (x',p')} \right) =$$

$$= \left[T_{\widetilde{\lambda}_{\eta'\to\eta}(x',p')}\, \gamma_{\eta'\to\eta} \right] \circ \left[T_{(x',p')}\, \widetilde{\lambda}_{\eta'\to\eta} \right] \left(\underline{\left[\widetilde{\pi}^*_{\eta'\to\eta}\, df_\eta \right]_{x',p'}} \right)$$

$$= 0\,,$$

since $\underline{\left[\widetilde{\pi}^*_{\eta'\to\eta}\, df_\eta \right]_{x',p'}} \in \left(\text{Ker}\, [T_{(x',p')}\, \widetilde{\pi}_{\eta'\to\eta}] \right)^\perp = \text{Ker} \left[T_{(x',p')}\, \widetilde{\lambda}_{\eta'\to\eta} \right]$.

Therefore, $[T_{(x',p')}\, \gamma_{\eta'}] \left(X_{f_\eta \circ \widetilde{\pi}_{\eta'\to\eta}, (x',p')} \right)$ only depends on x'. If we now define:

$$\forall x' \in \mathcal{C}_{\eta'},\ \overline{X}_{f_\eta \circ \widetilde{\pi}_{\eta'\to\eta}, x'} := [T_{(x',0)}\, \gamma_{\eta'}] \left(X_{f_\eta \circ \widetilde{\pi}_{\eta'\to\eta}, (x',0)} \right),$$

we have $\overline{X}_{f_\eta \circ \widetilde{\pi}_{\eta'\to\eta}} \in \mathcal{T}^\infty(\mathcal{C}_{\eta'})$ (the space of smooth vector fields on $\mathcal{C}_{\eta'}$) and:

$$\forall (x', p') \in \mathcal{M}_{\eta'},\ [T_{(x',p')}\, \gamma_{\eta'}] \left(X_{f_\eta \circ \widetilde{\pi}_{\eta'\to\eta}, (x',p')} \right) = \overline{X}_{f_\eta \circ \widetilde{\pi}_{\eta'\to\eta}, x'}\,.$$

Moreover, we have:

$$\forall x' \in \mathcal{C}_{\eta'},\ [T_{x'}\, \varphi_{\eta'\to\eta}] \left(\overline{X}_{f_\eta \circ \widetilde{\pi}_{\eta'\to\eta}, x'} \right) = \left(0,\ \overline{X}_{f_\eta, \pi_{\eta'\to\eta}(x')} \right). \tag{6.4.2}$$

In particular, the completeness of $\overline{X}_{f_\eta}$ implies that $\overline{X}_{f_\eta \circ \widetilde{\pi}_{\eta'\to\eta}}$ is complete as well, so $f_\eta \circ \widetilde{\pi}_{\eta'\to\eta}$ fulfills eq. (6.4.1).

On the other hand, for $f_\eta \in C^\infty(\mathcal{M}_\eta, \mathbb{R})$, if there exists $\eta' \succcurlyeq \eta$ such that $f_\eta \circ \widetilde{\pi}_{\eta'\to\eta}$ satisfy eq. (6.4.1), then, in the same way as above:

$$\forall (x', p') \in \mathcal{M}^{\eta'},\ \left[T_{\widetilde{\pi}_{\eta'\to\eta}(x',p')}\, \gamma_\eta \right] \left(X_{f_\eta, \widetilde{\pi}_{\eta'\to\eta}(x',p')} \right) = [T_{x'}\, \pi_{\eta'\to\eta}] \left(\overline{X}_{f_\eta \circ \widetilde{\pi}_{\eta'\to\eta}, x'} \right).$$

Now, the right-hand side does not depend on p', and we have, for any $x' \in \mathcal{C}_{\eta'}$:

$$\left\{ \left(\pi_{\eta'\to\eta}(x'),\ p \right) \mid p \in T_x(\mathcal{C}_\eta) \right\} = \{ \widetilde{\pi}_{\eta'\to\eta}(x', p') \mid p' \in T_{x'}(\mathcal{C}_{\eta'}) \}\,.$$

So, $\pi_{\eta'\to\eta}$ being surjective, there exists a smooth vector field $\overline{X}_{f_\eta}$ on $\mathcal{C}_\eta$ such that $\forall (x,p) \in \mathcal{M}_\eta$, $[T_{(x,p)}\, \gamma_\eta] \left(X_{f_\eta, (x,p)} \right) = \overline{X}_{f_\eta, x}$, and we again have eq. (6.4.2). Therefore, f_η also satisfy eq. (6.4.1).

Quantized observable. Let $f_\eta \in f \in \mathcal{O}^{\times, \text{Pos}}_{(\mathcal{L}, \mathcal{C}, \varphi)}$. From prop. B.14, f_η can be quantized into an essentially self-adjoint operator $\widehat{f}_\eta^{\mu_\eta}$ on $\mathcal{H}_\eta$ and one can check that both definitions coincides. We now want to determine $\iota_{\eta'\leftarrow\eta}\left(\widehat{f}_\eta^{\mu_\eta} \right)$ for $\eta' \succcurlyeq \eta$. We start by deriving an identity for $\text{div}_{\mu_\eta} \overline{X}_{f_\eta}$:

$$\left(\text{div}_{\mu_{\eta'}} \overline{X}_{f_\eta \circ \widetilde{\pi}_{\eta'\to\eta}} \right) \mu_{\eta'} = \mathfrak{L}_{\overline{X}_{f_\eta \circ \widetilde{\pi}_{\eta'\to\eta}}}\, \mu_{\eta'}$$

$$= \varphi^{-1}_{\eta'\to\eta, *} \left[\mathfrak{L}_{\varphi^{-1,*}_{\eta'\to\eta} \left(\overline{X}_{f_\eta \circ \widetilde{\pi}_{\eta'\to\eta}} \right)}\, \varphi_{\eta'\to\eta, *}\, \mu_{\eta'} \right]$$

$$= \varphi^{-1}_{\eta'\to\eta, *} \left[\mathfrak{L}_{\varphi^{-1,*}_{\eta'\to\eta} \left(\overline{X}_{f_\eta \circ \widetilde{\pi}_{\eta'\to\eta}} \right)}\, \mu_{\eta'\to\eta} \times \mu_\eta \right] \text{ (using def. 6.1.3)}$$

$$= \varphi^{-1}_{\eta' \to \eta,*} \left[\mathfrak{L}_{\left(0,\overline{X}_{f_\eta}\right)} \, \mu_{\eta' \to \eta} \times \mu_\eta\right] \text{ (using eq. (6.4.2))}$$

$$= \varphi^{-1}_{\eta' \to \eta,*} \left[\mu_{\eta' \to \eta} \times \left(\mathfrak{L}_{\overline{X}_{f_\eta}} \, \mu_\eta\right)\right]$$

$$= \left[\left(\mathrm{div}_{\mu_\eta} \overline{X}_{f_\eta}\right) \circ \pi_{\eta' \to \eta}\right] \mu_{\eta'} \,.$$

Hence, it follows:

$$\mathrm{div}_{\mu_{\eta'}} \overline{X}_{f_\eta \circ \widetilde{\pi}_{\eta' \to \eta}} = \left(\mathrm{div}_{\mu_\eta} \overline{X}_{f_\eta}\right) \circ \pi_{\eta' \to \eta} \,. \tag{6.4.3}$$

Now, let $\psi \in \Phi^{-1}_{\eta' \to \eta} \left\langle \mathcal{H}_{\eta' \to \eta} \otimes \mathcal{D}_\eta \right\rangle$ (where $\mathcal{D}_\eta$ is the domain of $\widehat{f_\eta}^{\mu_\eta}$ and the $\otimes$ is to be understood as a tensor product of vector spaces, that is without any completion in contrast to a tensor product of Hilbert spaces). Then, we have:

$$\forall (y, x) \in \mathcal{C}_{\eta' \to \eta} \times \mathcal{C}_\eta, \quad \left(\mathrm{id}_{\mathcal{H}_{\eta' \to \eta}} \otimes \widehat{f_\eta}^{\mu_\eta}\right) \circ \Phi_{\eta' \to \eta}(\psi)(y, x) =$$

$$= \psi \circ \varphi^{-1}_{\eta' \to \eta}(y, x) \; f_\eta(x, 0) + i \left[\partial_x \psi \circ \varphi^{-1}_{\eta' \to \eta}\right]_{(y,x)} \left(\overline{X}_{f_\eta, x}\right) + \frac{i}{2} \psi \circ \varphi^{-1}_{\eta' \to \eta}(y, x) \; \left(\mathrm{div}_{\mu_\eta} \overline{X}_{f_\eta}\right)(x)$$

$$= \psi \circ \varphi^{-1}_{\eta' \to \eta}(y, x) \; f_\eta \circ \widetilde{\pi}_{\eta' \to \eta}(\varphi^{-1}_{\eta' \to \eta}(y, x), 0) + i \left[d\psi\right]_{\varphi^{-1}_{\eta' \to \eta}(y,x)} \left(\overline{X}_{f_\eta \circ \widetilde{\pi}_{\eta' \to \eta}, \varphi^{-1}_{\eta' \to \eta}(y,x)}\right) +$$

$$+ \frac{i}{2} \psi \circ \varphi^{-1}_{\eta' \to \eta}(y, x) \; \left(\mathrm{div}_{\mu_{\eta'}} \overline{X}_{f_\eta \circ \widetilde{\pi}_{\eta' \to \eta}}\right) \circ \varphi^{-1}_{\eta' \to \eta}(y, x) \quad \text{(using eqs. (6.4.2) and (6.4.3))}$$

$$= \left(\Phi_{\eta' \to \eta} \circ \widehat{f_\eta \circ \widetilde{\pi}_{\eta' \to \eta}}^{\mu_{\eta'}}\right) (\psi)(y, x) \;\; .$$

Therefore, $\forall f_\eta, f_{\eta'} \in f \in \mathcal{O}^{\times,\mathrm{Pos}}_{(\mathcal{L},\mathcal{C},\varphi)}$, $\widehat{f_\eta}^{\mu_\eta} \sim \widehat{f_{\eta'}}^{\mu_{\eta'}}$. □

We close this subsection with an application of theorem 5.9: under the additional hypothesis that the measures are normalized to unity, we can construct an inductive limit of Hilbert spaces from the $\mathcal{H}_\eta$, whose space of states is naturally embedded in the one of the projective structure developed above. As long as all $\mathcal{C}_\eta$ have finite volume (hence in particular if they are compact), it is always possible to consistently normalize the measures to unity. Note however that, depending on the projective structure under consideration, it may not be possible to equip all $\mathcal{C}_\eta$ with normalizable measures fulfilling the factorization requirement def. 6.1.3 (see eg. the models considered in [65, 66], in particular the discussion in [65, section 1.1]).

As realized recently by Bianca Dittrich and Marc Geiller [23, section V.D], the natural classical precursor of an inductive limit of Hilbert spaces is a hybrid projective/injective limit of phase spaces, where the space attached to a coarser label η is obtained from the one attached to a finer label η' via *symplectic reduction* (aka. imposing constraints, see see [105, section 1.7] or def. A.1). Calling $\mathcal{M}_{\eta' \leftarrow \eta}$ the constraint surface (with respect to which this symplectic reduction takes place), we have a projection from $\mathcal{M}_{\eta' \leftarrow \eta}$ into $\mathcal{M}_\eta$ (to quotient out the gauge orbits), while $\mathcal{M}_{\eta' \leftarrow \eta}$ is naturally embedded in $\mathcal{M}_{\eta'}$ (as a submanifold), hence the hybrid character of the construction. At the quantum level, the identification of $\mathcal{H}_\eta$ with the subspace $\mathrm{Vect}(\zeta_{\eta' \to \eta}) \otimes \mathcal{H}_\eta$ of $\mathcal{H}_{\eta' \to \eta} \otimes \mathcal{H}_\eta \approx \mathcal{H}_{\eta'}$ (theorem 5.9) can then be seen as reflecting the imposition of these constraints. In the special case we are considering here,

$\zeta_{\eta'\to\eta}$ will be the uniform wave-function in $\mathcal{H}_{\eta'\to\eta} \approx L_2(\mathcal{C}_{\eta'\to\eta},\, d\mu_{\eta'\to\eta})$: the corresponding constraints set all impulsion variables in $\mathcal{M}_{\eta'\to\eta}$ to zero, thus making the configuration variables in $\mathcal{C}_{\eta'\to\eta}$ pure gauge. The inductive limit of Hilbert spaces therefore corresponds in this case to a projective limit of *configuration* spaces (as expressed by the formula for $\tau_{\eta'\leftarrow\eta}$ in Prop. 6.5 below, η is extracted from η' through the projection on the second Cartesian factor in $\mathcal{C}_{\eta'} \approx \mathcal{C}_{\eta'\to\eta} \times \mathcal{C}_\eta$). As we will see in subsection 12.2, this result allows to make contact with the Ashtekar-Lewandowski Hilbert space of LQG [19, 8], which is indeed known to arise from of a projective limit of configuration spaces [60, 92].

Proposition 6.5 We consider the same objects as in prop. 6.3, and we now additionally assume:

$$\forall \eta \in \mathcal{L},\ \mu_\eta(\mathcal{C}_\eta) = 1.$$

Then, we can also construct an Hilbert space $\mathcal{H}_\oplus$ as (the completion of) the inductive limit of $\left(\mathcal{L},\, \left(\mathcal{H}_\eta\right)_{\eta\in\mathcal{L}},\, \left(\tau_{\eta'\leftarrow\eta}\right)_{\eta\preccurlyeq\eta'}\right)$, where the injective maps $\tau_{\eta'\leftarrow\eta}$ are defined as:

$$\forall \eta \preccurlyeq \eta' \in \mathcal{L},\ \begin{array}{rl}\tau_{\eta'\leftarrow\eta} : \mathcal{H}_\eta & \to \mathcal{H}_{\eta'} \\ \psi & \mapsto \psi \circ s_{\eta'\to\eta} \circ \varphi_{\eta'\to\eta}\end{array},\ \text{with}\ \begin{array}{rl} s_{\eta'\to\eta} : \mathcal{C}_{\eta'\to\eta} \times \mathcal{C}_\eta & \to \mathcal{C}_\eta \\ (x, y) & \mapsto y\end{array}.$$

There exist maps $\sigma : \overline{\mathcal{S}}_\oplus \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ and $\alpha : \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \to \mathcal{A}_\oplus$ ($\overline{\mathcal{S}}_\oplus$ being the space of (self-adjoint) positive semi-definite, traceclass operators over $\mathcal{H}_\oplus$ and $\mathcal{A}_\oplus$ the algebra of bounded operators on $\mathcal{H}_\oplus$) such that:

1. $\forall \rho \in \overline{\mathcal{S}}_\oplus, \forall A \in \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)},\ \mathrm{Tr}_{\mathcal{H}_\oplus}(\rho\, \alpha(A)) = \mathrm{Tr}\,(\sigma(\rho)\, A)$;
2. σ is injective;
3. $\sigma\left\langle \mathcal{S}_\oplus \right\rangle = \left\{ \left(\rho_\eta\right)_{\eta\in\mathcal{L}} \,\middle|\, \sup_{\eta\in\mathcal{L}} \inf_{\eta'\succcurlyeq\eta} \int_{\mathcal{C}_{\eta'\to\eta}\times\mathcal{C}_{\eta'\to\eta}} dx dx' \int_{\mathcal{C}_\eta} dy\ \ \rho_{\eta'}\left(\varphi^{-1}_{\eta'\to\eta}(x, y); \varphi^{-1}_{\eta'\to\eta}(x', y)\right) = \mathrm{Tr}\,\rho = 1 \right\}$

   where $\mathcal{S}_\oplus$ is the space of density matrices over $\mathcal{H}_\oplus$ and $\rho_\eta(\,\cdot\,;\,\cdot\,)$ is the integral kernel of ρ_η.

Proof This is an application of theorem 5.9, where for $\eta \preccurlyeq \eta' \in \mathcal{L}$, we define:

$$\zeta_{\eta'\to\eta} \equiv 1 \in \mathcal{H}_{\eta'\to\eta}.$$

We have $\forall \eta \preccurlyeq \eta'$, $\|\zeta_{\eta'\to\eta}\| = 1$, since $\mu_{\eta'\to\eta}(\mathcal{C}_{\eta'\to\eta}) = \mu_{\eta'}(\mathcal{C}_{\eta'}) \,/\, \mu_\eta(\mathcal{C}_\eta) = 1$, and $\forall \eta \preccurlyeq \eta' \preccurlyeq \eta''$, $\Phi_{\eta''\to\eta'\to\eta}(\zeta_{\eta''\to\eta'}) \equiv 1 \equiv \zeta_{\eta''\to\eta'} \otimes \zeta_{\eta'\to\eta}$. □

Finally, note that, as far as the construction of the quantum projective state space and observables thereof is concerned, we can actually dispense from having a factorizing system of measures. Indeed, if we just have families $\left(\mu_\eta\right)_{\eta\in\mathcal{L}}$ and $\left(\mu_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'}$ of smooth measures, which do *not* satisfy the compatibility conditions from def. 6.1.3, we can rely on the canonical identification introduced in prop. B.15 to relate the position representation built on the measure $\mu_{\eta'}$ with the one built on the measure $\varphi^{-1}_{\eta'\to\eta,*}(\mu_{\eta'\to\eta} \times \mu_\eta)$. Provided this conversion is incorporated in the definition of the quantum projective structure, one can check that the three-spaces consistency condition still holds. In contrast, the consistency of the measures *is* essential for the inductive construction of prop. 6.5, where it ensures the necessary compatibility of the reference states $\zeta_{\eta'\to\eta}$.

6.2 Holomorphic representation

We now turn to the holomorphic representation. In order to get the scalar product right, we cannot spare, when doing holomorphic quantization, a formulation using a prequantum bundle $\boldsymbol{\mathcal{B}}_\eta$ (see [105, section 8] and appendix B, def. B.2) constructed over $\mathcal{M}_\eta$ (this differs from the previous subsection, for when dealing with configuration representation, the relevant part of the bundle structure is flat and, as a result, the prequantum bundle only needs to be taken into account if a unified context for describing various representations is demanded). Therefore, we begin by examining how to arrange prequantum bundles built on the $\mathcal{M}_\eta$ into a form of factorizing structure suitable for quantization. More precisely, we are looking for a way to connect the $\boldsymbol{\mathcal{B}}_\eta$ bundles that will provide the required tensor product factorizations of the corresponding L_2-spaces of bundle cross-sections (the prequantum Hilbert spaces, see [105, section 8] and def. B.3).

To address this question, we go back to the basics underlying the tensor product decomposition of the L_2-space of complex-valued functions over a Cartesian product $\mathcal{N}\times\mathcal{N}'$: there, the tensor product of a function on $\mathcal{N}$ with a function on $\mathcal{N}'$ is obtained as their pointwise product. Accordingly, what we need in the hermitian line bundle case is an operation to make the 'product' of a point in the bundle above $\mathcal{N}$ with a point in the one above $\mathcal{N}'$, and we want this operation to be valued in the bundle we happen to have above $\mathcal{N}\times\mathcal{N}'$.

Definition 6.6 Let $\mathcal{M}$, $\mathcal{N}'$ and $\mathcal{N}$ be three smooth, finite dimensional, manifolds, and let $\varphi : \mathcal{M} \to \mathcal{N}'\times\mathcal{N}$ be a diffeomorphism. Let $\boldsymbol{\mathcal{B}}_{\mathcal{M}}$, $\boldsymbol{\mathcal{B}}_{\mathcal{N}'}$ and $\boldsymbol{\mathcal{B}}_{\mathcal{N}}$ be hermitian line bundles (def. B.1), respectively with base $\mathcal{M}$, $\mathcal{N}'$ and $\mathcal{N}$. We call a smooth map $\zeta : \boldsymbol{\mathcal{B}}_{\mathcal{N}'}\times\boldsymbol{\mathcal{B}}_{\mathcal{N}} \to \boldsymbol{\mathcal{B}}_{\mathcal{M}}$ a factorization of $\boldsymbol{\mathcal{B}}_{\mathcal{M}}$ compatible with φ iff:

1. $\varphi\circ\boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}_{\mathcal{M}}}\circ\zeta = \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}_{\mathcal{N}'}}\times\boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}_{\mathcal{N}}}$, where $\boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}_{\mathcal{M}}}$, $\boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}_{\mathcal{N}'}}$ and $\boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}_{\mathcal{N}}}$ are the bundles projections of $\boldsymbol{\mathcal{B}}_{\mathcal{M}}$, $\boldsymbol{\mathcal{B}}_{\mathcal{N}'}$ and $\boldsymbol{\mathcal{B}}_{\mathcal{N}}$ respectively;
2. $\forall z'\in\boldsymbol{\mathcal{B}}_{\mathcal{N}'}, \forall z\in\boldsymbol{\mathcal{B}}_{\mathcal{N}},\ \ |\zeta(z', z)| = |z'|\,|z|$;
3. $\forall z'\in\boldsymbol{\mathcal{B}}_{\mathcal{N}'}, \forall z\in\boldsymbol{\mathcal{B}}_{\mathcal{N}}, \forall\lambda',\lambda\in\mathbb{C},\ \ \zeta(\lambda' z', \lambda z) = \lambda'\lambda\,\zeta(z', z)$.

Proposition 6.7 We consider the same objects as in def. 6.6. Moreover, we assume that $\mathcal{N}'$ and $\mathcal{N}$ are equipped with smooth measures $\mu_{\mathcal{N}'}$ and $\mu_{\mathcal{N}}$, and we equip $\mathcal{M}$ with the smooth measure $\mu_{\mathcal{M}} := \varphi_*^{-1}\left(\mu_{\mathcal{N}'}\times\mu_{\mathcal{N}}\right)$. Then, there exists a unique Hilbert space isomorphism:

$$\Phi_\zeta : L_2\left(\mathcal{M}\to\boldsymbol{\mathcal{B}}_{\mathcal{M}}, d\mu_{\mathcal{M}}\right) \to L_2\left(\mathcal{N}'\to\boldsymbol{\mathcal{B}}_{\mathcal{N}'}, d\mu_{\mathcal{N}'}\right)\otimes L_2\left(\mathcal{N}\to\boldsymbol{\mathcal{B}}_{\mathcal{N}}, d\mu_{\mathcal{N}}\right),$$

such that:

$$\forall s'\in L_2\left(\mathcal{N}'\to\boldsymbol{\mathcal{B}}_{\mathcal{N}'}, d\mu_{\mathcal{N}'}\right), \forall s\in L_2\left(\mathcal{N}\to\boldsymbol{\mathcal{B}}_{\mathcal{N}}, d\mu_{\mathcal{N}}\right),\ \ \Phi_\zeta\left(\widetilde{\zeta}(s', s)\right) = s'\otimes s,$$

where $\forall x', x\in\mathcal{N}'\times\mathcal{N}$, $\widetilde{\zeta}(s', s)\circ\varphi^{-1}(x', x) := \zeta\left(s'(x'), s(x)\right)$.

Proof We define $\mathcal{H}_{\mathcal{M}} := L_2\left(\mathcal{M}\to\boldsymbol{\mathcal{B}}_{\mathcal{M}}, d\mu_{\mathcal{M}}\right)$, and similarly $\mathcal{H}_{\mathcal{N}'}$ and $\mathcal{H}_{\mathcal{N}}$.

We first want to prove that $\mathrm{Vect}\left\{\widetilde{\zeta}(s', s)\,\middle|\, s'\in\mathcal{H}_{\mathcal{N}'}\ \ \&\ \ s\in\mathcal{H}_{\mathcal{N}}\right\}$ is dense in $\mathcal{H}_{\mathcal{M}}$. It is well-defined as a vector subspace of $\mathcal{H}_{\mathcal{M}}$ for $\forall s'\in\mathcal{H}_{\mathcal{M}}, \forall s\in\mathcal{H}_{\mathcal{N}},\ \widetilde{\zeta}(s', s)$ is a cross-section of

$\boldsymbol{\mathcal{B}}_{\mathcal{M}}$ (from def. 6.6.1) and $\left\|\widetilde{\zeta}(s',\, s)\right\|_{\mathcal{M}} = \|s'\|_{\mathcal{N}'}\, \|s\|_{\mathcal{N}}$ (from def. 6.6.2 and Fubini's theorem), hence $\widetilde{\zeta}(s',\, s) \in \mathcal{H}_{\mathcal{M}}$.

The cross-sections with compact support are dense in $\mathcal{H}_{\mathcal{M}}$, and, by partition of the unity, they are linear combinations of cross-sections with compact support of the form $W := \varphi^{-1}\left\langle V' \times V\right\rangle$, where V' is a trivialization patch for $\boldsymbol{\mathcal{B}}_{\mathcal{N}'}$ and V a trivialization patch for $\boldsymbol{\mathcal{B}}_{\mathcal{N}}$. Given a non-zero cross-section s, resp. s', of $\boldsymbol{\mathcal{B}}_{\mathcal{N}'}|_{V'}$, resp. $\boldsymbol{\mathcal{B}}_{\mathcal{N}}|_{V}$, we define a non-zero cross-section s'' of $\boldsymbol{\mathcal{B}}_{\mathcal{M}}|_{W}$ by:

$$\forall x',\, x \in V' \times V,\; s'' \circ \varphi(x', x) := \zeta\left(s'(x'),\, s(x)\right).$$

Thus, using the trivialization defined by s', resp. s, s'', to identify the vector subspace of $\mathcal{H}_{\mathcal{N}'}$, resp. $\mathcal{H}_{\mathcal{N}}$, $\mathcal{H}_{\mathcal{M}}$, of cross-sections with compact support in V', resp. V, W, with $L_2\left(V',\, d\mu_{\mathcal{N}'}|_{V'}\right)$, resp. $L_2\left(V,\, d\mu_{\mathcal{N}}|_{V}\right)$, $L_2\left(W,\, d\mu_{\mathcal{N}}|_{W}\right)$, the restriction $\widetilde{\zeta}_W$ of $\widetilde{\zeta}$ to these vector subspaces is given by:

$$\widetilde{\zeta}_W(f'\, s',\, f\, s) = \left(f' \otimes f\right) \circ \left(\varphi|_{W \to V' \times V}\right)\, s'' \text{ (from def. 6.6.3).}$$

Hence, its image is dense in $L_2\left(W,\, d\mu_{\mathcal{N}}|_{W}\right) = L_2\left(V',\, d\mu_{\mathcal{N}'}|_{V'}\right) \otimes L_2\left(W,\, d\mu_{\mathcal{N}}|_{W}\right)$ (which equality follows from def. 6.6.2 and Fubini's theorem).

Now, the application:

$$\begin{array}{rccl} \widetilde{\zeta} : & \mathcal{H}_{\mathcal{N}'} \times \mathcal{H}_{\mathcal{N}} & \to & \mathcal{H}_{\mathcal{M}} \\ & (s',\, s) & \mapsto & \widetilde{\zeta}(s',\, s) \end{array}$$

is a bilinear map (from def. 6.6.3) and satisfies (from def. 6.6.2 and Fubini's theorem):

$$\forall s', t' \in \mathcal{H}_{\mathcal{N}'},\, \forall s, t \in \mathcal{H}_{\mathcal{N}},\; \left\langle \widetilde{\zeta}(s',\, s),\, \widetilde{\zeta}(t',\, t)\right\rangle_{\mathcal{M}} = \langle s',\, t'\rangle_{\mathcal{N}'}\, \langle s,\, t\rangle_{\mathcal{N}}.$$

Hence, there exists a unique Hilbert space isomorphism $\Phi_{\zeta}^{-1} : \mathcal{H}_{\mathcal{N}'} \otimes \mathcal{H}_{\mathcal{N}} \to \overline{\mathrm{Vect}\,\mathrm{Im}\widetilde{\zeta}} = \mathcal{H}_{\mathcal{M}}$, such that $\forall s' \in \mathcal{H}_{\mathcal{N}'},\, \forall s \in \mathcal{H}_{\mathcal{N}},\; \Phi_{\zeta}^{-1}(s' \otimes s) = \widetilde{\zeta}(s',\, s)$. □

With this, we can now present the announced factorizing structure for prequantum bundles. As usual, we need to require an appropriate 'three-spaces consistency' that will support the corresponding consistency of the projective limits we are ultimately interested in (fig. 6.1 looks slightly different from what we had for factorizing system of phase spaces in fig. 2.2, because we are forced to define the maps ζ in the direction opposite to our standard convention for factorizing maps). Note that we also have a compatibility condition involving the connection of the prequantum bundles, that will come into play when (pre-)quantizing observables and expressing the holomorphic condition.

Definition 6.8 Let $(\mathcal{L},\, \mathcal{M},\, \varphi)^{\times}$ be a factorizing system of finite dimensional phase spaces (def. 2.12). A factorizing system of prequantum bundles for $(\mathcal{L},\, \mathcal{M},\, \varphi)^{\times}$ is a quadruple:

$$\left(\left(\boldsymbol{\mathcal{B}}_{\eta},\, \nabla_{\eta}\right)_{\eta \in \mathcal{L}},\, \left(\boldsymbol{\mathcal{B}}_{\eta' \to \eta},\, \nabla_{\eta' \to \eta}\right)_{\eta \preccurlyeq \eta'},\, \left(\zeta_{\eta' \to \eta}\right)_{\eta \preccurlyeq \eta'},\, \left(\zeta_{\eta'' \to \eta' \to \eta}\right)_{\eta \preccurlyeq \eta' \preccurlyeq \eta''}\right)$$

such that:

1. $\forall \eta \in \mathcal{L}$, $\left(\boldsymbol{\mathcal{B}}_{\eta},\, \nabla_{\eta}\right)$ is a prequantum bundle for $\mathcal{M}_{\eta}$ (def. B.2);

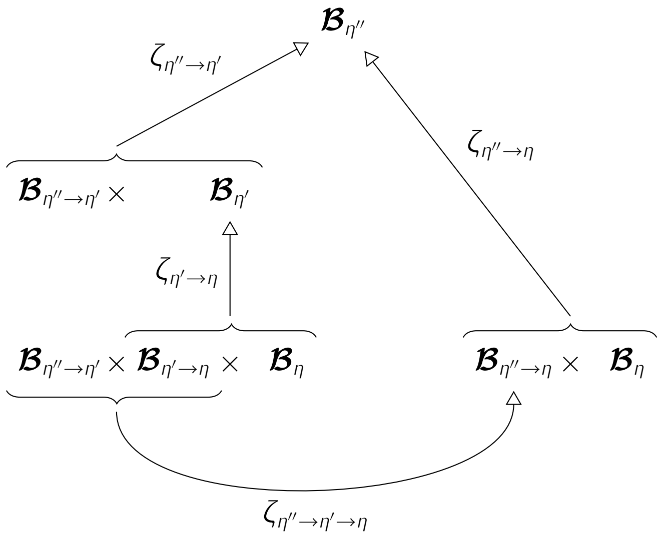


Figure 6.1 – Three-spaces consistency for factorizing systems of prequantum bundles

2. $\forall \eta \preccurlyeq \eta' \in \mathcal{L}$, $\left(\boldsymbol{\mathcal{B}}_{\eta'\to\eta}, \nabla_{\eta'\to\eta}\right)$ is a prequantum bundle for $\mathcal{M}_{\eta'\to\eta}$ (except for the case $\eta = \eta'$: $\mathcal{M}_{\eta\to\eta}$ has only one element and $\boldsymbol{\mathcal{B}}_{\eta\to\eta} = \mathbb{C}$);
3. $\forall \eta \preccurlyeq \eta' \in \mathcal{L}$, $\zeta_{\eta'\to\eta} : \boldsymbol{\mathcal{B}}_{\eta'\to\eta} \times \boldsymbol{\mathcal{B}}_{\eta} \to \boldsymbol{\mathcal{B}}_{\eta'}$ is a factorization of $\boldsymbol{\mathcal{B}}_{\eta'}$ compatible with $\varphi_{\eta'\to\eta} : \mathcal{M}_{\eta'} \to \mathcal{M}_{\eta'\to\eta} \times \mathcal{M}_{\eta}$;
4. $\forall \eta \preccurlyeq \eta' \in \mathcal{L}$, $\forall y \in \boldsymbol{\mathcal{B}}_{\eta'\to\eta}$, $\forall z \in \boldsymbol{\mathcal{B}}_{\eta}$,

$$\left[T_{y,z}\, \zeta_{\eta'\to\eta}\right] \left\langle \mathrm{Hor}_y(\boldsymbol{\mathcal{B}}_{\eta'\to\eta}, \nabla_{\eta'\to\eta}) \times \mathrm{Hor}_z(\boldsymbol{\mathcal{B}}_{\eta}, \nabla_{\eta}) \right\rangle = \mathrm{Hor}_{\zeta_{\eta'\to\eta}(y,z)}(\boldsymbol{\mathcal{B}}_{\eta'}, \nabla_{\eta'})\,, \tag{6.8.1}$$

where $\mathrm{Hor}_z(\boldsymbol{\mathcal{B}}_{\eta}, \nabla_{\eta})$ is defined as the ∇_η-horizontal subspace of $T_z(\boldsymbol{\mathcal{B}}_{\eta})$ for $z \in \boldsymbol{\mathcal{B}}_{\eta}$, and $\mathrm{Hor}_y(\boldsymbol{\mathcal{B}}_{\eta'\to\eta}, \nabla_{\eta'\to\eta})$ is defined similarly for $y \in \boldsymbol{\mathcal{B}}_{\eta'\to\eta}$;
5. $\forall \eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}$, $\zeta_{\eta''\to\eta'\to\eta} : \boldsymbol{\mathcal{B}}_{\eta''\to\eta'} \times \boldsymbol{\mathcal{B}}_{\eta'\to\eta} \to \boldsymbol{\mathcal{B}}_{\eta''\to\eta}$ is a smooth map such that:

$$\zeta_{\eta''\to\eta} \circ (\zeta_{\eta''\to\eta'\to\eta} \times \mathrm{id}_{\boldsymbol{\mathcal{B}}_\eta}) = \zeta_{\eta''\to\eta'} \circ (\mathrm{id}_{\boldsymbol{\mathcal{B}}_{\eta''\to\eta'}} \times \zeta_{\eta'\to\eta})\,. \tag{6.8.2}$$

Def. 6.8 seems to require a lot, so it is reassuring that, at least in the topologically trivial case, we can construct such a structure for any factorizing system of phase spaces satisfying nothing but the quantization rule [105, section 8.3], which is anyhow mandatory to ensure the existence of prequantum bundles for the $\mathcal{M}_\eta$.

Theorem 6.9 Let $(\mathcal{L}, \mathcal{M}, \varphi)^\times$ be a factorizing system of finite dimensional phase spaces such that:

1. $\forall \eta \in \mathcal{L}$, $\mathcal{M}_\eta$ is simply-connected;
2. $\forall \eta \in \mathcal{L}$, $\forall\, S$ a closed oriented 2-surface in $\mathcal{M}_\eta$, $\displaystyle\int_S \Omega_\eta \in 2\pi\,\mathbb{Z}$, where Ω_η is the symplectic structure of $\mathcal{M}_\eta$.

Then there exists a factorizing system of prequantum bundles for $(\mathcal{L}, \mathcal{M}, \varphi)^\times$.

Proof For $\eta \preccurlyeq \eta' \in \mathcal{L}$, we have that $\mathcal{M}_{\eta'\to\eta}$ is simply-connected, otherwise $\mathcal{M}_{\eta'} \approx \mathcal{M}_{\eta'\to\eta} \times \mathcal{M}_\eta$ would not be simply-connected.

Besides, for any oriented 2-surface $S_{\eta'}$ in $\mathcal{M}_{\eta'}$, we have:

$$\int_{S_{\eta'}} \Omega_{\eta'} = \int_{\varphi_{\eta'\to\eta} \circ S_{\eta'}} \varphi^{-1,*}_{\eta'\to\eta}\ \Omega_{\eta'}$$

$$= \int_{\varphi_{\eta'\to\eta} \circ S_{\eta'}} \Omega_{\eta'\to\eta} \times \Omega_\eta = \int_{S_{\eta'\to\eta}} \Omega_{\eta'\to\eta} + \int_{S_\eta} \Omega_\eta \,, \tag{6.9.1}$$

where $S_{\eta'\to\eta}$, resp. S_η, is the projection on $\mathcal{M}_{\eta'\to\eta}$, resp. $\mathcal{M}_\eta$, of $\varphi_{\eta'\to\eta} \circ S_{\eta'}$ (which is an oriented 2-surface in $\mathcal{M}_{\eta'\to\eta} \times \mathcal{M}_\eta$). In particular, if $S_{\eta'\to\eta}$ is a closed oriented 2-surface in $\mathcal{M}_{\eta'\to\eta}$, applying eq. (6.9.1) to $S_{\eta'} := \varphi^{-1}_{\eta'\to\eta} \circ \left(S_{\eta'\to\eta} \times \{x_\eta\}\right)$, where x_η is any point in $\mathcal{M}_\eta$, gives:

$$\int_{S_{\eta'\to\eta}} \Omega_{\eta'\to\eta} = \int_{S_{\eta'}} \Omega_{\eta'} \in 2\pi\,\mathbb{Z}\,.$$

Let $\left(x^o_\eta\right)_{\eta\in\mathcal{L}} \in \mathcal{S}^\times_{(\mathcal{L},\mathcal{M},\varphi)}$ and let $\eta \in \mathcal{L}$. The construction in [105, section 8.3] tells us that, thanks to the conditions 6.9.1 and 6.9.2, we can construct a prequantum bundle $\left(\boldsymbol{\mathcal{B}}_\eta,\, \nabla_\eta\right)$ for $\mathcal{M}_\eta$ in such a way that $\boldsymbol{\mathcal{B}}_\eta$ can be identified with the equivalence classes in:

$$\left\{\left(x_\eta, z_\eta, \gamma_\eta\right) \;\middle|\; x_\eta \in \mathcal{M}_\eta,\ z_\eta \in \mathbb{C},\ \gamma_\eta \text{ is a piecewise smooth path from } x^o_\eta \text{ to } x_\eta\right\},$$

for the equivalence relation:

$$\left(\left(x_\eta, z_\eta, \gamma_\eta\right) \simeq \left(x'_\eta, z'_\eta, \gamma'_\eta\right)\right) \;\Leftrightarrow\; \begin{cases} x_\eta &= x'_\eta \\ z'_\eta &= z_\eta \exp\left(-i \int_{\Sigma_\eta(\gamma_\eta,\gamma'_\eta)} \Omega_\eta\right) \end{cases},$$

where $\Sigma_\eta(\gamma_\eta, \gamma'_\eta)$ is any oriented 2-surface in $\mathcal{M}_\eta$ such that $\partial\Sigma_\eta(\gamma_\eta, \gamma'_\eta) = {\gamma'_\eta}^{-1} . \gamma_\eta$. Moreover, the ∇_η-parallel transport along some path γ'_η in $\mathcal{M}_\eta$ is then given by:

$$P^{\nabla_\eta}_{\gamma'_\eta}\left(\left[\gamma'_\eta(0), z_\eta, \gamma_\eta\right]_\simeq\right) = \left[\gamma'_\eta(1), z_\eta, \gamma'_\eta . \gamma_\eta\right]_\simeq .$$

Since we proved above that, for all $\eta \preccurlyeq \eta'$, $\mathcal{M}_{\eta'\to\eta}$ also fulfills these conditions, we can make the same construction to obtain a prequantum bundle $\left(\boldsymbol{\mathcal{B}}_{\eta'\to\eta},\, \nabla_{\eta'\to\eta}\right)$, using as origin the point $x^o_{\eta'\to\eta} \in \mathcal{M}_{\eta'\to\eta}$, defined by $\varphi_{\eta'\to\eta}(x^o_{\eta'}) = (x^o_{\eta'\to\eta},\, x^o_\eta)$.

Now, for $\eta \preccurlyeq \eta' \in \mathcal{L}$, we define $\zeta_{\eta'\to\eta} : \boldsymbol{\mathcal{B}}_{\eta'\to\eta} \times \boldsymbol{\mathcal{B}}_\eta \to \boldsymbol{\mathcal{B}}_{\eta'}$ by:

$$\zeta_{\eta'\to\eta}\left([x_{\eta'\to\eta}, z_{\eta'\to\eta}, \gamma_{\eta'\to\eta}]_\simeq\,,\, [x_\eta, z_\eta, \gamma_\eta]_\simeq\right) := \left[\varphi^{-1}_{\eta'\to\eta}(x_{\eta'\to\eta},\, x_\eta),\, z_{\eta'\to\eta}\, z_\eta,\, \varphi^{-1}_{\eta'\to\eta}(\gamma_{\eta'\to\eta},\, \gamma_\eta)\right]_\simeq .$$

This is a well-defined map, for we have, using eq. (6.9.1):

$$\exp\left(-i \int_{\Sigma_{\eta'}\left(\varphi^{-1}_{\eta'\to\eta}(\gamma_{\eta'\to\eta},\, \gamma_\eta),\, \varphi^{-1}_{\eta'\to\eta}(\gamma'_{\eta'\to\eta},\, \gamma'_\eta)\right)} \Omega_{\eta'}\right) =$$

$$= \exp\left(-i \int_{\Sigma_{\eta'\to\eta}\left(\gamma_{\eta'\to\eta},\, \gamma'_{\eta'\to\eta}\right)} \Omega_{\eta'\to\eta}\right) \exp\left(-i \int_{\Sigma_\eta\left(\gamma_\eta,\, \gamma'_\eta\right)} \Omega_\eta\right).$$

Moreover, we can check that it fulfills defs. 6.6.1 to 6.6.3.

Let $x_{\eta'} \in \mathcal{M}_{\eta'}$ and $(x_{\eta'\to\eta}, x_\eta) = \varphi_{\eta'\to\eta}(x_{\eta'})$. Let γ_η and $\gamma_{\eta'\to\eta}$ be piecewise smooth paths from x^o_η to x_η, and $x^o_{\eta'\to\eta}$ to $x_{\eta'\to\eta}$, respectively. We can choose local coordinates around $x_{\eta'\to\eta}$ in $\mathcal{M}_{\eta'\to\eta}$ and around x_η in $\mathcal{M}_\eta$. Hence, we have diffeomorphisms $\psi_{\eta'\to\eta} : [-1,1]^{d_{\eta'\to\eta}} \to U_{\eta'\to\eta}$, resp. $\psi_\eta : [-1,1]^{d_\eta} \to U_\eta$, where $d_{\eta'\to\eta} := \dim \mathcal{M}_{\eta'\to\eta}$, resp. $d_\eta := \dim \mathcal{M}_\eta$, and $U_{\eta'\to\eta}$, resp. U_η, is an open neighborhood of $x_{\eta'\to\eta}$ in $\mathcal{M}_{\eta'\to\eta}$, resp. of x_η in $\mathcal{M}_\eta$. This provides us local trivializations of the bundles $\boldsymbol{\mathcal{B}}_{\eta'}$, $\boldsymbol{\mathcal{B}}_{\eta'\to\eta}$ and $\boldsymbol{\mathcal{B}}_\eta$, by:

$$\forall r_{\eta'\to\eta} \in [-1,1]^{d_{\eta'\to\eta}}, \forall z_{\eta'\to\eta} \in \mathbb{C},$$

$$\Psi_{\eta'\to\eta}(r_{\eta'\to\eta}, z_{\eta'\to\eta}) = \left[\psi_{\eta'\to\eta}(r_{\eta'\to\eta}), z_{\eta'\to\eta}, \chi_{r_{\eta'\to\eta}} \cdot \gamma_{\eta'\to\eta}\right]_\simeq,$$

$$\forall r_\eta \in [-1,1]^{d_\eta}, \forall z_\eta \in \mathbb{C},$$

$$\Psi_\eta(r_\eta, z_\eta) = \left[\psi_\eta(r_\eta), z_\eta, \chi_{r_\eta} \cdot \gamma_\eta\right]_\simeq,$$

$$\forall r_{\eta'\to\eta}, r_\eta \in [-1,1]^{d_{\eta'\to\eta}} \times [-1,1]^{d_\eta}, \forall z_{\eta'} \in \mathbb{C},$$

$$\Psi_{\eta'}(r_{\eta'\to\eta}, r_\eta, z_{\eta'}) = \left[\varphi^{-1}_{\eta'\to\eta}\left(\psi_{\eta'\to\eta}(r_{\eta'\to\eta}), \psi_\eta(r_\eta)\right), z_{\eta'}, \varphi^{-1}_{\eta'\to\eta}\left(\chi_{r_{\eta'\to\eta}} \cdot \gamma_{\eta'\to\eta}, \chi_{r_\eta} \cdot \gamma_\eta\right)\right]_\simeq,$$

where $\chi_{r_{\eta'\to\eta}} : \tau \mapsto \psi_{\eta'\to\eta}(\tau\, r_{\eta'\to\eta})$ and $\chi_{r_\eta} : \tau \mapsto \psi_\eta(\tau\, r_\eta)$.
And we have:

$$\forall r_{\eta'\to\eta}, r_\eta \in [-1,1]^{d_{\eta'\to\eta}} \times [-1,1]^{d_\eta}, \forall z_{\eta'\to\eta}, z_\eta \in \mathbb{C},$$

$$\Psi^{-1}_{\eta'} \circ \zeta\left(\Psi_{\eta'\to\eta}(r_{\eta'\to\eta}, z_{\eta'\to\eta}), \Psi_\eta(r_\eta, z_\eta)\right) = (r_{\eta'\to\eta}, r_\eta, z_{\eta'\to\eta}\, z_\eta),$$

Therefore, $\zeta_{\eta'\to\eta}$ is smooth.

Then, for $\gamma'_{\eta'\to\eta}$ a path in $\mathcal{M}_{\eta'\to\eta}$, and γ'_η a path in $\mathcal{M}_\eta$, we have:

$$P^{\nabla_{\eta'}}_{\varphi^{-1}_{\eta'\to\eta}(\gamma'_{\eta'\to\eta}, \gamma'_\eta)} \circ \zeta_{\eta'\to\eta}\left(\left[\gamma'_{\eta'\to\eta}(0), z_{\eta'\to\eta}, \gamma_{\eta'\to\eta}\right]_\simeq, \left[\gamma'_\eta(0), z_\eta, \gamma_\eta\right]_\simeq\right) =$$

$$= \left[\varphi^{-1}_{\eta'\to\eta}\left(\gamma'_{\eta'\to\eta}(1), \gamma'_\eta(1)\right), z_{\eta'\to\eta}\, z_\eta, \varphi^{-1}_{\eta'\to\eta}\left(\gamma'_{\eta'\to\eta} \cdot \gamma_{\eta'\to\eta}, \gamma'_\eta \cdot \gamma_\eta\right)\right]_\simeq$$

$$= \zeta_{\eta'\to\eta}\left(P^{\nabla_{\eta'\to\eta}}_{\gamma'_{\eta'\to\eta}}\left(\left[\gamma'_{\eta'\to\eta}(0), z_{\eta'\to\eta}, \gamma_{\eta'\to\eta}\right]_\simeq\right), P^{\nabla_\eta}_{\gamma'_\eta}\left(\left[\gamma'_\eta(0), z_\eta, \gamma_\eta\right]_\simeq\right)\right),$$

hence $P^{\nabla_{\eta'}}_{\varphi^{-1}_{\eta'\to\eta}(\gamma'_{\eta'\to\eta}, \gamma'_\eta)} \circ \zeta_{\eta'\to\eta} = \zeta_{\eta'\to\eta} \circ \left(P^{\nabla_{\eta'\to\eta}}_{\gamma'_{\eta'\to\eta}}, P^{\nabla_\eta}_{\gamma'_\eta}\right)$. Therefore, eq. (6.8.1) is fulfilled.

Lastly, for $\eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}$, we can in a similar way define a map $\zeta_{\eta''\to\eta'\to\eta} : \boldsymbol{\mathcal{B}}_{\eta''\to\eta'} \times \boldsymbol{\mathcal{B}}_{\eta'\to\eta}$ and we have:

$$\zeta_{\eta''\to\eta} \circ (\zeta_{\eta''\to\eta'\to\eta} \times \mathrm{id}_{\boldsymbol{\mathcal{B}}_\eta})\left([x_{\eta''\to\eta'}, z_{\eta''\to\eta'}, \gamma_{\eta''\to\eta'}]_\simeq, [x_{\eta'\to\eta}, z_{\eta'\to\eta}, \gamma_{\eta'\to\eta}]_\simeq, [x_\eta, z_\eta, \gamma_\eta]_\simeq\right) =$$

$$= \left[\varphi^{-1}_{\eta''\to\eta}\left(\varphi^{-1}_{\eta''\to\eta'\to\eta}(x_{\eta''\to\eta'}, x_{\eta'\to\eta}), x_\eta\right), z_{\eta''\to\eta'}\, z_{\eta'\to\eta}\, z_\eta, \varphi^{-1}_{\eta''\to\eta}\left(\varphi^{-1}_{\eta''\to\eta'\to\eta}(\gamma_{\eta''\to\eta'}, \gamma_{\eta'\to\eta}), \gamma_\eta\right)\right]_\simeq$$

$$= \left[\varphi^{-1}_{\eta''\to\eta'}\left(x_{\eta''\to\eta'}, \varphi^{-1}_{\eta'\to\eta}(x_{\eta'\to\eta}, x_\eta)\right), z_{\eta''\to\eta'}\, z_{\eta'\to\eta}\, z_\eta, \varphi^{-1}_{\eta''\to\eta'}\left(\gamma_{\eta''\to\eta'}, \varphi^{-1}_{\eta'\to\eta}(\gamma_{\eta'\to\eta}, \gamma_\eta)\right)\right]_\simeq$$

(using eq. (2.11.1))

$$= \zeta_{\eta''\to\eta'} \circ \left(\mathrm{id}_{\boldsymbol{\mathcal{B}}_{\eta''\to\eta'}} \times \zeta_{\eta'\to\eta}\right) \left([x_{\eta''\to\eta'},\, z_{\eta''\to\eta'},\, y_{\eta''\to\eta'}]_{\simeq}\,,\, [x_{\eta'\to\eta},\, z_{\eta'\to\eta},\, y_{\eta'\to\eta}]_{\simeq}\,,\, [x_\eta,\, z_\eta,\, y_\eta]_{\simeq}\right),$$

therefore eq. (6.8.2) holds. □

The last ingredient we need in order to perform prequantization are measures on the $\mathcal{M}_\eta$ and $\mathcal{M}_{\eta'\to\eta}$, and they should be compatible, like we asked when setting up the configuration representation. But this is in fact something we can get automatically and in a very straightforward way from the structure $(\mathcal{L}, \mathcal{M}, \varphi)^\times$, since a symplectic form gives us a natural volume form and the compatibility of the symplectic forms is enough to ensure the compatibility of their associated volume form.

Proposition 6.10 Let $(\mathcal{L}, \mathcal{M}, \varphi)^\times$ be a factorizing system of finite dimensional phase spaces. We define:

1. for $\eta \in \mathcal{L}$, the volume form $\omega_\eta := \frac{1}{(d_\eta/2)!}\, \Omega_\eta \wedge \ldots \wedge \Omega_\eta = \frac{1}{(d_\eta/2)!}\, \Omega_\eta^{\wedge d_\eta/2}$ on $\mathcal{M}_\eta$ (where $d_\eta = \dim \mathcal{M}_\eta$) and the corresponding smooth measure μ_η on $\mathcal{M}_\eta$;
2. for $\eta \prec \eta' \in \mathcal{L}$, the volume form $\omega_{\eta'\to\eta} := \frac{1}{(d_{\eta'\to\eta}/2)!}\, \Omega_{\eta'\to\eta} \wedge \ldots \wedge \Omega_{\eta'\to\eta} = \frac{1}{(d_{\eta'\to\eta}/2)!}\, \Omega_{\eta'\to\eta}^{\wedge d_{\eta'\to\eta}/2}$ on $\mathcal{M}_{\eta'\to\eta}$ (where $d_{\eta'\to\eta} = \dim \mathcal{M}_{\eta'\to\eta}$) and the corresponding smooth measure $\mu_{\eta'\to\eta}$ on $\mathcal{M}_{\eta'\to\eta}$.

Then, this equips $(\mathcal{L}, \mathcal{M}, \varphi)^\times$ with a structure of factorizing system of measured manifolds (def. 6.1).

Proof That ω_η, resp. $\omega_{\eta'\to\eta}$, is a nowhere-vanishing top-dimensional form on $\mathcal{M}_\eta$, resp. $\mathcal{M}_{\eta'\to\eta}$, can be checked in local Darboux coordinates.

What is left to prove is the compatibility of these definitions of the volume forms with the maps $\varphi_{\eta'\to\eta}$ and $\varphi_{\eta''\to\eta'\to\eta}$ (def. 6.1.3). For $\eta \prec \eta'$, we have:

$$\varphi^{-1,*}_{\eta'\to\eta}\, \omega_{\eta'} = \frac{1}{(d_{\eta'}/2)!}\, \left(\varphi^{-1,*}_{\eta'\to\eta}\, \Omega_{\eta'}\right)^{\wedge d_{\eta'}/2}$$

$$= \frac{1}{(d_{\eta'}/2)!}\, \left(\Omega_{\eta'\to\eta} \times \Omega_\eta\right)^{\wedge d_{\eta'}/2} \text{ (since } \varphi_{\eta'\to\eta} \text{ is a symplectomorphism)}$$

$$= \frac{1}{(d_{\eta'\to\eta}/2)!\,(d_\eta/2)!}\, \left(\Omega_{\eta'\to\eta}\right)^{\wedge d_{\eta'\to\eta}/2} \wedge \left(\Omega_\eta\right)^{\wedge d_\eta/2}$$

$$= \omega_{\eta'\to\eta} \wedge \omega_\eta\,,$$

hence $\varphi_{\eta'\to\eta,*}\, \mu_{\eta'} = \mu_{\eta'\to\eta} \times \mu_\eta$, and similarly, for $\eta \prec \eta' \prec \eta''$:

$$\varphi_{\eta''\to\eta'\to\eta,*}\, \mu_{\eta''\to\eta} = \mu_{\eta''\to\eta'} \times \mu_{\eta'\to\eta}\,.$$

□

On the grounds of the preliminaries developed so far, the prequantization of a factorizing system of prequantum bundles is actually very similar to what we did for the position quantization, and, again, the link connecting the classical structure and the (pre-)quantum one is demonstrated by exposing the correspondence between observables.

Proposition 6.11 Let $(\mathcal{L}, \mathcal{M}, \varphi)^\times$ be a factorizing system of finite dimensional phase spaces, equipped with a structure of factorizing system of measured manifolds according to prop. 6.10, and let $\left(\left(\boldsymbol{\mathcal{B}}_\eta,\, \nabla_\eta\right)_{\eta\in\mathcal{L}},\, \left(\boldsymbol{\mathcal{B}}_{\eta'\to\eta},\, \nabla_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'},\, \left(\zeta_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'},\, \left(\zeta_{\eta''\to\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'\preccurlyeq\eta''}\right)$ be a factorizing system of prequantum bundles for $(\mathcal{L},\, \mathcal{M},\, \varphi)^\times$.

We define:

1. for $\eta \in \mathcal{L}$, $\mathcal{H}^{\text{preQ}}_\eta := L_2(\mathcal{M}_\eta \to \boldsymbol{\mathcal{B}}_\eta,\, d\mu_\eta)$;
2. for $\eta \prec \eta' \in \mathcal{L}$, $\mathcal{H}^{\text{preQ}}_{\eta'\to\eta} := L_2(\mathcal{M}_{\eta'\to\eta} \to \boldsymbol{\mathcal{B}}_{\eta'\to\eta},\, d\mu_{\eta'\to\eta})$, and $\Phi^{\text{preQ}}_{\eta'\to\eta} := \Phi_{\zeta_{\eta'\to\eta}} : \mathcal{H}^{\text{preQ}}_{\eta'} \to \mathcal{H}^{\text{preQ}}_{\eta'\to\eta} \otimes \mathcal{H}^{\text{preQ}}_\eta$.

Then, we can complete these elements into a projective system of quantum state spaces $(\mathcal{L}, \mathcal{H}^{\text{preQ}}, \Phi^{\text{preQ}})^\otimes$.

Proof The proof works like the one for prop. 6.3, using prop. 6.7 and:

$$\forall \eta \prec \eta' \prec \eta'',\ \forall s'' \in \mathcal{H}^{\text{preQ}}_{\eta''\to\eta'},\ \forall s' \in \mathcal{H}^{\text{preQ}}_{\eta'\to\eta},\ \forall s \in \mathcal{H}^{\text{preQ}}_\eta,$$

$$\Phi^{-1}_{\zeta_{\eta''\to\eta'}} \circ \left(\text{id}_{\mathcal{H}^{\text{preQ}}_{\eta''\to\eta'}} \otimes \Phi^{-1}_{\zeta_{\eta'\to\eta}}\right)(s'' \otimes s' \otimes s) = \widetilde{\zeta}_{\eta''\to\eta'}\left(s'',\, \widetilde{\zeta}_{\eta'\to\eta}(s', s)\right)$$

$$= \widetilde{\zeta}_{\eta''\to\eta}\left(\widetilde{\zeta}_{\eta''\to\eta'\to\eta}(s'', s'),\, s\right) \text{ (from eq. (6.8.2) and eq. (2.11.1))}$$

$$= \Phi^{-1}_{\zeta_{\eta''\to\eta}} \circ \left(\Phi^{-1}_{\zeta_{\eta''\to\eta'\to\eta}} \otimes \text{id}_{\mathcal{H}^{\text{preQ}}_\eta}\right)\left(s'' \otimes s' \otimes s\right).$$

□

Proposition 6.12 We consider the same objects as in prop. 6.11. Let $f_\eta \in C^\infty(\mathcal{M}_\eta, \mathbb{R})$ such that X_{f_η} is a *complete* vector field on $\mathcal{M}_\eta$. Let $f_{\eta'} \in C^\infty(\mathcal{M}_{\eta'}, \mathbb{R})$, such that $f_\eta \sim f_{\eta'}$ (the equivalence relation is defined in eq. (2.4.1), where we use $\pi_{\eta'\to\eta}$ from eq. (2.13.1)). Then, $X_{f_{\eta'}}$ is a *complete* vector field on $\mathcal{M}_{\eta'}$. Defining the prequantization $\widehat{f_\eta}$ of f_η, resp. $\widehat{f_{\eta'}}$ of $f_{\eta'}$, as a densely defined, essentially self-adjoint, operator on $\mathcal{H}^{\text{preQ}}_\eta$, resp. $\mathcal{H}^{\text{preQ}}_{\eta'}$ (def. B.3 and prop. B.4), we moreover have $\widehat{f_\eta} \sim \widehat{f_{\eta'}}$ (with the equivalence relation defined in eq. (5.3.2)).

Hence, defining:

$$\mathcal{O}^{\times,\text{preQ}}_{(\mathcal{L},\mathcal{M},\varphi)} := \left\{f \in \mathcal{O}^\times_{(\mathcal{L},\mathcal{M},\varphi)} \,\middle|\, \exists\, f_\eta \in f \,/\, X_{f_\eta} \text{ is complete}\right\}$$

(where $\mathcal{O}^\times_{(\mathcal{L},\mathcal{M},\varphi)}$ has been introduced in prop. 2.13), we can map a classical observable $f = \left[f_\eta\right]_\sim \in \mathcal{O}^{\times,\text{preQ}}_{(\mathcal{L},\mathcal{M},\varphi)}$ to a prequantum observable $\widehat{f} := \left[\widehat{f_\eta}\right]_\sim \in \mathcal{O}^\otimes_{\left(\mathcal{L},\mathcal{H}^{\text{preQ}},\Phi^{\text{preQ}}\right)}$ (prop. 5.5).

Proof Let $f_\eta \in C^\infty(\mathcal{M}_\eta, \mathbb{R})$ and let $\eta' \succcurlyeq \eta$. Using the definition of $\pi_{\eta'\to\eta}$ in terms of the symplectomorphism $\varphi_{\eta'\to\eta}$, we have $X_{f_\eta\circ\pi_{\eta'\to\eta}} = T\varphi^{-1}_{\eta'\to\eta}\left(0,\, X_{f_\eta}\right)$. Hence, X_{f_η} is a complete vector field iff

$X_{f_\eta \circ \pi_{\eta' \to \eta}}$ is.

Now, let $s' \in \Phi^{\mathrm{preQ},-1}_{\eta' \to \eta} \left\langle \mathcal{H}^{\mathrm{preQ}}_{\eta' \to \eta} \otimes \mathcal{D}^{\mathrm{preQ}}_{\eta} \right\rangle$ (where $\mathcal{D}^{\mathrm{preQ}}_{\eta} \subset \mathcal{H}^{\mathrm{preQ}}_{\eta}$ is the dense domain of $\widehat{f}_\eta$ and the $\otimes$ is to be understood as a tensor product of vector spaces). We define:

$$\Phi^{\mathrm{preQ}}_{\eta' \to \eta}(s') =: \sum_\alpha t^\alpha \otimes s^\alpha \text{, with } \forall \alpha,\ s^\alpha \in \mathcal{D}^{\mathrm{preQ}}_{\eta}.$$

Then, we have:

$$\forall (y, x) \in \mathcal{M}_{\eta' \to \eta} \times \mathcal{M}_\eta,\ \left[\Phi^{\mathrm{preQ},-1}_{\eta' \to \eta} \circ \left(\mathrm{id}_{\mathcal{H}^{\mathrm{preQ}}_{\eta' \to \eta}} \otimes \widehat{f}_\eta\right) \circ \Phi^{\mathrm{preQ}}_{\eta' \to \eta}(s')\right] \circ \varphi^{-1}_{\eta' \to \eta}(y, x) =$$

$$= \sum_\alpha \zeta_{\eta' \to \eta}\left[t^\alpha(y),\ \left(f_\eta(x)\, s^\alpha(x) + i \nabla_{\eta, X_{f_\eta}} s^\alpha(x)\right)\right]$$

$$= \sum_\alpha f_\eta \circ \pi_{\eta' \to \eta} \circ \varphi^{-1}_{\eta' \to \eta}(y, x) \left[\widetilde{\zeta}_{\eta' \to \eta}\left(t^\alpha,\ s^\alpha\right)\right] \circ \varphi^{-1}_{\eta' \to \eta}(y, x) +$$

$$+ i \sum_\alpha \left[\nabla_{\eta',\, T\varphi^{-1}_{\eta' \to \eta}(0, X_{f_\eta})}\, \widetilde{\zeta}_{\eta' \to \eta}\left(t^\alpha,\ s^\alpha\right)\right] \circ \varphi^{-1}_{\eta' \to \eta}(y, x) \ \text{(using eq. (6.8.1) and def. 6.6.3)}$$

$$= \left[\left(f_\eta \circ \pi_{\eta' \to \eta}\right) s' + i\, \nabla_{\eta',\, X_{f_\eta \circ \pi_{\eta' \to \eta}}} s'\right] \circ \varphi^{-1}_{\eta' \to \eta}(y, x)$$

$$= \left[\widehat{f_\eta \circ \pi_{\eta' \to \eta}}\,(s')\right] \circ \varphi^{-1}_{\eta' \to \eta}(y, x).$$

Therefore, we have $\forall f_\eta \in C^\infty(\mathcal{M}_\eta, \mathbb{R})$, $\forall \eta' \succcurlyeq \eta$, $\widehat{f}_\eta \sim \widehat{f_\eta \circ \pi_{\eta' \to \eta}}$.

Hence, for any $\eta, \eta' \in \mathcal{L}$ and any $f_\eta \in C^\infty(\mathcal{M}_\eta, \mathbb{R})$, $f_{\eta'} \in C^\infty(\mathcal{M}_{\eta'}, \mathbb{R})$ such that $f_\eta \sim f_{\eta'}$, X_{f_η} is complete iff $X_{f_{\eta'}}$ is, and we have $\widehat{f}_\eta \sim \widehat{f}_{\eta'}$. □

Finally, we obtain the advertised holomorphic representation for a choice of Kähler structure on the symplectic manifolds $\mathcal{M}_\eta$. Requiring the factorizing maps to be holomorphic is enough to ensure that the holomorphic subspaces of the prequantum Hilbert spaces $\mathcal{H}_\eta$ set up above will correctly decompose over the already arranged tensor product factorizations, as can be shown by proving the corresponding factorizing properties of the orthogonal projections on these (closed) vector subspaces.

Definition 6.13 A factorizing system of Kähler manifolds is a factorizing system of phase spaces $(\mathcal{L}, \mathcal{M}, \varphi)^\times$ (def. 2.12) such that:

1. for all $\eta \in \mathcal{L}$, $\mathcal{M}_\eta$ is equipped with a complex structure J_η such that $\left(\mathcal{M}_\eta,\ \Omega_\eta,\ J_\eta\right)$ is a Kähler manifold (def. B.5);
2. for all $\eta \prec \eta' \in \mathcal{L}$, $\mathcal{M}_{\eta' \to \eta}$ is equipped with a complex structure $J_{\eta' \to \eta}$ such that $\left(\mathcal{M}_{\eta' \to \eta},\ \Omega_{\eta' \to \eta},\ J_{\eta' \to \eta}\right)$ is a Kähler manifold;
3. for all $\eta \prec \eta' \in \mathcal{L}$, $\varphi_{\eta' \to \eta}$ is holomorphic, and for all $\eta \prec \eta' \prec \eta'' \in \mathcal{L}$, $\varphi_{\eta'' \to \eta' \to \eta}$ is holomorphic.

Proposition 6.14 We consider the same objects as in prop. 6.11, but we now moreover assume that

$(\mathcal{L}, \mathcal{M}, \varphi)^\times$ is a factorizing system of Kähler manifolds. We define:

1. for $\eta \in \mathcal{L}$, $\mathcal{H}^{\text{Holo}}_\eta := \mathcal{H}^{\text{preQ}}_\eta \cap \text{Holo}\left(\mathcal{M}_\eta \to \boldsymbol{\mathcal{B}}_\eta\right)$ (prop. B.6);
2. for $\eta \prec \eta' \in \mathcal{L}$, $\mathcal{H}^{\text{Holo}}_{\eta'\to\eta} := \mathcal{H}^{\text{preQ}}_{\eta'\to\eta} \cap \text{Holo}\left(\mathcal{M}_{\eta'\to\eta} \to \boldsymbol{\mathcal{B}}_{\eta'\to\eta}\right)$ and for $\eta \in \mathcal{L}$, $\mathcal{H}^{\text{Holo}}_{\eta\to\eta} := \mathcal{H}^{\text{preQ}}_{\eta\to\eta} = \mathbb{C}$.

Then, for all $\eta \preccurlyeq \eta'$, $\Phi^{\text{preQ}}_{\eta'\to\eta}\left\langle \mathcal{H}^{\text{Holo}}_{\eta'}\right\rangle = \mathcal{H}^{\text{Holo}}_{\eta'\to\eta} \otimes \mathcal{H}^{\text{Holo}}_\eta$ and for all $\eta \preccurlyeq \eta' \preccurlyeq \eta''$, $\Phi^{\text{preQ}}_{\eta''\to\eta'\to\eta}\left\langle \mathcal{H}^{\text{Holo}}_{\eta''\to\eta}\right\rangle = \mathcal{H}^{\text{Holo}}_{\eta''\to\eta'} \otimes \mathcal{H}^{\text{Holo}}_{\eta'\to\eta}$. Hence, defining:

3. for $\eta \preccurlyeq \eta' \in \mathcal{L}$, $\Phi^{\text{Holo}}_{\eta'\to\eta} := \Phi^{\text{preQ}}_{\eta'\to\eta}\Big|_{\mathcal{H}^{\text{Holo}}_{\eta'}\to\mathcal{H}^{\text{Holo}}_{\eta'\to\eta}\otimes\mathcal{H}^{\text{Holo}}_\eta}$;
4. and for $\eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}$, $\Phi^{\text{Holo}}_{\eta''\to\eta'\to\eta} := \Phi^{\text{preQ}}_{\eta''\to\eta'\to\eta}\Big|_{\mathcal{H}^{\text{Holo}}_{\eta''\to\eta}\to\mathcal{H}^{\text{Holo}}_{\eta''\to\eta'}\otimes\mathcal{H}^{\text{Holo}}_{\eta'\to\eta}}$;

$(\mathcal{L}, \mathcal{H}^{\text{Holo}}, \Phi^{\text{Holo}})^\otimes$ is a projective system of quantum state spaces.

Proof First, for every $\eta \in \mathcal{L}$, we can define a complex structure on $\boldsymbol{\mathcal{B}}_\eta$ in the following way: for $z \in \boldsymbol{\mathcal{B}}_\eta$, with $x = \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}_\eta}(z)$, we have $T_z\left(\boldsymbol{\mathcal{B}}_\eta\right) = \text{Hor}_z\left(\boldsymbol{\mathcal{B}}_\eta,\, \nabla_\eta\right) \oplus T_z\left(\boldsymbol{\pi}^{-1}_{\boldsymbol{\mathcal{B}}_\eta}(z)\right)$, where $\text{Hor}_z\left(\boldsymbol{\mathcal{B}}_\eta,\, \nabla_\eta\right)$ can be identified $T_x(\mathcal{M}_\eta)$, and thus equipped with the lift of the complex structure $J_{\eta,x}$, while on $T_z\left(\boldsymbol{\pi}^{-1}_{\boldsymbol{\mathcal{B}}_\eta}(z)\right)$, the multiplication by i provide a natural complex structure. With this, a cross-section of $\boldsymbol{\mathcal{B}}_\eta$ is holomorphic if and only if it is holomorphic as a map $\mathcal{M}_\eta \to \boldsymbol{\mathcal{B}}_\eta$.

Similarly, for every $\eta \preccurlyeq \eta' \in \mathcal{L}$, we have a complex structure on $\boldsymbol{\mathcal{B}}_{\eta'\to\eta}$. Eq. (6.8.1), together with defs. 6.6.1, 6.6.3 and the holomorphicity of $\varphi_{\eta'\to\eta}$, ensures that $\zeta_{\eta'\to\eta}$ is holomorphic as a map $\boldsymbol{\mathcal{B}}_{\eta'\to\eta} \times \boldsymbol{\mathcal{B}}_\eta \to \boldsymbol{\mathcal{B}}_{\eta'}$.

For $\eta \in \mathcal{L}$, $\Phi^{\text{preQ}}_{\eta\to\eta}$ is a trivial identification, hence the desired result holds. Thus, we consider $\eta \prec \eta' \in \mathcal{L}$. We first want to prove that $\Phi^{\text{preQ}}_{\eta'\to\eta} \circ \Pi^{\text{Holo}}_{\eta'} \circ \Phi^{\text{preQ},-1}_{\eta'\to\eta} = \Pi^{\text{Holo}}_{\eta'\to\eta} \otimes \Pi^{\text{Holo}}_\eta$, where $\Pi^{\text{Holo}}_{\eta'} : \mathcal{H}^{\text{preQ}}_{\eta'} \to \mathcal{H}^{\text{Holo}}_{\eta'}$ is the orthogonal projection on the closed vector subspace $\mathcal{H}^{\text{Holo}}_{\eta'}$ in $\mathcal{H}^{\text{preQ}}_{\eta'}$, and Π^{Holo}_η, $\Pi^{\text{Holo}}_{\eta'\to\eta}$ are defined analogously.

Let $t \in \mathcal{H}^{\text{preQ}}_{\eta'\to\eta}$, $s \in \mathcal{H}^{\text{preQ}}_\eta$ and define $\bar{t} = \Pi^{\text{Holo}}_{\eta'\to\eta}\, t$ and $\bar{s} = \Pi^{\text{Holo}}_\eta\, s$. By definition of $\Phi^{\text{preQ}}_{\eta'\to\eta}$, we have:

$$\forall y, x \in \mathcal{M}_{\eta'\to\eta} \times \mathcal{M}_\eta,\ \Phi^{\text{preQ},-1}_{\eta'\to\eta}\left(\bar{t} \otimes \bar{s}\right) \circ \varphi^{-1}_{\eta'\to\eta}(y, x) = \zeta_{\eta'\to\eta}(\bar{t}(y),\, \bar{s}(x))\,.$$

But, as a composition of holomorphic maps, $(y, x) \mapsto \zeta_{\eta'\to\eta}(\bar{t}(y),\, \bar{s}(x))$ is holomorphic. Hence, $\varphi_{\eta'\to\eta}$ being holomorphic, $\Phi^{\text{preQ},-1}_{\eta'\to\eta}\left(\bar{t} \otimes \bar{s}\right) \in \mathcal{H}^{\text{Holo}}_{\eta'}$.

Let $s' \in \mathcal{H}^{\text{Holo}}_{\eta'}$. Using the volume-preserving property of $\varphi_{\eta'\to\eta}$, we compute:

$$\left\langle s',\, \Pi^{\text{Holo}}_{\eta'} \circ \Phi^{\text{preQ},-1}_{\eta'\to\eta}(t \otimes s)\right\rangle_{\mathcal{H}_{\eta'}} = \left\langle s',\, \widetilde{\zeta}_{\eta'\to\eta}(t,\, s)\right\rangle_{\mathcal{H}_{\eta'}}$$

$$= \int_{\mathcal{M}_{\eta'\to\eta}} d\mu_{\eta'\to\eta}(y) \int_{\mathcal{M}_\eta} d\mu_\eta(x) \quad \left\langle s' \circ \varphi^{-1}_{\eta'\to\eta}(y, x),\, \zeta_{\eta'\to\eta}(t(y),\, s(x))\right\rangle_{\boldsymbol{\mathcal{B}}_{\eta'}}.$$

For $y, x \in \mathcal{M}_{\eta'\to\eta} \times \mathcal{M}_\eta$ we define $u^y(x) \in \boldsymbol{\mathcal{B}}^x_\eta$ such that:

$$\forall u' \in \boldsymbol{\mathcal{B}}^x_\eta,\ \left\langle s' \circ \varphi^{-1}_{\eta'\to\eta}(y, x),\, \zeta_{\eta'\to\eta}(t(y),\, u')\right\rangle_{\boldsymbol{\mathcal{B}}_{\eta'}} = \left\langle u^y(x),\, u'\right\rangle_{\boldsymbol{\mathcal{B}}_\eta},$$

$u^y(x)$ is well-defined, since the left-hand side is a $\mathbb{C}$-linear function of u' (from def. 6.6.3).

Since $s' \in \mathcal{H}^{\text{Holo}}_{\eta'}$, the map $x \mapsto s' \circ \varphi^{-1}_{\eta' \to \eta}(y, x)$ is holomorphic, and for any (local) anti-holomorphic cross-section u' of $\boldsymbol{\mathcal{B}}_\eta$, the map:

$$x \mapsto \left\langle \zeta_{\eta' \to \eta}(t(y),\, u'),\, s' \circ \varphi^{-1}_{\eta' \to \eta}(y, x) \right\rangle_{\boldsymbol{\mathcal{B}}_{\eta'}},$$

is holomorphic (for the connection is a $\mathcal{U}(1)$-connection, so the parallel transport preserve the scalar product). Therefore, the cross-section u^y is holomorphic. Moreover, using def. 6.6.2:

$$\forall x \in \mathcal{M}_\eta,\ \left|u^y(x)\right| \leqslant \left|s' \circ \varphi^{-1}_{\eta' \to \eta}(y, x)\right|\, \left|t(y)\right|,$$

thus, by Fubini theorem, for almost every y in $\left(\mathcal{M}_{\eta' \to \eta},\, d\mu_{\eta' \to \eta}\right)$, $u^y \in \mathcal{H}^{\text{Holo}}_\eta$.

Hence, for almost every $y \in \mathcal{M}_{\eta' \to \eta}$:

$$\int_{\mathcal{M}_\eta} d\mu_\eta(x) \quad \left\langle s' \circ \varphi^{-1}_{\eta' \to \eta}(y, x),\, \zeta_{\eta' \to \eta}(t(y),\, s(x)) \right\rangle_{\boldsymbol{\mathcal{B}}_{\eta'}} = \int_{\mathcal{M}_\eta} d\mu_\eta(x) \quad \left\langle u^y(x),\, s(x) \right\rangle_{\boldsymbol{\mathcal{B}}_\eta}$$

$$= \left\langle u^y,\, s \right\rangle_{\mathcal{H}_\eta} = \left\langle u^y,\, \Pi^{\text{Holo}}_\eta s \right\rangle_{\mathcal{H}_\eta} = \int_{\mathcal{M}_\eta} d\mu_\eta(x) \quad \left\langle s' \circ \varphi^{-1}_{\eta' \to \eta}(y, x),\, \zeta_{\eta' \to \eta}(t(y),\, \overline{s}(x)) \right\rangle_{\boldsymbol{\mathcal{B}}_{\eta'}}.$$

And we can prove in a similar way that, for almost every $x \in \mathcal{M}_{\eta' \to \eta}$:

$$\int_{\mathcal{M}_{\eta' \to \eta}} d\mu_{\eta' \to \eta}(y) \quad \left\langle s' \circ \varphi^{-1}_{\eta' \to \eta}(y, x),\, \zeta_{\eta' \to \eta}(t(y),\, \overline{s}(x)) \right\rangle_{\boldsymbol{\mathcal{B}}_{\eta'}} =$$

$$= \int_{\mathcal{M}_{\eta' \to \eta}} d\mu_{\eta' \to \eta}(y) \quad \left\langle s' \circ \varphi^{-1}_{\eta' \to \eta}(y, x),\, \zeta_{\eta' \to \eta}(\overline{t}(y),\, \overline{s}(x)) \right\rangle_{\boldsymbol{\mathcal{B}}_{\eta'}}.$$

Therefore, we arrive at:

$$\left\langle s',\, \Pi^{\text{Holo}}_{\eta'} \circ \Phi^{\text{preQ},-1}_{\eta' \to \eta}(t \otimes s) \right\rangle_{\mathcal{H}_{\eta'}} = \left\langle s',\, \Phi^{\text{preQ},-1}_{\eta' \to \eta}(\overline{t} \otimes \overline{s}) \right\rangle_{\mathcal{H}_{\eta'}}.$$

Since this holds for all $s' \in \mathcal{H}^{\text{Holo}}_{\eta'}$ and we have already proved that $\Phi^{\text{preQ},-1}_{\eta' \to \eta}\left(\overline{t} \otimes \overline{s}\right) \in \mathcal{H}^{\text{Holo}}_{\eta'}$, we have:

$$\Pi^{\text{Holo}}_{\eta'} \circ \Phi^{\text{preQ},-1}_{\eta' \to \eta}(t \otimes s) = \Phi^{\text{preQ},-1}_{\eta' \to \eta}(\overline{t} \otimes \overline{s}),$$

which gives us the announced result:

$$\Phi^{\text{preQ}}_{\eta' \to \eta} \circ \Pi^{\text{Holo}}_{\eta'} \circ \Phi^{\text{preQ},-1}_{\eta' \to \eta} = \Pi^{\text{Holo}}_{\eta' \to \eta} \otimes \Pi^{\text{Holo}}_\eta.$$

Hence, $\Phi^{\text{preQ}}_{\eta' \to \eta} \left\langle \mathcal{H}^{\text{Holo}}_{\eta'} \right\rangle = \Phi^{\text{preQ}}_{\eta' \to \eta} \circ \Pi^{\text{Holo}}_{\eta'} \circ \Phi^{\text{preQ},-1}_{\eta' \to \eta} \left\langle \mathcal{H}^{\text{preQ}}_{\eta' \to \eta} \otimes \mathcal{H}^{\text{preQ}}_\eta \right\rangle = \Pi^{\text{Holo}}_{\eta' \to \eta} \otimes \Pi^{\text{Holo}}_\eta \left\langle \mathcal{H}^{\text{preQ}}_{\eta' \to \eta} \otimes \mathcal{H}^{\text{preQ}}_\eta \right\rangle = \mathcal{H}^{\text{Holo}}_{\eta' \to \eta} \otimes \mathcal{H}^{\text{Holo}}_\eta$. And the relation involving $\Phi^{\text{preQ}}_{\eta'' \to \eta' \to \eta}$ can be proved in a similar way. □

Note that in an holomorphic representation, the evaluation of the holomorphic wave-function at a given point in phase space is a bounded linear form (via an argument similar to the proof of prop. B.6), hence is dual to a vector in the Hilbert space: this defines the coherent state centered around this classical point. Now, if we choose an element in the projective family of symplectic manifolds we started from, ie. a projective family of points $\left(x_\eta\right)_{\eta \in \mathcal{L}}$, we can form a projective family of quantum states, by considering, in each $\mathcal{H}_\eta^{\text{Holo}}$, the coherent state centered around x_η. This family of states will moreover be of the form considered in theorem 5.9, so we can apply this result to get a corresponding inductive limit Hilbert space and characterize its range in the quantum projective state space (for example, the Fock representation we will consider in prop. 16.17 can be obtained in this manner).

Here, the classical precursor for this inductive limit of Hilbert spaces (in the spirit of the the discussion preceding prop. 6.5) is an inductive limit of phase spaces: the restriction to the subspace $\mathrm{Vect}(\zeta_{\eta'\to\eta})\otimes\mathcal{H}_\eta$ of $\mathcal{H}_{\eta'\to\eta}\otimes\mathcal{H}_\eta \approx \mathcal{H}_{\eta'}$ can be seen as the quantum implementation of the second class constraints selecting $\left\{x_{\eta'\to\eta}\right\} \times \mathcal{M}_\eta$ in $\mathcal{M}_{\eta'\to\eta} \otimes \mathcal{M}_\eta \approx \mathcal{M}_{\eta'}$ (with $x_{\eta'\to\eta}$ defined through $(x_{\eta'\to\eta}\,,\, x_\eta) := \varphi_{\eta'\to\eta}(x_{\eta'})$).

7. Discussion: toward a systematic quantization framework

While the quantization schemes laid out in section 6 cover the most common use cases and, indeed, will be sufficient for the rest of the present work, it appears likely that they could be further developed. The goal would be to have general quantization prescriptions, building upon geometric quantization [105], to quantize any projective system of classical phase spaces, as soon as we give us a consistent family of polarizations thereon.

In particular, it should be possible to relax the requirement of having a global factorization. Recall that in our discussion of the local factorization result (prop. 2.10), we had identified two different kinds of obstructions that could prevent it from holding globally. In both cases, we can sketch a route for proceeding to quantization nevertheless.

The first kind of obstruction is realized when $\mathcal{M}_{\eta'}$ cannot be written as a Cartesian product, but at least can be seen as an open subset of a bigger manifold $\widetilde{\mathcal{M}}_{\eta'} := \mathcal{M}_{\eta'\to\eta} \times \mathcal{M}_\eta$. This suggests to deal with this situation by a slight generalization of def. 5.1, allowing $\mathcal{H}_{\eta'}$ to be a closed vector subspace in a bigger Hilbert space $\widetilde{\mathcal{H}}_{\eta'} := \mathcal{H}_{\eta'\to\eta} \otimes \mathcal{H}_\eta$. Then, the density matrices over $\mathcal{H}_{\eta'}$ could be seen as density matrices over $\widetilde{\mathcal{H}}_{\eta'}$, with support restricted to $\mathcal{H}_{\eta'}$. Thus, it would still be possible define a map $\mathrm{Tr}_{\eta'\to\eta} : \mathcal{S}_{\eta'} \to \mathcal{S}_\eta$ by first embedding $\mathcal{S}_{\eta'}$ in $\widetilde{\mathcal{S}}_{\eta'}$ and then tracing over $\mathcal{H}_{\eta'\to\eta}$. While such a map could no longer be seen as a partial trace over a tensor factor in $\mathcal{H}_{\eta'\to\eta}$, it should still retain the properties that we really need for the formalism to make sense (in particular, appropriate compatibility with the evaluation of expectation values).

The other obstacle for a global factorization is illustrated by taking $\mathcal{M}_{\eta'}$ as a covering space of $\mathcal{M}_\eta$: in this case we still have the option of writing $\mathcal{M}_{\eta'} \simeq \mathcal{M}_{\eta'\to\eta} \times \mathcal{M}_\eta$ with a discrete space $\mathcal{M}_{\eta'\to\eta}$, but we have to accept that the identification will not be everywhere smooth: there will be cuts, and the disposition of these cuts will, when going over to the quantum theory, be imprinted in the precise interpretation of the observables. For example if $\mathcal{M}_{\eta'} = \mathcal{U}(1) = \mathcal{M}_\eta$ (aka. circles), and the projection $\pi_{\eta'\to\eta}$ wraps $\mathcal{M}_{\eta'}$ n times around $\mathcal{M}_\eta$, so that the length of $\mathcal{M}_\eta$ is ℓ while the length of $\mathcal{M}_{\eta'}$ is $n\,\ell$, a suitable factorization of $\mathcal{H}_{\eta'} := L_2(\mathcal{M}_{\eta'})$ can be written in terms of the respective momentum eigenbasis as $\left|{}^{2\pi k'}/_{n\ell}\right\rangle_{\eta'} = \left|r\right\rangle_{\eta'\to\eta} \otimes \left|{}^{2\pi k}/_{\ell}\right\rangle_{\eta}$, where $k' = n\,k + r$ (Euclidean division).

It might even be possible to go further and relax the need to choose *consistent* polarizations on the classical projective system. For example, if the small phase spaces $\mathcal{M}_\eta$ are linear (or affine), different choices of linear complex structures on $\mathcal{M}_\eta$ (making $\mathcal{M}_\eta$ into a complex vector or affine space) yield unitarily equivalent holomorphic quantizations (see [105, sections 9.5 and 10.4] as well as [49]): one could then, when trying to relate a finer label η' and a coarser one η, use these identifications to switch to compatible choices of complexes structures on $\mathcal{M}_{\eta'}$ and $\mathcal{M}_\eta$ (in a way similar to the option, outlined at the end of subsection 6.1, to dispense from the choice of a consistent family of measures in the context of position quantization).

The bottom line is that quantizing a classical projective system will often be a fairly automatic procedure, and, in practice, most of the work goes into setting up this classical system in the first place. This is markedly illustrated by the next part, devoted to the design of projective quantum state spaces for a certain class of classical theories: while it will require appropriate forethought to arrange the classical factorizing system and check its validity (section 10), the quantization step (subsection 12.1) will be a straightforward application of subsection 6.1.

In particular, there tends to be a conflict, when preparing a projective or factorizing system along the lines of section 2, between the three-spaces consistency condition (fig. 2.1) and the requirement for the label set to be directed:

- when the three-spaces consistency fails, it is generally a sign that some information is missing from the labels: if we cannot unambiguously attach a precise physical meaning to each label (in the form of a specific sub-algebra of observables), we cannot guarantee that a given coarser/smaller label will always be extracted identically from a finer/bigger one, no matter in how many successive steps this extraction is done;
- on the other hand, the richer the structure of the labels is, the harder it gets to order them in a way that do full justice to the information they carry, while ensuring directedness.

These opposing considerations will play a prominent role in section 10, as we will have to carefully select a suitable label set (subsection 10.1), balancing the implications of directedness (subsection 10.2) with the need for the labels to fully specify the subset of observables to which they refer (subsection 10.3).

Finally, while we have discussed extensively how the classical structures presented in section 2 can be converted into their quantum analogues, we have not yet formalized how to infer from the strategy exposed in section 3 a program for dealing with constraints at the quantum level. Nevertheless, an example will be considered in subsection 16.2, suggesting how such a program could look like.

Projective Structure for the Holonomy-Flux Algebra

Chapter 3 – Classical Theory

8. Introduction

Theories of connections are field theories in which the field (aka. configuration variable) is a connection in a principal bundle [50, section I.5] (whose base is the spatial slice and whose structure group is the gauge group of the theory), like eg. the electromagnetic potential (with gauge group $G = \mathcal{U}(1)$). The corresponding conjugate variable is then analogous to the electric field of electrodynamics.

Our interest in such theories stems from the reformulation of General Relativity (GR) in terms of the Ashtekar variables [3, 4, 12]. Recall from the introduction of the present work, that the canonical version of GR (aka. the ADM formalism) provided us with a phase space using the 3-metric on a spatial slice as configuration variable and the extrinsic curvature of this slice as momentum variable. The key step is then to express the 3-metric on the spatial slice in terms of a triad field, that maps, at each point, an internal 3-dimensional Euclidean space into the tangent space. This introduces an additional gauge invariance, since rotations within this internal space do not affect the reconstructed metric. Therefore, we will have, besides the diffeomorphism and Hamiltonian constraints already mentioned in the main introduction, additional constraints implementing this gauge invariance, referred to as 'Gauss' constraints, owing to their analogy with the Gauss law of electromagnetism.

To reconstruct a connection in this context, we start from the parallel transport prescribed by the 3-metric on the *tangent bundle* of the spatial slice (as determined by its Christoffel symbols), convert it into a parallel transport between the *internal spaces* (using the triad that solders, at each point, the internal space to the tangent space), and add to the thus obtained connection a term proportional to the extrinsic curvature of the slice: this yields an object that still transforms as a connection under gauge transformation (because the connections form an affine space, see def. 9.2), while being canonically conjugate with the triad field (because the metric and the extrinsic curvature were canonically conjugate). This allows the triad field (more precisely, its densitized version, obtained through renormalization by the volume element) to play the role of the 'electric field' in the resulting theory of connections. This works because, in dimension $d = 3$, the Lie algebra $\mathfrak{su}(2)$ of the group of rotations (in whose dual the electric field should be valued) can be identified with the 3-dimensional Euclidean space (on which the triad is indexed). Generalizations of this construction to higher dimensions [15] require the addition of extra constraints to compensate for the mismatch of dimensions between the defining and adjoint representations of the corresponding rotation group.

Beware that, following the prevalent usage in the LQG literature, we will, from now on, refer to the connection as the position variable, and to the electric field as its conjugate momentum variable. This is reasonable in the context of usual gauge theory, where the connection is the gauge field, but somewhat at odds with the physical interpretation of the Ashtekar variables sketched above, since the intrinsic geometry along the spatial slice is encoded by the electric field while its extrinsic curvature enters the definition of the connection. The reason for this convention is that the quantum theory (section 12) is more easily formulated in the 'connection representation', which we will think of as a position representation.

To arrive at this quantum theory, we first have to decide what our elementary observables will be (recall the discussion in section 1). Smearing the (densitized) triad along 2-dimensional surfaces yields 'flux' observables, taking value in the dual of $\mathfrak{su}(2)$, while the parallel transport prescribed by the Ashtekar connection along 1-dimensional edges gives rise to $\mathcal{SU}(2)$-valued 'holonomies' (see [92, sections II.6.2 and II.6.3] or prop. 9.9 below). Those flux and holonomy observables have the advantage of being defined purely in terms of geometric objects (namely surfaces and edges), without having to refer to any fixed background metric: this makes them particularly suitable as the basic ingredients of a *diffeomorphism invariant* quantum theory. Still, their Poisson-brackets are finite (see [92, section II.6.4] or prop. 9.10 below) and the 'holonomy-flux algebra' they generate is simple enough to allow a representation as operators on an Hilbert space (the Ashtekar-Lewandowski Hilbert space [19, 8] to which we will come back in subsection 12.2).

As we will see, a projective quantum state space can also be build from this algebra. Inspired by previous works (in [68, section 5] and [69], achieving the application of the projective framework to models involving real-valued connections), we will, once equipped with an appropriate label set (theorem 10.14), proceed, in theorem 10.25, to set up a factorizing system of configuration spaces: as argued in subsection 6.1, and in particular in theorem 6.2, this is a fairly generic way of writing down a projective limit of phase spaces, provided that a family of real polarizations can be chosen consistently across all partial phase spaces, and it paves the way for its quantization in the position representation, that we will perform in subsection 12.1.

9. Theories of connections

The aim of this section is to briefly recall the canonical formulation of classical theories of connections, as well as the derivation of the holonomy-flux algebra mentioned above, to set the stage for the construction of a projective description thereof (in section 10). The material we will be reviewing is very standard, as it constitutes the foundation of LQG [4, 8], and a comprehensive exposition thereof can be found in [92, sections II.6 and IV.33].

9.1 Phase space

To make the classical theory well-defined, we extend the space of smooth connections into a reflexive Banach space (this will ensure the non-degeneracy of the symplectic structure on the corresponding phase space). This is achieved using a Sobolev space (with $p = 2$, $k = 0$) of bundle cross-sections [96, 72]: in contrast to the usual ($p = 2$) Sobolev spaces of functions on $\mathbb{R}^n$, it does *not* carry a canonical Hilbert structure, but (partial) scalar products can be defined locally using coordinates patches, and, in the case of a *compact* spatial slice Σ, the topologies induced by these coordinate-dependent scalar products fit together. In this way, we get a configuration space having the *topology* of an Hilbert space, and this is all what we will need.

Note that the restriction to a compact spatial slice is physically somewhat artificial: we make this assumption in the present section for mathematical convenience, to avoid having to carefully impose suitable boundary conditions. We will anyway be able to drop this restriction as soon as we will turn to the construction of a projective state space for this theory. In addition, we also require Σ to be oriented. This is again for convenience, and we could dispense from this assumption by working below with densities instead of volume forms: the former always exist and they can be integrated without reference to an overall orientation of Σ [31, section 11.4].

Let $(\boldsymbol{\mathcal{P}}, G, \lhd)$ be a smooth principal fiber bundle over Σ [50, section I.5]. In this whole section 9, we assume that Σ is *compact* and *oriented*.

Definition 9.1 Let $\boldsymbol{\mathcal{V}}$ be a vector bundle on Σ. The Sobolev space $\mathrm{H}_o(\Sigma \to \boldsymbol{\mathcal{V}})$ on $\boldsymbol{\mathcal{V}}$ can be defined intrinsically (ie. without any choice of coordinates or bundle trivialization) with the following properties [96]:

1. It has the *topology* of a Hilbert space, hence it is in particular an (infinite-dimensional) smooth manifold modeled on a reflexive Banach space [20, chapter VII].
2. The space of $C^\infty(\Sigma \to \boldsymbol{\mathcal{V}})$ of smooth cross-sections of $\boldsymbol{\mathcal{V}}$ is dense in $\mathrm{H}_o(\Sigma \to \boldsymbol{\mathcal{V}})$.
3. The *topological* dual of $\mathrm{H}_o(\Sigma \to \boldsymbol{\mathcal{V}})$ is canonically identified with $\mathrm{H}_o\left(\Sigma \to \Lambda^d T^*(\Sigma) \otimes \boldsymbol{\mathcal{V}}^*\right)$, where $\Lambda^d T^*(\Sigma)$ denotes the volume form bundle on Σ and $\otimes$ the vector bundle tensor product. More generally, for any $k \in \{0, \dots, d\}$, the topological dual of $\mathrm{H}_o(\Sigma \to \Lambda^k T^*(\Sigma) \otimes \boldsymbol{\mathcal{V}})$ is canonically identified with $\mathrm{H}_o\left(\Sigma \to \Lambda^{d-k} T^*(\Sigma) \otimes \boldsymbol{\mathcal{V}}^*\right)$ and we have, for any $Q \in C^\infty\left(\Sigma \to \Lambda^k T^*(\Sigma) \otimes \boldsymbol{\mathcal{V}}\right)$

and any $F \in C^\infty\left(\Sigma \to \Lambda^{d-k}T^*(\Sigma) \otimes \boldsymbol{\mathcal{V}}^*\right)$:

$$F(Q) := \int_\Sigma Q \wedge F\,,$$

where $\wedge : \left(\Lambda^k T^*(\Sigma) \otimes \boldsymbol{\mathcal{V}}\right) \times \left(\Lambda^l T^*(\Sigma) \otimes \boldsymbol{\mathcal{V}}^*\right) \to \Lambda^{k+l}T^*(\Sigma)$ denotes the contraction of a $\boldsymbol{\mathcal{V}}$-valued k-form with a $\boldsymbol{\mathcal{V}}^*$-valued l-form (note that this expression fixes our sign convention for the identification $\Lambda^d T^*(\Sigma) \otimes \Lambda^k T(\Sigma) \approx \Lambda^{d-k}T^*(\Sigma)$).

The space of smooth connections on $\boldsymbol{\mathcal{P}}$ is an *affine* space, the linear part of which consists – modulo the choice of a local cross-section – of $\mathfrak{g}$-valued 1-forms. The corresponding momentum space thus consists – again up to (local) gauge choice – of $\mathfrak{g}^*$-valued $(d-1)$-forms (or more precisely densitized vector fields, which, in the case of an oriented slice Σ, can be converted into $(d-1)$-forms). We are careful here to formulate the definitions in a gauge- and coordinate-independent language, so it does come as a surprise that automorphisms of the bundle give raise to symplectomorphisms on the thus defined phase space (these automorphisms are combinations of diffeomorphisms and gauge transformations; a split between pure diffeomorphisms vs. pure gauge shifts can be made with respect to a reference cross-section).

Definition 9.2 Let $\boldsymbol{\mathcal{G}}$ be the $\boldsymbol{\mathcal{P}}$-associated vector bundle arising from the adjoint action of G on $\mathfrak{g}$ [50, section I.5], that is the quotient of $\boldsymbol{\mathcal{P}} \times \mathfrak{g}$ by the equivalence relation:

$$(p, u) \sim (p', u') \quad \Leftrightarrow \quad \exists\, g \in G \,\big/\, p' = p \triangleleft g^{-1} \quad \& \quad u' = \mathrm{ad}_g(u)\,.$$

A smooth connection on $\boldsymbol{\mathcal{P}}$ [50, section II.1] is a smooth $\mathfrak{g}$-valued 1-form A on $\boldsymbol{\mathcal{P}}$ satisfying:

1. $\forall p \in \boldsymbol{\mathcal{P}},\ \ A_p \circ [T_{\mathbf{1}}\, p \triangleleft \cdot\,] = \mathrm{id}_{\mathfrak{g}}$

   (where $[T_{\mathbf{1}}\, p \triangleleft \cdot\,]$ denotes the tangent map at $\mathbf{1}$ of the map $G \to \boldsymbol{\mathcal{P}},\ g \mapsto p \triangleleft g$);
2. $\forall p \in \boldsymbol{\mathcal{P}},\ \forall g \in G,\ \ A_{p \triangleleft g} \circ [T_p\ \cdot \triangleleft g] = \mathrm{ad}_{g^{-1}} \circ A_p$

   (where $[T_p\ \cdot \triangleleft g]$ denotes the tangent map at p of the map $\boldsymbol{\mathcal{P}} \to \boldsymbol{\mathcal{P}},\ p' \mapsto p' \triangleleft g$).

Given a smooth connection A and a smooth section Q of $T^*(\Sigma) \otimes \boldsymbol{\mathcal{G}}$, there is a unique smooth connection $A' =: A + Q$ such that:

$$\forall p \in \boldsymbol{\mathcal{P}},\ \forall v \in T_p(\boldsymbol{\mathcal{P}}),\ Q_{\boldsymbol{\pi}(p)} \circ [T_p \boldsymbol{\pi}](v) = \left[\left(p,\ A'_p(v) - A_p(v)\right)\right]_\sim,$$

where $\boldsymbol{\pi}$ denotes the bundle projection of $\boldsymbol{\mathcal{P}}$.

The space of smooth connections on $(\boldsymbol{\mathcal{P}},\ G,\ \triangleleft)$ is thus an *affine* space over $C^\infty(\Sigma \to T^*(\Sigma) \otimes \boldsymbol{\mathcal{G}})$. It can then be extended into an affine space $\mathcal{C}_\Sigma$ over $T\mathcal{C}_\Sigma := \mathrm{H}_o(\Sigma \to T^*(\Sigma) \otimes \boldsymbol{\mathcal{G}})$, in which it forms a dense subspace.

Definition 9.3 Let $\mathcal{M}_\Sigma := T'(\mathcal{C}_\Sigma) \approx \mathcal{C}_\Sigma \times \mathcal{P}_\Sigma$, where $\mathcal{P}_\Sigma := \mathrm{H}_o\left(\Sigma \to \Lambda^{d-1}T^*(\Sigma) \otimes \boldsymbol{\mathcal{G}}^*\right)$ and let $\Omega_{\mathcal{M}_\Sigma}$ be the canonical symplectic structure on $\mathcal{M}_\Sigma$, which can be defined through (see [20, section VII.2], but beware that we are using an opposite sign convention here, to match eq. (2.16.1)):

$$\forall (A, E) \in \mathcal{M}_\Sigma,\ \forall Q, Q' \in T_A(\mathcal{C}_\Sigma) = T\mathcal{C}_\Sigma,\ \forall F, F' \in T_E(\mathcal{P}_\Sigma) = \mathcal{P}_\Sigma \approx (T\mathcal{C}_\Sigma)',$$
$$\Omega_{\mathcal{M}_\Sigma,(A,E)}\left((Q, F),\ (Q', F')\right) := F'(Q) - F(Q')\,,$$

where we have used the identification provided by def. 9.1.3. Since $\mathcal{C}_\Sigma$ is a smooth manifold modeled on a reflexive Banach space, $\Omega_{\mathcal{M}_\Sigma}$ is a *strong* symplectic structure (ie. for each $(A, E) \in \mathcal{M}_\Sigma$, $(Q, F) \mapsto \Omega_{\mathcal{M}_\Sigma,(A,E)}\left((Q, F), \cdot\right)$ is an isomorphism $T_{(A,E)}(\mathcal{M}_\Sigma) \to T'_{(A,E)}(\mathcal{M}_\Sigma)$).

Definition 9.4 Let Φ be a smooth bundle automorphism of $\boldsymbol{\mathcal{P}}$, namely a smooth diffeomorphism $\boldsymbol{\mathcal{P}} \to \boldsymbol{\mathcal{P}}$ satisfying:

1. there exists a smooth diffeomorphism $\phi : \Sigma \to \Sigma$ such that $\boldsymbol{\pi} \circ \Phi = \phi \circ \boldsymbol{\pi}$;
2. $\forall g \in G$, $(\,\cdot \lhd g) \circ \Phi = \Phi \circ (\,\cdot \lhd g)$.

Φ induces a diffeomorphism $\Phi_{\boldsymbol{\mathcal{G}}}$ on $\boldsymbol{\mathcal{G}}$ such that:

$$\forall p \in \boldsymbol{\mathcal{P}},\ \forall u \in \mathfrak{g},\ \Phi_{\boldsymbol{\mathcal{G}}}\left([(p,\, u)]_\sim\right) = [(\Phi(p),\, u)]_\sim .$$

For any smooth connections A, A' and any $Q \in C^\infty(\Sigma \to T^*(\Sigma) \otimes \boldsymbol{\mathcal{G}})$ such that $A' = A + Q$ (def. 9.2), we have $\Phi^{*,-1} A' = \Phi^{*,-1} A + T\Phi^{*,-1} Q$, where $\Phi^{*,-1} A$ (resp. $\Phi^{*,-1} A$) denotes the pullback of the 1-form A (resp. A') by Φ^{-1} and $T\Phi^{*,-1} Q$ is *defined* by:

$$\forall x \in \Sigma,\ (T\Phi^{*,-1} Q)_x = \Phi_{\boldsymbol{\mathcal{G}}} \circ Q_{\phi^{-1}(x)} \circ \left[T_x \phi^{-1}\right].$$

$\Phi^{*,-1}$, resp. $T\Phi^{*,-1}$, can be extended by continuity to an affine bijective map $\Phi_{\mathcal{C}_\Sigma}$ on $\mathcal{C}_\Sigma$, resp. to a linear isomorphism $T\Phi_{\mathcal{C}_\Sigma}$ on $T\mathcal{C}_\Sigma$. Finally, it can be lifted to a map $\Phi_{\mathcal{M}_\Sigma}$ on $\mathcal{M}_\Sigma$:

$$\forall (A, E) \in \mathcal{M}_\Sigma \approx \mathcal{C}_\Sigma \times (T\mathcal{C}_\Sigma)',\ \Phi_{\mathcal{M}_\Sigma}(A, E) := \left(\Phi_{\mathcal{C}_\Sigma}(A),\ E \circ T\Phi^{-1}_{\mathcal{C}_\Sigma}\right),$$

and, by construction, $\Phi_{\mathcal{M}_\Sigma}$ is a symplectomorphism (ie. $\Phi^*_{\mathcal{M}_\Sigma} \Omega_{\mathcal{M}_\Sigma} = \Omega_{\mathcal{M}_\Sigma}$).

The evaluation of the connection or electric field at a given point of Σ is not a well-defined observable on this phase space (it would be only densely defined on the Sobolev space we are considering). To get smooth observables we need to smear the basic variables in all d dimensions. This is the reason why the Poisson algebra of the holonomies and fluxes, that we will introduce in the next subsection as smearing on lower dimensional geometrical objects, will have to be regularized.

Definition 9.5 Let $U, \boldsymbol{s}$ be a smooth local cross-section of $\boldsymbol{\mathcal{P}}$ and let f, resp. q, be a smooth, $\mathfrak{g}^*$-valued, $(d-1)$-form, resp. a smooth, $\mathfrak{g}$-valued, 1-form, on Σ, with $\operatorname{supp} f$, $\operatorname{supp} q \subset U$. For any smooth connection A on $\boldsymbol{\mathcal{P}}$ and any $E \in C^\infty(\Sigma \to \Lambda^{d-1} T^*(\Sigma) \otimes \boldsymbol{\mathcal{G}}^*)$, we define:

$$\widehat{\mathrm{X}}_{f,\boldsymbol{s}}(A, E) := \int_U A_{\boldsymbol{s}} \wedge f \quad \& \quad \widehat{\mathrm{P}}_{q,\boldsymbol{s}}(A, E) := \int_U q \wedge E_{\boldsymbol{s}},$$

where:

$$\forall x \in \Sigma,\ \ A_{\boldsymbol{s};x} := A_{\boldsymbol{s}(x)} \circ [T_x \boldsymbol{s}] \quad \& \quad E_x(\,\cdot\,, \ldots, \cdot\,) =: \left[\left(\boldsymbol{s}(x),\ E_{\boldsymbol{s};x}(\,\cdot\,, \ldots, \cdot\,)\right)\right]_\sim .$$

$\widehat{\mathrm{X}}_{f,\boldsymbol{s}}$ and $\widehat{\mathrm{P}}_{q,\boldsymbol{s}}$ can be extended by continuity to $\mathbb{R}$-valued smooth affine maps on $\mathcal{M}_\Sigma$, and we have:

$$\left\{\widehat{\mathrm{P}}_{q,\boldsymbol{s}},\ \widehat{\mathrm{X}}_{f,\boldsymbol{s}}\right\}_{\mathcal{M}_\Sigma} \equiv \int_U q \wedge f .$$

Moreover, for any smooth bundle automorphism Φ of $\boldsymbol{\mathcal{P}}$, we have:

$$\widehat{X}_{f,\boldsymbol{s}} \circ \Phi_{\mathcal{M}_\Sigma} = \widehat{X}_{\phi^* f,\, \Phi^{-1} \circ \boldsymbol{s} \circ \phi} \quad \& \quad \widehat{P}_{q,\boldsymbol{s}} \circ \Phi_{\mathcal{M}_\Sigma} = \widehat{P}_{\phi^* q,\, \Phi^{-1} \circ \boldsymbol{s} \circ \phi} .$$

9.2 Holonomy-flux algebra

To fix our notations and definitions, we begin by writing down what we mean precisely by the edges and surfaces, that will label the elementary observables of the holonomy-flux algebra. An important precision is that these geometric objects will be required to be *analytic*. While this is not needed for the definition of the holonomies and fluxes themselves, it comes into play for the regularization of their Poisson brackets (see the comments preceding prop. 9.10 below), and it will be crucial for the construction of an associated projective system, where it will ensure the *directedness* of the label set (lemmas 10.5 and 10.8). For similar reasons, analyticity is also required for the construction of the Ashtekar-Lewandowski representation of the holonomy-flux algebra [4, 8].

On the other hand, the class of edges and surfaces we will be using here is actually more restrictive than strictly necessary: we are considering only fully analytic edges and fully analytic, disc-shaped surfaces, embedded in a single analytic coordinate patch, in contrast to the class of *semi-analytic* edges and surfaces commonly used in LQG [92, section IV.20]. This is purely for convenience, and as far as the construction of the quantum state space is concerned, it has absolutely no incidence: the label set we will build from these restricted classes of edges and surfaces (def. 10.12) is a cofinal part of the one we could build from a broader, semi-analytic class, so prop. 5.6 allows to identify the corresponding projective state spaces (and a similar result holds at the level of the inductive limit construction underlying the Ashtekar-Lewandowski Hilbert space, see subsection 12.2 and in particular the proof of theorem 12.11 for more details).

Definition 9.6 An analytic, encharted edge in Σ is an analytic diffeomorphism $\breve{e} : U \to V$, where U is an open neighborhood of $[0,1] \times \{0\}^{d-1}$ in $\mathbb{R}^d$, and V is an open subset of Σ. We call $\breve{\mathcal{L}}_{\text{edges}}$ the set of all encharted edges, and for $\breve{e} \in \breve{\mathcal{L}}_{\text{edges}}$ we define its starting point $b(\breve{e}) := \breve{e}(0,0)$, its ending point $f(\breve{e}) := \breve{e}(1,0)$ and its range $r(\breve{e}) := \breve{e}\left\langle [0,1] \times \{0\}^{d-1} \right\rangle$.

We say that $\breve{e}_1, \breve{e}_2 \in \breve{\mathcal{L}}_{\text{edges}}$ are equivalent, and we write $\breve{e}_1 \sim \breve{e}_2$, iff:

$$r(\breve{e}_1) = r(\breve{e}_1) \quad \& \quad b(\breve{e}_1) = b(\breve{e}_2) .$$

This defines an equivalence relation on $\breve{\mathcal{L}}_{\text{edges}}$. Its set of equivalence classes will be denoted by $\mathcal{L}_{\text{edges}}$. An element $e \in \mathcal{L}_{\text{edges}}$ is called an edge, and we can define its starting point $b(e)$, its ending point $f(e)$ and its range $r(e)$, since these are the same for any representative of e.

Note. If $d = 1$, $\mathbb{R}^{d-1} = \{0\}$, and $[0,\, 1] \times \{0\}^{d-1} \approx [0,1] \subset \mathbb{R}$.

Definition 9.7 An analytic, encharted surface in Σ is an analytic diffeomorphism $\breve{S} : U \to V$, where U is an open neighborhood of $\{0\} \times B^{(d-1)}$ in $\mathbb{R}^d$ ($B^{(d-1)}$ being the closed unit ball of $\mathbb{R}^{d-1}$), and V is an open subset of Σ. We call $\breve{\mathcal{L}}_{\text{surfcs}}$ the set of all encharted surfaces, and for $\breve{S} \in \breve{\mathcal{L}}_{\text{surfcs}}$ we define its range $r(\breve{S}) := \breve{S}\left\langle \{0\} \times B^{(d-1)} \right\rangle$.

We say that $\breve{S}_1 : U_1 \to V_1, \breve{S}_2 : U_2 \to V_2 \in \breve{\mathcal{L}}_{\text{surfcs}}$ are equivalent, and we write $\breve{S}_1 \sim \breve{S}_2$, iff:

$$r(\check{S}_1) = r(\check{S}_2) \quad \& \quad \check{S}_1 \left\langle U_1' \cap \left(\mathbb{R}_+ \times \mathbb{R}^{d-1}\right)\right\rangle = \check{S}_2 \left\langle U_2' \cap \left(\mathbb{R}_+ \times \mathbb{R}^{d-1}\right)\right\rangle,$$

where U_1' (resp. U_2') is an open neighborhood of $\{0\} \times B^{(d-1)}$ in U_1 (resp. in U_2), and $\mathbb{R}_+$ is the set of non-negative reals. This defines an equivalence relation on $\check{\mathcal{L}}_{\text{surfcs}}$. Its set of equivalence classes will be denoted by $\mathcal{L}_{\text{surfcs}}$. An element $S \in \mathcal{L}_{\text{surfcs}}$ is called a surface, and we can define its range $r(S)$, since it is the same for any representative of S.

Note. If $d = 1$, $\mathbb{R}^{d-1} = \{0\}$, and $\{0\} \times B^{(d-1)} \approx \{0\} \subset \mathbb{R}$.

As hinted above, the holonomy and flux observables are not smooth functions on the continuum, infinite dimensional phase space of def. 9.3 (in fact, they are not even everywhere *defined*). On the other hand, by smearing the connection, which is a 1-form, on a 1-dimensional object, respectively the electric field, which is $(d-1)$-form, on a $(d-1)$-dimensional object, we ensure that these observables are *diffeomorphism-covariant*: they can be expressed in the language of coordinates-free differential geometry and depend only on the geometric objects indexing them. This purely geometric dependence is also the guarantee that their Poisson brackets can be *regularized* in an intrinsic way (see [4], [92, section II.6.4] or prop. 9.10 below), ie. *without* having to depend on any auxiliary supporting structure (such as a particular choice of coordinates or a background, fiducial metric).

As a technical side note, the concept of orientation that matters for the definition of a flux operator is the orientation of the *normal* of its supporting surface: what we are really doing is to integrate on the *unoriented* surface the density obtained by contracting its oriented normal (which is a 1-form) with the electric field (which, properly, is a densitized vector field). Although this aspect is somewhat obscured as we take the shortcut to rely on a bulk orientation on Σ (converting an orientation of the normal into an orientation of the surface itself, and densitized vector fields into $(d-1)$-forms), this is the reason for the precise definition of the equivalence relation in def. 9.7.

Definition 9.8 Let $(\tau_i)_{1\leqslant i\leqslant k}$ (with $k := \dim G$) be a basis of $\mathfrak{g}$ and $(\tau^i)_{1\leqslant i\leqslant k}$ be the corresponding dual basis in $\mathfrak{g}^*$. Let $(\delta_\epsilon)_{\epsilon>0}$ be a smooth regularization of the δ-distribution in $\mathbb{R}^d$, ie. a familly of smooth maps $\delta_\epsilon : \mathbb{R}^d \to \mathbb{R}_+$ such that, for any $\epsilon > 0$:

1. $\displaystyle\int dx_1 \dots dx_d \;\; \delta_\epsilon(x_1, \dots, x_d) = 1\,,$
2. $\operatorname{supp} \delta_\epsilon \subset B^{(d)}_\epsilon$ (where $B^{(d)}_\epsilon$ is the closed ball of radius ϵ and center 0 in $\mathbb{R}^d$).

For any $x \in \mathbb{R}^d$, we define $\delta_{\epsilon,x} : x' \mapsto \delta_\epsilon(x' - x)$.

Let $V, \boldsymbol{s}$ be a smooth local cross-section of $\boldsymbol{\mathcal{P}}$ and let $\check{e} : U_1 \to V_1 \subset V$ be an encharted edge in Σ. Since U_1 is an open neighborhood of the compact subset $[0, 1] \times \{0\}^{d-1}$ of $\mathbb{R}^d$, there exists ϵ_1 such that:

$$\forall t \in [0, 1],\; B^{(d)}_{\epsilon_1, (t,0)} \subset U_1$$

(where $B^{(d)}_{\epsilon,x}$ denotes the closed ball of center x and radius ϵ in $\mathbb{R}^d$). For any $\epsilon \in \,]0, \epsilon_1[\,$, $t \in [0, 1]$ and $i \in \{1, \dots, k\}$, we define:

$$f^{\check{e},i}_{\epsilon,t} := \begin{cases} \check{e}^{-1,*}\left(\mathrm{sgn}(\check{e})\, \delta_{\epsilon,(t,0)}\, dx_2 \wedge \ldots \wedge dx_d\right)\ \tau^i \\ 0 \ \text{ outside of } \ V_1 \end{cases},$$

where $\mathrm{sgn}(\check{e}) := +1$ if $\check{e}$ is orientation-preserving, -1 if it is orientation-reversing (with respect to the canonical orientation on $U_1 \subset \mathbb{R}^d$ and to the chosen orientation on $V_1 \subset \Sigma$). We can then, for any smooth map $m : G \to \mathbb{R}$ and any $\epsilon \in]0,\ \epsilon_1[$, define:

$$\begin{array}{rccl} \mathrm{h}^{(\check{e},\boldsymbol{s},m)}_\epsilon : & \mathcal{M}_\Sigma & \to & \mathbb{R} \\ & (A, E) & \mapsto & m\left[\mathcal{P}\mathrm{exp}\left(\int_0^1 dt\ \ \widehat{\mathrm{X}}_{f^{\check{e},i}_{\epsilon,t},\boldsymbol{s}}(A, E)\, \tau_i\right)\right] \end{array},$$

where we implicitly sum over $i \in \{1, \ldots, k\}$ and $\mathcal{P}\mathrm{exp}$ denotes the G-valued, path-ordered (with respect to t), exponential.

Similarly, let $\check{S} : U_2 \to V_2 \subset V$ be an encharted surface in Σ and let ϵ_2 such that:

$$\forall y \in B^{(d-1)},\ B^{(d)}_{\epsilon_2,\,(0,y)} \subset U_2\,.$$

For any $\epsilon \in]0,\ \epsilon_2[$, $y \in B^{(d-1)}$ and $i \in \{1, \ldots, k\}$, we define:

$$q^{\check{S}}_{\epsilon,y,i} := \begin{cases} \check{S}^{-1,*}\left(\delta_{\epsilon,(0,y)}\, dx_1\right)\ \tau_i \\ 0 \ \text{ outside of } \ V_2 \end{cases}.$$

We can then, for any $u \in \mathfrak{g}$ and any $\epsilon \in]0,\ \epsilon_2[$, define:

$$\begin{array}{rccl} \mathrm{P}^{(\check{S},\boldsymbol{s},u)}_\epsilon : & \mathcal{M}_\Sigma & \to & \mathbb{R} \\ & (A, E) & \mapsto & \int_{B^{(d-1)}} dy\ \ \widehat{\mathrm{P}}_{q^{\check{S}}_{\epsilon,y,i},\boldsymbol{s}}(A, E)\, \tau^i(u) \end{array}.$$

Proposition 9.9 We consider the same objects as in def. 9.8 and we fix a *smooth* connection A on $\boldsymbol{\mathcal{P}}$ as well as a *smooth* $\boldsymbol{\mathcal{G}}^*$-valued $(d-1)$-form E. For any smooth map $m : G \to \mathbb{R}$, we have:

$$\mathrm{h}^{(e,\boldsymbol{s},m)}(A, E) := \lim_{\epsilon \to 0} \mathrm{h}^{(\check{e},\boldsymbol{s},m)}_\epsilon(A, E) = m\left[\mathcal{P}\mathrm{exp}\left(\int_0^1\ \gamma^*_{\check{e},\boldsymbol{s}} A\right)\right],$$

where $\gamma_{\check{e},\boldsymbol{s}} : [0, 1] \to \mathbb{R},\ t \mapsto \boldsymbol{s} \circ \check{e}(t, 0)$. Note that $\mathrm{h}^{(e,\boldsymbol{s},m)}(A, E)$ only depends on the equivalence class e of $\check{e}$.

Similarly, for any $u \in \mathfrak{g}$, we have:

$$\mathrm{P}^{(S,\boldsymbol{s},u)}(A, E) := \lim_{\epsilon \to 0} \mathrm{P}^{(\check{S},\boldsymbol{s},u)}_\epsilon(A, E) = \int_{\mathring{r}(S)} (\iota^* E)(u_{\boldsymbol{s}})\,,$$

where $\mathring{r}(S) := \check{S}\left\langle \{0\} \times \mathring{B}^{(d-1)} \right\rangle$ (with $\mathring{B}^{(d-1)}$ the open ball of center 0 and radius 1 in $\mathbb{R}^{d-1}$), ι is the canonical injection $\mathring{r}(S) \to \Sigma$, $\mathring{r}(S)$ is oriented so that:

$$\forall y \in \mathring{r}(S),\ \forall \upsilon \in \Lambda^{d-1} T^*_y(\Sigma),\quad \iota^*\upsilon > 0 \Leftrightarrow \check{S}^{-1,*}(dx_1) \wedge \upsilon > 0\,,$$

and $u_{\boldsymbol{s}} \in C^\infty(V \to \boldsymbol{\mathcal{G}})$ is defined by:

$$\forall x \in V,\ u_{\boldsymbol{s}}(x) = \left[(\boldsymbol{s}(x),\ u)\right]_\sim.$$

Again, $\mathrm{P}^{(S,\boldsymbol{s},u)}$ only depends on the equivalence class S of $\check{S}$.

Proof Putting all definitions together, we have, for any smooth map $m : G \to \mathbb{R}$ and any $\epsilon \in]0,\ \epsilon_1[$:

$$\widehat{X}_{f^{\check{e},i}_{\epsilon,t},\boldsymbol{s}}(A,E) = \int_{U_1} \delta_{\epsilon,(t,0)}\, \widetilde{A}^i_{\check{e},\boldsymbol{s}} \wedge dx_2 \wedge \ldots \wedge dx_d\,,$$

where U_1 carries the canonical orientation of $\mathbb{R}^d$ and $\widetilde{A}^i_{\check{e},\boldsymbol{s}}$ the 1-form is defined by:

$$\forall x \in U_1,\, \widetilde{A}^i_{\check{e},\boldsymbol{s};x} = \tau^i \circ A_{\boldsymbol{s}\circ\check{e}(x)} \circ [T_x\, \boldsymbol{s} \circ \check{e}].$$

This can be rewritten as the following unoriented integral:

$$\widehat{X}_{f^{\check{e},i}_{\epsilon,t},\boldsymbol{s}}(A,E) = \int_{U_1} dx_1 \ldots dx_d \quad \delta_{\epsilon,(t,0)}(x)\, \widetilde{A}^i_{\check{e},\boldsymbol{s}}(\partial_{x_1})\,.$$

Since $\widetilde{A}^i_{\check{e},\boldsymbol{s}}$ is a smooth 1-form on U_1 and $\bigcup_{t\in[0,1]} B_{\epsilon_1,\,(t,0)}$ is a compact subset of $U_1 \subset \mathbb{R}^d$, we have, *uniformly* in t:

$$\lim_{\epsilon\to 0} \widehat{X}_{f^{\check{e},i}_{\epsilon,t},\boldsymbol{s}}(A,E) = \widetilde{A}^i_{\check{e},\boldsymbol{s};(t,0)}(\partial_{x_1}) = \tau^i \circ A_{\boldsymbol{s}\circ\check{e}(t,0)} \left(\frac{d}{dt'} \boldsymbol{s} \circ \check{e}(t',0)\bigg|_{t'=t} \right) = \tau^i \circ A_{\gamma_{\check{e},\boldsymbol{s}}(t)} \left(\gamma'_{\check{e},\boldsymbol{s}}(t)\right).$$

Thus, we get:

$$\lim_{\epsilon\to 0} \mathrm{h}^{(\check{e},\boldsymbol{s},m)}_{\epsilon}(A,E) = m\left[\mathcal{P}\mathrm{exp}\left(\int_0^1 dt \quad A_{\gamma_{\check{e},\boldsymbol{s}}(t)}\left(\gamma'_{\check{e},\boldsymbol{s}}(t)\right)\right)\right] = m\left[\mathcal{P}\mathrm{exp}\left(\textstyle\int_0^1 \gamma^*_{\check{e},\boldsymbol{s}} A\right)\right].$$

Similarly, for any $u \in \mathfrak{g}$ and any $\epsilon \in \,]0,\, \epsilon_2[$:

$$\widehat{\mathrm{P}}_{q^{\check{S}}_{\epsilon,y,i},\boldsymbol{s}}(A,E)\, \tau^i(u) := \int_{U_2} dx_1 \ldots dx_d \quad \delta_{\epsilon,(0,y)}(x)\, \widetilde{E}_{\check{S},\boldsymbol{s},u}(x)$$

where $\widetilde{E}_{\check{S},\boldsymbol{s},u}$ is the smooth map $U_2 \to \mathbb{R}$ defined by:

$$\forall x \in U_2,\, \widetilde{E}_{\check{S},\boldsymbol{s},u}(x) := \mathrm{sgn}(\check{S}) \quad E_{\check{S}(x)}\left(\left[T_x\check{S}\right](\partial_{x_2}), \ldots, \left[T_x\check{S}\right](\partial_{x_d})\right)\left(\left[\left(\boldsymbol{s}\circ\check{S}(x),\, u\right)\right]_{\sim}\right).$$

Thus, we get:

$$\lim_{\epsilon\to 0} \mathrm{P}^{(\check{S},\boldsymbol{s},u)}_{\epsilon} = \int_{B^{(d-1)}} dy_2 \ldots d_{y_d} \quad \widetilde{E}_{\check{S},\boldsymbol{s},u}(0,y)$$

We now switch back to an oriented integral, taking care of the orientation:

$$\lim_{\epsilon\to 0} \mathrm{P}^{(\check{S},\boldsymbol{s},u)}_{\epsilon} = \int_{\mathring{B}^{(d-1)}} \mathrm{sgn}(\check{S}) \quad (\check{S}\circ\widetilde{\imath})^* E_{\check{S},\boldsymbol{s},u}$$

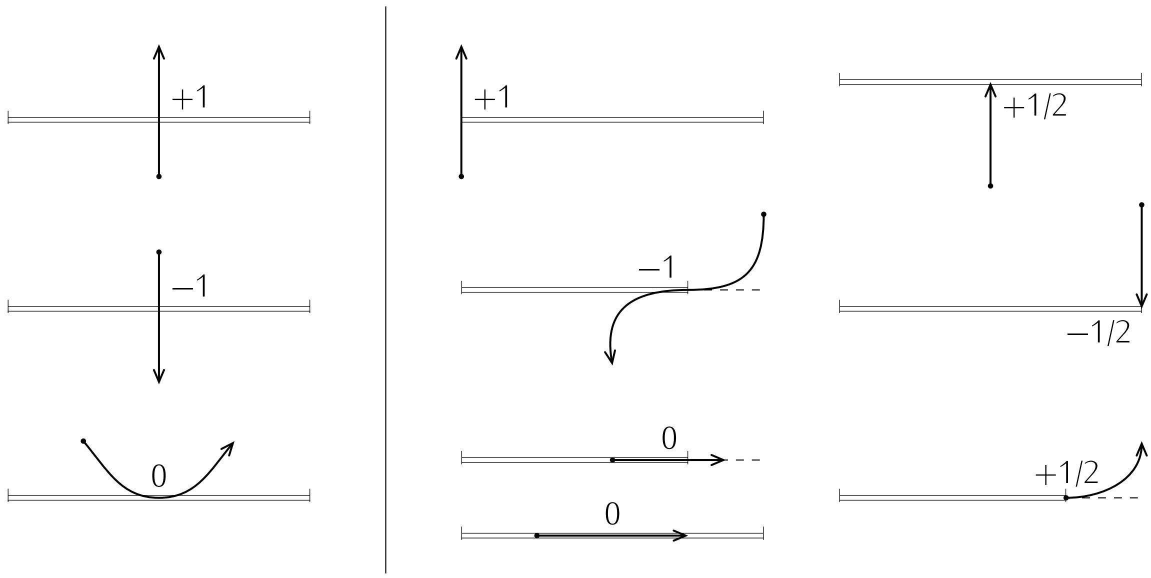
where $\widetilde{\imath}$ denotes the canonical injection $\mathring{B}^{(d-1)} \approx \{0\} \times \mathring{B}^{(d-1)} \to \mathbb{R}^d$ and the $(d-1)$-form $E_{\check{S},\boldsymbol{s},u}$ is defined by:

$$\forall y \in \mathring{r}(S),\, \forall w_2, \ldots, w_d \in T_y\left(\mathring{r}(S)\right),$$
$$E_{S,\boldsymbol{s},u;\,y}\left(w_2,\ldots,w_d\right) = E_y\left(w_2,\ldots,w_d\right)\left(\left[(\boldsymbol{s}(y),\, u)\right]_{\sim}\right).$$

Orienting $\mathring{r}(S)$ so that $\mathrm{sgn}(\check{S}\circ\widetilde{\imath}) = \mathrm{sgn}(\check{S})$, we obtain the final expression:

$$\lim_{\epsilon\to 0} \mathrm{P}^{(\check{S},\boldsymbol{s},u)}_{\epsilon} = \int_{\mathring{r}(S)} E_{\check{S},\boldsymbol{s},u} = \int_{\mathring{r}(S)} (\iota^* E)(u_{\boldsymbol{s}})\,.$$

Note that this orientation on $\mathring{r}(S)$ does not depend on the choice of the representative $\check{S}$ of S thanks



We represent edges by arrows (going from $b(e)$ to $f(e)$) and surfaces by double lines.
The plane $\check{S}\left\langle\{0\} \times \mathbb{R}^{d-1}\right\rangle$ is represented by a dashed line when its disposition is relevant,
and all surfaces are assumed to be oriented upward.

Figure 9.1 – Sign factors for generic intersections and our conventions for non-generic ones

to the way the equivalence class was defined in def. 9.7. □

We now come to the regularization of the Poisson Poisson brackets between the holonomy and flux observables. Thanks to the analyticity, this computation can be broken down to a few elementary cases: indeed, given an edge and a surface, we can cut the edge into parts having a simple intersection with the surface, and express the holonomy along the full edge as the composition of the holonomies along these parts (see [92, section II.6.4] or the slight reformulation of this result, in a form suitable for the projective construction, in prop. 10.7 and fig. 10.2).

In the two generic cases – an edge part that either passes cleanly through the surface or does not intersect it, the regularization give an unambiguous prescription for the Poisson brackets. There are also a few non-generic cases: edge parts that are entirely included in the surface, cross the boundary of the surface or start/end on the surface (note that the case of an edge hitting the surface tangentially is rescued by the analyticity and can be treated among the generic cases). A well-defined regularization for those can only be obtained by fine-tunning how the limit in prop. 9.10 should be taken and/or by imposing additional conditions on the δ-regularization from def. 9.8. We will not detail these considerations here, and simply accept the resulting Poisson brackets as a matter of conventions: the prescriptions we will adopt in the following are the ones commonly used in LQG, and are summarized in fig. 9.1.

Finally, we need to fix the remaining commutation relations to get the advertised holonomy-flux algebra. While the Poisson brackets of the holonomies with each others can be set to zero, in accordance with the fact that they all only depend on the connection, the same does *not* hold for the fluxes (unless the gauge group G is Abelian). Although the latter were obtained in prop. 9.9 as limits of mutually Poisson-commuting functions on the phase space, their Poisson brackets with the holonomies enforce *non-vanishing* commutators, in order to get a valid algebra of observables, namely an algebra satisfying the Jacobi identity. More precisely, the Poisson bracket of two fluxes, supported by *intersecting* surfaces S and S', and indexed by two vectors u, $u' \in \mathfrak{g}$, should be a flux-like observable (in terms its commutation relations with the holonomies), supported on the intersection $S \cap S'$ and indexed by the Lie bracket $[u, u']_{\mathfrak{g}}$. This non-commutation of the fluxes will play quite an important role throughout the development of the next section and we will discuss it in more depth in the light of prop. 10.28.

Proposition 9.10 We consider the same objects as in prop. 9.9. For any smooth map $m : G \to \mathbb{R}$ and any $u \in \mathfrak{g}$, we define:

$$\left\{\mathsf{P}^{(S,\boldsymbol{s},u)}, \mathsf{h}^{(e,\boldsymbol{s},m)}\right\}_{\lim}(A,E) := \lim_{\epsilon\to 0}\left\{\mathsf{P}^{(\check{S},\boldsymbol{s},u)}_{\epsilon}, \mathsf{h}^{(\check{e},\boldsymbol{s},m)}_{\epsilon}\right\}_{\mathcal{M}_\Sigma}(A,E),$$

whenever the limit on the right hand side exists and is independent of the δ-regularization $(\delta_\epsilon)_{\epsilon>0}$.

We have:

1. if $r(e) \cap r(S) = \varnothing$, then:

$$\left\{\mathsf{P}^{(S,\boldsymbol{s},u)}, \mathsf{h}^{(e,\boldsymbol{s},m)}\right\}_{\lim}(A,E) = 0,$$

2. if $r(e) \cap r(S) = \{\check{e}(c,0)\}$ with $c \in]0,1[$ and $\check{e}(c,0) \in \mathring{r}(S)$, then:

$$\left\{\mathsf{P}^{(S,\boldsymbol{s},u)}, \mathsf{h}^{(e,\boldsymbol{s},m)}\right\}_{\lim}(A,E) = \operatorname{sgn}(e,S)\ \frac{d}{dr} m\left[\mathcal{P}\!\exp\left(\textstyle\int_c^1 \gamma^*_{\check{e},\boldsymbol{s}}A\right).\, e^{r\,u}.\,\mathcal{P}\!\exp\left(\textstyle\int_0^c \gamma^*_{\check{e},\boldsymbol{s}}A\right)\right]\Bigg|_{r=0},$$

where $\operatorname{sgn}(e,S) = \pm 1$ if there exists $\epsilon \in]0, \epsilon_1[$ such that:

$$\check{e}\left\langle [c, c+\epsilon]\times\{0\}\right\rangle \subset \check{S}\left\langle U_{2,\pm}\right\rangle \quad \& \quad \check{e}\left\langle [c-\epsilon, c]\times\{0\}\right\rangle \subset \check{S}\left\langle U_{2,\mp}\right\rangle,$$

(with $U_{2,\pm} := U_2 \cap \left(\mathbb{R}_\pm \times \mathbb{R}^{d-1}\right)$) and $\operatorname{sgn}(e,S) = 0$ otherwise (note that this sign does not depend on the choice of the representatives $\check{e}$ of e, resp. $\check{S}$ of S).

Proof *Poisson brackets at finite ϵ.* Let $\epsilon \in]0, \min(\epsilon_1,\epsilon_2)[$. For any $\psi \in C^\infty\big([0,1]\times[-1,1], \mathfrak{g}\big)$, we have:

$$\frac{d}{dr} m\left[\mathcal{P}\!\exp\left(\int_0^1 dt\ \ \psi(t,r)\right)\right]\Bigg|_{r=0} =$$

$$= \int_0^1 dm\ \ \frac{d}{dr} m\left[\mathcal{P}\!\exp\left(\int_m^1 dt\ \ \psi(t,0)\right).\, e^{\psi(m,r)}.\,\mathcal{P}\!\exp\left(\int_0^m dt\ \ \psi(t,0)\right)\right]\Bigg|_{r=0}$$

$$= \int_0^1 dm\ \left[T_{\mathbb{1}}\, m\left[\mathcal{P}\!\exp\left(\int_m^1 dt\ \ \psi(t,0)\right).\ \cdot\ .\,\mathcal{P}\!\exp\left(\int_0^m dt\ \ \psi(t,0)\right)\right]\right]\big(\partial_r\psi(m,0)\big).$$

Let $Q \in \mathsf{H}_o(\Sigma \to T^*(\Sigma)\otimes\boldsymbol{\mathcal{G}})$ and $F \in \mathsf{H}_o(\Sigma \to \Lambda^{d-1}T^*(\Sigma)\otimes\boldsymbol{\mathcal{G}}^*)$. Applying this general formula to $\psi(t,r) = \widehat{\mathsf{X}}_{f^{\check{e},i}_{\epsilon,t},\boldsymbol{s}}(A + r\,Q, E + r\,F)\,\tau_i$ yields:

$$\left[T_{(A,E)}\mathsf{h}^{(\check{e},\boldsymbol{s},m)}_{\epsilon}\right](Q,F) = \int_0^1 dm\ \ H^m_\epsilon(\tau_i)\ \left[T_{(A,E)}\widehat{\mathsf{X}}_{f^{\check{e},i}_{\epsilon,m},\boldsymbol{s}}\right](Q,F)$$

where, for any $m \in [0,1]$, $H^m_\epsilon \in \mathfrak{g}^*$ is defined by:

$$H^m_\epsilon := \left[T_{\mathbb{1}}\, m\left[\mathcal{P}\!\exp\left(\int_m^1 dt\ \ \widehat{\mathsf{X}}_{f^{\check{e},i}_{\epsilon,t},\boldsymbol{s}}(A,E)\,\tau_i\right).\ \cdot\ .\,\mathcal{P}\!\exp\left(\int_0^m dt\ \ \widehat{\mathsf{X}}_{f^{\check{e},i}_{\epsilon,t},\boldsymbol{s}}(A,E)\,\tau_i\right)\right]\right].$$

Also, by the dominated convergence theorem, we have, for any $Q \in \mathrm{H}_o(\Sigma \to T^*(\Sigma) \otimes \boldsymbol{\mathcal{G}})$ and any $F \in \mathrm{H}_o(\Sigma \to \Lambda^{d-1} T^*(\Sigma) \otimes \boldsymbol{\mathcal{G}}^*)$:

$$\left[T_{(A,E)} \mathrm{P}_\epsilon^{(\check{S},\boldsymbol{s},u)}\right](Q,F) = \int_{B^{(d-1)}} dy \quad \tau^j(u) \left[T_{(A,E)} \widehat{\mathrm{P}}_{q^{\check{S}}_{\epsilon,y,j},\boldsymbol{s}}(A,E)\right](Q,F)\,.$$

Together with the Poisson bracket expression from def. 9.5, this implies:

$$\begin{aligned}
&\left\{\mathrm{P}_\epsilon^{(\check{S},\boldsymbol{s},u)},\, \mathrm{h}_\epsilon^{(\check{e},\boldsymbol{s},m)}\right\}_{\mathcal{M}_\Sigma}(A,E) = \\
&= \int_{B^{(d-1)}} dy \quad \tau^j(u) \int_0^1 dm \quad H_\epsilon^m(\tau_i) \left\{\widehat{\mathrm{X}}_{f^{\check{e},i}_{\epsilon,m},\boldsymbol{s}},\, \widehat{\mathrm{P}}_{q^{\check{S}}_{\epsilon,y,j},\boldsymbol{s}}\right\}_{\mathcal{M}_\Sigma}(A,E) \\
&= \int_{B^{(d-1)}} dy \int_0^1 dm \quad \tau^j(u)\, H_\epsilon^m(\tau_i) \int_{V_o} q^{\check{S}}_{\epsilon,y,j} \wedge f^{\check{e},i}_{\epsilon,m} \\
&= \int_{B^{(d-1)}} dy \int_0^1 dm \quad H_\epsilon^m(u) \int_{V_o} \check{S}^{-1,*}(\delta_{\epsilon,(0,y)} dx_1) \wedge \check{e}^{-1,*}\left(\mathrm{sgn}(\check{e})\, \delta_{\epsilon,(m,0)}\, dx_2 \wedge \ldots \wedge dx_d\right) \\
&= \int_{B^{(d-1)}} dy \int_0^1 dm \quad H_\epsilon^m(u) \int_{U_o} \delta_{\epsilon,(m,0)} \quad (\check{S}^{-1} \circ \check{e})^*(\delta_{\epsilon,(0,y)} dx_1) \wedge dx_2 \wedge \ldots \wedge dx_d \\
&= \int_{B^{(d-1)}} dy \int_0^1 dm \quad H_\epsilon^m(u) \int_{U_o} dx \quad \delta_{\epsilon,(m,0)}(x)\, \delta_{\epsilon,(0,y)}\left(\check{S}^{-1} \circ \check{e}(x)\right) \left[(\check{S}^{-1} \circ \check{e})^*\, dx_1\right]_x (\partial_{x_1})\,, \quad (9.10.1)
\end{aligned}$$

where $V_o := V_1 \cap V_2$ and $U_o := \check{e}^{-1}\langle V_o \rangle$ (oriented as subsets of Σ, resp. $\mathbb{R}^d$).

Limit in the case $r(e) \cap r(S) = \varnothing$. If $r(e) \cap r(S) = \varnothing$, then, by compacity, there exists $\epsilon_o \in \,]0, \min(\epsilon_1, \epsilon_2)[$ such that:

$$\check{e}\left\langle \bigcup_{m \in [0,1]} B^{(d)}_{\epsilon_o,(m,0)} \right\rangle \cap \check{S}\left\langle \bigcup_{y \in B^{(d-1)}} B^{(d)}_{\epsilon_o,(0,y)} \right\rangle = \varnothing\,.$$

Hence, for any $\epsilon < \epsilon_o$, $\left\{\mathrm{P}_\epsilon^{(\check{S},\boldsymbol{s},u)},\, \mathrm{h}_\epsilon^{(\check{e},\boldsymbol{s},m)}\right\}_{\mathcal{M}_\Sigma}(A,E) = 0$. Therefore, $\left\{\mathrm{P}^{(S,\boldsymbol{s},u)},\, \mathrm{h}^{(e,\boldsymbol{s},m)}\right\}_{\lim}(A,E) = 0$.

Limit in the case $r(e) \cap r(S) = \{\check{e}(c,0)\}$. We now assume that there exist $c \in \,]0,1[$ and $b \in \mathring{B}^{(d-1)}$ such that $r(e) \cap r(S) = \{\check{e}(c,0)\} = \left\{\check{S}(0,b)\right\}$. We define $\alpha : U_o \to \mathbb{R}$ and $\beta : U_o \to \mathbb{R}^{d-1}$ by:

$$\forall x \in U_o\,,\; \check{S}^{-1} \circ \check{e}(x) = \left(\alpha(x),\, \beta(x)\right).$$

Inserting this definition in eq. (9.10.1) and reordering the integrals yields:

$$\begin{aligned}
&\left\{\mathrm{P}_\epsilon^{(\check{S},\boldsymbol{s},u)},\, \mathrm{h}_\epsilon^{(\check{e},\boldsymbol{s},m)}\right\}_{\mathcal{M}_\Sigma}(A,E) \\
&\qquad = \int_{U_o} dx_1\, d\widetilde{x} \int_{[0,1] \times B^{(d-1)}} dm\, dy \quad H_\epsilon^m(u)\, \delta_\epsilon(x_1 - m, \widetilde{x})\, \delta_\epsilon\left(\alpha(x_1,\widetilde{x}), \beta(x_1,\widetilde{x}) - y\right) \partial_{x_1}\alpha(x_1,\widetilde{x})\,.
\end{aligned}$$

Next, performing the change of variables $(m,\, y) \mapsto (y_1,\, \widetilde{y}) := \left(x_1 - m,\, \beta(x_1,\widetilde{x}) - y\right)$, we get:

$$\left\{\mathrm{P}_\epsilon^{(\check{S},\boldsymbol{s},u)},\, \mathrm{h}_\epsilon^{(\check{e},\boldsymbol{s},m)}\right\}_{\mathcal{M}_\Sigma}(A,E) = \int_{U_o} dx_1\, d\widetilde{x} \int_{[x_1-1,\, x_1] \times B^{(d-1)}_{1,\beta(x_1,\widetilde{x})}} dy_1\, d\widetilde{y} \quad H_\epsilon^{x_1 - y_1}(u)\, \delta_\epsilon(y_1,\widetilde{x})\, \delta_\epsilon\left(\alpha(x_1,\widetilde{x}), \widetilde{y}\right) \partial_{x_1}\alpha(x_1,\widetilde{x})\,.$$

(9.10.2)

U_o is an open neighborhood of $(c, 0)$ in $\mathbb{R}^d$. Let $\epsilon_3 \in \left]0,\, \min(c, 1-c, 1-\|b\|)\right[$ such that:

$$B^{(d)}_{\epsilon_3,(c,0)} \subset U_o \quad \& \quad \forall x \in B^{(d)}_{\epsilon_3,(c,0)}\,, \quad \|\beta(x) - (0, b)\| < \frac{1 - \|b\|}{3}\,.$$

In particular, for any $x_1 \in \left]c - \epsilon_3,\, c + \epsilon_3\right[$, $(x_1, 0) \in U_o$ and $\|\beta(x_1, 0)\| < 1$. On the other hand, we have:

$$\{x_1 \in [0, 1] \mid (x_1, 0) \in U_o,\, \alpha(x_1, 0) = 0 \quad \& \quad \|\beta(x_1, 0)\| \leqslant 1\} = \{(0, c)\}\,.$$

Therefore, for any $x_1 \in \left]c - \epsilon_3,\, c + \epsilon_3\right[$, $\alpha(x_1, 0) = 0 \Leftrightarrow x_1 = c$. Since α is analytic, there exists $N \geqslant 1$ such that:

$$\partial^N_{x_1} \alpha(c, 0) \neq 0\,.$$

Hence, there exists $\epsilon_4 \in \left]0,\, \epsilon_3\right[$ such that:

$$\forall (x_1, \widetilde{x}) \in B^{(d)}_{\epsilon_4,(c,0)}\,, \quad \partial^N_{x_1} \alpha(x_1, \widetilde{x}) \neq 0$$

Let $\tau > 0$ and let $\epsilon_5 \in \left]0,\, \epsilon_4\right[$ such that:

$$\forall m \in [c - \epsilon_5,\, c + \epsilon_5]\,, \forall \epsilon \in \left]0,\, \epsilon_5\right[\,, \quad \left\| H^m_\epsilon(u) - H^c_o(u) \right\| < \frac{\tau}{N}\,,$$

where $H^c_o \in \mathfrak{g}^*$ is defined by:

$$H^c_o := \left[T_{\mathbf{1}} m \left[\mathcal{P}\exp\left(\int_c^1 dt \;\; \gamma^*_{\breve{e},\boldsymbol{s}} A \right) \cdot\; \cdot\; . \mathcal{P}\exp\left(\int_0^c dt \;\; \gamma^*_{\breve{e},\boldsymbol{s}} A \right) \right] \right].$$

Finally, $\breve{e}\left\langle [0, c - \epsilon_5/3] \cup [c + \epsilon_5/3, 1] \right\rangle$ and $r(S)$ are compact subsets of Σ and do not intersect, hence, there exists $\epsilon_6 \in \left]0,\, \min(\epsilon_1, \epsilon_2)\right[$ such that:

$$\breve{e} \left\langle \bigcup_{m \in [0, c - \epsilon_5/3] \cup [c + \epsilon_5/3, 1]} B^{(d)}_{\epsilon_6,\,(m,0)} \right\rangle \cap \breve{S} \left\langle \bigcup_{y \in B^{(d-1)}} B^{(d)}_{\epsilon_6,\,(0,y)} \right\rangle = \varnothing\,.$$

Note that this in particular implies $\epsilon_6 < \epsilon_5/3$ and we have, for any $\epsilon \in \left]0,\, \epsilon_6\right[$:

$$\forall (x_1,\, \widetilde{x}) \in U_o \setminus \left([c - 2\epsilon_5/3,\, c + 2\epsilon_5/3] \times B^{(d-1)}_{\epsilon_5/3} \right),$$
$$\forall m \in [0, 1]\,, \forall y \in B^{(d-1)}, \quad (x_1,\, \widetilde{x}) \in B^{(d)}_{\epsilon,(m,0)} \Rightarrow \breve{S}^{-1} \circ \breve{e}(x_1,\, \widetilde{x}) \notin B^{(d)}_{\epsilon,\,(0,y)}\,.$$

Moreover, for any $(x_1,\, \widetilde{x}) \in [c - 2\epsilon_5/3,\, c + 2\epsilon_5/3] \times B^{(d-1)}_{\epsilon_5/3}$, we have $(x_1,\, \widetilde{x}) \in B^{(d)}_{\epsilon_5,\,(c,0)} \subset B^{(d)}_{\epsilon_4,\,(c,0)} \subset B^{(d)}_{\epsilon_3,\,(c,0)}$, and, by definition of ϵ_3:

$$[-\epsilon_5/3,\, \epsilon_5/3] \times B^{(d-1)}_{\epsilon_5/3} \subset [x_1 - 1,\, x_1] \times B^{(d-1)}_{1,\beta(x_1,\widetilde{x})}\,.$$

For $\epsilon \in \left]0,\, \epsilon_6\right[$, we can thus rewrite eq. (9.10.2) as:

$$\left\{ \mathsf{P}^{(\breve{S},\boldsymbol{s},u)}_\epsilon,\, \mathsf{h}^{(\breve{e},\boldsymbol{s},m)}_\epsilon \right\}_{\mathcal{M}_\Sigma} (A, E)$$
$$= \int_{c-2\epsilon_5/3}^{c+2\epsilon_5/3} dx_1 \int_{B^{(d-1)}_{\epsilon_5/3}} d\widetilde{x} \int_{[-\epsilon_5/3,\, \epsilon_5/3] \times B^{(d-1)}_{\epsilon_5/3}} dy_1\, d\widetilde{y} \;\; H^{x_1 - y_1}_\epsilon(u)\, \delta_\epsilon(y_1, \widetilde{x})\, \delta_\epsilon\left(\alpha(x_1, \widetilde{x}), \widetilde{y}\right) \partial_{x_1} \alpha(x_1, \widetilde{x})$$

$$= \int_{B^{(d-1)}_{\epsilon_5/3}} d\widetilde{x} \int_{-\epsilon_5/3}^{\epsilon_5/3} dy_1 \;\; \delta_\epsilon(y_1, \widetilde{x}) \int_{c-2\epsilon_5/3}^{c+2\epsilon_5/3} dx_1\, \partial_{x_1}\alpha(x_1, \widetilde{x}) \;\; H_\epsilon^{x_1-y_1}(u) \int_{B^{(d-1)}_{\epsilon_5/3}} d\widetilde{y} \;\; \delta_\epsilon\big(\alpha(x_1, \widetilde{x}), \widetilde{y}\big) \,.$$

Since $\{x_1 \in [c - 2\epsilon_5/3,\ c + 2\epsilon_5/3] \mid \alpha(x_1, 0) = 0\} = \{c\}$, we have:

$$\begin{array}{llll} \alpha(c - 2\epsilon_5/3,\ 0) < 0 & \& \ \ \alpha(c + 2\epsilon_5/3,\ 0) > 0 & \text{if} & \operatorname{sgn}(e, S) = +1 \\ \alpha(c - 2\epsilon_5/3,\ 0) > 0 & \& \ \ \alpha(c + 2\epsilon_5/3,\ 0) < 0 & \text{if} & \operatorname{sgn}(e, S) = +1 \\ \alpha(c \pm 2\epsilon_5/3,\ 0) > 0 & \text{or} \ \ \alpha(c \pm 2\epsilon_5/3,\ 0) < 0 & \text{if} & \operatorname{sgn}(e, S) = 0 \end{array} .$$

Let $\epsilon_7 \in \,]0,\ \epsilon_6[$ such that:

$$\forall \widetilde{x} \in B^{(d-1)}_{\epsilon_7}\,,\ \big|\alpha(c \pm 2\epsilon_5/3,\ \widetilde{x}) - \alpha(c \pm 2\epsilon_5/3,\ 0)\big| < \frac{\big|\alpha(c \pm 2\epsilon_5/3,\ 0)\big|}{2}\,,$$

and let $\epsilon_o := \min\big(\epsilon_7,\ \frac{1}{2}\big|\alpha(c + 2\epsilon_5/3,\ 0)\big|,\ \frac{1}{2}\big|\alpha(c - 2\epsilon_5/3,\ 0)\big|\big)$. Then, for any $\epsilon \in \,]0,\ \epsilon_o[$ and any $\widetilde{x} \in B^{(d-1)}_{\epsilon}$, we get:

$$\int_{c-2\epsilon_5/3}^{c+2\epsilon_5/3} dx_1\, \partial_{x_1}\alpha(x_1, \widetilde{x}) \int_{B^{(d-1)}_{\epsilon_5/3}} d\widetilde{y} \;\; \delta_\epsilon\big(\alpha(x_1, \widetilde{x}), \widetilde{y}\big) = \int_{\alpha(c-2\epsilon_5/3,\, \widetilde{x})}^{\alpha(c+2\epsilon_5/3,\, \widetilde{x})} d\alpha \int_{B^{(d-1)}_{\epsilon_5/3}} d\widetilde{y} \;\; \delta_\epsilon\big(\alpha, \widetilde{y}\big) = \operatorname{sgn}(e, S)\,,$$

by performing a change of variable in the *oriented* integral and using the properties of δ_ϵ.

Therefore, for any $\epsilon \in \,]0,\ \epsilon_o[$, we have:

$$\begin{aligned} &\left|\left\{\mathrm{P}_\epsilon^{(\check{S},\boldsymbol{s},u)},\ \mathrm{h}_\epsilon^{(\check{e},\boldsymbol{s},m)}\right\}_{\mathcal{M}_\Sigma}(A, E) - \operatorname{sgn}(e, S)\, H_o^c(u)\right| \\ &\quad \leqslant \int_{B^{(d-1)}_{\epsilon_5/3}} d\widetilde{x} \int_{-\epsilon_5/3}^{\epsilon_5/3} dy_1 \;\; \delta_\epsilon(y_1, \widetilde{x}) \int_{c-2\epsilon_5/3}^{c+2\epsilon_5/3} dx_1 \left|\partial_{x_1}\alpha(x_1, \widetilde{x})\right| \; \left|H_\epsilon^{x_1-y_1}(u) - H_o^c(u)\right| \int_{B^{(d-1)}_{\epsilon_5/3}} d\widetilde{y} \;\; \delta_\epsilon\big(\alpha(x_1, \widetilde{x}), \widetilde{y}\big) \\ &\quad \leqslant \frac{\tau}{N} \int_{B^{(d-1)}_{\epsilon_5/3}} d\widetilde{x} \int_{-\epsilon_5/3}^{\epsilon_5/3} dy_1 \;\; \delta_\epsilon(y_1, \widetilde{x}) \int_{[c-2\epsilon_5/3,\, c+2\epsilon_5/3]} dx_1 \;\left|\partial_{x_1}\alpha(x_1, \widetilde{x})\right| \int_{B^{(d-1)}_{\epsilon_5/3}} d\widetilde{y} \;\; \delta_\epsilon\big(\alpha(x_1, \widetilde{x}), \widetilde{y}\big) \,. \end{aligned}$$

To perform the change of variable $x_1 \mapsto \alpha(x_1,\ \widetilde{x})$ in the *unoriented* integral over x_1, we need to decompose its domain $[c - 2\epsilon_5/3,\ c + 2\epsilon_5/3]$ into pieces over which $\alpha(\,\cdot\,,\ \widetilde{x})$ is injective. Since, for any $\widetilde{x} \in B^{(d-1)}_{\epsilon_5/3}$, $\partial^N_{x_1}\alpha(\,\cdot\,,\ \widetilde{x})$ is non-zero over this domain (by definition of ϵ_4), $\partial_{x_1}\alpha(\,\cdot\,,\ \widetilde{x})$ changes its sign at most $N - 1$ times. Hence, we need at most N such pieces. Therefore, we get:

$$\forall \epsilon \in \,]0,\ \epsilon_o[\,,\quad \left|\left\{\mathrm{P}_\epsilon^{(\check{S},\boldsymbol{s},u)},\ \mathrm{h}_\epsilon^{(\check{e},\boldsymbol{s},m)}\right\}_{\mathcal{M}_\Sigma}(A, E) - \operatorname{sgn}(e, S)\, H_o^c(u)\right| \leqslant \tau \,.$$

As τ was arbitrary, this concludes the proof. □

The claim made many times above that the holonomy-flux algebra is particularly suitable for the description of a *diffeomorphism-invariant* theory of connections can be substantiated by observing that diffeomorphisms act on holonomies and fluxes simply by shifting the supporting edges and surfaces. More precisely, keeping in mind that we need to preserve analyticity, what we have is an action of the *semi-analytic* diffeomorphisms [92, section IV.20] on the algebra generated by the holonomy and flux observables (regardless of whether fully analytic or semi-analytic edges and surfaces are used, because, as hinted above, the *generated algebra* is the same in both cases, see subsection 12.2). This is sufficient to implement diffeomorphism invariance because arbitrary diffeomorphisms can be approximated by semi-analytic ones (by contrast, fully analytic diffeomorphisms, that can be completely characterized by their restriction to any small open subset of Σ, do *not* form a dense subgroup of the group of all diffeomorphisms; see the extended discussion in section 19 exploring what is needed for a satisfactory implementation of diffeomorphism invariance).

Actually[1], one could argue that the algebra generated by the linear smearings introduced in def. 9.5 (say compactly supported ones if we want to switch back to a possibly non-compact slice Σ) also carry an action of the group of diffeomorphisms: the observable labeled by a certain $(d-1)$- or 1-form is transformed into the observable labeled by the *pullback* of this form. In addition, a projective state space could be easily derived for this algebra: the configuration spaces spanned by finitely many smearings of the connection are *affine* spaces, so we would be in the context of [68] (that we covered in theorem 6.2 and that we will reexamine from a different perspective in subsection 18.1). However, the behavior of these linear observables under *gauge* transformation is not very convenient: in particular, if we want to express the transformation of a smearing of the connection in terms of a smearing in the *same* reference cross-section $\boldsymbol{s}$, we need to include an extra contribution (because connections only transform as cross-sections of $\boldsymbol{\mathcal{G}}$ up to an additional constant, coming from the derivative of the gauge transformation under consideration). By contrast, the transformation of an holonomy is perfectly transparent, as it only depends on the gauge shifts at the beginning and at the end of the edge.

On the other hand, fluxes in a *fixed* cross-section $\boldsymbol{s}$ do not transform well under gauge transformation: unless the considered gauge transformation is *constant* (with respect to $\boldsymbol{s}$) along the surface supporting a flux observable, the transformation of this observable will *not* be of the flux type (at least not relative to this cross-section $\boldsymbol{s}$). Instead, it will be a *weighted* flux, obtained by integrating the electric field weighted by a ($\mathfrak{g}$-valued) function on the surface (so that the non-constant gauge transformation can be absorbed into the weight function). This is not a problem in the context of the Ashtekar-Lewandowski Hilbert space $\mathcal{H}_{\text{AL}}$ because, as stressed in the main introduction, the nature of the Ashtekar-Lewandowski vacuum enforces discreteness of the geometry along the slice (which is encoded by the electric field): more precisely, the flux operator through a surface S, when evaluated on states in $\mathcal{H}_{\text{AL}}$, is effectively supported on *discrete points* of S. Thus, the full algebra generated by holonomies and *weighted* fluxes can be represented on $\mathcal{H}_{\text{AL}}$ without overhead. Unfortunately, weighted fluxes *cannot* be incorporated in the projective structure we will setup in the subsequent sections (basically because, as argued above, fluxes on intersecting surfaces do not Poisson-commute, so that the algebra generated by two different weighted fluxes supported on the *same* surface could be infinite-dimensional, which would prevent us from assembling the continuum theory out of small, finite-dimensional phase spaces).

Because of this, the projective state space we will construct will *not* be gauge-invariant. In particular, we will, from now on, assume that a *global* cross-section $\boldsymbol{s}$ can be chosen in $\boldsymbol{\mathcal{P}}$ (dropping from the notations the explicit $\boldsymbol{s}$-labeling of all observables): both the algebra of observables and the state space will *depend* on this choice. This is a severe shortcoming of the construction, and we will sketch in sections 13 and 20 a path to overcome it.

1. Thanks to Jerzy Lewandowski for drawing my attention to this possibility.

10. Factorizing system

The construction employed in LQG to obtain the Ashtekar-Lewandowski representation of the holonomy-flux algebra uses an inductive limit of 'small' Hilbert spaces, with building blocks labeled by *graphs* (see [60, 8], [92, section II.8.2] or subsection 12.2). Equivalently, it can be understood as relying on a projective limit of finite dimensional configuration spaces (recall that we have underlined the correspondence around prop. 6.5). Each graph corresponds to a selection of position variables, namely the holonomies along its edges: it thus defines a small configuration space, that can be identified with G^N (one group variable per edge).

In principle, we could associate to any such graph a corresponding phase space, as the cotangent bundle on its configuration space. However, if we now consider a big graph γ' and a subgraph γ of γ', there is no preferred way of defining a projection from the phase space $\mathcal{M}_{\gamma'}$ thus associated to γ' into the phase space $\mathcal{M}_\gamma$ associated to γ. In order to define unambiguously such a projection, we would indeed have to specify how the impulsion variables described by $\mathcal{M}_\gamma$ should be transported to $\mathcal{M}_{\gamma'}$ (having in mind that a downward projection between the phase spaces is dual to an upward injection between the algebras of observables). We pointed out in prop. 2.10 that a projection between the phase spaces encapsulates, at least locally, the same information as a factorization of $\mathcal{M}_{\gamma'}$ into a Cartesian product of $\mathcal{M}_\gamma$ times a complementary phase space $\mathcal{M}_{\gamma'\to\gamma}$ (remember that the complementary degrees of freedom were characterized by their vanishing Poisson brackets with the retained ones). In addition, if the projection we are considering is compatible with the splitting of the phase spaces into position and momentum variables (like in the setup we considered in theorem 6.2), this factorization of $\mathcal{M}_{\gamma'}$ should go down to a factorization $\mathcal{C}_{\gamma'} \approx \mathcal{C}_{\gamma'\to\gamma} \times \mathcal{C}_\gamma$ of the underlying configuration spaces.

As stressed before def. 2.15, it is essential to get such a preferred choice of complementary configuration variables (spanning $\mathcal{C}_{\gamma'\to\gamma}$) that we specify not only which configuration variables are to be retained by γ, but also which momentum variables: if we are only provided with a projection between configuration spaces, we cannot single out suitable complementary variables within $\mathcal{C}_{\gamma'}$. These considerations suggest that the desired projective structure should rely on labels that are made not only of edges but also of surfaces, whose role will be, for each label η, to select which fluxes are to be the momentum variables associated to η. Besides, such mixed labels clearly sounds promising in view of giving the holonomies and fluxes a more symmetric status (in support of the goal, put forward in the main introduction, to help study of the semi-classical regime of the theory).

The need to include surfaces in the labels was already recognized by Okołów in [68, 69]. The label set he was using is however not immediately applicable to the non-Abelian case, which requires, as we will see, to impose more restrictive conditions on the relative disposition of the edges and surfaces. The reason why complications emerge in the non-Abelian case is the following. As mentioned above, a projection from the phase spaces associated to a finer label η' into the one associated to a coarser label η is dual to an embedding of the algebra of observables selected by η into the algebra of observables selected by η'. Moreover, this embedding is linear and preserves the Poisson brackets (prop. 2.2), ie. it is an injective algebra morphism. But this requires that the vector space generated by the observables associated to η, within the algebra of η', should be closed under Poisson brackets, and that the algebra structure thus induced by η' should match the one seen from η.

This is a rather harmless requirement in the Abelian case, for there the Poisson bracket of a flux variable with an holonomy variable is just a constant (possibly 0 depending on the intersection of the corresponding edge and surface) and the flux operators commute with each other (see the comment preceding prop. 9.10), so the set of observables associated to a collection of edges and surfaces will automatically be closed under Poisson brackets. The aim of the present section will therefore be to determine which collections of edges and surfaces are admissible when the gauge group is arbitrary, and to check that the label set they are forming, although much reduced, is still directed.

Actually there do exist possibilities to write the state space of a theory of connection in projective form while using labels made of edges only: two such models have been for example proposed in [66] and, at the classical level, in [89]. In subsection 12.2, we will discuss in more details how a non ambiguous choice of complementary variables is achieved in these proposals without explicitly referring to the momentum variables in the definition of the labels, and why the thus obtained projective structures would altogether not fit our purpose.

10.1 Definition of the label set

We begin by recalling a few elementary properties on the subject of edges and surfaces [92, section II.6], that we will use again and again in the following.

Subedges, and in particular the splitting of an edge into parts, will play an important role in the construction, because the holonomy along a composed edge is just the composition of the holonomies on its parts.

Proposition 10.1 Let $e \in \mathcal{L}_{\text{edges}}$ and let $p \neq p'$ be two distinct points in $r(e)$. Then, there exists a unique edge $e_{[p,p']} \in \mathcal{L}_{\text{edges}}$ such that:

$$r\left(e_{[p,p']}\right) \subset r(e)\,, \quad b\left(e_{[p,p']}\right) = p \quad \& \quad f\left(e_{[p,p']}\right) = p'\,. \tag{10.1.1}$$

We denote by e^{-1} the reversed edge $e^{-1} := e_{[f(e),b(e)]}$. We also define a strict, total order on the points of $r(e)$ by:

$$\begin{aligned} &\forall p \in r(e), \;\; b(e) <_{(e)} p \;\Leftrightarrow\; b(e) \neq p \\ &\& \;\; \forall p, p' \in r(e) \setminus \{b(e)\}\,, \;\; p <_{(e)} p' \;\Leftrightarrow\; r\left(e_{[b(e),p]}\right) \subsetneq r\left(e_{[b(e),p']}\right) \end{aligned} \tag{10.1.2}$$

For any $p_1 \neq p_4 \in r(e)$ and any $p_2 \neq p_3 \in r\left(e_{[p_1,p_4]}\right)$, we have:

1. $\left(e_{[p_1,p_4]}\right)_{[p_2,p_3]} = e_{[p_2,p_3]}$, so in particular $\left(e^{-1}\right)^{-1} = e$, $\left(e^{-1}\right)_{[p_1,p_4]} = e_{[p_1,p_4]}$ and $\left(e_{[p_1,p_4]}\right)^{-1} = e_{[p_4,p_1]}$;

2. $r\left(e_{[p_1,p_4]}\right) = \begin{cases} \{p \in r(e) \mid p_1 \leqslant_{(e)} p \leqslant_{(e)} p_4\} & \text{if } p_1 <_{(e)} p_4 \\ \{p \in r(e) \mid p_4 \leqslant_{(e)} p \leqslant_{(e)} p_1\} & \text{if } p_1 >_{(e)} p_4 \end{cases}$;

3. $p_2 <_{(e_{[p_1,p_4]})} p_3 \;\Leftrightarrow\; \begin{cases} p_2 <_{(e)} p_3 & \text{if } p_1 <_{(e)} p_4 \\ p_2 >_{(e)} p_3 & \text{if } p_1 >_{(e)} p_4 \end{cases}$.

Proof *Existence and uniqueness.* Let $p \neq p' \in r(e)$ and let $\check{e} : U \to V$ be a representative of e. Let $t \neq t' \in [0,1]$ such that $\check{e}(t,0) = p$ and $\check{e}(t',0) = p'$. The map φ_1, defined by:

$$\begin{array}{rrcl} \varphi_1 : & U & \to & \mathbb{R}^d \approx \mathbb{R} \times \mathbb{R}^{d-1} \\ & (\tau,\, x) & \mapsto & (t + \tau\,(t'-t),\, x) \end{array},$$

is an analytic diffeomorphism onto its image $W_1 := \varphi_1 \langle U \rangle$, with $\varphi_1(0,0) = (t,0)$, $\varphi_1(1,0) = (t',0)$ and $\varphi_1 \left\langle [0,1] \times \{0\}^{d-1} \right\rangle = [t,t'] \times \{0\}^{d-1}$. Next, U is an open neighborhood of $[0,1] \times \{0\}^{d-1}$ in $\mathbb{R}^d$, thus also of $[t,t'] \times \{0\}^{d-1}$. Hence, $U_1 := \varphi_1^{-1} \langle U \rangle$ is an open neighborhood of $[0,1] \times \{0\}^{d-1}$ in $\mathbb{R}^d$. Defining $W_1' = \varphi_1 \langle U_1 \rangle$ and $V_1 := \check{e} \langle W_1' \rangle$, $\check{e}_1 := \check{e}|_{W_1' \to V_1} \circ \varphi_1|_{U_1 \to W_1'}$ is an encharted edge, with $b(\check{e}_1) = p$, $f(\check{e}_1) = p'$ and $r(\check{e}_1) = \check{e} \left\langle [t,t'] \times \{0\}^{d-1} \right\rangle \subset r(\check{e})$.

Let $\check{e}_2 : U_2 \to V_2$ be an encharted edge such that $b(\check{e}_2) = p$, $f(\check{e}_2) = p'$ and $r(\check{e}_2) \subset r(\check{e})$. Since V is an open neighborhood of $r(\check{e})$ in Σ, $W_2 := (\check{e}_2)^{-1} \langle V \rangle$ is an open neighborhood of $[0,1] \times \{0\}^{d-1}$ in $\mathbb{R}^d$ and $\varphi_2 = \left(\check{e}^{-1}\right) \circ \left(\check{e}_2|_{W_2}\right) : W_2 \to U$ is an analytic diffeomorphism onto its image. Moreover, we have $\varphi_2(0,0) = (t,0)$, $\varphi_2(1,0) = (t',0)$ and $\varphi_2 \left\langle [0,1] \times \{0\}^{d-1} \right\rangle \subset [0,1] \times \{0\}^{d-1}$. So, by the intermediate value theorem, $\varphi_2 \left\langle [0,1] \times \{0\}^{d-1} \right\rangle = [t,t'] \times \{0\}^{d-1}$, and therefore $r(\check{e}_1) = r(\check{e}_2)$. Since we also have $b(\check{e}_2) = p = b(\check{e}_1)$, $\check{e}_1$ and $\check{e}_2$ are two representative of the same edge $e_{[p,p']}$.

Prop. 10.1.1 then follows immediately from eq. (10.1.1).

Order on $r(e)$. That eq. (10.1.2) unambiguously defines is a strict order on $r(e)$ (ie. an irreflexive and transitive relation) can be checked directly. Moreover, if $\check{e}$ is a representative of e and $t, t' \in [0,1]$ are such that $\check{e}(t,0) = p$ and $\check{e}(t',0) = p'$, we have, from the previous point:

$$p <_{(e)} p' \;\Leftrightarrow\; t < t' .$$

In particular, $<_{(e)}$ is therefore a total order.

Let $p_1 \neq p_4 \in r(e)$ and $p_2 \neq p_3 \in r\left(e_{[p_1,p_4]}\right)$. Using the explicit expression above for a representative of $e_{[p_1,p_4]}$, there exist $t_1 \neq t_4 \in [0,1]$ and $t_2 \neq t_3 \in [t_1,t_4]$ such that $\forall i \leqslant 4,\ \check{e}(t_i,0) = p_i$ and we have:

$$r\left(e_{[p_1,p_4]}\right) = \check{e} \left\langle [t_1,t_4] \times \{0\}^{d-1} \right\rangle \quad \text{and} \quad p_2 <_{(e_{[p_1,p_4]})} p_3 \;\Leftrightarrow\; \left(\frac{t_2 - t_1}{t_4 - t_1} < \frac{t_3 - t_1}{t_4 - t_1} \right),$$

which yields props. 10.1.2 and 10.1.3. □

Proposition 10.2 We say that $e_1, \ldots, e_n \in \mathcal{L}_{\text{edges}}$ are composable iff there exist an edge $e \in \mathcal{L}_{\text{edges}}$ and points $p_0, p_1, \ldots, p_n$ in $r(e)$ such that:

$$b(e) = p_0 <_{(e)} p_1 <_{(e)} \ldots <_{(e)} p_{n-1} <_{(e)} p_n = f(e) \quad \& \quad \forall i \in \{1,\ldots,n\},\ e_i = e_{[p_{i-1},p_i]} . \quad (10.2.1)$$

Then e is uniquely determined by $e_1, \ldots, e_n$ and we write $e = e_n \circ \ldots \circ e_1$. Moreover, the following properties holds:

1. $e_n^{-1}, \ldots, e_1^{-1}$ are composable and $e^{-1} = e_1^{-1} \circ \ldots \circ e_n^{-1}$;
2. $\forall i \leqslant j \in \{1,\ldots,n\}$, $e_i, \ldots, e_j$ are composable and $e_{[b(e_i),f(e_j)]} = e_j \circ \ldots \circ e_i$;

3. if, for all $i \in \{1, \ldots, n\}$, there exist composable edges $e_{i,1}, \ldots, e_{i,m_i} \in \mathcal{L}_{\text{edges}}$ such that $e_i = e_{i,m_i} \circ \ldots \circ e_{i,1}$, then $e_{1,1}, \ldots, e_{1,m_1}, \ldots, \ldots, e_{n,1}, \ldots, e_{n,m_n}$ are composable and:
$$e = e_{n,m_n} \circ \ldots \circ e_{n,1} \circ \ldots \circ \ldots \circ e_{1,m_1} \circ \ldots \circ e_{1,1} \,.$$

Proof Let $e_1, \ldots, e_n \in \mathcal{L}_{\text{edges}}$ and let $e \in \mathcal{L}_{\text{edges}}$ and $p_0, p_1, \ldots, p_n \in r(e)$ be as in eq. (10.2.1). Let $\check{e}$ be a representative of e and $t_0, t_1, \ldots, t_n \in [0,1]$ such that $\check{e}(t_i, 0) = p_i$ for all $i \leqslant n$. Then, using auxiliary results from the proof of prop. 10.1, eq. (10.2.1) can be rewritten as:
$$0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1$$
$$\text{and} \quad \forall i \in \{1, \ldots, n\}\,,\ \check{e}\left\langle [t_{i-1}, t_i] \times \{0\}^{d-1} \right\rangle = r(e_i) \quad \& \quad \check{e}(t_{i-1}, 0) = b(e_i)\,.$$
Thus, $r(e) = \bigcup_{i=1}^{n} r(e_i)$ and $b(e) = b(e_1)$, and therefore e is uniquely determined by $e_1, \ldots, e_n$.

Then, props. 10.2.1 to 10.2.3 can be checked using props. 10.1.1 to 10.1.3. ☐

Definition 10.3 A graph is a *finite* set of edges $\gamma \subset \mathcal{L}_{\text{edges}}$ such that:
$$\forall e \neq e' \in \gamma,\ r(e) \cap r(e') \subset \{b(e),\ f(e)\} \cap \{b(e'),\ f(e')\}\,.$$

We denote the set of graphs by $\mathcal{L}_{\text{graphs}}$ and we equip it with the preorder (reflexive and transitive relation):
$$\forall \gamma, \gamma' \in \mathcal{L}_{\text{graphs}}, \quad \gamma \preccurlyeq \gamma' \Leftrightarrow \left(\forall e \in \gamma,\ \exists\, e_1, \ldots, e_n \in \gamma',\ \exists\, \epsilon_1, \ldots, \epsilon_n \in \{\pm 1\} \;/\; e = e_n^{\epsilon_n} \circ \ldots \circ e_1^{\epsilon_1}\right). \tag{10.3.1}$$
(The transitivity of $\preccurlyeq$ follows from props. 10.2.1 and 10.2.3.)

As a warming up for the more difficult proof of directedness that we will carry out in subsection 10.2 (where we will be dealing with labels that are made of edges and surfaces), we recall here why the set of analytic graphs $\mathcal{L}_{\text{graphs}}$ is directed [60, 92]. Note that it is only in lemma 10.5 (and in its analogue for the intersection of an edge with a surface, viz. lemma 10.8) that the analyticity actually plays a role. Hence, any class of edges (and surfaces) that could provide such an intersection property would do as well for the whole construction [10].

Proposition 10.4 Let $\widetilde{\gamma}$ be a finite set of edges. Then, there exists $\gamma \in \mathcal{L}_{\text{graphs}}$ such that $\forall e \in \widetilde{\gamma},\ \{e\} \preccurlyeq \gamma$.

In particular, $\mathcal{L}_{\text{graphs}}, \preccurlyeq$ is a directed preordered set.

Lemma 10.5 Let $e, e' \in \mathcal{L}_{\text{edges}}$ such that:
$$\forall p \in r(e) \setminus \{b(e)\}\,,\ \exists\, p' \in r(e) \;/\; b(e) <_{(e)} p' <_{(e)} p \quad \& \quad p' \in r(e')\,. \tag{10.5.1}$$
Then, there exists $p \in r(e) \setminus \{b(e)\}$ such that $r\left(e_{[b(e),p]}\right) \subset r(e')$.

Proof Let $\check{e} : U \to V$, resp. $\check{e}' : U' \to V'$, be a representative of e, resp. e'. Eq. (10.5.1) can be rewritten:

$$\forall t \in \,]0,\, 1]\,,\, \exists\, t' \in \,]0,\, t\,[\quad / \quad \breve{e}(t', 0) \in r(e')\,. \tag{10.5.2}$$

U being an open neighborhood of $[0, 1] \times \{0\}^{d-1}$ in $\mathbb{R}^d$ there exists $\epsilon \in]0, 1]$ such that $]-\epsilon, \epsilon[\, \times \{0\}^{d-1} \subset U$. Now, the map $t' \mapsto \breve{e}(t', 0)$ is continuous from $]-\epsilon, \epsilon[$ into Σ and $r(e')$ is compact, so $b(e) = \breve{e}(0, 0) \in r(e') \subset V'$. Hence, since V' is an open subset of Σ, there exists $\epsilon' \in]0, \epsilon]$ such that $\breve{e}\left\langle]-\epsilon', \epsilon'[\times \{0\}^{d-1} \right\rangle \subset V'$.

Thus, we can define a map $\psi :]-\epsilon', \epsilon'[\to \mathbb{R}^{d-1}$ by:

$$\begin{array}{rcl} \psi \;:\;]-\epsilon', \epsilon'[& \to & \mathbb{R}^{d-1} \\ t & \mapsto & s \circ \left(\breve{e'}\right)^{-1} \circ \breve{e}(t, 0) \end{array},$$

where $s : \mathbb{R}^d \approx \mathbb{R} \times \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ is the projection map on the second Cartesian factor. ψ is analytic as a composition of analytic maps and, from eq. (10.5.2), 0 is an accumulation point of $\psi^{-1}\langle 0 \rangle$, hence $\psi \equiv 0$.

Next, we define the map $\psi' :]-\epsilon', \epsilon'[\to \mathbb{R}$ by:

$$\begin{array}{rcl} \psi' \;:\;]-\epsilon', \epsilon'[& \to & \mathbb{R} \\ t & \mapsto & p \circ \left(\breve{e'}\right)^{-1} \circ \breve{e}(t, 0) \end{array},$$

where $p : \mathbb{R}^d \approx \mathbb{R} \times \mathbb{R}^{d-1} \to \mathbb{R}$ is the projection map on the first Cartesian factor. ψ' is a continuous, injective map (combining $\psi \equiv 0$ with the bijectivity of $\breve{e}$ and $\breve{e'}$), $\psi'(0) \in [0, 1]$, and from eq. (10.5.2) there exists $\epsilon'' \in]0, \epsilon'[$ such that $\psi'(\epsilon'') \in [0, 1]$, hence by the intermediate value theorem $\psi'\langle [0, \epsilon''] \rangle \subset [0, 1]$. In other words, defining $p := \breve{e}(0, \epsilon'') \in r(e) \setminus \{b(e)\}$, we have $r\left(e_{[b(e), p]}\right) \subset r(e')$.
□

Proof of prop. 10.4 *Intersection of 2 edges.* Let $e_1, e_2 \in \mathcal{L}_{\text{edges}}$. We define:

$$C(e_1,\, e_2) := \{e_3 \in \mathcal{L}_{\text{edges}} \mid r(e_3) \subset r(e_1) \cap r(e_2)\}\,,$$

and:

$$c(e_1,\, e_2) := \left\{p \in r(e_1) \cap r(e_2) \;\middle|\; \forall e_3 \in C(e_1,\, e_2),\;\; p \in r(e_3) \;\Rightarrow\; p \in \{b(e_3),\, f(e_3)\}\right\}.$$

Then, for any $p \in r(e_1) \setminus \{b(e_1)\}$ we have, by applying lemma 10.5 to $e = e_{1,[p, b(e_1)]}$, $e' = e_2$ and using prop. 10.1:

$$\exists\, p' <_{(e_1)} p \;\;/\;\; \left(\forall p'' \in r(e_1),\;\; p' <_{(e_1)} p'' <_{(e_1)} p \Rightarrow p'' \notin r(e_2)\right) \;\text{ or }\; r\left(e_{1,[p', p]}\right) \subset r(e_2)\,,$$

and therefore:

$$\exists\, p' <_{(e_1)} p \;\;/\;\; \forall p'' \in r(e_1),\;\; p' <_{(e_1)} p'' <_{(e_1)} p \Rightarrow p'' \notin c(e_1,\, e_2)\,.$$

Similarly, for any $p \in r(e_1) \setminus \{f(e_1)\}$, applying lemma 10.5 to $e = e_{1,[p, f(e_1)]}$, $e' = e_2$ yields:

$$\exists\, p' >_{(e_1)} p \;\;/\;\; \forall p'' \in r(e_1),\;\; p <_{(e_1)} p'' <_{(e_1)} p' \Rightarrow p'' \notin c(e_1,\, e_2)\,.$$

Hence, choosing a representative of e_1 and using the explicit form of $<_{(e_1)}$ from the proof of prop. 10.1, we can, for any $p \in r(e_1)$, construct an open neighborhood V_p of p in $r(e_1)$ such that $c(e_1, e_2) \cap V_p \subset \{p\}$. Since $r(e_1)$ is compact, we thus have that $c(e_1, e_2)$ is finite.

Let $p \in r(e_1) \cap r(e_2) \setminus c(e_1,\, e_2)$. Using prop. 10.1.2 together with the definition of $c(e_1,\, e_2)$, there exists $p' \in r(e_1)$ such that:

$$p' <_{(e_1)} p \quad \text{and} \quad r\left(e_{1,[p',p]}\right) \subset r(e_2)\,.$$

Thus, using again the explicit form of $<_{(e_1)}$ in terms of some representative of e_1, we can define $p_{\text{inf}} \in r(e_1)$ by:

$$p_{\text{inf}} = \inf_{<_{(e_1)}} \left\{p' \in r(e_1) \;\middle|\; p' <_{(e_1)} p \quad \& \quad r\left(e_{1,[p',p]}\right) \subset r(e_2)\right\},$$

and $p_{\text{inf}} \in r(e_2)$ (for $r(e_1) \cap r(e_2)$ is closed in $r(e_1)$). Moreover, for any $p'' \in r(e_1)$ with $p_{\text{inf}} <_{(e_1)} p'' <_{(e_1)} p$, there exists $p' \in r(e_1)$ such that:

$$p' <_{(e_1)} p'' <_{(e_1)} p \quad \text{and} \quad r\left(e_{1,[p',p]}\right) \subset r(e_2)\,,$$

therefore $p'' \in r(e_2)$. Hence, $r\left(e_{1,[p_{\text{inf}},p]}\right) \subset r(e_2)$. On the other hand, if there exists $e_3 \in C(e_1,\, e_2)$ with $p_{\text{inf}} \in r(e_3)$, then $e_3 = e_{1,[b(e_3),f(e_3)]}$, thus there exists $p'' \in \{b(e_3),\, f(e_3)\}$ such that:

$$p'' \leqslant_{(e_1)} p_{\text{inf}} <_{(e_1)} p \quad \text{and} \quad r\left(e_{1,[p'',p]}\right) \subset r(e_2)\,,$$

so $p_{\text{inf}} = p''$. Therefore, $p_{\text{inf}} \in c(e_1,\, e_2)$. Similarly, we can construct $p_{\text{sup}} \in c(e_1,\, e_2)$ such that $p <_{(e_1)} p_{\text{sup}}$ and $r\left(e_{1,[p,p_{\text{sup}}]}\right) \subset r(e_2)$. To summarize, we have proved:

$$\begin{aligned} &\forall p \in r(e_1) \cap r(e_2) \setminus c(e_1,\, e_2), \\ &\qquad \exists\, p_{\text{inf}},\, p_{\text{sup}} \in c(e_1,\, e_2) \,/\, p_{\text{inf}} <_{(e_1)} p <_{(e_1)} p_{\text{sup}} \quad \& \quad r\left(e_{1,[p_{\text{inf}},p_{\text{sup}}]}\right) \subset r(e_2)\,. \end{aligned} \tag{10.5.3}$$

Intersection of 3 edges. Let $e_1, e_2, e_3 \in \mathcal{L}_{\text{edges}}$ and consider $p \in r(e_1) \cap r(e_2) \cap r(e_3)$ with $p \notin c(e_1, e_3) \cup c(e_2, e_3)$. Then, for $i = 1, 2$, there exist $p'_i, p''_i \in r(e_3)$ such that:

$$p'_i <_{(e_3)} p <_{(e_3)} p''_i \quad \text{and} \quad r\left(e_{3,[p'_i,p''_i]}\right) \subset r(e_i)\,.$$

Hence, defining $p' := \max_{<_{(e_3)}} \left(p'_1,\, p'_2\right)$ and $p'' := \min_{<_{(e_3)}} \left(p''_1,\, p''_2\right)$, we have:

$$p' <_{(e_3)} p <_{(e_3)} p'' \quad \text{and} \quad e_{3,[p',p'']} \in C(e_1,\, e_2)\,.$$

Thus, $p \notin c(e_1,\, e_2)$. In other words, we get:

$$c(e_1,\, e_2) \cap r(e_3) \subset c(e_1, e_3) \cup c(e_2, e_3)\,.$$

Directedness of $\mathcal{L}_{\text{graphs}}$. Let $\widetilde{\gamma}$ be a finite subset of $\mathcal{L}_{\text{edges}}$, and define:

$$c(\widetilde{\gamma}) := \bigcup_{e_1, e_2 \in \widetilde{\gamma}} c(e_1,\, e_2)\,.$$

Let $e_1 \in \widetilde{\gamma}$. From the previous point, we have:

$$c(e_1,\, \widetilde{\gamma}) := c(\widetilde{\gamma}) \cap r(e_1) = \bigcup_{e_2 \in \widetilde{\gamma}} c(e_1,\, e_2)\,,$$

and since all $c(e_1,\, e_2)$ are finite, so is $c(e_1,\, \widetilde{\gamma})$. Moreover, $\{b(e_1),\, f(e_1)\} = c(e_1,\, e_1) \subset c(e_1,\, \widetilde{\gamma})$, so there exist $n_{e_1} \geqslant 1$ and $p_0^{e_1}, \ldots, p_{n_{e_1}}^{e_1} \in r(e_1)$ such that:

$$b(e_1) = p_0^{e_1} <_{(e_1)} p_1^{e_1} <_{(e_1)} \ldots <_{(e_1)} p_{n_{e_1}}^{e_1} = f(e_1) \quad \text{and} \quad c(e_1, \widetilde{\gamma}) = \left\{p_0^{e_1}, \ldots, p_{n_{e_1}}^{e_1}\right\}.$$

Hence, $e_1 = e_{1,n_{e_1}} \circ \ldots \circ e_{1,1}$, where $e_{1,i} := e_{1,[p_{i-1}^{e_1}, p_i^{e_1}]}$.

Let $e_1, e_2 \in \widetilde{\gamma}$ and $i \in \{1, \ldots, n_{e_1}\}$. Suppose that there exists $p \in r(e_{1,i}) \setminus \{p_{i-1}^{e_1}, p_i^{e_1}\}$ such that $p \in r(e_2)$. By definition of $e_{1,i}$, we then have $p \in r(e_1) \cap r(e_2) \setminus c(e_1, e_2)$, so from eq. (10.5.3),

$$\exists\, k \leqslant i-1,\ \exists\, k' \geqslant i \ \ /\ \ r\left(e_{1,[p_k^{e_1}, p_{k'}^{e_1}]}\right) \subset r(e_2).$$

Thus, we have in particular $r(e_{1,i}) \subset r(e_2)$. Moreover, $r(e_{1,i}) \cap c(e_2, \widetilde{\gamma}) = r(e_{1,i}) \cap c(\widetilde{\gamma}) = r(e_{1,i}) \cap c(e_1, \widetilde{\gamma}) = \{p_{i-1}^{e_1}, p_i^{e_1}\}$. Hence, there exist $j \in \{1, \ldots, n_{e_2}\}$ such that:

$$\left(p_{i-1}^{e_1} = p_{j-1}^{e_2} \quad \& \quad p_i^{e_1} = p_j^{e_2}\right) \quad \text{or} \quad \left(p_{i-1}^{e_1} = p_j^{e_2} \quad \& \quad p_i^{e_1} = p_{j-1}^{e_2}\right).$$

In other words, we have proved:

$$\begin{aligned} &\forall e_1, e_2 \in \widetilde{\gamma},\ \ \forall i \in \{1, \ldots, n_{e_1}\}, \\ &\qquad \left(r(e_{1,i}) \cap r(e_2) \subset \{p_{i-1}^{e_1}, p_i^{e_1}\}\right) \ \text{ or } \ \left(\exists\, j \in \{1, \ldots, n_{e_2}\},\ \exists\, \epsilon = \pm 1 \,\big/\, e_{1,i} = e_{2,j}^{\epsilon}\right). \end{aligned}$$

Moreover, we have from prop. 10.1.2:

$$\forall e_1 \in \widetilde{\gamma},\ \ \forall i \neq j \in \{1, \ldots, n_{e_1}\},\ \ r(e_{1,i}) \cap r(e_{1,j}) \subset \{p_{i-1}^{e_1}, p_i^{e_1}\},$$

so we get:

$$\begin{aligned} &\forall e_1, e_2 \in \widetilde{\gamma},\ \ \forall i \in \{1, \ldots, n_{e_1}\},\ \ \forall j \in \{1, \ldots, n_{e_2}\}, \\ &\qquad \left(r(e_{1,i}) \cap r(e_{2,j}) \subset \{p_{i-1}^{e_1}, p_i^{e_1}\}\right) \ \text{ or } \ \left(\exists\, \epsilon = \pm 1 \,\big/\, e_{1,i} = e_{2,j}^{\epsilon}\right). \end{aligned} \tag{10.5.4}$$

Finally, we define the finite subset $\gamma' := \left\{e_{1,i} \,\middle|\, e_1 \in \widetilde{\gamma},\ i \in \{1, \ldots, n_{e_1}\}\right\} \subset \mathcal{L}_{\text{edges}}$ and we can construct $\gamma \subset \gamma'$, such that:

$$\forall e \in \gamma',\ \ \exists!\, \epsilon = \pm 1 \,\big/\, e^{\epsilon} \in \gamma.$$

From eq. (10.5.4), we then have:

$$\forall e, e' \in \gamma,\ \ \left(r(e) \cap r(e') \subset \{b(e), f(e)\}\right) \ \text{ or } \ \left(e = e'\right).$$

Therefore $\gamma \in \mathcal{L}_{\text{graphs}}$ and, by construction, $\forall e_1 \in \widetilde{\gamma},\ \{e_1\} \preccurlyeq \gamma$.

In particular, for any $\gamma, \gamma' \in \mathcal{L}_{\text{graphs}}$, there exists $\gamma'' \in \mathcal{L}_{\text{graphs}}$, such that $\forall e \in \gamma \cup \gamma',\ \{e\} \preccurlyeq \gamma''$, hence $\gamma, \gamma' \preccurlyeq \gamma''$. □

We now bring the surfaces into play. In accordance with prop. 9.10 (and the conventions summarized on fig. 9.1), the symplectic structure of the 'small' phase spaces spanned by finitely many holonomies and fluxes (and therefore, on the quantum side, the action of the flux operators in the position representation) will be specified by the relative positioning of the corresponding edges and surfaces.

It will make the construction in subsection 10.3 appreciably simpler to consider 'one-sided' fluxes, that only interact with the edges reaching the surface from one side (flux associated to a surface in the 'continuum' picture of def. 9.8 will then be recovered as the half-difference of the one-sided fluxes on both sides of this surface, see prop. 10.28). Also, we will impose that all edges having a

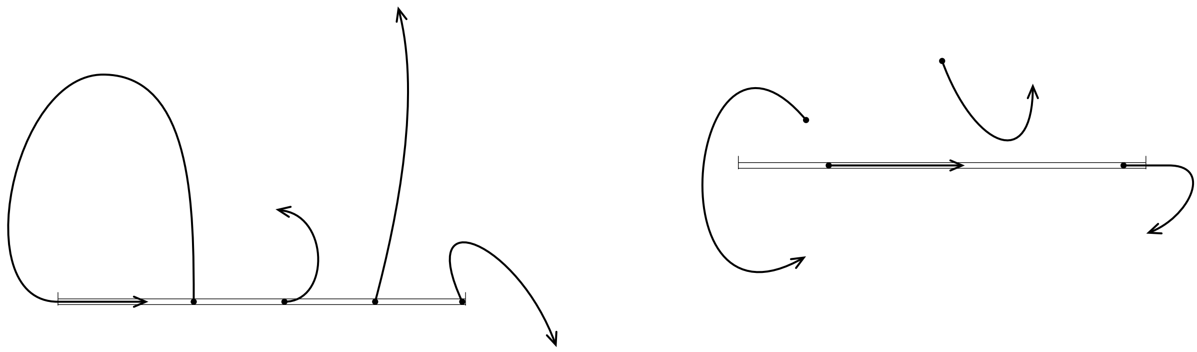


Figure 10.1 – Examples of edges above (on the left, assuming the surface is oriented upward) and indifferent to a surface (on the right)

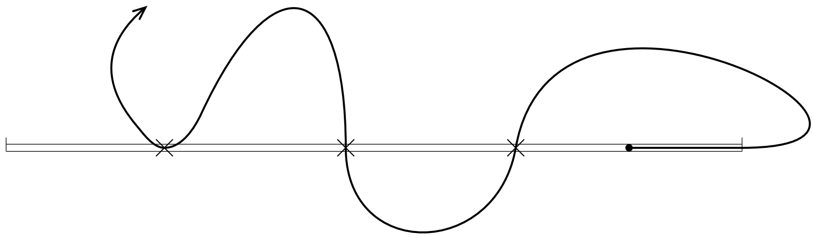
non-trivial interaction with a given surface should *start* from that surface (reorienting them if need be), so that flux operators always act at the beginning of edges. Thus, we classify the edges adapted to a surface as being above, below or indifferent to it (instead of the slightly different classification as outside/inside/up/down [92, section II.6.4]).

Since the surfaces we are considering are closed, one might be worried that an edge hitting some surface precisely on its boundary would have an unclear positioning: this is however not the case, for our surfaces have been defined in def. 9.7 as being embedded within an open analytic plane, which extends beyond the surface itself and allows to distinguish between above and below in its neighborhood. In particular, an edge intersecting the boundary of a surface can only be indifferent to that surface if it runs along the same analytic plane.

Examples of well-positioned edges are shown in fig. 10.1. Note that all figures will be drawn in the case $d = 2$ (where both edges and surfaces are one-dimensional), as this is sufficient to illustrate most aspects of the construction (we will comment on subtleties arising in the physically more relevant case $d = 3$ when appropriate).

Proposition 10.6 Let $e \in \mathcal{L}_{\text{edges}}$ and $S \in \mathcal{L}_{\text{surfcs}}$. We say that:

1. e is indifferent to S, and we write $e \rightsquigarrow S$, if there exist a representative $\check{S} : U \to V$ of S and $e_1, \dots, e_n \in \mathcal{L}_{\text{edges}}$ such that:
$$e = e_n \circ \dots \circ e_1 \quad \& \quad \forall i \in \{1, \dots, n\}\,, \quad r(e_i) \cap r(S) = \varnothing \text{ or } r(e_i) \subset \check{S}\left\langle U_o \right\rangle,$$
where $U_o := U \cap \left(\{0\} \times \mathbb{R}^{d-1}\right)$;
2. e is above S, and we write $e \uparrow S$, if there exist a representative $\check{S} : U \to V$ of S and $e_1, e_2 \in \mathcal{L}_{\text{edges}}$ such that:
$$e = e_2 \circ e_1, \quad e_2 \rightsquigarrow S, \quad r(e_1) \cap r(S) = \{b(e)\} \quad \& \quad r(e_1) \setminus \{b(e)\} \subset \check{S}\left\langle U_+ \setminus U_o \right\rangle,$$
where $U_+ := U \cap \left(\mathbb{R}_+ \times \mathbb{R}^{d-1}\right)$;
3. e is below S, and we write $e \downarrow S$, if there exist a representative $\check{S} : U \to V$ of S and $e_1, e_2 \in \mathcal{L}_{\text{edges}}$ such that:



The points in $c(e, S)$ are marked by crosses.

Figure 10.2 – Adapting an edge to a given surface

$$e = e_2 \circ e_1, \quad e_2 \curvearrowright S, \quad r(e_1) \cap r(S) = \{b(e)\} \quad \& \quad r(e_1) \setminus \{b(e)\} \subset \check{S}\left\langle U_- \setminus U_o \right\rangle,$$

where $U_- := U \cap \left(\mathbb{R}_- \times \mathbb{R}^{d-1}\right)$ and $\mathbb{R}_-$ is the set of non-positive reals.

We have the following properties:

4. these 3 cases are mutually disjoint;
5. if $e \curvearrowright S$, then $e^{-1} \curvearrowright S$;
6. if $e_1, e_2 \in \mathcal{L}_{\text{edges}}$ are such that $e = e_2 \circ e_1$, then, for any $\diamond \in \left\{\curvearrowright, \uparrow, \downarrow\right\}$:

$$e \diamond S \quad \Leftrightarrow \quad e_1 \diamond S \quad \& \quad e_2 \curvearrowright S.$$

Proof Assertion 10.6.4 follows from the definition of the equivalence relation in def. 9.7, assertions 10.6.5 and 10.6.6 from the properties of subedges (prop. 10.1) and edge compositions (prop. 10.2). □

Given some surface, we mentioned in the comments preceding prop. 9.10 that any edge can be subdivided into parts adapted to that surface [92, section II.6.4]. As announced, this is the second place where the requirement for analyticity plays a critical role: it ensures that an edge cannot cross the plane of the surface more than finitely many times (fig. 10.2). Thus, we can cut this edge at each intersection point, and, splitting again each section in two parts, we can reorient these parts so that they start from the surface.

Proposition 10.7 For any $e \in \mathcal{L}_{\text{edges}}$ and any $S \in \mathcal{L}_{\text{surfcs}}$, there exist $e_1, \ldots, e_n \in \mathcal{L}_{\text{edges}}$ and $\epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$ such that:

$$e = e_n^{\epsilon_n} \circ \ldots \circ e_1^{\epsilon_1} \quad \& \quad \forall i \in \{1, \ldots, n\}, \quad e_i \diamond_i S \quad \text{with} \quad \diamond_i \in \left\{\curvearrowright, \uparrow, \downarrow\right\}.$$

Lemma 10.8 Let $e \in \mathcal{L}_{\text{edges}}$ and $S \in \mathcal{L}_{\text{surfcs}}$. Then, there exists $p \in r(e) \setminus \{b(e)\}$ such that $e_{[b(e),p]} \diamond S$ with $\diamond \in \left\{\curvearrowright, \uparrow, \downarrow\right\}$.

Proof If $b(e) \notin r(S)$, then there exists an open neighborhood of $b(e)$ in $r(e)$ that does not intersects $r(S)$, for $r(S)$ is compact. Hence, choosing some representative of $\check{e}$ and using the explicit expression for the range of a subedge (from the proof of prop. 10.1), there exists $p \in r(e) \setminus \{b(e)\}$ such that $r\left(e_{[b(e),p]}\right) \cap r(S) = \varnothing$, so $e_{[b(e),p]} \curvearrowright S$.

We now assume $b(e) \in r(S)$ and we pick out representatives $\check{e} : U \to V$ of e and $\check{S} : U' \to V'$

of S. U is an open neighborhood of $[0,1]\times\{0\}^{d-1}$ in $\mathbb{R}^d$, V' an open neighborhood of $r(S)$ in Σ, and $\check{e}(0,0)\in r(S)$. Hence, as in the proof of lemma 10.5, there exists $\epsilon\in]0,1]$ such that $\check{e}\left\langle]-\epsilon,\epsilon[\times\{0\}^{d-1}\right\rangle\subset V'$ and we can define an analytic map $\psi:]-\epsilon,\epsilon[\to\mathbb{R}$ by:

$$\begin{array}{rcl}\psi :]-\epsilon,\epsilon[& \to & \mathbb{R} \\ t & \mapsto & p\circ\left(\check{S}\right)^{-1}\circ\check{e}(t,0)\end{array},$$

where $p:\mathbb{R}^d\approx\mathbb{R}\times\mathbb{R}^{d-1}\to\mathbb{R}$ is the projection map on the first Cartesian factor.

If there exists $\epsilon'\in]0,\epsilon[$ such that $\forall t\in]0,\epsilon'[,\psi(t)>0$ (resp. <0), we can take $p:=\check{e}(\epsilon'',0)$, with some $\epsilon''\in]0,\epsilon'[$, and we have $e_{[b(e),p]}\uparrow S$ (resp. $\downarrow S$). Hence, there only remains to consider:

$$\forall\epsilon'\in]0,\epsilon[,\ \exists t_1,t_2\in]0,\epsilon'[\ /\ \psi(t_1)\geqslant 0\quad\&\quad\psi(t_2)\leqslant 0\,.$$

The intermediate value theorem yields:

$$\forall\epsilon'\in]0,\epsilon[,\ \exists t\in]0,\epsilon'[\ /\ \psi(t)=0\,.$$

Therefore, 0 is an accumulation point of $\psi^{-1}\langle 0\rangle$, so $\psi\equiv 0$. Defining $p=\check{e}(\epsilon',0)$ for some $\epsilon'\in]0,\epsilon[$, we thus get $e_{[b(e),p]}\curvearrowright S$. □

Proof of prop. 10.7 *Transversal crossings $c(e,S)$.* Let $e\in\mathcal{L}_{\text{edges}}$ and $S\in\mathcal{L}_{\text{surfcs}}$. We define:

$$c(e,S):=\left\{p\in r(e)\ \middle|\ \exists p'\in r(e)\setminus\{p\}\ /\ e_{[p,p']}\uparrow S\text{ or }e_{[p,p']}\downarrow S\right\}. \tag{10.8.1}$$

Let $p\in r(e)\setminus\{b(e)\}$. Applying lemma 10.8 to $e_{[p,b(e)]}$ and S, there exists $p'<_{(e)}p$ such that $e_{[p,p']}\diamond_p S$ with $\diamond_p\in\left\{\curvearrowright,\uparrow,\downarrow\right\}$. Let $q\in r(e)$ such that $p'<_{(e)}q<_{(e)}p$ and assume that there exists $q'\in r(e)\setminus\{q\}$ such that $e_{[q,q']}\diamond_q S$ with $\diamond_q\in\left\{\curvearrowright,\uparrow,\downarrow\right\}$. Since $q\in r(e_{[p,p']})\setminus\{p,p'\}$, there exists $q''\in\left(r(e_{[q,q']})\setminus\{q,q'\}\right)\cap\left(r(e_{[p,p']})\setminus\{p,p'\}\right)$. From prop. 10.6.6, $e_{[q,q'']}\diamond_q S$. On the other hand, we have either $e_{[p,p']}=e_{[q'',p']}\circ e_{[q,q'']}\circ e_{[p,q]}$ (if $q''<_{(e)}q$) or $e_{[p,p']}=e_{[q,p']}\circ e_{[q'',q]}\circ e_{[p,q'']}$ (if $q''>_{(e)}q$), so using twice prop. 10.6.6 (together with prop. 10.6.5), we get $e_{[q,q'']}\curvearrowright S$. Therefore, prop. 10.6.4 yields $\diamond_q=\curvearrowright$.

Thus, for any $p\in r(e)\setminus\{b(e)\}$, there exists $p'<_{(e)}p$ such that:

$$\forall q\in r(e),\ \ p'<_{(e)}q<_{(e)}p\quad\Rightarrow\quad q\notin c(e,S)\,.$$

Similarly, for any $p\in r(e)\setminus\{f(e)\}$, there exists $p'>_{(e)}p$ such that:

$$\forall q\in r(e),\ \ p<_{(e)}q<_{(e)}p'\quad\Rightarrow\quad q\notin c(e,S)\,.$$

As in the proof of prop. 10.4, this ensures that $c(e,S)$ is finite.

Subedges with no transversal crossing are indifferent. Let $p\neq p'\in r(e)$ such that $r(e')\cap c(e,S)=\varnothing$, with $e':=e_{[p,p']}$. Applying lemma 10.8 to e' and recalling the definition of $c(e,S)$, there exists $p''\in r(e')\setminus\{p\}$ such that $e_{[p,p'']}\curvearrowright S$. This allows to define $p_{\text{sup}}\in r(e')\setminus\{p\}$ by:

$$p_{\text{sup}}:=\sup_{<_{(e')}}\left\{p''\in r(e')\setminus\{p\}\ \middle|\ e_{[p,p'']}\curvearrowright S\right\}.$$

Applying lemma 10.8 to $e_{[p_{\sup},p]}$, there exists p'' such that:

$$p \leqslant_{(e')} p'' <_{(e')} p_{\sup} \text{ and } e_{[p_{\sup},p'']} \curvearrowright S\,.$$

On the other hand, by definition of $p_{\sup}$, there exists p''' such that:

$$p'' <_{(e')} p''' <_{(e')} p_{\sup} \text{ and } e_{[p,p''']} \curvearrowright S\,.$$

Combining props. 10.6.5 and 10.6.6, we thus get $e_{[p,p_{\sup}]} \curvearrowright S$.

Let $p'' \in r(e') \setminus \{p, p'\}$ such that $e_{[p,p'']} \curvearrowright S$. Applying lemma 10.8 to $e_{[p'',p']}$, there exists p''' such that:

$$p'' <_{(e')} p''' \leqslant_{(e')} p' \text{ and } e_{[p'',p''']} \curvearrowright S\,,$$

hence $e_{[p,p''']} \curvearrowright S$ (using again prop. 10.6.6), and therefore $p'' <_{(e')} p_{\sup}$. So we have $p_{\sup} = p'$. Together with the previous point, this implies:

$$\forall p \neq p' \in r(e),\ \ r\left(e_{[p,p']}\right) \cap c(e,\, S) = \varnothing \;\Rightarrow\; e_{[p,p']} \curvearrowright S\,. \tag{10.8.2}$$

Well-positioned subedges. Let $p \neq p' \in r(e)$ such that $r\left(e_{[p,p']}\right) \cap c(e,\, S) \subset \{p\}$. Applying lemma 10.8 to $e_{[p,p']}$, there exists $p'' \in r\left(e_{[p,p']}\right) \setminus \{p\}$ such that $e_{[p,p'']} \diamond S$ with $\diamond \in \left\{\curvearrowright, \uparrow, \downarrow\right\}$. If $p'' = p'$, then $e_{[p,p']} \diamond S$. Otherwise, we have $r\left(e_{[p'',p']}\right) \cap c(e,\, S) = \varnothing$, so from eq. (10.8.2), $e_{[p'',p']} \curvearrowright S$, and therefore $e_{[p,p']} \diamond S$ (using $e_{[p,p']} = e_{[p'',p']} \circ e_{[p,p'']}$ together with prop. 10.6.6). Thus, we have proved:

$$\forall p \neq p' \in r(e),\ \ r\left(e_{[p,p']}\right) \cap c(e,\, S) \subset \{p\} \;\Rightarrow\; e_{[p,p']} \diamond S \text{ with } \diamond \in \left\{\curvearrowright, \uparrow, \downarrow\right\}. \tag{10.8.3}$$

Decomposition of e adapted to S. Since $c(e,\, S)$ is finite there exists $n \geqslant 1$, $\kappa \in \{0, 1\}$ and $p_0, p_1, \ldots, p_n \in r(e)$ such that:

$$b(e) = p_0 <_{(e)} p_1 <_{(e)} \ldots <_{(e)} p_{n-1} <_{(e)} p_n = f(e)\,,$$

and:

$$c(e,\, S) = \{p_{2k+\kappa} \mid k \in \mathbb{N},\, 2k + \kappa \leqslant n\}\,. \tag{10.8.4}$$

For $i \in \{1, \ldots, n\}$, we define:

$$\epsilon_i = \begin{cases} +1 & \text{if } i + \kappa \text{ is odd} \\ -1 & \text{if } i + \kappa \text{ is even} \end{cases}\,,$$

and $e_i = e_{[p_{i-1},p_i]}^{\epsilon_i}$. From eqs. (10.8.3) and (10.8.4), there exists $\diamond_i \in \left\{\curvearrowright, \uparrow, \downarrow\right\}$ such that $e_i \diamond_i S$. Moreover, we have $e = e_n^{\epsilon_n} \circ \ldots \circ e_2^{\epsilon_2} \circ e_1^{\epsilon_1}$. □

As argued at the beginning of the present section, a satisfactory projective limit of phase spaces for conjugate holonomy and flux variables requires labels containing not only edges but also surfaces. The difficulty is that we cannot prevent the surfaces in a label to intersect wildly, for this would void the hopes for directedness: if a surface S_1 belongs to some label, and a surface S_2 belongs to some other label, there has to be, in a directed label set, a label containing both S_1 and S_2 at the same time. On the other hand, the set of variables described by a label should be closed under Poisson brackets: as already stressed, the algebra of observables associated to some label η will be mounted by pullback into the algebra of any finer label η' in a way that preserves the Poisson brackets (prop. 2.2), so the brackets between two variables should be correct all the way down from the very first label in which these two variables appear. Since we know from subsection 9.2 that fluxes associated to intersecting surfaces do not commute (at least as soon as the gauge group G is non-Abelian), the additional variables arising as their Poisson brackets should therefore be included as soon as these surfaces are considered.

Thus, whenever the surfaces in a label η intersect, the flux variables supported on their intersection are naturally among the observables selected by η. Accordingly, the momentum variables assigned to this label are not attached to the individual surfaces in η, but rather to so called 'faces', which enumerate all possible non trivial ways of positioning an edge with respect to these surfaces. It might be that different collections of surfaces actually result in the same set of momentum variables, which motivates the equivalence relation introduced in def. 10.10. Also, the ordering of the corresponding equivalence classes is prescribed by the comparison of their associated algebras of momentum variables (as will become clear in prop. 10.26).

This also means that the flux operators we are retaining as observables (prop. 10.27) are not just the ones supported on the 'round' surfaces described in def. 9.7, but also on all finite intersections and differences thereof. This is the reason why we could afford to start form a fairly limited class of surfaces (although relaxing our definition, eg. by cutting an arbitrary compact piece out of an analytic plane, instead of only considering disk-shaped surfaces, would be relatively harmless).

Proposition 10.9 Let $\widetilde{\lambda}$ be a *finite* set of surfaces. For $\diamond : \widetilde{\lambda} \to \left\{ \curvearrowright, \uparrow, \downarrow \right\}$, $S \mapsto \diamond_S$, we define:

$$F_\diamond(\widetilde{\lambda}) := \left\{ e \in \mathcal{L}_{\text{edges}} \,\middle|\, \forall S \in \widetilde{\lambda},\, e \diamond_S S \right\}.$$

In particular (abusing notations by writing $\curvearrowright$ for the constant map $S \mapsto \curvearrowright$), we have the set of all edges that are indifferent to every surface in $\widetilde{\lambda}$:

$$F_\curvearrowright(\widetilde{\lambda}) = \left\{ e \in \mathcal{L}_{\text{edges}} \,\middle|\, \forall S \in \widetilde{\lambda},\, e \curvearrowright S \right\}.$$

The set of faces in $\widetilde{\lambda}$ is defined as:

$$\mathcal{F}(\widetilde{\lambda}) := \left\{ F_\diamond(\widetilde{\lambda}) \,\middle|\, \diamond : \widetilde{\lambda} \to \left\{ \curvearrowright, \uparrow, \downarrow \right\} \;\; / \;\; F_\diamond(\widetilde{\lambda}) \neq \varnothing \;\;\&\;\; \diamond \not\equiv \curvearrowright \right\},$$

(where $\diamond \not\equiv \curvearrowright$ stands for $\{S \mid \diamond_S \neq \curvearrowright\} \neq \varnothing$). In addition, we define:

$$F_{\text{any}}(\widetilde{\lambda}) := \bigcup_{F \in \mathcal{F}(\widetilde{\lambda})} F,$$

and, for $F, F' \subset \mathcal{L}_{\text{edges}}$:

$$F' \circ F := \{e_2 \circ e_1 \mid e_1 \in F,\, e_2 \in F', \text{ and } e_1, e_2 \text{ are composable}\}.$$

We have the following properties:

1. the elements of $\mathcal{F}(\widetilde{\lambda})$ are disjoints;
2. for any $F \in \mathcal{F}(\widetilde{\lambda}) \cup \left\{ F_\curvearrowright(\widetilde{\lambda}) \right\}$, $F_\curvearrowright(\widetilde{\lambda}) \circ F = F$;

3. $F_{\rightturn}(\widetilde{\lambda}) = \left\{ e \in \mathcal{L}_{\text{edges}} \;\middle|\; \forall p \neq p' \in r(e),\;\; e_{[p,p']} \notin F_{\text{any}}(\widetilde{\lambda}) \right\}$;

4. for any $e \in \mathcal{L}_{\text{edges}}$ there exist $e_1, \ldots, e_n \in \mathcal{L}_{\text{edges}}$ and $\epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$ such that:

$$e = e_n^{\epsilon_n} \circ \ldots \circ e_1^{\epsilon_1} \text{ and } \forall i \in \{1, \ldots, n\}, \, e_i \in F_{\text{any}}(\widetilde{\lambda}) \cup F_{\rightturn}(\widetilde{\lambda}).$$

Proof Assertions 10.9.1 and 10.9.2 follow from prop. 10.6.4 and 10.6.6 respectively.

Assertion 10.9.3. From prop. 10.6, we have:

$$F_{\rightturn}(\widetilde{\lambda}) \subset \left\{ e \in \mathcal{L}_{\text{edges}} \;\middle|\; \forall p \neq p' \in r(e),\;\; e_{[p,p']} \notin F_{\text{any}}(\widetilde{\lambda}) \right\}.$$

We now want to prove the reverse inclusion.

Let $e \in \mathcal{L}_{\text{edges}}$ such that for any $p \neq p' \in r(e)$, $e_{[p,p']} \notin F_{\text{any}}(\widetilde{\lambda})$. Let $S_o \in \widetilde{\lambda}$ and $p \neq p_o \in r(e)$ such that $e_{[p,p_o]} \diamond_o S_o$ with $\diamond_o \in \left\{ \rightturn, \uparrow, \downarrow \right\}$. Choose an ordering of the finitely many remaining surfaces $\widetilde{\lambda} \setminus S = \{S_1, \ldots, S_m\}$, and define p_i and $\diamond_i$ for $i \in \{1, \ldots, n\}$ such that:

$$p_i \in r\left(e_{[p,p_{i-1}]}\right) \setminus \{p\} \text{ and } e_{[p,p_i]} \diamond_i S_i,$$

by applying inductively lemma 10.8 to $e_{[p,p_{i-1}]}$ and S_i. From prop. 10.6.6, $e_{[p,p_m]} \in F_{\diamond}(\widetilde{\lambda})$ where $\diamond \colon \widetilde{\lambda} \to \left\{ \rightturn, \uparrow, \downarrow \right\}$ is defined by $\forall i \in \{0, \ldots, m\}, \diamond_{S_i} := \diamond_i$. Since $e_{[p,p_m]} \notin F_{\text{any}}(\widetilde{\lambda})$, $\diamond \equiv \rightturn$, therefore $\diamond_o = \rightturn$.

Hence, $c(e, S_o) = \varnothing$ (where $c(e, S_o)$ has been defined in eq. (10.8.1)). So, using eq. (10.8.2) with $p = b(e)$ and $p' = f(e)$, $e \rightturn S_o$. As this holds for any $S_o \in \widetilde{\lambda}$, $e \in F_{\rightturn}(\widetilde{\lambda})$.

Assertion 10.9.4. Let $e \in \mathcal{L}_{\text{edges}}$. We define:

$$c(e, \widetilde{\lambda}) := \bigcup_{S \in \widetilde{\lambda}} c(e, S).$$

$c(e, \widetilde{\lambda})$ is finite, for $\widetilde{\lambda}$ is finite and each $c(e, S)$ is finite. Moreover, eq. (10.8.3) becomes:

$$\forall p \neq p' \in r(e),\;\; r\left(e_{[p,p']}\right) \cap c(e, \widetilde{\lambda}) \subset \{p\} \;\Rightarrow\; e_{[p,p']} \in F_{\text{any}}(\widetilde{\lambda}) \cup F_{\rightturn}(\widetilde{\lambda}).$$

Thus we can form a decomposition of e adapted to $\widetilde{\lambda}$ exactly like in the last step of the proof of prop. 10.7. □

Definition 10.10 We define on the set of finite subsets of $\mathcal{L}_{\text{surfcs}}$ an equivalence relation by:

$$\widetilde{\lambda} \sim \widetilde{\lambda}' \Leftrightarrow \mathcal{F}(\widetilde{\lambda}) = \mathcal{F}(\widetilde{\lambda}').$$

Its set of equivalence classes will be denoted by $\mathcal{L}_{\text{profls}}$. An element $\lambda \in \mathcal{L}_{\text{profls}}$ is called a profile, and we can define its set of faces $\mathcal{F}(\lambda)$ and the corresponding set of indifferent edges $F_{\rightturn}(\lambda)$, since these are the same for any representative of λ (thanks to prop. 10.9.3).

Proposition 10.11 We equip $\mathcal{L}_{\text{profls}}$ with the binary relation:

$$\forall \lambda, \lambda' \in \mathcal{L}_{\text{profls}},$$

$$\lambda \preccurlyeq \lambda' \Leftrightarrow \left(\forall F \in \mathcal{F}(\lambda),\, \exists F_1, \ldots, F_m \in \mathcal{F}(\lambda') \,\big/\, F = F_{\circlearrowright}(\lambda) \circ \bigcup_{i=1}^{m} F_i\right).$$

For any two finite sets of surfaces $\widetilde{\lambda}, \widetilde{\lambda}'$, we have:

$$\left[\widetilde{\lambda}\right]_{\text{profl}} \preccurlyeq \left[\widetilde{\lambda} \cup \widetilde{\lambda}'\right]_{\text{profl}},$$

where $[\,\cdot\,]_{\text{profl}}$ denotes the equivalence class in $\mathcal{L}_{\text{profls}}$.

In particular, $\mathcal{L}_{\text{profls}}, \preccurlyeq$ is a directed preordered set.

Proof Let $\widetilde{\lambda}, \widetilde{\lambda}'$ be two finite sets of surfaces and $F \in \mathcal{F}(\widetilde{\lambda})$. There exists $\diamond\colon \widetilde{\lambda} \to \left\{\circlearrowright, \uparrow, \downarrow\right\}$, with $\diamond \not\equiv \circlearrowright$, such that $F = F_\diamond(\widetilde{\lambda})$. We define:

$$\mathcal{F}(\widetilde{\lambda}, \widetilde{\lambda}', \diamond) := \left\{F_{\diamond'}(\widetilde{\lambda} \cup \widetilde{\lambda}') \,\middle|\, F_{\diamond'}(\widetilde{\lambda} \cup \widetilde{\lambda}') \in \mathcal{F}(\widetilde{\lambda} \cup \widetilde{\lambda}') \quad \& \quad \diamond'|_{\widetilde{\lambda}} = \diamond\right\}.$$

Let $e \in F$ ($F \neq \varnothing$ by definition of $\mathcal{F}(\widetilde{\lambda})$). By applying inductively lemma 10.8 to the surfaces in $\widetilde{\lambda}'$ and using prop. 10.6.6 (as in the proof of prop. 10.9.3), there exists $p \in r(e) \setminus \{b(e)\}$ and an extension $\diamond'\colon \widetilde{\lambda} \cup \widetilde{\lambda}' \to \left\{\circlearrowright, \uparrow, \downarrow\right\}$ of $\diamond$ such that $e_{[b(e),p]} \in F_{\diamond'}(\widetilde{\lambda} \cup \widetilde{\lambda}')$. Let $p' \in r\left(e_{[b(e),p]}\right) \setminus \{b(e), p\}$. From prop. 10.6.6, we have $e_{[b(e),p']} \in F_{\diamond'}(\widetilde{\lambda} \cup \widetilde{\lambda}')$ and $e_{[p',f(e)]} \in F_{\circlearrowright}(\widetilde{\lambda})$. Therefore, $e \in F_{\circlearrowright}(\widetilde{\lambda}) \circ F_{\diamond'}(\widetilde{\lambda} \cup \widetilde{\lambda}')$.

In particular, $F_{\diamond'}(\widetilde{\lambda} \cup \widetilde{\lambda}') \neq \varnothing$ and, since $\diamond \not\equiv \circlearrowright$, we also have $\diamond' \not\equiv \circlearrowright$. Thus, $F_{\diamond'}(\widetilde{\lambda} \cup \widetilde{\lambda}') \in \mathcal{F}(\widetilde{\lambda}, \widetilde{\lambda}', \diamond)$. So $\mathcal{F}(\widetilde{\lambda}, \widetilde{\lambda}', \diamond) \neq \varnothing$ and there exists $F_1, \ldots, F_m \in \mathcal{F}(\widetilde{\lambda} \cup \widetilde{\lambda}')$ such that:

$$\mathcal{F}(\widetilde{\lambda}, \widetilde{\lambda}', \diamond) = \{F_1, \ldots, F_m\}.$$

and we just proved that $F \subset F_{\circlearrowright}(\widetilde{\lambda}) \circ \bigcup_{i=1}^{m} F_i$.

Now, let e_1, e_2 be composable edges such that $e_1 \in F_i$ for some $i \in \{1, \ldots, m\}$ and $e_2 \in F_{\circlearrowright}(\widetilde{\lambda})$. By definition of $\mathcal{F}(\widetilde{\lambda}, \widetilde{\lambda}', \diamond)$, $e_1 \in F$, hence, by 10.9.2, $e_2 \circ e_1 \in F$. Therefore, $F_{\circlearrowright}(\widetilde{\lambda}) \circ \bigcup_{i=1}^{m} F_i \subset F$.

So, we have $\left[\widetilde{\lambda}\right]_{\text{profl}} \preccurlyeq \left[\widetilde{\lambda} \cup \widetilde{\lambda}'\right]_{\text{profl}}$.

To prove that $\mathcal{L}_{\text{profls}}, \preccurlyeq$ is a directed preordered set, only the transitivity of $\preccurlyeq$ remains to be checked. Let $\lambda, \lambda', \lambda'' \in \mathcal{L}_{\text{profls}}$ with $\lambda \preccurlyeq \lambda'$ and $\lambda' \preccurlyeq \lambda''$. Using the definition of $\preccurlyeq$ on $\mathcal{L}_{\text{profls}}$ together with prop. 10.9.3, we have:

$$F_{\circlearrowright}(\lambda'') \subset F_{\circlearrowright}(\lambda') \subset F_{\circlearrowright}(\lambda).$$

Then, for any $F \in \mathcal{F}(\lambda'')$, we can use prop. 10.9.2 to write:

$$F_{\circlearrowright}(\lambda) \circ F_{\circlearrowright}(\lambda'') \circ F = F_{\circlearrowright}(\lambda) \circ F = F_{\circlearrowright}(\lambda) \circ F_{\circlearrowright}(\lambda) \circ F,$$

so we get:

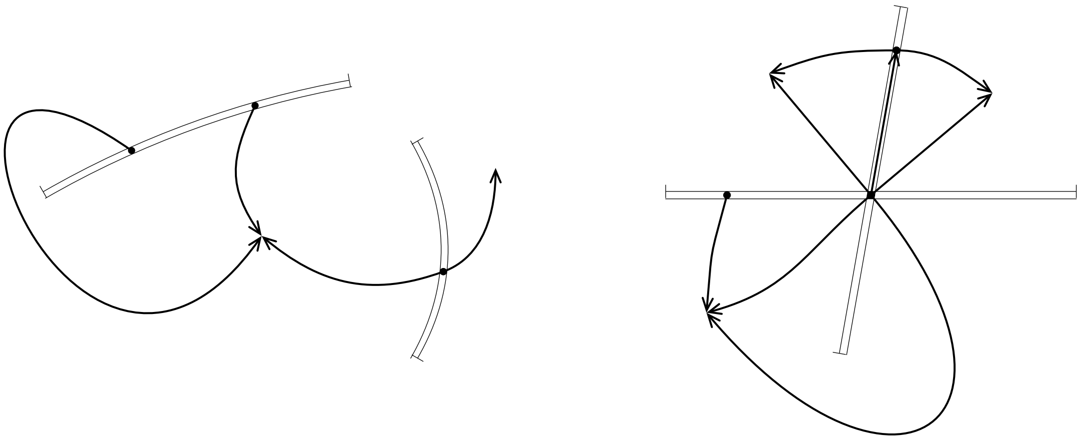


Figure 10.3 – Examples of valid labels

$$F_{\daleth}(\lambda) \circ F_{\daleth}(\lambda') \circ F = F_{\daleth}(\lambda) \circ F\,, \tag{10.11.1}$$

and therefore $\lambda \preccurlyeq \lambda''$. □

Finally, we are ready to describe what our labels should be, keeping in mind that each label is meant to be associated with a small, finite dimensional phase space, on which the observables it selects can be represented. As underlined many times, this phase space should be big enough for their Poisson algebra to be correctly reproduced. Yet it should not be too big either, otherwise the projection between the phase spaces corresponding to two labels $\eta \preccurlyeq \eta'$ would not be uniquely characterized by the sole prescription of how its pullback should mount the observables from η to η'.

These considerations reveal that edges and faces should comes in conjugate pairs. In particular, if the label contains intersecting surfaces, it should also contain edges that will probe the intersection from every side, so that the additional momentum variables promoted above are supplied with suitable conjugate configuration variables (fig. 10.3).

Definition 10.12 We define the label set $\mathcal{L}_{\text{HF}}$ by:

$$\mathcal{L}_{\text{HF}} := \left\{ (\gamma, \lambda) \in \mathcal{L}_{\text{graphs}} \times \mathcal{L}_{\text{profls}} \;\middle|\; \exists \chi : \gamma \to \mathcal{F}(\lambda) \text{ bijective } \;/\; \forall e \in \gamma,\; e \in \chi(e) \right\}.$$

For $\eta = (\gamma, \lambda)$ we define its underlying graph $\gamma(\eta) := \gamma$ and profile $\lambda(\eta) := \lambda$, its set of faces $\mathcal{F}(\eta) := \mathcal{F}(\lambda)$ and its set of indifferent edges $F_{\daleth}(\eta) := F_{\daleth}(\lambda)$, as well as the unique bijective map $\chi_\eta : \gamma(\eta) \to \mathcal{F}(\eta)$ such that $\forall e \in \gamma(\eta)$, $e \in \chi_\eta(e)$ (uniqueness follows from the fact that the faces in $\mathcal{F}(\eta)$ are disjoints, see prop. 10.9.1).

We equip $\mathcal{L}_{\text{HF}}$ with the product preorder, defined by:

$$\forall \eta, \eta' \in \mathcal{L}_{\text{HF}},\; \eta \preccurlyeq \eta' \;\Leftrightarrow\; \big(\gamma(\eta) \preccurlyeq \gamma(\eta') \quad \& \quad \lambda(\eta) \preccurlyeq \lambda(\eta')\big)\,.$$

10.2 Directedness

It is of critical importance for the intended construction of a projective state space that the label set $\mathcal{L}_{HF}$ should be directed (eg. the pivotal 'three-spaces consistency' condition gets truly useful in combination with the directedness of the label set). Since both $\mathcal{L}_{graphs}$ and $\mathcal{L}_{profls}$ are directed on their own, so is $\mathcal{L}_{graphs} \times \mathcal{L}_{profls}$, thus it is sufficient to show that $\mathcal{L}_{HF}$ is a cofinal part of $\mathcal{L}_{graphs} \times \mathcal{L}_{profls}$. In other words, given some arbitrary graph γ and profile λ, we want to construct a finer graph γ' and a finer profile λ' that are adapted to each other in the sense of def. 10.12.

For this, we will proceed in successive steps. First we will subdivide the edges of γ to adapt them to λ in the sense of prop. 10.7. Next we will add a bunch of small surfaces to ensure there is never more than one edge belonging to a given face, and we will add a few small edges to populate the faces that do not yet contain one edge. Finally we will add a few more small surfaces so that every edge has its fellow face. Note that the order of these steps is important, for we have to ensure that what has been achieved at a given point will be preserved by the subsequent steps. Also, we should take care that the whole procedure only requires a finite sequence of operations: graphs have been defined as *finite* sets of edges, while profiles arise from *finite* sets of surfaces, thus adding infinitely many edges or surfaces, or subdividing an edge into infinitely many parts, would not lead to a valid label.

Definition 10.13 For any $\gamma \in \mathcal{L}_{graphs}$ and any $\lambda \in \mathcal{L}_{profls}$, we define:

1. $M^{(1)}_{(\gamma,\lambda)} := \left\{\chi : \gamma \to \mathcal{F}(\lambda) \cup \left\{F_{\urcorner}(\lambda)\right\} \,\middle|\, \forall e \in \gamma,\ e \in \chi(e)\right\}$;
2. $M^{(2)}_{(\gamma,\lambda)} := \left\{\chi \in M^{(1)}_{(\gamma,\lambda)} \,\middle|\, \forall F \in \mathcal{F}(\lambda),\ \forall e, e' \in \chi^{-1}\left\langle F\right\rangle,\ e = e'\right\}$;
3. $M^{(3)}_{(\gamma,\lambda)} := \left\{\chi \in M^{(2)}_{(\gamma,\lambda)} \,\middle|\, \forall F \in \mathcal{F}(\lambda),\ \chi^{-1}\left\langle F\right\rangle \neq \varnothing\right\}$;
4. $M^{(4)}_{(\gamma,\lambda)} := \left\{\chi \in M^{(3)}_{(\gamma,\lambda)} \,\middle|\, \chi^{-1}\left\langle F_{\urcorner}(\lambda)\right\rangle = \varnothing\right\} = \left\{\chi : \gamma \to \mathcal{F}(\lambda) \,\middle|\, \chi \text{ bijective } \ \&\ \ \forall e \in \gamma, e \in \chi(e)\right\}$.

Theorem 10.14 $\mathcal{L}_{HF}$, $\preccurlyeq$ is a directed preordered set.

Lemma 10.15 Let $\gamma \in \mathcal{L}_{graphs}$ and $\lambda \in \mathcal{L}_{profls}$. Then, there exists $\gamma' \in \mathcal{L}_{graphs}$, such that $\gamma \preccurlyeq \gamma'$ and $M^{(1)}_{(\gamma',\lambda)} \neq \varnothing$.

Proof This follows from prop. 10.9.4 and the definition of $\preccurlyeq$ on $\mathcal{L}_{graphs}$ (def. 10.3). □

The next step is to deal with the faces that contain more than one edge of the graph. The key idea here is to add some small surfaces with respect to which these edges have distinct positionings: thus, they will not belong to the same face any more. If they have different starting points, we can simply add, for each of them, a small surface going through its starting point, with respect to witch it is, say, above (as we do for the edge on the right in fig. 10.4). If many edges start from the same point of the face, we will add, for each e among these, a small surface arranged so that e is indifferent to it, while all other edges starting from that point are either above or below it: this is another way of ensuring that two different edges will have a distinct positioning with respect to at least one of the added surfaces (see the two edges on the left in fig. 10.4).

Figure 10.4 – Adding surfaces to separate the edges into different faces

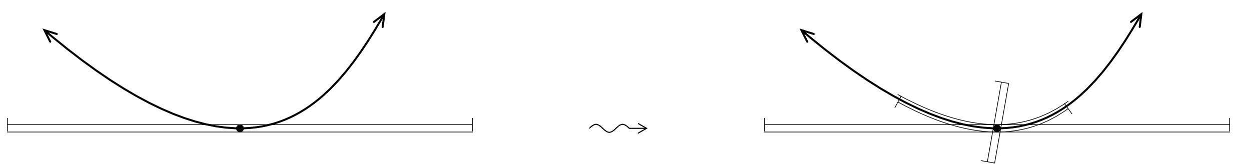


Figure 10.5 – Only a transversal surface can separate an edge from its analytic continuation

Given an edge e and a bunch of other edges $\{e'\}$ starting from the same point, we therefore want to construct a surface that contains an initial subedge of e and intersect each e' transversally at the common starting point $b(e) = b(e')$: in two dimensions, where "surfaces" are one-dimensional, we can cut such a surface out of the very analytic curve that provides e; in higher dimension, the family of surfaces that contains e is parametrized by continuous parameters, so we just need to pick out one that passes in between the finitely many edges starting from $b(e)$. However, if the analytic extension of e beyond $b(e)$ is among the edges $\{e'\}$, it will automatically be in the analytic plane of any surface containing an initial subedge of e, and thus indifferent to that surface, as illustrated on fig. 10.5. Note that it may seems at first a very unlucky special case, that precisely the analytic extension of e also belongs to the graph, but we should keep in mind that this very situation is produced in great numbers when we subdivides the edges of γ to adapt them to the given λ (fig. 10.2). To deal with this case we need to also include an additional small surface with respect to which e is of the "above" type, because its analytic continuation will then be of the "below" type: this will ensure that these two edges do not belong to the same face at the end. To avoid laborious case distinctions in the proof, we add surfaces quite liberally, and, for any initial face F containing more than one edge, and any edge e belonging to F, we will systematically add two small surfaces, one along e and the other transverse to e. Proceeding this way we probably end up with much more faces that would have been strictly necessary to separate the edges into distinct faces: we generate a lot of new faces that do not contain any edge at all. On the other hand, we will have to deal with such faces in a latter step anyway, so it does not cost us more to add more of those in the present step.

It is by contrast crucial that we preserve what has been achieved in the *previous* step, namely that all edges of the graph are adapted to the new profile. When adding a surface for an edge e, we can make it as small as we want around $b(e)$ without prejudice to the requirements above. In particular, we can ensure that it does not intersect any edge of the graph that does not go through

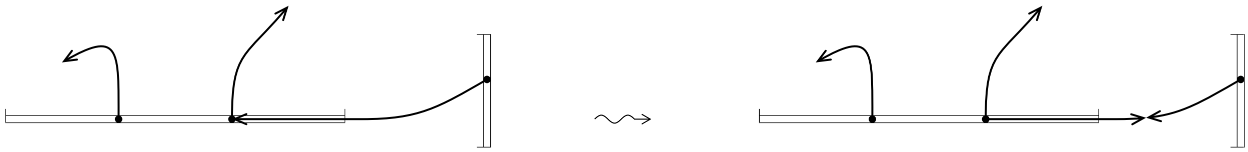


Figure 10.6 – Dealing with an edge that ends precisely where surfaces will have to be added

$b(e)$. Moreover, for any edge e' that start at $b(e)$, the analyticity of edges and surfaces will ensure that, provided the surface is chosen small enough around $b(e)$, e' will either be above or below it or indifferent to it. The only edges e' that might be problematic here are therefore those that contains $b(e)$ but not as starting point. Since e and e' are edges of a graph, $b(e)$ should then be $f(e')$, and we deal preventively with this potential source of difficulties by subdividing e' and reorienting the second part so that it now *starts* from $b(e)$ (see fig. 10.6). Note that, by doing so, we admittedly increase the number of edges, but not the number of those for which we will then need to add small surfaces, for the first part of e' takes the place of e' in $\chi(e')$, while its second part is in $F_{\rightharpoondown}(\lambda)$. Thus this does not cause infinite recursion, and the total number of surfaces added during the present step is finite, as it should.

Lemma 10.16 Let $\gamma \in \mathcal{L}_{\text{graphs}}$ and $\lambda \in \mathcal{L}_{\text{profls}}$ such that $M^{(1)}_{(\gamma,\lambda)} \neq \varnothing$. Then, there exists $\gamma' \in \mathcal{L}_{\text{graphs}}$ and $\lambda' \in \mathcal{L}_{\text{profls}}$, such that $\gamma \preccurlyeq \gamma'$, $\lambda \preccurlyeq \lambda'$ and $M^{(2)}_{(\gamma',\lambda')} \neq \varnothing$.

Proof *One-dimensional case.* We first consider the case $d = 1$. Let $\chi \in M^{(1)}_{(\gamma,\lambda)}$, $F \in \mathcal{F}(\lambda)$, and $e, e' \in \chi^{-1}\langle F\rangle$. By definition of $\mathcal{F}(\lambda)$, there exists $S \in \mathcal{L}_{\text{surfcs}}$ and $\diamond \in \{\uparrow, \downarrow\}$ such that $e, e' \diamond S$. By reversing the orientation of S if necessary, we can assume that $\diamond = \uparrow$. Let $\breve{S} : U \to V$ be a representative of S. Then, $b(e) = \breve{S}(0) = b(e')$ (for $r(S) = \{\breve{S}(0)\}$ when $d = 1$) and there exists $p \in r(e) \setminus \{b(e)\}$, resp. $p' \in r(e') \setminus \{b(e')\}$, such that $r\left(e_{[b(e),p]}\right)$, $r\left(e'_{[b(e'),p']}\right) \subset \breve{S}\left(U \cap \mathbb{R}_+\right)$.

Let $\breve{e}$, resp. $\breve{e}'$, be a representative of e, resp. e', and let t, resp. t', such that $p = \breve{e}(t)$, resp. $p' = \breve{e}'(t')$. The map $\varphi : [0,1] \to U$, $\tau \mapsto \breve{S}^{-1} \circ \breve{e}(\tau t)$ is injective as a composition of injective maps. It is everywhere postive and $\varphi(0) = 0$, so there exists $s > 0$ such that $\varphi(1) = s$ and $\varphi\langle [0,1]\rangle = [0,s]$, ie. $r\left(e_{[b(e),p]}\right) = \breve{S}\langle [0,s]\rangle$, with $\breve{S}(s) = p$. Similarly, there exists $s' > 0$ such that $r\left(e'_{[b(e'),p']}\right) = \breve{S}\langle [0,s']\rangle$, with $\breve{S}(s') = p'$. Let $s'' \in \,]0, \min(s,s')[$. We have $\breve{S}(s'') \in r(e) \cap r(e') \setminus \{b(e), f(e), f(e')\}$, hence $e = e'$ (from def. 10.3). Thus, $\chi \in M^{(2)}_{(\gamma,\lambda)}$, and in particular $M^{(2)}_{(\gamma,\lambda)} \neq \varnothing$. For the rest of this proof, we will therefore assume $d > 1$.

Construction of γ'. Let $\chi \in M^{(1)}_{(\gamma,\lambda)}$. We define:

$$\gamma_{(1,\chi)} := \left\{ e \in \gamma \;\middle|\; \exists\, e' \neq e \,/\, \chi(e) = \chi(e') \in \mathcal{F}(\lambda) \right\},$$

and:

$$B\left(\gamma_{(1,\chi)}\right) := \left\{ b(e) \mid e \in \gamma_{(1,\chi)} \right\}.$$

Let $e \in \gamma$. Since γ is a graph, $r(e) \cap B\left(\gamma_{(1,\chi)}\right) \subset \{b(e), f(e)\}$. If $f(e) \notin r(e) \cap B\left(\gamma_{(1,\chi)}\right)$, we define $\gamma'_e := \{e\}$. Otherwise, we choose some point $p \in r(e) \setminus \{b(e), f(e)\}$ and we define $\gamma'_e := \left\{e_{[b(e),p]},\ e_{[f(e),p]}\right\}$.

$\gamma' := \bigcup_{e\in\gamma}\gamma'_e$ is a graph such that $\gamma \preccurlyeq \gamma'$ and, from prop. 10.6, there exists $\chi' \in M^{(1)}_{(\gamma',\lambda)}$, satisfying $B\left(\gamma'_{(1,\chi')}\right) = B\left(\gamma_{(1,\chi)}\right)$. Moreover, we now have for any $e \in \gamma'$, $r(e) \cap B\left(\gamma'_{(1,\chi')}\right) \subset \{b(e)\}$.

Construction of λ'. Let $e \in \gamma'_{(1,\chi')}$ and define:

$$\gamma'_{(e)} := \{e' \in \gamma' \mid b(e) \in r(e')\} = \{e' \in \gamma' \mid b(e) = b(e')\}.$$

We choose a representative $\check{e} : U \to V$ of e and a real $\epsilon > 0$ such that $B^{(d)}_\epsilon \subset U$ (where $B^{(d)}_\epsilon$ is the closed ball of radius ϵ and center 0 in $\mathbb{R}^d$). We define:

$$\begin{array}{rcll} \check{S}_{\check{e},\epsilon} & : & U_\epsilon & \to & V \\ & & x & \mapsto & \check{e}(\epsilon\, x) \end{array} \quad \text{with} \quad U_\epsilon := \left\{x \in \mathbb{R}^d \;\middle|\; \epsilon\, x \in U\right\}.$$

Since $\{0\} \times B^{(d-1)} \subset B^{(d)} \subset U_\epsilon$, $\check{S}_{\check{e},\epsilon}$ is an analytic, encharted surface in Σ. We denote by $S_{\check{e},\epsilon}$ the corresponding surface (def. 9.7). We also define, for any $\theta \in S^{(d-2)}$ (with $S^{(d-2)}$ the unit sphere in $\mathbb{R}^{d-1}$):

$$\begin{array}{rcll} R_\theta & : & \mathbb{R} \times \mathbb{R}^{d-1} & \to & \mathbb{R} \times \mathbb{R}^{d-1} \\ & & (t, y) & \mapsto & (-\theta.y,\ y + (t - \theta.y)\,\theta) \end{array}.$$

R_θ is an analytic diffeomorphism $\mathbb{R}^d \to \mathbb{R}^d$ and:

$$R_\theta\left\langle \{0\} \times B^{(d-1)}\right\rangle \subset R_\theta\left\langle B^{(d)}\right\rangle = B^{(d)} \subset U_\epsilon.$$

Hence, $\check{S}_{\check{e},\epsilon,\theta}$ defined by $\check{S}_{\check{e},\epsilon,\theta} := \check{S}_{\check{e},\epsilon} \circ R_\theta : R_\theta^{-1}\left\langle U_\epsilon\right\rangle \to V$ is an analytic, encharted surface in Σ. We denote by $S_{\check{e},\epsilon,\theta}$ the corresponding surface.

Let $e' \in \gamma'_{(e)} \setminus \{e\}$. We define:

$$K_{(e,e')} = \left\{\theta \in S^{(d-2)} \;\middle|\; \exists\, p \in r(e') \setminus \{b(e')\} \;/\; e'_{[b(e'),p]} \curvearrowright S_{\check{e},\epsilon,\theta}\right\}.$$

Since $b(e') = b(e) = \check{e}(0) \in r\left(S_{\check{e},\epsilon,\theta}\right)$ for any $\theta \in S^{(d-2)}$, we have from prop. 10.6 (together with the definition of the equivalence relation in def. 9.7):

$$\forall \theta \in K_{(e,e')},\ \exists\, p' \in r(e') \setminus \{b(e')\} \;/\; r\left(e'_{[b(e'),p']}\right) \subset \check{e}\left\langle U \cap R_\theta\left\langle \{0\} \times \mathbb{R}^{d-1}\right\rangle\right\rangle,$$

where we have used:

$$\check{S}_{\check{e},\epsilon,\theta}\left\langle R_\theta^{-1}\left\langle U_\epsilon\right\rangle \cap \left(\{0\} \times \mathbb{R}^{d-1}\right)\right\rangle = \check{e}\left\langle U \cap R_\theta\left\langle \{0\} \times \mathbb{R}^{d-1}\right\rangle\right\rangle.$$

Suppose now that there exist $d-1$ vectors $\theta_1, \dots, \theta_{d-1} \in S^{(d-2)}$, linearly independent in $\mathbb{R}^{d-1}$, such that $\forall i \leqslant d-1$, $\theta_i \in K_{(e,e')}$. Then, there exists $p'' \in r(e') \setminus \{b(e')\}$ such that:

$$r\left(e'_{[b(e'),p'']}\right) \subset \check{e}\left\langle U \cap \bigcap_{i=1}^{d-1} R_{\theta_i}\left\langle \{0\} \times \mathbb{R}^{d-1}\right\rangle\right\rangle = \check{e}\left\langle U \cap \left(\mathbb{R} \times \{0\}\right)\right\rangle.$$

Since $b(e') = \check{e}(0)$ and $r(e) \cap r(e') \subset \{b(e), f(e)\}$ (for $e \neq e'$ and both belong to the graph γ'), we have, by the intermediate value theorem, $r\left(e'_{[b(e'),p'']}\right) = \check{e}\left\langle[-\alpha, 0] \times \{0\}\right\rangle$ for some $\alpha > 0$. Thus,

$e'_{[b(e'),p'']} \downarrow S_{\check{e},\epsilon}$.

Accordingly, we define:

$$\gamma'_{(e,0)} := \{e\} \cup \left\{ e' \in \gamma'_{(e)} \,\middle|\, \exists\, p'' \in r(e') \setminus \{b(e')\} \;/\; e'_{[b(e'),p'']} \downarrow S_{\check{e},\epsilon} \right\},$$

and $\gamma'_{(e,1)} := \gamma'_{(e)} \setminus \gamma'_{(e,0)}$. Then, for any $e' \in \gamma'_{(e,1)}$, $K_{(e,e')}$ has measure zero in $S^{(d-2)}$ (for example with respect to the standard measure on $S^{(d-2)}$ if $d \geqslant 3$, and with respect to the counting measure if $d = 2$). Hence, there exists $\theta_1 \in S^{(d-2)}$ such that $\forall e' \in \gamma'_{(e,1)}$, $\theta_1 \notin K_{(e,e')}$.

Now, from lemma 10.8 and prop. 10.6.6, there exist, for any $e' \in \gamma'_{(e)}$, $p_{e'} \in r(e') \setminus \{b(e')\}$ and $\diamond_{e',e,0}, \diamond_{e',e,1} \in \left\{ \curvearrowright, \uparrow, \downarrow \right\}$ such that:

$$e'_{[b(e'),p_{e'}]} \diamond_{e',e,0} S_{\check{e},\epsilon} \text{ and } e'_{[b(e'),p_{e'}]} \diamond_{e',e,1} S_{\check{e},\epsilon,\theta_1}.$$

By construction, we have:

$$\forall e' \in \gamma'_{(e,0)} \setminus \{e\},\ \diamond_{e',e,0} = \downarrow \quad \text{and} \quad \forall e' \in \gamma'_{(e,1)},\ \diamond_{e',e,1} \neq \curvearrowright.$$

On the other hand, we can check from the definition of $S_{\check{e},\epsilon}$ and $S_{\check{e},\epsilon,\theta_1}$ that:

$$\diamond_{e,e,0} = \uparrow \quad \text{and} \quad \diamond_{e,e,1} = \curvearrowright.$$

Thus, we get $\forall e' \in \gamma'_{(e)} \setminus \{e\}$, $\left(\diamond_{e',e,0}, \diamond_{e',e,1}\right) \neq \left(\diamond_{e,e,0}, \diamond_{e,e,1}\right)$.

Next, using def. 9.7 and prop. 10.6, there exists $p'_{e'} \in r\left(e'_{[b(e'),p_{e'}]}\right) \setminus \{b(e'), p_{e'}\}$ such that:

$$r\left(e'_{[b(e'),p'_{e'}]}\right) \setminus \{b(e')\} \subset \check{S}_{\check{e},\epsilon} \left\langle U_\epsilon \cap D_{\diamond_{e',e,0}} \cap R_{\theta_1} \left\langle D_{\diamond_{e',e,1}} \right\rangle \right\rangle, \tag{10.16.1}$$

where $D_{\curvearrowright} := \{0\} \times \mathbb{R}^{d-1}$, $D_\uparrow := (\mathbb{R}_+ \setminus \{0\}) \times \mathbb{R}^{d-1}$ and $D_\downarrow := (\mathbb{R}_- \setminus \{0\}) \times \mathbb{R}^{d-1}$. Since:

$$\bigcup_{e' \in \gamma' \setminus \gamma'_{(e)}} r(e') \cup \bigcup_{e' \in \gamma'_{(e)}} r\left(e'_{[p'_{e'},f(e')]}\right)$$

is a compact set of Σ that does not contain $b(e)$, there exists an open neighborhood W of $b(e)$ in V such that:

$$\forall e' \in \gamma' \setminus \gamma'_{(e)},\ r(e') \cap W = \varnothing \text{ and } \forall e' \in \gamma'_{(e)},\ r\left(e'_{[p'_{e'},f(e')]}\right) \cap W = \varnothing.$$

$\check{e}^{-1}\langle W \rangle$ is an open neighborhood of 0 in $\mathbb{R}^d$, hence there exists $\epsilon' \in]0,\epsilon]$ such that $B^{(d)}_{\epsilon'} \subset \check{e}^{-1}\langle W \rangle$. We define:

$$S_{e,0} := S_{\check{e},\epsilon'} \text{ and } S_{e,1} := S_{\check{e},\epsilon',\theta_1}.$$

For $k \in \{0,1\}$, we have $r\left(S_{e,k}\right) \subset W$, therefore:

$$\forall e' \in \gamma' \setminus \gamma'_{(e)},\ e' \curvearrowright S_{e,k} \text{ and } \forall e' \in \gamma'_{(e)},\ e'_{[p'_{e'},f(e')]} \curvearrowright S_{e,k}.$$

In addition, we get from eq. (10.16.1):

$$\forall e' \in \gamma'_{(e)},\ e'_{[b(e'),p'_{e'}]} \diamond_{e',e,k} S_{\check{e},k},$$

thus, using prop. 10.6.6:

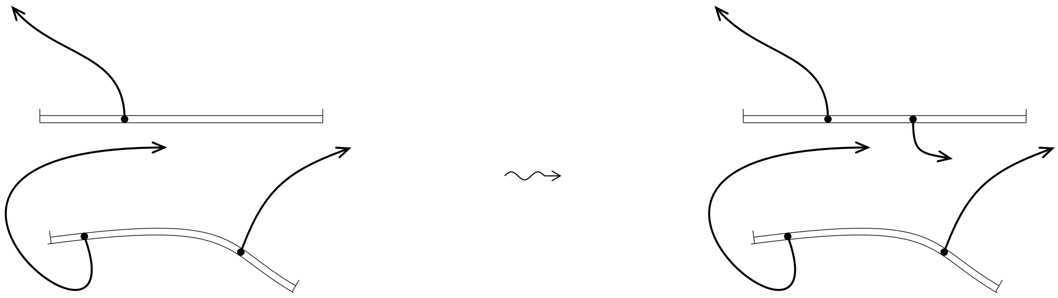


Figure 10.7 – Adding a small edge to populate a face that was empty

$$\forall e' \in \gamma'_{(e)},\ e' \diamond_{e',e,k} S_{\check{e},k}\,.$$

To summarize, we have proven that, for any $e \in \gamma'_{(1,\chi')}$ there exist a finite set of surfaces $\widetilde{\lambda}'_e := \{S_{e,0}, S_{e,1}\}$ and a map $\chi'_e : \gamma' \to \mathfrak{F}\big(\widetilde{\lambda}'_e\big) \cup F_{\daleth}\big(\widetilde{\lambda}'_e\big)$ such that:

$$\forall e' \in \gamma',\ e' \in \chi'_e(e') \quad \& \quad \forall e' \in \gamma' \setminus \{e\}\,,\ \chi'_e(e') \neq \chi'_e(e)\,.$$

Finally, we choose $\widetilde{\lambda} \subset \mathcal{L}_{\text{surfcs}}$ such that $\lambda = \big[\widetilde{\lambda}\big]_{\text{profl}}$ and we define a profile λ' by:

$$\lambda' := \left[\widetilde{\lambda} \cup \bigcup_{e \in \gamma'_{(1,\chi')}} \widetilde{\lambda}_e\right]_{\text{profl}}\,.$$

Then, $\lambda \preccurlyeq \lambda'$ (prop. 10.11) and the map $\chi'' : \gamma' \to \mathfrak{F}(\lambda') \cup \big\{F_{\daleth}(\lambda')\big\}$, given by:

$$\forall e' \in \gamma',\ \chi''(e') = \chi'(e') \cap \bigcap_{e \in \gamma'_{(1,\chi')}} \chi'_e(e'),$$

belongs to $M^{(2)}_{(\gamma',\lambda')}$. □

We will now turn to populating faces that do not contain any edge yet. The basic idea is to pick some edge in the concerned face and to add it to our graph (remember that faces have been defined as non empty set of edges in prop. 10.9). Since the edge we add is chosen in a face of λ, the result of the first step is preserved, and since we add a single edge per face, and only for those faces that do not already contain an edge of the graph, there is no problem with the second step either. The only precaution required here is therefore to make sure that the added edges, together with the already present ones, form a graph, ie. intersects only at their extremities. Now, if we have picked some edge e in a face F, any initial subedge of edge of e will also be in F, and will do just as well for our purpose. This way, if e intersects another edge e' (either an edge of the graph, or one of the to-be-added edges), and if this intersection takes place *away* from $b(e)$, we can simply make e shorter to avoid it (fig. 10.7 shows an edge being added to a face that were initially empty: we make the added edge small enough so that it stays away from any preexisting edges). Moreover, it cannot be that e has a common initial subedge with another edge e' for this would mean that e' belongs to F, in contradiction with the preserved validity of step two (eg. the fact that we do not get more that one edge per face, as underlined above).

The only kind of intersection that cannot be fixed by shortening e is therefore the situation in

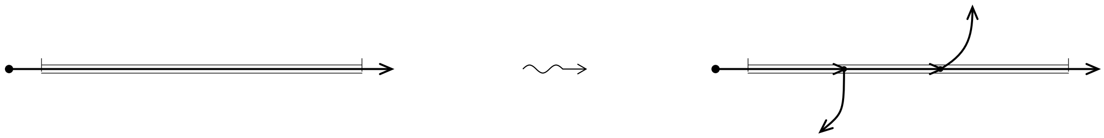


Figure 10.8 – An edge that runs along a surface (or an intersection of surfaces) may need to be subdivided when populating the corresponding faces

which $e \cap e' = \{b(e)\}$. As illustrated in fig. 10.8, it might not be possible to prevent this by a careful choice of e, because $b(e)$ has to belong to every surface involved in the face F (this is indeed a necessary condition for e to belongs to F): it might force $b(e)$ to be in the interior of some edge e'. Not that this concern is *not* an artifact of the dimension two: even when the surfaces have dimension $d - 1 > 1$, there are faces that arises at the intersection of surfaces, and this intersection could be one-dimensional, with an edge running along it. Thus, the present step might require, besides the addition of new edges, also the subdivision of preexisting edges of the graph. Such a subdivision does not, however, threaten the two previous steps, nor does it lead to infinite recursion in the present step, for the first part of the subdivided edge e' will belong to $\chi(e')$, and every subsequent parts will be of the $F_{\daleth}(\lambda)$ type.

Lemma 10.17 Let $\gamma \in \mathcal{L}_{\text{graphs}}$ and $\lambda \in \mathcal{L}_{\text{profls}}$ such that $M^{(2)}_{(\gamma,\lambda)} \neq \varnothing$. Then, there exists $\gamma' \in \mathcal{L}_{\text{graphs}}$, such that $\gamma \preccurlyeq \gamma'$ and $M^{(3)}_{(\gamma',\lambda)} \neq \varnothing$.

Proof *Auxiliary result: Intersection of edges belonging to different faces.* Let $e \in \mathcal{L}_{\text{edges}}$ and $F \in \mathcal{F}(\lambda)$ such that $e \in F$. Suppose that there exists $p \neq q \in r(e)$ and $F' \in \mathcal{F}(\lambda) \cup \left\{F_{\daleth}(\lambda)\right\}$ such that the subedge $e_{[p,q]} \in F'$. Then, from prop. 10.6.6, there exists $q' \in r\left(e_{[p,q]}\right) \setminus \{p, q\}$ such that $e_{[p,q']} \in F'$. Since $q' \notin \{b(e), f(e)\}$, we are in one of the following situations:

$$\begin{array}{ll} e = e_{[q',f(e)]} \circ e_{[p,q']} & \text{if} \quad b(e) = p <_{(e)} q' <_{(e)} f(e) \\ e = e_{[q',f(e)]} \circ e_{[p,q']} \circ e_{[b(e),p]} & \text{if} \quad b(e) <_{(e)} p <_{(e)} q' <_{(e)} f(e) \\ e = \left(e_{[p,q']}\right)^{-1} \circ e_{[b(e),q']} & \text{if} \quad b(e) <_{(e)} q' <_{(e)} p = f(e) \\ e = e_{[p,f(e)]} \circ \left(e_{[p,q']}\right)^{-1} \circ e_{[b(e),q']} & \text{if} \quad b(e) <_{(e)} q' <_{(e)} p <_{(e)} f(e) \end{array}.$$

Hence, from props. 10.6.5 and 10.6.6, we have either $F' = F$ (if $p = b(e)$) or $F' = F_{\daleth}(\lambda)$ (otherwise).

Let $e_1, e_2 \in \mathcal{L}_{\text{edges}}$ and $F_1, F_2 \in \mathcal{F}(\lambda)$ such that $e_1 \in F_1$, $e_2 \in F_2$ and:

$$\forall p \in r(e_1) \setminus \{b(e)\}, \exists p' \in r(e_1) \mathrel{/} b(e_1) <_{(e_1)} p' <_{(e_1)} p \quad \& \quad p' \in r(e_2).$$

Then, from lemma 10.5, there exists $q \in r(e_1) \setminus \{b(e_1)\}$ such that $r\left(e_{1,[b(e_1),q]}\right) \subset r(e_2)$, hence $e_{2,[b(e_1),q]} = e_{1,[b(e_1),q]}$. Since $e_{1,[b(e_1),q]} \in F_1 \neq F_{\daleth}(\lambda)$ (by definition of $\mathcal{F}(\lambda)$), the previous argument applied to the subedge $e_{2,[b(e_1),q]}$ of e_2, together with prop. 10.9.1, implies that $F_1 = F_2$.

Thus, we have proven that, for any $e_1, e_2 \in \mathcal{L}_{\text{edges}}$ and any $F_1 \neq F_2 \in \mathcal{F}(\lambda)$ such that $e_1 \in F_1$ and $e_2 \in F_2$, there exists $p \in r(e_1) \setminus \{b(e_1)\}$ such that $r\left(e_{1,[b(e_1),p]}\right) \cap r(e_2) \subset \{b(e_1), p\}$. Hence, there exists $p' \in r\left(e_{1,[b(e_1),p]}\right) \setminus \{b(e_1)\}$ such that $r\left(e_{1,[b(e_1),p']}\right) \cap r(e_2) \subset \{b(e_1)\}$.

Construction of γ'. Let $\chi \in M^{(2)}_{(\gamma,\lambda)}$ and define:

$$\mathcal{F}_{(2,\chi)}(\lambda) := \left\{F \in \mathcal{F}(\lambda) \,\middle|\, \chi^{-1}\langle F\rangle = \varnothing\right\}.$$

For any $F \in \mathcal{F}_{(2,\chi)}(\lambda)$, we choose an edge $e_F \in F$, and we define:

$$\gamma_{(2,\chi)} := \{e_F \mid F \in \mathcal{F}_{(2,\chi)}(\lambda)\}.$$

Let $e \in \gamma_{(2,\chi)}$ and $F \in \mathcal{F}_{(2,\chi)}(\lambda)$ such that $e \in F$. For any $\widetilde{e} \in \gamma$, $e \in \widetilde{F}$ with $\widetilde{F} = \chi(\widetilde{e}) \in \mathcal{F}(\lambda) \setminus \mathcal{F}_{(2,\chi)}(\lambda)$. And for any $\widetilde{e} \in \gamma_{(2,\chi)} \setminus \{e\}$, there exists $\widetilde{F} \in \mathcal{F}_{(2,\chi)}(\lambda) \setminus \{F\}$ such that $\widetilde{e} \in \widetilde{F}$. Hence, for any $\widetilde{e} \in \gamma \cup \gamma_{(2,\chi)} \setminus \{e\}$, there exists $p_{e,\widetilde{e}} \in r(e) \setminus \{b(e)\}$ such that $r\left(e_{[b(e),p_{e,\widetilde{e}}]}\right) \cap r(\widetilde{e}) \subset \{b(e)\}$. Since $\gamma \cup \gamma_{(2,\chi)} \setminus \{e\}$ is a finite set, there exists p_e such that:

$$\forall \widetilde{e} \in \gamma \cup \gamma_{(2,\chi)} \setminus \{e\},\ r\left(e_{[b(e),p_e]}\right) \cap r(\widetilde{e}) \subset \{b(e)\}.$$

We define $\gamma'_e := \left\{e_{[b(e),p_e]}\right\}$ and $\chi'_e : \gamma'_e \to \mathcal{F}(\lambda) \cup \left\{F_{\daleth}(\lambda)\right\}$, $e_{[b(e),p_e]} \mapsto F$. Then, for any $e' \in \gamma'_e$, we have $r(e') \subset r(e)$, $e' \in \chi'_e(e')$ and:

$$\forall \widetilde{e} \in \gamma \cup \gamma_{(2,\chi)} \setminus \{e\},\ r(e') \cap r(\widetilde{e}) \subset \{b(e'),\, f(e')\}.$$

Let $e \in \gamma$. The set $\{b(\widetilde{e}) \mid \widetilde{e} \in \gamma_{(2,\chi)} \ \ \&\ \ b(\widetilde{e}) \in r(e)\}$ is finite, hence there exist $n_e \geqslant 1$ and composable edges $e_1, \dots, e_{n_e} \in \mathcal{L}_{\text{edges}}$ such that $e = e_{n_e} \circ \dots \circ e_1$ and:

$$\{b(\widetilde{e}) \mid \widetilde{e} \in \gamma_{(2,\chi)} \ \ \&\ \ b(\widetilde{e}) \in r(e)\} \subset \bigcup_{i=1}^{n_e} \{b(e_i),\, f(e_i)\}.$$

We define $\gamma'_e := \{e_1, \dots, e_{n_e}\}$. We also define the map $\chi'_e : \gamma'_e \to \mathcal{F}(\lambda) \cup \left\{F_{\daleth}(\lambda)\right\}$ by:

$$\chi'_e(e_1) := \chi(e) \quad \& \quad \forall k \in \{2, \dots, n_e\},\ \chi'_e(e_k) := F_{\daleth}(\lambda).$$

Then, for any $e' \in \gamma'_e$, we have $r(e') \subset r(e)$, $e' \in \chi'_e(e')$ (combining $e \in \chi(e)$ with props. 10.6.6 and 10.2.3) and:

$$\forall \widetilde{e} \in \gamma \setminus \{e\},\ r(e') \cap r(\widetilde{e}) \subset r(e') \cap \{b(e),\, f(e)\} \subset \{b(e'),\, f(e')\},$$

where we have used that $\gamma \in \mathcal{L}_{\text{graphs}}$. For any $\widetilde{e} \in \gamma_{(2,\chi)}$, we have by definition of γ'_e and $\gamma'_{\widetilde{e}}$:

$$\forall e' \in \gamma'_e,\ \forall \widetilde{e}' \in \gamma'_{\widetilde{e}},\ r(e') \cap r(\widetilde{e}') \subset r(e') \cap \bigcup_{i=1}^{n_e} \{b(e_i),\, f(e_i)\} \subset \{b(e'),\, f(e')\}.$$

Additionaly, we have from props. 10.2 and 10.1:

$$\forall e' \neq e'' \in \gamma'_e,\ r(e') \cap r(e'') \subset \{b(e'),\, f(e')\}.$$

Finally, we construct a finite set of edges γ' as $\gamma' := \bigcup_{e \in \gamma \cup \gamma_{(2,\chi)}} \gamma'_e$, and we define a map $\chi' : \gamma' \to \mathcal{F}(\lambda) \cup \left\{F_{\daleth}(\lambda)\right\}$ by:

$$\forall e \in \gamma \cup \gamma_{(2,\chi)},\ \forall e' \in \gamma'_e,\ \chi'(e') := \chi'_e(e')$$

(χ' is well-defined since the γ'_e for different e are disjoints by construction). By putting together what we have proven above, we get:

$$\forall e' \neq \widetilde{e}' \in \gamma',\ r(e') \cap r(\widetilde{e}') \subset \{b(e'),\, f(e')\},$$

thus $\gamma' \in \mathcal{L}_{\text{graphs}}$, and, by definition of γ'_e for $e \in \gamma$, $\gamma \preccurlyeq \gamma'$. We also have $\forall e' \in \gamma'$, $e' \in \chi'(e')$ and:

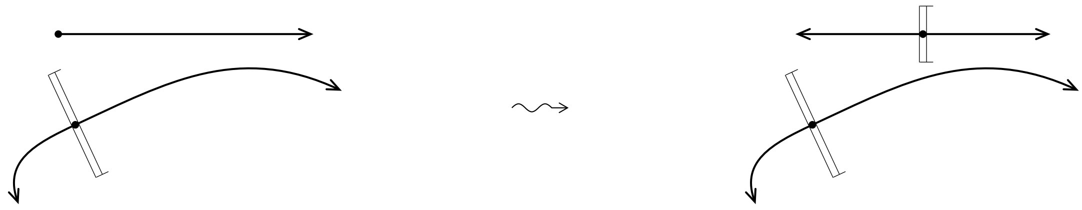


Figure 10.9 – Adding a small surface through the middle of a still unpaired edge

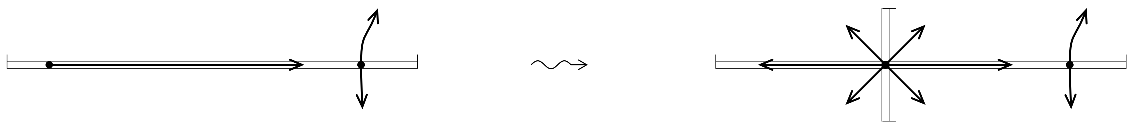


Figure 10.10 – Accidental extra faces need to be populated

$$\forall F \in \chi\left\langle\gamma\right\rangle, \chi'^{-1}\left\langle F\right\rangle = \left\{e_1 \,\middle|\, e \in \chi^{-1}\left\langle F\right\rangle\right\} \quad \& \quad \forall F \in \mathcal{F}_{(2,\chi)}(\lambda), \chi'^{-1}\left\langle F\right\rangle = \gamma'_{e_F}\,.$$

Hence, using $\mathcal{F}(\lambda) = \mathcal{F}_{(2,\chi)}(\lambda) \sqcup \left(\chi\left\langle\gamma\right\rangle \setminus \left\{F_{\urcorner}(\lambda)\right\}\right)$ and $\chi \in M^{(2)}_{(\gamma,\lambda)}$, we obtain that $\chi' \in M^{(3)}_{(\gamma,\lambda)}$. □

Finally, we want to consider those edges that do not yet belong to any face. If e is such an edge, we will let a small surface cut it through the middle, subdivide e accordingly, and reorient the parts so that they start from the added surface (fig. 10.9): thus one part will be above the surface, and the other below. Since this surface goes through an interior point p_e of e, we can ensure, by making it small enough, that it does not cross any other edges of the graph (by definition of a graph, p_e can not belong to any other edge). Thus, the results of the first two steps are preserved.

However, if e lies inside some preexisting surface, the achievement of the third step might have to be restored at this point: besides the two faces now populated by the two parts of e, there might be additional new faces corresponding to intersections of the added small surface with the preexisting ones. If this occurs, we will need to add a few edges to populate those faces (fig. 10.10), but we can make sure that all added edges, together with the two parts of e, intersects at most at their starting points. Also, by making the added edges shorter if required, we can prevent them to intersect any other edge of the graph. Thus, there is no need for further subdivision of the edges (in contrast to the situation depicted in fig. 10.8 that could arise in the previous step), and we have achieved the goal announced at the beginning of the present subsection.

Lemma 10.18 Let $\gamma \in \mathcal{L}_{\text{graphs}}$ and $\lambda \in \mathcal{L}_{\text{profls}}$ such that $M^{(3)}_{(\gamma,\lambda)} \neq \varnothing$. Then, there exists $\gamma' \in \mathcal{L}_{\text{graphs}}$ and $\lambda' \in \mathcal{L}_{\text{profls}}$, such that $\gamma \preccurlyeq \gamma'$, $\lambda \preccurlyeq \lambda'$ and $M^{(4)}_{(\gamma',\lambda')} \neq \varnothing$.

Proof Let $\widetilde{\lambda} \subset \mathcal{L}_{\text{surfcs}}$ such that $\lambda = \left[\widetilde{\lambda}\right]_{\text{profl}}$ and $\chi \in M^{(3)}_{(\gamma,\lambda)}$. We define:

$$\gamma_{(3,\chi)} := \left\{e \in \gamma \,\middle|\, \chi(e) = F_{\urcorner}(\lambda)\right\} \quad \& \quad \gamma'_0 := \gamma \setminus \gamma_{(3,\chi)}\,.$$

Thus, $\chi\left\langle\gamma'_0\right\rangle \subset \mathcal{F}(\lambda)$ and, by definition of $M^{(3)}_{(\gamma,\lambda)}$, $\chi_0 := \chi|_{\gamma'_0 \to \mathcal{F}(\lambda)}$ is bijective.

For each $e \in \gamma_{(3,\chi)}$, we choose a representative $\breve{e} : U_e \to V_e$ of e and define $p_e := \breve{e}(0.5, 0)$. Since, for any $e \in \gamma_{(3,\chi)}$, $\bigcup_{e' \in \gamma \setminus \{e\}} r(e')$ is a compact set that does not contain p_e (for $p_e \in r(e) \setminus \{b(e), f(e)\}$ and $\gamma \in \mathcal{L}_{\text{graphs}}$), there exists an open neighborhood W_e of p_e in V_e such that:

$$\forall e' \in \gamma \setminus \{e\},\ r(e') \cap W_e = \varnothing .$$

Next, $\{p_e \mid e \in \gamma_{(3,\chi)}\}$ is finite and Σ is Hausdorff, hence there exists a family $\left(W'_e\right)_{e \in \gamma_{(3,\chi)}}$ of *disjoint* open subsets of Σ such that, for any $e \in \gamma_{(3,\chi)}$, W'_e is an open neighborhood of p_e in W_e. Thus, we have:

$$\forall e \in \gamma_{(3,\chi)},\ W'_e \cap \left(\bigcup_{e' \in \gamma \setminus \{e\}} r(e') \cup \bigcup_{e' \in \gamma_{(3,\chi)} \setminus \{e\}} W'_{e'} \right) = \varnothing . \qquad (10.18.1)$$

Let $e \in \gamma_{(3,\chi)}$. There exists $\epsilon > 0$ such that $\{0.5\} \times B^{(d-1)}_{\epsilon} \subset \check{e}^{-1}\left\langle W'_e \right\rangle$ and we define:

$$\begin{array}{rccl} \check{S}_e & : & U'_e & \to & V_e \\ & & t, y & \mapsto & \check{e}(t + 0.5,\ \epsilon\, y) \end{array} \quad \text{with } U'_e := \{(t, y) \mid (t + 0.5,\ \epsilon\, y) \in U_e\} .$$

Since $\{0\} \times B^{(d-1)} \subset U'_e$, $\check{S}_e$ is an analytic, encharted surface in Σ. We denote by S_e the corresponding surface. We can check from the definition of $\check{S}_e$ that $r(S_e) \subset W'_e$ and:

$$e_{\uparrow} := e_{[p_e, f(e)]} \in F_{e,\uparrow} \quad \& \quad e_{\downarrow} := e_{[p_e, b(e)]} \in F_{e,\downarrow} ,$$

where $F_{e,\uparrow} := \left\{ e' \in F_{\urcorner}(\lambda) \,\middle|\, e' \uparrow S_e \right\}$ and $F_{e,\downarrow} := \left\{ e' \in F_{\urcorner}(\lambda) \,\middle|\, e' \downarrow S_e \right\}$ (using $e \in \chi(e) = F_{\urcorner}(\lambda)$ together with props. 10.6.5 and 10.6.6). Defining $\gamma'_{e,0} := \{e_{\uparrow},\ e_{\downarrow}\}$, we moreover have:

$$\forall e' \in \gamma'_{e,0},\ r(e') \cap \{b(e),\ f(e)\} \subset \{f(e')\} , \qquad (10.18.2)$$

$$\text{and} \quad \forall e' \neq \widetilde{e}' \in \gamma'_{e,0}, r(e') \cap r(\widetilde{e}') \subset \{b(e')\} = \{b(\widetilde{e}')\} . \qquad (10.18.3)$$

Next, we define:

$$\mathcal{F}_{(e,\uparrow)}(\lambda) := \left\{ F \in \mathcal{F}(\lambda) \,\middle|\, \exists\, e' \in F \,/\, e' \uparrow S_e \right\} \quad \& \quad \mathcal{F}_{(e,\downarrow)}(\lambda) := \left\{ F \in \mathcal{F}(\lambda) \,\middle|\, \exists\, e' \in F \,/\, e' \downarrow S_e \right\} .$$

For each $F \in \mathcal{F}_{(e,\uparrow)}(\lambda)$ (resp. $F \in \mathcal{F}_{(e,\downarrow)}(\lambda)$), we define:

$$F_{e,F,\uparrow} := \left\{ e' \in F \,\middle|\, e' \uparrow S_e \right\} \quad (\text{resp. } F_{e,F,\downarrow} := \left\{ e' \in F \,\middle|\, e' \downarrow S_e \right\}),$$

and we choose an edge $e_{e,F,\uparrow} \in F_{e,F,\uparrow}$ (resp. $e_{e,F,\downarrow} \in F_{e,F,\downarrow}$). We also define:

$$\gamma'_{e,2} := \{e_{e,F,\uparrow} \mid F \in \mathcal{F}_{(e,\uparrow)}(\lambda)\} \cup \{e_{e,F,\downarrow} \mid F \in \mathcal{F}_{(e,\downarrow)}(\lambda)\} .$$

The sets $\{F_{e,\uparrow},\ F_{e,\downarrow}\}$, $\{F_{e,F,\uparrow} \mid F \in \mathcal{F}_{(e,\uparrow)}(\lambda)\}$ and $\{F_{e,F,\downarrow} \mid F \in \mathcal{F}_{(e,\downarrow)}(\lambda)\}$ are disjoints subsets of $\mathcal{F}\left(\widetilde{\lambda} \cup \{S_e\}\right)$. Hence, defining:

$$\mathcal{F}_{(e)}\left(\widetilde{\lambda} \cup \{S_e\}\right) := \{F_{e,\uparrow},\ F_{e,\downarrow}\} \cup \{F_{e,F,\uparrow} \mid F \in \mathcal{F}_{(e,\uparrow)}(\lambda)\} \cup \{F_{e,F,\downarrow} \mid F \in \mathcal{F}_{(e,\downarrow)}(\lambda)\} ,$$

$\mathcal{F}_{(e)}\left(\widetilde{\lambda} \cup \{S_e\}\right) \subset \mathcal{F}\left(\widetilde{\lambda} \cup \{S_e\}\right)$ and there exists a *bijective* map $\chi_{e,2} : \gamma'_{e,0} \cup \gamma'_{e,2} \to \mathcal{F}_{(e)}\left(\widetilde{\lambda} \cup \{S_e\}\right)$ such that, for any $e' \in \gamma'_{e,0} \cup \gamma'_{e,2}$, $e' \in \chi_{e,2}(e')$.

Let $e' \in \gamma'_{e,2}$. There exists $F' := \chi_{e,2}(e') \in \mathcal{F}\left(\widetilde{\lambda} \cup \{S_e\}\right)$ such that $e' \in F'$. Moreover, for any $\widetilde{e}' \in \gamma'_{e,0} \cup \gamma'_{e,2} \setminus \{e'\}$, there exists $\widetilde{F}' := \chi_{e,2}(\widetilde{e}') \in \mathcal{F}\left(\widetilde{\lambda} \cup \{S_e\}\right) \setminus \{F'\}$ such that $\widetilde{e}' \in \widetilde{F}'$. Thus, using the auxiliary result from the proof of lemma 10.17, there exists, for any $\widetilde{e}' \in \gamma'_{e,0} \cup \gamma'_{e,2} \setminus \{e'\}$, a point $q_{e',\widetilde{e}'} \in r(e') \setminus \{b(e')\}$ such that $r\left(e'_{[b(e'),q_{e',\widetilde{e}'}]}\right) \cap r(\widetilde{e}') \subset \{b(e')\}$. Hence, there exists $q_{e'} \in r(e') \setminus \{b(e')\}$ such that:

$$\forall \widetilde{e}' \in \gamma'_{e,0} \cup \gamma'_{e,2} \setminus \{e'\},\ r\left(e'_{[b(e'),q_{e'}]}\right) \cap r(\widetilde{e}') \subset \{b(e')\}.$$

Since W'_e is an open subset of Σ containing $b(e')$ (for $b(e') \in r(S_e) \subset W'_e$), there exists $q'_{e'} \in r\left(e'_{[b(e'),q_{e'}]}\right) \setminus \{b(e')\}$ such that $r\left(e'_{[b(e'),q'_{e'}]}\right) \subset W'_e$. Also, we have $r\left(e'_{[b(e'),q'_{e'}]}\right) \subset r(e')$ and, from prop. 10.6.6, $e'_{[b(e'),q'_{e'}]} \in F'$.

Defining $\gamma'_{e,1} := \left\{e'_{[b(e'),q'_{e'}]} \,\middle|\, e' \in \gamma'_{e,2}\right\}$ and $\gamma'_e := \gamma'_{e,0} \cup \gamma'_{e,1}$, there exists therefore a bijective map $\chi_e : \gamma'_e \to \mathcal{F}_{(e)}\left(\widetilde{\lambda} \cup \{S_e\}\right)$ such that, for any $e' \in \gamma'_e$, $e' \in \chi_e(e')$.
In addition, we have:

$$\forall e' \in \gamma'_{e,0},\ r(e') \subset r(e) \quad \& \quad \forall e' \in \gamma'_{e,1},\ r(e') \subset W'_e, \tag{10.18.4}$$

$$\text{and} \quad \forall e' \in \gamma'_{e,1},\ \forall \widetilde{e}' \in \gamma'_e \setminus \{e'\},\ r(e') \cap r(\widetilde{e}') \subset \{b(e')\}. \tag{10.18.5}$$

But, since for any $e' \in \gamma'_{e,1}$, $b(e') \in r(S_e)$, and for any $\widetilde{e}' \in \gamma'_{e,0}$, $r(\widetilde{e}') \cap r(S_e) \subset \{b(\widetilde{e}')\}$, eq. (10.18.5) together with eq. (10.18.3) implies:

$$\forall e' \neq \widetilde{e}' \in \gamma'_e,\ r(e') \cap r(\widetilde{e}') \subset \{b(e')\}. \tag{10.18.6}$$

Now, we define:

$$\gamma' := \gamma'_0 \cup \bigcup_{e \in \gamma_{(3,\chi)}} (\gamma'_e) \quad \& \quad \widetilde{\lambda}' := \widetilde{\lambda} \cup \bigcup_{e \in \gamma_{(3,\chi)}} \{S_e\}.$$

The fact that γ is a graph, together with eqs. (10.18.1), (10.18.2), (10.18.4) and (10.18.6), ensures that γ' is again a graph. Moreover, by definition of $\gamma'_{e,0}$ for $e \in \gamma_{(3,\chi)}$, $\gamma \preccurlyeq \gamma'$, and, from prop. 10.11, $\lambda \preccurlyeq \lambda'$ where $\lambda' := \left[\widetilde{\lambda}'\right]_{\text{profl}}$. Next, we define:

$$\mathcal{F}_{(0)}(\widetilde{\lambda}') := \left\{F \cap F_{\urcorner}(\widetilde{\lambda}' \setminus \widetilde{\lambda}) \,\middle|\, F \in \mathcal{F}(\widetilde{\lambda})\right\},$$

$$\text{and, for any } e \in \gamma_{(3,\chi)},\ \mathcal{F}_{(e)}(\widetilde{\lambda}') := \left\{F \cap F_{\urcorner}\left(\widetilde{\lambda}' \setminus (\widetilde{\lambda} \cup \{S_e\})\right) \,\middle|\, F \in \mathcal{F}_{(e)}(\widetilde{\lambda} \cup \{S_e\})\right\}.$$

Since, for any $e \in \gamma_{(3,\chi)}$, $r(S_e) \subset W'_e$, we have:

$$\begin{aligned}\left\{e' \in \mathcal{L}_{\text{edges}} \,\middle|\, \exists\, \diamond_e \in \left\{\uparrow, \downarrow\right\} / e' \diamond_e S_e\right\} &\subset \{e' \in \mathcal{L}_{\text{edges}} \mid b(e') \in r(S_e)\} \\ &\subset \{e' \in \mathcal{L}_{\text{edges}} \mid b(e') \in W'_e\}.\end{aligned} \tag{10.18.7}$$

Therefore, for any $e \neq \widetilde{e} \in \gamma_{(3,\chi)}$, we get, using $W'_e \cap W'_{\widetilde{e}} = \varnothing$:

$$\left\{e' \in \mathcal{L}_{\text{edges}} \,\middle|\, \exists\, \diamond_e, \diamond_{\widetilde{e}} \in \left\{\uparrow, \downarrow\right\} \,/\, e' \diamond_e S_e \quad \& \quad e' \diamond_{\widetilde{e}} S_{\widetilde{e}}\right\} = \varnothing\,.$$

This, together with the definition of $\mathcal{F}_{(e)}(\widetilde{\lambda} \cup \{S_e\})$ for $e \in \gamma_{(3,\chi)}$, implies:

$$\mathcal{F}(\widetilde{\lambda'}) \quad \subset \bigcup_{e \in \{0\} \sqcup \gamma_{(3,\chi)}} \mathcal{F}_{(e)}(\widetilde{\lambda'})\,.$$

In addition, we can define for any $e \in \{0\} \sqcup \gamma_{(3,\chi)}$ a bijective map $\chi'_e : \gamma'_e \to \mathcal{F}_{(e)}(\widetilde{\lambda'})$ by:

$$\forall e' \in \gamma'_0,\, \chi'_0(e') := \chi_0(e') \cap F_{\urcorner}(\widetilde{\lambda'} \setminus \widetilde{\lambda}) \quad \& \quad \forall e' \in \gamma'_e,\, \chi'_e(e') := \chi_e(e') \cap F_{\urcorner}\big(\widetilde{\lambda'} \setminus (\widetilde{\lambda} \cup \{S_e\})\big)\,.$$

For any $e \in \{0\} \sqcup \gamma_{(3,\chi)}$, χ'_e satisfies:

$$\forall e' \in \gamma'_e,\, e' \in \chi'_e(e')\,,$$

as follows from the corresponding property of χ_e together with eqs. (10.18.1), (10.18.4) and (10.18.7). In particular, we thus have:

$$\forall e \in \{0\} \sqcup \gamma_{(3,\chi)},\, \forall F \in \mathcal{F}_{(e)}(\widetilde{\lambda'}),\, F \neq \varnothing\,,$$

so, we obtain:

$$\mathcal{F}(\widetilde{\lambda'}) \quad = \bigcup_{e \in \{0\} \sqcup \gamma_{(3,\chi)}} \mathcal{F}_{(e)}(\widetilde{\lambda'})\,.$$

Since the domains of the bijective maps χ'_e for $e \in \{0\} \sqcup \gamma_{(3,\chi)}$ are disjoints, as well as their images, they can be combined into a well-defined bijective map $\chi' : \gamma' \to \mathcal{F}(\lambda')$. Hence, $M^{(4)}_{(\gamma',\lambda')} \neq \varnothing$. □

Proof of theorem 10.14 Let (γ, λ), $(\gamma', \lambda') \in \mathcal{L}_{\text{HF}}$. Since $\mathcal{L}_{\text{graphs}}, \preccurlyeq$ and $\mathcal{L}_{\text{profls}}, \preccurlyeq$ are directed sets (props. 10.4 and 10.11), there exists $(\gamma_1, \lambda_1) \in \mathcal{L}_{\text{graphs}} \times \mathcal{L}_{\text{profls}}$ such that $\gamma, \gamma' \preccurlyeq \gamma_1$ and $\lambda, \lambda' \preccurlyeq \lambda_1$. By chaining lemmas 10.15 to 10.18 and using the transitivity of $\preccurlyeq$ on $\mathcal{L}_{\text{graphs}}$ and $\mathcal{L}_{\text{profls}}$, there exists $(\gamma'', \lambda'') \in \mathcal{L}_{\text{graphs}} \times \mathcal{L}_{\text{profls}}$ such that $\gamma_1 \preccurlyeq \gamma''$, $\lambda_1 \preccurlyeq \lambda''$ and $M^{(4)}_{(\gamma'',\lambda'')} \neq \varnothing$. Thus, $(\gamma'', \lambda'') \in \mathcal{L}_{\text{HF}}$ and (γ, λ), $(\gamma', \lambda') \preccurlyeq (\gamma'', \lambda'')$. □

10.3 Factorization maps

The labels introduced in subsection 10.1 are meant to identify corresponding small algebras of observables, and the ordering has been chosen such that, whenever $\eta \preccurlyeq \eta'$, there is a natural injection of the algebra labeled by η into the one labeled by η'. By carefully adjusting under which conditions a collection of edges and surfaces can be turned into a label, we have ensured that these algebras of observables can be represented on small phase spaces $\mathcal{M}_\eta$ and $\mathcal{M}_{\eta'}$ respectively, and that the identification between observables on $\mathcal{M}_\eta$ and $\mathcal{M}_{\eta'}$ unambiguously prescribes a suitable projection from $\mathcal{M}_{\eta'}$ into $\mathcal{M}_\eta$.

Moreover, this projection is compatible with the symplectic structures (def. 2.1), and is actually of the form that was considered in theorem 6.2 (as will be shown in prop. 10.26). Thus, we expect from this theorem that the obtained projective system of symplectic manifolds goes down to a factorizing system on the underlying configuration spaces (def. 2.15). Indeed, we will prove that it is the case by giving the explicit expressions for the factorization maps (with the added benefit that no further restriction need to be imposed on the finite-dimensional Lie group G, while theorem 6.2 have been derived in the case of simply-connected groups). Once we have such a factorizing system of configuration spaces, it will be straightforward to quantize it into a projective quantum state space (along the lines of [68] and subsection 6.1), as we will do in subsection 12.1.

We start by attaching to each label η a configuration space $\mathcal{C}_\eta$: $\mathcal{C}_\eta$ is nothing but the configuration space routinely associated in LQG to the graph $\gamma(\eta)$, with one group var per edge (corresponding to the value of the holonomy along this edge). We also attach to η a momentum space $\mathcal{P}_\eta$: as announced in subsection 10.1, momentum variables are assigned to the faces of the label (eventually, we will, in prop. 10.26, rely on left translations to identify the cotangent bundle $T^*(G)$ with $G \times \mathfrak{g}^*$, and thus $T^*(\mathcal{C}_\eta)$ with $\mathcal{C}_\eta \times \mathcal{P}_\eta$, using the pairing χ_η between edges and their conjugate faces provided by def. 10.12).

Also, we introduce a few notations to discuss how the holonomy along an edge, or the flux through a face, can be related to the variables in $\mathcal{C}_\eta$ and $\mathcal{P}_\eta$ (provided η is fine enough to describe the desired observable). Since we want to deduce the correct projective structure from the interpretation of the labels in terms of observables, it is particularly important that the relation between these observables and the variables in the small phase spaces should be unambiguous. This will ensure that the factorization maps are well-defined, and will be essential for proving three-spaces consistency in the resulting factorizing system (eq. (2.11.1) and fig. 2.2): as stressed many times above, this consistency condition, combined with the directedness of $\mathcal{L}_{\text{HF}}$, indeed expresses the concern for an unequivocal meaning of the variables attached to a label.

For a clean labeling of the flux observables we formulate in prop. 10.21 a notion of 'free-standing' faces. In prop. 10.9 the different faces corresponding to a certain collection of surfaces $\{S\}$ have been defined as particular sets of edges, and each such face F can only contain edges that are adapted to every surface in $\{S\}$. In particular, an edge can be prevented from belonging to F simply because it crosses transversally some surface of the collection, even this surface is in reality unrelated to the face F. Thus, a given flux operator is described in different profiles by different set of edges, and its characterization by a more intrinsic set is only obtained after compensating this effect.

Definition 10.19 For any $\eta \in \mathcal{L}_{\text{HF}}$, we define its associated configuration space:

$$\mathcal{C}_\eta := \{h : \gamma(\eta) \to G\} \approx G^{\#\gamma(\eta)},$$

where $\#\gamma(\eta) < \infty$ denotes the number of edges in $\gamma(\eta)$. Since G is a finite-dimensional Lie group, $\mathcal{C}_\eta$ is a finite-dimensional smooth manifold.

Similarly, we define the corresponding momentum space:

$$\mathcal{P}_\eta := \{P : \mathcal{F}(\eta) \to \mathfrak{g}^*\} \approx (\mathfrak{g}^*)^{\#\mathcal{F}(\eta)},$$

which is a finite-dimensional real vector space.

Proposition 10.20 Let $e \in \mathcal{L}_{\text{edges}}$. We define:

$$\mathcal{L}_{\text{graphs}/e} := \{\gamma \in \mathcal{L}_{\text{graphs}} \mid \{e\} \preccurlyeq \gamma\}.$$

For any $\gamma \in \mathcal{L}_{\text{graphs}/e}$, there exists a *unique* map $a_{\gamma\to e} : \{1, \dots, n_{\gamma\to e}\} \to \gamma$ (with $n_{\gamma\to e} \geqslant 1$) such that:

$$e = a_{\gamma\to e}(n_{\gamma\to e})^{\epsilon_{\gamma\to e}(n_{\gamma\to e})} \circ \ldots \circ a_{\gamma\to e}(1)^{\epsilon_{\gamma\to e}(1)}\,, \tag{10.20.1}$$

where, for any $k \in \{1, \ldots, n_{\gamma\to e}\}$:

$$\epsilon_{\gamma\to e}(k) := \begin{cases} +1 & \text{if } b\left(a_{\gamma\to e}(k)\right) <_{(e)} f\left(a_{\gamma\to e}(k)\right) \\ -1 & \text{if } b\left(a_{\gamma\to e}(k)\right) >_{(e)} f\left(a_{\gamma\to e}(k)\right) \end{cases}\,. \tag{10.20.2}$$

Moreover, $a_{\gamma\to e}$ then induces a bijection $\{1, \ldots, n_{\gamma\to e}\} \to H_{\gamma\to e}$, where:

$$H_{\gamma\to e} := \{e' \in \gamma \mid r(e') \subset r(e)\}\,. \tag{10.20.3}$$

Proof Let $\gamma \in \mathcal{L}_{\text{graphs}/e}$. Since $\{e\} \preccurlyeq \gamma$, there exist $n_{\gamma\to e} \geqslant 1$ and a map $a_{\gamma\to e} : \{1, \ldots, n_{\gamma\to e}\} \to \gamma$ such that:

$$e = a_{\gamma\to e}(n_{\gamma\to e})^{\epsilon_{\gamma\to e}(n_{\gamma\to e})} \circ \ldots \circ a_{\gamma\to e}(1)^{\epsilon_{\gamma\to e}(1)}\,,$$

where, for any $k \in \{1, \ldots, n_{\gamma\to e}\}$, $\epsilon_{\gamma\to e}(k)$ is defined as in eq. (10.20.2). By definition of the composition of edges (prop. 10.2), $a_{\gamma\to e}$ is injective and for any $k \in \{1, \ldots, n_{\gamma\to e}\}$, $a_{\gamma\to e}(k) \in H_{\gamma\to e}$.

Next, let $e' \in H_{\gamma\to e}$, and let $p \in r(e') \setminus \{b(e'), f(e')\}$. Since $p \in r(e)$, there exits $k \in \{1, \ldots, n_{\gamma\to e}\}$ such that $p \in a_{\gamma\to e}(k)$, so, γ being a graph, $e' = a_{\gamma\to e}(k)$. Thus, $a_{\gamma\to e}$ induces a bijection $\{1, \ldots, n_{\gamma\to e}\} \to H_{\gamma\to e}$. In particular, $n_{\gamma\to e} = \#H_{\gamma\to e}$.

Finally, defining the map $\mu : H_{\gamma\to e} \to r(e)$ by:

$$\forall e' \in H_{\gamma\to e},\, \mu(e') := \begin{cases} b(e') & \text{if } b(e') <_{(e)} f(e') \\ f(e') & \text{if } b(e') >_{(e)} f(e') \end{cases}\,,$$

prop. 10.2 implies that $\mu \circ a_{\gamma\to e}$ is strictly increasing (using $<$ on $\{1, \ldots, n_{\gamma\to e}\}$ and $<_{(e)}$ on $r(e)$), hence the uniqueness. □

Proposition 10.21 For any $F \subset \mathcal{L}_{\text{edges}}$, we define:

$$F^{\perp} := \left\{e \in \mathcal{L}_{\text{edges}} \,\middle|\, \forall p \neq p' \in r(e),\, e_{[p,p']} \notin F\right\},$$

and we will denote by $\mathcal{L}_{\text{faces}}$ the set:

$$\mathcal{L}_{\text{faces}} := \bigcup_{\lambda\in\mathcal{L}_{\text{profls}}} \left\{F^{\perp} \circ F \,\middle|\, F \in \mathfrak{F}(\lambda)\right\}.$$

In addition, we define for any $\overline{F} \in \mathcal{L}_{\text{faces}}$:

$$\mathcal{L}_{\text{profls}/\overline{F}} := \left\{\lambda' \in \mathcal{L}_{\text{profls}} \,\middle|\, \exists\, \lambda \preccurlyeq \lambda',\, \exists\, F \in \mathfrak{F}(\lambda) \,/\, \overline{F} = F^{\perp} \circ F\right\}.$$

Then, for any $\overline{F} \in \mathcal{L}_{\text{faces}}$ and any $\lambda' \in \mathcal{L}_{\text{profls}/\overline{F}}$, we have:

$$\overline{F} = \overline{F}^{\perp} \circ \bigcup_{F'\in H_{\lambda'\to\overline{F}}} F'\,, \tag{10.21.1}$$

where $H_{\lambda'\to\overline{F}} := \left\{F' \in \mathfrak{F}(\lambda') \,\middle|\, F' \subset \overline{F}\right\}$.

Proof Let $\lambda \in \mathcal{L}_{\text{profls}}$, $F \in \mathfrak{F}(\lambda)$, $\overline{F} := F^{\perp} \circ F$ and $\lambda' \in \mathcal{L}_{\text{profls}}$ such that $\lambda' \succcurlyeq \lambda$. From props. 10.6.5

and 10.6.6, we have:

$$F^{\perp} = F^{\perp} \circ F^{\perp} \quad \& \quad \overline{F}^{\perp} = \left(F^{\perp} \circ F\right)^{\perp} = F^{\perp}.$$

Hence, for any $F' \in H_{\lambda' \to \overline{F}}$:

$$\overline{F}^{\perp} \circ F' \subset \overline{F}^{\perp} \circ \overline{F} = F^{\perp} \circ F^{\perp} \circ F = \overline{F},$$

therefore $\overline{F}^{\perp} \circ \bigcup_{F' \in H_{\lambda' \to \overline{F}}} F' \subset \overline{F}$.

Now, by definition of the preorder $\preccurlyeq$ on $\mathcal{L}_{\text{profls}}$ (prop. 10.11), there exist $F_1', \ldots, F_m' \in \mathcal{F}(\lambda')$ ($m \geqslant 1$) such that:

$$F = F_{\urcorner}(\lambda) \circ \bigcup_{i=1}^{m} F_i'.$$

For any $i \in \{1, \ldots, m\}$, $F_{\urcorner}(\lambda) \circ F_i' \subset F$ implies $F_i' \subset F \subset \overline{F}$ (using again props. 10.6.5 and 10.6.6), so $F_i' \in H_{\lambda' \to \overline{F}}$. And from prop. 10.9.3, $F_{\urcorner}(\lambda) \subset F^{\perp}$, therefore:

$$\overline{F} \subset \overline{F}^{\perp} \circ \bigcup_{i=1}^{m} F_i' \subset \overline{F}^{\perp} \circ \bigcup_{F' \in H_{\lambda' \to \overline{F}}} F'.$$

□

Props. 10.20 and 10.21 make it possible to unambiguously attach a physical interpretation to the variables in $\mathcal{C}_{\eta}$ and $\mathcal{P}_{\eta}$, and are therefore at the root of the relation between the variables assigned to a label η and the ones assigned to a finer label η'. Although we will directly give an analytic expression for the factorization map $\varphi_{\eta' \to \eta} : \mathcal{C}_{\eta'} \to \mathcal{C}_{\eta' \to \eta} \times \mathcal{C}_{\eta}$ and we will not explicitly make use of theorem 6.2, the proof we gave for this result (following ideas from [68, section 3.4]) provides the right hints regarding why such a factorization map does exist, and how it should be defined so that it leads to the desired projection between the phase spaces.

The key idea is that the momentum variables assigned to the label η can be mounted into the phase space associated to η', and therefore correspond to certain vector fields on $\mathcal{C}_{\eta'}$. Each orbit under the finite transformations generated by these vector fields can then be naturally identified with $\mathcal{C}_{\eta}$ (for the relation between the configuration variables in $\mathcal{C}_{\eta}$ and $\mathcal{C}_{\eta'}$ yields a projection from $\mathcal{C}_{\eta'}$ into $\mathcal{C}_{\eta}$ which intertwines the action of these transformations). The complementary space $\mathcal{C}_{\eta' \to \eta}$ can thus be taken as the corresponding quotient space, which itself can be identified with the preimage of some point in $\mathcal{C}_{\eta}$ (eg. the function mapping every edge in $\gamma(\eta)$ to the identity element in G) under the projection $\mathcal{C}_{\eta'} \to \mathcal{C}_{\eta}$.

This prescription can equivalently be expressed as the realization that the space $\mathcal{C}_{\eta' \to \eta}$, together with the projection from $\mathcal{C}_{\eta'}$ into $\mathcal{C}_{\eta' \to \eta}$, can be completely specified by identifying a maximal set of variables in $\mathcal{C}_{\eta'}$ which are not acted upon by the fluxes retained in the label η. On the other hand, the projection from $\mathcal{C}_{\eta'}$ into $\mathcal{C}_{\eta}$ is obtained by writing down the edges in η as compositions of edges in η'. Thus, the first step toward the determination of the factorization map is to state precisely how the edges and faces of the label η lie within η'. For this, we will classify the edges of η' into various categories depending on whether they belong to some face and/or are part of some edge of η. Note that no edge in $\gamma(\eta')$ can be a subedge of two different edges in $\gamma(\eta)$, nor can it belong to

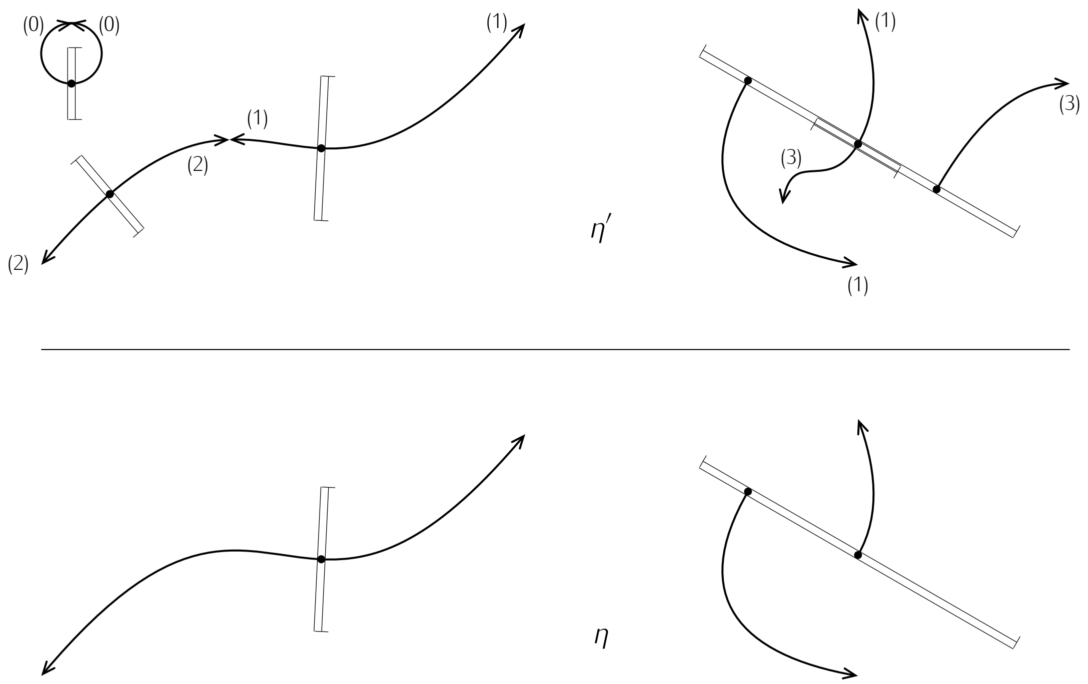


Figure 10.11 – Classification of the edges in $\eta' \succcurlyeq \eta$: a tag $^{(\kappa)}$ denotes an edge belonging to $H^{(\kappa)}_{\eta' \to \eta}$ (prop. 10.22)

two different faces in $\mathcal{F}(\eta)$, nor can a subedge of an edge $e \in \gamma(\eta)$ belongs to any face in $\mathcal{F}(\eta)$ but $\chi(e)$. So we are left with the 4 options listed in prop. 10.22 and depicted in fig. 10.11.

Clearly, the group variables corresponding to edges of η' of type '0' or '2' (those that do not belong to any face among $\mathcal{F}(\eta)$) qualify as complementary variables (they are not acted upon by the fluxes retained in η). Also, knowing the holonomy along some edge of η, as well as the holonomies along all edges of η' that compose it *except* the first one, the holonomy along this first part (which is of type '1') can be reconstructed. Hence, the group variables corresponding to edges of type '1' can be safely droped when extracting $\mathcal{C}_{\eta' \to \eta}$ (fig. 10.12).

Dealing with the group variables corresponding to edges of type '3' is slightly more subtle. Such an edge $e' \in \gamma(\eta')$ belongs to some face $F \in \mathcal{F}(\eta)$ without being the initial part of the conjugate edge $e \in \gamma(\eta)$. To build a group variable invariant under the flux corresponding to F, we will compose the holonomy along e^{-1} (which ends in F) followed by the holonomy along e' (which starts in F). In this way the action of the flux through F cancel out, while the variable in $\mathcal{C}_{\eta'}$ that corresponds to the considered edge e' can still be reconstructed from the variables in $\mathcal{C}_{\eta' \to \eta}$ and $\mathcal{C}_{\eta}$ (fig. 10.13). Note that we could equally well take the composition of the holonomy along e_1^{-1} followed by the holonomy along e', with e_1 the first part of e (in its decomposition into edges of η'). These two alternatives only differ by a function of the group variables attached to the remaining parts of e, which already belongs to $\mathcal{C}_{\eta' \to \eta}$ (these remaining parts being edges of type '2' as outlined above), so it is nothing but a change of coordinates on $\mathcal{C}_{\eta' \to \eta}$, and therefore of no consequences for the construction.

Proposition 10.22 Let $\eta \preccurlyeq \eta' \in \mathcal{L}_{\text{HF}}$. We define:

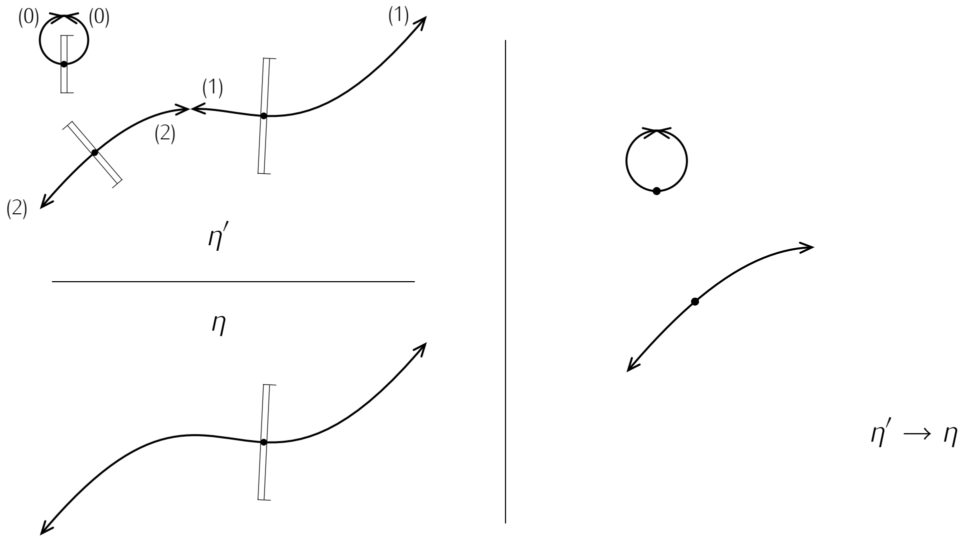


Figure 10.12 – Complementary variables related to edges of type '0' or '2', edges of type '1' do not demand any extra complementary variable

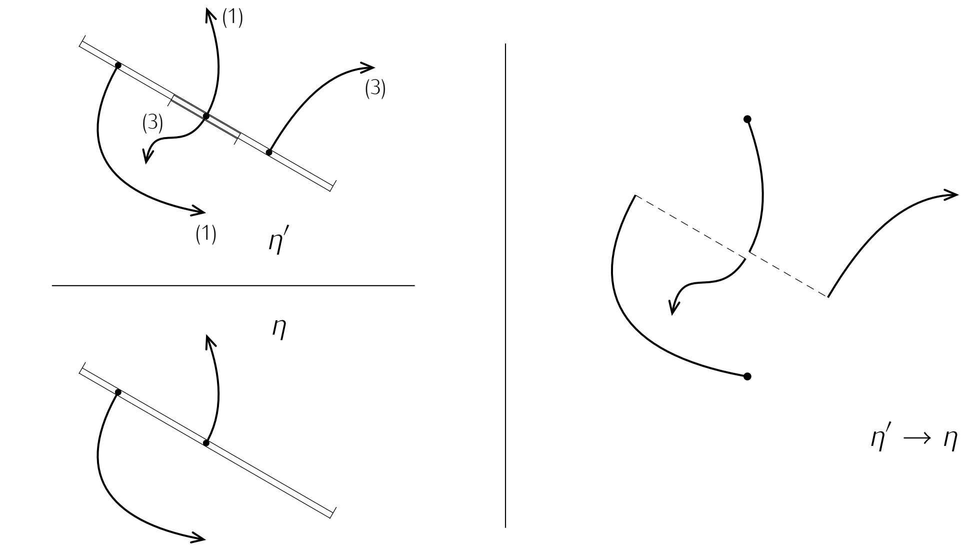


Figure 10.13 – Complementary variables related to edges of type '3'

1. $H^{(0)}_{\eta'\to\eta} := \{e' \in \gamma(\eta') \mid \forall e \in \gamma(\eta),\ \ r(e') \not\subset r(e) \ \ \&\ \ e' \notin \chi_\eta(e)\}$;

and, for any $e \in \gamma(\eta)$:

2. $H^{(1)}_{\eta'\to\eta,e} := \{e' \in \gamma(\eta') \mid r(e') \subset r(e) \ \ \&\ \ e' \in \chi_\eta(e)\}$;
3. $H^{(2)}_{\eta'\to\eta,e} := \{e' \in \gamma(\eta') \mid r(e') \subset r(e) \ \ \&\ \ e' \notin \chi_\eta(e)\}$;
4. $H^{(3)}_{\eta'\to\eta,e} := \{e' \in \gamma(\eta') \mid r(e') \not\subset r(e) \ \ \&\ \ e' \in \chi_\eta(e)\}$;

Then,

$$\left\{H^{(0)}_{\eta'\to\eta}\right\} \cup \bigcup_{e\in\gamma(\eta)} \left\{H^{(1)}_{\eta'\to\eta,e},\, H^{(2)}_{\eta'\to\eta,e},\, H^{(3)}_{\eta'\to\eta,e}\right\}$$

is a partition of $\gamma(\eta')$.

Additionally, we define:

$$\forall \kappa \in \{1,2,3\}\,,\ H^{(\kappa)}_{\eta'\to\eta} := \bigcup_{e\in\gamma(\eta)} H^{(\kappa)}_{\eta'\to\eta,e}\,.$$

Proof For any $e \in \gamma(\eta)$, $\left\{H^{(1)}_{\eta'\to\eta,e},\, H^{(2)}_{\eta'\to\eta,e},\, H^{(3)}_{\eta'\to\eta,e}\right\}$ is a partition of:

$$H^{(4)}_{\eta'\to\eta,e} := H^{(1)}_{\eta'\to\eta,e} \cup H^{(2)}_{\eta'\to\eta,e} \cup H^{(3)}_{\eta'\to\eta,e} = \{e' \in \gamma(\eta') \mid r(e') \subset r(e) \ \text{or}\ e' \in \chi_\eta(e)\}\,.$$

Since we have:

$$H^{(0)}_{\eta'\to\eta} = \left\{e' \in \gamma(\eta') \,\middle|\, \forall e \in \gamma(\eta),\ \ e' \notin H^{(4)}_{\eta'\to\eta,e}\right\},$$

there only remains to prove that the $H^{(4)}_{\eta'\to\eta,e}$ for $e \in \gamma(\eta)$ are mutually disjoint.

Let $e \in \gamma(\eta)$ and let $e' \in \gamma(\eta')$ such that $r(e') \subset r(e)$. There exists $p \in r(e')\setminus\{b(e'),\, f(e')\}$. From prop. 10.1, we get $p \notin \{b(e),\, f(e)\}$, thus $\forall \widetilde{e} \in \gamma(\eta) \setminus \{e\}\,,\ p \notin r(\widetilde{e})$, for $\gamma(\eta)$ is a graph. Therefore, $\forall \widetilde{e} \in \gamma(\eta) \setminus \{e\}\,,\ r(e') \not\subset r(\widetilde{e})$. Moreover, $e \in \chi_\eta(e)$ and for any $\widetilde{e} \in \gamma(\eta) \setminus \{e\}$, $\chi_\eta(\widetilde{e}) \neq \chi_\eta(e)$, so using the auxiliary result at the beginning of the proof of lemma 10.17, $e' \notin \chi_\eta(\widetilde{e})$. Hence, we have proved:

$$\forall e \neq \widetilde{e} \in \gamma(\eta),\ \forall e' \in \gamma(\eta'),\ \left(r(e') \subset r(e) \ \Rightarrow\ e' \notin H^{(4)}_{\eta'\to\eta,\widetilde{e}}\right).$$

Now let $e \in \gamma(\eta)$ and let $e' \in \gamma(\eta')$ such that $e' \in \chi_\eta(e)$. Then, from the previous point, we get $\forall \widetilde{e} \in \gamma(\eta) \setminus \{e\}\,,\ r(e') \not\subset r(\widetilde{e})$. And, since the elements of $\mathcal{F}(\eta)$ are disjoints and χ_η is bijective, we also have $\forall \widetilde{e} \in \gamma(\eta) \setminus \{e\}\,,\ r(e') \notin \chi_\eta(\widetilde{e})$. Therefore, we obtain the desired result:

$$\forall e \neq \widetilde{e} \in \gamma(\eta),\ H^{(4)}_{\eta'\to\eta,e} \cap H^{(4)}_{\eta'\to\eta,\widetilde{e}} = \varnothing\,.$$

□

Proposition 10.23 Let $\eta \preccurlyeq \eta' \in \mathcal{L}_{\text{HF}}$ and $e \in \gamma(\eta)$. We have $\{e\} \preccurlyeq \gamma(\eta')$, and making use of prop. 10.20, we define:

$$n_{\eta'\to\eta,e} := n_{\gamma(\eta')\to e}\,,\quad a_{\eta'\to\eta,e} := a_{\gamma(\eta')\to e} \quad \& \quad \epsilon_{\eta'\to\eta,e} := \epsilon_{\gamma(\eta')\to e}\,.$$

We then have:

$$\epsilon_{\eta'\to\eta,e}(1) = +1\,,\quad H^{(1)}_{\eta'\to\eta,e} = \{a_{\eta'\to\eta,e}(1)\} \quad \& \quad H^{(2)}_{\eta'\to\eta,e} = \{a_{\eta'\to\eta,e}(k) \mid k > 1\}\,, \tag{10.23.1}$$

therefore $n_{\eta'\to\eta,e} = \#H^{(2)}_{\eta'\to\eta,e} + 1$ and $a_{\eta'\to\eta,e}$ induces a bijection $\{1,\ldots,n_{\eta'\to\eta,e}\} \to H^{(1)}_{\eta'\to\eta,e} \cup H^{(2)}_{\eta'\to\eta,e}$.

Also, for any $F \in \mathfrak{F}(\eta)$, we have, using the notations of prop. 10.21 with $\overline{F} = F^{\perp} \circ F$, $\lambda(\eta') \in \mathcal{L}_{\text{profls}/\overline{F}}$ and:

$$H_{\lambda(\eta')\to\overline{F}} = H^{(1,3)}_{\eta'\to\eta,F} := \left\{F' \in \mathfrak{F}(\eta') \mid \chi^{-1}_{\eta'}(F') \in F\right\} = \left\{\chi_{\eta'}(e') \,\middle|\, e' \in H^{(1)}_{\eta'\to\eta,\chi^{-1}_{\eta}(F)} \cup H^{(3)}_{\eta'\to\eta,\chi^{-1}_{\eta}(F)}\right\}. \tag{10.23.2}$$

Proof Let $e \in \gamma(\eta)$. From eq. (10.20.3) $a_{\eta'\to\eta,e}$ induces a bijection into its image:

$$H_{\gamma(\eta')\to e} = H^{(1)}_{\eta'\to\eta,e} \cup H^{(2)}_{\eta'\to\eta,e}\,,$$

and from eq. (10.20.1) together with props. 10.6.5 and 10.6.6:

$$a_{\eta'\to\eta,e}(1)^{\epsilon_{\eta'\to\eta,e}(1)} \in \chi_{\eta}(e) \quad \& \quad \forall k > 1,\ a_{\eta'\to\eta,e}(k) \notin \chi_{\eta}(e).$$

Then, writing $a_{\eta'\to\eta,e}(1)^{\epsilon_{\eta'\to\eta,e}(1)} = e_{[p,p']}$, there exists $p'' \in r\left(e_{[p,p']}\right) \setminus \{p,p'\} \subset r(e) \setminus \{b(e), f(e)\}$ such that $e_{[p,p'']} \in \chi_{\eta}(e)$. This can only hold if $p = b(e)$, ie. $\epsilon_{\eta'\to\eta,e}(1) = +1$. Thus, we get:

$$a_{\eta'\to\eta,e}(1) \in H^{(1)}_{\eta'\to\eta,e} \quad \& \quad \forall k > 1,\ a_{\eta'\to\eta,e}(k) \in H^{(2)}_{\eta'\to\eta,e}.$$

Let $F \in \mathfrak{F}(\eta)$ and $\overline{F} = F^{\perp} \circ F$. For any $F' \in H_{\lambda(\eta')\to\overline{F}}$, we have $\chi^{-1}_{\eta'}(F') \in F' \subset \overline{F}$, hence there exist $e_1 \in F$ and $e_2 \in F^{\perp}$ such that $\chi^{-1}_{\eta'}(F') = e_2 \circ e_1$. But from prop. 10.6.6 together with $F_{\gimel}(\eta') \subset F_{\gimel}(\eta)$ (as in the proof of prop. 10.11), we have $e_2 \in F_{\gimel}(\eta') \subset F_{\gimel}(\eta)$, so $\chi^{-1}_{\eta'}(F') \in F$. Thus, we get $H_{\lambda(\eta')\to\overline{F}} \subset H^{(1,3)}_{\eta'\to\eta,F}$. Reciprocally, let $F' \in \mathfrak{F}(\eta')$ such that $\chi^{-1}_{\eta'}(F') \in F \subset \overline{F}$. Then, from eq. (10.21.1), there exist $e_1 \in F''$, for some $F'' \in H_{\lambda(\eta')\to\overline{F}}$, and $e_2 \in \overline{F}^{\perp}$, such that $\chi^{-1}_{\eta'}(F') = e_2 \circ e_1$. And since $\chi^{-1}_{\eta'}(F') \in F'$, we also have $e_1 \in F'$, therefore $F' = F'' \in H_{\lambda(\eta')\to\overline{F}}$ (for the elements of $\mathfrak{F}(\eta')$ are disjoint from prop. 10.9.1). This proves $H^{(1,3)}_{\eta'\to\eta,F} \subset H_{\lambda(\eta')\to\overline{F}}$, hence $H_{\lambda(\eta')\to\overline{F}} = H^{(1,3)}_{\eta'\to\eta,F}$. $\square$

Proposition 10.24 Let $\eta \preccurlyeq \eta' \in \mathcal{L}_{\text{HF}}$ and define:

$$\mathcal{C}_{\eta'\to\eta} := \left\{h^{(0)} : H^{(0)}_{\eta'\to\eta} \to G\right\} \times \left\{h^{(2)} : H^{(2)}_{\eta'\to\eta} \to G\right\} \times \left\{h^{(3)} : H^{(3)}_{\eta'\to\eta} \to G\right\}.$$

Like $\mathcal{C}_{\eta}$, $\mathcal{C}_{\eta'\to\eta}$ is a finite-dimensional smooth manifold.

To any $h_{\eta'} \in \mathcal{C}_{\eta'}$ we associate maps $h_{\eta} : \gamma(\eta) \to G$, $h^{(0)}_{\eta'\to\eta} : H^{(0)}_{\eta'\to\eta} \to G$, $h^{(2)}_{\eta'\to\eta} : H^{(2)}_{\eta'\to\eta} \to G$, $h^{(3)}_{\eta'\to\eta} : H^{(3)}_{\eta'\to\eta} \to G$ by:

1. $\forall e' \in H^{(0)}_{\eta'\to\eta}$, $h^{(0)}_{\eta'\to\eta}(e') := h_{\eta'}(e')$;
2. $\forall e \in \gamma(\eta)$, $h_{\eta}(e) := \left(\prod_{k=2}^{n_{\eta'\to\eta,e}} \left[h_{\eta'} \circ a_{\eta'\to\eta,e}(k)\right]^{\epsilon_{\eta'\to\eta,e}(k)}\right) . \left[h_{\eta'} \circ a_{\eta'\to\eta,e}(1)\right]$

   with the convention that products of group elements are ordered from right to left:

   $$\forall g_1,\ldots,g_n \in G,\ \prod_{k=1}^{n} g_k := g_n \,.\, \ldots \,.\, g_1\,;$$

3. $\forall e' \in H^{(2)}_{\eta'\to\eta}$, $h^{(2)}_{\eta'\to\eta}(e') := h_{\eta'}(e')$;

4. $\forall e \in \gamma(\eta)$, $\forall e' \in H^{(3)}_{\eta'\to\eta,e}$, $h^{(3)}_{\eta'\to\eta}(e') := h_{\eta'}(e') \, . \, \big(h_\eta(e)\big)^{-1}$ (with $h_\eta(e)$ from 10.24.2).

Then, the map $\varphi_{\eta'\to\eta} : h_{\eta'} \mapsto h^{(0)}_{\eta'\to\eta} \, , \, h^{(2)}_{\eta'\to\eta} \, , \, h^{(3)}_{\eta'\to\eta} \, ; \, h_\eta$ is a diffeomorphism $\mathcal{C}_{\eta'} \to \mathcal{C}_{\eta'\to\eta} \times \mathcal{C}_\eta$.

Proof $\varphi_{\eta'\to\eta}$ is smooth for G is a Lie group. Next, for any $j_\eta \in \mathcal{C}_\eta$ and any $\left(j^{(0)}_{\eta'\to\eta}, j^{(2)}_{\eta'\to\eta}, j^{(3)}_{\eta'\to\eta}\right) \in \mathcal{C}_{\eta'\to\eta}$ we define a map $j_{\eta'}$ by:

5. $\forall e' \in H^{(0)}_{\eta'\to\eta}$, $j_{\eta'}(e') := j^{(0)}_{\eta'\to\eta}(e')$;

6. $\forall e \in \gamma(\eta)$, $\forall e' \in H^{(1)}_{\eta'\to\eta,e}$, $j_{\eta'}(e') := \left(\prod\limits_{k=n_{\eta'\to\eta,e}}^{2} \left[j^{(2)}_{\eta'\to\eta} \circ a_{\eta'\to\eta,e}(k) \right]^{-\epsilon_{\eta'\to\eta,e}(k)} \right) . \, j_\eta(e)$;

7. $\forall e' \in H^{(2)}_{\eta'\to\eta}$, $j_{\eta'}(e') := j^{(2)}_{\eta'\to\eta}(e')$;

8. $\forall e \in \gamma(\eta)$, $\forall e' \in H^{(3)}_{\eta'\to\eta,e}$, $j_{\eta'}(e') := j^{(3)}_{\eta'\to\eta}(e') \, . \, j_\eta(e)$.

Having a partition of $\gamma(\eta')$ (prop. 10.22) ensures that $j_{\eta'}$ is well-defined and, again because G is a Lie group, the map $\widetilde{\varphi}_{\eta'\to\eta} : j^{(0)}_{\eta'\to\eta} \, , \, j^{(2)}_{\eta'\to\eta} \, , \, j^{(3)}_{\eta'\to\eta} \, ; \, j_\eta \mapsto j_{\eta'}$ is a smooth map $\mathcal{C}_{\eta'\to\eta} \times \mathcal{C}_\eta \to \mathcal{C}_{\eta'}$. We can check that $\widetilde{\varphi}_{\eta'\to\eta} \circ \varphi_{\eta'\to\eta} = \mathrm{id}_{\mathcal{C}_{\eta'}}$ and $\varphi_{\eta'\to\eta} \circ \widetilde{\varphi}_{\eta'\to\eta} = \mathrm{id}_{\mathcal{C}_{\eta'\to\eta} \times \mathcal{C}_\eta}$, thus $\varphi_{\eta'\to\eta}$ is a diffeomorphism. □

In order for the previously defined factorization maps to provide a valid factorizing system, they should fulfill the three-spaces consistency condition (eq. (2.11.1) and fig. 2.2). As detailed in subsection 2.3, we need a map $\varphi_{\eta''\to\eta'\to\eta}$ identifying $\mathcal{C}_{\eta''\to\eta}$ with $\mathcal{C}_{\eta''\to\eta'} \times \mathcal{C}_{\eta'\to\eta}$, in agreement with the factorization maps $\varphi_{\eta''\to\eta}$, $\varphi_{\eta''\to\eta'}$ and $\varphi_{\eta'\to\eta}$: given three labels $\eta \preccurlyeq \eta' \preccurlyeq \eta''$, the variables discarded when going down in one step from η'' to η should be the same as the ones discarded when going, in two successive steps, first from η'' to η' and then from η' to η. This will ensure that there is no ambiguity as to how the variables associated to the label η are to be extracted from the phase space corresponding to η''.

To ascertain that this is indeed the case, we have to distinguish the different ways, for an edge e'' of η'', to be positioned with respect to the edges and faces of η' and η: this yields 13 inequivalent possibilities, as depicted in fig. 10.14. Only one of them is compatible with e'' being of type '1' for transition $\eta'' \to \eta$, while, in the 12 others case, the group variable attached to e'' contributes to the variables of $\mathcal{C}_{\eta''\to\eta}$, in terms of which $\varphi_{\eta''\to\eta'\to\eta}$ then needs to be appropriately specified (see the points 10.25.5 to 10.25.10 of the proof below).

Note that we could have made use of prop. 2.17 to obtain the three-spaces consistency of the factorization maps from a similar condition formulated at the level of the projections between the small phase spaces def. 2.3 and fig. 2.1: that these projections fulfills such a condition can indeed be read out from their expression (that will be given in prop. 10.26). Yet, using this result would require G to be connected: a restriction that appears quite artificial when the factorization map $\varphi_{\eta'\to\eta}$ in prop. 10.24 has been expressed solely in terms of the group operations (multiplication and inverse). Preferably, the three-spaces consistency can be obtained in full generality directly at the level of the factorization maps: once the correct explicit expression for $\varphi_{\eta''\to\eta'\to\eta}$ has been deduced from the one for $\varphi_{\eta'\to\eta}$, it is a straightforward (albeit rather fastidious) check that eq. (2.11.1) holds.

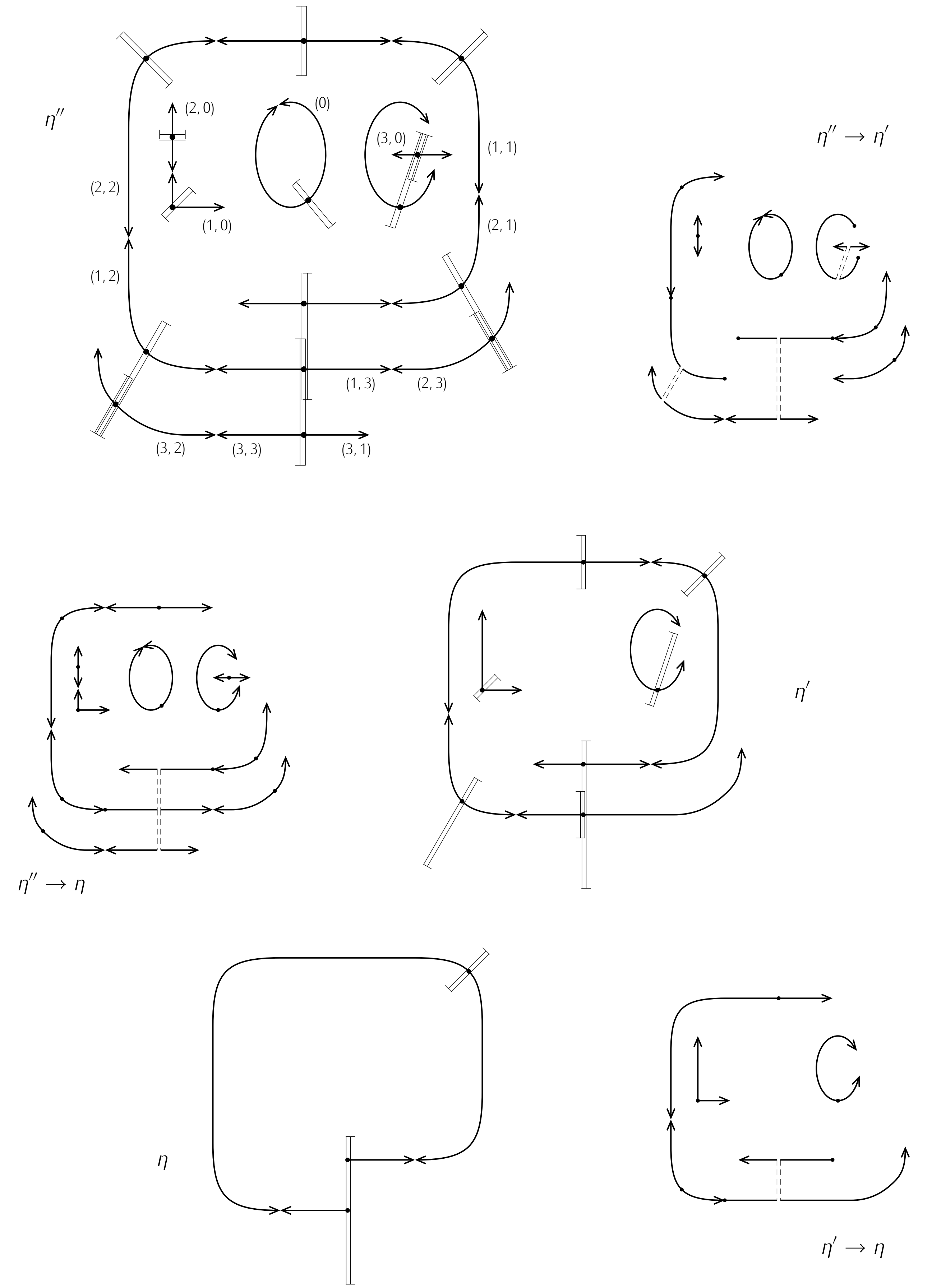


Figure 10.14 – Three-spaces consistency: in the illustration of η'', a tag $^{(\kappa',\kappa)}$ denotes an edge belonging to $H^{(\kappa')}_{\eta''\to\eta',e}$ for some e in $H^{(\kappa)}_{\eta'\to\eta}$ and a tag $^{(0)}$ denotes an edge belonging to $H^{(0)}_{\eta''\to\eta'}$ (for better readability we have tagged only one edge of each type)

Theorem 10.25 Let $\eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}_{\text{HF}}$. There exists a diffeomorphism $\varphi_{\eta''\to\eta'\to\eta} : \mathcal{C}_{\eta''\to\eta} \to \mathcal{C}_{\eta''\to\eta'} \times \mathcal{C}_{\eta'\to\eta}$ such that:

$$\left(\varphi_{\eta''\to\eta'\to\eta} \times \mathrm{id}_{\mathcal{C}_\eta}\right) \circ \varphi_{\eta''\to\eta} = \left(\mathrm{id}_{\mathcal{C}_{\eta''\to\eta'}} \times \varphi_{\eta'\to\eta}\right) \circ \varphi_{\eta''\to\eta'} \quad \text{(aka. eq. (2.11.1))}.$$

This provides a factorizing system of smooth manifolds $(\mathcal{L}_{\text{HF}}, \mathcal{C}, \varphi)^\times$ (def. 2.15).

Proof Let $\eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}_{\text{HF}}$. Let $e \in \gamma(\eta)$ and, for any $k \in \{1, \dots, n_{\eta'\to\eta,e} + 1\}$, define:

$$s^{(k)}_{\eta''\to\eta'\to\eta,e} := \sum_{r=1}^{k-1} n_{\eta''\to\eta', a_{\eta'\to\eta,e}(r)} \,.$$

Using prop. 10.2 together with the uniqueness of $a_{\eta''\to\eta,e}$ (prop. 10.20), we then get $n_{\eta''\to\eta,e} = s^{(n_{\eta'\to\eta,e}+1)}_{\eta''\to\eta'\to\eta,e}$ and, for any $k \in \{1, \dots, n_{\eta'\to\eta,e}\}$ and any $l \in \left\{s^{(k)}_{\eta''\to\eta'\to\eta,e} + 1, \dots, s^{(k+1)}_{\eta''\to\eta'\to\eta,e}\right\}$:

$$a_{\eta''\to\eta,e}(l) = \begin{cases} a_{\eta''\to\eta', a_{\eta'\to\eta,e}(k)}\left(l - s^{(k)}_{\eta''\to\eta'\to\eta,e}\right) & \text{if } k = 1 \text{ or } \epsilon_{\eta'\to\eta,e}(k) = +1 \\ a_{\eta''\to\eta', a_{\eta'\to\eta,e}(k)}\left(s^{(k+1)}_{\eta''\to\eta'\to\eta,e} + 1 - l\right) & \text{otherwise} \end{cases},$$

$$\epsilon_{\eta''\to\eta,e}(l) = \begin{cases} \epsilon_{\eta''\to\eta', a_{\eta'\to\eta,e}(k)}\left(l - s^{(k)}_{\eta''\to\eta'\to\eta,e}\right) & \text{if } k = 1 \text{ or } \epsilon_{\eta'\to\eta,e}(k) = +1 \\ -\epsilon_{\eta''\to\eta', a_{\eta'\to\eta,e}(k)}\left(s^{(k+1)}_{\eta''\to\eta'\to\eta,e} + 1 - l\right) & \text{otherwise} \end{cases}.$$

Next, for any $F \in \mathfrak{F}(\eta)$, we have:

$$H^{(1,3)}_{\eta''\to\eta,F} = \bigcup_{F' \in H^{(1,3)}_{\eta'\to\eta,F}} H^{(1,3)}_{\eta''\to\eta',F'}$$

(the proof is similar to the one for the second part of prop. 10.23).

In particular, using eqs. (10.23.1) and (10.23.2), we then get, for any $e \in \gamma(\eta)$:

1. $H^{(0)}_{\eta''\to\eta} = H^{(0)}_{\eta''\to\eta'} \cup \bigcup_{e' \in H^{(0)}_{\eta'\to\eta}} H^{(4)}_{\eta''\to\eta',e'} \cup \bigcup_{e' \in H^{(2)}_{\eta'\to\eta}} H^{(3)}_{\eta''\to\eta',e'} \cup \bigcup_{e' \in H^{(3)}_{\eta'\to\eta}} H^{(2)}_{\eta''\to\eta',e'}$;

2. $H^{(1)}_{\eta''\to\eta,e} = \bigcup_{e' \in H^{(1)}_{\eta'\to\eta,e}} H^{(1)}_{\eta''\to\eta',e'}$;

3. $H^{(2)}_{\eta''\to\eta,e} = \bigcup_{e' \in H^{(1)}_{\eta'\to\eta,e}} H^{(2)}_{\eta''\to\eta',e'} \cup \bigcup_{e' \in H^{(2)}_{\eta'\to\eta,e}} \left(H^{(1)}_{\eta''\to\eta',e'} \cup H^{(2)}_{\eta''\to\eta',e'}\right)$;

4. $H^{(3)}_{\eta''\to\eta,e} = \bigcup_{e' \in H^{(1)}_{\eta'\to\eta,e}} H^{(3)}_{\eta''\to\eta',e'} \cup \bigcup_{e' \in H^{(3)}_{\eta'\to\eta,e}} \left(H^{(1)}_{\eta''\to\eta',e'} \cup H^{(3)}_{\eta''\to\eta',e'}\right)$;

Now, for any $\left(h^{(0)}_{\eta''\to\eta}, h^{(2)}_{\eta''\to\eta}, h^{(3)}_{\eta''\to\eta}\right) \in \mathcal{C}_{\eta''\to\eta}$, we define:

5. $\forall e' \in H^{(0)}_{\eta'\to\eta}, \quad j^{(0)}_{\eta'\to\eta}(e') := \left(\prod_{k=2}^{n_{\eta''\to\eta',e'}} \left[h^{(0)}_{\eta''\to\eta} \circ a_{\eta''\to\eta',e'}(k)\right]^{\epsilon_{\eta''\to\eta',e'}(k)}\right) \cdot \left[h^{(0)}_{\eta''\to\eta} \circ a_{\eta''\to\eta',e'}(1)\right]$

   (well-defined since $H^{(1)}_{\eta''\to\eta',e'} \cup H^{(2)}_{\eta''\to\eta',e'} \subset H^{(0)}_{\eta''\to\eta}$);

6. $\forall e' \in H^{(2)}_{\eta'\to\eta}\,,\ \ j^{(2)}_{\eta'\to\eta}(e') := \left(\prod_{k=2}^{n_{\eta''\to\eta',e'}} \left[h^{(2)}_{\eta''\to\eta} \circ a_{\eta''\to\eta',e'}(k)\right]^{\epsilon_{\eta''\to\eta',e'}(k)}\right) . \left[h^{(2)}_{\eta''\to\eta} \circ a_{\eta''\to\eta',e'}(1)\right]$

(well-defined since $H^{(1)}_{\eta''\to\eta',e'} \cup H^{(2)}_{\eta''\to\eta',e'} \subset H^{(2)}_{\eta''\to\eta}$);

7. $\forall e' \in H^{(3)}_{\eta'\to\eta}\,,\ \ j^{(3)}_{\eta'\to\eta}(e') := \left(\prod_{k=2}^{n_{\eta''\to\eta',e'}} \left[h^{(0)}_{\eta''\to\eta} \circ a_{\eta''\to\eta',e'}(k)\right]^{\epsilon_{\eta''\to\eta',e'}(k)}\right) . \left[h^{(3)}_{\eta''\to\eta} \circ a_{\eta''\to\eta',e'}(1)\right]$

(well-defined since $H^{(1)}_{\eta''\to\eta',e'} \subset H^{(3)}_{\eta''\to\eta}$ and $H^{(2)}_{\eta''\to\eta',e'} \cup H^{(0)}_{\eta''\to\eta',e'} \subset H^{(2)}_{\eta''\to\eta}$);

8. $\forall e'' \in H^{(0)}_{\eta''\to\eta'}\,,\ \ j^{(0)}_{\eta''\to\eta'}(e'') := h^{(0)}_{\eta''\to\eta}(e'')$

(well-defined since $H^{(0)}_{\eta''\to\eta'} \subset H^{(0)}_{\eta''\to\eta}$);

9. $\forall e'' \in H^{(2)}_{\eta''\to\eta'}\,,\ \ j^{(2)}_{\eta''\to\eta'}(e'') := \begin{cases} h^{(0)}_{\eta''\to\eta}(e'') & \text{if } e'' \in H^{(0)}_{\eta''\to\eta} \\ h^{(2)}_{\eta''\to\eta}(e'') & \text{if } e'' \in H^{(2)}_{\eta''\to\eta} \end{cases}$

(well-defined since $H^{(2)}_{\eta''\to\eta'} \subset H^{(0)}_{\eta''\to\eta} \cup H^{(2)}_{\eta''\to\eta}$);

10. $\forall e' \in \gamma(\eta'),\ \forall e'' \in H^{(3)}_{\eta''\to\eta',e'}\,,$

$$j^{(3)}_{\eta''\to\eta'}(e'') := \begin{cases} h^{(0)}_{\eta''\to\eta}(e'') \,.\, \left(j^{(0)}_{\eta'\to\eta}(e')\right)^{-1} & \text{if } e' \in H^{(0)}_{\eta''\to\eta} \\ h^{(0)}_{\eta''\to\eta}(e'') \,.\, \left(j^{(2)}_{\eta'\to\eta}(e')\right)^{-1} & \text{if } e' \in H^{(2)}_{\eta''\to\eta} \\ h^{(3)}_{\eta''\to\eta}(e'') \,.\, \prod_{k=2}^{n_{\eta'\to\eta,e}} \left(j^{(2)}_{\eta'\to\eta} \circ a_{\eta'\to\eta,e}(k)\right)^{\epsilon_{\eta'\to\eta,e}(k)} & \text{if } e' \in H^{(1)}_{\eta''\to\eta,e} \text{ (with } e \in \gamma(\eta)\text{)} \\ h^{(3)}_{\eta''\to\eta}(e'') \,.\, \left(j^{(3)}_{\eta'\to\eta}(e')\right)^{-1} & \text{if } e' \in H^{(3)}_{\eta''\to\eta} \end{cases}$$

(using $j^{(0)}_{\eta'\to\eta}$, $j^{(2)}_{\eta'\to\eta}$ and $j^{(3)}_{\eta'\to\eta}$ from 10.25.5 to 10.25.7; well-defined since $H^{(3)}_{\eta''\to\eta',e'} \subset H^{(0)}_{\eta''\to\eta}$ in the first two cases, and $H^{(3)}_{\eta''\to\eta',e'} \subset H^{(3)}_{\eta''\to\eta}$ in the last two cases).

Then, the map $\varphi_{\eta''\to\eta'\to\eta} : h^{(0)}_{\eta''\to\eta}\,,\, h^{(2)}_{\eta''\to\eta}\,,\, h^{(3)}_{\eta''\to\eta} \mapsto j^{(0)}_{\eta''\to\eta'}\,,\, j^{(2)}_{\eta''\to\eta'}\,,\, j^{(3)}_{\eta''\to\eta'}\,;\, j^{(0)}_{\eta'\to\eta}\,,\, j^{(2)}_{\eta'\to\eta}\,,\, j^{(3)}_{\eta'\to\eta}$ is smooth $\mathcal{C}_{\eta''\to\eta} \to \mathcal{C}_{\eta''\to\eta'} \times \mathcal{C}_{\eta'\to\eta}$.

Let $h_{\eta''} \in \mathcal{C}_{\eta''}$, and define:

11. $\left(h^{(0)}_{\eta''\to\eta'}\,,\, h^{(2)}_{\eta''\to\eta'}\,,\, h^{(3)}_{\eta''\to\eta'}\,;\, h_{\eta'}\right) := \varphi_{\eta''\to\eta'}\left(h_{\eta''}\right) \in \mathcal{C}_{\eta''\to\eta'} \times \mathcal{C}_{\eta'}$;

12. $\left(h^{(0)}_{\eta'\to\eta}\,,\, h^{(2)}_{\eta'\to\eta}\,,\, h^{(3)}_{\eta'\to\eta}\,;\, h_{\eta}\right) := \varphi_{\eta'\to\eta}\left(h_{\eta'}\right) \in \mathcal{C}_{\eta'\to\eta} \times \mathcal{C}_{\eta}$;

13. $\left(h^{(0)}_{\eta''\to\eta}\,,\, h^{(2)}_{\eta''\to\eta}\,,\, h^{(3)}_{\eta''\to\eta}\,;\, j_{\eta}\right) := \varphi_{\eta''\to\eta}\left(h_{\eta''}\right) \in \mathcal{C}_{\eta''\to\eta} \times \mathcal{C}_{\eta}$;

14. $\left(j^{(0)}_{\eta''\to\eta'}\,,\, j^{(2)}_{\eta''\to\eta'}\,,\, j^{(3)}_{\eta''\to\eta'}\,;\, j^{(0)}_{\eta'\to\eta}\,,\, j^{(2)}_{\eta'\to\eta}\,,\, j^{(3)}_{\eta'\to\eta}\right) := \varphi_{\eta''\to\eta'\to\eta'}\left(h^{(0)}_{\eta''\to\eta}\,,\, h^{(2)}_{\eta''\to\eta}\,,\, h^{(3)}_{\eta''\to\eta}\right)$.

Using the definitions of $\varphi_{\eta'\to\eta}$ (from prop. 10.24) and $\varphi_{\eta''\to\eta'\to\eta'}$, we can check that:

$$\left(j^{(0)}_{\eta''\to\eta'}\,,\, j^{(2)}_{\eta''\to\eta'}\,,\, j^{(3)}_{\eta''\to\eta'}\,;\, j^{(0)}_{\eta'\to\eta}\,,\, j^{(2)}_{\eta'\to\eta}\,,\, j^{(3)}_{\eta'\to\eta}\,;\, j_{\eta}\right) = \\ = \left(h^{(0)}_{\eta''\to\eta'}\,,\, h^{(2)}_{\eta''\to\eta'}\,,\, h^{(3)}_{\eta''\to\eta'}\,;\, h^{(0)}_{\eta'\to\eta}\,,\, h^{(2)}_{\eta'\to\eta}\,,\, h^{(3)}_{\eta'\to\eta}\,;\, h_{\eta}\right).$$

In other words, eq. (2.11.1) is fulfilled. But this also ensures that $\varphi_{\eta''\to\eta'\to\eta}$ is a diffeomorphism for $\varphi_{\eta''\to\eta'}$, $\varphi_{\eta'\to\eta}$, and $\varphi_{\eta''\to\eta}$ are diffeomorphisms (prop. 10.24). Together with the directedness of $\mathcal{L}_{\text{HF}}$

proved in theorem 10.14, this yields the desired factorizing system (def. 2.15). □

Finally, we wrap up the classical side of the construction by writing down the symplectic manifold $\mathcal{M}_\eta$ attached to a label η, the projection from $\mathcal{M}_{\eta'}$ into $\mathcal{M}_\eta$ (as prescribed by the factorization $\varphi_{\eta' \to \eta}$, for $\eta \preccurlyeq \eta'$), and the expression of the holonomy and flux observables over $\mathcal{M}_\eta$. These explicit formulas allow to check (prop. 10.28) that we indeed reproduce the holonomy-flux algebra that was derived in prop. 9.10 in the context of the infinite phase space description of a theory of connections, thus validating a posteriori the intuitive arguments that were repeatedly asserted above (on account of the underlying physical interpretation of the labels).

Note that, as announced at the beginning of the present section, the projections $\mathcal{M}_{\eta'} \to \mathcal{M}_\eta$ are of the form that was considered in theorem 6.2 (making $\mathcal{C}_\eta$ into a Lie group via pointwise multiplication and inverse).

Proposition 10.26 To any $\eta \in \mathcal{L}_{\text{HF}}$ we associate the symplectic manifold $\mathcal{M}_\eta := T^*(\mathcal{C}_\eta)$ (with its canonical symplectic structure as a cotangent bundle), which we identify with $\mathcal{C}_\eta \times \mathcal{P}_\eta$ (def. 10.19) via:

$$\begin{array}{rrcl} L_\eta : & \mathcal{M}_\eta & \to & \mathcal{C}_\eta \times \mathcal{P}_\eta \\ & h,\, p & \mapsto & h,\, \left(F \mapsto p \circ \left[T_1 \left(t \mapsto R^{(F)}_{\eta,t} h\right)\right]\right) \end{array},$$

the map $R^{(F)}_{\eta,t} h \in \mathcal{C}_\eta$ being defined for $F \in \mathcal{F}(\eta)$, $t \in G$, and $h \in \mathcal{C}_\eta$ by:

$$\forall e \in \gamma(\eta),\, R^{(F)}_{\eta,t} h(e) := \begin{cases} h(e)\,.\,t & \text{if } \chi_\eta(e) = F \\ h(e) & \text{else} \end{cases}.$$

For any $\eta \preccurlyeq \eta' \in \mathcal{L}_{\text{HF}}$, we define $\pi_{\eta' \to \eta} : \mathcal{M}_{\eta'} \to \mathcal{M}_\eta$ as $\pi_{\eta' \to \eta} = \widetilde{s}_{\eta' \to \eta} \circ \widetilde{\varphi}_{\eta' \to \eta}$, where $\widetilde{s}_{\eta' \to \eta} : T^*(\mathcal{C}_{\eta' \to \eta} \times \mathcal{C}_\eta) \approx T^*(\mathcal{C}_{\eta' \to \eta}) \times T^*(\mathcal{C}_\eta) \to T^*(\mathcal{C}_\eta)$ is the projection on the second Cartesian factor and $\widetilde{\varphi}_{\eta' \to \eta} : T^*(\mathcal{C}_{\eta'}) \to T^*(\mathcal{C}_{\eta' \to \eta} \times \mathcal{C}_\eta)$ is the cotangent lift of $\varphi_{\eta' \to \eta}$. Then, $(\mathcal{L}_{\text{HF}}, \mathcal{M}, \pi)^\downarrow$ is a projective system of phase spaces (props. 2.16 and 2.13).

Moreover, we have:

$$\begin{array}{rrcl} L_\eta \circ \pi_{\eta' \to \eta} \circ L^{-1}_{\eta'} : & \mathcal{C}_{\eta'} \times \mathcal{P}_{\eta'} & \to & \mathcal{C}_\eta \times \mathcal{P}_\eta \\ & h_{\eta'}\,,\, P_{\eta'} & \mapsto & h_\eta\,,\, P_\eta \end{array},$$

where h_η and P_η are given in terms of $h_{\eta'}$ and $P_{\eta'}$ by:

$$\forall e \in \gamma(\eta),\, h_\eta(e) = \left(\prod_{k=2}^{n_{\eta' \to \eta, e}} [h_{\eta'} \circ a_{\eta' \to \eta, e}(k)]^{\epsilon_{\eta' \to \eta, e}(k)}\right) . [h_{\eta'} \circ a_{\eta' \to \eta, e}(1)], \tag{10.26.1}$$

$$\text{and} \quad \forall F \in \mathcal{F}(\eta),\, P_\eta(F) = \sum_{F' \in H^{(1,3)}_{\eta' \to \eta, F}} P_{\eta'}(F'). \tag{10.26.2}$$

Proof For any η, L_η is a diffeomorphism $\mathcal{M}_\eta \to \mathcal{C}_\eta \times \mathcal{P}_\eta$ by definition of χ_η (def. 10.12).

Let $(h_{\eta'}\,,\, p_{\eta'}) \in \mathcal{M}_{\eta'}$ and $(h_{\eta'}\,,\, P_{\eta'}) = L_{\eta'}(h_{\eta'}\,,\, p_{\eta'})$. We have:

$$\widetilde{\varphi}_{\eta'\to\eta}\left(h_{\eta'}\,;\,p_{\eta'}\right) = \left(h^{(0)}_{\eta'\to\eta}\,,\,h^{(2)}_{\eta'\to\eta}\,,\,h^{(3)}_{\eta'\to\eta}\,,\,h_\eta\,;\,p_{\eta'}\circ\left[T_{\varphi_{\eta'\to\eta}(h_{\eta'})}\varphi^{-1}_{\eta'\to\eta}\right]\right),$$

with $h^{(0)}_{\eta'\to\eta}\,,\,h^{(2)}_{\eta'\to\eta}\,,\,h^{(3)}_{\eta'\to\eta}\,,\,h_\eta$ constructed from $h_{\eta'}$ as in prop. 10.24 (in particular, h_η is given by eq. (10.26.1)). Thus:

$$\pi_{\eta'\to\eta}\left(h_{\eta'}\,;\,p_{\eta'}\right) = \left(h_\eta\,;\,p_{\eta'}\circ\left[T_{h_\eta}\varphi^{-1}_{\eta'\to\eta}\left(h^{(0)}_{\eta'\to\eta}\,,\,h^{(2)}_{\eta'\to\eta}\,,\,h^{(3)}_{\eta'\to\eta}\,,\,\cdot\,\right)\right]\right),$$

and finally $L_\eta\circ\pi_{\eta'\to\eta}\left(h_{\eta'}\,;\,p_{\eta'}\right) = \left(h_\eta\,;\,P_\eta\right)$, where P_η is given in terms of $h_{\eta'}\,,\,p_{\eta'}$ by:

$$\forall F\in\mathcal{F}(\eta),\ P_\eta(F) = p_{\eta'}\circ\left[T_1\,t\mapsto\varphi^{-1}_{\eta'\to\eta}\left(h^{(0)}_{\eta'\to\eta}\,,\,h^{(2)}_{\eta'\to\eta}\,,\,h^{(3)}_{\eta'\to\eta}\,,\,R^{(F)}_{\eta,t}h_\eta\right)\right].$$

Now, for any $F\in\mathcal{F}(\eta)$, $t\in G$ and $e'\in\gamma(\eta')$, we get, using the explicit expression for $\varphi^{-1}_{\eta'\to\eta}$ from the proof of prop. 10.24:

$$\varphi^{-1}_{\eta'\to\eta}\left(h^{(0)}_{\eta'\to\eta}\,,\,h^{(2)}_{\eta'\to\eta}\,,\,h^{(3)}_{\eta'\to\eta}\,,\,R^{(F)}_{\eta,t}h_\eta\right)(e') = \begin{cases} h_{\eta'}(e')\,.\,t & \text{if } \chi_{\eta'}(e')\in H^{(1,3)}_{\eta'\to\eta,F} \\ h_{\eta'}(e') & \text{else}\end{cases}.$$

Therefore, $\forall F\in\mathcal{F}(\eta),\ P_\eta(F) = \sum_{F'\in H^{(1,3)}_{\eta'\to\eta,F}} P_{\eta'}(F')$. □

Proposition 10.27 Let $e\in\mathcal{L}_{\text{edges}}$ and let $\eta\in\mathcal{L}_{\text{HF}}$ such that $\gamma(\eta)\in\mathcal{L}_{\text{graphs}/e}$ (prop. 10.20). For any $m\in C^\infty(G,\mathbb{R})$ (with $C^\infty(G,\mathbb{R})$ the set of smooth functions $G\to\mathbb{R}$), we define $\mathrm{h}^{(e,m)}_\eta\in C^\infty(\mathcal{M}_\eta\,,\,\mathbb{R})$ by:

$$\forall(h,p)\in\mathcal{M}_\eta\,,\quad \mathrm{h}^{(e,m)}_\eta(h,p) := m\left(\prod_{k=1}^{n_{\gamma(\eta)\to e}}\left[h\circ a_{\gamma(\eta)\to e}(k)\right]^{\epsilon_{\gamma(\eta)\to e}(k)}\right).$$

Then, for any $\eta,\eta'\in\mathcal{L}_{\text{HF}}$ such that $\gamma(\eta),\ \gamma(\eta')\in\mathcal{L}_{\text{graphs}/e}$, we have:

$$\forall m\in C^\infty(G,\mathbb{R}),\ \mathrm{h}^{(e,m)}_\eta\sim\mathrm{h}^{(e,m)}_{\eta'},$$

with the equivalence relation of eq. (2.4.1). Moreover $\left\{\eta\in\mathcal{L}_{\text{HF}}\,\middle|\,\gamma(\eta)\in\mathcal{L}_{\text{graphs}/e}\right\}\neq\varnothing$, thus we can associate to any $m\in C^\infty(G,\mathbb{R})$ a well-defined observable $\mathrm{h}^{(e,m)}\in\mathcal{O}^{\downarrow}_{(\mathcal{L}_{\text{HF}},\mathcal{M},\pi)}$ (def. 2.4).

Let $\overline{F}\in\mathcal{L}_{\text{faces}}$ and let $\eta\in\mathcal{L}_{\text{HF}}$ such that $\lambda(\eta)\in\mathcal{L}_{\text{profls}/\overline{F}}$ (prop. 10.21). For any $u\in\mathfrak{g}$, we define $\mathrm{P}^{(\overline{F},u)}_\eta\in C^\infty(\mathcal{M}_\eta\,,\,\mathbb{R})$ by:

$$\forall(h,p)\in\mathcal{M}_\eta\,,\quad \mathrm{P}^{(\overline{F},u)}_\eta(h,p) := \sum_{F'\in H_{\lambda(\eta)\to\overline{F}}} p\circ\left[T_1\left(t\mapsto R^{(F')}_{\eta,t}h\right)\right](u).$$

Then, for any $\eta,\eta'\in\mathcal{L}_{\text{HF}}$ such that $\lambda(\eta),\ \lambda(\eta')\in\mathcal{L}_{\text{profls}/\overline{F}}$, we have:

$$\forall u\in\mathfrak{g},\ \mathrm{P}^{(\overline{F},u)}_\eta\sim\mathrm{P}^{(\overline{F},u)}_{\eta'}.$$

And since $\left\{\eta\in\mathcal{L}_{\text{HF}}\,\middle|\,\lambda(\eta)\in\mathcal{L}_{\text{profls}/\overline{F}}\right\}\neq\varnothing$, we can associate to any $u\in\mathfrak{g}$ a well-defined observable $\mathrm{P}^{(\overline{F},u)}\in\mathcal{O}^{\downarrow}_{(\mathcal{L}_{\text{HF}},\mathcal{M},\pi)}$.

Proof Let $e\in\mathcal{L}_{\text{edges}}$. By chaining lemma 10.15 to 10.18, there exists $(\gamma,\lambda)\in\mathcal{L}_{\text{HF}}$ such that $\{e\}\preccurlyeq\gamma$ (and $[\varnothing]_{\text{profl}}\preccurlyeq\lambda$ where $[\,\cdot\,]_{\text{profl}}$ denotes the equivalence class in $\mathcal{L}_{\text{profls}}$), hence $\left\{\eta\in\mathcal{L}_{\text{HF}}\,\middle|\,\gamma(\eta)\in\mathcal{L}_{\text{graphs}/e}\right\}$

$\neq \varnothing$. Let η such that $\gamma(\eta) \in \mathcal{L}_{\text{graphs}/e}$ and let $\eta' \succcurlyeq \eta$. Then, $\gamma(\eta') \in \mathcal{L}_{\text{graphs}/e}$. Moreover, we can express $a_{\gamma(\eta')\to e}$ in terms of $a_{\gamma(\eta)\to e}$ and of the $a_{\eta'\to\eta,e'}$ for e' in the image of $a_{\gamma(\eta)\to e}$ (like we did above in the proof of theorem 10.25). Together with eq. (10.26.1), we readily check that:

$$\forall m \in C^\infty(G,\mathbb{R}),\ \mathrm{h}_\eta^{(e,m)} \circ \pi_{\eta'\to\eta} = \mathrm{h}_{\eta'}^{(e,m)} .$$

Finally, for any η, η' such that $\gamma(\eta), \gamma(\eta') \in \mathcal{L}_{\text{graphs}/e}$, there exists $\eta'' \succcurlyeq \eta, \eta'$, hence:

$$\forall m \in C^\infty(G,\mathbb{R}),\ \mathrm{h}_\eta^{(e,m)} \circ \pi_{\eta''\to\eta} = \mathrm{h}_{\eta''}^{(e,m)} = \mathrm{h}_{\eta'}^{(e,m)} \circ \pi_{\eta''\to\eta'} ,$$

ie. for any $m \in C^\infty(G,\mathbb{R})$, $\mathrm{h}_\eta^{(e,m)} \sim \mathrm{h}_{\eta'}^{(e,m)}$.

Let $\lambda \in \mathcal{L}_{\text{profls}}$, $F \in \mathcal{F}(\lambda)$ and $\overline{F} = F^\perp \circ F$. Again, there exists $(\gamma', \lambda') \in \mathcal{L}_{\text{HF}}$ such that $\lambda \preccurlyeq \lambda'$ (and $\varnothing \preccurlyeq \gamma'$), hence $\left\{\eta \in \mathcal{L}_{\text{HF}} \,\middle|\, \lambda(\eta) \in \mathcal{L}_{\text{profls}/\overline{F}}\right\} \neq \varnothing$. Let η such that $\lambda(\eta) \in \mathcal{L}_{\text{profls}/\overline{F}}$ and let $\eta' \succcurlyeq \eta$. Then, $\lambda(\eta') \in \mathcal{L}_{\text{profls}/\overline{F}}$ and we have (through a reasoning similar to the proof for the second part of prop. 10.23):

$$H_{\lambda(\eta')\to\overline{F}} = \bigcup_{F' \in H_{\lambda(\eta)\to\overline{F}}} H^{(1,3)}_{\eta'\to\eta,F'} . \tag{10.27.1}$$

Combining this with eq. (10.26.2), we get:

$$\forall u \in \mathfrak{g},\ \mathrm{P}_\eta^{(\overline{F},u)} \circ \pi_{\eta'\to\eta} = \mathrm{P}_{\eta'}^{(\overline{F},u)} .$$

Like above, this ensures that for any η, η' such that $\lambda(\eta), \lambda(\eta') \in \mathcal{L}_{\text{profls}/\overline{F}}$ and for any $u \in \mathfrak{g}$, $\mathrm{P}_\eta^{(\overline{F},u)} \sim \mathrm{P}_{\eta'}^{(\overline{F},u)}$. □

Proposition 10.28 Let e be an analytical edge in Σ, S be an analytic surface in Σ, $m \in C^\infty(G,\ \mathbb{R})$, and $u \in \mathfrak{g}$. We define:

$$\forall \diamond \in \{\uparrow, \downarrow\},\quad S_\diamond := \{e' \in \mathcal{L}_{\text{edges}} \mid e' \diamond S\} \quad \& \quad \overline{S}_\diamond := S_\diamond^\perp \circ S_\diamond \in \mathcal{L}_{\text{faces}}$$

(using the notations of prop. 10.21), as well as:

$$\mathrm{P}^{(S,u)} := \frac{1}{2}\left(\mathrm{P}^{(\overline{S}_\uparrow,\,u)} - \mathrm{P}^{(\overline{S}_\downarrow,\,u)}\right),$$

and we have:

1. if $e \curvearrowright S$, then:

$$\left\{\mathrm{P}^{(S,u)},\ \mathrm{h}^{(e,m)}\right\} = 0 ,$$

where the Poisson brakets $\{\,\cdot\,,\,\cdot\,\}$ between observables over $\mathcal{S}^\downarrow_{(\mathcal{L}_{\text{HF}},\,\mathcal{M},\,\pi)}$ have been defined in def. 2.4;

2. if $e = e_2 \circ e_1^{-1}$ with e_1, $e_2 \diamond S$ and $\diamond \in \{\uparrow, \downarrow\}$, then:

$$\left\{\mathrm{P}^{(S,u)},\ \mathrm{h}^{(e,m)}\right\} = 0 ;$$

3. if $e = e_2 \circ e_1^{-1}$ with $e_1 \diamond_1 S$, $e_2 \diamond_2 S$ and $\{\diamond_1, \diamond_2\} = \{\uparrow, \downarrow\}$, then, defining:

$$\gamma := \{e_1\,,\ e_2\} \in \mathcal{L}_{\text{graphs}} ,\quad \lambda := \left[\,\{S\}\,\right]_\sim \in \mathcal{L}_{\text{profls}} ,\quad \eta := (\gamma,\ \lambda) \in \mathcal{L}_{\text{HF}} ,$$

we get, for any $(h, p) \in \mathcal{M}_\eta$:

$$\left\{\mathrm{P}_\eta^{(S,u)}\,,\ \mathrm{h}_\eta^{(e,m)}\right\}_{\mathcal{M}_\eta}(h, p) = \operatorname{sgn}(e, S)\ \frac{d}{dr} m\left[h(e_2).\, e^{r\,u}.\, h(e_1)^{-1}\right]\Big|_{r=0},$$

where $\operatorname{sgn}(e, S)$ is given by:

$$\operatorname{sgn}(e, S) := \begin{cases} +1 & \text{if } (\diamond_1\,,\ \diamond_2) = (\downarrow,\ \uparrow) \\ -1 & \text{if } (\diamond_1\,,\ \diamond_2) = (\uparrow,\ \downarrow) \end{cases}.$$

Proof *Poisson brackets between holonomy and flux observables.* Let $\eta \in \mathcal{L}_{\text{HF}}$ and let $(h,\ p) \in \mathcal{M}_\eta$. Let $P \in \mathcal{P}_\eta$ such that $(h,\ P) = L_\eta(h,\ p)$, where $L_\eta : \mathcal{M}_\eta \to \mathcal{C}_\eta \times \mathcal{P}_\eta$ has been defined in prop. 10.26. Using left-translated exponential coordinates around $h \in \mathcal{C}_\eta$ (cf. the proof of theorem 6.2), we can show that the symplectic structure Ω_η on $\mathcal{M}_\eta$ is given by:

$$\forall v, v' \in T_h(\mathcal{C}_\eta),\ \forall w, w' \in T_P(\mathcal{P}_\eta) \approx \mathcal{P}_\eta\,,\quad \left(L_\eta^{-1,*}\Omega_\eta\right)_{(h,P)}\left((v, w),\, (v', w')\right) =$$

$$= \sum_{e \in \gamma(\eta)} w'\big(\chi_\eta(e)\big) \circ \ell^{(e)}_{\eta,h^{-1}}(v) - w\big(\chi_\eta(e)\big) \circ \ell^{(e)}_{\eta,h^{-1}}(v') + P\big(\chi_\eta(e)\big)\left(\left[\ell^{(e)}_{\eta,h^{-1}}(v),\ \ell^{(e)}_{\eta,h^{-1}}(v')\right]_{\mathfrak{g}}\right), \tag{10.28.1}$$

where, for any $e \in \gamma(\eta)$:

$$\forall h \in \mathcal{C}_\eta\,,\ \ell^{(e)}_{\eta,h^{-1}} := \left[T_h\left(j \mapsto h(e)^{-1}.\, j(e)\right)\right] : T_h\mathcal{C}_\eta \to \mathfrak{g}\,.$$

Now, for any $\overline{F} \in \mathcal{L}_{\text{faces}}$ such that $\lambda(\eta) \in \mathcal{L}_{\text{profls}/\overline{F}}$ and for any $u \in \mathfrak{g}$, we define:

$$\left(V^{(\overline{F},u)},\ W^{(\overline{F},u)}\right) := \left[T_{(h,p)}L_\eta\right]\left(X_{\mathrm{P}_\eta^{(\overline{F},u)},(h,p)}\right),$$

where $X_{\mathrm{P}_\eta^{(\overline{F},u)}}$ denotes the Hamiltonian vector field of $\mathrm{P}_\eta^{(\overline{F},u)}$. Eq. (10.28.1) then yields:

$$\forall e \in \gamma(\eta),\quad \ell^{(e)}_{\eta,h^{-1}}\left(V^{(\overline{F},u)}\right) = Y^{(\overline{F},u)}(e) \quad \& \quad W^{(\overline{F},u)}(\chi_\eta(e)) = P\big(\chi_\eta(e)\big)\left(\left[Y^{(\overline{F},u)}(e),\ \cdot\ \right]_{\mathfrak{g}}\right),$$

where $Y^{(\overline{F},u)} : \gamma(\eta) \to \mathfrak{g}$ is given by:

$$\forall e \in \gamma(\eta),\quad Y^{(\overline{F},u)}(e) = \begin{cases} u & \text{if } e \in \overline{F} \\ 0 & \text{else} \end{cases},$$

and we have used that:

$$\left\{e \in \gamma(\eta)\ \middle|\ e \in \overline{F}\right\} = \left\{e \in \gamma(\eta)\ \middle|\ \exists\, F' \in H_{\lambda(\eta) \to \overline{F}} \,/\, \chi_\eta(e) = F'\right\} \tag{10.28.2}$$

(through a reasoning similar to the proof for the second part of prop. 10.23). Applying the map $\left[T_1\big(t \mapsto R^{(\chi_\eta(e))}_{\eta,t}\, h\big)\right] : \mathfrak{g} \to T_h(\mathcal{C}_\eta)$ (from prop. 10.26) on both sides of the first equation allows to solve for $V^{(\overline{F},u)}$:

$$V^{(\overline{F},u)} = \left[T_1\big(t \mapsto R^{(\overline{F})}_{\eta,t}\, h\big)\right](u)\,,$$

where $R^{(\overline{F})}_{\eta,t} h \in \mathcal{C}_\eta$ is given, for any $t \in G$, by:

$$\forall e \in \gamma(\eta),\quad R^{(\overline{F})}_{\eta,t} h(e) := \begin{cases} h(e)\,.\,t & \text{if } e \in \overline{F} \\ h(e) & \text{else} \end{cases} .$$

Thus, we get, for any $e \in \mathcal{L}_{\text{edges}}$ such that $\gamma(\eta) \in \mathcal{L}_{\text{graphs}/e}$, and any $m \in C^\infty(G, \mathbb{R})$:

$$\forall (h,p) \in \mathcal{M}_\eta\,,\quad \left\{\mathrm{P}^{(\overline{F},u)}_\eta,\, \mathrm{h}^{(e,m)}_\eta\right\}_{\mathcal{M}_\eta}(h,p) = \left[T_{\mathbf{1}}\left(t \mapsto \mathrm{h}^{(e,m)}_\eta\left(R^{(\overline{F})}_{\eta,t}\, h, p\right)\right)\right](u)\,, \tag{10.28.3}$$

where we have used that, for any $(h,p) \in \mathcal{M}_\eta$, $\mathrm{h}^{(e,m)}_\eta(h,p) = \mathrm{h}^{(e,m)}_\eta(h,0) = \mathrm{h}^{(e,m)}_\eta \circ L^{-1}_\eta(h,0)$.

Faces of a surface. Let S be an analytic surface in Σ and $u \in \mathfrak{g}$. From the definitions in prop. 10.9, we get:

$$\mathcal{F}\big(\{S\}\big) = \left\{S_\uparrow\,,\, S_\downarrow\right\} \tag{10.28.4}$$

(note that $S_\uparrow \neq \varnothing$, as can be ascertained by choosing a representative $\check{S}$ of S and pre-composing $\check{S}$ with a suitable rescaling of $\mathbb{R}^d$ to define an encharted edge; similarly $S_\downarrow \neq \varnothing$, by pre-composing in addition with a reflection). In particular, $\overline{S}_\uparrow\,,\, \overline{S}_\downarrow \in \mathcal{L}_{\text{faces}}$.

Edge indifferent to the surface or only on one side. Let e be an analytical edge in Σ such that $e = e_2 \circ e_1^{-1}$ with $e_1\,,\, e_2 \diamond S$ and $\diamond \in \left\{\rightsquigarrow, \uparrow, \downarrow\right\}$ (note that this includes the case $e \rightsquigarrow S$ by writing $e = e_{[p,f(e)]} e^{-1}_{[p,b(e)]}$ for some $p \in r(e) \setminus \{b(e), f(e)\}$ and using prop. 10.6.6). Let $m \in C^\infty(G, \mathbb{R})$. From lemma 10.15 there exists a label $\eta \in \mathcal{L}_{\text{HF}}$ such that:

$$\{e_1\,,\, e_2\} \preccurlyeq \gamma(\eta) \quad \& \quad \big[\,\{S\}\,\big]_\sim \preccurlyeq \lambda(\eta)\,,$$

and we have (relying on the uniqueness part of prop. 10.20):

$$\forall (h,p) \in \mathcal{M}_\eta\,,\quad \mathrm{h}^{(e,m)}_\eta(h,p) := m\left(\prod_{k=1}^{n_{\gamma(\eta)\to e_2}} \left[h \circ a_{\gamma(\eta)\to e_2}(k)\right]^{\epsilon_{\gamma(\eta)\to e_2}(k)} . \prod_{l=n_{\gamma(\eta)\to e_1}}^{1} \left[h \circ a_{\gamma(\eta)\to e_1}(l)\right]^{-\epsilon_{\gamma(\eta)\to e_1}(l)}\right),$$

with:

$$a_{\gamma(\eta)\to e_1}(1),\, a_{\gamma(\eta)\to e_2}(1) \diamond_S,\quad \forall k > 1,\, a_{\gamma(\eta)\to e_2}(k) \rightsquigarrow S \quad \& \quad \forall l > 1,\, a_{\gamma(\eta)\to e_1}(l) \rightsquigarrow S.$$

Thus, we obtain:

$$\forall (h,p) \in \mathcal{M}_\eta\,, \forall t \in G\,,\quad \mathrm{h}^{(e,m)}_\eta\left(R^{(\overline{S}_\uparrow)}_{\eta,t}\, h, p\right) = \mathrm{h}^{(e,m)}_\eta\left(R^{(\overline{S}_\downarrow)}_{\eta,t}\, h, p\right) = \mathrm{h}^{(e,m)}_\eta\big(h,p\big)\,,$$

so that:

$$\forall (h,p) \in \mathcal{M}_\eta\,,\quad \left\{\mathrm{P}^{(S,u)}_\eta,\, \mathrm{h}^{(e,m)}_\eta\right\}_{\mathcal{M}_\eta}(h,p) = 0\,,$$

and therefore $\left\{\mathrm{P}^{(S,u)},\, \mathrm{h}^{(e,m)}\right\} = 0$.

Edge crossing the surface. Let $e \in \mathcal{L}_{\text{edges}}$ such that $e = e_2 \circ e_1^{-1}$ with $e_1 \diamond_1 S$, $e_2 \diamond_2 S$ and $\{\diamond_1, \diamond_2\} = \{\,\uparrow, \downarrow\,\}$. Eq. (10.28.4) then ensures that $\eta = (\gamma, \lambda) \in \mathcal{L}_{\text{HF}}$, with $\gamma := \{e_1\,,\, e_2\}$ and $\lambda := \big[\,\{S\}\,\big]_\sim$, and we have:

$$\forall h \in \mathcal{C}_\eta\,,\ \forall t \in G,\ \left(R^{(\overline{S}_\uparrow)}_{\eta,t}\, h(e_2)\right) . \left(R^{(\overline{S}_\uparrow)}_{\eta,t}\, h(e_1)\right)^{-1} = h(e_2)\,.\,t^{\mathrm{sgn}(e,\,S)}\,.\,h(e_1)^{-1}\,,$$

as well as:

$$\forall h \in \mathcal{C}_\eta\,,\ \forall t \in G,\ \left(R^{(\overline{S}_\downarrow)}_{\eta,t}\, h(e_2)\right) . \left(R^{(\overline{S}_\downarrow)}_{\eta,t}\, h(e_1)\right)^{-1} = h(e_2)\,.\,t^{-\mathrm{sgn}(e,\,S)}\,.\,h(e_1)^{-1}\,.$$

Let $m \in C^\infty(G, \mathbb{R})$. We obtain:

$$\forall (h,p) \in \mathcal{M}_\eta\,,\ \ \left\{\mathrm{P}^{(S,u)}_\eta,\, \mathrm{h}^{(e,m)}_\eta\right\}_{\mathcal{M}_\eta}(h,p) = \mathrm{sgn}(e,\,S)\ \frac{d}{dr} m\left(h(e_2)\,.\,e^{r\,u}\,.\,h(e_1)^{-1}\right)\bigg|_{r=0}\,.$$

□

While we successfully reproduce the Poisson brackets from prop. 9.10, the projective state space set up in prop. 10.26 regrettably cannot be displayed as the rendering (def. 2.6) of a continuous classical theory of connections. If we were to define projection maps $\pi_{\Sigma\to\eta}$ from the infinite dimensional phase space $\mathcal{M}_\Sigma$ (def. 9.3) into the various small phase spaces $\mathcal{M}_\eta$, these maps would not fulfill def. 2.1:

- They would not be smooth, and in fact, they would not even be everywhere defined on $\mathcal{M}_\Sigma$. As discussed in subsection 9.2, the holonomy, resp. flux, variables are obtained by smearing the connection, resp. its conjugate electric field, along singular geometrical objects (respectively 1- and $(d-1)$-dimensional), while a smearing by a smooth function on the d-dimensional manifold Σ would be required to get well-defined observables on $\mathcal{M}_\Sigma$.
- They would not be surjective onto $\mathcal{M}_\eta$. The 'one-sided' flux variables $\mathrm{P}^{(\overline{S}_\downarrow\,,\,u)}$ and $\mathrm{P}^{(\overline{S}_\uparrow\,,\,u)}$ that constitute *independent* variables in $\mathcal{M}_\eta$ have no equivalent in terms of appropriate smearings of the electric field: only the combination $\mathrm{P}^{(S,u)}$ has. Moreover, the fluxes attached to submanifolds of dimension less than $d-2$ (arising at the intersection of surfaces) should vanish according to the continuous theory.
- They would not exactly transport the Poisson brackets (compatibility with the symplectic structure as expressed by eq. (2.1.1) does not directly make sense for non smooth maps, but it would be desirable to at least recover its transcription at the level of the observables, aka. prop. 2.2). We stressed when reviewing the regularization of the Poisson brackets (prop. 9.10) that it forces non-vanishing commutators between fluxes on intersecting surfaces: this anomaly in the holonomy-flux algebra is the price for rescuing its Jacobi identity. While the Poisson bracket of fluxes on *transversally* intersecting surfaces (ie. surfaces whose intersection is of dimension $d-2$ or less) is an observable that actually vanishes on $\mathcal{M}_\Sigma$ (as just underlined), in accordance with the fluxes being commuting in the continuum theory, the non-commutation of fluxes on *overlapping* surfaces (with an intersection of dimension $d-1$) genuinely conflicts with the symplectic structure on $\mathcal{M}_\Sigma$.

It is open to debate whether the last point, in particular, should be considered a bug (aka. disturbing quantum anomaly), a feature (aka. interesting quantum-geometrical effect) or simply an artifact (that could be eliminated by expressing the relation between the continuous and the discretized theories in terms of more appropriate variables, see eg. [89, section 6] or [23, appendix C]). Note that this issue is strictly speaking one of *regularization* rather than *quantization*: the concern is whether the two descriptions we have for the classical continuum theory (on an infinite-dimensional phase space vs. on a projective limit of finite-dimensional phase spaces) lead to the *same* physical predictions. A way to analyze this question could be to try and reproduce, on the projective side (along the lines of section 3), the dynamics of the original theory of connections (see sections 13 and 20 for ideas in this direction, as well as sections 15 and 16 for examples of how the dynamics of a classical field theory can be recovered in the projective setting).

Chapter 4 – Quantum Theory

11. Introduction

Quantizing the classical formalism set up in section 10 is a straightforward application of the prescriptions for quantization in the position representation that were detailed in props. 6.3 and 6.4. While the relation between this projective formalism and the continuum phase space description of subsection 9.1 has some unclear aspects (as just discussed at the end of section 10), its relation with the well-established Ashtekar-Lewandowski representation [6, 60] of the holonomy-flux algebra can be understood very precisely using the general results derived in subsection 5.2.

As already mentioned, the Ashtekar-Lewandowski Hilbert space $\mathcal{H}_{AL}$ is obtained as an inductive limit of building block Hilbert spaces labeled by *graphs*. Vectors in $\mathcal{H}_{AL}$ that happen to belong to the small Hilbert subspace $\mathcal{H}_\gamma$ labeled by a graph γ are said to be *supported* on γ: in such a state, any flux operator associated to a surface that does not intersect the graph is strictly zero, while any holonomy along an edge that does not belong to the graph is uniformly distributed (as can be checked from the definition of the observables on $\mathcal{H}_{AL}$, that we will recall in props. 12.8 and 12.9; see also [8]). This is the property to which we were referring, when we pointed out, in the main introduction, that states build on top of the Ashtekar-Lewandowski vacuum exhibit a discrete geometry along the spatial slice (remembering from section 8 that the spatial geometry is encoded in the electric field, which is itself measured by the flux observables), and it is closely related to the fact that the inductive limit structure of $\mathcal{H}_{AL}$ arises from a projective limit of partial *configuration* spaces on the classical side [60, 92]. As we will develop below, this inductive limit construction underlying $\mathcal{H}_{AL}$ is also the reason why it only exists if the gauge group is compact.

By contrast, for the construction we will be presenting in subsection 12.1, the gauge group does not need to be compact, nor do we have to impose any particular restriction beyond the assumption of a finite-dimensional Lie group (should it be of any use, even countable discrete groups fall into this category, as the 0-dimensional case). Of course, the just highlighted results concerning the relation with $\mathcal{H}_{AL}$ only apply when the latter can at all be defined, in which case they incidentally ensure that the projective quantum state space is not empty (because it contains all quantum states that can be defined on $\mathcal{H}_{AL}$, and even more, see prop. 12.13). The proof of non-emptiness in the non-compact case seems less obvious, and we will come back to this concern at the end of section 18.

The possibility of setting up quantum state spaces for theories of connections in the case of a non-compact gauge group, although not in the focus of our interest, might find applications in the treatment of the *complex* Ashtekar variables [3, 4], that require $G = \mathcal{SL}(2, \mathbb{C})$ (aka. the universal cover of the Lorentz group) as gauge group and whose geometric interpretation is more transparent (see eg. the discussion in [66] and references therein). There are hints coming from various directions [13, 95] that a quantum gravity theory based on these variables could have nicer physical properties than one using the *real* Ashtekar variables (with gauge group $G = \mathcal{SU}(2)$, as described in section 8).

Note that the close ties of the state space proposed here with the Ashtekar-Lewandowski one means that many technical tools developed for the latter can probably be imported in the former, in particular those that operate at the *building block level*. Some of these techniques, such as the use of so-called spin networks to write down gauge-invariant states [79] could be called upon to help the resolution of the constraints in the projective setting (recall from the main introduction and section 8 that the constraints are the means to realize gauge-invariance, general covariance and dynamics in a canonical formulation of general relativity). On the other hand, the very aspect that the projective formalism is meant to overcome, namely the unbalanced treatment of configuration and momentum variables, does play a crucial part in the resolution of the Gauss and diffeomorphism constraints over $\mathcal{H}_{\text{AL}}$, so that new ingredients will be needed, now that we are treating these variable on equal footing. In section 13, we will outline possible paths to deal with the specific problems that hinder an easy implementation of the constraints over the projective quantum state space.

12. Quantum state space

Following the procedure laid out in subsection 6.1, we can quantize each partial phase space $\mathcal{M}_\eta = T^*(\mathcal{C}_\eta)$ along its position polarization to get a partial Hilbert space $\mathcal{H}_\eta = L_2(\mathcal{C}_\eta\,,\, d\mu_\eta)$ and the consistency of these polarizations with the projection maps (that was shown in prop. 10.26) ensures that these Hilbert spaces $\mathcal{H}_\eta$ can eventually be arranged into a projective system of quantum state spaces. Equivalently, the tensor product factorizations needed for the quantum projective system can be read out directly from the classical factorizing system (prop. 6.3). Note that the resulting partial traces $\text{Tr}_{\eta'\to\eta}$ are reminiscent of the coarse graining maps considered in earlier works [56].

On the other hand, we suggested in subsection 2.3 that any factorizing system of configuration spaces yields, by forgetting about the more precise information it provides regarding the links between the partial theories, an associated projective system of configuration spaces. If a family of measures μ_η can be found on this projective system that are at the same time *normalizable* and *compatible with the projections* (and this will be the case if the group G happens to be compact), the small Hilbert spaces $\mathcal{H}_\eta$ can alternatively be arranged into an inductive limit: this is the situation we considered in prop. 6.5, where we proved that we are then provided with a natural injection, embedding the space of density matrices on the resulting inductive limit Hilbert space into the projective state space. In the perspective of this result, the compactness of the group G ensures that normalizable measures $\mu_{\eta'\to\eta}$ can be chosen on the complementary configuration spaces $\mathcal{C}_{\eta'\to\eta}$, which in turn allows to pick out a natural, constant, 'reference state' $\zeta_{\eta'\to\eta}$ in each complementary Hilbert space $\mathcal{H}_{\eta'\to\eta} = L_2(\mathcal{C}_{\eta'\to\eta}\,,\, d\mu_{\eta'\to\eta})$: these reference states assemble precisely into the Ashtekar-Lewandowski vacuum. Although probability measures could be constructed on the spaces $\mathcal{C}_{\eta' \to \eta}$ even in the case of a non-compact group, they would not fulfill the factorization condition from def. 6.1.3, required for the inductive limit to be well-defined (see the discussion around prop. 6.5, as well as complementary views on this problem in [65, 66]).

We will make use of this device to understand how the state space we are proposing here *extends* the Ashtekar-Lewandowski Hilbert space $\mathcal{H}_{\text{AL}}$ (theorem 12.11, props. 12.12 and 12.13). It may at first seem surprising that an inductive limit obtained this way can in our case be identified with $\mathcal{H}_{\text{AL}}$, since the former is made of building blocks labeled by edges and surfaces, while the latter use labels which are just graphs. The trick here is that the injections defining this inductive limit in fact do not depend on the disposition of the surfaces in the labels. This is in fact the very observation that was spelled out before def. 2.15: projections between configuration spaces (which are the ingredients of an inductive limit construction) are less specific than factorizations, and it was precisely in order to distinguish between different possible factorizations corresponding to the same projection map that we had to introduce surfaces in the labels. The injection that mounts the Hilbert space associated to a label η into the one associated to a finer label η' thus turns out to only depend on the underlying graphs $\gamma(\eta)$ and $\gamma(\eta')$, so that the inductive system labeled by elements of $\mathcal{L}_{\text{HF}}$ actually collapses into an inductive system simply labeled by graphs (via the same mechanism as the one discussed at the classical level in prop. 2.8).

12.1 Quantization in the position representation

The expectation that the projective approach could allow for a more balanced treatment of the configuration and momentum variables than the Ashtekar-Lewandowski representation is supported by the respective classical precursors of both formalisms (in accordance with the general picture exposed in chaps. 1 and 2): as underlined above, the inductive limit construction underlying $\mathcal{H}_{\text{AL}}$ can be seen as emerging from a projective limit of *configuration* spaces, while the projective quantum state space in prop. 12.1 below will be obtained as the quantization of the projective limit of *phase* spaces from prop. 10.26.

One might be worried that relying on the position representation to perform the quantization of the individual partial theories would reintroduce an unwanted singularization of the configuration variables. This is however not the case, because they only play a special role as far as the quantization of the *finite dimensional* small phase spaces is concerned: this is therefore comparable to the choice of a representation in quantum *mechanics* (in opposition to quantum field theory), which is known to be rather innocuous [98] (viz. the thoughts in section 7 to dispense altogether from a choice of polarization). Indeed, the small phase spaces $\mathcal{M}_\eta$ are cotangent bundles on Lie groups, whose position representation (namely, the L_2 space on the considered group) can be shown to be unitarily equivalent to a suitably defined momentum representation [32, 11].

Thus, while working in the position representation means that we are using the same building blocks as the ones used for the construction of $\mathcal{H}_{\text{AL}}$ (which is convenient to study the relation between the two approaches along the lines sketched above), the critical difference comes from the alternative way of gluing them together to compose the state space of the full theory: design choices in this regard are precisely the most likely to have irreversible consequences on the final quantum (field) theory.

Proposition 12.1 Let μ be a right-invariant Haar measure on G. For $\eta \in \mathcal{L}_{\text{HF}}$, we can, using the natural identification $\mathcal{C}_\eta \approx G^{\#\gamma(\eta)}$, define a measure $\mu_\eta \approx \mu^{\#\gamma(\eta)}$ on $\mathcal{C}_\eta$ (actually the identification $\mathcal{C}_\eta \approx G^{\#\gamma(\eta)}$ is not unique, since it depends on an ordering of the edges in $\gamma(\eta)$; yet the measure μ_η is independent of this choice).

Then, there exists a family of smooth measures $\mu_{\eta'\to\eta}$ on each $\mathcal{C}_{\eta'\to\eta}$ for $\eta \preccurlyeq \eta' \in \mathcal{L}_{\text{HF}}$, such that $(\mathcal{L}_{\text{HF}}, (\mathcal{C}, \mu), \varphi)^\times$ is a factorizing system of measured manifolds (def. 6.1). Defining:

1. $\forall \eta \in \mathcal{L}_{\text{HF}},\ \mathcal{H}_\eta := L_2(\mathcal{C}_\eta,\ d\mu_\eta)$;
2. $\forall \eta \preccurlyeq \eta' \in \mathcal{L}_{\text{HF}},\ \mathcal{H}_{\eta'\to\eta} := L_2(\mathcal{C}_{\eta'\to\eta},\ d\mu_{\eta'\to\eta})$ and:

$$\begin{array}{rccl} \Phi_{\eta'\to\eta} & : \mathcal{H}_{\eta'} & \to & \mathcal{H}_{\eta'\to\eta} \otimes \mathcal{H}_\eta \approx L_2(\mathcal{C}_{\eta'\to\eta} \times \mathcal{C}_\eta,\ d\mu_{\eta'\to\eta} \times d\mu_\eta) \\ & \psi & \mapsto & \psi \circ \varphi^{-1}_{\eta'\to\eta} \end{array};$$

there exists a family of Hilbert spaces isomorphisms $\left(\Phi_{\eta''\to\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'\preccurlyeq\eta''}$ such that $(\mathcal{L}_{\text{HF}},\ \mathcal{H},\ \Phi)^\otimes$ is a projective system of quantum state spaces (def. 5.1).

Proof The right-invariant Haar measure μ comes from a smooth, right-invariant volume form on G. Hence, for any η there exists a smooth, right-invariant volume form ω_η on $\mathcal{C}_\eta$ such that μ_η is the measure associated to ω_η (ω_η is not unique, as there is a freedom in the orientation of $\mathcal{C}_\eta$; however, it is sufficient here to just pick an ω_η for each $\mathcal{C}_\eta$). In particular, μ_η is therefore a smooth measure on $\mathcal{C}_\eta$. Defining:

$$\begin{array}{rccl} R_{\eta,h} & : \mathcal{C}_\eta & \to & \mathcal{C}_\eta \\ & j & \mapsto & (e \mapsto j(e)\,.\,h(e)) \end{array},$$

the right-invariance of ω_η can be written as:

$$\forall h \in \mathcal{C}_\eta,\ R^*_{\eta,h}\,\omega_\eta = \omega_\eta\,. \tag{12.1.1}$$

Let $\eta \preccurlyeq \eta' \in \mathcal{L}_{\text{HF}}$, $d_\eta := \dim \mathcal{C}_\eta$, $d_{\eta'} := \dim \mathcal{C}_{\eta'}$ and $d_{\eta'\to\eta} := d_{\eta'} - d_\eta$. We choose a basis $u_1, \ldots, u_{d_\eta}$ in $T_{\mathbf{1}}(\mathcal{C}_\eta)$ (where $\mathbf{1} \in \mathcal{C}_\eta$ is the map $e \mapsto \mathbf{1} \in G$) and we define a smooth volume form $\omega_{\eta'\to\eta}$ on $\mathcal{C}_{\eta'\to\eta}$ by:

$$\forall q \in \mathcal{C}_{\eta'\to\eta},\ \forall w_1, \ldots, w_{d_{\eta'\to\eta}} \in T_q(\mathcal{C}_{\eta'\to\eta}),$$

$$\omega_{\eta'\to\eta,q}(w_1, \ldots, w_{d_{\eta'\to\eta}}) := \frac{\left[\varphi^{-1,*}_{\eta'\to\eta}\,\omega_{\eta'}\right]_{(q,\mathbf{1})}\Big((w_1, 0), \ldots, (w_{d_{\eta'\to\eta}}, 0), (0, u_1), \ldots, (0, u_{d_\eta})\Big)}{\omega_{\eta,\mathbf{1}}(u_1, \ldots, u_{d_\eta})}\,.$$

Now, for any $h \in \mathcal{C}_\eta$, we define $\widetilde{h} \in \mathcal{C}_{\eta'}$ by:

$$\forall e \in \gamma(\eta),\ \forall e' \in H^{(1)}_{\eta'\to\eta,e} \cup H^{(3)}_{\eta'\to\eta,e},\ \widetilde{h}(e') = h(e)\,;$$

$$\text{and} \quad \forall e' \in H^{(0)}_{\eta'\to\eta} \cup H^{(2)}_{\eta'\to\eta},\ \widetilde{h}(e') = \mathbf{1}\,.$$

Using the explicit expression for $\varphi^{-1}_{\eta'\to\eta}$ from the proof of prop. 10.24, we can check that:

$$\forall h \in \mathcal{C}_\eta,\ R_{\eta',\tilde{h}} \circ \varphi^{-1}_{\eta'\to\eta} = \varphi^{-1}_{\eta'\to\eta} \circ \left(\mathrm{id}_{\mathcal{C}_{\eta'\to\eta}} \times R_{\eta,h}\right).$$

Applying eq. (12.1.1), we thus get, for any $h \in \mathcal{C}_\eta$:

$$\omega_{\eta,\mathbf{1}}(u_1, \ldots, u_{d_\eta}) = \omega_{\eta,h}(U_{1,h}, \ldots, U_{d_\eta,h})$$

$$\text{and } \forall q \in \mathcal{C}_{\eta'\to\eta},\ \forall w_1, \ldots, w_{d_{\eta'\to\eta}} \in T_q(\mathcal{C}_{\eta'\to\eta}),$$

$$\left[\varphi^{-1,*}_{\eta'\to\eta}\,\omega_{\eta'}\right]_{(q,\mathbf{1})} \Big((w_1, 0), \ldots, (w_{d_{\eta'\to\eta}}, 0), (0, u_1), \ldots, (0, u_{d_\eta})\Big)$$

$$= \left[\varphi^{-1,*}_{\eta'\to\eta}\,\omega_{\eta'}\right]_{(q,h)} \Big((w_1, 0), \ldots, (w_{d_{\eta'\to\eta}}, 0), (0, U_{1,h}), \ldots, (0, U_{d_\eta,h})\Big),$$

where $\forall i \in \{1, \ldots, d_\eta\}$, $U_{i,h} := [T_{\mathbf{1}}R_{\eta,h}](u_i) \in T_h(\mathcal{C}_\eta)$. And since $U_{1,h}, \ldots, U_{d_\eta,h}$ is a basis of $T_h(\mathcal{C}_\eta)$, we get:

$$\varphi^{-1,*}_{\eta'\to\eta}\,\omega_{\eta'} = \omega_{\eta'\to\eta} \times \omega_\eta\,.$$

Therefore, defining, for any $\eta \preccurlyeq \eta'$, $\mu_{\eta'\to\eta}$ to be the smooth measure on $\mathcal{C}_{\eta'\to\eta}$ associated to the volume form $\omega_{\eta'\to\eta}$, we have:

$$\forall \eta \preccurlyeq \eta' \in \mathcal{L}_{\text{HF}},\ \varphi_{\eta'\to\eta,*}\,\mu_{\eta'} = \mu_{\eta'\to\eta} \times \mu_\eta\,,$$

and, using the 3-spaces consistency condition eq. (2.11.1) in the factorizing system $(\mathcal{L}_{\text{HF}}, \mathcal{C}, \varphi)^\times$, this also implies:

$$\forall \eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}_{\text{HF}},\ \varphi_{\eta''\to\eta'\to\eta,*}\,\mu_{\eta''\to\eta} = \mu_{\eta''\to\eta'} \times \mu_{\eta'\to\eta}\,.$$

Thus, $(\mathcal{L}_{\text{HF}}, (\mathcal{C}, \mu), \varphi)^\times$ is a factorizing system of measured manifolds, from which a projective system of quantum state spaces $(\mathcal{L}_{\text{HF}}, \mathcal{H}, \Phi)^\otimes$ can be constructed as described in prop. 6.3. □

Proposition 12.2 Let $e \in \mathcal{L}_{\text{edges}}$ and let $\eta \in \mathcal{L}_{\text{HF}}$ such that $\gamma(\eta) \in \mathcal{L}_{\text{graphs}/e}$. For any $m \in C^\infty(G, \mathbb{R})$, $\mathrm{h}^{(e,m)}_\eta$ fulfills the quantization condition eq. (6.4.1) and can be quantized as a densely defined, essentially self-adjoint operator $\widehat{\mathrm{h}^{(e,m)}_\eta}$ on $\mathcal{H}_\eta$ (with dense domain $\mathcal{D}_\eta$), given by:

$$\forall \psi \in \mathcal{D}_\eta,\ \forall h \in \mathcal{C}_\eta,\ \left(\widehat{\mathrm{h}^{(e,m)}_\eta}\psi\right)(h) := m\left(\prod_{k=1}^{n_{\gamma(\eta)\to e}} \left[h \circ a_{\gamma(\eta)\to e}(k)\right]^{\epsilon_{\gamma(\eta)\to e}(k)}\right) \psi(h)\,. \tag{12.2.1}$$

Moreover, we have for any $\eta, \eta' \in \mathcal{L}_{\text{HF}}$ such that $\gamma(\eta), \gamma(\eta') \in \mathcal{L}_{\text{graphs}/e}$:

$$\forall m \in C^\infty(G, \mathbb{R}),\ \widehat{\mathrm{h}^{(e,m)}_\eta} \sim \widehat{\mathrm{h}^{(e,m)}_{\eta'}}\,,$$

with the equivalence relation of eq. (5.3.2), thus we can associate to any $e \in \mathcal{L}_{\text{edges}}$ and any $m \in C^\infty(G, \mathbb{R})$ a well-defined observable $\widehat{\mathrm{h}^{(e,m)}} \in \mathcal{O}^\otimes_{(\mathcal{L}_{\text{HF}}, \mathcal{H}, \Phi)}$ (prop. 5.5).

Let $\overline{F} \in \mathcal{L}_{\text{faces}}$ and let $\eta \in \mathcal{L}_{\text{HF}}$ such that $\lambda(\eta) \in \mathcal{L}_{\text{profls}/\overline{F}}$. For any $u \in \mathfrak{g}$, $\mathrm{P}^{(\overline{F},u)}_\eta$ fulfills the quantization condition and can be quantized as a densely defined, essentially self-adjoint operator $\widehat{\mathrm{P}^{(\overline{F},u)}_\eta}$ on $\mathcal{H}_\eta$, given by:

$$\forall \psi \in \mathcal{D}_\eta\,,\ \forall h \in \mathcal{C}_\eta\,,\ \big(\widehat{\mathrm{P}_\eta^{(\overline{F},u)}}\psi\big)(h) := i\,\left[T_1\, t \mapsto \psi\left(R^{(\overline{F})}_{\eta,t} h\right)\right](u)\,, \tag{12.2.2}$$

the map $R^{(\overline{F})}_{\eta,t} h \in \mathcal{C}_\eta$ being defined for $t \in G$ and $h \in \mathcal{C}_\eta$ by:

$$\forall e \in \gamma(\eta),\ R^{(\overline{F})}_{\eta,t} h(e) := \begin{cases} h(e)\,.\,t & \text{if } e \in \overline{F} \\ h(e) & \text{else} \end{cases}\,.$$

Moreover, we have for any $\eta, \eta' \in \mathcal{L}_{\text{HF}}$ such that $\lambda(\eta), \lambda(\eta') \in \mathcal{L}_{\text{profls}/\overline{F}}$:

$$\forall u \in \mathfrak{g},\ \widehat{\mathrm{P}_\eta^{(\overline{F},u)}} \sim \widehat{\mathrm{P}_{\eta'}^{(\overline{F},u)}}\,,$$

thus we can associate to any $\overline{F} \in \mathcal{L}_{\text{faces}}$ and any $u \in \mathfrak{g}$ a well-defined observable $\widehat{\mathrm{P}^{(\overline{F},u)}} \in \mathcal{O}^{\otimes}_{(\mathcal{L}_{\text{HF}},\mathcal{H},\Phi)}$.

Proof *Hamiltonian vector field projected on* $\mathcal{C}_\eta$. Let $\eta \in \mathcal{L}_{\text{HF}}$. Let $\mathrm{r} \in C^\infty(\mathcal{M}_\eta\,, \mathbb{R})$. For any $(h, P) \in \mathcal{C}_\eta \times \mathcal{P}_\eta$ and any $w \in \mathcal{P}_\eta$, eq. (10.28.1) yields:

$$\begin{aligned}
&\left[d\mathrm{r} \circ L_\eta^{-1}\right]_{(h,P)}(0, w) = \\
&\qquad = \Omega_{\eta, L_\eta^{-1}(h,P)}\left(X_{\mathrm{r}, L_\eta^{-1}(h,P)}\,,\ \left[T_{(h,P)} L_\eta^{-1}\right](0, w)\right) \\
&\qquad = \left(L_\eta^{-1,*}\Omega_\eta\right)_{h,P}\left(\left[T_{L_\eta^{-1}(h,P)} L_\eta\right]\left(X_{\mathrm{r}, L_\eta^{-1}(h,P)}\right)\,,\ (0, w)\right) \\
&\qquad = \sum_{e \in \gamma(\eta)} w\left(\chi_\eta(e)\right) \circ \ell^{(e)}_{\eta, h^{-1}}\left(\left[T_{L_\eta^{-1}(h,P)}\gamma_\eta\right]\left(X_{\mathrm{r}, L_\eta^{-1}(h,P)}\right)\right),
\end{aligned} \tag{12.2.3}$$

where X_r is the Hamiltonian vector field of r and $\gamma_\eta : \mathcal{M}_\eta \to \mathcal{C}_\eta$ is the bundle projection in $\mathcal{M}_\eta = T^*(\mathcal{C}_\eta)$.

Now, for any $h, j \in \mathcal{C}_\eta$ and any $e \in \gamma(\eta)$, we have:

$$\left(t \mapsto R^{(\chi_\eta(e)})_{\eta,t} h\right) \circ \left(j' \mapsto h(e)^{-1}\,.\,j'(e)\right)(j) = R^{(\chi_\eta(e)})_{\eta, h(e)^{-1}\,.\,j(e)} h : e' \mapsto \begin{cases} j(e') & \text{if } e' = e \\ h(e') & \text{else} \end{cases}\,.$$

Therefore, we get for any $h \in \mathcal{C}_\eta$:

$$\sum_{e \in \gamma(\eta)} \left[T_1\ \ t \mapsto R^{(\chi_\eta(e)})_{\eta,t} h\right] \circ \ell^{(e)}_{\eta,h^{-1}} = \mathrm{id}_{T_h(\mathcal{C}_\eta)}\,.$$

For any $h \in \mathcal{C}_\eta$ and any $\upsilon \in T_h^*(\mathcal{C}_\eta)$, we define $w_\upsilon^{(h)} \in \mathcal{P}_\eta$ by:

$$\forall F \in \mathcal{F}(\eta),\ w_\upsilon^{(h)}(F) := \upsilon \circ \left[T_1\ \ t \mapsto R^{(F)}_{\eta,t} h\right],$$

so that eq. (12.2.3) becomes:

$$\forall (h, P) \in \mathcal{C}_\eta \times \mathcal{P}_\eta\,,\ \upsilon \circ \left[T_{L_\eta^{-1}(h,P)}\gamma_\eta\right]\left(X_{\mathrm{r}, L_\eta^{-1}(h,P)}\right) = \left[d\mathrm{r} \circ L_\eta^{-1}\right]_{(h,P)}(0, w_\upsilon^{(h)})\,.$$

Thus, r fulfills the quantization condition eq. (6.4.1) if and only if there exists a smooth, complete vector field $\overline{X}_\mathrm{r}$ on $\mathcal{C}_\eta$ such that:

$$\forall (h,P) \in \mathcal{C}_\eta \times \mathcal{P}_\eta\,,\ \forall \upsilon \in T^*_h(\mathcal{C}_\eta),\ \left[d\mathrm{r} \circ L_\eta^{-1}\right]_{(h,P)}(0, w^{(h)}_\upsilon) = \upsilon\left(\overline{X}_{\mathrm{r},h}\right). \tag{12.2.4}$$

If this is the case, we can then define $\widehat{\mathrm{r}}$ as a densely defined, essentially self-adjoint operator on $\mathcal{H}_\eta$ by (prop. B.14):

$$\forall \psi \in \mathcal{D}_\eta\,,\ \forall h \in \mathcal{C}_\eta\,,\ \left(\widehat{\mathrm{r}}\,\psi\right)(h) := \left(\mathrm{r}(h,0) + \frac{i}{2}\,\mathrm{div}_{\mu_\eta}\overline{X}_\mathrm{r}(h)\right)\,\psi(h) + i\,\left[d\mathrm{r} \circ L_\eta^{-1}\right]_{(h,P)}(0, w^{(h)}_{d\psi_h})\,,$$

where $\mathrm{div}_{\mu_\eta}\overline{X}_\mathrm{r} \in C^\infty(\mathcal{C}_\eta)$ is defined by $\mathfrak{L}_{\overline{X}_\mathrm{r}}\mu_\eta = \left(\mathrm{div}_{\mu_\eta}\overline{X}_\mathrm{r}\right)\,\mu_\eta$ (def. B.12).

Holonomies. Let $e \in \mathcal{L}_{\mathrm{edges}}$ and let $\eta \in \mathcal{L}_{\mathrm{HF}}$ such that $\gamma(\eta) \in \mathcal{L}_{\mathrm{graphs}/e}$. For any $m \in C^\infty(G,\mathbb{R})$, we have:

$$\forall (h,P) \in \mathcal{C}_\eta \times \mathrm{P}_\eta\,,\ \mathrm{h}^{(e,m)}_\eta \circ L_\eta^{-1}(h,P) = m\left(\prod_{k=1}^{n_{\gamma(\eta)\to e}} \left[h \circ a_{\gamma(\eta)\to e}(k)\right]^{\epsilon_{\gamma(\eta)\to e}(k)}\right),$$

hence $\mathrm{h}^{(e,m)}_\eta$ fulfills the quantization condition with $\overline{X}_{\mathrm{h}^{(e,m)}_\eta} := 0$. This yields the expression in eq. (12.2.1) for $\widehat{\mathrm{h}^{(e,m)}_\eta}$. Finally, for any $\eta,\eta' \in \mathcal{L}_{\mathrm{HF}}$ such that $\gamma(\eta),\gamma(\eta') \in \mathcal{L}_{\mathrm{graphs}/e}$, we have $\mathrm{h}^{(e,m)}_\eta \sim \mathrm{h}^{(e,m)}_{\eta'}$ (prop. 10.27), so prop. 6.4 ensures that $\widehat{\mathrm{h}^{(e,m)}_\eta} \sim \widehat{\mathrm{h}^{(e,m)}_{\eta'}}$, and therefore that an observable $\widehat{\mathrm{h}^{(e,m)}}$ can be consistently defined on $\mathcal{S}^\otimes_{(\mathcal{H},\mathcal{L}_{\mathrm{HF}},\Phi)}$.

Fluxes. Let $\overline{F} \in \mathcal{L}_{\mathrm{faces}}$ and let $\eta \in \mathcal{L}_{\mathrm{HF}}$ such that $\lambda(\eta) \in \mathcal{L}_{\mathrm{profls}/\overline{F}}$. For any $u \in \mathfrak{g}$, we have:

$$\forall (h,P) \in \mathcal{C}_\eta \times \mathrm{P}_\eta\,,\ \mathrm{P}^{(\overline{F},u)}_\eta \circ L_\eta^{-1}(h,P) = \sum_{F' \in H_{\lambda(\eta)\to\overline{F}}} P(F')(u)\,,$$

hence eq. (12.2.4) yields:

$$\forall h \in \mathcal{C}_\eta\,,\ \overline{X}_{\mathrm{P}^{(\overline{F},u)}_\eta,h} := \sum_{F' \in H_{\lambda(\eta)\to\overline{F}}} \left[T_\mathbf{1}\ \ t \mapsto R^{(F')}_{\eta,t} h\right](u) = \sum_{e \in \overline{F}} \left[T_\mathbf{1}\ \ t \mapsto R^{(\chi_\eta(e))}_{\eta,t} h\right](u)\,,$$

where the second equality comes from eq. (10.28.2). Therefore:

$$\forall h \in \mathcal{C}_\eta\,,\ \overline{X}_{\mathrm{P}^{(\overline{F},u)}_\eta,h} = \left[T_\mathbf{1}\ \ t \mapsto R^{(\overline{F})}_{\eta,t} h\right](u)\,,$$

and in particular $\overline{X}_{\mathrm{P}^{(\overline{F},u)}_\eta}$ is a complete vector field on $\mathcal{C}_\eta$. Thus, $\mathrm{P}^{(\overline{F},u)}_\eta$ fulfills the quantization condition, with the expression in eq. (12.2.2) for $\widehat{\mathrm{P}^{(\overline{F},u)}_\eta}$, since the flow $\tau \mapsto R^{(\overline{F})}_{\eta,e^{\tau u}}$ of $\overline{X}_{\mathrm{P}^{(\overline{F},u)}_\eta}$ preserves the right-invariant measure μ_η on $\mathcal{C}_\eta$. Finally, like above, prop. 10.27 together with prop. 6.4 ensures that the $\widehat{\mathrm{P}^{(\overline{F},u)}_\eta}$, defined for each $\eta \in \mathcal{L}_{\mathrm{HF}}$ such that $\lambda(\eta) \in \mathcal{L}_{\mathrm{profls}/\overline{F}}$, can be consistently assembled into an observable $\widehat{\mathrm{P}^{(\overline{F},u)}}$ on $\mathcal{S}^\otimes_{(\mathcal{H},\mathcal{L}_{\mathrm{HF}},\Phi)}$. □

12.2 Relation to the Ashtekar-Lewandowski Hilbert space

We now want to investigate in which sense the quantum projective state space from prop. 12.1 can be understood as a reasonable extension of the space of states defined over the Ashtekar-Lewandowski Hilbert space $\mathcal{H}_{\text{AL}}$. We start by recalling the construction of $\mathcal{H}_{\text{AL}}$, in a form suitable to make contact with section 10 and subsection 12.1, using the tools developed in subsection 5.2. As stressed in section 11, this analysis has to to be carried out in the case of a compact group G, since this is a prerequisite for $\mathcal{H}_{\text{AL}}$ to exist.

Results similar to prop. 2.5 (for classical projective systems) or prop. 5.6 (for quantum ones) can be formulated for inductive systems of Hilbert spaces, and, in particular, the limit of an inductive system is not affected if one restricts its label set to some cofinal part. This is the reason why we have so far only considered graphs with fully analytic edges (viz. the discussion before def. 9.6). Still, the use of graphs with semi-analytic edges [92, sections II.6 and IV.20] is favored in LQG, for, although they yields the same Hilbert space, they present it in a form more convenient for writing the action of semi-analytic diffeomorphisms (which, unlike fully analytic ones, can be local; see the comments in this respect at the end of subsection 9.2 and at the beginning of subsection 19.2). Therefore, we briefly sketch below how to switch back to the semi-analytic class.

In this subsection the gauge group G is assumed to be *compact*, and the measure μ (introduced in prop. 12.1) is taken to be the normalized Haar measure on G.

Definition 12.3 Let $k \in \{1, 2, \ldots, \infty\}$. We define the set $\mathcal{L}^{(k)}_{\text{edges}}$ of (k)-edges like in def. 9.6, by requiring encharted (k)-edges to be C^k-diffeomorphisms instead of analytic ones. In analogy to props. 10.1 and 10.2, we define the range $r(e)$, the beginning and ending points $b(e)$ and $f(e)$, (k)-subedges $e_{[p,p']}$ (for $p \neq p' \in r(e)$), the reversed (k)-edge e^{-1}, and the order $<_{(e)}$ on the range of a (k)-edge e, as well as the composition of (k)-composable (k)-edges. These have the same properties as in the analytic case, since the proofs of props. 10.1 and 10.2 did not made use of the analyticity.

An analytic encharted edge is also an encharted (k)-edge, and two analytic encharted edge are equivalent in $\breve{\mathcal{L}}_{\text{edges}}$ iff they are equivalent in $\breve{\mathcal{L}}^{(k)}_{\text{edges}}$. Thus, we have a natural injection of $\mathcal{L}_{\text{edges}}$ into $\mathcal{L}^{(k)}_{\text{edges}}$. In the following, we will always identify $\mathcal{L}_{\text{edges}}$ with the image of this injection and write $\mathcal{L}_{\text{edges}} \subset \mathcal{L}^{(k)}_{\text{edges}}$.

Proposition 12.4 We define AL-graphs as finite sets of (1)-edges $\gamma \subset \mathcal{L}^{(1)}_{\text{edges}}$ such that:

1. $\forall e \in \gamma,\ \exists\, e_1, \ldots, e_n \in \mathcal{L}_{\text{edges}} \subset \mathcal{L}^{(1)}_{\text{edges}}$, (1)-composable $/\, e = e_n \circ \ldots \circ e_1$;
2. $\forall e \neq e' \in \gamma,\ r(e) \cap r(e') \subset \{b(e), f(e)\} \cap \{b(e'), f(e')\}$.

We denote the set of AL-graphs by $\mathcal{L}_{\text{AL}}$ and we equip it with the preorder $\preccurlyeq$ defined as in def. 10.3.

Then, $\mathcal{L}_{\text{graphs}}, \preccurlyeq$ is a cofinal subset of $\mathcal{L}_{\text{AL}}, \preccurlyeq$, so in particular $\mathcal{L}_{\text{AL}}, \preccurlyeq$ is a directed preordered set.

Proof By construction $\mathcal{L}_{\text{graphs}} \subset \mathcal{L}_{\text{AL}}$, and the order $\preccurlyeq$ between two elements of $\mathcal{L}_{\text{graphs}}$ coincides with their order as elements of $\mathcal{L}_{\text{AL}}$ (indeed, if an analytic edge is the composition of (1)-composable analytic edges, then, by definition, these edges are composable in $\mathcal{L}_{\text{edges}}$).

Next, let $\gamma \in \mathcal{L}_{\text{AL}}$ and for any $e \in \gamma$, choose $\gamma_e = \{e_1, \ldots, e_n\} \subset \mathcal{L}_{\text{edges}}$ such that $e = e_n \circ \ldots \circ e_1$. We have:

$$\forall e \in \gamma,\ \forall \widetilde{e} \neq \widetilde{e}' \in \gamma_e,\ r(\widetilde{e}) \cap r(\widetilde{e}') \subset \{b(\widetilde{e}), f(\widetilde{e})\} \cap \{b(\widetilde{e}'), f(\widetilde{e}')\},$$

$$\begin{aligned}
\text{and} \quad & \forall e \neq e' \in \gamma,\ \forall \widetilde{e} \in \gamma_e,\ \forall \widetilde{e}' \in \gamma_{e'}, \\
& r(\widetilde{e}) \cap r(\widetilde{e}') \subset r(\widetilde{e}) \cap r(\widetilde{e}') \cap r(e) \cap r(e') \\
& \qquad \subset r(\widetilde{e}) \cap \{b(e), f(e)\} \cap r(\widetilde{e}') \cap \{b(e'), f(e')\} \\
& \qquad \subset \{b(\widetilde{e}), f(\widetilde{e})\} \cap \{b(\widetilde{e}'), f(\widetilde{e}')\},
\end{aligned}$$

by definition of the composition of edges. Therefore $\gamma' := \bigcup_{e \in \gamma} \gamma_e \in \mathcal{L}_{\text{edges}}$, and we have $\gamma \preccurlyeq \gamma'$. □

We now rewrite the classical construction underlying the Ashtekar-Lewandowski Hilbert space, namely the design of a projective limit of configuration spaces, using a presentation parallel to that of subsection 10.3. This allows us to ascertain that the projection maps involved here match exactly the ones induced by the projections between phase spaces considered in prop. 10.26 (modulo the straightforward extension to semi-analytic graphs). As stressed at the beginning of the present section, since we are not setting up an actual projective limit of symplectic manifolds, the momentum variables do not come into play in this context, so we can directly use graphs as labels, without having to decorate them with faces.

Proposition 12.5 Let $\gamma \preccurlyeq \gamma' \in \mathcal{L}_{\text{AL}}$. Then, for any $e \in \gamma$, there exists a *unique* map $a_{\gamma' \to \gamma, e} : \{1, \ldots, n_{\gamma' \to \gamma, e}\} \to \gamma'$ (with $n_{\gamma' \to \gamma, e} \geqslant 1$) such that:

$$e = a_{\gamma' \to \gamma, e}(n_{\gamma' \to \gamma, e})^{\epsilon_{\gamma' \to \gamma, e}(n_{\gamma' \to \gamma, e})} \circ \ldots \circ a_{\gamma' \to \gamma, e}(1)^{\epsilon_{\gamma' \to \gamma, e}(1)},$$

where, for any $k \in \{1, \ldots, n_{\gamma' \to \gamma, e}\}$, $\epsilon_{\gamma' \to \gamma, e}(k)$ is defined from $a_{\gamma' \to \gamma, e}$ as in eq. (10.20.2).

Proof The proof works exactly as in the analytic case (prop. 10.20). □

Proposition 12.6 For any $\gamma \in \mathcal{L}_{\text{AL}}$ we define the finite-dimensional smooth configuration space $\mathcal{C}_\gamma := \{h : \gamma \to G\}$ (like in def. 10.19). And for any $\gamma \preccurlyeq \gamma' \in \mathcal{L}_{\text{AL}}$, we define the map $\pi_{\gamma' \to \gamma} : \mathcal{C}_{\gamma'} \to \mathcal{C}_\gamma$ by:

$$\forall h' \in \mathcal{C}_{\gamma'},\ \forall e \in \gamma,\ \pi_{\gamma' \to \gamma}(h')(e) := \left(\prod_{k=1}^{n_{\gamma' \to \gamma, e}} [h' \circ a_{\gamma' \to \gamma, e}(k)]^{\epsilon_{\gamma' \to \gamma, e}(k)} \right).$$

$\pi_{\gamma' \to \gamma}$ is smooth, for G is a Lie group, and it is moreover surjective.

In addition, for any $\gamma \preccurlyeq \gamma' \preccurlyeq \gamma'' \in \mathcal{L}_{\text{AL}}$, we have:

$$\pi_{\gamma' \to \gamma} \circ \pi_{\gamma'' \to \gamma'} = \pi_{\gamma'' \to \gamma}. \tag{12.6.1}$$

Proof Let $\gamma \preccurlyeq \gamma'$ and let $h \in \mathcal{C}_\gamma$. For any $e \neq e' \in \gamma$, we have:

$$\begin{aligned}
& \forall \widetilde{e} \in a_{\gamma' \to \gamma, e} \langle \{1, \ldots, n_{\gamma' \to \gamma, e}\} \rangle,\ \forall \widetilde{e}' \in a_{\gamma' \to \gamma, e'} \langle \{1, \ldots, n_{\gamma' \to \gamma, e'}\} \rangle, \\
& \qquad r(\widetilde{e}) \cap r(\widetilde{e}') \subset \{b(\widetilde{e}), f(\widetilde{e})\} \cap \{b(\widetilde{e}'), f(\widetilde{e}')\},
\end{aligned}$$

as in the proof of prop. 12.4, hence $a_{\gamma' \to \gamma, e} \left\langle \{1, \ldots, n_{\gamma' \to \gamma, e}\} \right\rangle \cap a_{\gamma' \to \gamma, e'} \left\langle \{1, \ldots, n_{\gamma' \to \gamma, e'}\} \right\rangle = \varnothing$. Moreover, $a_{\gamma' \to \gamma, e}$ is injective (by definition of the composition of edges), hence the map $h' \in \mathcal{C}_{\gamma'}$ given by:

$$\forall e \in \gamma,\ h' \circ a_{\gamma' \to \gamma, e}(1) = h(e)^{\epsilon_{\gamma' \to \gamma, e}(1)}\,,$$

$$\forall e' \in \gamma' / e' \notin \{a_{\gamma' \to \gamma, e}(1) | e \in \gamma\}\,,\ h'(e') = \mathbf{1}\,,$$

is well-defined and is such that $\pi_{\gamma' \to \gamma}(h') = h$. Thus, $\pi_{\gamma' \to \gamma}$ is surjective.
Finally, eq. (12.6.1) follows from the uniqueness of $a_{\gamma'' \to \gamma, e}$ as in the proof of theorem 10.25. □

As mentioned earlier, there are many projections between the phase spaces $T^*(\mathcal{C}_{\gamma'})$ and $T^*(\mathcal{C}_{\gamma})$ (with $\gamma \preccurlyeq \gamma'$) that can reproduce a given projection between the configuration spaces $\mathcal{C}_{\gamma'}$ and $\mathcal{C}_{\gamma}$, or, equivalently, many ways of choosing in $\mathcal{C}_{\gamma'}$ a set of variables complementary to the ones coming from $\mathcal{C}_{\gamma}$. In the proof of prop. 12.7, we choose, for each pair of graphs $\gamma \preccurlyeq \gamma'$, a factorization of $\mathcal{C}_{\gamma'}$ as $\mathcal{C}_{\gamma' - \gamma} \times \mathcal{C}_{\gamma}$ consistent with the projection $\pi_{\gamma' \to \gamma}$ defined in prop. 12.6. The injective maps $\tau_{\gamma' \leftarrow \gamma}$ between the Hilbert spaces $\mathcal{H}_{\gamma}$ that serve as building blocks for $\mathcal{H}_{\text{AL}}$ can then be understood as arising from the corresponding factorization $\mathcal{H}_{\gamma'} \approx \mathcal{H}_{\gamma' - \gamma} \otimes \mathcal{H}_{\gamma}$, via the selection of a 'reference state' $\zeta_{\gamma' - \gamma}$ in $\mathcal{H}_{\gamma' - \gamma}$. Because this reference state is taken as the constant function $\zeta_{\gamma' - \gamma} \equiv 1$ on $\mathcal{C}_{\gamma' - \gamma}$, the thus obtained identification of $\mathcal{H}_{\gamma}$ with a vector subspace of $\mathcal{H}_{\gamma'}$ in the end does not depend on which particular factorization of $\mathcal{C}_{\gamma'}$ has been used, but solely on the projection $\pi_{\gamma' \to \gamma}$ from $\mathcal{C}_{\gamma'}$ into $\mathcal{C}_{\gamma}$. Also, as announced earlier, the need for a *compact* gauge group G manifests itself in this approach as a condition for $\zeta_{\gamma' - \gamma}$ to be a normalizable element of $\mathcal{H}_{\gamma' - \gamma}$ (otherwise the map $\tau_{\gamma' \leftarrow \gamma}$ would not be well-defined).

Of course, it is not really necessary for assembling the inductive limit $\mathcal{H}_{\text{AL}}$ to ever introduce such factorization maps (a more standard path being to directly check that $\pi_{\gamma' \to \gamma, *}\, \mu_{\gamma'} = \mu_{\gamma}$). Still, it is worth looking closer at the particular family of factorizations elected below. The projections between phase spaces to which they give rise turn out to be precisely the ones that were considered in [89, def. 3.8] and although they do not fulfill the three-spaces consistency needed for a projective system (expressed in def. 2.3, or, at the level of the factorization maps, in eq. (2.11.1)), this can be quickly fixed: by somewhat tightening the ordering among graphs (requiring, in addition to eq. (10.3.1), that the first part of an edge $e \in \gamma$, in its decomposition into edges of γ', should be oriented like e itself, ie. that $\epsilon_1 = +1$), we can, without voiding the directedness of $\mathcal{L}_{\text{AL}}$, rescue this pivotal consistency condition [85] .

So why did we go through the intricacy of carefully setting up a label set with edges and surfaces if an apparently valid projective system could readily have been built over $\mathcal{L}_{\text{AL}}$? If we examine carefully the structure of observables that would arise from such a projective structure (and in particular, if we compare it to the one obtained in the previous subsection), we realize that its momentum variables are fluxes carried by single edge germs (defining the germ of an edge e as the equivalence class consisting of all edges sharing an initial subedge with e).

This sheds light on how a projective limit of *phase* spaces can at all be constructed using labels that seems to only know about *configuration* variables. What makes it possible, is the availability of a preferred pairing of conjugate variables, binding each configuration variable to its own particular momentum variable (eg. the holonomy along an edge with the flux carried by its germ). This pairing is such that whenever an edge e belongs to some graph γ, its companion flux does not act on any other edge of γ (indeed, edges in a graph cannot share a common subedge). In this way, any graph, as it selects specific holonomy variables, selects at the same time their conjugate flux variables (and the slight sharpening of the ordering described above ensures that if $\gamma' \succcurlyeq \gamma$ the fluxes thus attached to γ are also in γ').

A similar mechanism underlies the projective quantum state space built in [66]. In this work, holonomies along analytic loops were used as a complete set of *independent* configuration variables, which again provides an a priori pairing of conjugate variables, since the selected configuration variables can be thought as coordinates and mapped to their dual differential operator.

In both cases, the resulting factorizing systems leads to a theory whose basic momentum variables have no equivalent in the continuum classical theory. The trouble is then that fluxes associated to non-degenerate surfaces (aka. $(d-1)$-dimensional ones) cannot be represented on the corresponding quantum projective state space. If we however insist on using regular holonomies and fluxes as our primary variables (so as to implement an algebra of observables that separates the points in the continuum classical theory), then there is no way of choosing beforehand a canonical pairing that would, as described above, automatically provides a suitable set of canonically conjugate variables for any arbitrary graph γ.

Proposition 12.7 For any $\gamma \in \mathcal{L}_{\text{AL}}$, we define the measure $\mu_\gamma \approx \mu^{\#\gamma}$ on $\mathcal{C}_\gamma \approx G^{\#\gamma}$ (as in prop. 12.1) and the Hilbert space $\mathcal{H}_\gamma := L_2(\mathcal{C}_\gamma\,,\, d\mu_\gamma)$. Next, for any $\gamma \preccurlyeq \gamma' \in \mathcal{L}_{\text{AL}}$, we define the map $\tau_{\gamma' \leftarrow \gamma}$ by:

$$\begin{array}{rccc} \tau_{\gamma'\leftarrow\gamma} : & \mathcal{H}_\gamma & \to & \mathcal{H}_{\gamma'} \\ & \psi & \mapsto & \psi \circ \pi_{\gamma'\to\gamma} \end{array} .$$

$\tau_{\gamma'\leftarrow\gamma}$ is an isometry and, for any $\gamma \preccurlyeq \gamma' \preccurlyeq \gamma'' \in \mathcal{L}_{\text{AL}}$:

$$\tau_{\gamma''\leftarrow\gamma'} \circ \tau_{\gamma'\leftarrow\gamma} = \tau_{\gamma''\leftarrow\gamma} . \tag{12.7.1}$$

We define the Ashtekar-Lewandowski Hilbert space $\mathcal{H}_{\text{AL}}$ as the (norm completion of) the inductive limit of the system $\left(\left(\mathcal{H}_\gamma\right)_{\gamma\in\mathcal{L}_{\text{AL}}}\,,\,\left(\tau_{\gamma'\leftarrow\gamma}\right)_{\gamma\preccurlyeq\gamma'}\right)$. For $\gamma \in \mathcal{L}_{\text{AL}}$, we will denote by $\tau_{\text{AL}\leftarrow\gamma}$ the natural isometric injection of $\mathcal{H}_\gamma$ in $\mathcal{H}_{\text{AL}}$.

Proof Let $\gamma \preccurlyeq \gamma' \in \mathcal{L}_{\text{AL}}$. We define:

$$\gamma' - \gamma := \gamma' \setminus \{a_{\gamma'\to\gamma,e}(1) \mid e \in \gamma\} ,$$

as well as the finite dimensional smooth manifold $\mathcal{C}_{\gamma'-\gamma} := \left\{\widetilde{h} : (\gamma'-\gamma) \to G\right\}$. The smooth map $\varphi_{\gamma'-\gamma}$ given by:

$$\begin{array}{rccc} \varphi_{\gamma'-\gamma} : & \mathcal{C}_{\gamma'} & \to & \mathcal{C}_{\gamma'-\gamma} \times \mathcal{C}_\gamma \\ & h' & \mapsto & h'\big|_{\gamma'-\gamma}\,,\, \pi_{\gamma'\to\gamma}(h') \end{array} ,$$

is a diffeomorphism (as can be checked by expressing its inverse like in the proof of prop. 10.24). Moreover, defining for any $h \in \mathcal{C}_\gamma$ the maps $T^{(\gamma)}_{\gamma',h} : \mathcal{C}_{\gamma'} \to \mathcal{C}_{\gamma'}$ and $R_{\gamma,h} : \mathcal{C}_\gamma \to \mathcal{C}_\gamma$ by:

$\forall j' \in \mathcal{C}_{\gamma'}\,,\, \forall e' \in \gamma'$,

$$\left[T^{(\gamma)}_{\gamma',h}(j')\right](e') = \begin{cases} j'(e')\,.\,h(e) & \text{if } \exists e \in \gamma \,/\, e' = a_{\gamma'\to\gamma,e}(1) \ \ \& \ \ \epsilon_{\gamma'\to\gamma,e} = +1 \\ h(e)^{-1}\,.\,j'(e') & \text{if } \exists e \in \gamma \,/\, e' = a_{\gamma'\to\gamma,e}(1) \ \ \& \ \ \epsilon_{\gamma'\to\gamma,e} = -1 \\ j'(e') & \text{else} \end{cases},$$

and $\forall j \in \mathcal{C}_\gamma\,,\ \forall e \in \gamma,\ [R_{\gamma,h}(j)](e) = j(e)\,.\,h(e)\,,$

we have $\varphi_{\gamma'-\gamma} \circ T^{(\gamma)}_{\gamma',h} = \left(\mathrm{id}_{\mathcal{C}_{\gamma'-\gamma}} \times R_{\gamma,h}\right) \circ \varphi_{\gamma'-\gamma}$. Let ω_γ, resp. $\omega_{\gamma'}$, be a right invariant volume form on $\mathcal{C}_\gamma$, resp. $\mathcal{C}_{\gamma'}$, such that μ_γ, resp. $\mu_{\gamma'}$, is the corresponding measure. Because the Haar measure on a compact group is left-invariant as well as right-invariant, there exists a smooth map $\varepsilon : \mathcal{C}_\gamma \to \{+1, -1\}$ such that, for any $h \in \mathcal{C}_\gamma$, $T^{(\gamma),*}_{\gamma',h}\omega_{\gamma'} = \varepsilon(h)\,\omega_{\gamma'}$. Then, we can, like in the proof of prop. 12.1, construct a smooth volume form $\omega_{\gamma'-\gamma}$ on $\mathcal{C}_{\gamma'-\gamma}$ such that $\left[\varphi^{-1,*}_{\gamma'-\gamma}\,\omega_{\gamma'}\right] = \omega_{\gamma'-\gamma} \times \left(\varepsilon\omega_\gamma\right)$. Therefore, there exists a smooth measure $\mu_{\gamma'-\gamma}$ on $\mathcal{C}_{\gamma'-\gamma}$ such that $\varphi_{\gamma'-\gamma,*}\,\mu_{\gamma'} = \mu_{\gamma'-\gamma} \times \mu_\gamma$. And from $\mu_\gamma\left(\mathcal{C}_\gamma\right) = 1 = \mu_{\gamma'}\left(\mathcal{C}_{\gamma'}\right)$, we get $\mu_{\gamma'-\gamma}\left(\mathcal{C}_{\gamma'-\gamma}\right) = 1$.

Thus, for any measurable function $\zeta : \mathcal{C}_\gamma \to \mathbb{R}_+$, we have, by Fubini's theorem:

$$\int_{\mathcal{C}_\gamma} d\mu_\gamma(h)\,\zeta(h) = \int_{\mathcal{C}_{\gamma'}} d\mu_{\gamma'}(h')\,\zeta \circ \pi_{\gamma'-\gamma}(h')\,,$$

so that $\tau_{\gamma'\leftarrow\gamma}$ is well-defined as a map $\mathcal{H}_\gamma \to \mathcal{H}_{\gamma'}$ and is an isometry. Finally, eq. (12.7.1) follows from eq. (12.6.1).

Note. Denoting by γ'_o the AL-graph $\gamma' - \gamma \subset \gamma'$, we have $\mu_{\gamma'-\gamma} = \mu_{\gamma'_o}$. Indeed, for any measurable function $\zeta : \mathcal{C}_{\gamma'-\gamma} \to \mathbb{R}_+$:

$$\begin{aligned} \int_{\mathcal{C}_{\gamma'-\gamma}} d\mu_{\gamma'-\gamma}(j)\ \ \zeta(j) &= \int_{\mathcal{C}_{\gamma'-\gamma}\times\mathcal{C}_\gamma} d\mu_{\gamma'-\gamma}(j)\,d\mu_\gamma(h)\ \ \zeta(j) \quad (\text{for } \mu_\gamma(\mathcal{C}_\gamma) = 1) \\ &= \int_{\mathcal{C}_{\gamma'}} d\mu_{\gamma'}(h')\ \ \zeta\left(\left.h'\right|_{\gamma_o}\right) \\ &= \int_{\mathcal{C}_{\gamma'_o}\times\mathcal{C}_{\gamma'_1}} d\mu_{\gamma'_o}(h'_o)\,d\mu_{\gamma'_1}(h'_1)\ \ \zeta(h'_o) \end{aligned}$$

(with $\gamma'_1 := \gamma' \setminus \gamma'_o \in \mathcal{L}_{\text{AL}}$; $\gamma' = \gamma'_o \cup \gamma'_1$ implies $\left(\mathcal{C}_{\gamma'}\,,\ d\mu_{\gamma'}\right) \approx \left(\mathcal{C}_{\gamma'_o}\,,\ d\mu_{\gamma'_o}\right) \times \left(\mathcal{C}_{\gamma'_1}\,,\ d\mu_{\gamma'_1}\right)$)

$$= \int_{\mathcal{C}_{\gamma'_o}} d\mu_{\gamma'_o}(h'_o)\ \ \zeta(h'_o) \quad (\text{for } \mu_{\gamma'_1}(\mathcal{C}_{\gamma'_1}) = 1).$$

□

Finally, we also recall the definition of the holonomy and flux operators on $\mathcal{H}_{\text{AL}}$ [8, 4, 92]. Indeed, if we want to investigate the relation between the Ashtekar-Lewandowski construction and the just developed projective formalism, it is not enough to produce a map σ between the state spaces: we should also check that σ is dual to the map α that transports the observables according to their physical interpretation. Actually, the second part of prop. 12.10 shows that the map σ is *uniquely* specified as soon as we require it to intertwine the implementation of holonomies and exponentiated fluxes on both sides.

Proposition 12.8 Let $e \in \mathcal{L}_{\text{edges}}$ and define:

$$\mathcal{L}_{\text{AL}/e} := \{\gamma \in \mathcal{L}_{\text{AL}} \mid \{e\} \preccurlyeq \gamma\} .$$

For any $\gamma \in \mathcal{L}_{\text{AL}/e}$ and any $m \in C^\infty(G, \mathbb{R})$, we have on $\mathcal{H}_\gamma$ a densely defined, essentially self-adjoint operator $\widehat{\mathrm{h}_\gamma^{(e,m)}}$ on $\mathcal{H}_\gamma$ (with dense domain $\mathcal{D}_\gamma$) given by:

$$\forall \psi \in \mathcal{D}_\gamma \,,\, \forall h \in \mathcal{C}_\gamma \,,\, \left(\widehat{\mathrm{h}_\gamma^{(e,m)}} \psi\right)(h) := m\left(\prod_{k=1}^{n_{\gamma\to\{e\},e}} \left[h \circ a_{\gamma\to\{e\},e}(k)\right]^{\epsilon_{\gamma\to\{e\},e}(k)}\right) \psi(h) .$$

Moreover, for any $\gamma' \succcurlyeq \gamma$, we have $\gamma' \in \mathcal{L}_{\text{AL}/e}$ and:

$$\forall \psi \in \mathcal{D}_\gamma \,,\, \tau_{\gamma'\leftarrow\gamma}(\psi) \in \mathcal{D}_{\gamma'} \quad \& \quad \widehat{\mathrm{h}_{\gamma'}^{(e,m)}} \circ \tau_{\gamma'\leftarrow\gamma}(\psi) = \tau_{\gamma'\leftarrow\gamma} \circ \widehat{\mathrm{h}_\gamma^{(e,m)}}(\psi) .$$

Thus, the family $\left(\widehat{\mathrm{h}_\gamma^{(e,m)}}\right)_{\gamma\in\mathcal{L}_{\text{AL}/e}}$ provides a densely defined, essentially self-adjoint operator $\widehat{\mathrm{h}_{\text{AL}}^{(e,m)}}$ on $\mathcal{H}_{\text{AL}}$.

Proof Let $\gamma \in \mathcal{L}_{\text{AL}/e}$ and $m \in C^\infty(G, \mathbb{R})$. Taking $\mathcal{D}_\gamma = C^\infty(\mathcal{C}_\gamma, \mathbb{C}) \subset \mathcal{H}_\gamma$ (this matches the compactly supported smooth functions used in prop. 12.2 since $\mathcal{C}_\gamma$ is compact), $\widehat{\mathrm{h}_\gamma^{(e,m)}}$ is well-defined and essentially self-adjoint (actually, it is a bounded operator). Moreover, we have:

$$\forall h \in \mathcal{C}_\gamma \,,\, m\left(\prod_{k=1}^{n_{\gamma\to\{e\},e}} \left[h \circ a_{\gamma\to\{e\},e}(k)\right]^{\epsilon_{\gamma\to\{e\},e}(k)}\right) = m\left(\left[\pi_{\gamma\to\{e\}}(h)\right](e)\right) . \tag{12.8.1}$$

Now let $\gamma' \succcurlyeq \gamma$. By transitivity of $\succcurlyeq$ on $\mathcal{L}_{\text{AL}}$, $\gamma' \in \mathcal{L}_{\text{AL}/e}$. For any $\psi \in \mathcal{D}_\gamma$, $\tau_{\gamma'\leftarrow\gamma}(\psi) \in \mathcal{D}_{\gamma'}$ (for $\pi_{\gamma'\to\gamma}$ is smooth) and, using eq. (12.6.1):

$$\forall h' \in \mathcal{C}_{\gamma'} \,,\, \left[\widehat{\mathrm{h}_{\gamma'}^{(e,m)}} \circ \tau_{\gamma'\leftarrow\gamma}(\psi)\right](h') = m\left(\left[\pi_{\gamma'\to\{e\}}(h')\right](e)\right) \left[\psi \circ \pi_{\gamma'\to\gamma}\right](h') = \left[\tau_{\gamma'\leftarrow\gamma} \circ \widehat{\mathrm{h}_\gamma^{(e,m)}}(\psi)\right](h') .$$

$\mathcal{L}_{\text{AL}/e}$ is a cofinal part of $\mathcal{L}_{\text{AL}}$ (for $\mathcal{L}_{\text{AL}}$ is directed), and this allows us to construct a symmetric operator $\widehat{\mathrm{h}_{\text{AL}}^{(e,m)}}$ on the vector subspace $\mathcal{D}_{\text{AL}} \subset \mathcal{H}_{\text{AL}}$, defined as the inductive limit of vector spaces $\left(\mathcal{D}_\gamma\right)_{\gamma\in\mathcal{L}_{\text{AL}}}$, $\left(\tau_{\gamma'\leftarrow\gamma}\big|_{\mathcal{D}_\gamma\to\mathcal{D}_{\gamma'}}\right)_{\gamma\preccurlyeq\gamma'}$ (without any completion). $\mathcal{D}_{\text{AL}}$ is dense in $\mathcal{H}_{\text{AL}}$ and $\widehat{\mathrm{h}_{\text{AL}}^{(e,m)}}$ is bounded, hence essentially self-adjoint. □

Proposition 12.9 Let $\overline{F} \in \mathcal{L}_{\text{faces}}$ (prop. 10.21) and define:

$$\mathcal{L}_{\text{AL}/\overline{F}} := \left\{\gamma \in \mathcal{L}_{\text{AL}} \;\middle|\; \forall e \in \gamma,\, \forall p \neq p' \in r(e),\, \left(e_{[p,p']} \in \overline{F} \;\Rightarrow\; p \in \{b(e),\, f(e)\}\right)\right\} .$$

Let $\gamma \in \mathcal{L}_{\text{AL}/\overline{F}}$ and define, for any $h \in \mathcal{C}_\gamma$ and any $t \in G$, the map $T^{(\overline{F})}_{\gamma,t} h \in \mathcal{C}_\gamma$ by:

$$\forall e \in \gamma, \quad T^{(\overline{F})}_{\gamma,t} h(e) := \begin{cases} h(e)\,.\,t & \text{if } c(e, \overline{F}) = \{b(e)\} \\ t^{-1}\,.\,h(e) & \text{if } c(e, \overline{F}) = \{f(e)\} \\ t^{-1}\,.\,h(e)\,.\,t & \text{if } c(e, \overline{F}) = \{b(e),\, f(e)\} \\ h(e) & \text{if } c(e, \overline{F}) = \varnothing \end{cases} ,$$

where $c(e, \overline{F}) := \left\{p \in r(e) \;\middle|\; \exists p' \neq p \,/\, e_{[p,p']} \in \overline{F}\right\} \subset \{b(e),\, f(e)\}$. Then, for any $u \in \mathfrak{g}$, we have a densely defined, essentially self-adjoint operator $\widehat{\mathsf{P}_\gamma^{(\overline{F},u)}}$ on $\mathcal{H}_\gamma$ (with dense domain $\mathcal{D}_\gamma$) given by:

$$\forall \psi \in \mathcal{D}_\gamma\,,\ \forall h \in \mathcal{C}_\gamma\,,\ \left(\widehat{\mathsf{P}_\gamma^{(\overline{F},u)}}\psi\right)(h) := i\left[T_1\, t \mapsto \psi\left(T_{\gamma,t}^{(\overline{F})} h\right)\right](u)\,,$$

and, for any $t \in G$, we have a unitary operator $\widehat{\mathsf{T}_\gamma^{(\overline{F},t)}}$ on $\mathcal{H}_\gamma$ given by:

$$\forall \psi \in \mathcal{H}_\gamma\,,\ \forall h \in \mathcal{C}_\gamma\,,\ \left(\widehat{\mathsf{T}_\gamma^{(\overline{F},t)}}\psi\right)(h) := \psi\left(T_{\gamma,t}^{(\overline{F})} h\right).$$

Moreover, $\mathcal{L}_{\text{AL}/\overline{F}}$ is a cofinal part of $\mathcal{L}_{\text{AL}}$ and for any $\gamma \preccurlyeq \gamma' \in \mathcal{L}_{\text{AL}/\overline{F}}$:

$$\forall \psi \in \mathcal{D}_\gamma\,,\ \tau_{\gamma' \leftarrow \gamma}(\psi) \in \mathcal{D}_{\gamma'} \quad \& \quad \widehat{\mathsf{P}_{\gamma'}^{(\overline{F},u)}} \circ \tau_{\gamma' \leftarrow \gamma}(\psi) = \tau_{\gamma' \leftarrow \gamma} \circ \widehat{\mathsf{P}_\gamma^{(\overline{F},u)}}(\psi)\,,$$

$$\text{and} \quad \forall \psi \in \mathcal{H}_\gamma\,,\ \widehat{\mathsf{T}_{\gamma'}^{(\overline{F},t)}} \circ \tau_{\gamma' \leftarrow \gamma}(\psi) = \tau_{\gamma' \leftarrow \gamma} \circ \widehat{\mathsf{T}_\gamma^{(\overline{F},t)}}(\psi)\,.$$

Thus, the family $\left(\widehat{\mathsf{P}_\gamma^{(\overline{F},u)}}\right)_{\gamma \in \mathcal{L}_{\text{AL}/\overline{F}}}$ provides a densely defined, essentially self-adjoint operator $\widehat{\mathsf{P}_{\text{AL}}^{(\overline{F},u)}}$ on $\mathcal{H}_{\text{AL}}$, while the family $\left(\widehat{\mathsf{T}_\gamma^{(\overline{F},t)}}\right)_{\gamma \in \mathcal{L}_{\text{AL}/\overline{F}}}$ provides a unitary operator $\widehat{\mathsf{T}_{\text{AL}}^{(\overline{F},t)}}$.

Proof Let $\gamma \in \mathcal{L}_{\text{AL}/\overline{F}}$ and $u \in \mathfrak{g}$. Taking as dense domain $\mathcal{D}_\gamma = C^\infty(\mathcal{C}_\gamma\,, \mathbb{C}) \subset \mathcal{H}_\gamma$, $\widehat{\mathsf{P}_\gamma^{(\overline{F},u)}}$ is well-defined, and, using the invariance of the measure μ_γ under left- and right-translations, we can check that it is a symmetric operator with $\text{Ker}\left(\widehat{\mathsf{P}_\gamma^{(\overline{F},u)}}^{\dagger} \pm i\right) = \{0\}$ (like in the proof of prop. 12.2), therefore it is an essentially self-adjoint operator. The left- and right-invariance of the measure also ensures that, for any $t \in G$, $T_{\gamma,t,*}^{(\overline{F})}\,\mu_\gamma = \mu_\gamma$, so $\widehat{\mathsf{T}_\gamma^{(\overline{F},t)}}$ is a well-defined unitary operator on $\mathcal{H}_\gamma$.

Now, let $\gamma \in \mathcal{L}_{\text{AL}}$ and let $\lambda \in \mathcal{L}_{\text{profls}}$ such that there exists $F \in \mathcal{F}(\lambda)$ with $\overline{F} = F^\perp \circ F$. Since $\mathcal{L}_{\text{graphs}}$ is cofinal in $\mathcal{L}_{\text{AL}}$ (prop. 12.4), there exists $\gamma' \in \mathcal{L}_{\text{graphs}}$ with $\gamma' \succcurlyeq \gamma$, and by subsection 10.2, there exists $(\gamma'', \lambda'') \in \mathcal{L}_{\text{HF}}$ with $\gamma'' \succcurlyeq \gamma'$ and $\lambda'' \succcurlyeq \lambda$. Next, let $e'' \in \gamma''$ and suppose that there exists $p \neq p' \in r(e'')$ such that $e''_{[p,p']} \in \overline{F}$. Then, using $\overline{F} = F^\perp \circ F$ together with $\lambda'' \succcurlyeq \lambda$, there exists $p'' \in r\left(e''_{[p,p']}\right) \setminus \{p,\, p'\} \subset r(e'') \setminus \{b(e''),\, f(e'')\}$ such that $e''_{[p,p'']} \in F''$ for some $F'' \in \mathcal{F}(\lambda'')$. But since $e'' \in \chi_{(\gamma'',\lambda'')}(e'')$, this can only be the case if $p = b(e)$ (props. 10.6.5 and 10.6.6). Therefore $\gamma'' \in \mathcal{L}_{\text{AL}/\overline{F}}$. Thus, $\mathcal{L}_{\text{AL}/\overline{F}}$ is cofinal in $\mathcal{L}_{\text{AL}}$.

Let $e \in \mathcal{L}_{\text{edges}}^{(1)}$ such that $c(e, \overline{F}) \subset \{b(e), f(e)\}$ and let $e_1\,, \ldots, e_n \in \mathcal{L}_{\text{edges}}^{(1)}$, $\epsilon_1\,, \ldots, \epsilon_n \in \{\pm 1\}$ such that $e = e_n^{\epsilon_n} \circ \ldots \circ e_1^{\epsilon_1}$. Then, using:

$$\forall \widetilde{e} \in \overline{F},\ \forall p \in r(\widetilde{e}) \setminus \{b(e)\}\,,\ \widetilde{e}_{[b(e),p]} \in \overline{F}\,,$$

we have:

$$b\left(e_i^{\epsilon_i}\right) \in c(e_i, \overline{F}) \Leftrightarrow \left(i = 1 \quad \& \quad b(e) \in c(e, \overline{F})\right),$$

$$f\left(e_i^{\epsilon_i}\right) \in c(e_i, \overline{F}) \Leftrightarrow \left(i = n \quad \& \quad f(e) \in c(e, \overline{F})\right).$$

Thus, for any $\gamma \preccurlyeq \gamma' \in \mathcal{L}_{\text{AL}/\overline{F}}$, we can check that:

$$\forall h' \in \mathcal{C}_{\gamma'}, \forall t \in G, \quad \pi_{\gamma' \to \gamma}\left(T^{(\overline{F})}_{\gamma',t} h'\right) = T^{(\overline{F})}_{\gamma,t}\left(\pi_{\gamma' \to \gamma}(h')\right). \tag{12.9.1}$$

Hence, for any $t \in G$, the unitary operators $\widehat{\mathsf{T}^{(\overline{F},t)}_{\gamma}}$ defined on each $\mathcal{H}_\gamma$ for $\gamma \in \mathcal{L}_{\text{AL}/\overline{F}}$ can be assembled into a unitary operator $\widehat{\mathsf{T}^{(\overline{F},t)}_{\text{AL}}}$ on $\mathcal{H}_{\text{AL}}$, while, for any $u \in \mathfrak{g}$, a densely defined, symmetric operator $\widehat{\mathsf{P}^{(\overline{F},u)}_{\text{AL}}}$ can be constructed on the dense subspace $\mathcal{D}_{\text{AL}} \subset \mathcal{H}_{\text{AL}}$, defined as the inductive limit of vector spaces $\left(\mathcal{D}_\gamma\right)_{\gamma \in \mathcal{L}_{\text{AL}}}$, $\left(\left.\tau_{\gamma' \leftarrow \gamma}\right|_{\mathcal{D}_\gamma \to \mathcal{D}_{\gamma'}}\right)_{\gamma \preccurlyeq \gamma'}$.

Finally, let $\psi \in \text{Ker}\left(\widehat{\mathsf{P}^{(\overline{F},u)}_{\text{AL}}}^{\dagger} \pm i\right)$ and define, for $\gamma \in \mathcal{L}_{\text{AL}/\overline{F}}$, $\psi_\gamma \in \mathcal{H}_\gamma$ such that $\tau_{\text{AL} \leftarrow \gamma}\left(\psi_\gamma\right)$ is the orthogonal projection of ψ on the closed vector subspace $\tau_{\text{AL} \leftarrow \gamma}\left\langle \mathcal{H}_\gamma \right\rangle$. Then, $\psi_\gamma \in \text{Ker}\left(\widehat{\mathsf{P}^{(\overline{F},u)}_{\gamma}}^{\dagger} \pm i\right) = \{0\}$, so:

$$\psi \in \left(\bigcup_{\gamma \in \mathcal{L}_{\text{AL}/\overline{F}}} \tau_{\text{AL} \leftarrow \gamma}\left\langle \mathcal{H}_\gamma \right\rangle\right)^{\perp} = \left(\bigcup_{\gamma \in \mathcal{L}_{\text{AL}}} \tau_{\text{AL} \leftarrow \gamma}\left\langle \mathcal{H}_\gamma \right\rangle\right)^{\perp} = \{0\}.$$

Hence, $\widehat{\mathsf{P}^{(\overline{F},u)}_{\text{AL}}}$ is essentially self-adjoint. □

Proposition 12.10 Let $\overline{F} \in \mathcal{L}_{\text{faces}}$ and $t \in G$. For any $\eta \in \mathcal{L}_{\text{HF}}$ such that $\lambda(\eta) \in \mathcal{L}_{\text{profls}/\overline{F}}$, we define a unitary operator $\widehat{\mathsf{T}^{(\overline{F},t)}_{\eta}}$ on $\mathcal{H}_\eta$ by:

$$\forall \psi \in \mathcal{H}_\eta, \forall h \in \mathcal{C}_\eta, \left(\widehat{\mathsf{T}^{(\overline{F},t)}_{\eta}} \psi\right)(h) := \psi\left(R^{(\overline{F})}_{\eta,t} h\right),$$

where $R^{(\overline{F})}_{\eta,t} : \mathcal{C}_\eta \to \mathcal{C}_\eta$ was defined in prop. 12.2. Then, for any $\eta, \eta' \in \mathcal{L}_{\text{HF}}$ such that $\lambda(\eta), \lambda(\eta') \in \mathcal{L}_{\text{profls}/\overline{F}}$:

$$\widehat{\mathsf{T}^{(\overline{F},t)}_{\eta}} \sim \widehat{\mathsf{T}^{(\overline{F},t)}_{\eta'}},$$

so this provides an element $\widehat{\mathsf{T}^{(\overline{F},t)}} \in \mathcal{A}^{\otimes}_{(\mathcal{L}_{\text{HF}},\mathcal{H},\Phi)}$ (def. 5.3).

Let $\eta \in \mathcal{L}_{\text{HF}}$ and denote by $\mathcal{I}_\eta$ the algebra of bounded operators on $\mathcal{H}_\eta$ generated by:

$$\left\{\widehat{\mathsf{h}^{(e,m)}_{\eta}} \,\middle|\, m \in C^\infty(G, \mathbb{R}) \;\;\&\;\; e \in \gamma(\eta)\right\} \cup \left\{\widehat{\mathsf{T}^{(F^{\perp} \circ F,t)}_{\eta}} \,\middle|\, t \in G \;\;\&\;\; F \in \mathcal{F}(\eta)\right\}.$$

If ρ, ρ' are (self-adjoint) positive semi-definite, traceclass operators on $\mathcal{H}_\eta$ such that:

$$\forall A \in \mathcal{I}_\eta, \text{Tr}_{\mathcal{H}_\eta} \rho A = \text{Tr}_{\mathcal{H}_\eta} \rho' A,$$

then $\rho = \rho'$.

Proof *Definition of* $\widehat{\mathsf{T}^{(\overline{F},t)}}$. For any $\eta \in \mathcal{L}_{\text{HF}}$ such that $\lambda(\eta) \in \mathcal{L}_{\text{profls}/\overline{F}}$, we have $R^{(\overline{F})}_{\eta,t,*}\mu_\eta = \mu_\eta$ (for μ is invariant under right-translations) hence $\widehat{\mathsf{T}^{(\overline{F},t)}_{\eta}}$ is a well-defined and unitary operator on $\mathcal{H}_\eta$.

Next, for any η such that $\lambda(\eta) \in \mathcal{L}_{\text{profls}/\overline{F}}$ and any $\eta' \succcurlyeq \eta$, we have $\lambda(\eta') \in \mathcal{L}_{\text{profls}/\overline{F}}$. Moreover, using eqs. (10.28.2), (10.27.1) and (10.23.2), we can check that:

$$\left(\mathrm{id}_{\mathcal{C}_{\eta'\to\eta}} \times R^{(\overline{F})}_{\eta,t}\right) \circ \varphi_{\eta'\to\eta} = \varphi_{\eta'\to\eta} \circ R^{(\overline{F})}_{\eta',t}\,.$$

Thus, we get $\Phi_{\eta'\to\eta} \circ \widehat{\mathrm{T}^{(\overline{F},t)}_{\eta'}} = \left(\mathrm{id}_{\mathcal{H}_{\eta'\to\eta}} \times \widehat{\mathrm{T}^{(\overline{F},t)}_{\eta}}\right) \circ \Phi_{\eta'\to\eta}$. Therefore, the directedness of $\mathcal{L}_{\text{HF}}$ ensures that, for any $\eta,\ \eta' \in \mathcal{L}_{\text{HF}}$ such that $\lambda(\eta),\ \lambda(\eta') \in \mathcal{L}_{\text{profls}/\overline{F}}$, $\widehat{\mathrm{T}^{(\overline{F},t)}_{\eta}} \sim \widehat{\mathrm{T}^{(\overline{F},t)}_{\eta'}}$ (with the equivalence relation defined in eq. (5.3.2)), so we can define an element $\widehat{\mathrm{T}^{(\overline{F},t)}} \in \mathcal{A}^{\otimes}_{(\mathcal{L}_{\text{HF}},\mathcal{H},\Phi)}$.

Definition of $\widehat{\mathrm{h}^{(M)}_{\eta}}$ *and* $\widehat{\mathrm{T}^{(j)}_{\eta}}$. Let $\eta \in \mathcal{L}_{\text{HF}}$ and $M : \gamma(\eta) \to C^\infty(G,\ \mathbb{R})$. Then we can define an element $\widehat{\mathrm{h}^{(M)}_{\eta}} \in \mathfrak{I}_\eta$ by:

$$\widehat{\mathrm{h}^{(M)}_{\eta}} := \prod_{e\in\gamma(\eta)} \widehat{\mathrm{h}^{(e,M(e))}}_{\eta}\,.$$

The right-hand side does not depend on the ordering of the product and we have:

$$\forall \psi \in \mathcal{H}_\eta\,,\ \forall h \in \mathcal{C}_\eta\,,\ \left(\widehat{\mathrm{h}^{(M)}_{\eta}}\psi\right)(h) = \left(\prod_{e\in\gamma(\eta)} \left[M(e)\circ h\right](e)\right)\ \psi(h)\,.$$

Next, for any $F \in \mathcal{F}(\eta)$ and any $t \in G$, we have $\lambda(\eta) \in \mathcal{L}_{\text{profls}/F^\perp\circ F}$ and, with the help of prop. 10.6:

$$\forall h \in \mathcal{C}_\eta\,,\ \forall e \in \gamma(\eta)\,,\ \left[R^{(F^\perp\circ F)}_{\eta,t}\,h\right](e) := \begin{cases} h(e)\,.\,t & \text{if } \chi_\eta(e) = F \\ h(e) & \text{else} \end{cases}\,.$$

Hence, for any $j \in \mathcal{C}_\eta$ we can define a unitary operator $\widehat{\mathrm{T}^{(j)}_{\eta}} \in \mathfrak{I}_\eta$ by:

$$\widehat{\mathrm{T}^{(j)}_{\eta}} := \prod_{F\in\mathcal{F}(\eta)} \widehat{\mathrm{T}^{(F^\perp\circ F,\,j\circ\chi_\eta^{-1}(F))}_{\eta}}\,,$$

and we have (the product ordering being again irrelevant):

$$\forall \psi \in \mathcal{H}_\eta\,,\ \forall h \in \mathcal{C}_\eta\,,\ \left(\widehat{\mathrm{T}^{(j)}_{\eta}}\psi\right)(h) = \psi(h\,.\,j)\,,$$

where, for any $h,\ h' \in \mathcal{C}_\eta$, $h\,.\,h'$ denotes their pointwise multiplication as maps $\gamma(\eta) \to G$.

Characterization of $\rho \in \overline{\mathcal{S}}_\eta$ *by its evaluations over* $\mathfrak{I}_\eta$. Let ρ, ρ' be non-negative traceclass operators on $\mathcal{H}_\eta$ and $\varphi, \varphi' \in \mathcal{H}_\eta$. Let $\epsilon > 0$ and define $\epsilon_o := \frac{\epsilon}{10}$, as well as:

$$\epsilon_1 := \min\left(\frac{\epsilon_o}{1 + \mathrm{Tr}_{\mathcal{H}_\eta}\,\rho}\,,\ \frac{\epsilon_o}{1 + \mathrm{Tr}_{\mathcal{H}_\eta}\,\rho'}\,,\ 1\right) > 0\,.$$

The $\mathbb{C}$-vector space generated by $C^\infty(G,\mathbb{R})$ is dense in $L_2(G,\ d\mu)$ ($C^\infty(G,\mathbb{R}) \subset L_2(G,\ d\mu)$ for G is compact), hence the $\mathbb{C}$-vector space generated by $\{\otimes_{e\in\gamma(\eta)}M(e) \mid M : \gamma(\eta) \to C^\infty(G,\mathbb{R})\}$ is dense in $\bigotimes_{e\in\gamma(\eta)} L_2(G,\ d\mu) \approx \mathcal{H}_\eta$ (using the isometric identification provided by Fubini's theorem). Thus, there exist a finite family of maps $\left(M_l : \gamma(\eta) \to C^\infty(G, \mathbb{R})\right)_{1 \leqslant l \leqslant L}$ and a finite family of complex numbers $(\mu_l)_{1 \leqslant l \leqslant L}$ such that:

$$\left\| \varphi - \sum_{l=1}^{L} \mu_l \left[\otimes_{\gamma(\eta)} M_l\right] \right\|_{\mathcal{H}_\eta} < \frac{\epsilon_1}{1 + \|\varphi'\|_{\mathcal{H}_\eta}} \ ,$$

where, for any $M : \gamma(\eta) \to C^\infty(G,\ \mathbb{R})$, the vector $\otimes_{\gamma(\eta)} M \in \mathcal{H}_\eta$ is defined by:

$$\forall h \in \mathcal{C}_\eta\,,\ [\otimes_{\gamma(\eta)} M](h) := \prod_{e \in \gamma(\eta)} [M(e) \circ h](e)\,.$$

Similarly, there exist $\left(M'_{l'} : \gamma(\eta) \to C^\infty(G, \mathbb{R})\right)_{1 \leqslant l' \leqslant L'}$ and $\left(\mu'_{l'}\right)_{1 \leqslant l' \leqslant L'}$ ($0 \leqslant L, L' < \infty$) such that:

$$\left\| \varphi' - \sum_{l'=1}^{L'} \mu'_{l'} \left[\otimes_{\gamma(\eta)} M'_{l'}\right] \right\|_{\mathcal{H}_\eta} < \frac{\epsilon_1}{1 + \|\varphi\|_{\mathcal{H}_\eta}} \ .$$

Thus, we have:

$$\begin{aligned} &\left| \langle \varphi' \mid \rho \mid \varphi \rangle - \sum_{l=1}^{L} \sum_{l'=1}^{L'} {\mu'_{l'}}^* \mu_l \, \langle \otimes_{\gamma(\eta)} M'_{l'} \mid \rho \mid \otimes_{\gamma(\eta)} M_l \rangle \right| \\ &\qquad < \frac{\epsilon_1 \ \|\varphi\|_{\mathcal{H}_\eta} \ \mathrm{Tr}_{\mathcal{H}_\eta} \rho}{1 + \|\varphi\|_{\mathcal{H}_\eta}} + \frac{\|\varphi'\|_{\mathcal{H}_\eta} \ \epsilon_1 \ \mathrm{Tr}_{\mathcal{H}_\eta} \rho}{1 + \|\varphi'\|_{\mathcal{H}_\eta}} + \frac{\epsilon_1^2 \, \mathrm{Tr}_{\mathcal{H}_\eta} \rho}{\left(1 + \|\varphi\|_{\mathcal{H}_\eta}\right) \left(1 + \|\varphi'\|_{\mathcal{H}_\eta}\right)} \leqslant 2\, \epsilon_o\,, \end{aligned} \tag{12.10.1}$$

and similarly for ρ'. We define:

$$\epsilon_2 := \frac{\epsilon_o}{1 + \sum_{l=1}^{L} \sum_{l'=1}^{L'} \left(|\mu_l| \ |\mu'_{l'}| \ \prod_{e \in \gamma(\eta)} \|M_l(e)\|_\infty \ \|M'_{l'}(e)\|_\infty \right)} > 0\,,$$

where, for any $m \in C^\infty(G,\ \mathbb{R})$, $\|m\|_\infty := \sup_{t \in G} |m(t)|$ ($< \infty$ for G is compact). Note that, for any $M : \gamma(\eta) \to C^\infty(G,\ \mathbb{R})$, we have the bound:

$$\left\| \widehat{\mathrm{h}_\eta^{(M)}} \right\|_{\mathcal{A}_\eta} \leqslant \prod_{e \in \gamma(\eta)} \|M(e)\|_\infty\,,$$

where $\|\cdot\|_{\mathcal{A}_\eta}$ denotes the operator norm on the algebra $\mathcal{A}_\eta$ of bounded operators over $\mathcal{H}_\eta$ (def. 5.3), as well as:

$$\|\otimes_{\gamma(\eta)} M\|_{\mathcal{H}_\eta} \leqslant \prod_{e \in \gamma(\eta)} \|M(e)\|_\infty$$

(using the fact that μ_η is normalized).

Now, from the spectral theorem, together with the non-negative and traceclass conditions, there exist an orthonormal family $\left(\widetilde{\psi}_k\right)_{1 \leqslant k \leqslant \widetilde{K}}$ in $\mathcal{H}_\eta$ (with $0 \leqslant \widetilde{K} \leqslant \infty$) and a family of strictly positive reals $(p_k)_{1 \leqslant k \leqslant \widetilde{K}}$ such that:

$$\rho = \sum_{k=1}^{\widetilde{K}} p_k \left| \widetilde{\psi}_k \right\rangle \left\langle \widetilde{\psi}_k \right| \quad \& \quad \sum_{k=1}^{\widetilde{K}} p_k =: \mathrm{Tr}_{\mathcal{H}_\eta} \rho < \infty\,.$$

Let $K < \infty$ such that $\left|\operatorname{Tr}_{\mathcal{H}_\eta} \rho - \sum_{k=1}^{K} p_k\right| < \frac{\epsilon_2}{4}$. Then, for any $k \leqslant K$, there exists $\psi_k \in C^\infty(\mathcal{C}_\eta\,, \mathbb{C}) \subset \mathcal{H}_\eta$ such that:

$$\left\|\widetilde{\psi}_k - \psi_k\right\|_{\mathcal{H}_\eta} < \min\left(1,\ \frac{\epsilon_2}{4\operatorname{Tr}_{\mathcal{H}_\eta} \rho + 1}\right).$$

Thus, we have:

$$\begin{aligned}&\left|\sum_{l=1}^{L}\sum_{l'=1}^{L'} {\mu'_{l'}}^{*}\mu_l \left\langle \otimes_{\gamma(\eta)} M'_{l'} \mid \rho \mid \otimes_{\gamma(\eta)} M\right\rangle - \sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{l'=1}^{L'} p_k\, {\mu'_{l'}}^{*}\mu_l \left\langle \otimes_{\gamma(\eta)} M'_{l'} \mid \psi_k\right\rangle \left\langle \psi_k \mid \otimes_{\gamma(\eta)} M\right\rangle\right| \\ &\qquad < \sum_{l=1}^{L}\sum_{l'=1}^{L'} |\mu'_{l'}|\ |\mu_l|\ \|\otimes_{\gamma(\eta)} M'_{l'}\|_{\mathcal{H}_\eta}\ \|\otimes_{\gamma(\eta)} M\|_{\mathcal{H}_\eta} \left[\frac{\epsilon_2}{4} + \sum_{k=1}^{K} p_k \frac{3\epsilon_2}{4\operatorname{Tr}_{\mathcal{H}_\eta} \rho + 1}\right] \leqslant \epsilon_o\,, \end{aligned} \tag{12.10.2}$$

and, for any $j \in \mathcal{C}_\eta$:

$$\begin{aligned}&\left|\sum_{l=1}^{L}\sum_{l'=1}^{L'} {\mu'_{l'}}^{*}\mu_l \operatorname{Tr}_{\mathcal{H}_\eta}\left(\rho\, \widehat{\mathrm{h}_\eta^{(M_l)}}\ \widehat{\mathrm{T}_\eta^{(j)}}\ \widehat{\mathrm{h}_\eta^{(M'_{l'})}}\right) - \sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{l'=1}^{L'} p_k\, {\mu'_{l'}}^{*}\mu_l \left\langle \psi_k \,\middle|\, \widehat{\mathrm{h}_\eta^{(M_l)}}\ \widehat{\mathrm{T}_\eta^{(j)}}\ \widehat{\mathrm{h}_\eta^{(M'_{l'})}} \,\middle|\, \psi_k\right\rangle\right| \\ &\qquad < \sum_{l=1}^{L}\sum_{l'=1}^{L'} |\mu'_{l'}|\ |\mu_l| \left\|\widehat{\mathrm{h}_\eta^{(M_l)}}\ \widehat{\mathrm{T}_\eta^{(j)}}\ \widehat{\mathrm{h}_\eta^{(M'_{l'})}}\right\|_{\mathcal{A}_\eta} \left[\frac{\epsilon_2}{4} + \sum_{k=1}^{K} p_k \frac{3\,\epsilon_2}{4\operatorname{Tr}_{\mathcal{H}_\eta} \rho + 1}\right] \leqslant \epsilon_o\,. \end{aligned} \tag{12.10.3}$$

Similarly, there exist a finite family $\left(\psi'_k\right)_{1\leqslant k\leqslant K'}$ of functions in $C^\infty(\mathcal{C}_\eta\,, \mathbb{C})$ (with $0 \leqslant K' < \infty$) and a finite family $\left(p'_k\right)_{1\leqslant k\leqslant K'}$ of strictly positive reals such that the equivalents of eqs. (12.10.2) and (12.10.3) are fulfilled for ρ'. We define:

$$\epsilon_3 := \frac{\epsilon_o}{1 + \sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{l'=1}^{L'} p_k\ |\mu_l|\ |\mu'_{l'}|\ \|\psi_k\|_\infty\ \|\otimes_{\gamma(\eta)} M_l\|_\infty} > 0\,,$$

where, for any $\zeta \in C^\infty(\mathcal{C}_\eta\,, \mathbb{C})$, $\|\zeta\|_\infty := \sup_{j\in\mathcal{C}_\eta} |\zeta(j)|$. Similarly, we define ϵ'_3 using $\left(\psi'_k\right)_{1\leqslant k\leqslant K'}$ and $\left(p'_k\right)_{1\leqslant k\leqslant K'}$.

Let $l' \leqslant L'$ and $k \leqslant K$. The function $\zeta_{l',k}$, defined on $\mathcal{C}_\eta \times \mathcal{C}_\eta$ by:

$$\forall h \in \mathcal{C}_\eta\,,\ \forall j \in \mathcal{C}_\eta\,,\ \zeta_{l',k}(h, j) := [\otimes_{\gamma(\eta)} M'_{l'}](h.j)\, \psi_k(h.j)\,,$$

is smooth by construction. Hence, for any $h \in \mathcal{C}_\eta$, there exists an open, measurable neighborhood $V^{(h)}_{k,l'}$ of $\mathbf{1}$ in $\mathcal{C}_\eta$ and an open, measurable neighborhood $W^{(h)}_{k,l'}$ of h in $\mathcal{C}_\eta$ such that:

$$\forall h' \in W^{(h)}_{k,l'}\,,\ \forall j \in V^{(h)}_{k,l'}\,,\ |\zeta_{l',k}(h', j) - \zeta_{l',k}(h, \mathbf{1})| < \frac{\epsilon_3}{2}\,,$$

and therefore:

$$\forall h' \in W^{(h)}_{k,l'}\,,\ \forall j \in V^{(h)}_{k,l'}\,,\ |\zeta_{l',k}(h', j) - \zeta_{l',k}(h', \mathbf{1})| < \epsilon_3\,.$$

Now, since $\mathcal{C}_\eta$ is compact, there exists a finite subset $H_{k,l'}$ of $\mathcal{C}_\eta$ such that $\mathcal{C}_\eta \subset \bigcup_{h\in H_{k,l'}} W^{(h)}_{k,l'}$, so, defining $V_{k,l'} := \bigcap_{h\in H_{k,l'}} V^{(h)}_{k,l'}$, $V_{k,l'}$ is an open, measurable neighborhood of $\mathbf{1}$ in $\mathcal{C}_\eta$ and we have:

$$\forall h' \in \mathcal{C}_\eta\,,\ \forall j \in V_{k,l'}\,,\ |\zeta_{l',k}(h',j) - \zeta_{l',k}(h',\mathbf{1})| < \epsilon_3\,.$$

Next, defining $V := \bigcap_{k=1}^{K}\bigcap_{l'=1}^{L'} V_{k,l'}$, we get, for any measurable subset $\widetilde{V} \subset V$, any $k \leqslant K$ and any $l' \leqslant L'$:

$$\forall h \in \mathcal{C}_\eta\,,\ \left|\left(\int_{\widetilde{V}} d\mu_\eta(j)\,[\otimes_{\gamma(\eta)}M'_{l'}](h.j)\,\psi_k(h.j)\right) - \left(\mu(\widetilde{V})\,[\otimes_{\gamma(\eta)}M'_{l'}](h)\,\psi_k(h)\right)\right| < \mu(\widetilde{V})\,\epsilon_3\,. \tag{12.10.4}$$

Similarly, we have an open, measurable neighborhood V' of $\mathbf{1}$ in $\mathcal{C}_\eta$ such that, for any measurable subset $\widetilde{V} \subset V'$, any $k' \leqslant K'$ and any $l' \leqslant L'$, the equivalent of eq. (12.10.4) is fulfilled for $\psi'_{k'}$ (with respect to ϵ'_3 instead of ϵ_3). We define $V_o := V \cap V'$.

For any $j \in \mathcal{C}_\eta$, $j\,.\,V_o$ is an open, measurable neighborhood of j in $\mathcal{C}_\eta$, hence, from the compacity of $\mathcal{C}_\eta$, there exist $j_1, \dots, j_M$ in $\mathcal{C}_\eta$ ($1 \leqslant M < \infty$) such that $\mathcal{C}_\eta \subset \bigcup_{m=1}^{M} j_m\,.\,V_o$. For $m \leqslant M$ we define:

$$V_m := (j_m\,.\,V_o) \cap \left(\mathcal{C}_\eta \setminus \bigcup_{n=1}^{m-1} j_n\,.\,V_o\right).$$

Thus, $(V_m)_{1\leqslant m\leqslant M}$ is a partition of $\mathcal{C}_\eta$ into finitely many measurable parts. Moreover, we have, for any $m \leqslant M$:

$$j_m^{-1}\,.\,V_m \subset V_o\,,$$

so that, for any $k \leqslant K$ and any $l' \leqslant L'$, eq. (12.10.4) yields:

$$\begin{aligned}
&\forall h \in \mathcal{C}_\eta\,,\ \left|\langle \otimes_{\gamma(\eta)}M'_{l'} \mid \psi_k\rangle - \sum_{m=1}^{M} \mu(V_m)\left[\widehat{\mathrm{T}_\eta^{(j_m)}}\,\widehat{\mathrm{h}_\eta^{(M'_{l'})}}\,\psi_k\right](h)\right| = \\
&= \left|\left(\int_{\mathcal{C}_\eta} d\mu_\eta(j)\,[\otimes_{\gamma(\eta)}M'_{l'}](h.j)\,\psi_k(h.j)\right) - \sum_{m=1}^{M}\left(\mu(V_m)\,[\otimes_{\gamma(\eta)}M'_{l'}](h.j_m)\,\psi_k(h.j_m)\right)\right| \\
&\leqslant \sum_{m=1}^{M}\left|\left(\int_{j_m^{-1}\,.\,V_m} d\mu_\eta(j)\,[\otimes_{\gamma(\eta)}M'_{l'}](h.j_m\,.\,j)\,\psi_k(h.j_m\,.\,j)\right) - \left(\mu(j_m^{-1}\,.\,V_m)\,[\otimes_{\gamma(\eta)}M'_{l'}](h.j_m)\,\psi_k(h.j_m)\right)\right| \\
&< \epsilon_3\,.
\end{aligned}$$

Therefore, for any $k \leqslant K$, any $l \leqslant L$ and any $l' \leqslant L'$, we get:

$$\begin{aligned}
&\left|\langle \psi_k \mid \otimes_{\gamma(\eta)}M_l\rangle\,\langle \otimes_{\gamma(\eta)}M'_{l'} \mid \psi_k\rangle - \sum_{m=1}^{M}\mu(V_m)\left\langle \psi_k \middle| \widehat{\mathrm{h}_\eta^{(M_l)}}\,\widehat{\mathrm{T}_\eta^{(j_m)}}\,\widehat{\mathrm{h}_\eta^{(M'_{l'})}} \middle| \psi_k\right\rangle\right| \\
&\leqslant \left|\int_{\mathcal{C}_\eta} d\mu(h)\,\psi_k^*(h)\,[\otimes_{\gamma(\eta)}M_l](h)\,\left(\langle \otimes_{\gamma(\eta)}M'_{l'} \mid \psi_k\rangle - \sum_{m=1}^{M}\mu(V_m)\left[\widehat{\mathrm{T}_\eta^{(j_m)}}\,\widehat{\mathrm{h}_\eta^{(M'_{l'})}}\,\psi_k\right](h)\right)\right| \\
&< \epsilon_3\,\|\psi_k\|_\infty\,\|\otimes_{\gamma(\eta)}M_l\|_\infty\,.
\end{aligned}$$

Now, using eqs. (12.10.1) to (12.10.3) together with the definition of ϵ_3, this implies:

$$\left|\langle \varphi' \mid \rho \mid \varphi\rangle - \sum_{l=1}^{L}\sum_{l'=1}^{L'}\sum_{m=1}^{M} {\mu'_{l'}}^{*}\,\mu_l\,\mu(V_m)\,\mathrm{Tr}_{\mathcal{H}_\eta}\left(\rho\,\widehat{\mathrm{h}_\eta^{(M_l)}}\,\widehat{\mathrm{T}_\eta^{(j_m)}}\,\widehat{\mathrm{h}_\eta^{(M'_{l'})}}\right)\right| < 5\,\epsilon_o\,,$$

and the same holds for ρ'.

Finally, if ρ, ρ' are such that:

$$\forall A \in \mathcal{I}_\eta\,,\ \mathrm{Tr}_{\mathcal{H}_\eta}\, \rho\, A = \mathrm{Tr}_{\mathcal{H}_\eta}\, \rho' A\,,$$

we thus have:

$$\forall \varphi,\, \varphi' \in \mathcal{H}_\eta\,,\ \forall \epsilon > 0,\ \left|\langle \varphi' \mid \rho \mid \varphi \rangle - \langle \varphi' \mid \rho' \mid \varphi \rangle\right| < \epsilon\,,$$

and therefore $\rho = \rho'$. □

We can now formulate the relation between the inductive construction just reviewed and the projective construction from section 10 and subsection 12.1, by displaying how an arbitrary state over $\mathcal{H}_{\text{AL}}$ can be unambiguously identified with a projective family of density matrices over the Hilbert spaces $\mathcal{H}_\eta$. Note that, as stressed above, we are not merely stating that there exists *some* injective map σ between the state spaces (which would only be an assertion about the respective cardinalities, with very little physical content): we also make sure that the mapping of the states considered here intertwines the evaluation of observables in agreement with their physical interpretation.

By using prop. 6.5 (which is itself a straightforward application of the more general result in theorem 5.9), we first obtain a map from the state space of an inductive limit of Hilbert spaces built over the label set $\mathcal{L}_{\text{HF}}$. As previously announced, the insensitivity of the injections with respect to the faces in each label then allows to collapse this inductive system into a simpler one, built over a subset of $\mathcal{L}_{\text{graphs}}$ (namely those analytic graphs that are the underlying graph of some label). Finally, since this set of graphs is cofinal in $\mathcal{L}_{\text{AL}}$ (as follows from $\mathcal{L}_{\text{HF}}$ being cofinal in $\mathcal{L}_{\text{graphs}} \times \mathcal{L}_{\text{profls}}$ and $\mathcal{L}_{\text{graphs}}$ in $\mathcal{L}_{\text{AL}}$), the corresponding inductive limit can be identified with $\mathcal{H}_{\text{AL}}$.

Theorem 12.11 There exist maps $\sigma : \overline{\mathcal{S}}_{\text{AL}} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}_{\text{HF}},\mathcal{H},\Phi)}$ and $\alpha : \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}_{\text{HF}},\mathcal{H},\Phi)} \to \mathcal{A}_{\text{AL}}$ (where $\overline{\mathcal{S}}_{\text{AL}}$ is the space of self-adjoint, positive semi-definite, traceclass operators over $\mathcal{H}_{\text{AL}}$, $\mathcal{A}_{\text{AL}}$ is the space of bounded operators on $\mathcal{H}_{\text{AL}}$, and $\overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}_{\text{HF}},\mathcal{H},\Phi)}$ and $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}_{\text{HF}},\mathcal{H},\Phi)}$ were defined respectively in def. 5.2 and prop. 5.4) such that:

1. α is a C^*-algebra morphism;
2. for any $e \in \mathcal{L}_{\text{edges}}$ and any $m \in C^\infty(G, \mathbb{R})$, $\alpha\left(\widehat{\mathrm{h}^{(e,m)}}\right) = \widehat{\mathrm{h}^{(e,m)}_{\text{AL}}}$, while for any $\overline{F} \in \mathcal{L}_{\text{faces}}$ and any $t \in G$, $\alpha\left(\widehat{\mathrm{T}^{(\overline{F},t)}}\right) = \widehat{\mathrm{T}^{(\overline{F},t)}_{\text{AL}}}$;
3. for any $\rho \in \overline{\mathcal{S}}_{\text{AL}}$ and any $A \in \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}_{\text{HF}},\mathcal{H},\Phi)}$, $\mathrm{Tr}_{\mathcal{H}_{\text{AL}}}\left(\rho\, \alpha(A)\right) = \mathrm{Tr}\left(\sigma(\rho)\, A\right)$;
4. σ is an injective map;
5. $\sigma\left\langle \mathcal{S}_{\text{AL}} \right\rangle = \left\{ \left(\rho_\eta\right)_{\eta \in \mathcal{L}_{\text{HF}}} \;\middle|\; \sup_{\eta \in \mathcal{L}_{\text{HF}}} \inf_{\eta' \succcurlyeq \eta} \mathrm{Tr}_{\mathcal{H}_{\eta'}}\, \rho_{\eta'}\, \Theta_{\eta'|\eta} = \mathrm{Tr}\, \rho = 1 \right\}$, where $\mathcal{S}_{\text{AL}}$ is the space of density matrices over $\mathcal{H}_{\text{AL}}$ and the bounded operator $\Theta_{\eta'|\eta}$ is defined on $\mathcal{H}_{\eta'}$ by:

$$\forall \psi \in \mathcal{H}_{\eta'}\,,\ \forall j \in \mathcal{C}_{\eta' \to \eta}\,,\ \forall h \in \mathcal{C}_\eta\,,\ \left[\Theta_{\eta'|\eta}\, \psi\right] \circ \varphi^{-1}_{\eta' \to \eta}(j,\, h) := \int_{\mathcal{C}_{\eta' \to \eta}} d\mu_{\eta' \to \eta}(\widetilde{j})\ \ \psi \circ \varphi^{-1}_{\eta' \to \eta}(\widetilde{j},\, h)\,.$$

Proof *Auxiliary inductive limit of Hilbert spaces.* For any $\eta \preccurlyeq \eta' \in \mathcal{L}_{\text{HF}}$, we define the map $\tau_{\eta'\leftarrow\eta} : \mathcal{H}_\eta \to \mathcal{H}_{\eta'}$ by:

$$\forall \psi \in \mathcal{H}_\eta \,, \forall h_{\eta'} \in \mathcal{C}_{\eta'} \,,$$

$$\big[\tau_{\eta'\leftarrow\eta}(\psi)\big](h_{\eta'}) := \psi \left(e \mapsto \left[\prod_{k=2}^{n_{\eta'\to\eta,e}} \left[h_{\eta'} \circ a_{\eta'\to\eta,e}(k)\right]^{\epsilon_{\eta'\to\eta,e}(k)} \right] . \left[h_{\eta'} \circ a_{\eta'\to\eta,e}(1)\right] \right) .$$

As was shown in prop. 6.5, $\tau_{\eta'\leftarrow\eta}$ is well-defined, and $\left(\mathcal{L}_{\text{HF}}, \left(\mathcal{H}_\eta\right)_{\eta\in\mathcal{L}_{\text{HF}}}, \left(\tau_{\eta'\leftarrow\eta}\right)_{\eta\preccurlyeq\eta'}\right)$ is an inductive system of Hilbert spaces whose limit we denote by $\mathcal{H}_{\widetilde{\text{AL}}}$. For any $\eta \in \mathcal{L}_{\text{HF}}$, we call $\tau_{\widetilde{\text{AL}}\leftarrow\eta}$ the natural injection of $\mathcal{H}_\eta$ into $\mathcal{H}_{\widetilde{\text{AL}}}$. Also by prop. 6.5, there exist maps $\widetilde{\sigma} : \overline{\mathcal{S}}_{\widetilde{\text{AL}}} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}_{\text{HF}},\mathcal{H},\Phi)}$ and $\widetilde{\alpha} : \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}_{\text{HF}},\mathcal{H},\Phi)} \to \mathcal{A}_{\widetilde{\text{AL}}}$ (where $\overline{\mathcal{S}}_{\widetilde{\text{AL}}}$, resp. $\mathcal{A}_{\widetilde{\text{AL}}}$, denote the space of non-negative traceclass operators, resp. of bounded operators, over $\mathcal{H}_{\widetilde{\text{AL}}}$) satisfying:

$$\forall \rho \in \overline{\mathcal{S}}_{\widetilde{\text{AL}}} \,, \forall A \in \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}_{\text{HF}},\mathcal{H},\Phi)} \,, \operatorname{Tr}_{\mathcal{H}_{\widetilde{\text{AL}}}}\big(\rho\, \widetilde{\alpha}(A)\big) = \operatorname{Tr}\big(\widetilde{\sigma}(\rho)\, A\big) .$$

Moreover, $\widetilde{\sigma}$ is injective and:

$$\widetilde{\sigma}\left\langle \mathcal{S}_{\widetilde{\text{AL}}} \right\rangle = \left\{ \left(\rho_\eta\right)_{\eta\in\mathcal{L}_{\text{HF}}} \;\middle|\; \sup_{\eta\in\mathcal{L}_{\text{HF}}} \inf_{\eta'\succcurlyeq\eta} \int_{\mathcal{C}_{\eta'\to\eta}\times\mathcal{C}_{\eta'\to\eta}} d^{(2)}\mu_{\eta'\to\eta}(j,\, j') \int_{\mathcal{C}_\eta} d\mu_\eta(h) \;\; \rho_{\eta'}\left(\varphi^{-1}_{\eta'\to\eta}(j,h),\, \varphi^{-1}_{\eta'\to\eta}(j',h)\right) = 1 \right\},$$

where $\mathcal{S}_{\widetilde{\text{AL}}}$ is the space of density matrices over $\mathcal{H}_{\widetilde{\text{AL}}}$ and, for any density matrix ρ_η over $\mathcal{H}_\eta$, $\rho_\eta(\,\cdot\,,\,\cdot\,)$ denotes the integral kernel of ρ_η.

To further specify $\widetilde{\alpha}$, we now fetch its explicit definition from the proof of theorem 5.9. First, we can define, for any $\eta \in \mathcal{L}_{\text{HF}}$, an Hilbert space $\mathcal{H}_{\widetilde{\text{AL}}\to\eta}$, and an Hilbert space isomorphism $\Phi_{\widetilde{\text{AL}}\to\eta} : \mathcal{H}_{\widetilde{\text{AL}}} \to \mathcal{H}_{\widetilde{\text{AL}}\to\eta} \otimes \mathcal{H}_\eta$. $\mathcal{H}_{\widetilde{\text{AL}}\to\eta}$ is given as an inductive limit and we have, for any $\kappa \succcurlyeq \eta \in \mathcal{L}_{\text{HF}}$, a natural isometric injection $\tau_{\widetilde{\text{AL}}\leftarrow\kappa\to\eta} : \mathcal{H}_{\kappa\to\eta} \to \mathcal{H}_{\widetilde{\text{AL}}\to\eta}$ satisfying:

$$(\tau_{\widetilde{\text{AL}}\leftarrow\kappa\to\eta} \otimes \mathrm{id}_{\mathcal{H}_\eta}) \circ \Phi_{\kappa\to\eta} = \Phi_{\widetilde{\text{AL}}\to\eta} \circ \tau_{\widetilde{\text{AL}}\leftarrow\kappa} .$$

Then, $\widetilde{\alpha}$ is the C^*-algebra morphism $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}_{\text{HF}},\mathcal{H},\Phi)} \to \mathcal{A}_{\widetilde{\text{AL}}}$ such that, for any $\eta \in \mathcal{H}_\eta$ and any bounded operator A_η on $\mathcal{H}_\eta$:

$$\widetilde{\alpha}\left([A_\eta]_\sim\right) = \Phi^{-1}_{\widetilde{\text{AL}}\to\eta} \circ \left(\mathrm{id}_{\mathcal{H}_{\widetilde{\text{AL}}\to\eta}} \otimes A_\eta\right) \circ \Phi_{\widetilde{\text{AL}}\to\eta} ,$$

where $[A_\eta]_\sim$ denotes the equivalence class of A_η in $\mathcal{A}^{\otimes}_{(\mathcal{L}_{\text{HF}},\mathcal{H},\Phi)}$ (eq. (5.3.2)). In particular, for any $\kappa \succcurlyeq \eta$, we have:

$$\widetilde{\alpha}\left([A_\eta]_\sim\right) \circ \tau_{\widetilde{\text{AL}}\leftarrow\kappa} = \tau_{\widetilde{\text{AL}}\leftarrow\kappa} \circ \Phi^{-1}_{\kappa\to\eta} \circ \left(\mathrm{id}_{\mathcal{H}_{\kappa\to\eta}} \otimes A_\eta\right) \circ \Phi_{\kappa\to\eta} .$$

Identification of $\mathcal{H}_{\widetilde{\text{AL}}}$ *with* $\mathcal{H}_{\text{AL}}$. For any $\eta \in \mathcal{L}_{\text{HF}}$, we define $\Gamma_{\eta\to\gamma(\eta)} : \mathcal{H}_\eta \to \mathcal{H}_{\gamma(\eta)}$ as the identity map on $\mathcal{H}_\eta = L_2(\mathcal{C}_\eta\,, d\mu_\eta) = L_2(\mathcal{C}_{\gamma(\eta)}\,, d\mu_{\gamma(\eta)}) = \mathcal{H}_{\gamma(\eta)}$. Then, for any $\eta \preccurlyeq \eta' \in \mathcal{L}_{\text{HF}}$, we have $\gamma(\eta) \preccurlyeq \gamma(\eta')$ and (from props. 10.20 and 12.5):

$$\Gamma_{\eta'\to\gamma(\eta')} \circ \tau_{\eta'\leftarrow\eta} = \tau_{\gamma(\eta')\leftarrow\gamma(\eta)} \circ \Gamma_{\eta\to\gamma(\eta)} .$$

Thus, there exists an isometric injection $\Gamma_{\widetilde{\text{AL}}\to\text{AL}} : \mathcal{H}_{\widetilde{\text{AL}}} \to \mathcal{H}_{\text{AL}}$ satisfying:

$$\forall \eta \in \mathcal{L}_{\text{HF}},\, \Gamma_{\widetilde{\text{AL}}\to\text{AL}} \circ \tau_{\widetilde{\text{AL}}\leftarrow\eta} = \tau_{\text{AL}\leftarrow\gamma(\eta)} \circ \Gamma_{\eta\to\gamma(\eta)} .$$

Moreover, $\left\{\gamma \in \mathcal{L}_{\text{AL}} \,\middle|\, \exists \eta \in \mathcal{L}_{\text{HF}} \,/\, \gamma(\eta) = \gamma\right\}$ is cofinal in $\mathcal{L}_{\text{AL}}$ (from subsection 10.2 and prop. 12.4), hence:

$$\bigcup_{\eta \in \mathcal{L}_{\text{HF}}} \Gamma_{\widetilde{\text{AL}} \to \text{AL}} \circ \tau_{\widetilde{\text{AL}} \leftarrow \eta} \left\langle \mathcal{H}_\eta \right\rangle = \bigcup_{\gamma \in \mathcal{L}_{\text{AL}}} \tau_{\text{AL} \leftarrow \gamma} \left\langle \mathcal{H}_\gamma \right\rangle ,$$

is dense in $\mathcal{H}_{\text{AL}}$, and therefore $\Gamma_{\widetilde{\text{AL}} \to \text{AL}}$ is an Hilbert space isomorphism.

Now, we define, for any $\rho \in \overline{\mathcal{S}}_{\text{AL}}$:

$$\sigma(\rho) := \widetilde{\sigma}\left(\Gamma^{-1}_{\widetilde{\text{AL}} \to \text{AL}} \circ \rho \circ \Gamma_{\widetilde{\text{AL}} \to \text{AL}}\right),$$

and for any $A \in \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}_{\text{HF}}, \mathcal{H}, \Phi)}$:

$$\alpha(A) := \Gamma_{\widetilde{\text{AL}} \to \text{AL}} \circ \widetilde{\alpha}(A) \circ \Gamma^{-1}_{\widetilde{\text{AL}} \to \text{AL}} .$$

The points 12.11.1, 12.11.3 and 12.11.4 follow from the corresponding properties of $\widetilde{\sigma}$ and $\widetilde{\alpha}$. Moreover, we have, for any $\eta \in \mathcal{L}_{\text{HF}}$ and any bounded operator A_η on $\mathcal{H}_\eta$:

$$\alpha\left([A_\eta]_\sim\right) \circ \tau_{\text{AL} \leftarrow \gamma(\eta)} = \tau_{\text{AL} \leftarrow \gamma(\eta)} \circ \Gamma_{\eta \to \gamma(\eta)} \circ A_\eta \circ \Gamma^{-1}_{\eta \to \gamma(\eta)} . \tag{12.11.1}$$

Let $e \in \mathcal{L}_{\text{edges}}$, $m \in C^\infty(G, \mathbb{R})$ and $\eta \in \mathcal{L}_{\text{HF}}$ such that $\gamma(\eta) \in \mathcal{L}_{\text{graphs}/e}$. Then, $\gamma(\eta) \in \mathcal{L}_{\text{AL}/e}$ and, using again props. 10.20 and 12.5, we can check that:

$$\Gamma_{\eta \to \gamma(\eta)} \circ \widehat{\mathrm{h}^{(e,m)}_\eta} \circ \Gamma^{-1}_{\eta \to \gamma(\eta)} = \widehat{\mathrm{h}^{(e,m)}_{\gamma(\eta)}} ,$$

hence:

$$\alpha\left(\widehat{\mathrm{h}^{(e,m)}}\right) \circ \tau_{\text{AL} \leftarrow \gamma(\eta)} = \tau_{\text{AL} \leftarrow \gamma(\eta)} \circ \widehat{\mathrm{h}^{(e,m)}_{\gamma(\eta)}} = \widehat{\mathrm{h}^{(e,m)}_{\text{AL}}} \circ \tau_{\text{AL} \leftarrow \gamma(\eta)} .$$

Now, α is a C^*-algebra isomorphism, so in particular an isometry, and $\left\{\gamma(\eta) \,\middle|\, \eta \in \mathcal{L}_{\text{HF}} \,/\, \gamma(\eta) \in \mathcal{L}_{\text{graphs}/e}\right\}$ is cofinal in $\mathcal{L}_{\text{AL}}$ so that:

$$\bigcup_{\eta \in \mathcal{L}_{\text{HF}} \,/\, \gamma(\eta) \in \mathcal{L}_{\text{graphs}/e}} \tau_{\text{AL} \leftarrow \gamma(\eta)} \left\langle \mathcal{H}_{\gamma(\eta)} \right\rangle = \bigcup_{\gamma \in \mathcal{L}_{\text{AL}}} \tau_{\text{AL} \leftarrow \gamma} \left\langle \mathcal{H}_\gamma \right\rangle$$

is dense in $\mathcal{H}_{\text{AL}}$. Therefore, $\alpha\left(\widehat{\mathrm{h}^{(e,m)}}\right) = \widehat{\mathrm{h}^{(e,m)}_{\text{AL}}}$.

Next, let $\overline{F} \in \mathcal{L}_{\text{faces}}$, $t \in G$ and $\eta \in \mathcal{L}_{\text{HF}}$ such that $\lambda(\eta) \in \mathcal{L}_{\text{profls}/\overline{F}}$. From eq. (10.21.1) and prop. 10.6, we get:

$$\forall e \in \gamma(\eta), \forall p \neq p' \in r(e), \left(e_{[p,p']} \in \overline{F} \Leftrightarrow \left[p = b(e) \;\;\&\;\; \chi_\eta(e) \in H_{\lambda(\eta) \to \overline{F}}\right]\right).$$

Hence, $\gamma(\eta) \in \mathcal{L}_{\text{AL}/\overline{F}}$ and, using eq. (10.28.2):

$$\forall h \in \mathcal{C}_\eta = \mathcal{C}_{\gamma(\eta)} ,\; T^{(\overline{F})}_{\gamma(\eta),t} h = R^{(\overline{F})}_{\eta,t} h .$$

Thus, we get:

$$\Gamma_{\eta \to \gamma(\eta)} \circ \widehat{\mathrm{T}^{(\overline{F},t)}_\eta} \circ \Gamma^{-1}_{\eta \to \gamma(\eta)} = \widehat{\mathrm{T}^{(\overline{F},t)}_{\gamma(\eta)}} .$$

Like above, this ensures that $\alpha\left(\widehat{\mathrm{T}^{(\overline{F},t)}}\right) = \widehat{\mathrm{T}^{(\overline{F},t)}_{\text{AL}}}$, for $\left\{\gamma(\eta) \,\middle|\, \eta \in \mathcal{L}_{\text{HF}} \,/\, \lambda(\eta) \in \mathcal{L}_{\text{profls}/\overline{F}}\right\}$ is cofinal in $\mathcal{L}_{\text{AL}}$.

Finally, for any $\eta \preccurlyeq \eta' \in \mathcal{L}_{\text{HF}}$, we define $\zeta_{\eta' \to \eta} : \mathcal{C}_{\eta' \to \eta} \to \mathbb{C}$ by:

$$\forall j \in \mathcal{C}_{\eta'\to\eta}\,,\ \zeta_{\eta'\to\eta}(j) = 1\,.$$

Observe that $\|\zeta_{\eta'\to\eta}\|_{\mathcal{H}_{\eta'\to\eta}} = \mu_{\eta'\to\eta}\left(\mathcal{C}_{\eta'\to\eta}\right) = \mu_{\eta'}\left(\mathcal{C}_{\eta'}\right) / \mu_{\eta}\left(\mathcal{C}_{\eta}\right) = 1$. Hence, the operator $\Theta_{\eta'|\eta}$ is well-defined and bounded on $\mathcal{H}_{\eta'}$, for it is given by:

$$\Theta_{\eta'|\eta} = \Phi^{-1}_{\eta'\to\eta} \circ \left(|\zeta_{\eta'\to\eta}\,\rangle\langle\,\zeta_{\eta'\to\eta}| \otimes \mathrm{id}_{\mathcal{H}_\eta}\right) \circ \Phi_{\eta'\to\eta}\,. \tag{12.11.2}$$

And, since for any non-negative traceclass operator $\rho_{\eta'}$ on $\mathcal{H}_{\eta'}$ we have:

$$\int_{\mathcal{C}_{\eta'\to\eta}\times\mathcal{C}_{\eta'\to\eta}} d^{(2)}\mu_{\eta'\to\eta}(j,j') \int_{\mathcal{C}_\eta\times\mathcal{C}_\eta} d^{(2)}\mu_\eta(h,h')\,\delta_{\mu_\eta}(h,h')\,\rho_{\eta'}\left(\varphi^{-1}_{\eta'\to\eta}(j,h),\,\varphi^{-1}_{\eta'\to\eta}(j',h')\right) =$$
$$= \sum_{\psi\in\mathrm{ONB}_\eta} \left\langle \Phi^{-1}_{\eta'\to\eta}(\zeta_{\eta'\to\eta}\otimes\psi)\,\middle|\,\rho_{\eta'}\,\middle|\,\Phi^{-1}_{\eta'\to\eta}(\zeta_{\eta'\to\eta}\otimes\psi)\right\rangle,$$

(with ONB_η some orthonormal basis of $\mathcal{H}_\eta$), we get:

$$\sigma\left\langle \mathcal{S}_{\mathrm{AL}}\right\rangle = \widetilde{\sigma}\left\langle \mathcal{S}_{\widetilde{\mathrm{AL}}}\right\rangle = \left\{ \left(\rho_\eta\right)_{\eta\in\mathcal{L}_{\mathrm{HF}}} \,\middle|\, \sup_{\eta\in\mathcal{L}_{\mathrm{HF}}} \inf_{\eta'\succcurlyeq\eta} \mathrm{Tr}_{\mathcal{H}_{\eta'}}\,\rho_{\eta'}\,\Theta_{\eta'|\eta} = 1 \right\}.$$

□

The injective map σ allows to identify the space $\mathcal{S}_{\mathrm{AL}}$ of states over $\mathcal{H}_{\mathrm{AL}}$ with a certain subset in the space $\mathcal{S}^{\otimes}_{(\mathcal{L}_{\mathrm{HF}},\mathcal{H},\Phi)}$ of all projective states. The results below suggests that $\mathcal{S}^{\otimes}_{(\mathcal{L}_{\mathrm{HF}},\mathcal{H},\Phi)}$ can even be thought as a closure of $\mathcal{S}_{\mathrm{AL}}$. Indeed, $\mathcal{S}^{\otimes}_{(\mathcal{L}_{\mathrm{HF}},\mathcal{H},\Phi)}$ is complete, in the sense that a net of projective states admits a limit as soon as it converges over each $\mathcal{H}_\eta$, and $\sigma\left\langle\mathcal{S}_{\mathrm{AL}}\right\rangle$ is dense in the same sense, namely the restrictions of its elements over each $\mathcal{H}_\eta$ fill a dense subset of the associated space $\mathcal{S}_\eta$ of density matrices (in fact, they fill all $\mathcal{S}_\eta$). This is somewhat reminiscent of the Fell's theorem [29], but, while the Fell's theorem tells us that $\mathcal{S}_{\mathrm{AL}}$ is dense in the space of all states [41, part III, def. 2.2.8] over the C^*-algebra $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}_{\mathrm{HF}},\mathcal{H},\Phi)}$ (prop. 5.4), with respect to the weakest topology that makes the evaluation maps $\rho \mapsto \mathrm{Tr}\,\rho A$ continuous, we show here that $\mathcal{S}_{\mathrm{AL}}$ is dense in the (presumably smaller) set of all projective states with respect to a much stronger topology.

Proposition 12.12 For any $\eta\in\mathcal{L}_{\mathrm{HF}}$, we define on $\mathcal{S}^{\otimes}_{(\mathcal{L}_{\mathrm{HF}},\mathcal{H},\Phi)}$ (def. 5.2) the semi-metric (aka. possibly degenerate metric, see [25, section IX.10]) $d^{(\eta)}$ by:

$$\forall\left(\rho_{\eta'}\right)_{\eta'\in\mathcal{L}_{\mathrm{HF}}},\ \left(\rho'_{\eta'}\right)_{\eta'\in\mathcal{L}_{\mathrm{HF}}} \in \mathcal{S}^{\otimes}_{(\mathcal{L}_{\mathrm{HF}},\mathcal{H},\Phi)},\quad d^{(\eta)}\left[\left(\rho_{\eta'}\right)_{\eta'\in\mathcal{L}_{\mathrm{HF}}},\ \left(\rho'_{\eta'}\right)_{\eta'\in\mathcal{L}_{\mathrm{HF}}}\right] := \left\|\rho_\eta - \rho'_\eta\right\|_1,$$

where $\|\cdot\|_1$ denotes the trace norm on the space of traceclass operators over $\mathcal{H}_\eta$ (see lemma 5.10 and/or [82]).

The family of semi-metrics $\left(d^{(\eta)}\right)_{\eta\in\mathcal{L}_{\mathrm{HF}}}$ equips $\mathcal{S}^{\otimes}_{(\mathcal{L}_{\mathrm{HF}},\mathcal{H},\Phi)}$ with the structure of a *complete* uniform space [25, sections IX.11 and XIV.9], and $\sigma\left\langle\mathcal{S}_{\mathrm{AL}}\right\rangle$ is dense in the induced topology.

Proof For any η, the space $\mathcal{S}_\eta$ of density matrices over $\mathcal{H}_\eta$ equipped with the metric $d^{(\eta)}_\eta$ defined by:

$$\forall \rho_\eta\,,\ \rho_{\eta'} \in \mathcal{S}_\eta\,,\quad d^{(\eta)}_\eta\left[\rho_\eta\,,\ \rho_{\eta'}\right] := \left\|\rho_\eta - \rho'_\eta\right\|_1,$$

is complete (for the traceclass operators over $\mathcal{H}_\eta$ form, with respect to the trace norm, a Banach space [82] in which $\mathcal{S}_\eta$ is a closed subset). Hence, $\prod_{\eta \in \mathcal{L}_{HF}} \mathcal{S}_\eta$ has a structure of complete and Hausdorff uniform space as a Cartesian product of complete metric spaces [25, theorem XIV.9.4].

Let $\eta \preccurlyeq \eta' \in \mathcal{L}_{HF}$. For any self-adjoint traceclass operator δ on $\mathcal{H}_{\eta'}$, we have, writing $\delta = \delta^{(+)} - \delta^{(-)}$ with $\delta^{(+)}$ and $(-\delta^{(-)})$ respectively the positive and negative parts of δ:

$$\begin{aligned} \left\| \mathrm{Tr}_{\eta' \to \eta} \delta \right\|_1 &\leqslant \left\| \mathrm{Tr}_{\eta' \to \eta} \delta^{(+)} \right\|_1 + \left\| \mathrm{Tr}_{\eta' \to \eta} \delta^{(-)} \right\|_1 \\ &= \mathrm{Tr}_{\mathcal{H}_\eta} \mathrm{Tr}_{\eta' \to \eta} \delta^{(+)} + \mathrm{Tr}_{\mathcal{H}_\eta} \mathrm{Tr}_{\eta' \to \eta} \delta^{(-)} \\ &= \mathrm{Tr}_{\mathcal{H}_{\eta'}} \delta^{(+)} + \mathrm{Tr}_{\mathcal{H}_{\eta'}} \delta^{(-)} = \left\| \delta \right\|_1 . \end{aligned}$$

Thus, $\mathrm{Tr}_{\eta' \to \eta}$ is a continuous map between the metric spaces $\left(\mathcal{S}_{\eta'}\,,\, d^{(\eta')}_{\eta'}\right)$ and $\left(\mathcal{S}_{\eta}\,,\, d^{(\eta)}_{\eta}\right)$, so its graph is closed in their Cartesian product. The projective limit $\mathcal{S}^{\otimes}_{(\mathcal{L}_{HF},\mathcal{H},\Phi)}$ is therefore a closed subset of $\prod_{\eta \in \mathcal{L}_{HF}} \mathcal{S}_\eta$, hence inherits a structure of complete and Hausdorff uniform space, which is precisely the one induced by the family of semi-metrics $\left(d^{(\eta)}\right)_{\eta \in \mathcal{L}_{HF}}$.

Let $\rho = \left(\rho_\eta\right)_{\eta \in \mathcal{L}_{HF}} \in \mathcal{S}^{\otimes}_{(\mathcal{L}_{HF},\mathcal{H},\Phi)}$. For any $\eta \in \mathcal{L}_{HF}$, we define $\rho^{(\eta)} \in \mathcal{S}_{AL}$ by:

$$\rho^{(\eta)} := \tau_{AL \leftarrow \gamma(\eta)} \circ \Gamma_{\eta \to \gamma(\eta)} \circ \rho_\eta \circ \Gamma^{-1}_{\eta \to \gamma(\eta)} \circ \tau^{+}_{AL \leftarrow \gamma(\eta)} .$$

From theorem 12.11.3 and eq. (12.11.1), we have:

$$\forall A_\eta \in \mathcal{A}_\eta\,,\ \mathrm{Tr}_{\mathcal{H}_\eta} \left(\left[\sigma(\rho^{(\eta)})\right]_\eta A_\eta \right) = \mathrm{Tr}_{\mathcal{H}_{AL}} \left(\rho^{(\eta)}\, \alpha \left([A_\eta]_\sim \right) \right) = \mathrm{Tr}_{\mathcal{H}_\eta} \left(\rho_\eta A_\eta \right),$$

hence $\left[\sigma\left(\rho^{(\eta)}\right)\right]_\eta = \rho_\eta$. Thus, the net $\left(\sigma\left(\rho^{(\kappa)}\right)\right)_{\kappa \in \mathcal{L}_{HF}}$ converges to ρ with respect to the family of semimetrics $\left(d^{(\eta)}\right)_{\eta \in \mathcal{L}_{HF}}$ (like in the proof of prop. 2.7). Therefore, $\sigma \left\langle \mathcal{S}_{AL} \right\rangle$ is dense in $\mathcal{S}^{\otimes}_{(\mathcal{L}_{HF},\mathcal{H},\Phi)}$ for the corresponding topology. □

Finally, we want to check that the projective quantum state space $\mathcal{S}^{\otimes}_{(,\mathcal{H},\Phi)}$ is not a mere rewriting of the space of density matrices over $\mathcal{H}_{AL}$, ie. that σ, while being injective, is *not* surjective, and yields a *strict* embedding of $\mathcal{S}_{AL}$ in $\mathcal{S}^{\otimes}_{(,\mathcal{H},\Phi)}$. To exhibit a projective state that cannot be realized as a density matrix on $\mathcal{H}_{AL}$, we will, in line with the previous result, consider a sequence of states over $\mathcal{H}_{AL}$, which, although it does not converge in $\mathcal{S}_{AL}$, does converge in $\mathcal{S}^{\otimes}_{(\mathcal{L}_{HF},\mathcal{H},\Phi)}$.

To this intend, we cut some analytic edge e into infinitely many pieces (with an accumulation point at one extremity of the edge) and we denote by $\psi^{(n)}$ the state in $\mathcal{H}_{AL}$ that assigns to the first n pieces a given, non-trivial, sample wave-function χ. It can then be shown that the sequence $\left[\sigma\left(\left|\psi^{(n)}\right\rangle\left\langle\psi^{(n)}\right|\right)\right]_\eta$ (as a sequence in n, with η held fixed) is eventually constant: indeed the evaluations on $\psi^{(n)}$ of the observables covered by the label η freeze as soon as n exceeds a certain (η-dependent) threshold n_η (fig. 12.1), and we have proved in prop. 12.10 that these evaluations completely specify a state over $\mathcal{H}_\eta$.

Next, we need to confirm that the thus constructed projective state is not in the image of σ. Let us look closer at the characterization of this image given in theorem 12.11.5. The orthogonal projection $\Theta_{\eta'|\eta}$ selects in $\mathcal{H}_{\eta'}$ those states that do not depend on the complementary variables

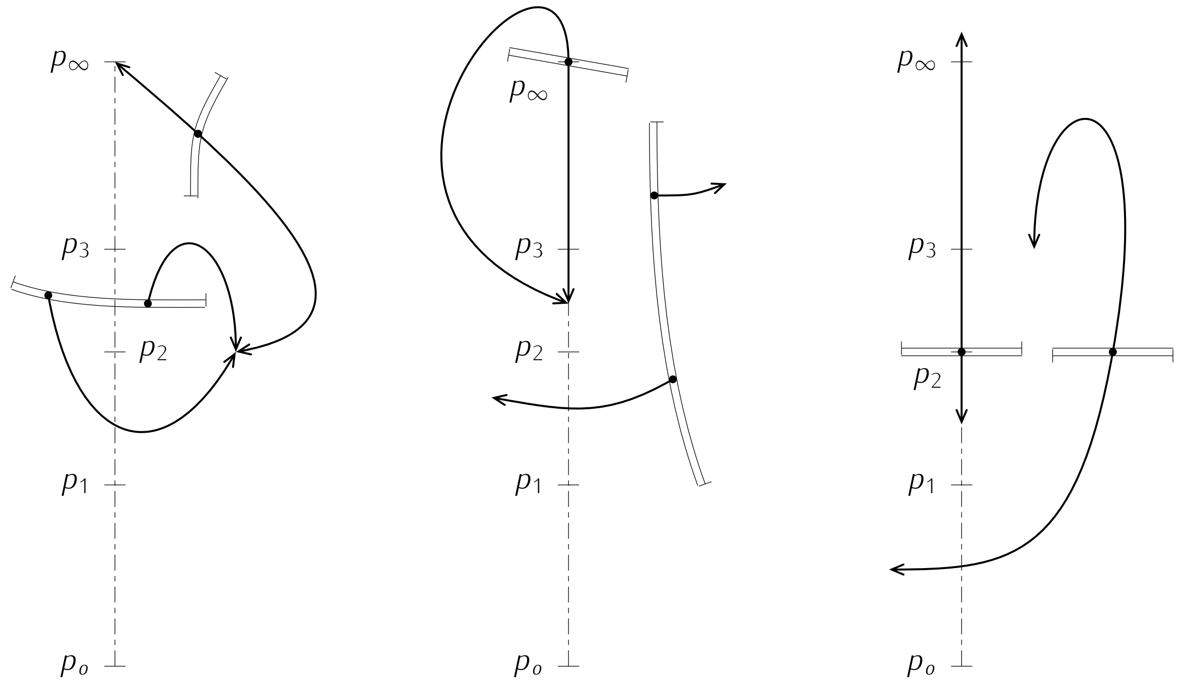


Figure 12.1 – A few labels having 3 as the smallest possible choice for n_η (the dashed line indicates the base edge e)

$\eta' \to \eta$. Letting the *upper* label η' become infinitely fine corresponds, in $\mathcal{H}_{\text{AL}}$, to the orthogonal projection on the fixed-graph subspace $\mathcal{H}_{\gamma(\eta)}$. Thus, if we now let the *lower* label η get finer and finer, we will recover the state we started from, provided it was a state on $\mathcal{H}_{\text{AL}}$ to begin with. By contrast, for a state that is *not* realizable on $\mathcal{H}_{\text{AL}}$, chopping those parts of the state that depart from the Ashtekar-Lewandowski vacuum over the $\eta' \to \eta$ degrees of freedom and taking first the net limit on η', we loose a significant part of the state, no matter how fine η: such states are not just excitations around the AL-vacuum, but differ from it all the way down to infinitely fine labels.

Now, for any label η and any integer M, there exists a set of labels $\eta' \succcurlyeq \eta$, which is cofinal in $\mathcal{L}_{\text{HF}}$ and such that, for each η' in this set, the holonomies along M distinct pieces of e can be discovered among the complementary variables $\eta' \to \eta$. Since the state we are considering here attributes to each such piece a distribution χ distinct from the uniform one, its agreement with the AL-vacuum, as far as the $\eta' \to \eta$ degrees of freedom are concerned, can thus be bounded by an exponentially decreasing function of M. Taking first the limit on finer and finer η', we can then let M goes to infinity, so that the overlap, over the degrees of freedom beyond η, between this state and the AL-vacuum is actually zero. If it would be a state in $\mathcal{S}_{\text{AL}}$ its projection on any finite-graph subspace $\mathcal{H}_{\gamma(\eta)}$ should therefore vanish, in contradiction with the state having unit trace.

Proposition 12.13 The map σ is *not* surjective.

Proof *A family of states in $\mathcal{H}_{\text{AL}}$.* Let $e \in \mathcal{L}_{\text{edges}}$ and let $\breve{e} : U \to V$ be a representative of e. We define $p_o = b(e)$, $p_\infty = f(e)$ and, for any $n \geqslant 1$:

$$p_n = \breve{e}\left(\frac{n}{n+1}, 0\right).$$

For any $n \geqslant 1$, we define:

$$\gamma^{(n)} := \left\{ e_{[p_{k-1}\,,\,p_k]} \;\middle|\; 1 \leqslant k \leqslant n \right\} \in \mathcal{L}_{\text{AL}}\,.$$

Next, we choose $\chi \in L_2(G, d\mu)$ such that $\|\chi\| = 1$ and $\int_G d\mu(g)\, |\chi(g)| < 1$ (for example, we can take $\chi := 1/\sqrt{v}\,\mathbb{1}_V$ where $\mathbb{1}_V$ is the indicator function of a measurable region of G with $0 < v := \mu(V) < 1$). For any $n \geqslant 1$, we define $\psi^{(n)}_{\gamma^{(n)}} \in \mathcal{H}_{\gamma^{(n)}}$ by:

$$\forall h \in \mathcal{C}_{\gamma^{(n)}}\,,\; \psi^{(n)}_{\gamma^{(n)}}(h) := \prod_{k=1}^{n} [\chi \circ h]\left(e_{[p_{k-1}\,,\,p_k]}\right),$$

and $\psi^{(n)} := \tau_{\text{AL}\leftarrow\gamma^{(n)}}\left(\psi^{(n)}_{\gamma^{(n)}}\right) \in \mathcal{H}_{\text{AL}}$. We have $\left\|\psi^{(n)}\right\|_{\mathcal{H}_{\text{AL}}} = \left\|\psi^{(n)}_{\gamma^{(n)}}\right\|_{\mathcal{H}_{\gamma^{(n)}}} = 1$.

Evaluation of the η-observables is (n)-eventually-constant. Let $\eta \in \mathcal{L}_{\text{HF}}$. From prop. 10.9.4, there exist $p \in r(e) \setminus \{f(e)\}$, $\epsilon = \pm 1$ and $F \in \mathfrak{F}(\eta) \cup \left\{F_{\daleth}(\eta)\right\}$ such that $e^{\epsilon}_{[p,f(e)]} \in F$. Let $p' \in r\left(e_{[p,f(e)]}\right) \setminus \{p,\, f(e)\}$. Then, $p' <_{(e)} f(e)$ and, from prop. 10.6:

$$e_{[f(e),p']} \in F_{\daleth}(\eta) \quad \text{or} \quad e_{[f(e),p']} \in F \in \mathfrak{F}(\eta)\,.$$

Moreover, for any $e' \in \gamma(\eta)$, applying lemma 10.5 to e^{-1} and e' yields:

$$\exists q \in r(e) \setminus \{f(e)\} \;/\; \left(r\left(e_{[f(e),q]}\right) \subset r(e') \quad \text{or} \quad r\left(e_{[f(e),q]}\right) \cap r(e') \subset \{f(e)\}\right).$$

Thus, there exists $q' <_{(e)} f(e)$ such that:

$$\forall e' \in \gamma(\eta),\; \left(r\left(e_{[f(e),q']}\right) \subset r(e') \quad \text{or} \quad r\left(e_{[f(e),q']}\right) \cap r(e') \subset \{f(e)\}\right).$$

Therefore, there exists $n_\eta \geqslant 1$ such that (fig. 12.1):

$$r\left(e_{[f(e),p_{n_\eta}]}\right) \cap \left(\bigcup_{e' \in \gamma(\eta)} r(e')\right) \subset \{f(e)\} \quad \text{or} \quad \exists\, e' \in \gamma(\eta) \,\big/\, r\left(e_{[f(e),p_{n_\eta}]}\right) \subset r(e')\,,$$

$$\text{and: } e_{[f(e),p_{n_\eta}]} \in F_{\daleth}(\eta) \quad \text{or} \quad \exists\, F \in \mathfrak{F}(\eta) \,\big/\, e_{[f(e),p_{n_\eta}]} \in F\,.$$

Let $m > n \geqslant n_\eta$. We have:

$$r\left(e_{[p_\infty,p_n]}\right) \cap \left(\bigcup_{e' \in \gamma(\eta)} r(e')\right) \subset \{p_\infty\} \quad \text{or} \quad \exists\, e' \in \gamma(\eta) \,\big/\, r\left(e_{[p_\infty,p_n]}\right) \subset r(e')\,, \tag{12.13.1}$$

$$\text{and: } e_{[p_\infty,p_n]} \in F_{\daleth}(\eta) \quad \text{or} \quad \exists\, F \in \mathfrak{F}(\eta) \,\big/\, e_{[p_\infty,p_n]} \in F\,. \tag{12.13.2}$$

Let $\gamma := \gamma^{(m)} \cup \left\{e_{[p_\infty,p_m]}\right\} \in \mathcal{L}_{\text{AL}}$. For any $e' \in \gamma(\eta)$, $\mathcal{L}_{\text{AL}/e'}$ is cofinal in $\mathcal{L}_{\text{AL}}$ and, for any $F \in \mathfrak{F}(\eta)$, $\mathcal{L}_{\text{AL}/F^{\perp}\circ F}$ is cofinal in $\mathcal{L}_{\text{AL}}$, hence there exists $\gamma' \in \mathcal{L}_{\text{AL}}$, with $\gamma \preccurlyeq \gamma'$, such that:

$$\forall e' \in \gamma(\eta),\; \gamma' \in \mathcal{L}_{\text{AL}/e'} \quad \& \quad \forall F \in \mathfrak{F}(\eta),\; \gamma' \in \mathcal{L}_{\text{AL}/F^{\perp}\circ F}\,.$$

Next, let $\widetilde{\gamma} := \gamma^{(n)} \cup \left\{e_{[p_\infty,p_n]}\right\} \in \mathcal{L}_{\text{AL}}$. We have $\widetilde{\gamma} \preccurlyeq \gamma$ for:

$$\gamma^{(n)} \subset \gamma^{(m)} \quad \& \quad e_{[p_\infty,p_n]} = e^{-1}_{[p_n,p_{n+1}]} \circ \ldots \circ e^{-1}_{[p_{m-1},p_m]} \circ e_{[p_\infty,p_m]}\,.$$

Thus, we can define $\varphi_{\gamma'-(m)-(n)} := s_{\gamma'-(m)-(n)} \circ \left(\text{id}_{\mathcal{C}_{\gamma'-\gamma}} \times \varphi_{\gamma-\widetilde{\gamma}}\right) \circ \varphi_{\gamma'-\gamma}$, where $\varphi_{\gamma_2-\gamma_1} : \mathcal{C}_{\gamma_2} \to$

$\mathcal{C}_{\gamma_2-\gamma_1} \times \mathcal{C}_{\gamma_1}$ has been defined for any $\gamma_1 \preccurlyeq \gamma_2 \in \mathcal{L}_{\mathrm{AL}}$ in the proof of prop. 12.7 and $s_{\gamma'-(m)-(n)}$ is given by:

$$\begin{array}{rcl} s_{\gamma'-(m)-(n)} : \mathcal{C}_{\gamma'-\gamma} \times \mathcal{C}_{\gamma-\widetilde{\gamma}} \times \mathcal{C}_{\widetilde{\gamma}} & \to & \mathcal{C}_{\gamma'-(m)-(n)} \times \mathcal{C}_{(m)-(n)} \\ j', h, j & \mapsto & (j', j), h \end{array},$$

with $\mathcal{C}_{\gamma'-(m)-(n)} := \mathcal{C}_{\gamma'-\gamma} \times \mathcal{C}_{\widetilde{\gamma}}$ and $\mathcal{C}_{(m)-(n)} := \mathcal{C}_{\gamma-\widetilde{\gamma}}$. Using the definition of $\varphi_{\gamma_2-\gamma_1}$ for $\gamma_1 \preccurlyeq \gamma_2$ together with eq. (12.6.1), we get:

$$\forall h' \in \mathcal{C}_{\gamma'} ,\ \varphi_{\gamma'-(m)-(n)}(h') = \left(h'\big|_{\gamma'-\gamma} ,\ \pi_{\gamma'\to\widetilde{\gamma}}(h') \right) ,\ \pi_{\gamma'\to\gamma}(h')\big|_{\gamma^{(m)-(n)}} , \tag{12.13.3}$$

where $\gamma^{(m)-(n)} := \gamma - \widetilde{\gamma} = \left\{ e_{[p_{k-1}, p_k]} \,\middle|\, n+1 \leqslant k \leqslant m \right\}$.

We define $\mathcal{H}_{\gamma'-(m)-(n)} := L_2\left(\mathcal{C}_{\gamma'-\gamma} , d\mu_{\gamma'-\gamma}\right) \otimes L_2\left(\mathcal{C}_{\widetilde{\gamma}} , d\mu_{\widetilde{\gamma}}\right)$ and $\mathcal{H}_{(m)-(n)} := L_2\left(\mathcal{C}_{\gamma-\widetilde{\gamma}} , d\mu_{\gamma-\widetilde{\gamma}}\right)$, so that $\varphi_{\gamma'-(m)-(n)}$ provides a unitary map $\Phi_{\gamma'-(m)-(n)}$:

$$\begin{array}{rcl} \Phi_{\gamma'-(m)-(n)} : \mathcal{H}_{\gamma'} & \to & \mathcal{H}_{\gamma'-(m)-(n)} \otimes \mathcal{H}_{(m)-(n)} \\ \psi & \mapsto & \psi \circ \varphi^{-1}_{\gamma'-(m)-(n)} \end{array}.$$

Since $\gamma^{(n)}, \gamma^{(m)} \preccurlyeq \gamma'$, we have:

$$\psi^{(n)} = \tau_{\mathrm{AL}\leftarrow\gamma'} \circ \tau_{\gamma'\leftarrow\gamma^{(n)}} \left(\psi^{(n)}_{\gamma^{(n)}}\right) \quad \& \quad \psi^{(n)} = \tau_{\mathrm{AL}\leftarrow\gamma'} \circ \tau_{\gamma'\leftarrow\gamma^{(m)}} \left(\psi^{(m)}_{\gamma^{(m)}}\right),$$

and eq. (12.13.3) yields, for any $(j', j) \in \mathcal{C}_{\gamma'-(m)-(n)}$ and any $h \in \mathcal{C}_{(m)-(n)}$:

$$\begin{aligned} \Phi_{\gamma'-(m)-(n)} \circ \tau_{\gamma'\leftarrow\gamma^{(n)}} \left(\psi^{(n)}_{\gamma^{(n)}}\right) ((j', j),\ h) &= \psi^{(n)}_{\gamma^{(n)}} \circ \pi_{\gamma'\to\gamma^{(n)}} \circ \varphi^{-1}_{\gamma'-(m)-(n)}((j', j),\ h) \\ &= \psi^{(n)}_{\gamma^{(n)}} \circ \pi_{\widetilde{\gamma}\to\gamma^{(n)}}(j) = \psi^{(n)}_{\gamma^{(n)}} \left(j\big|_{\gamma^{(n)}} \right) \\ &= \prod_{k=1}^{n} [\chi \circ j] \left(e_{[p_{k-1}, p_k]}\right), \end{aligned}$$

as well as:

$$\begin{aligned} \Phi_{\gamma'-(m)-(n)} \circ \tau_{\gamma'\leftarrow\gamma^{(m)}} \left(\psi^{(m)}_{\gamma^{(m)}}\right) ((j', j),\ h) &= \prod_{k=1}^{m} \left[\chi \circ \left(\pi_{\gamma'\to\gamma^{(m)}} \circ \varphi^{-1}_{\gamma'-(m)-(n)}((j', j),\ h)\right)\right] \left(e_{[p_{k-1}, p_k]}\right) \\ &= \prod_{k=1}^{n} [\chi \circ j] \left(e_{[p_{k-1}, p_k]}\right) \prod_{k=n+1}^{m} [\chi \circ h] \left(e_{[p_{k-1}, p_k]}\right). \end{aligned}$$

Thus there exist $\psi_{\gamma'-(m)-(n)} \in \mathcal{H}_{\gamma'-(m)-(n)}$ and $\zeta_{(m)-(n)} , \chi_{(m)-(n)} \in \mathcal{H}_{(m)-(n)}$ such that:

$$\begin{aligned} &\psi^{(n)} = \tau_{\mathrm{AL}\leftarrow\gamma'} \circ \Phi^{-1}_{\gamma'-(m)-(n)} \left(\psi_{\gamma'-(m)-(n)} \otimes \zeta_{(m)-(n)}\right), \\ \& \quad &\psi^{(m)} = \tau_{\mathrm{AL}\leftarrow\gamma'} \circ \Phi^{-1}_{\gamma'-(m)-(n)} \left(\psi_{\gamma'-(m)-(n)} \otimes \chi_{(m)-(n)}\right). \end{aligned} \tag{12.13.4}$$

In addition, $\left\|\zeta_{(m)-(n)}\right\|_{\mathcal{H}_{(m)-(n)}} = \left\|\chi_{(m)-(n)}\right\|_{\mathcal{H}_{(m)-(n)}}$, for $\left\|\psi^{(n)}\right\|_{\mathcal{H}_{\mathrm{AL}}} = 1 = \left\|\psi^{(m)}\right\|_{\mathcal{H}_{\mathrm{AL}}}$.

Now, let $\gamma'' := \left\{ e'' \in \gamma' \,\middle|\, r(e'') \subset r\left(e_{[p_o, p_n]}\right) \right\} \cup \left\{ e_{[p_\infty, p_n]} \right\} \in \mathcal{L}_{\mathrm{AL}}$. We have $\widetilde{\gamma} \preccurlyeq \gamma'' \preccurlyeq \gamma'$, hence:

$$\forall h' \in \mathcal{C}_{\gamma'} ,\ \varphi_{\gamma''-\widetilde{\gamma}} \circ \pi_{\gamma'\to\gamma''}(h') = \pi_{\gamma'\to(\gamma''-\widetilde{\gamma})}(h') ,\ \pi_{\gamma'\to\widetilde{\gamma}}(h'),$$

and, since $(\gamma'' - \widetilde{\gamma}) \subset (\gamma' - \gamma)$, we get, for any $(j', j) \in \mathcal{C}_{\gamma'-(m)-(n)}$ and any $h \in \mathcal{C}_{(m)-(n)}$:

$$\pi_{\gamma'\to\gamma''} \circ \varphi^{-1}_{\gamma'-(m)-(n)}((j', j),\, h) = \varphi^{-1}_{\gamma''-\widetilde{\gamma}}\left(j'|_{\gamma''-\widetilde{\gamma}}\,,\, j\right). \tag{12.13.5}$$

Next, for any $e' \in \gamma(\eta)$, we define:

$$\gamma'_{(e')} = \{e'' \in \gamma' \mid r(e'') \subset r(e') \;\;\&\;\; r(e'') \not\subset r(e)\} \;\;\cup\;\; \{e'' \in \gamma'' \mid r(e'') \subset r(e')\} \in \mathcal{L}_{\mathrm{AL}}\,.$$

$\gamma'_{(e')} \preccurlyeq \gamma'$ and, from eq. (12.13.1), $\gamma'_{(e')} \in \mathcal{L}_{\mathrm{AL}/e'}$, so $\pi_{\gamma'\to\{e'\}} = \pi_{\gamma'_{(e')}\to\{e'\}} \circ \pi_{\gamma'\to\gamma'_{(e')}}$. Moreover, since $\gamma'_{(e')} \subset (\gamma' - \gamma) \cup \gamma''$, we have, for any $(j', j) \in \mathcal{C}_{\gamma'-(m)-(n)}$ and any $h \in \mathcal{C}_{(m)-(n)}$:

$$\forall e'' \in \gamma'_{(e')}\,,\; \left[\pi_{\gamma'\to\gamma'_{(e')}} \circ \varphi^{-1}_{\gamma'-(m)-(n)}((j', j),\, h)\right](e'') =$$

$$= \begin{cases} \left[\pi_{\gamma'\to(\gamma'-\gamma)} \circ \varphi^{-1}_{\gamma'-(m)-(n)}((j', j),\, h)\right](e'') = j'(e'') & \text{if } e'' \in (\gamma' - \gamma) \\ \left[\pi_{\gamma'\to\gamma''} \circ \varphi^{-1}_{\gamma'-(m)-(n)}((j', j),\, h)\right](e'') \quad = \left[\varphi^{-1}_{\gamma''-\widetilde{\gamma}}\left(j'|_{\gamma''-\widetilde{\gamma}}\,,\, j\right)\right](e'') & \text{if } e'' \in \gamma'' \end{cases}.$$

Thus, $\pi_{\gamma'\to\gamma'_{(e')}} \circ \varphi^{-1}_{\gamma'-(m)-(n)}((j', j),\, h)$ does not depend on h. Therefore, using eq. (12.8.1), there exists, for any $m \in C^\infty(G,\, \mathbb{R})$, an operator $A \in \mathcal{A}_{\gamma'-(m)-(n)}$ (with $\mathcal{A}_{\gamma'-(m)-(n)}$ the algebra of bounded operators on $\mathcal{H}_{\gamma'-(m)-(n)}$) such that:

$$\widehat{\mathsf{h}^{(e',m)}_{\gamma'}} = \Phi^{-1}_{\gamma'-(m)-(n)} \circ \left[A \otimes \mathrm{id}_{\mathcal{H}_{(m)-(n)}}\right] \circ \Phi_{\gamma'-(m)-(n)}\,.$$

Let $F \in \mathfrak{F}(\eta)$. Since $\widetilde{\gamma} \preccurlyeq \gamma'' \preccurlyeq \gamma'$, eq. (12.13.3) implies:

$$\forall h' \in \mathcal{C}_{\gamma'}\,,\; \varphi_{\gamma'-(m)-(n)}(h') = \left(\pi_{\gamma'\to(\gamma'-\gamma)}(h'),\, \pi_{\gamma''\to\widetilde{\gamma}} \circ \pi_{\gamma'\to\gamma''}(h')\right),\, \pi_{\gamma'\to\gamma^{(m)-(n)}}(h')\,.$$

But since $\gamma' \in \mathcal{L}_{\mathrm{AL}/F^\perp\circ F}$ and $e_{[p_\infty,p_n]} \in F'$ with $F' \in \mathfrak{F}(\eta) \cup \left\{F_{\urcorner}(\eta)\right\}$, we have $(\gamma' - \gamma)$, γ'' & $\gamma^{(m)-(n)} \in \mathcal{L}_{\mathrm{AL}/F^\perp\circ F}$ and, moreover:

$$\forall e' \in \gamma^{(m)-(n)},\; c(e',\, F^\perp \circ F) = \varnothing\,,$$

so that $\forall t \in G$, $\forall h \in \mathcal{C}_{(m)-(n)}$, $T^{(F^\perp\circ F)}_{\gamma^{(m)-(n)},t} h = h$. Thus, using eq. (12.9.1), we get, for any $t \in G$ and any $h' \in \mathcal{C}_{\gamma'}$:

$$\varphi_{\gamma'-(m)-(n)}\left(T^{(F^\perp\circ F)}_{\gamma',t} h'\right) = \left(T^{(F^\perp\circ F)}_{(\gamma'-\gamma),t} \circ \pi_{\gamma'\to(\gamma'-\gamma)}(h'),\, \pi_{\gamma''\to\widetilde{\gamma}} \circ T^{(F^\perp\circ F)}_{\gamma'',t} \circ \pi_{\gamma'\to\gamma''}(h')\right),\, \pi_{\gamma'\to\gamma^{(m)-(n)}}(h')\,.$$

Hence, eq. (12.13.5) yields, for any $(j', j) \in \mathcal{C}_{\gamma'-(m)-(n)}$ and any $h \in \mathcal{C}_{(m)-(n)}$:

$$T^{(F^\perp\circ F)}_{\gamma',t}\left(\varphi^{-1}_{\gamma'-(m)-(n)}\left((j', j),\, h\right)\right) = \varphi^{-1}_{\gamma'-(m)-(n)}\left(T^{(F^\perp\circ F)}_{(\gamma'-(m)-(n)),t}(j', j),\, h\right),$$

$$\text{where } T^{(F^\perp\circ F)}_{(\gamma'-(m)-(n)),t}(j', j) := \left(T^{(F^\perp\circ F)}_{(\gamma'-\gamma),t} j',\, \pi_{\gamma''\to\widetilde{\gamma}} \circ T^{(F^\perp\circ F)}_{\gamma'',t} \circ \varphi^{-1}_{\gamma''\to\widetilde{\gamma}}\left(j'|_{\gamma''-\widetilde{\gamma}}\,,\, j\right)\right).$$

Therefore, there exists, for any $t \in G$, an operator $A \in \mathcal{A}_{\gamma'-(m)-(n)}$ such that:

$$\widehat{\mathsf{T}^{(F^\perp\circ F,t)}_{\gamma'}} = \Phi^{-1}_{\gamma'-(m)-(n)} \circ \left[A \otimes \mathrm{id}_{\mathcal{H}_{(m)-(n)}}\right] \circ \Phi_{\gamma'-(m)-(n)}\,.$$

So we have proved that for any A_{AL} in:

$$\left\{\, \widehat{\mathsf{h}^{(e',m)}_{\mathrm{AL}}} \,\middle|\, m \in C^\infty(G,\, \mathbb{R}) \;\;\&\;\; e' \in \gamma(\eta) \,\right\} \cup \left\{\, \widehat{\mathsf{T}^{(F^\perp\circ F,t)}_{\mathrm{AL}}} \,\middle|\, t \in G \;\;\&\;\; F \in \mathfrak{F}(\eta) \,\right\},$$

there exists $A \in \mathcal{A}_{\gamma'-(m)-(n)}$ such that:

$$A_{\text{AL}} \circ \tau_{\text{AL}\leftarrow\gamma'} = \tau_{\text{AL}\leftarrow\gamma'} \circ \Phi^{-1}_{\gamma'-(m)-(n)} \circ \left[A \otimes \mathrm{id}_{\mathcal{H}_{(m)-(n)}}\right] \circ \Phi_{\gamma'-(m)-(n)} .$$

Theorem 12.11.1 and 12.11.2 then implies that for any $A_\eta \in \mathfrak{I}_\eta$, there exists $A \in \mathcal{A}_{\gamma'-(m)-(n)}$ such that:

$$\alpha\left([A_\eta]_\sim\right) \circ \tau_{\text{AL}\leftarrow\gamma'} = \tau_{\text{AL}\leftarrow\gamma'} \circ \Phi^{-1}_{\gamma'-(m)-(n)} \circ \left[A \otimes \mathrm{id}_{\mathcal{H}_{(m)-(n)}}\right] \circ \Phi_{\gamma'-(m)-(n)} ,$$

hence, using the expression for $\psi^{(n)}$ and $\psi^{(m)}$ from eq. (12.13.4):

$$\forall A_\eta \in \mathfrak{I}_\eta ,\ \left\langle \psi^{(n)},\ \alpha\left([A_\eta]_\sim\right)\ \psi^{(n)} \right\rangle_{\mathcal{H}_{\text{AL}}} = \left\langle \psi^{(m)},\ \alpha\left([A_\eta]_\sim\right)\ \psi^{(m)} \right\rangle_{\mathcal{H}_{\text{AL}}} .$$

Constructing a projective state from the $\psi^{(n)}$. For each $\eta \in \mathcal{L}_{\text{HF}}$, we choose n_η as above, and we define:

$$\rho_\eta := \left[\sigma\Big(\ \big|\psi^{(n_\eta)}\ \big\rangle \big\langle\ \psi^{(n_\eta)}\big|\ \Big)\right]_\eta .$$

Let $\eta \preccurlyeq \eta' \in \mathcal{L}_{\text{HF}}$ and let $m \geqslant n_\eta ,\ n_{\eta'}$. From the previous point, together with theorem 12.11.3, we have, for any $A_\eta \in \mathfrak{I}_\eta$:

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}_\eta} \rho_\eta A_\eta &= \left\langle \psi^{(n_\eta)}\ \middle|\ \alpha\left([A_\eta]_\sim\right)\ \psi^{(n_\eta)} \right\rangle_{\mathcal{H}_{\text{AL}}} \\ &= \left\langle \psi^{(m)}\ \middle|\ \alpha\left([A_\eta]_\sim\right)\ \psi^{(m)} \right\rangle_{\mathcal{H}_{\text{AL}}} = \mathrm{Tr}_{\mathcal{H}_\eta}\left[\sigma\Big(\ \big|\psi^{(m)}\ \big\rangle \big\langle\ \psi^{(m)}\big|\ \Big)\right]_\eta A_\eta .\end{aligned}$$

Hence, the second part of prop. 12.10 implies:

$$\rho_\eta = \left[\sigma\Big(\ \big|\psi^{(m)}\ \big\rangle \big\langle\ \psi^{(m)}\big|\ \Big)\right]_\eta ,$$

and similarly for $\rho_{\eta'}$. But $\sigma\big(\,|\,\psi^{(m)}\,\rangle\langle\,\psi^{(m)}\,|\big) \in \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}_{\text{HF}},\mathcal{H},\Phi)}$ (def. 5.2), so we get:

$$\mathrm{Tr}_{\eta'\to\eta}\, \rho_{\eta'} = \mathrm{Tr}_{\eta'\to\eta}\left[\sigma\Big(\ \big|\psi^{(m)}\ \big\rangle \big\langle\ \psi^{(m)}\big|\ \Big)\right]_{\eta'} = \left[\sigma\Big(\ \big|\psi^{(m)}\ \big\rangle \big\langle\ \psi^{(m)}\big|\ \Big)\right]_\eta = \rho_\eta ,$$

and therefore $\rho := \left(\rho_\eta\right)_{\eta\in\mathcal{L}_{\text{HF}}} \in \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}_{\text{HF}},\mathcal{H},\Phi)}$.

ρ is not in the image of σ. Let $\eta \in \mathcal{L}_{\text{HF}}$ and $n = n_\eta \geqslant 1$. Let $M \geqslant 1$ be an *odd* integer and $m = n + M$. By definition of n_η we have, for any l such that $0 \leqslant 2l \leqslant M - 1$:

$$p_{n+2l+1} \notin \bigcup_{e'\in\gamma_\sharp} r(e') ,$$

where $\gamma_\sharp := \gamma(\eta) \setminus \left\{e' \in \gamma(\eta)\ \middle|\ r(e_{[p_\infty,p_n]}) \subset r(e')\right\}$. If there exists $e' \in \gamma(\eta)$ such that $r(e_{[p_\infty,p_n]}) \subset r(e')$ (note that there can be as much one such e' for $\gamma(\eta)$ is a graph), we define $\gamma_\flat \in \mathcal{L}_{\text{graph}}$ by:

$$\gamma_\flat = \begin{cases} \left\{e'_{[b(e'),p]},\ e'_{[q,f(e')]}\right\} & \text{if } b(e') <_{(e')} p <_{(e')} q <_{(e')} f(e') \\ \left\{e'_{[b(e'),p]}\right\} & \text{if } b(e') <_{(e')} p <_{(e')} q = f(e') \\ \left\{e'_{[q,f(e')]}\right\} & \text{if } b(e') = p <_{(e')} q <_{(e')} f(e') \\ \varnothing & \text{if } b(e') = p <_{(e')} q = f(e') \end{cases} \quad \text{with } \{p,q\} = \{p_\infty, p_n\} ,$$

otherwise we define $\gamma_\flat = \varnothing$. By construction, we have $\gamma\setminus\gamma_\sharp \preccurlyeq \gamma_\flat \cup \left\{e_{[p_\infty,p_n]}\right\}$ and, for any l such that

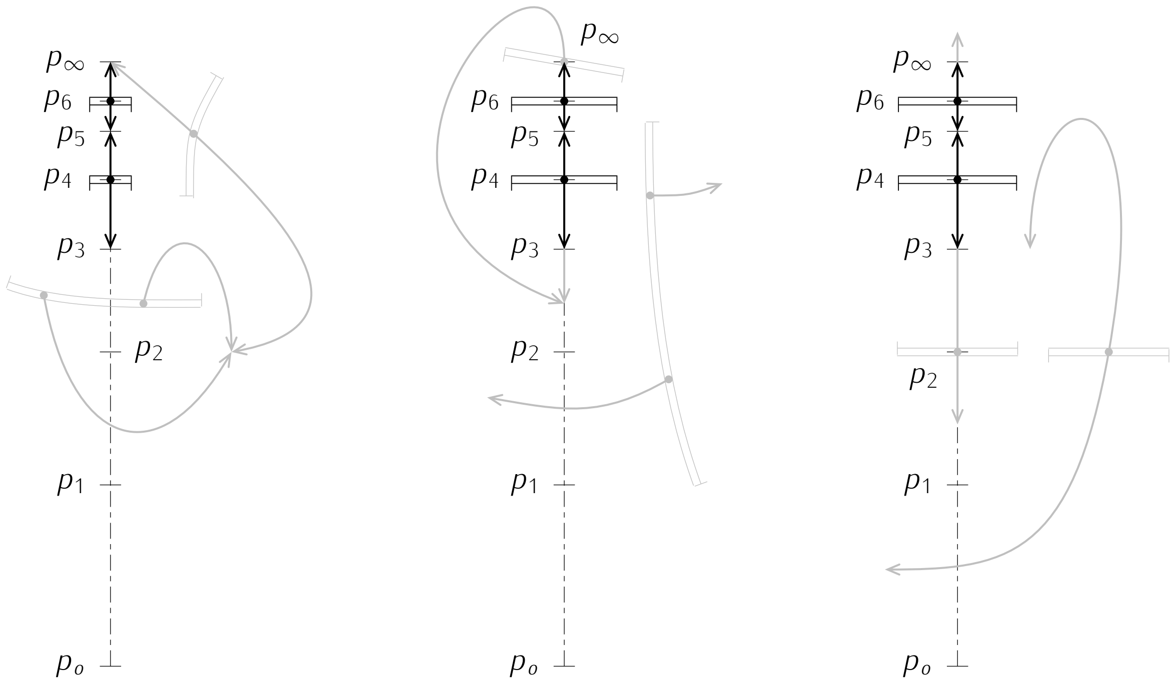


Figure 12.2 – Construction of the auxiliary label $\widetilde{\eta}$ with $M = 3$ for the labels of fig. 12.1 (recalled in light gray)

$0 \leqslant 2l \leqslant M-1$, $p_{n+2l+1} \notin \bigcup_{e'' \in \gamma_\flat} r(e'')$. Since $\bigcup_{e' \in \gamma_\sharp \cup \gamma_\flat} r(e')$ is compact, $\check{e}^{-1}\left\langle V \setminus \bigcup_{e' \in \gamma_\sharp \cup \gamma_\flat} r(e') \right\rangle \subset U$ is an open neighborhood of $\left\{ \frac{n+2l+1}{n+2l+2} \,\middle|\, 0 \leqslant 2l \leqslant M-1 \right\} \times \{0\}^{d-1}$ in $\mathbb{R}^d$ so there exists $R > 0$ such that, for any l in $\{0, \dots, (M-1)/2\}$:

$$\left\{ \frac{n+2l+1}{n+2l+2} \right\} \times B_R^{(d-1)} \subset U \quad \& \quad \check{e}\left\langle \left\{ \frac{n+2l+1}{n+2l+2} \right\} \times B_R^{(d-1)} \right\rangle \cap \bigcup_{e' \in \gamma_\sharp \cup \gamma_\flat} r(e') = \varnothing$$

(where $B_R^{(d-1)}$ is the closed ball of radius R and center 0 in $\mathbb{R}^{d-1}$). Thus, this allows us to construct a label $\widetilde{\eta} \in \mathcal{L}_{\text{HF}}$ such that (fig. 12.2):

$$\gamma(\widetilde{\eta}) = \widetilde{\gamma} := \left\{ e_{[p_m, p_{m-1}]}, \, e_{[p_m, p_\infty]} \right\} \cup \bigcup_{0 \leqslant 2l < M-1} \left\{ e_{[p_{n+2l+1}, p_{n+2l}]}, \, e_{[p_{n+2l+1}, p_{n+2l+2}]} \right\}$$

$$\& \quad \forall e' \in \gamma_\sharp \cup \gamma_\flat \cup \left\{ e_{[p_o, p_n]} \right\}, \; e' \in F_{\gamma}(\widetilde{\eta}).$$

We have $e_{[p_\infty, p_n]} = e_{[p_{n+1}, p_n]} \circ e^{-1}_{[p_{n+1}, p_{n+2}]} \circ \dots \circ e_{[p_m, p_{m-1}]} \circ e^{-1}_{[p_m, p_\infty]}$, hence $\left\{ e_{[p_\infty, p_n]} \right\} \preccurlyeq \widetilde{\gamma}$ and:

$$\widetilde{\gamma}_\sharp := \widetilde{\gamma} - \left\{ e_{[p_\infty, p_n]} \right\} = \widetilde{\gamma} \setminus \left\{ e_{[p_m, p_\infty]} \right\}.$$

Note that $\gamma^{(m)-(n)} = \left\{ e_{[p_{k-1}, p_k]} \,\middle|\, n+1 \leqslant k \leqslant m \right\} \preccurlyeq \widetilde{\gamma}_\sharp$ with $\widetilde{\gamma}_\sharp - \gamma^{(m)-(n)} = \varnothing$, so that $\pi_{\widetilde{\gamma}_\sharp \to \gamma^{(m)-(n)}}$ is a diffeomorphism $\mathcal{C}_{\widetilde{\gamma}_\sharp} \to \mathcal{C}_{\gamma^{(m)-(n)}} = \mathcal{C}_{(m)-(n)}$.

Next, let $\eta' \succcurlyeq \eta, \widetilde{\eta}$. Since $\left\{ e_{[p_m, p_{m-1}]} \right\} \preccurlyeq \widetilde{\gamma} \preccurlyeq \gamma(\eta')$, there exists $e' \in \gamma(\eta')$ such that $r(e') \subset r\left(e_{[p_m, p_{m-1}]} \right)$, so $n' := n_{\eta'}$ has to be bigger than m. Also, since $\mathcal{L}_{\text{graphs}}$ is directed, there exists $\gamma' \succcurlyeq \gamma(\eta'), \gamma^{(n')}$. So, applying the construction from subsection 10.2 to $(\gamma', \lambda(\eta')) \in \mathcal{L}_{\text{graphs}} \times \mathcal{L}_{\text{profls}}$, there exists $\eta'' \succcurlyeq \eta'$ with $\gamma(\eta'') \succcurlyeq \gamma^{(n')}$. We define a diffeomorphism $\varphi_{\eta'' \to (m)-(n)} : \mathcal{C}_{\eta''} \to \mathcal{C}_{\eta'' \to (m)-(n)} \times \mathcal{C}_{(m)-(n)}$ by:

$$\varphi_{\eta''\to(m)-(n)} = \left[\widetilde{j}', (\widetilde{h}, \widetilde{j}) \mapsto (\widetilde{j}', \widetilde{j}),\ \pi_{\widetilde{\gamma}_\sharp \to \gamma^{(m)-(n)}}(\widetilde{h})\right] \circ \left(\mathrm{id}_{\mathcal{C}_{\eta''\to\widetilde{\eta}}} \times \varphi_{\widetilde{\gamma}-\left\{e_{[p_\infty,p_n]}\right\}}\right) \circ \varphi_{\eta''\to\widetilde{\eta}}\,,$$

with $\mathcal{C}_{\eta''\to(m)-(n)} := \mathcal{C}_{\eta''\to\widetilde{\eta}} \times \mathcal{C}_{\left\{e_{[p_\infty,p_n]}\right\}}$.

Let $h'' \in \mathcal{C}_{\eta''}$ and define:

$$\widetilde{j}', \widetilde{h}, \widetilde{j} = \left(\mathrm{id}_{\mathcal{C}_{\eta''\to\widetilde{\eta}}} \times \varphi_{\widetilde{\gamma}-\left\{e_{[p_\infty,p_n]}\right\}}\right) \circ \varphi_{\eta''\to\widetilde{\eta}}(h'')\,,$$

$$\&\quad j', j, h = \left(\mathrm{id}_{\mathcal{C}_{\eta''\to\eta'}} \times \varphi_{\eta'\to\eta}\right) \circ \varphi_{\eta''\to\eta'}(h'')\,.$$

Using eq. (2.11.1) (which was proved in theorem 10.25), we have:

$$\begin{aligned}
j' &= [j'_\circ\,, h' \mapsto j'_\circ] \circ \varphi_{\eta''\to\eta'}(h'') \\
&= [j'_\circ\,, h' \mapsto j'_\circ] \circ \varphi_{\eta''\to\eta'} \circ \varphi^{-1}_{\eta''\to\widetilde{\eta}} \circ \varphi_{\eta''\to\widetilde{\eta}}(h'') \\
&= [j'_\circ\,, h' \mapsto j'_\circ] \circ \left(\mathrm{id}_{\mathcal{C}_{\eta''\to\eta'}} \times \varphi^{-1}_{\eta'\to\widetilde{\eta}}\right) \circ \left(\varphi_{\eta''\to\eta'\to\widetilde{\eta}} \times \mathrm{id}_{\mathcal{C}_{\widetilde{\eta}}}\right) \circ \varphi_{\eta''\to\widetilde{\eta}}(h'') \\
&= \left[j'_\circ\,, \widetilde{j}''\,, \widetilde{h}' \mapsto j'_\circ\right] \circ \left(\varphi_{\eta''\to\eta'\to\widetilde{\eta}} \times \mathrm{id}_{\mathcal{C}_{\widetilde{\eta}}}\right) \circ \varphi_{\eta''\to\widetilde{\eta}}(h'') \\
&= \left[j'_\circ\,, \widetilde{j}'' \mapsto j'_\circ\right] \circ \left(\varphi_{\eta''\to\eta'\to\widetilde{\eta}}\right)(\widetilde{j}')\,,
\end{aligned}$$

and:

$$\begin{aligned}
h &= [j'_\circ, j_\circ, h_\circ \mapsto h_\circ] \circ \left(\varphi_{\eta''\to\eta'\to\eta} \times \mathrm{id}_{\mathcal{C}_\eta}\right) \circ \varphi_{\eta''\to\eta}(h'') \\
&= [j'', h_\circ \mapsto h_\circ] \circ \varphi_{\eta''\to\eta}(h'') \\
&= e' \mapsto \left(\prod_{k=1}^{n_{\eta''\to\eta,e'}} [h'' \circ a_{\eta''\to\eta,e'}(k)]^{\epsilon_{\eta''\to\eta,e'}(k)}\right) \\
&= \pi_{\gamma(\eta'')\to\gamma(\eta)}(h'')\,.
\end{aligned}$$

Let $e' \in \gamma_\sharp$. By construction, we have, for any $k \in \{1, \dots, n_{\eta''\to\eta,e'}\}$, $a_{\eta''\to\eta,e'}(k) \in F_{\urcorner}(\widetilde{\eta})$, as well as $\forall \widetilde{e} \in \widetilde{\gamma}$, $r\left(a_{\eta''\to\eta,e'}(k)\right) \not\subset r(\widetilde{e})$ (for $r(e') \cap r\left(e_{[p_\infty,p_n]}\right) \subset \{p_\infty, p_n\}$). Hence, $a_{\eta''\to\eta,e'}(k) \in H^{(0)}_{\eta''\to\widetilde{\eta}}$. Writing $\widetilde{j}' =: \left(\widetilde{j}'^{(0)}, \widetilde{j}'^{(2)}, \widetilde{j}'^{(3)}\right)$, we thus get:

$$\forall k \in \{1, \dots, n_{\eta''\to\eta,e'}\}\,,\ h'' \circ a_{\eta''\to\eta,e'}(k) = \widetilde{j}'^{(0)} \circ a_{\eta''\to\eta,e'}(k).$$

So, for any $e' \in \gamma_\sharp$, $[\pi_{\gamma(\eta'')\to\gamma(\eta)}(h'')](e')$ only depends on h'' via $\widetilde{j}'^{(0)}$.

If there exists $e' \in \gamma(\eta)$ such that $r\left(e_{[p_\infty,p_n]}\right) \subset r(e')$, then we have at the same time $\{e'\} \preccurlyeq \gamma(\eta) \preccurlyeq \gamma(\eta')$ and $\left\{e_{[p_\infty,p_n]}\right\} \preccurlyeq \gamma(\widetilde{\eta}) \preccurlyeq \gamma(\eta')$, so $\gamma_\flat \preccurlyeq \gamma(\eta') \preccurlyeq \gamma(\eta'')$ must hold (we can check this by writing both e' and $e_{[p_\infty,p_n]}$ as compositions of edges in $\gamma(\eta')$, and by showing that the edges that appears in the decomposition of $e_{[p_\infty,p_n]}$ should appear in the decomposition of e' as well, for $\gamma(\eta')$ is a graph; then, the remaining edges from the decomposition of e' build up precisely the edges in $\gamma_\flat$). With the same argument as for $\gamma_\sharp$ above, we have for any $e'' \in \gamma_\flat$ and any $k \in \{1, \dots, n_{\gamma(\eta'')\to e''}\}$, $a_{\gamma(\eta'')\to e''}(k) \in H^{(0)}_{\eta''\to\widetilde{\eta}}$, and therefore $h'' \circ a_{\gamma(\eta'')\to e''}(k) = \widetilde{j}'^{(0)} \circ a_{\gamma(\eta'')\to e''}(k)$. On the other hand, we have $\pi_{\gamma(\eta'')\to\left\{e_{[p_\infty,p_n]}\right\}}(h'') = \pi_{\widetilde{\gamma}\to\left\{e_{[p_\infty,p_n]}\right\}} \circ \pi_{\gamma(\eta'')\to\gamma(\widetilde{\eta})}(h'') = \widetilde{j}$. Thus, $\pi_{\gamma(\eta'')\to\gamma_\flat\cup\left\{e_{[p_\infty,p_n]}\right\}}(h'')$ only depends on

h'' via $\widetilde{j}'^{(0)}$ and $\widetilde{j}$, and the same holds for $[\pi_{\gamma(\eta'')\to\gamma(\eta)}(h'')](e')$ since:

$$[\pi_{\gamma(\eta'')\to\gamma(\eta)}(h'')](e') = \left[\pi_{\gamma(\eta'')\to\{e'\}}(h'')\right](e') = \left[\pi_{\gamma_\flat\cup\left\{e_{[p_\infty,p_n]}\right\}\to\{e'\}} \circ \pi_{\gamma(\eta'')\to\gamma_\flat\cup\left\{e_{[p_\infty,p_n]}\right\}}(h'')\right](e')\,.$$

Hence, we have proved so far that there exist two maps $\theta_{\eta''\to\eta'} : \mathcal{C}_{\eta''\to(m)-(n)} \to \mathcal{C}_{\eta''\to\eta'}$ and $\theta_\eta : \mathcal{C}_{\eta''\to(m)-(n)} \to \mathcal{C}_\eta$ such that:

$$j' = \theta_{\eta''\to\eta'}\left(\widetilde{j}', \widetilde{j}\right) \quad \& \quad h = \theta_\eta\left(\widetilde{j}', \widetilde{j}\right).$$

Now, we define an Hilbert space isomorphism $\Phi_{\eta''\to(m)-(n)} : \mathcal{H}_{\eta''} \to \mathcal{H}_{\eta''\to(m)-(n)} \otimes \mathcal{H}_{(m)-(n)}$ by:

$$\begin{array}{rccl} \Phi_{\eta''\to(m)-(n)} & : & \mathcal{H}_{\eta''} & \to & \mathcal{H}_{\eta''\to(m)-(n)} \otimes \mathcal{H}_{(m)-(n)} \\ & & \psi & \mapsto & \psi \circ \varphi^{-1}_{\eta''\to(m)-(n)} \end{array},$$

with $\mathcal{H}_{\eta''\to(m)-(n)} := \mathcal{H}_{\eta''\to\widetilde{\eta}} \otimes \mathcal{H}_{\left\{e_{[p_\infty,p_n]}\right\}}$ and $\mathcal{H}_{(m)-(n)} := \mathcal{H}_{\gamma^{(m)-(n)}}$. $\Phi_{\eta''\to(m)-(n)}$ being an Hilbert space isomorphism follows from the note at the end of the proof of prop. 12.7 and from the fact that $\pi_{\widetilde{\gamma}_\sharp\to\gamma^{(m)-(n)},*}\,\mu_{\widetilde{\gamma}_\sharp} = \mu_{\gamma^{(m)-(n)}}$ (for the Haar measure on a compact group is invariant under taking the inverse). Then, we have, for any $\psi_{\eta''\to\eta'} \in \mathcal{H}_{\eta''\to\eta'}$ and any $\psi_\eta \in \mathcal{H}_\eta$:

$$\Phi^{-1}_{\eta''\to\eta'}\circ\left(\mathrm{id}_{\mathcal{H}_{\eta''\to\eta'}} \otimes \Phi^{-1}_{\eta'\to\eta}\right)\left(\psi_{\eta''\to\eta'} \otimes \zeta_{\eta'\to\eta} \otimes \psi_\eta\right) = \Phi^{-1}_{\eta''\to(m)-(n)}\left([\psi_{\eta''\to\eta'} \circ \theta_{\eta''\to\eta'}\, \psi_\eta \circ \theta_\eta] \otimes \zeta_{(m)-(n)}\right),$$

where $\zeta_{\eta'\to\eta} \equiv 1$ and $\zeta_{(m)-(n)} \equiv 1$. Since $\Phi_{\eta''\to(m)-(n)}$, $\Phi_{\eta''\to\eta'}$ and $\Phi_{\eta'\to\eta}$ are Hilbert space isomorphism, $\Phi_{\eta''\to(m)-(n)} \circ \Phi^{-1}_{\eta''\to\eta'} \circ \left(\mathrm{id}_{\mathcal{H}_{\eta''\to\eta'}} \otimes \Phi^{-1}_{\eta'\to\eta}\right)$ thus induces a unitary (injective) map from:

$$\overline{\mathrm{Vect}\left\{\psi_{\eta''\to\eta'} \otimes \zeta_{\eta'\to\eta} \otimes \psi_\eta \,\middle|\, \psi_{\eta''\to\eta'} \in \mathcal{H}_{\eta''\to\eta'}\,,\ \psi_\eta \in \mathcal{H}_\eta\right\}}$$

(where $\overline{\ \cdot\ }$ denotes the completion) into $\mathcal{H}_{\eta''\to(m)-(n)} \otimes \{\zeta_{(m)-(n)}\}$. Therefore, we get, using the characterization of $\Theta_{\eta'|\eta}$ from the proof of theorem 12.11 (eq. (12.11.2)) together with def. 5.3:

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{H}_{\eta'}}\, \rho_{\eta'}\, \Theta_{\eta'|\eta} = \\
&\quad = \mathrm{Tr}_{\mathcal{H}_{\eta'}}\left[\sigma\left(\left|\psi^{(n')}\right\rangle\left\langle\psi^{(n')}\right|\right)\right]_{\eta'} \Theta_{\eta'|\eta} \\
&\quad = \mathrm{Tr}_{\mathcal{H}_{\eta''}}\left[\sigma\left(\left|\psi^{(n')}\right\rangle\left\langle\psi^{(n')}\right|\right)\right]_{\eta''} \Phi^{-1}_{\eta''\to\eta'}\left(\mathrm{id}_{\mathcal{H}_{\eta''\to\eta'}} \otimes \Theta_{\eta'|\eta}\right)\Phi_{\eta''\to\eta'} \\
&\quad = \mathrm{Tr}_{\mathcal{H}_{\eta''}}\left[\sigma\left(\left|\psi^{(n')}\right\rangle\left\langle\psi^{(n')}\right|\right)\right]_{\eta''} \Phi^{-1}_{\eta''\to\eta'} \circ \left(\mathrm{id}_{\mathcal{H}_{\eta''\to\eta'}} \otimes \Phi^{-1}_{\eta'\to\eta}\right) \\
&\qquad\qquad \left(\mathrm{id}_{\mathcal{H}_{\eta''\to\eta'}} \otimes |\zeta_{\eta'\to\eta}\rangle\langle\zeta_{\eta'\to\eta}| \otimes \mathrm{id}_{\mathcal{H}_\eta}\right)\left(\mathrm{id}_{\mathcal{H}_{\eta''\to\eta'}} \otimes \Phi_{\eta'\to\eta}\right) \circ \Phi_{\eta''\to\eta'} \\
&\quad \leqslant \mathrm{Tr}_{\mathcal{H}_{\eta''}}\left[\sigma\left(\left|\psi^{(n')}\right\rangle\left\langle\psi^{(n')}\right|\right)\right]_{\eta''} \Phi^{-1}_{\eta''\to(m)-(n)}\left(\mathrm{id}_{\mathcal{H}_{\eta''\to(m)-(n)}} \otimes |\zeta_{(m)-(n)}\rangle\langle\zeta_{(m)-(n)}|\right)\Phi_{\eta''\to(m)-(n)}\,.
\end{aligned}$$

This implies, using theorem 12.11.3 and eq. (12.11.1):

$$\mathrm{Tr}_{\mathcal{H}_{\eta'}}\, \rho_{\eta'}\, \Theta_{\eta'|\eta} \leqslant \left\langle \Phi_{\eta''\to(m)-(n)}\, \Gamma^{-1}_{\eta''\to\gamma(\eta'')}\, \tau_{\gamma(\eta'')\leftarrow\gamma^{(n')}}\, \psi^{(n')}_{\gamma^{(n')}} \,\middle|\, \left(\mathrm{id}_{\mathcal{H}_{\eta''\to(m)-(n)}} \otimes |\zeta_{(m)-(n)}\rangle\langle\zeta_{(m)-(n)}|\right) \right|$$

$$\left| \Phi_{\eta'' \to (m)-(n)} \, \Gamma^{-1}_{\eta'' \to \gamma(\eta'')} \, \tau_{\gamma(\eta'') \leftarrow \gamma^{(n')}} \, \psi^{(n')}_{\gamma^{(n')}} \right\rangle_{\mathcal{H}_{\eta'' \to (m)-(n)} \otimes \mathcal{H}_{(m)-(n)}} . \qquad (12.13.6)$$

Let $h'' \in \mathcal{C}_{\eta''}$ and $\left(\left(\widetilde{j}'^{(0)}, \widetilde{j}'^{(2)}, \widetilde{j}'^{(3)} \right), \widetilde{j}; \widetilde{h} \right) := \varphi_{\eta'' \to (m)-(n)}(h'')$. Let $\widetilde{\gamma}^{(0)} := H^{(0)}_{\eta'' \to \widetilde{\eta}}$ and $\widetilde{\gamma}^{(2)} := H^{(2)}_{\eta'' \to \widetilde{\eta}}$. Since $\widetilde{\gamma}^{(0)}, \widetilde{\gamma}^{(2)} \subset \gamma(\eta'') \in \mathcal{L}_{\text{graphs}}$, we have $\widetilde{\gamma}^{(0)}, \widetilde{\gamma}^{(2)} \in \mathcal{L}_{\text{graphs}} \subset \mathcal{L}_{\text{AL}}$ and $\widetilde{\gamma}^{(0)}, \widetilde{\gamma}^{(2)} \preccurlyeq \gamma(\eta'')$. Moreover, the expression for $\varphi_{\eta'' \to \widetilde{\eta}}$ (prop. 10.24) together with the uniqueness part of prop. 12.5 yields:

$$\widetilde{j}'^{(0)} = h''|_{\widetilde{\gamma}^{(0)}} = \pi_{\gamma(\eta'') \to \widetilde{\gamma}^{(0)}}(h''), \quad \widetilde{j}'^{(2)} = h''|_{\widetilde{\gamma}^{(2)}} = \pi_{\gamma(\eta'') \to \widetilde{\gamma}^{(2)}}(h''),$$

$$\& \quad \pi^{-1}_{\widetilde{\gamma}_\sharp \to \gamma^{(m)-(n)}}(\widetilde{h}), \, \widetilde{j} = \varphi_{\widetilde{\gamma} - \left\{ e_{[p_\infty, p_n]} \right\}} \circ \pi_{\gamma(\eta'') \to \widetilde{\gamma}}(h'').$$

Using eq. (12.6.1) and the expression for $\varphi_{\gamma_2 - \gamma_1}$ from the proof of prop. 12.7, the last relation above becomes:

$$\widetilde{h} = \pi_{\gamma(\eta'') \to \gamma^{(m)-(n)}}(h'') \quad \& \quad \widetilde{j} = \pi_{\gamma(\eta'') \to \left\{ e_{[p_\infty, p_n]} \right\}}(h'').$$

Next, since $e_{[p_o, p_n]} \in F_{\daleth}(\widetilde{\eta})$ and $\bigcup_{\widetilde{e} \in \widetilde{\gamma}} r(\widetilde{e}) = r(e_{[p_n, p_\infty]})$, we have, for any $e' \in \gamma^{(n)}$ and any $r \in \left\{ 1, \dots, n_{\gamma(\eta'') \to \gamma^{(n)}, e'} \right\}$:

$$a_{\gamma(\eta'') \to \gamma^{(n)}, e'}(r) \in F_{\daleth}(\widetilde{\eta}) \quad \& \quad \forall \widetilde{e} \in \widetilde{\gamma}, \, r\left[a_{\gamma(\eta'') \to \gamma^{(n)}, e'}(r) \right] \not\subset r(\widetilde{e}).$$

Hence $a_{\gamma(\eta'') \to \gamma^{(n)}, e'}(r) \in H^{(0)}_{\eta'' \to \widetilde{\eta}}$, so that $\gamma^{(n)} \preccurlyeq \widetilde{\gamma}^{(0)}$. Similarly, if $n' \geqslant m + 1$, $e_{[p_m, p_\infty]} \in \widetilde{\gamma}$ implies $\left\{ e_{[p_{m+1}, p_\infty]} \right\} \preccurlyeq \widetilde{\gamma}^{(2)}$ (we have $\left\{ e_{[p_{m+1}, p_\infty]} \right\} \preccurlyeq \gamma(\eta'')$ for $\left\{ e_{[p_m, p_{m+1}]} \right\} \preccurlyeq \gamma^{(n')} \preccurlyeq \gamma(\eta'')$ and $\left\{ e_{[p_m, p_\infty]} \right\} \preccurlyeq \widetilde{\gamma} \preccurlyeq \gamma(\eta'')$), as well as $\gamma^{(n')-(m+1)} \preccurlyeq \widetilde{\gamma}^{(2)}$ (if n' happens to be bigger than $m + 2$). Then, using repeatedly eq. (12.6.1), we get:

$$\forall k \in \{1, \dots, n\}, \, \pi_{\gamma(\eta'') \to \gamma^{(n')}}(h'') \left(e_{[p_{k-1}, p_k]} \right) = \pi_{\widetilde{\gamma}^{(0)} \to \gamma^{(n)}}(\widetilde{j}'^{(0)}) \left(e_{[p_{k-1}, p_k]} \right),$$

$$\forall k \in \{n+1, \dots, m\}, \, \pi_{\gamma(\eta'') \to \gamma^{(n')}}(h'') \left(e_{[p_{k-1}, p_k]} \right) = \widetilde{h} \left(e_{[p_{k-1}, p_k]} \right)$$

(note that $m \leqslant n'$ as underlined earlier),

$$\forall k \in \{m+2, \dots, n'\}, \, \pi_{\gamma(\eta'') \to \gamma^{(n')}}(h'') \left(e_{[p_{k-1}, p_k]} \right) = \pi_{\widetilde{\gamma}^{(2)} \to \gamma^{(n')-(m+1)}}(\widetilde{j}'^{(2)}) \left(e_{[p_{k-1}, p_k]} \right)$$

(of course, this only applies if $n' \geqslant m + 2$).

If $n' > m$ we also need the evaluation of $\pi_{\gamma(\eta'') \to \gamma^{(n')}}(h'')$ on $e_{[p_m, p_{m+1}]}$. For this, we notice that $\left\{ e_{[p_\infty, p_n]} \right\} \preccurlyeq \gamma^{[m, m+1]} \preccurlyeq \gamma(\eta'')$, where $\gamma^{[m, m+1]} := \left\{ e_{[p_n, p_m]}, e_{[p_m, p_{m+1}]}, e_{[p_{m+1}, p_\infty]} \right\}$. Using the explicit expression for $\pi_{\gamma^{[m, m+1]} \to \left\{ e_{[p_\infty, p_n]} \right\}}$ together with $\left\{ e_{[p_n, p_m]} \right\} \preccurlyeq \gamma^{(m)-(n)}$ and $\left\{ e_{[p_{m+1}, p_\infty]} \right\} \preccurlyeq \widetilde{\gamma}^{(2)}$ (and again repeatedly applying eq. (12.6.1)), we get:

$$\begin{aligned}
\widetilde{j}(e_{[p_\infty, p_n]}) &= \pi_{\gamma(\eta'') \to \left\{ e_{[p_\infty, p_n]} \right\}}(h'')(e_{[p_\infty, p_n]}) \\
&= \pi_{\gamma(\eta'') \to \gamma^{[m, m+1]}}(h'')(e_{[p_n, p_m]})^{-1} . \, \pi_{\gamma(\eta'') \to \gamma^{[m, m+1]}}(h'')(e_{[p_m, p_{m+1}]})^{-1} . \, \pi_{\gamma(\eta'') \to \gamma^{[m, m+1]}}(h'')(e_{[p_{m+1}, p_\infty]})^{-1} \\
&= \pi_{\gamma^{(m)-(n)} \to \left\{ e_{[p_n, p_m]} \right\}}(\widetilde{h})(e_{[p_n, p_m]})^{-1} . \, \pi_{\gamma(\eta'') \to \gamma^{(n')}}(h'')(e_{[p_m, p_{m+1}]})^{-1} . \, \pi_{\widetilde{\gamma}^{(2)} \to \left\{ e_{[p_{m+1}, p_\infty]} \right\}}(\widetilde{j}'^{(2)})(e_{[p_{m+1}, p_\infty]})^{-1} \\
&= \left[\prod_{k=n+1}^{m} \widetilde{h}(e_{[p_{k-1}, p_k]}) \right]^{-1} . \, \pi_{\gamma(\eta'') \to \gamma^{(n')}}(h'')(e_{[p_m, p_{m+1}]})^{-1} . \, \pi_{\widetilde{\gamma}^{(2)} \to \left\{ e_{[p_{m+1}, p_\infty]} \right\}}(\widetilde{j}'^{(2)})(e_{[p_{m+1}, p_\infty]})^{-1},
\end{aligned}$$

so that:

$$\pi_{\gamma(\eta'')\to\gamma^{(n')}}(h'')(e_{[p_m,p_{m+1}]}) = \left[\prod_{k=n+1}^{m} \widetilde{h}(e_{[p_{k-1},p_k]}) \,.\, \widetilde{j}(e_{[p_\infty,p_n]}) \,.\, \pi_{\widetilde{\gamma}^{(2)}\to\left\{e_{[p_{m+1},p_\infty]}\right\}}(\widetilde{j}'^{(2)})(e_{[p_{m+1},p_\infty]})\right]^{-1} .$$

We want to use the thus obtained relations between $\pi_{\gamma(\eta'')\to\gamma^{(n')}}$ and $\varphi_{\eta''\to(m)-(n)}$ in order to reformulate eq. (12.13.6). We first consider the case $n' = m$. Then, we have:

$$\Phi_{\eta''\to(m)-(n)}\,\Gamma^{-1}_{\eta''\to\gamma(\eta'')}\,\tau_{\gamma(\eta'')\leftarrow\gamma^{(n')}}\,\psi^{(n')}_{\gamma^{(n')}} = \psi_{\eta''\to(m)-(n)} \otimes \chi_{(m)-(n)}\,,$$

where:

$$\forall\left(\widetilde{j}'^{(0)}, \widetilde{j}'^{(2)}, \widetilde{j}'^{(3)}\right), \widetilde{j} \in \mathcal{C}_{\eta''\to(m)-(n)}\,,$$

$$\psi_{\eta''\to(m)-(n)}\left(\widetilde{j}'^{(0)}, \widetilde{j}'^{(2)}, \widetilde{j}'^{(3)}; \widetilde{j}\right) := \prod_{k=1}^{n}\left[\chi\circ\pi_{\widetilde{\gamma}^{(0)}\to\gamma^{(n)}}(\widetilde{j}'^{(0)})\right](e_{[p_{k-1},p_k]})$$

$$\&\quad \forall\,\widetilde{h}\in\mathcal{C}_{(m)-(n)}\,,\ \chi_{(m)-(n)}\big(\widetilde{h}\big) := \prod_{k=n+1}^{m}\big[\chi\circ\widetilde{h}\big](e_{[p_{k-1},p_k]})\,.$$

Thus, eq. (12.13.6) becomes:

$$\mathrm{Tr}_{\mathcal{H}_{\eta'}}\,\rho_{\eta'}\,\Theta_{\eta'|\eta} \leqslant \|\psi_{\eta''\to(m)-(n)}\|^2_{\mathcal{H}_{\eta''\to(m)-(n)}}\left|\langle\zeta_{(m)-(n)}\mid\chi_{(m)-(n)}\rangle_{\mathcal{H}_{(m)-(n)}}\right|^2 .$$

And, since $\left\|\psi^{(n')}_{\gamma^{(n')}}\right\|_{\mathcal{H}_{\gamma^{(n')}}} = 1$, we get:

$$\mathrm{Tr}_{\mathcal{H}_{\eta'}}\,\rho_{\eta'}\,\Theta_{\eta'|\eta} \leqslant \frac{\left|\langle\zeta_{(m)-(n)}\mid\chi_{(m)-(n)}\rangle_{\mathcal{H}_{(m)-(n)}}\right|^2}{\|\chi_{(m)-(n)}\|^2_{\mathcal{H}_{(m)-(n)}}} = \frac{\left|\int_G d\mu(g)\,\chi(g)\right|^{2M}}{\|\chi\|^{2M}} \leqslant \left(\int_G d\mu(g)\ |\chi(g)|\right)^{2M} .$$

We now consider the case $n' > m$. Here, we get:

$$\forall\big(\widetilde{j}', \widetilde{j}\big)\in\mathcal{C}_{\eta''\to(m)-(n)}\,,\ \forall\widetilde{h}\in\mathcal{C}_{(m)-(n)}\,,$$

$$\Phi_{\eta''\to(m)-(n)}\,\Gamma^{-1}_{\eta''\to\gamma(\eta'')}\,\tau_{\gamma(\eta'')\leftarrow\gamma^{(n')}}\,\psi^{(n')}_{\gamma^{(n')}}\left[\widetilde{j}', \widetilde{j}; \widetilde{h}\right] =$$
$$= \psi_{\eta''\to\widetilde{\eta}}\big(\widetilde{j}'\big)\ \ \chi_{(m)-(n)}\big(\widetilde{h}\big)\ \ \chi\left[\beta_{\eta''\to\widetilde{\eta}}\big(\widetilde{j}'\big)\,.\,\widetilde{j}(e_{[p_\infty,p_n]})^{-1}\,.\,\beta_{(m)-(n)}\big(\widetilde{h}\big)\right],$$

where:

$$\forall\left(\widetilde{j}'^{(0)}, \widetilde{j}'^{(2)}, \widetilde{j}'^{(3)}\right)\in\mathcal{C}_{\eta''\to\widetilde{\eta}}\,,$$

$$\psi_{\eta''\to\widetilde{\eta}}\left(\widetilde{j}'^{(0)}, \widetilde{j}'^{(2)}, \widetilde{j}'^{(3)}\right)$$
$$:= \left(\prod_{k=1}^{n}\left[\chi\circ\pi_{\widetilde{\gamma}^{(0)}\to\gamma^{(n)}}(\widetilde{j}'^{(0)})\right](e_{[p_{k-1},p_k]})\right)\ \left(\prod_{k=m+2}^{n'}\left[\chi\circ\pi_{\widetilde{\gamma}^{(2)}\to\gamma^{(n')-(m+1)}}(\widetilde{j}'^{(2)})\right](e_{[p_{k-1},p_k]})\right),$$

$$\beta_{\eta''\to\widetilde{\eta}}\left(\widetilde{j}'^{(0)}, \widetilde{j}'^{(2)}, \widetilde{j}'^{(3)}\right) := \left[\pi_{\widetilde{\gamma}^{(2)}\to\left\{e_{[p_{m+1},p_\infty]}\right\}}(\widetilde{j}'^{(2)})(e_{[p_{m+1},p_\infty]})\right]^{-1},$$

$$\&\quad \forall\,\widetilde{h}\in\mathcal{C}_{(m)-(n)}\,,\ \beta_{(m)-(n)}(\widetilde{h}) := \prod_{k=m}^{n+1}\widetilde{h}(e_{[p_{k-1},p_k]})^{-1} .$$

Thus, eq. (12.13.6) now reads:

$$\mathrm{Tr}_{\mathcal{H}_{\eta'}} \rho_{\eta'}\, \Theta_{\eta'|\eta} \leqslant$$

$$\int_{\mathcal{C}_{\eta''\to\widetilde{\eta}}} d\mu_{\eta''\to\widetilde{\eta}}(\widetilde{j}') \quad \left|\psi_{\eta''\to\widetilde{\eta}}\left(\widetilde{j}'\right)\right|^2 \int_{\mathcal{C}_{(m)-(n)}\times\mathcal{C}_{(m)-(n)}} d\mu^{(2)}_{(m)-(n)}(\widetilde{h},\, \widetilde{h}') \;\; \chi^*_{(m)-(n)}\left(\widetilde{h}\right) \chi_{(m)-(n)}\left(\widetilde{h}'\right)$$

$$\int_G d\mu(t) \;\; \chi^*\left[\beta_{\eta''\to\widetilde{\eta}}\left(\widetilde{j}'\right) . \, t^{-1} . \, \beta_{(m)-(n)}\left(\widetilde{h}\right)\right] \chi\left[\beta_{\eta''\to\widetilde{\eta}}\left(\widetilde{j}'\right) . \, t^{-1} . \, \beta_{(m)-(n)}\left(\widetilde{h}'\right)\right],$$

while the normalization condition $\left\|\psi^{(n')}_{\gamma^{(n')}}\right\|_{\mathcal{H}_{\gamma^{(n')}}} = 1$ yields:

$$1 = \int_{\mathcal{C}_{\eta''\to\widetilde{\eta}}} d\mu_{\eta''\to\widetilde{\eta}}(\widetilde{j}') \quad \left|\psi_{\eta''\to\widetilde{\eta}}\left(\widetilde{j}'\right)\right|^2 \int_{\mathcal{C}_{(m)-(n)}} d\mu_{(m)-(n)}(\widetilde{h}) \quad \left|\chi_{(m)-(n)}\left(\widetilde{h}\right)\right|^2 \int_G d\mu(t) \quad \left|\chi(t)\right|^2$$

(the measure μ being invariant under the transformation $t \mapsto t_1 . t^{-1} . t_2$ for any $t_1 ,\ t_2 \in G$)

$$= \int_{\mathcal{C}_{\eta''\to\widetilde{\eta}}} d\mu_{\eta''\to\widetilde{\eta}}(\widetilde{j}') \quad \left|\psi_{\eta''\to\widetilde{\eta}}\left(\widetilde{j}'\right)\right|^2 \qquad \text{(since } \|\chi\| = 1\text{)}.$$

Moreover, for any $t_1 ,\ t_2 ,\ t_2' \in G$, the Cauchy-Schwarz inequality ensures that:

$$\left|\int_G d\mu(t) \;\; \chi^*\left[t_1 . \, t^{-1} . \, t_2\right] \chi\left[t_1 . \, t^{-1} . \, t_2'\right]\right| \leqslant \|\chi\|^2 = 1\,,$$

so we again get:

$$\mathrm{Tr}_{\mathcal{H}_{\eta'}} \rho_{\eta'}\, \Theta_{\eta'|\eta} \leqslant \int_{\mathcal{C}_{\eta''\to\widetilde{\eta}}} d\mu_{\eta''\to\widetilde{\eta}}(\widetilde{j}') \quad \left|\psi_{\eta''\to\widetilde{\eta}}\left(\widetilde{j}'\right)\right|^2 \int_{\mathcal{C}_{(m)-(n)}\times\mathcal{C}_{(m)-(n)}} d\mu^{(2)}_{(m)-(n)}(\widetilde{h},\, \widetilde{h}') \quad \left|\chi^*_{(m)-(n)}\left(\widetilde{h}\right)\right| \left|\chi_{(m)-(n)}\left(\widetilde{h}'\right)\right|$$

$$= \left(\int_G d\mu(g)\; |\chi(g)|\right)^{2M}.$$

Since χ has been chosen so that $\int_G d\mu(g)\; |\chi(g)| < 1$, there exists, for any $\epsilon > 0$, an odd integer $M \geqslant 1$ with $\left(\int_G d\mu(g)\; |\chi(g)|\right)^{2M} < \epsilon$. Thus, there exists, for any $\eta \in \mathcal{L}_{\text{HF}}$ and any $\epsilon > 0$, $\eta' \succcurlyeq \eta$ such that:

$$\mathrm{Tr}_{\mathcal{H}_{\eta'}} \rho_{\eta'}\, \Theta_{\eta'|\eta} < \epsilon\,.$$

Hence, for any $\eta \in \mathcal{L}_{\text{HF}}$, $\inf_{\eta'\succcurlyeq\eta} \mathrm{Tr}_{\mathcal{H}_{\eta'}} \rho_{\eta'}\, \Theta_{\eta'|\eta} = 0$, and, therefore, $\sup_{\eta\in\mathcal{L}_{\text{HF}}} \inf_{\eta'\succcurlyeq\eta} \mathrm{Tr}_{\mathcal{H}_{\eta'}} \rho_{\eta'}\, \Theta_{\eta'|\eta} = 0$. On the other hand theorem 12.11.5 implies:

$$\sigma\left\langle\overline{\mathcal{S}}_{\text{AL}}\right\rangle = \left\{\rho' = \left(\rho'_\eta\right)_{\eta\in\mathcal{L}_{\text{HF}}} \;\middle|\; \sup_{\eta\in\mathcal{L}_{\text{HF}}} \inf_{\eta'\succcurlyeq\eta} \mathrm{Tr}_{\mathcal{H}_{\eta'}} \rho'_{\eta'}\, \Theta_{\eta'|\eta} = \mathrm{Tr}\,\rho'\right\},$$

and we have $\mathrm{Tr}\,\rho = \mathrm{Tr}_{\mathcal{H}_\eta} \rho_\eta = \left\|\psi^{(n_\eta)}\right\|_{\mathcal{H}_{\text{AL}}} = 1$ (for some $\eta \in \mathcal{L}_{\text{HF}}$), so $\rho \notin \sigma\left\langle\overline{\mathcal{S}}_{\text{AL}}\right\rangle$. □

So, we have obtained a clear picture of how the projective state space set up in prop. 12.1 relates to the more conventional Ashtekar-Lewandowski state space. Recall that our motivation for this construction was to extend the latter, to try and cure the difficulties arising in the search for good semi-classical states. The observation of prop. 12.13, confirming that we indeed have gained new states in the process, is in this respect particularly important.

The state we used to prove this result can even be seen as a first step toward the design of satisfactory semi-classical states. Indeed, if we take as 'pattern' χ a coherent state (eg. a Hall state [42], which is the generalization over a compact Lie group of a Gaussian state), we obtain a projective state yielding a narrow distribution for infinitely many holonomies (namely the ones along the infinitely many pieces of the base edge e), while such a state could not exist over $\mathcal{H}_{AL}$.

Still, this would not yet be a state suitable for the study of the semi-classical limit, where we would need states presenting narrow distributions for a full set of holonomies and fluxes. There remain in fact further obstructions to this endeavor, the understanding and overcoming of which will be the topic of chap. 6.

13. Discussion: imposing the constraints (1/2)

An issue that will have to be addressed thoroughly before the projective formalism can provide a serious alternative to the successful inductive one, is how to solve at least the Gauss and diffeomorphism constraints (see the brief overview of the ADM formalism in the main introduction, and of the Ashtekar variables in section 8), as those can be readily solved on $\mathcal{H}_{AL}$ [4, 8, 92]. In section 3 we proposed a strategy to deal with constraints in the projective context, with the help of a suitably defined regularization scheme (this proposal was developed at the classical level, ie. in the setting of a projective limit of symplectic manifolds, however we will display on an example, in subsection 16.2, how a similar approach could be implemented at the quantum level).

Note that while the Gauss constraints are well-adapted to the inductive structure underlying the Ashtekar-Lewandowski Hilbert space (ie. they leave the fixed-graph subspaces $\mathcal{H}_\gamma$ invariant, which allows for their straightforward resolution in $\mathcal{H}_{AL}$), they are *not* adapted (in the sense of subsection 3.1) to the projective structure we have introduced: while gauge transformations preserve the algebra of holonomies attached to a graph, they do *not* preserve the algebra of fluxes attached to a profile. In fact, as stressed at the end of subsection 9.2, fluxes do not at all transform nicely under gauge transformations (except when the gauge transformation happens to be constant on the surface supporting the considered flux). A popular method to circumvent this difficulty is to use, instead of the standard fluxes, appropriately 'anchored' ones [89, def. 3.5]: by choosing, for each face, a supporting system of paths, we can parallel transport the electric field at each point of the face back to a common root, thus forming an observable with better transformations properties. Fluxes of this kind have for example been used in [23] to build a projective structure of *momentum* spaces. Yet, a complete solution based on this device will require some more work, because again one needs to ensure, at the same time, that the labels we are using can be properly associated to an algebra of observables (ie. that the observables assigned to a given label form a subset closed under Poisson brackets), and that they build a directed preordered set (with a preorder that respects the relations between the associated algebras of observables). Specifically, we will have to keep track, in each label, of the system of paths used to define its anchored fluxes, thus putting the directedness at stake (as underlined in section 7, the richer the label structure, the harder it is to arrange for the label set to be directed). It turns out that the tools we will develop in section 19 could help designing a suitable label set in this context, and we will come back to this point in section 20.

Similarly, the techniques developed for the resolution of the diffeomorphism constraints in $\mathcal{H}_{AL}$ [8] cannot be directly imported into the projective formalism because they critically rely on having states built from discrete excitations. Still, they suggest that the building blocks for the space of solutions should be indexed by equivalence classes of labels under diffeomorphisms, in other words that we should try to quotient out all embedding knowledge from the projective structure. However, defining the projection between two labels is subtle, when there is no embedding telling us their respective disposition: if one takes a small embeddingless label η and a larger one η', it will in general not be possible to unequivocally identify η with a sublabel of η' (this is sidestepped in the construction of the diffeomorphism invariant version of $\mathcal{H}_{AL}$, because distinct graphs can there be made to label mutually orthogonal sectors, with no need to relate their respective degrees of freedom). A prospective way out could be to position η within η' *depending* on the state we happen to have over η'. More precisely, we would like to select, among all the inequivalent sublabels of η' that can be identified with η, the one that captures the most relevant information about this state. We can regard a prescription of this kind as a means to give diffeomorphism-invariant meaning to the observables attached to a given label, by deparametrizing them using as reference field the geometric data itself. As such it is similar to the deployment of *intrinsic* coordinates systems that are defined purely in terms of the geometry [57, 58].

While the solution of the Gauss and diffeomorphism constraints on $\mathcal{H}_{AL}$ is explicitly known, the resolution of the Hamiltonian ones is less understood. These constraints *can* be regularized [88] to define corresponding operators on $\mathcal{H}_{AL}$, and, with the help of the Master Constraint program [91, 36], the *existence* of a Hilbert space collecting their solutions can be established, yet there remain some computational hurdles. One can therefore speculate whether this known regularization scheme for the Hamiltonian constraints could be combined with the strategy from section 3 in order to arrive at a constructive description of a space of states solving the quantum dynamics of gravity. Note that the ideas sketched above to define the diffeomorphism-invariant projective structure would have the added benefit of pushing the most interesting information about the states down toward the 'coarsest' labels: this could in turn improve the convergence properties of a subsequent regularization of the Hamiltonian constraints, as it would make it easier to access the data we need to correctly evolve the spatial slice.

We will continue this discussion in section 20, in the light of the developments covered in chaps. 5 and 6.

Applications and Toy-Models

Chapter 5 – Examples of Constraints Regularizations

14. Introduction

In section 3, we introduced a strategy to deal with dynamical constraints in a projective limit of symplectic manifolds. After having convinced ourselves that a regularization of these constraints will in general be necessary, since we cannot expect them to be adapted to the projective system, we adopted the perspective that a dynamical state can be identified with the family of successive approximations approaching an exact solution of the dynamics. On the one hand, this allows us to put the dynamical state space into a projective form. On the other hand, it also provides a suitable ground for a notion of convergence, that will make it possible to define meaningful physical observables on this state space.

However, applying this procedure demands that one sets up a regularization scheme fulfilling a number of restrictive properties (summarized in prop. 3.23), which raises the question of its practicability. Hence, we now want to discuss two simple examples, meant as 'proofs of concept' that such schemes can indeed be designed.

Note that the framework in section 3 was purely classical. We have not yet undertaken to formulate a general procedure regarding the resolution of dynamical constraints in projective systems of quantum state spaces. Nevertheless, our second example will explore how analogous ideas can be implemented at the quantum level, and will give us the opportunity to delineate an appropriate course and to underline possible difficulties.

15. Linear constraints on a Kähler vector space

This first example is arguably mostly artificial and does not pretend to have great physical relevance. Our motivation here is to illustrate the concepts introduced in sections 2 and 3 in the simplest possible setup. We consider an infinite dimensional Hilbert space $\mathcal{H}$ (which is nothing but a linear Kähler manifold) and form its rendering by a projective structure of finite dimensional Hilbert spaces (to prevent any confusion: the Hilbert spaces in discussion here are the phase spaces of classical systems, there will be nothing quantum in the present section). This rendering is built from an Hilbert basis of $\mathcal{H}$ by considering all the vector subspaces of $\mathcal{H}$ spanned by a finite number of basis vectors and linking them by orthogonal projections (a more satisfactory rendering for $\mathcal{H}$, namely one that does not require the choice of a preferred basis, will be presented in section 16; however we do not want to use it here, since the constraints we will be looking at could be directly formulated as an elementary reduction over a cofinal part of its label set, and it would therefore not be appropriate as an example for the regularization procedure).

Proposition 15.1 Let $\mathcal{H}, \langle\,\cdot\,,\,\cdot\,\rangle$ be a complex Hilbert space and define:

1. $\forall v \in \mathcal{H},\ J\,v := i\,v$;
2. $\forall v, w \in \mathcal{H},\ \Omega(v,\ w) := 2\,\mathrm{Im}\left(\langle v,\ w\rangle\right)$.

Then, $\mathcal{H},\ \Omega,\ J$ is a Kähler manifold.

Proof The real scalar product $\mathrm{Re}\,\langle\,\cdot\,,\,\cdot\,\rangle$ equips $\mathcal{H}$ (seen as a real vector space) with a structure of real Hilbert space, therefore, any bounded real-valued real-linear form on $\mathcal{H}$ can be written as $\mathrm{Re}\,\langle v,\ \cdot\,\rangle = 2\,\mathrm{Im}\left\langle -\frac{i}{2}\,v,\ \cdot\,\right\rangle = \Omega\left(-\frac{i}{2}\,v,\ \cdot\,\right)$ for some $v \in \mathcal{H}$. Hence, Ω is a strong symplectic structure.

Next, J is by construction a complex structure on $\mathcal{H}$. We have $\forall v, w \in \mathcal{H},\ \Omega(i\,v,\ i\,w) = \Omega(v,\ w)$, and $v \mapsto \Omega(v,\ i\,v) = 2\,\mathrm{Re}\,\langle v,\ v\rangle$ is positive definite.

The integrability conditions for Ω and J are trivially satisfied since we actually have a Kähler *vector space*. □

Proposition 15.2 Let $\mathcal{H}$ be a separable, infinite dimensional Hilbert space (equipped with the strong symplectic structure Ω defined in prop. 15.1) and let $(e_i)_{i\in\mathbb{N}}$ be an Hilbert basis of $\mathcal{H}$. We define:

1. $\mathcal{L} := \{I \subset \mathbb{N} \mid 0 < \#I < \infty\}$ equipped with the preorder defined by $\subset$;
2. $\forall I \in \mathcal{L},\ \mathcal{H}_I := \mathrm{Vect}\,\{e_i \mid i \in I\}$ equipped with the induced symplectic structure Ω_I (which is also the natural symplectic structure on $\mathcal{H}_I$ as a finite dimensional Hilbert space);
3. $\forall I \subset I' \in \mathcal{L},\ \pi_{I'\to I} := \Pi_I|_{\mathcal{H}_{I'}\to\mathcal{H}_I}$ where Π_I is the orthogonal projection on $\mathcal{H}_I$;
4. $\mathcal{H}_{\mathbb{N}} := \mathcal{H}$ and $\forall I \in \mathcal{L},\ \pi_{\mathbb{N}\to I} := \Pi_I|_{\mathcal{H}\to\mathcal{H}_I}$.

Then, this defines a rendering (def. 2.6) of the symplectic manifold $\mathcal{H}$ by the projective system of phase spaces $(\mathcal{L},\ \mathcal{H},\ \pi)^{\downarrow}$. We define $\sigma_{\downarrow} : \mathcal{H} \to \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{H},\pi)}$ as in def. 2.6.

Additionally, defining the dense vector subspace of $\mathcal{H}$, $\mathcal{D} := \mathrm{Vect}\,\{e_i \mid i \in \mathbb{N}\}$ (without completion, ie. the space of finite linear combinations of the e_i), we have a bijective antilinear map

$\zeta : \mathcal{D}^* \to \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{H},\pi)}$ such that $\zeta^{-1} \circ \sigma_{\downarrow} : \mathcal{H} \to \mathcal{D}^*$ is the canonical identification of $\mathcal{H}$ with $\mathcal{D}' \subset \mathcal{D}^*$ (where $\mathcal{D}^*$ is the algebraical dual of $\mathcal{D}$ and $\mathcal{D}'$ the topological one).

Proof $\mathcal{L}$ is a directed set, since $\forall l, l' \in \mathcal{L}$, $l \cup l' \in \mathcal{L}$ and $l, l' \subset l \cup l'$.

Let $l, l' \in \mathcal{L} \sqcup \{\mathbb{N}\}$ with $l \subset l'$. $\pi_{l' \to l}$ is surjective by construction. Next, since $\mathcal{H}_l$ is closed, we have, for any bounded real-valued real-linear form υ on $\mathcal{H}_l$, a vector $\underline{\upsilon} \in \mathcal{H}_l$ such that:

$$\forall v \in \mathcal{H}_l,\ \upsilon(v) = \Omega_l(\underline{\upsilon}, v) = \text{Re}\ \langle 2i\, \underline{\upsilon}, v \rangle_l .$$

Hence, since Π_l is the $\mathbb{C}$-orthogonal projection on the complex vector subspace $\mathcal{H}_l$, it is also the $\mathbb{R}$-orthogonal projection on the real vector subspace $\mathcal{H}_l$, and we have:

$$\forall v \in \mathcal{H}_{l'},\ \upsilon \circ \pi_{l' \to l}(v) = \text{Re}\ \langle 2i\, \underline{\upsilon}, \Pi_l\, v \rangle_l = \text{Re}\ \langle 2i\, \underline{\upsilon}, v \rangle_{l'} = \Omega_{l'}(\underline{\upsilon}, v)\,,$$

and therefore $\pi_{l' \to l}\left(\underline{\upsilon \circ \pi_{l' \to l}}\right) = \pi_{l' \to l}\left(\underline{\upsilon}\right) = \underline{\upsilon}$.

Clearly for $l \in \mathcal{L}$, we have $\pi_{l \to l} = \text{id}_{\mathcal{H}_l}$ and for $l, l', l'' \in \mathcal{L} \sqcup \{\mathbb{N}\}$ with $l \subset l' \subset l''$, $\pi_{l' \to l} \circ \pi_{l'' \to l'} = \pi_{l'' \to l}$.

Lastly, we define:

$$\begin{array}{rcl} \zeta\ :\ \mathcal{D}^* & \to & \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{H},\pi)} \\ \upsilon & \to & \left(\overline{\upsilon|_{\mathcal{H}_l}}\right)_{l \in \mathcal{L}} \end{array} .$$

where for all $l \in \mathcal{L}$, $\overline{(\,\cdot\,)} : \mathcal{H}_l^* \to \mathcal{H}_l$ is the canonical identification provided by the complex Hilbert space structure on $\mathcal{H}_l$ ($\mathcal{H}_l$ is finite dimensional, hence $\mathcal{H}_l^* = \mathcal{H}_l'$).

The map ζ is well-defined, since $\forall l \subset l' \in \mathcal{L}$, $\forall v \in \mathcal{H}_l$, $\left\langle \pi_{l' \to l}\left(\overline{\upsilon|_{\mathcal{H}_{l'}}}\right), v \right\rangle_l = \left\langle \overline{\upsilon|_{\mathcal{H}_{l'}}}, v \right\rangle_{l'} = \upsilon(v) = \left\langle \overline{\upsilon|_{\mathcal{H}_l}}, v \right\rangle_l$, hence $\pi_{l' \to l}\left(\overline{\upsilon|_{\mathcal{H}_{l'}}}\right) = \overline{\upsilon|_{\mathcal{H}_l}}$.

On the other hand, we define $\widetilde{\zeta} : \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{H},\pi)} \to \mathcal{D}^*$, by:

$$\forall\, (v_l)_{l \in \mathcal{L}}\,,\ \forall w \in \mathcal{D},\ \widetilde{\zeta}\left((v_l)_{l \in \mathcal{L}}\right)(w) = \langle v_l, w \rangle_l \text{ for any } l \in \mathcal{L} \text{ such that } w \in \mathcal{H}_l.$$

The map $\widetilde{\zeta}$ is well-defined since $\mathcal{D} = \bigcup_{l \in \mathcal{L}} \mathcal{H}_l$ and if $l, l' \in \mathcal{L}$ are such that $w \in \mathcal{H}_l \cap \mathcal{H}_{l'}$, then there exists $l'' \in \mathcal{L}$ such that $l, l' \subset l''$ and:

$$\langle v_l, w \rangle_l = \langle \pi_{l'' \to l}(v_{l''}), w \rangle_l = \langle v_{l''}, w \rangle_{l''} = \langle v_{l'}, w \rangle_{l'} .$$

Now, we have $\widetilde{\zeta} \circ \zeta = \text{id}_{\mathcal{D}^*}$, $\zeta \circ \widetilde{\zeta} = \text{id}_{\mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{H},\pi)}}$ and $\forall v \in \mathcal{H}$, $\forall l \in \mathcal{L}$, $\forall w \in \mathcal{H}_l \subset \mathcal{D}$, $\widetilde{\zeta} \circ \sigma_{\downarrow}(v)\,(w) = \langle \pi_{\mathbb{N} \to l}(v), w \rangle_l = \langle v, w \rangle_{\mathcal{H}}$. □

We now present the constraint surface of interest, as a real vector subspace of $\mathcal{H}$ admitting a description of a specific form (alternatively, we could characterize it by of a family of linear holomorphic second class constraints and a family of linear first class constraints). Additionally, we anticipate on the regularization of the constraints by providing a rendering (similar to the one we adopted for $\mathcal{H}$) for the corresponding reduced phase space.

Proposition 15.3 We consider the same objects as in prop. 15.2. Let $(f_j)_{j\in\mathbb{N}}$ and $(g_k)_{k\in\mathbb{N}}$ be two, mutually orthogonal, orthonormal families in $\mathcal{H}$. We define:

1. $\mathcal{J} := \overline{\mathrm{Vect}_{\mathbb{C}}\left\{f_j \mid j \in \mathbb{N}\right\}}$ (equipped with the induced symplectic structure $\Omega_{\mathcal{J}}$) and $\mathcal{K}_{\mathbb{R}} := \overline{\mathrm{Vect}_{\mathbb{R}}\left\{g_k \mid k \in \mathbb{N}\right\}}$;
2. $\delta : \mathcal{J} \oplus \mathcal{K}_{\mathbb{R}} \to \mathcal{J}$ by $\delta := \Pi_{\mathcal{J}}|_{\mathcal{J}\oplus\mathcal{K}_{\mathbb{R}}\to\mathcal{J}}$ where $\Pi_{\mathcal{J}}$ is the orthogonal projection on $\mathcal{J}$.

Then $(\mathcal{J},\, \mathcal{J} \oplus \mathcal{K}_{\mathbb{R}},\, \delta)$ is a phase space reduction of $\mathcal{H}$ (def. A.1).

Additionally, we define:

3. $\forall J \in \mathcal{L},\ \mathcal{J}_J := \mathrm{Vect}_{\mathbb{C}}\left\{f_j \mid j \in J\right\}$ equipped with the induced symplectic structure Ω'_J;
4. $\forall K \in \mathcal{L},\ \mathcal{K}_K := \mathrm{Vect}_{\mathbb{C}}\left\{g_k \mid k \in K\right\}$ & $\mathcal{K}_{K,\mathbb{R}} := \mathrm{Vect}_{\mathbb{R}}\left\{g_k \mid k \in K\right\}$;
5. $\forall J \subset J' \in \mathcal{L},\ \pi'_{J'\to J} := \Pi'_J|_{\mathcal{J}_{J'}\to\mathcal{J}_J}$ where Π'_J is the orthogonal projection on $\mathcal{J}_J$;
6. $\mathcal{J}_{\mathbb{N}} := \mathcal{J}$ and $\forall J \in \mathcal{L},\ \pi'_{\mathbb{N}\to J} := \Pi'_J|_{\mathcal{J}\to\mathcal{J}_J}$.

As in prop. 15.2, this provides a rendering of $\mathcal{J}$ by $(\mathcal{L},\, \mathcal{J},\, \pi')^{\downarrow}$ and we define $\sigma'_{\downarrow} : \mathcal{J} \to \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{J},\pi')}$ as well as the bijective antilinear map $\zeta' : \mathcal{F}^* \to \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{J},\pi')}$ where $\mathcal{F} := \mathrm{Vect}\left\{f_j \mid j \in \mathbb{N}\right\}$.

Proof δ is a surjective linear map and for $v \in \mathcal{J}$, we have $\delta^{-1}\langle v\rangle = v + \mathcal{K}_{\mathbb{R}}$, hence $\delta^{-1}\langle v\rangle$ is connected. For $v, w \in \mathcal{J} \oplus \mathcal{K}_{\mathbb{R}}$, we write $v = v' + v''$ and $w = w' + w''$ with $v', w' \in \mathcal{J}$ and $v'', w'' \in \mathcal{K}_{\mathbb{R}}$. Then, we have:

$\Omega(v, w) = 2\,\mathrm{Im}\,\langle v,\, w\rangle_{\mathcal{H}} = 2\,\mathrm{Im}\,\langle v',\, w'\rangle_{\mathcal{H}} + 2\,\mathrm{Im}\,\langle v'',\, w''\rangle_{\mathcal{H}}$ (since $\mathcal{J} \perp \mathcal{K}_{\mathbb{R}}$)

$= 2\,\mathrm{Im}\,\langle v',\, w'\rangle_{\mathcal{J}} = \Omega_{\mathcal{J}}(\delta(v),\, \delta(w))$ (since $\mathcal{K}_{\mathbb{R}}$ is the real vector subspace generated by an orthonormal family).

Hence, $(\mathcal{J},\, \mathcal{J} \oplus \mathcal{K}_{\mathbb{R}},\, \delta)$ is a phase space reduction of $\mathcal{H}$. □

We are ready to turn to the core of the regularization procedure, namely formulating a set of approached implementations of the constraints (indexed by a label set $\mathcal{E}$), endowing $\mathcal{E}$ with an appropriate preorder, and linking together the approximate dynamics by supplying projecting maps between their reduced phase spaces.

Here we choose $\mathcal{E}$ to enumerate a large class of approximate solutions, ordered by comparing how good they are at approximating the exact solution (the precise definition of $\mathcal{E}$ may at first seem to arise from nowhere but will become transparent when we will actually detail the corresponding approximate constraint surfaces). This way of composing $\mathcal{E}$ will make the study the convergence mostly inexpensive: a large part of the work is actually done beforehand when checking that $\mathcal{E}$ with this preorder is really a directed set.

It also has the advantage of partially getting rid of the arbitrariness inherent of working with an approximating scheme. The philosophy is that an explicit, concretely implemented, approximating scheme will correspond to a specific cofinal part of $\mathcal{E}$, but that we have the option of considering all such particular schemes at the same time, by arranging them into a (huge) set $\mathcal{E}$, provided we carefully tailor its preorder to our purpose.

Besides, note that being quite broad in recruiting suitable approximate theories is, up to a certain extent, forced upon us by the fact that we are dealing with an unphysical and not further specified system, since, in a more realistic example, we could probably, from the physics of the system, infer guiding principles to be more selective.

On the other hand, we could fear that such a loose label set $\mathcal{E}$ will leave us with a disproportionately complicated projective structure for the dynamical theory. But, in fact, this dynamical structure (on $\mathcal{E}\mathcal{L}$) gets spontaneously quotiented down to the projective structure we had already introduced above for the dynamical state space. The idea is that we can transparently match two partial dynamical theories as soon as they have a common ancestor out of which they are carved in the same way (recall this mechanism was presented at the end of subsection 2.2, and expressed precisely in props. 2.8 and 2.9).

Definition 15.4 We consider the same objects as in prop. 15.3 and we define $\mathcal{E}$ as the set of all sextuples $\big(I,\, I',\, J,\, K,\, \varphi,\, \epsilon\big)$ such that:

1. $I \subset I' \in \mathcal{L} \quad \& \quad J,\, K \in \mathcal{L}$;
2. $\varphi : \mathcal{J}_J \oplus \mathcal{K}_K \to \mathcal{H}_{I'}$ is a linear application and $\varphi|_{\mathcal{J}_J\oplus\mathcal{K}_K\to\mathrm{Im}\varphi}$ is a unitary map;
3. $\epsilon > 0$ and $\forall v \in \mathcal{J}_J,\ \|v - \varphi(v)\| \leqslant \epsilon\, \|v\|$;
4. $\Pi_I \left\langle \varphi \left\langle \mathcal{K}_{K,\mathbb{R}} \right\rangle \right\rangle = \Pi_I \left\langle \mathcal{K}_{K,\mathbb{R}} \right\rangle$.

On $\mathcal{E}$ we define a preorder $\preccurlyeq$ by $\big(I_1,\, I'_1,\, J_1,\, K_1,\, \varphi_1,\, \epsilon_1\big) \preccurlyeq \big(I_2,\, I'_2,\, J_2,\, K_2,\, \varphi_2,\, \epsilon_2\big)$ iff:

5. $I_1 \subset I_2,\ I'_1 \subset I'_2,\ J_1 \subset J_2 \quad \& \quad K_1 \subset K_2$;
6. $\epsilon_2 \leqslant \epsilon_1$.

Proposition 15.5 We consider the same objects as in def. 15.4. Let $I \in \mathcal{L}$ and $\epsilon > 0$. Let $J,\, K \in \mathcal{L}$ such that:

$$\dim \Pi_I \left\langle \mathcal{J}_J \oplus \mathcal{K}_K \right\rangle = \dim \left(\mathcal{J}_J \oplus \mathcal{K}_K\right).$$

Then, there exist $I' \in \mathcal{L}$ and a linear application $\varphi : \mathcal{J}_J \oplus \mathcal{K}_K \to \mathcal{H}_{I'}$ such that $\big(I,\, I',\, J,\, K,\, \varphi,\, \epsilon\big) \in \mathcal{E}$.

Lemma 15.6 Let $\mathcal{H}$ be a Hilbert space and let $F,\, G$ be two finite dimensional vector subspaces of $\mathcal{H}$, such that $\dim \Pi_G \left\langle F \right\rangle = \dim F$, where Π_G denotes the orthogonal projection on G.

Then, there exists a unique linear application $\varphi_{F\to G} : F \to G$ satisfying:

1. $\varphi_{F\to G}|_{F\to\mathrm{Im}\varphi_{F\to G}}$ is a unitary map;
2. $\displaystyle\int_{S_F} d\mu_{S_F}(e)\ \|e - \varphi_{F\to G}(e)\|^2$ is minimal, where S_F is the unit sphere of F equipped with the measure induced by the euclidean structure of F.

For $v \in F$, $\displaystyle\|v - \varphi_{F\to G}(v)\| \leqslant 2\dim F\ \|v\| \sup_{\substack{e\in F\\ \|e\|=1}} \|e - \Pi_G(e)\|$

Proof *Existence and uniqueness.* Let $f = \dim F$. From $\dim \Pi_G \left\langle F \right\rangle = f$, Π_G induces a bijective map $F \to \Pi_G \left\langle F \right\rangle$, hence $\left\langle \Pi_G\left(\,\cdot\,\right),\, \Pi_G\left(\,\cdot\,\right) \right\rangle_G$ defines a positive definite sesquilinear map on F. Therefore, there exists an orthonormal basis $(e_i)_{i\in\{1,\ldots,f\}}$ such that:

$$\forall i,j\in\{1,\ldots,f\}\,,\ \left\langle \Pi_G(e_i),\,\Pi_G(e_j)\right\rangle = \lambda_i\,\delta_{ij} \text{ with } \lambda_i>0\,.$$

Let φ be a linear application $F\to G$ such that $\varphi|_{F\to\mathrm{Im}\varphi}$ is a unitary map. We define $B_{ij}\in\mathbb{C}$ for $i,j\in\{1,\ldots,f\}$ and $w_i\in G\cap(\Pi_G\langle F\rangle)^\perp$ for $i\in\{1,\ldots,f\}$ by:

$$\forall i\in\{1,\ldots,f\}\,,\ \varphi(e_i)=\frac{1}{\sqrt{\lambda_i}}\Pi_G(e_i)+\sum_j B_{ij}\,\frac{1}{\sqrt{\lambda_j}}\Pi_G(e_j)+w_i\,.$$

From $\left\langle\varphi(e_i),\,\varphi(e_j)\right\rangle_G=\delta_{ij}$, we have:

$$\forall i,j\in\{1,\ldots,f\}\,,\ B_{ij}^*+B_{ji}+\sum_k B_{ik}^*B_{jk}+\left\langle w_i,\,w_j\right\rangle=0\,. \tag{15.6.1}$$

With these notations, we have:

$$\begin{aligned}\int_{S_F} d\mu_{S_F}(e)\ \|e-\varphi(e)\|^2 &= \int_{S_F} d\mu_{S_F}(e)\ \|\Pi_G(e)-\varphi(e)\|^2+\|e-\Pi_G(e)\|^2\\
&=\sum_{i,j}\left(\int_{S_{\mathbb{C}^f}} d\mu_{S_{\mathbb{C}^f}}(x)\,x_i^*\,x_j\right)\left\langle\Pi_G(e_i)-\varphi(e_i),\,\Pi_G(e_j)-\varphi(e_j)\right\rangle+\int_{S_F}d\mu_{S_F}(e)\ \|e-\Pi_G(e)\|^2\\
&=\sum_i \mathrm{Vol}\,(S_{\mathbb{C}^f})\left[1+\lambda_i-2\sqrt{\lambda_i}\,\mathrm{Re}\,(1+B_{ii})\right]+\int_{S_F}d\mu_{S_F}(e)\ \|e-\Pi_G(e)\|^2\\
&=\sum_i \mathrm{Vol}\,(S_{\mathbb{C}^f})\left[1+\lambda_i-2\sqrt{\lambda_i}+\sqrt{\lambda_i}\,\sum_k|B_{ik}|^2+\sqrt{\lambda_i}\,\|w_i\|^2\right]+\int_{S_F}d\mu_{S_F}(e)\ \|e-\Pi_G(e)\|^2\end{aligned}$$

(using eq. (15.6.1)).

Hence, this expression is minimal if and only if $\forall i,j\in\{1,\ldots,f\}\,,\ B_{ij}=0$ and $\forall i\in\{1,\ldots,f\}\,,\ w_i=0$. Therefore, we define $\varphi_{F\to G}$ by:

$$\forall i\in\{1,\ldots,f\}\,,\ \varphi_{F\to G}(e_i)=\frac{1}{\sqrt{\lambda_i}}\Pi_G(e_i)\,.$$

Bound on $\|v-\varphi_{F\to G}(v)\|$. Let $v=\sum_{j=1}^f v_j\,e_j\in F$. We have:

$$\|v-\varphi_{F\to G}(v)\|\leqslant\sum_{j=1}^f\left|v_j\right|\,\left\|e_j-\varphi_{F\to G}(e_j)\right\|\leqslant f\ \|v\|\,\sup_j\left\|e_j-\varphi_{F\to G}(e_j)\right\|\,.$$

Then, for $j\in\{1,\ldots,f\}$, $\left\|e_j-\Pi_G(e_j)\right\|^2+\lambda_j=1$ implies:

$$\left|1-\sqrt{\lambda_j}\right|=\left\|e_j-\Pi_G(e_j)\right\|\frac{\left\|e_j-\Pi_G(e_j)\right\|}{1+\sqrt{\lambda_j}}\leqslant\left\|e_j-\Pi_G(e_j)\right\|,$$

therefore:

$$\left\|e_j - \varphi_{F\to G}(e_j)\right\| \leqslant \left\|e_j - \Pi_G(e_j)\right\| + \left\|\Pi_G(e_j) - \varphi_{F\to G}(e_j)\right\|$$

$$= \left\|e_j - \Pi_G(e_j)\right\| + \left|\sqrt{\lambda_j} - 1\right| \leqslant 2\,\left\|e_j - \Pi_G(e_j)\right\|.$$

Hence, $\|v - \varphi_{F\to G}(v)\| \leqslant 2f\ \|v\| \sup_j \left\|e_j - \Pi_G(e_j)\right\| \leqslant 2f\ \|v\| \sup_{\substack{e\in F\\ \|e\|=1}} \|e - \Pi_G(e)\|$. □

Proof of prop. 15.5 Since $(e_i)_{i\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$ and $\mathcal{J}_J$ has finite dimension, we can find $l_1' \in \mathcal{L}$ such that:

1. $\sup_{\substack{e\in\mathcal{J}_J\\ \|e\|=1}} \left\|e - \Pi_{l_1'}(e)\right\| \leqslant \dfrac{\epsilon}{2\,\#J}$;

and l_2' such that:

2. $l_2' \cap l = \varnothing \quad \& \quad \dim \Pi_{l_2'} \langle \mathcal{J}_J \rangle + \dim\,(\mathcal{J}_J \cap \mathcal{H}_l) = \dim \mathcal{J}_J$;
3. $\dim\mathcal{H}_{l_2'} \geqslant \dim \Pi_{l_2'} \langle \mathcal{J}_J \rangle + \dim\mathcal{K}_K$.

Let $l' := l \cup l_1' \cup l_2'$ and $l_3' := l' \setminus l$. We have $\dim \Pi_{l_3'} \langle \mathcal{J}_J \rangle + \dim\,(\mathcal{J}_J \cap \mathcal{H}_l) = \dim \mathcal{J}_J$ and $\mathcal{K}_K \perp \mathcal{J}_J$, hence for all $k \in K$, there exists $g_k' \in \Pi_{l_3'} \langle \mathcal{J}_J \rangle$ such that:

$$\forall j \in J,\ \left\langle \Pi_l(f_j),\, \Pi_l(g_k)\right\rangle_l + \left\langle \Pi_{l_3'}(f_j),\, g_k'\right\rangle_{l_3'} = 0\,.$$

This holds because, for all families of coefficients $\left(\alpha^j\right)_{j\in J}$ such that $\sum_j \alpha^j \left\langle \Pi_{l_3'}(f_j),\, \cdot\,\right\rangle_{l_3'} = 0$, we have $\sum_j \alpha^{j*} f_j \in \mathcal{H}_l$ (from point 15.5.2 above), and therefore $\sum_j \alpha^j \left\langle \Pi_l(f_j),\, \Pi_l(g_k)\right\rangle_l = \left\langle \sum_j \alpha^{j*} f_j,\, g_k\right\rangle_{\mathcal{H}} = 0$. So, for all $k \in K$, $\left(\left\langle \Pi_l(f_j),\, \Pi_l(g_k)\right\rangle_l\right)_{j\in J}$ is in the image of $\left(\left\langle \Pi_{l_3'}(f_j),\, \cdot\,\right\rangle_{l_3'}\right)_{j\in J}$.

Next, using $\dim\mathcal{H}_{l_3'} \geqslant \dim \Pi_{l_3'} \langle \mathcal{J}_J \rangle + \dim\mathcal{K}_K$, there exists a family of vectors $g_k'' \in \mathcal{H}_{l_3'} \cap \left(\Pi_{l_3'} \langle \mathcal{J}_J \rangle\right)^{\perp}$ for all $k \in K$ such that:

$$\forall k, l \in K,\ \langle \Pi_l(g_k),\, \Pi_l(g_l)\rangle_l + \langle g_k',\, g_l'\rangle_{l_3'} + \langle g_k'',\, g_l''\rangle_{l_3'} = 0\,.$$

Now, we define $\varphi : \mathcal{J}_J \oplus \mathcal{K}_K \to \mathcal{H}_{l'}$ by:

$\forall j \in J,\ \varphi(f_j) := \varphi_{\mathcal{J}_J \to \mathcal{H}_{l'}}(f_j)$ (where $\varphi_{\mathcal{J}_J \to \mathcal{H}_{l'}}$ is defined as in lemma 15.6),

and $\forall k \in K,\ \varphi(g_k) := \dfrac{\Pi_l(g_k) + g_k' + g_k''}{\|\Pi_l(g_k) + g_k' + g_k''\|}$.

From the proof of lemma 15.6, $\varphi_{\mathcal{J}_J \to \mathcal{H}_{l'}} \langle \mathcal{J}_J \rangle = \Pi_{l'} \langle \mathcal{J}_J \rangle$, hence, for all $k \in K$, we have, by construction of g_k' and g_k'', $\varphi(g_k) \perp \varphi \langle \mathcal{J}_J \rangle$. Also by construction of g_k'', we have, for all $k, l \in K$, $\langle \varphi(g_k),\, \varphi(g_l)\rangle = \delta_{kl}$. Therefore φ induces an Hilbert space isomorphism $\mathcal{J}_J \oplus \mathcal{K}_K \to \mathrm{Im}\varphi$.

Finally, we can check that defs. 15.4.3 and 15.4.4 are fulfilled. □

Proposition 15.7 With the notations of def. 15.4, $\mathcal{E}$, $\preccurlyeq$ is a directed set.

Proof Let $\left(l_1,\, l_1',\, J_1,\, K_1,\, \varphi_1,\, \epsilon_1\right) \in \mathcal{E}$ and $\left(l_2,\, l_2',\, J_2,\, K_2,\, \varphi_2,\, \epsilon_2\right) \in \mathcal{E}$. We define $\widetilde{l} = l_1 \cup l_2$, $J = J_1 \cup J_2$, $K = K_1 \cup K_2$ and $\epsilon = \min(\epsilon_1,\, \epsilon_2) > 0$. Then, since $(e_i)_{i\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$, we can find $l \in \mathcal{L}$ such that $\widetilde{l} \subset l$ and $\dim \Pi_l \langle \mathcal{J}_J \oplus \mathcal{K}_K \rangle = \dim (\mathcal{J}_J \oplus \mathcal{K}_K)$.

From prop. 15.5, there exist $\widetilde{l'} \in \mathcal{L}$ and $\widetilde{\varphi} : \mathcal{J}_J \oplus \mathcal{K}_K \to \mathcal{H}_{\widetilde{l'}}$ such that $\left(l, \widetilde{l'}, J, K, \widetilde{\varphi}, \epsilon\right) \in \mathcal{E}$. We define $l' = l'_1 \cup l'_2 \cup \widetilde{l'}$ and $\varphi : \mathcal{J}_J \oplus \mathcal{K}_K \to \mathcal{H}_{l'}$ by:

$$\forall v \in \mathcal{J}_J \oplus \mathcal{K}_K,\ \varphi(v) := \widetilde{\varphi}(v)\,.$$

Then, $\left(l, l', J, K, \varphi, \epsilon\right) \in \mathcal{E}$ and $\left(l_1, l'_1, J_1, K_1, \varphi_1, \epsilon_1\right), \left(l_2, l'_2, J_2, K_2, \varphi_2, \epsilon_2\right) \preccurlyeq \left(l, l', J, K, \varphi, \epsilon\right)$. □

Proposition 15.8 We consider the same objects as in def. 15.4. Let $\varepsilon = \left(l, l', J, K, \varphi, \epsilon\right) \in \mathcal{E}$. We define:

1. $\mathcal{L}^\varepsilon := \{l'' \in \mathcal{L} \mid l' \subset l''\}$;
2. $\forall l'' \in \mathcal{L}^\varepsilon \sqcup \{\mathbb{N}\}$, $\mathcal{J}^\varepsilon_{l''} := \mathcal{J}_J$, equipped with the induced symplectic structure Ω'_J;
3. $\forall l'' \in \mathcal{L}^\varepsilon \sqcup \{\mathbb{N}\}$, $\mathcal{H}^\varepsilon_{l''} := \varphi \langle \mathcal{J}_J \oplus \mathcal{K}_{K,\mathbb{R}} \rangle \subset \mathcal{H}_{l'} \subset \mathcal{H}_{l''}$;
4. $\forall l'' \in \mathcal{L}^\varepsilon \sqcup \{\mathbb{N}\}$, $\delta^\varepsilon_{l''} := \left(\Pi'_J\big|_{\mathcal{J}_J \oplus \mathcal{K}_{K,\mathbb{R}} \to \mathcal{J}_J}\right) \circ \left(\varphi\big|_{\mathcal{J}_J \oplus \mathcal{K}_{K,\mathbb{R}} \to \varphi\langle \mathcal{J}_J \oplus \mathcal{K}_{K,\mathbb{R}}\rangle}\right)^{-1}$ where Π'_J is the orthogonal projection on $\mathcal{J}_J$;
5. $\forall l''_1, l''_2 \in \mathcal{L}^\varepsilon \sqcup \{\mathbb{N}\} \,/\, l''_1 \subset l''_2$, $\pi^{\varepsilon \to \varepsilon}_{l''_2 \to l''_1} := \mathrm{id}_{\mathcal{J}_J}$.

Then, $\mathcal{L}^\varepsilon$ is cofinal in $\mathcal{L}$ and $\left((\mathcal{J}^\varepsilon_{l''})_{l'' \in \mathcal{L}^\varepsilon \sqcup \{\mathbb{N}\}}, (\mathcal{H}^\varepsilon_{l''})_{l'' \in \mathcal{L}^\varepsilon \sqcup \{\mathbb{N}\}}, \left(\pi^{\varepsilon \to \varepsilon}_{l''_2 \to l''_1}\right)_{l''_1 \subset l''_2}, (\delta^\varepsilon_{l''})_{l'' \in \mathcal{L}^\varepsilon \sqcup \{\mathbb{N}\}}\right)$ is an elementary reduction (def. 3.7) of $(\mathcal{L}^\varepsilon \sqcup \{\mathbb{N}\}, \mathcal{H}, \pi)^\downarrow$.

Proof For $\widetilde{l} \in \mathcal{L}$, $\widetilde{l} \cup l' \in \mathcal{L}^\varepsilon$, hence $\mathcal{L}^\varepsilon$ is cofinal in $\mathcal{L}$, so in particular it is directed (and so is $\mathcal{L}^\varepsilon \sqcup \{\mathbb{N}\}$ since it has a greatest element).

Then, it is clear from the definitions that $(\mathcal{L}^\varepsilon \sqcup \{\mathbb{N}\}, \mathcal{J}^\varepsilon, \pi^{\varepsilon \to \varepsilon})$ is a projective system of phase spaces.

Replicating the proof of prop. 15.3, we can show that $\left(\mathcal{J}_J, \mathcal{J}_J \oplus \mathcal{K}_{K,\mathbb{R}}, \Pi'_J\big|_{\mathcal{J}_J \oplus \mathcal{K}_{K,\mathbb{R}} \to \mathcal{J}_J}\right)$ is a phase space reduction of $\mathcal{J}_J \oplus \mathcal{K}_K$. But since $\varphi\big|_{\mathcal{J}_J \oplus \mathcal{K}_K \to \mathrm{Im}\varphi}$ is unitary, $(\mathcal{J}_J, \mathcal{H}^\varepsilon_{l''}, \delta^\varepsilon_{l''})$ is a phase space reduction of $\mathrm{Im}\varphi$, hence of $\mathcal{H}_{l''}$, for all $l'' \in \mathcal{L}^\varepsilon \sqcup \{\mathbb{N}\}$.

Let $l''_1, l''_2 \in \mathcal{L}^\varepsilon \sqcup \{\mathbb{N}\}$ such that $l''_1 \subset l''_2$. We have $\pi_{l''_2 \to l''_1} \left\langle \mathcal{H}^\varepsilon_{l''_2} \right\rangle = \Pi_{l''_1} \left\langle \mathcal{H}^\varepsilon_{l''_2} \right\rangle = \mathcal{H}^\varepsilon_{l''_1}$ since $\mathcal{H}^\varepsilon_{l''_2} = \mathcal{H}^\varepsilon_{l''_1} \subset \mathcal{H}_{l''_1}$. Lastly, for $x_1 \in \mathcal{H}^\varepsilon_{l''_1}$, $y_2 \in \mathcal{J}^\varepsilon_{l''_2} = \mathcal{J}_J$, we have:

$$\left(\exists\, x_2 \in \mathcal{H}^\varepsilon_{l''_2} \,/\, \delta^\varepsilon_{l''_2}(x_2) = y_2 \quad \& \quad \pi_{l''_2 \to l''_1}(x_2) = x_1\right) \Leftrightarrow \left(\exists\, x_2 \in \mathcal{H}^\varepsilon_{l''_1} \,/\, \delta^\varepsilon_{l''_1}(x_2) = y_2 \quad \& \quad x_2 = x_1\right)$$

$$\Leftrightarrow \left(\delta^\varepsilon_{l''_1}(x_1) = y_2\right),$$

therefore def. 3.7.3 is fulfilled. □

Proposition 15.9 We consider the same objects as in prop. 15.8. We define:

1. $\widetilde{\mathcal{E}} := \mathcal{E} \sqcup \{\mathbb{N}\}$ (we extend the preorder by $\forall \varepsilon \in \mathcal{E}, \varepsilon \prec \mathbb{N}$), $\forall \varepsilon \in \mathcal{E}$, $\widetilde{\mathcal{L}}^\varepsilon := \mathcal{L}^\varepsilon \sqcup \{\mathbb{N}\}$, and $\widetilde{\mathcal{L}}^{\mathbb{N}} := \{\mathbb{N}\}$

2. $\forall \varepsilon = \left(I,\, I',\, J,\, K,\, \varphi,\, \epsilon\right) \in \mathcal{E},\ \forall I'' \in \widetilde{\mathcal{L}}^{\varepsilon},\ \ell(\varepsilon, I'') := J$, and $\ell(\mathbb{N}, \mathbb{N}) := \mathbb{N}$;

3. $\mathcal{J}^{\mathbb{N}}_{\mathbb{N}} := \mathcal{J}_{\mathbb{N}} = \mathcal{J}$, $\mathcal{H}^{\mathbb{N}}_{\mathbb{N}} := \mathcal{J} \oplus \mathcal{K}_{\mathbb{R}}$ and $\delta^{\mathbb{N}}_{\mathbb{N}} := \delta$;

4. $\forall (\varepsilon_1, I''_1),\ (\varepsilon_2, I''_2) \in \widetilde{\mathcal{E}}\widetilde{\mathcal{L}} \,/\, \varepsilon_1 \preccurlyeq \varepsilon_2 \quad \& \quad I''_1 \subset I''_2,\ \ \pi^{\varepsilon_2 \to \varepsilon_1}_{I''_2 \to I''_1} := \pi'_{\ell(\varepsilon_2, I''_2) \to \ell(\varepsilon_1, I''_1)}$ ($\pi'_{J' \to J}$ for $J, J' \in \mathcal{L} \sqcup \{\mathbb{N}\}$ with $J \subset J'$ has been defined in defs. 15.3.5 and 15.3.6).

Then, $\left(\widetilde{\mathcal{E}},\, \left(\widetilde{\mathcal{L}}^{\varepsilon}\right)_{\varepsilon \in \widetilde{\mathcal{E}}},\, (\mathcal{J}^{\varepsilon}_{I''})_{(\varepsilon, I'') \in \widetilde{\mathcal{E}}\widetilde{\mathcal{L}}},\, (\mathcal{H}^{\varepsilon}_{I''})_{(\varepsilon, I'') \in \widetilde{\mathcal{E}}\widetilde{\mathcal{L}}},\, \left(\pi^{\varepsilon_2 \to \varepsilon_1}_{I''_2 \to I''_1}\right)_{(\varepsilon_1, I''_1) \preccurlyeq (\varepsilon_2, I''_2)},\, (\delta^{\varepsilon}_{I''})_{(\varepsilon, I'') \in \widetilde{\mathcal{E}}\widetilde{\mathcal{L}}}\right)$ is a regularized reduction (def. 3.16) of $(\mathcal{L} \sqcup \{\mathbb{N}\}, \mathcal{H}, \pi)^{\downarrow}$.

Additionaly, we have a bijective map $\kappa : \mathcal{S}^{\downarrow}_{(\widetilde{\mathcal{E}}\widetilde{\mathcal{L}}, \mathcal{J}, \pi)} \to \mathcal{S}^{\downarrow}_{(\mathcal{L} \sqcup \{\mathbb{N}\}, \mathcal{J}, \pi')}$.

Proof $\widetilde{\mathcal{E}}$ is a directed set (for it has a greatest element) and $\left(\widetilde{\mathcal{L}}^{\varepsilon}\right)_{\varepsilon \in \widetilde{\mathcal{E}}}$ is a family of decreasing cofinal parts of $\mathcal{L} \sqcup \{\mathbb{N}\}$.

Next, we have $\forall (\varepsilon, I'') \in \widetilde{\mathcal{E}}\widetilde{\mathcal{L}},\ \mathcal{J}^{\varepsilon}_{I''} = \mathcal{J}_{\ell(\varepsilon, I'')}$ hence for $(\varepsilon_1, I''_1) \preccurlyeq (\varepsilon_2, I''_2) \in \widetilde{\mathcal{E}}\widetilde{\mathcal{L}}$, $\pi^{\varepsilon_2 \to \varepsilon_1}_{I''_2 \to I''_1}$ is well-defined as a surjective map $\mathcal{J}^{\varepsilon_2}_{I''_2} \to \mathcal{J}^{\varepsilon_1}_{I''_1}$ and it is compatible with the symplectic structures.

Moreover, for $\varepsilon_1 = \varepsilon_2 \in \mathcal{E}$, this definition coincides with the map $\pi^{\varepsilon \to \varepsilon}_{I''_2 \to I''_1}$ that has been introduced in prop. 15.8. Hence, for all $\varepsilon \in \mathcal{E}$, we have from prop. 15.8 that $(\mathcal{J}^{\varepsilon}, \mathcal{H}^{\varepsilon}, \pi^{\varepsilon \to \varepsilon}, \delta^{\varepsilon})$ is an elementary reduction of $\left(\widetilde{\mathcal{L}}^{\varepsilon}, \mathcal{H}, \pi\right)^{\downarrow}$.

And $\left(\mathcal{J}^{\mathbb{N}}, \mathcal{H}^{\mathbb{N}}, \pi^{\mathbb{N} \to \mathbb{N}}, \delta^{\mathbb{N}}\right)$ is an elementary reduction of $\left(\widetilde{\mathcal{L}}^{\mathbb{N}}, \mathcal{H}, \pi\right)^{\downarrow}$ since $\mathcal{L}^{\mathbb{N}\prime}$ has only one element and $\left(\mathcal{J}^{\mathbb{N}}_{\mathbb{N}}, \mathcal{H}^{\mathbb{N}}_{\mathbb{N}}, \delta^{\mathbb{N}}_{\mathbb{N}}\right) = (\mathcal{J}, \mathcal{J} \oplus \mathcal{K}_{\mathbb{R}}, \delta)$ is a phase space reduction of $\mathcal{H}_{\mathbb{N}} = \mathcal{H}$ (prop. 15.3).

Lastly, using $\ell : \widetilde{\mathcal{E}}\widetilde{\mathcal{L}} \to \mathcal{L} \sqcup \{\mathbb{N}\}$ (ℓ satisfies that $\ell\left\langle \widetilde{\mathcal{E}}\widetilde{\mathcal{L}} \right\rangle$ is cofinal in $\mathcal{L} \sqcup \{\mathbb{N}\}$ since it contains $\mathbb{N}$, which is a greatest element in $\mathcal{L} \sqcup \{\mathbb{N}\}$), we have by prop. 2.9 that $\left(\widetilde{\mathcal{E}}\widetilde{\mathcal{L}}, \mathcal{J}, \pi\right)^{\downarrow}$ is a projective system of phase spaces, thus $\left(\widetilde{\mathcal{E}},\, \left(\widetilde{\mathcal{L}}^{\varepsilon}\right)_{\varepsilon \in \widetilde{\mathcal{E}}},\, (\mathcal{J}^{\varepsilon}_{I''})_{(\varepsilon, I'') \in \widetilde{\mathcal{E}}\widetilde{\mathcal{L}}},\, (\mathcal{H}^{\varepsilon}_{I''})_{(\varepsilon, I'') \in \widetilde{\mathcal{E}}\widetilde{\mathcal{L}}},\, \left(\pi^{\varepsilon_2 \to \varepsilon_1}_{I''_2 \to I''_1}\right)_{(\varepsilon_1, I''_1) \preccurlyeq (\varepsilon_2, I''_2)},\, (\delta^{\varepsilon}_{I''})_{(\varepsilon, I'') \in \widetilde{\mathcal{E}}\widetilde{\mathcal{L}}}\right)$ is a regularized reduction of $(\mathcal{L} \sqcup \{\mathbb{N}\}, \mathcal{H}, \pi)^{\downarrow}$. And, in addition, there exists a bijective map $\kappa : \mathcal{S}^{\downarrow}_{(\widetilde{\mathcal{E}}\widetilde{\mathcal{L}}, \mathcal{J}, \pi)} \to \mathcal{S}^{\downarrow}_{(\mathcal{L} \sqcup \{\mathbb{N}\}, \mathcal{J}, \pi')}$. □

Lastly, we can investigate the convergence and check that we are indeed in the optimal situation discussed at the end of subsection 3.2 (more precisely in prop. 3.23). As announced above, the key ingredient for the convergence is the auxiliary result from prop. 15.5, that we proved in the process of establishing the directedness of $\mathcal{E}$.

Theorem 15.10 We consider the same objects as in prop. 15.9. Let $\psi \in \mathcal{J} = \mathcal{J}^{\mathbb{N}}_{\mathbb{N}}$. For $\varepsilon \in \mathcal{E}$, we define:

1. $\psi^{\varepsilon} := (\delta^{\varepsilon}_{\mathbb{N}})^{-1} \left\langle \pi^{\mathbb{N} \to \varepsilon}_{\mathbb{N} \to \mathbb{N}}(\psi) \right\rangle \subset \mathcal{H}^{\varepsilon}_{\mathbb{N}} \subset \mathcal{H}_{\mathbb{N}} = \mathcal{H}$;

2. $\Psi^{\varepsilon} := \widehat{\sigma}_{\downarrow}(\psi^{\varepsilon})$, where $\widehat{\sigma}_{\downarrow} : \mathcal{P}(\mathcal{H}) \to \widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L}, \mathcal{H}, \pi)}$ is defined as in prop. 3.23.

Then, the net $\left(\Psi^{\varepsilon}\right)_{\varepsilon\in\mathcal{E}}$ converges in $\widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L},\mathcal{H},\pi)}$ to $\widehat{\sigma}_{\downarrow}\left(\delta^{-1}\left\langle\psi\right\rangle\right)$ (def. 3.21).

Proof For $\varepsilon = \left(I, I', J, K, \varphi, \epsilon\right) \in \mathcal{E}$, we have, by putting all definitions together:

$$\psi^{\varepsilon} = \varphi\left\langle\Pi'_J(\psi) + \mathcal{K}_{K,\mathbb{R}}\right\rangle, \text{ where } \Pi'_J \text{ is the orthogonal projection on } \mathcal{J}_J,$$

hence, for all $I_o \in \mathcal{L}$:

$$\left[\Psi^{\varepsilon}\right]_{I_o} = \pi_{\mathbb{N}\to I_o}\left\langle\psi^{\varepsilon}\right\rangle = \Pi_{I_o}\left\langle\varphi\left\langle\Pi'_J(\psi) + \mathcal{K}_{K,\mathbb{R}}\right\rangle\right\rangle \subset \mathcal{H}_{I_o}.$$

Let $I_o \in \mathcal{L}$ and let U be an open set in $\mathcal{H}_{I_o}$ such that $U \cap \Pi_{I_o}\left\langle\psi + \mathcal{K}_{\mathbb{R}}\right\rangle \neq \varnothing$. Let $\psi' \in U \cap \Pi_{I_o}\left\langle\psi + \mathcal{K}_{\mathbb{R}}\right\rangle$ and let $\epsilon_1 > 0$ such that $\forall \psi'' \in \mathcal{H}_{I_o},\ \|\psi'' - \psi'\| \leqslant 3\,\epsilon_1\left(\|\psi\| + 1\right) \Rightarrow \psi'' \in U$.

Next, there exits $\chi \in \mathcal{K}_{\mathbb{R}}$ such that $\psi' = \Pi_{I_o}\left\langle\psi + \chi\right\rangle$. We choose $J_1, K_1 \in \mathcal{L}$ and $\chi' \in \mathcal{K}_{K_1,\mathbb{R}}$ such that $\left\|\psi - \Pi'_{J_1}(\psi)\right\| \leqslant \epsilon_1\left(\|\psi\| + 1\right)$ and $\|\chi - \chi'\| \leqslant \epsilon_1\left(\|\psi\| + 1\right)$. And we can find $I_1 \in \mathcal{L}$ with $I_1 \supset I_o$ such that $\dim \Pi_{I_1}\left\langle\mathcal{J}_{J_1} \oplus \mathcal{K}_{K_1}\right\rangle = \dim\left(\mathcal{J}_{J_1} + \mathcal{K}_{K_1}\right)$. So, from prop. 15.5, there exist $I'_1 \in \mathcal{L}$ and $\varphi_1 : \mathcal{J}_{J_1} \oplus \mathcal{K}_{K_1} \to \mathcal{H}_{I'_1}$ such that $\varepsilon_1 := \left(I_1, I'_1, J_1, K_1, \varphi_1, \epsilon_1\right) \in \mathcal{E}$.

Let $\varepsilon_2 = \left(I_2, I'_2, J_2, K_2, \varphi_2, \epsilon_2\right) \in \mathcal{E}$ with $\varepsilon_2 \succcurlyeq \varepsilon_1$. Then, we have:

$$\begin{aligned}
\left\|\Pi_{I_o}(\psi) - \Pi_{I_o} \circ \varphi_2 \circ \Pi'_{J_2}(\psi)\right\| &\leqslant \left\|\psi - \varphi_2 \circ \Pi'_{J_2}(\psi)\right\| \\
&\leqslant \left\|\psi - \Pi'_{J_2}(\psi)\right\| + \left\|\Pi'_{J_2}(\psi) - \varphi_2 \circ \Pi'_{J_2}(\psi)\right\| \\
&\leqslant \left\|\psi - \Pi'_{J_1}(\psi)\right\| + \epsilon_2\left\|\Pi'_{J_2}(\psi)\right\| \\
&\leqslant 2\epsilon_1\left(\|\psi\| + 1\right).
\end{aligned}$$

Moreover, we have $\chi' \in \mathcal{K}_{K_1,\mathbb{R}} \subset \mathcal{K}_{K_2,\mathbb{R}}$, so there exists $\chi'' \in \mathcal{K}_{K_2,\mathbb{R}}$ such that $\Pi_{I_2} \circ \varphi_2(\chi'') = \Pi_{I_2}(\chi')$ and we have:

$$\begin{aligned}
\left\|\Pi_{I_o}(\chi) - \Pi_{I_o} \circ \varphi_2\left(\chi''\right)\right\| &\leqslant \left\|\Pi_{I_2}(\chi) - \Pi_{I_2} \circ \varphi_2\left(\chi''\right)\right\| \text{ (since } I_o \subset I_1 \subset I_2) \\
&\leqslant \left\|\Pi_{I_2}(\chi) - \Pi_{I_2}(\chi')\right\| \leqslant \epsilon_1\left(\|\psi\| + 1\right).
\end{aligned}$$

Therefore, $\psi'' := \Pi_{I_o} \circ \varphi_2\left(\Pi'_{J_2}(\psi) + \chi''\right) \in U$, but since $\psi'' \in \left[\Psi^{\varepsilon_2}\right]_{I_o}$, we have $\forall \varepsilon_2 \succcurlyeq \varepsilon_1$, $\left[\Psi^{\varepsilon_2}\right]_{I_o} \cap U \neq \varnothing$.

Let K be a compact set in $\mathcal{H}_{I_o}$ such that $K \cap \Pi_{I_o}\left\langle\psi + \mathcal{K}_{\mathbb{R}}\right\rangle = \varnothing$. Hence, there exists $\epsilon_1 > 0$ such that:

$$\forall \psi'' \in \mathcal{H}_{I_o},\ \left(\exists \psi' \in \Pi_{I_o}\left\langle\psi + \mathcal{K}_{\mathbb{R}}\right\rangle \,/\, \|\psi' - \psi''\| \leqslant 2\epsilon_1\left(\|\psi\| + 1\right)\right) \Rightarrow \left(\psi'' \notin K\right).$$

As above, we can, using prop. 15.5, construct $\varepsilon_1 := \left(I_1, I'_1, J_1, K_1, \varphi_1, \epsilon_1\right) \in \mathcal{E}$ with $I_o \subset I_1$ and $\left\|\psi - \Pi'_{J_1}(\psi)\right\| \leqslant \epsilon_1\left(\|\psi\| + 1\right)$. Let $\varepsilon_2 = \left(I_2, I'_2, J_2, K_2, \varphi_2, \epsilon_2\right) \in \mathcal{E}$ with $\varepsilon_2 \succcurlyeq \varepsilon_1$ and let $\psi'' \in \left[\Psi^{\varepsilon_2}\right]_{I_o}$. Then, there exists $\chi'' \in \mathcal{K}_{K_2,\mathbb{R}}$ such that $\psi'' = \Pi_{I_o} \circ \varphi_2\left(\Pi'_{J_2}(\psi) + \chi''\right)$ and there exists $\chi' \in \mathcal{K}_{K_2,\mathbb{R}} \subset \mathcal{K}_{\mathbb{R}}$ such that $\Pi_{I_2} \circ \varphi_2(\chi'') = \Pi_{I_2}(\chi')$. Moreover, we have again:

$$\left\|\Pi_{I_o}(\psi) - \Pi_{I_o} \circ \varphi_2 \circ \Pi'_{J_2}(\psi)\right\| \leqslant 2\epsilon_1\left(\|\psi\| + 1\right).$$

We define $\psi' = \Pi_{I_o}\left(\psi + \chi'\right) = \Pi_{I_o}(\psi) + \Pi_{I_o} \circ \varphi_2\left(\chi''\right)$ (since $I_o \subset I_1 \subset I_2$) and we have

$\|\psi'' - \psi'\| \leqslant 2\epsilon_1 \left(\|\psi\| + 1\right)$, hence $\psi'' \notin K$, and therefore $\forall \varepsilon_2 \succcurlyeq \varepsilon_1$, $[\Psi^{\varepsilon_2}]_{l_o} \cap K = \varnothing$.

So, the net $\left([\Psi^{\varepsilon}]_{l_o}\right)_{\varepsilon \in \mathcal{E}}$ converges in $\mathcal{P}(\mathcal{H}_{l_o})$ to $\sqcap_{l_o} \langle \psi + \mathcal{K}_{\mathbb{R}} \rangle = \pi_{\mathbb{N} \rightarrow l_o} \left\langle \delta^{-1} \langle \psi \rangle \right\rangle$, thus, the net $(\Psi^{\varepsilon})_{\varepsilon \in \mathcal{E}}$ converges in $\widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L},\mathcal{H},\pi)}$ to $\widehat{\sigma}_{\downarrow}\left(\delta^{-1} \langle \psi \rangle\right)$. □

16. Second quantization of the Schrödinger equation

In this section we want to apply the projective state space formalism to the second quantization of the Schrödinger equation. In other words, we will consider the one-particle quantum mechanics defined on an Hilbert space $\mathcal{H}$ as a classical field theory (looking at the wave function as a classical field, whose evolution is described by a linear partial differential equation, namely the Schrödinger equation), and we will discuss how this field theory can be quantized. The standard way of doing this leads to the bosonic Fock space build on $\mathcal{H}$ [41, section I.3.4]. Here we want to compare this trusted path with the strategy outlined in the first part of the present work (chaps. 1 and 2): first, look for a rendering (def. 2.6) of the classical field theory by a collection of finite dimensional partial theories, then come up with a regularizing procedure to implement the dynamics (subsection 3.2), and, last but not least, take advantage of this classical insight to build a quantization of the theory (section 6), thus obtaining a projective system of quantum state spaces (subsection 5.1). In particular, we want to use this example to illustrate how the classical regularization of the dynamics lays the stage for a corresponding procedure at the quantum level.

16.1 Classical theory

In section 3, we only considered dynamics specified by constraints, whereas here we have a theory originally formulated with a 'true' Hamiltonian. However, this is quickly fixed, since there exists a routine trick (discussed in [105, section 1.8] and similar to the more general procedure presented in [52]), that can be physically interpreted as introducing an artificial time parametrization, and allows to transform any theory on $\mathcal{H}$ with an non-vanishing Hamiltonian into a theory on $\mathcal{H} \times \mathbb{R}^2$ with an Hamiltonian constraint (the $\mathbb{R}^2$ part holds the time coordinate and its conjugate momentum, aka. the energy variable).

Note that there is a technical subtlety arising when we try to write the theory on an infinite dimensional symplectic manifold in the naive setup of def. A.1, and we are forced to require the one-particle quantum Hamiltonian to be a bounded operator (we cannot simply restrict the constraint surface so that it is included in an appropriate dense subspace, for it would then cost the reduced phase space its strong symplectic structure, by spoiling the needed non-degeneracy property). However, we will be able to lift this restriction without great efforts when switching to the projective state space formalism.

Proposition 16.1 Let $\mathcal{H}$ be a separable, infinite dimensional Hilbert space and H be a bounded self-adjoint operator on $\mathcal{H}$. We equip $\mathcal{M}^{\text{KIN}} := \mathcal{H} \times \mathbb{R}^2$ with the strong symplectic structure:

1. $\forall (\varphi_1, u_1, v_1),\ (\varphi_2, u_2, v_2) \in \mathcal{M}^{\text{KIN}}, \quad \Omega_{\text{KIN}}(\varphi_1, u_1, v_1; \varphi_2, u_2, v_2) := 2 \operatorname{Im}\left(\langle \varphi_1,\ \varphi_2 \rangle \right) + (u_2 v_1 - u_1 v_2)$.

   We define:

2. $\mathcal{M}^{\text{SHELL}} := \{ (\psi,\ t,\ E) \in \mathcal{M}^{\text{KIN}} \mid E = \langle \psi,\ H\psi \rangle \}$;
3. $\mathcal{M}^{\text{DYN}} := \mathcal{H}$ with symplectic structure $\Omega_{\text{DYN}} := 2 \operatorname{Im} \langle \cdot\,,\ \cdot \rangle$;
4. $$\begin{array}{rccc} \delta\ : & \mathcal{M}^{\text{SHELL}} & \to & \mathcal{M}^{\text{DYN}} \\ & (\psi,\ t,\ E) & \mapsto & \exp(i\,t\,H)\ \psi \end{array}.$$

   Then, $(\mathcal{M}^{\text{DYN}},\ \mathcal{M}^{\text{SHELL}},\ \delta)$ is a phase space reduction of $\mathcal{M}^{\text{KIN}}$ (def. A.1).

Proof From prop. 15.1, Ω_{KIN}, resp. Ω_{DYN}, defines a strong symplectic structure on $\mathcal{M}^{\text{KIN}}$, resp. $\mathcal{M}^{\text{DYN}}$.

The map δ is surjective and, for $\psi_o \in \mathcal{M}^{\text{DYN}}$, $\delta^{-1} \langle \psi_o \rangle = \{ (\exp(-i\,t\,H)\ \psi_o,\ t,\ E_o) \mid t \in \mathbb{R} \}$, where $E_o := \langle \psi_o,\ H\,\psi_o \rangle$. So $\delta^{-1} \langle \psi_o \rangle$ is in particular connected.

Let $(\psi,\ t,\ E) \in \mathcal{M}^{\text{SHELL}}$. We have:

$$T_{(\psi,t,E)}\left(\mathcal{M}^{\text{SHELL}}\right) = \{ (\varphi,\ u,\ 2 \operatorname{Re} \langle \varphi,\ H\psi \rangle) \mid \varphi \in \mathcal{H},\ u \in \mathbb{R} \},$$

and:

$$\begin{array}{rccc} T_{(\psi,t,E)}\delta\ : & T_{(\psi,t,E)}\left(\mathcal{M}^{\text{SHELL}}\right) & \to & T_{e^{iHt}\psi}\left(\mathcal{M}^{\text{DYN}}\right) \\ & (\varphi,\ u,\ 2 \operatorname{Re} \langle \varphi,\ H\psi \rangle) & \mapsto & e^{iHt}\,\varphi + iu\,e^{iHt}\,H\psi \end{array}.$$

Hence, $T_{(\psi,t,E)}\delta$ is surjective and, for $(\varphi_1,\ u_1,\ 2 \operatorname{Re} \langle \varphi_1,\ H\psi \rangle)\,,\ (\varphi_2,\ u_2,\ 2 \operatorname{Re} \langle \varphi_2,\ H\psi \rangle) \in T_{(\psi,t,E)}\left(\mathcal{M}^{\text{SHELL}}\right)$, we have:

$$\Omega_{\text{KIN}}\left(\varphi_1,\ u_1,\ 2 \operatorname{Re} \langle \varphi_1,\ H\psi \rangle;\ \ \varphi_2,\ u_2,\ 2 \operatorname{Re} \langle \varphi_2,\ H\psi \rangle\right) =$$

$$= 2 \operatorname{Im} \langle \varphi_1,\ \varphi_2 \rangle + 2\,u_2 \operatorname{Im} \langle \varphi_1,\ iH\psi \rangle - 2\,u_1 \operatorname{Im} \langle \varphi_2,\ iH\psi \rangle + 2\,u_1 u_2 \operatorname{Im} \langle i\,H\psi,\ i\,H\psi \rangle$$

$$= 2 \operatorname{Im} \left\langle T_{(\psi,t,E)}\delta\left(\varphi_1,\ u_1,\ 2 \operatorname{Re} \langle \varphi_1,\ H\psi \rangle\right),\ T_{(\psi,t,E)}\delta\left(\varphi_2,\ u_2,\ 2 \operatorname{Re} \langle \varphi_2,\ H\psi \rangle\right)\right\rangle,$$

therefore $\Omega_{\text{KIN}}\big|_{T_{(\psi,t,E)}(\mathcal{M}^{\text{SHELL}})} = \left[\delta^* \,\Omega_{\text{DYN}}\right]_{(\psi,t,E)}$. □

On $\mathcal{H}$ viewed as a phase space, we can define some remarkable observables (this defines the algebra that we will latter endeavor to quantize): of interest are for us the scalar product with a vector $e \in \mathcal{H}$ (that will give rise in the quantum theory to the corresponding creation and annihilation operators) and the expectation value of an operator on $\mathcal{H}$. Additionally the Heisenberg (ie. time-dependent) operators of the first-quantized theory can be seen in a natural way as dynamical observables associated (in the sense of def. A.2) to particular kinematical observables (up to a technical artifact: we restrict the support of the considered observables to spheres in $\mathcal{H}$ because we had defined the map $(\,\cdot\,)^{\text{DYN}}$ translating a kinematical observable into its dynamical version only for bounded observables; note that, alternatively, we could just weaken this requirement, for it would be enough to only demand the kinematical observables to be bounded on orbits of the dynamics).

Proposition 16.2 We consider the same objects as in prop. 16.1. Let $e \in \mathcal{H}$. On $\mathcal{H}$ we can define the observables:

$$\begin{array}{rccc} \mathrm{a}_e & : & \mathcal{H} & \to & \mathbb{C} \\ & & \psi & \mapsto & \langle e,\, \psi\rangle \end{array} \quad \text{and} \quad \begin{array}{rccc} \mathrm{a}_e^* & : & \mathcal{H} & \to & \mathbb{C} \\ & & \psi & \mapsto & \langle \psi,\, e\rangle \end{array}.$$

We have, for all $e,\, f \in \mathcal{H}$:

$$\{\mathrm{a}_e,\, \mathrm{a}_f\}_{\mathcal{H}} = 0\,,\ \{\mathrm{a}_e^*,\, \mathrm{a}_f^*\}_{\mathcal{H}} = 0\,, \text{ and } \{\mathrm{a}_e,\, \mathrm{a}_f^*\}_{\mathcal{H}} = i\,\langle e,\, f\rangle\,.$$

Let A be a bounded self-adjoint operator on $\mathcal{H}$. We define on $\mathcal{H}$ the observable $\langle A\rangle$ by:

$$\forall \psi \in \mathcal{H},\ \langle A\rangle(\psi) := \langle \psi,\, A\,\psi\rangle\,.$$

We have, for all A, B bounded self-adjoint operators on $\mathcal{H}$ and $e \in \mathcal{H}$:

$$\big\{\,\langle A\rangle\,,\,\langle B\rangle\,\big\}_{\mathcal{H}} = i\,\big\langle [A,\, B]_{\mathcal{H}}\,\big\rangle\,,\ \big\{\mathrm{a}_e,\,\langle A\rangle\,\big\}_{\mathcal{H}} = i\,\mathrm{a}_{Ae}\,, \text{ and } \big\{\mathrm{a}_e^*,\,\langle A\rangle\,\big\}_{\mathcal{H}} = -i\,\mathrm{a}_{Ae}^*\,.$$

Lastly, for A a bounded self-adjoint operator on $\mathcal{H}$, $N > 0$ and $t_o \in \mathbb{R}$, we can define on $\mathcal{M}^{\text{KIN}}$ the observable:

$$\begin{array}{rccl} \langle A,\, N,\, t_o\rangle & : & \mathcal{M}^{\text{KIN}} & \to & \mathbb{R} \\ & & (\psi,\, t,\, E) & \mapsto & \begin{cases} \langle A\rangle(\psi) & \text{if} \quad \|\psi\| = N \quad \& \quad t = t_o \\ 0 & \text{else} \end{cases}, \end{array}$$

and we have:

$$\begin{aligned} \forall \psi_o \in \mathcal{M}^{\text{DYN}},\ \langle A,\, N,\, t_o\rangle^{\text{DYN}}(\psi_o) &:= \sup_{(\psi,t,E)\in\delta^{-1}\langle\psi_o\rangle} \langle A,\, N,\, t_o\rangle(\psi,\, t,\, E) \\ &= \begin{cases} \big\langle e^{it_oH}\,A\,e^{-it_oH}\big\rangle(\psi_o) & \text{if} \quad \|\psi_o\| = N \\ 0 & \text{else} \end{cases}. \end{aligned} \tag{16.2.1}$$

Proof In order to compute the Poisson brackets between observables of the type a_e and a_e^*, we have to be careful not to mix up the complex structure on $\mathcal{H}$ with the complex structure coming from a_e and a_e^* being $\mathbb{C}$-valued. Therefore, we will write $J\,\varphi$ for the scalar multiplication of φ by i (in $\mathcal{H}$ seen as a $\mathbb{C}$-vector space) and $\imath\,\varphi$ for the vector $(i \otimes_{\mathbb{R}} \varphi)$ in $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{H} \approx T_{\mathbb{C}}(\mathcal{H})$ (for $\mathcal{H}$ seen as a real manifold). Extending $\mathrm{Im}\,\langle\,\cdot\,,\,\cdot\,\rangle$ and $\mathrm{Re}\,\langle\,\cdot\,,\,\cdot\,\rangle$ by $\mathbb{C}$-bilinearity on $T_{\mathbb{C}}(\mathcal{H})$ (because we want $\{\,\cdot\,,\,\cdot\,\}_{\mathcal{H}}$ to be $\mathbb{C}$-bilinear), we then have:

$$\begin{aligned} & \mathrm{Im}\,\langle\varphi',\, J\,\varphi\rangle = -\mathrm{Im}\,\langle J\,\varphi',\, \varphi\rangle = \mathrm{Re}\,\langle\varphi',\, \varphi\rangle \\ \& \quad & \mathrm{Im}\,\langle\varphi',\, \imath\,\varphi\rangle = \mathrm{Im}\,\langle\imath\,\varphi',\, \varphi\rangle = i\,\mathrm{Im}\,\langle\varphi',\, \varphi\rangle\,. \end{aligned}$$

With this we can compute the Hamiltonian vector fields at $\psi \in \mathcal{H}$ of a_e and a_e^*, for $e \in \mathcal{H}$:

$$\begin{aligned} & [d\mathrm{a}_e]_\psi(\varphi) = \langle e,\, \varphi\rangle = 2\,\mathrm{Im}\,\left\langle -\frac{J\,e}{2} + \frac{\imath\,e}{2},\, \varphi\right\rangle \\ \& \quad & [d\mathrm{a}_e^*]_\psi(\varphi) = \langle \varphi,\, e\rangle = 2\,\mathrm{Im}\,\left\langle -\frac{J\,e}{2} - \frac{\imath\,e}{2},\, \varphi\right\rangle. \end{aligned}$$

Hence, for $e,\, f \in \mathcal{H}$:

$$\{\mathsf{a}_e,\, \mathsf{a}_f\}_{\mathcal{H},\psi} = 2\,\mathrm{Im}\,\langle X_{\mathsf{a}_f,\psi},\, X_{\mathsf{a}_e,\psi}\rangle = 2\,\mathrm{Im}\,\left\langle -\frac{J f}{2} + \frac{\mathrm{i}\, f}{2},\, -\frac{J e}{2} + \frac{\mathrm{i}\, e}{2}\right\rangle = 0\,,$$

$$\{\mathsf{a}^*_e,\, \mathsf{a}^*_f\}_{\mathcal{H},\psi} = 2\,\mathrm{Im}\,\left\langle X_{\mathsf{a}^*_f,\psi},\, X_{\mathsf{a}^*_e,\psi}\right\rangle = 2\,\mathrm{Im}\,\left\langle -\frac{J f}{2} - \frac{\mathrm{i}\, f}{2},\, -\frac{J e}{2} - \frac{\mathrm{i}\, e}{2}\right\rangle = 0\,,$$

$$\text{and}\ \ \{\mathsf{a}_e,\, \mathsf{a}^*_f\}_{\mathcal{H},\psi} = 2\,\mathrm{Im}\,\left\langle X_{\mathsf{a}^*_f,\psi},\, X_{\mathsf{a}_e,\psi}\right\rangle = 2\,\mathrm{Im}\,\left\langle -\frac{J f}{2} - \frac{\mathrm{i}\, f}{2},\, -\frac{J e}{2} + \frac{\mathrm{i}\, e}{2}\right\rangle$$

$$= i\,(\mathrm{Re}\,\langle f,\, e\rangle - i\,\mathrm{Im}\,\langle f,\, e\rangle) = i\,\langle e,\, f\rangle\,.$$

Similarly, we have for any A bounded self-adjoint operator on $\mathcal{H}$ and at every $\psi \in \mathcal{H}$:

$$[d\,\langle A\rangle]_\psi = 2\,\mathrm{Re}\,\langle A\,\psi,\, \varphi\rangle = 2\,\mathrm{Im}\,\langle -J\,A\,\psi,\, \varphi\rangle\,,$$

hence, for A, B bounded self-adjoint operators on $\mathcal{H}$, $e \in \mathcal{H}$, and $\psi \in \mathcal{H}$:

$$\{\,\langle A\rangle\,,\, \langle B\rangle\,\}_{\mathcal{H},\psi} = 2\,\mathrm{Im}\,\left\langle X_{\langle B\rangle,\psi},\, X_{\langle A\rangle,\psi}\right\rangle = 2\,\mathrm{Im}\,\langle -J\,B\,\psi,\, -J\,A\,\psi\rangle$$

$$= -i\,(\langle B\,\psi,\, A\,\psi\rangle - \langle A\,\psi,\, B\,\psi\rangle) = i\,\left\langle\, [A,\, B]_{\mathcal{H}}\,\right\rangle(\psi)\,,$$

$$\{\mathsf{a}_e,\, \langle A\rangle\,\}_{\mathcal{H},\psi} = 2\,\mathrm{Im}\,\left\langle X_{\langle A\rangle,\psi},\, X_{\mathsf{a}_e,\psi}\right\rangle = 2\,\mathrm{Im}\,\left\langle -J\,A\,\psi,\, -\frac{J e}{2} + \frac{\mathrm{i}\, e}{2}\right\rangle$$

$$= i\,(-i\,\mathrm{Im}\,\langle\psi,\, A\,e\rangle + \mathrm{Re}\,\langle\psi,\, A\,e\rangle) = i\,\mathsf{a}_{Ae}(\psi)\,,$$

$$\text{and}\ \ \{\mathsf{a}^*_e,\, \langle A\rangle\,\}_{\mathcal{H},\psi} = 2\,\mathrm{Im}\,\left\langle X_{\langle A\rangle,\psi},\, X_{\mathsf{a}^*_e,\psi}\right\rangle = 2\,\mathrm{Im}\,\left\langle -J\,A\,\psi,\, -\frac{J e}{2} - \frac{\mathrm{i}\, e}{2}\right\rangle$$

$$= -i\,(i\,\mathrm{Im}\,\langle\psi,\, A\,e\rangle + \mathrm{Re}\,\langle\psi,\, A\,e\rangle) = -i\,\mathsf{a}^*_{Ae}(\psi)\,.$$

Lastly, eq. (16.2.1) comes from:

$$\forall\psi_o \in \mathcal{M}^{\text{DYN}},\ \delta^{-1}\,\langle\psi_o\rangle = \{(\exp(-i\,t\,H)\ \psi_o,\ t,\ \langle H\rangle\,(\psi_o))\ |\ t \in \mathbb{R}\}\,.$$

□

The projective system we will use here differs significantly from the one we were using in the previous section (prop. 15.2), for we do not rely any more on the choice of a particular basis to define a family of vector subspaces: instead, we simply take as label set the set of *all* finite dimensional vector subspaces of $\mathcal{H}$ (this structure is of course more satisfactory from a physical point of view; as mentioned at the beginning of section 15, we could not use it in the previous example, for our aim was to illustrate the regularizing strategy, while this larger label set contains a cofinal family on which the linear constraints we were considering form an elementary reduction).

Note that the space of states of this projective system can be naturally identified with the algebraic dual on $\mathcal{H}$, in such a way that the injection of $\mathcal{H}$ into the projective state space (in the sense of a rendering, as introduced in def. 2.6) corresponds to the identification with its topological dual.

Proposition 16.3 Let $\mathcal{H}$ be a separable, infinite dimensional Hilbert space. We define $\mathcal{L}$ as the set of all finite dimensional vector subspaces of $\mathcal{H}$ and we equip it with the preorder $\subset$. We define:

1. $\forall \mathcal{I} \in \mathcal{L} \sqcup \{\mathcal{H}\}$, $\mathcal{M}^{\text{KIN}}_{\mathcal{I}} := \mathcal{I} \times \mathbb{R}^2$, equipped with the symplectic structure $\Omega_{\text{KIN},\mathcal{I}}$ induced from $\mathcal{M}^{\text{KIN}}$, Ω_{KIN};
2. $\forall \mathcal{I}, \mathcal{I}' \in \mathcal{L} \sqcup \{\mathcal{H}\}$, with $\mathcal{I} \subset \mathcal{I}'$, $\pi^{\text{KIN}}_{\mathcal{I}' \to \mathcal{I}} := \Pi_{\mathcal{I}}|_{\mathcal{I}' \to \mathcal{I}} \times \text{id}_{\mathbb{R}^2}$ where $\Pi_{\mathcal{I}}$ is the orthogonal projection on $\mathcal{I}$

Then, this defines a rendering (def. 2.6) of $\mathcal{M}^{\text{KIN}}$ by the projective system of phase spaces $(\mathcal{L}, \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^{\downarrow}$. We define $\sigma^{\text{KIN}}_{\downarrow} : \mathcal{M}^{\text{KIN}} \to \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ as in def. 2.6.

Additionally, we have a bijective map $\zeta^{\text{KIN}} : \mathcal{H}^* \times \mathbb{R}^2 \to \mathcal{S}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ such that $\zeta^{\text{KIN},-1} \circ \sigma^{\text{KIN}}_{\downarrow} : \mathcal{M}^{\text{KIN}} \to \mathcal{H}^* \times \mathbb{R}^2$ corresponds to the canonical identification of $\mathcal{H}$ with $\mathcal{H}' \subset \mathcal{H}^*$.

Proof The proof works in the same way as the proof of prop. 15.2. □

We are ready to go on to the formulation of an approximating scheme for the dynamics. The approximation here will take place in two different directions. First, we introduce a deformation of the constraint surface, controlled by a small parameter $\epsilon > 0$, to replace the non-compact orbits of the exact dynamics (going in time from $-\infty$ to $+\infty$) by compact orbits (running only through a finite time interval): the rough idea is that instead of having a 'free particle' in the energy-time variable, we put an harmonic oscillator, thus preventing the time variable to grow for ever. This will be more comfortable when switching to the quantum theory: having compact orbits is closely related to having well-normalized states solving the quantum constraints (heuristically, quantum solutions of the constraints have much in common with classical statistical states, supported by the constraint surface and constant on the gauge orbits, and these will only exist as properly normalized probability measures if the orbits are compact).

The other aspect of the approximation is what will allow us to build, for the approximated dynamics, a corresponding elementary reduction on a cofinal part of the projective system introduced previously. For this, we truncate the exact Hamiltonian H of the first-quantized theory as $\Pi_{\mathcal{J}}\, H\, \Pi_{\mathcal{J}}$ where $\mathcal{J}$ is a finite vector subspace of $\mathcal{H}$, such that H is bounded on $\mathcal{J}$ (from now on, we can indeed relax the requirement we had above, and we allow H to be an unbounded, densely defined, operator on $\mathcal{H}$). In other words, we project the Hamiltonian flow on the symplectic submanifold $\mathcal{J} \times \mathbb{R}^2$ of $\mathcal{H} \times \mathbb{R}^2$. Moreover, we include in the approximated dynamics additional second class constraints, forcing the wave function ψ to belong to $\mathcal{J}$ (by definition of the truncated Hamiltonian these additional constraints are preserved by the truncated evolution): the point is that it does not make sense to keep the degrees of freedom orthogonal to the subspace $\mathcal{J}$ since with the truncated Hamiltonian we would not evolve them at all and they would soon lie very far away from their correct values (note that the degrees of freedom along $\mathcal{J}$ are not evolved exactly either, but at least they are evolved approximately; the error comes from neglecting the backreaction of the degrees of freedoms along $\mathcal{J}^{\perp}$, due to the cross-terms of the exact Hamiltonian H between $\mathcal{J}$ and $\mathcal{J}^{\perp}$).

The side effect of these additional second class constraints is to make the approximated reduced phase space finite dimensional (aka. $\mathcal{M}^{\text{DYN},\mathcal{E}}_{\infty}$, using the notations of prop. 3.24): this is not needed for the construction (in general only the 'partial' reduced phase space $\mathcal{M}^{\text{DYN},\mathcal{E}}_{\eta}$, arising from the constraint surface projected on $\mathcal{M}^{\text{KIN}}_{\eta}$ for some $\eta \in \mathcal{L}^{\mathcal{E}}$, is expected to be finite dimensional), but it will simplify the structure of the dynamical projective system.

Definition 16.4 We consider the same objects as in prop. 16.3. Let H be a densely defined (possibly unbounded) self-adjoint operator on $\mathcal{H}$. We define $\mathcal{E}$ as the set of all pairs $(\mathcal{J},\, \epsilon)$ such that:

1. $\mathcal{J} \in \mathcal{L}$ and $\forall \psi \in \mathcal{J},\ \|H\psi\| < \infty$;
2. $\epsilon > 0$.

On $\mathcal{E}$ we will use the preorder:

3. $(\mathcal{J},\, \epsilon) \preccurlyeq (\mathcal{J}',\, \epsilon') \quad \Leftrightarrow \quad \left(\mathcal{J} \subset \mathcal{J}' \quad \& \quad \epsilon \geqslant \epsilon'\right)$.

Proposition 16.5 $\mathcal{E},\ \preccurlyeq$ is a directed preordered set.

Proof Let $(\mathcal{J}_1,\, \epsilon_1),\ (\mathcal{J}_2,\, \epsilon_2) \in \mathcal{E}$. Then, we have $\mathcal{J}_1 + \mathcal{J}_2 \in \mathcal{L}$, $\forall \psi \in \mathcal{J}_1 + \mathcal{J}_2$, $\|H\psi\| < \infty$ and $\min(\epsilon_1,\, \epsilon_2) > 0$. Hence, $(\mathcal{J}_1 + \mathcal{J}_2,\, \min(\epsilon_1,\, \epsilon_2)) \in \mathcal{E}$ and $(\mathcal{J}_1,\, \epsilon_1),\ (\mathcal{J}_2,\, \epsilon_2) \preccurlyeq (\mathcal{J}_1 + \mathcal{J}_2,\, \min(\epsilon_1,\, \epsilon_2))$. □

Proposition 16.6 We consider the same objects as in def. 16.4. Let $\varepsilon = (\mathcal{J},\, \epsilon) \in \mathcal{E}$. We define:

1. $\mathcal{L}^{\varepsilon} := \{\mathcal{I} \in \mathcal{L} \mid \mathcal{J} \subset \mathcal{I}\}$;
2. $\forall \mathcal{I} \in \mathcal{L}^{\varepsilon} \sqcup \{\mathcal{H}\}$, $\mathcal{M}^{\text{DYN},\varepsilon}_{\mathcal{I}} := \mathcal{J}$ equipped with the symplectic structure $\Omega_{\text{DYN},\mathcal{J}}$ induced from $\mathcal{M}^{\text{DYN}}$, Ω_{DYN};
3. $\forall \mathcal{I} \in \mathcal{L}^{\varepsilon} \sqcup \{\mathcal{H}\}$, $\mathcal{M}^{\text{SHELL},\varepsilon}_{\mathcal{I}} := \left\{(\psi,\, t,\, E) \in \mathcal{M}^{\text{KIN}}_{\mathcal{J}} \subset \mathcal{M}^{\text{KIN}}_{\mathcal{I}} \,\middle|\, (E - \langle \psi,\, H\psi \rangle)^2 + \epsilon^4\, t^2 = \epsilon^2\right\}$;
4. $\forall \mathcal{I} \in \mathcal{L}^{\varepsilon} \sqcup \{\mathcal{H}\}$, $\forall (\psi,\, t,\, E) \in \mathcal{M}^{\text{SHELL},\varepsilon}_{\mathcal{I}}$, $\quad \delta^{\varepsilon}_{\mathcal{I}}(\psi,\, t,\, E) := \exp(i\, t\, \Pi_{\mathcal{J}}\, H)\ \psi \in \mathcal{J}$;
5. $\forall \mathcal{I},\, \mathcal{I}' \in \mathcal{L}^{\varepsilon} \sqcup \{\mathcal{H}\}$, with $\mathcal{I} \subset \mathcal{I}'$, $\pi^{\text{DYN},\varepsilon\to\varepsilon}_{\mathcal{I}'\to\mathcal{I}} := \mathrm{id}_{\mathcal{J}}$.

Then, $\mathcal{L}^{\varepsilon}$ is cofinal in $\mathcal{L}$ and $\left(\left(\mathcal{M}^{\text{DYN},\varepsilon}_{\mathcal{I}}\right)_{\mathcal{I}\in\mathcal{L}^{\varepsilon}\sqcup\{\mathcal{H}\}},\, \left(\mathcal{M}^{\text{SHELL},\varepsilon}_{\mathcal{I}}\right)_{\mathcal{I}\in\mathcal{L}^{\varepsilon}\sqcup\{\mathcal{H}\}},\, \left(\pi^{\text{DYN},\varepsilon\to\varepsilon}_{\mathcal{I}'\to\mathcal{I}}\right)_{\mathcal{I}\subset\mathcal{I}'},\, (\delta^{\varepsilon}_{\mathcal{I}})_{\mathcal{I}\in\mathcal{L}^{\varepsilon}\sqcup\{\mathcal{H}\}}\right)$ is an elementary reduction (def. 3.7) of $(\mathcal{L}^{\varepsilon} \sqcup \{\mathcal{H}\}\, , \mathcal{M}^{\text{KIN}}, \pi^{\text{KIN}})^{\downarrow}$.

Proof For $\mathcal{I} \in \mathcal{L}$, $\mathcal{I} + \mathcal{J} \in \mathcal{L}^{\varepsilon}$, hence $\mathcal{L}^{\varepsilon}$ is cofinal in $\mathcal{L}$, so in particular it is directed (and so is $\mathcal{L}^{\varepsilon} \sqcup \{\mathcal{H}\}$ since it has a greatest element).

Then, it is clear from the definitions that $(\mathcal{L}^{\varepsilon} \sqcup \{\mathcal{H}\}\, ,\, \mathcal{M}^{\text{DYN},\varepsilon},\, \pi^{\text{DYN},\varepsilon\to\varepsilon})$ is a projective system of phase spaces.

Now, we define:

6. $\mathcal{M}^{\varepsilon} := \left\{(\psi,\, t,\, E) \in \mathcal{M}^{\text{KIN}}_{\mathcal{J}} \subset \mathcal{M}^{\text{KIN}} \,\middle|\, (E - \langle \psi,\, H\psi \rangle)^2 + \epsilon^4\, t^2 = \epsilon^2\right\}$;
7. $\forall (\psi,\, t,\, E) \in \mathcal{M}^{\varepsilon}$, $\quad \delta^{\varepsilon}(\psi,\, t,\, E) := \exp(i\, t\, \Pi_{\mathcal{J}}\, H)\ \psi \in \mathcal{J}$;

and we want to show that $(\mathcal{J},\, \mathcal{M}^{\varepsilon},\, \delta^{\varepsilon})$ is a phase space reduction of $\mathcal{M}^{\text{KIN}}$.

$\Pi_{\mathcal{J}}\, H|_{\mathcal{J}\to\mathcal{J}}$ is a bounded (by definition of $\mathcal{E}$), self-adjoint operator on $\mathcal{J}$. Therefore, the map δ^{ε} is surjective and, for $\psi_o \in \mathcal{J}$, $\delta^{\varepsilon,-1}\langle \psi_o \rangle = \left\{\left(e^{-\frac{i}{\epsilon}\sin\theta\, \Pi_{\mathcal{J}}\, H}\, \psi_o,\, \frac{1}{\epsilon}\sin\theta,\, E_o + \epsilon\cos\theta\right) \,\middle|\, \theta \in [0, 2\pi[\right\}$, where $E_o := \langle \psi_o,\, H\, \psi_o \rangle$. So $\delta^{\varepsilon,-1}\langle \psi_o \rangle$ is in particular connected.

Let $(\psi,\, t,\, E) \in \mathcal{M}^{\varepsilon}$. $T_{(\psi,t,E)}(\mathcal{M}^{\varepsilon})$ is given by:

$$\left\{(\varphi,\, u,\, v) \in \mathcal{M}^{\text{KIN}}_{\mathcal{J}} \,\middle|\, \varphi \in \mathcal{J} \quad \& \quad 2\epsilon\sqrt{1 - \epsilon^2\, t^2}\, (v - 2\,\mathrm{Re}\, \langle \varphi,\, H\psi \rangle) + 2\,\epsilon^4\, t\, u = 0\right\}, \tag{16.6.1}$$

and we have:

$$\begin{array}{rcl} T_{(\psi,t,E)}\, \delta^{\varepsilon} \ : \ T_{(\psi,t,E)}(\mathcal{M}^{\varepsilon}) & \to & \mathcal{J} \\ (\varphi,\, u,\, v) & \mapsto & i\, u e^{i\, t\, \Pi_{\mathcal{J}}\, H}\, \Pi_{\mathcal{J}}\, H\, \psi + e^{i\, t\, \Pi_{\mathcal{J}}\, H}\, \varphi \end{array} .$$

Hence, $T_{(\psi,t,E)}\,\delta^{\varepsilon}$ is surjective and, for $(\varphi_1, u_1, v_1)\,, (\varphi_2, u_2, v_2) \in T_{(\psi,t,E)}\,(\mathcal{M}^{\varepsilon})$, we have:

$$\Omega_{\text{KIN}}\,(\varphi_1, u_1, v_{1;}\;\; \varphi_2, u_2, v_2) =$$

$$= 2\,\mathrm{Im}\,\langle\varphi_1, \varphi_2\rangle + u_2\,\left(2\,\mathrm{Re}\,\langle\varphi_1, H\psi\rangle - \frac{\epsilon^3\,t}{\sqrt{1-\epsilon^2\,t^2}}\,u_1\right) - u_1\,\left(2\,\mathrm{Re}\,\langle\varphi_2, H\psi\rangle - \frac{\epsilon^3\,t}{\sqrt{1-\epsilon^2\,t^2}}\,u_2\right)$$

(using eq. (16.6.1))

$$= 2\,\mathrm{Im}\,\langle T_{(\psi,t,E)}\,\delta^{\varepsilon}\,(\varphi_1, u_1, v_1)\,, T_{(\psi,t,E)}\,\delta^{\varepsilon}\,(\varphi_2, u_2, v_2)\rangle$$ (like in the proof of prop. 16.1),

therefore $\Omega_{\text{KIN}}\big|_{T_{(\psi,t,E)}(\mathcal{M}^{\varepsilon})} = [\delta^{\varepsilon,*}\,\Omega_{\text{DYN},\mathcal{J}}]_{(\psi,t,E)}$.

Thus, for all $\mathcal{I} \in \mathcal{L}^{\varepsilon} \sqcup \{\mathcal{H}\}$, $\left(\mathcal{M}^{\text{DYN},\varepsilon}_{\mathcal{J}}, \mathcal{M}^{\text{SHELL},\varepsilon}_{\mathcal{J}}, \delta^{\varepsilon}_{\mathcal{J}}\right) = (\mathcal{J}, \mathcal{M}^{\varepsilon}, \delta^{\varepsilon})$ is a phase space reduction of $\mathcal{M}^{\text{KIN}}$, hence of $\mathcal{M}^{\text{KIN}}_{\mathcal{J}}$.

Let $\mathcal{I}, \mathcal{I}' \in \mathcal{L}^{\varepsilon} \sqcup \{\mathcal{H}\}$, with $\mathcal{I} \subset \mathcal{I}'$. We have:

$$\pi^{\text{KIN}}_{\mathcal{J}'\to\mathcal{J}}\,\langle\mathcal{M}^{\text{SHELL},\varepsilon}_{\mathcal{J}'}\rangle = \left\{(\Pi_{\mathcal{I}}\,\psi, t, E)\;\middle|\;\psi \in \mathcal{J} \quad \& \quad (E - \langle\psi, H\psi\rangle)^2 + \epsilon^4\,t^2 = \epsilon^2\right\} = \mathcal{M}^{\text{SHELL},\varepsilon}_{\mathcal{J}},$$

since $\mathcal{J} \subset \mathcal{I}$. Lastly, for $(\psi, t, E) \in \mathcal{M}^{\text{SHELL},\varepsilon}_{\mathcal{J}}$, $\psi'_o \in \mathcal{M}^{\text{DYN},\varepsilon}_{\mathcal{J}'} = \mathcal{J}$, we get:

$$\left(\exists\,(\psi', t', E') \in \mathcal{M}^{\text{SHELL},\varepsilon}_{\mathcal{J}'}\,/\,\delta^{\varepsilon}_{\mathcal{J}'}(\psi', t', E') = \psi'_o \quad \& \quad \pi^{\text{KIN}}_{\mathcal{J}'\to\mathcal{J}}(\psi', t', E') = (\psi, t, E)\right) \Leftrightarrow$$

$$\Leftrightarrow \left(\exists\,(\psi', t', E') \in \mathcal{M}^{\text{SHELL},\varepsilon}_{\mathcal{J}}\,/\,\delta^{\varepsilon}_{\mathcal{J}}(\psi', t', E') = \psi'_o \quad \& \quad (\psi', t', E') = (\psi, t, E)\right)$$

$$\Leftrightarrow \left(\delta^{\varepsilon}_{\mathcal{J}}(\psi, t, E) = \psi'_o = \pi^{\text{DYN},\varepsilon\to\varepsilon}_{\mathcal{J}'\to\mathcal{J}}(\psi'_o)\right),$$

therefore def. 3.7.3 is fulfilled. □

Now, as we did in the previous section (prop. 15.3), we introduce a more concise dynamical projective system, that we will be able to identify with the one on the label set $\mathcal{EL}$ using the facility developed in prop. 2.8. This dynamical projective system could be thought of as a rendering (def. 2.6) of the dense domain $\mathcal{D}$ of the operator H, except for the fact that $\mathcal{D}$ is actually *not* a strong symplectic manifold (unless H is bounded, in which case $\mathcal{D} = \mathcal{H}$). For the same reason, the assertion in prop. 16.8 could not be put in the form of prop. 3.24, since we are lacking a phase space reduction at the level of the infinite dimensional manifold $\mathcal{M}^{\text{KIN}} = \mathcal{H} \times \mathbb{R}^2$ when H is unbounded. Instead, we collect in prop. 16.9 a set of properties imitating the framework of prop. 3.24, and we will formulate the convergence on this substitute ground.

It is worth mentioning that here, as in the previous example, we are able to directly give a projective system rendering the space of dynamical states, and more generally being able to find a regularizing scheme in the sense of subsection 3.2 implies that one can construct such a dynamical projective structure. This perhaps requires a few comments. At first it sounds as if implementing and solving the constraints requires to already know completely the structure of the dynamical theory. However, one should keep in mind that solving the dynamics and obtaining the dynamical theory is *not* simply constructing the space of physical states: the more crucial part is to construct the dynamical observables, not simply as a space of functions on the reduced phase space, but as a family of *non functionally independent* elementary observables, each of which should be linked to a physical meaning (aka. an experimental protocol).

This point is transparently illustrated by the toy model we are studying in the present section.

The submanifold $t = 0$ is obviously a gauge fixing surface of the theory we are considering, and this is what allows us to obtain immediately a description of the reduced phase space. But, clearly, having realized this property of the dynamics does not mean we have solved the theory: if we want to know how a given system will evolve we need to define dynamical observables associated to kinematical ones with support on other constant time surfaces. Indeed, the dynamical observables associated with time $t = 0$ are the only ones that can be directly defined on the reduced phase space defined through the aforementioned gauge fixing. And, although they provide a parametrization of the dynamical state space, they do not allow us to compute predictions for any arbitrary experiment, since, as underlined many times in the discussion of the handling of constraints (section 3), the predictive content of the theory is encoded in the functional relations among an overcomplete set of dynamical observables, arising from functionally independent kinematical observables.

Note that in any theory admitting some obvious gauge fixing (which needs not to be singled out nor preferred in any sense: in the example at hand, selecting $t = 0$ rather than any other time surface is an arbitrary choice), we can use this gauge fixing surface as a starting point to design an approximating scheme: it provides an explicit description of the reduced phase space, and we can use it as a pivot to define projections between the successive approximated dynamical theories (for we can relate approximated orbits depending on their intersection with the gauge fixing surface, as we indeed do in the present example). In particular, this suggests that such approximating schemes could be obtained without many difficulties within the so called 'deparametrization' framework [35].

Proposition 16.7 Under the same hypotheses as in def. 16.4, we define:

1. $\mathcal{L}_H := \{\mathcal{J} \in \mathcal{L} \mid \forall \psi \in \mathcal{J},\ \|H\psi\| < \infty\}$ with the preorder defined by $\subset$;
2. $\forall \mathcal{J} \in \mathcal{L}_H$, $\mathcal{M}^{\text{DYN}}_{\mathcal{J}} := \mathcal{J}$, equipped with the symplectic structure $\Omega_{\text{DYN},\mathcal{J}}$ induced from $\mathcal{M}^{\text{DYN}}$, Ω_{DYN};
3. $\mathcal{D} := \{\psi \in \mathcal{H} \mid \|H\psi\| < \infty\}$ ($\mathcal{D}$ is the dense domain of the self-adjoint possibly unbounded operator H) and $\mathcal{M}^{\text{DYN}}_{\mathcal{D}} := \mathcal{D}$;
4. $\forall \mathcal{J}, \mathcal{J}' \in \mathcal{L}_H$, with $\mathcal{J} \subset \mathcal{J}'$, $\pi^{\text{DYN}}_{\mathcal{J}' \to \mathcal{J}} := \Pi_{\mathcal{J}}|_{\mathcal{J}' \to \mathcal{J}}$ where $\Pi_{\mathcal{J}}$ is the orthogonal projection on $\mathcal{J}$;
5. $\forall \mathcal{J} \in \mathcal{L}_H$, $\pi^{\text{DYN}}_{\mathcal{D} \to \mathcal{J}} := \Pi_{\mathcal{J}}|_{\mathcal{D} \to \mathcal{J}}$.

Then, $(\mathcal{L}_H, \mathcal{M}^{\text{DYN}}, \pi^{\text{DYN}})^{\downarrow}$ is a projective system of phase spaces and we can define (in analogy to def. 2.6) a map $\sigma^{\text{DYN}}_{\downarrow}$ as:

$$\begin{array}{rccl} \sigma^{\text{DYN}}_{\downarrow} & : & \mathcal{M}^{\text{DYN}}_{\mathcal{D}} & \to & \mathcal{S}^{\downarrow}_{(\mathcal{L}_H, \mathcal{M}^{\text{DYN}}, \pi^{\text{DYN}})} \\ & & \psi & \mapsto & \left(\pi^{\text{DYN}}_{\mathcal{D} \to \mathcal{J}}(\psi)\right)_{\mathcal{J} \in \mathcal{L}_H} \end{array}.$$

Additionally, we have a bijective antilinear map $\zeta^{\text{DYN}} : \mathcal{D}^* \to \mathcal{S}^{\downarrow}_{(\mathcal{L}_H, \mathcal{M}^{\text{DYN}}, \pi^{\text{DYN}})}$ such that $\zeta^{\text{DYN},-1} \circ \sigma^{\text{DYN}}_{\downarrow} : \mathcal{M}^{\text{DYN}}_{\mathcal{D}} \to \mathcal{D}^*$ is the restriction to $\mathcal{D}$ of the canonical identification of $\mathcal{H}$ with $\mathcal{D}' \subset \mathcal{D}^*$.

Proof We prove that $\mathcal{L}_H$ is directed like in the proof of prop. 16.5.

For $\mathcal{J} \in \mathcal{L}_H$, we have that $\pi^{\text{DYN}}_{\mathcal{D} \to \mathcal{J}}$ is surjective (since $\mathcal{J} \subset \mathcal{D}$) and, for $\mathcal{J} \subset \mathcal{J}' \in \mathcal{L}_H$, $\pi^{\text{DYN}}_{\mathcal{J}' \to \mathcal{J}} \circ \pi^{\text{DYN}}_{\mathcal{D} \to \mathcal{J}'} = \pi^{\text{DYN}}_{\mathcal{D} \to \mathcal{J}}$ (but speaking of compatibility with symplectic structure does not make sense for $\pi^{\text{DYN}}_{\mathcal{D} \to \mathcal{J}}$ since $\mathcal{D}$ is not a strong symplectic manifold).

The rest of the proof works as for prop. 15.2. □

Proposition 16.8 We consider the objects introduced in props. 16.6 and 16.7. We define:

1. $\forall \varepsilon \in \mathcal{E},\ \mathcal{L}'^{\varepsilon} := \mathcal{L}^{\varepsilon} \sqcup \{\mathcal{H}\}$;

2. $\forall \varepsilon = (\mathcal{J},\ \epsilon) \in \mathcal{E},\ \forall \mathcal{I} \in \mathcal{L}'^{\varepsilon},\ \ell^{\text{DYN}}(\varepsilon,\ \mathcal{I}) := \mathcal{J}$;

3. $\forall (\varepsilon_1,\ \mathcal{I}_1),\ (\varepsilon_2,\ \mathcal{I}_2) \in \mathcal{E}\mathcal{L}' \,/\, (\varepsilon_1,\ \mathcal{I}_1) \preccurlyeq (\varepsilon_2,\ \mathcal{I}_2),\quad \pi^{\text{DYN},\varepsilon_2\to\varepsilon_1}_{\mathcal{I}_2\to\mathcal{I}_1} := \pi^{\text{DYN}}_{\ell^{\text{DYN}}(\varepsilon_2,\mathcal{I}_2)\to\ell^{\text{DYN}}(\varepsilon_1,\mathcal{I}_1)}$.

Then, $\left(\mathcal{E},\ \left(\mathcal{L}'^{\varepsilon}\right)_{\varepsilon\in\mathcal{E}},\ \left(\mathcal{M}^{\text{DYN},\varepsilon}_{\mathcal{I}}\right)_{(\varepsilon,\mathcal{I})\in\mathcal{E}\mathcal{L}'},\ \left(\mathcal{M}^{\text{SHELL},\varepsilon}_{\mathcal{I}}\right)_{(\varepsilon,\mathcal{I})\in\mathcal{E}\mathcal{L}'},\ \left(\pi^{\text{DYN},\varepsilon_2\to\varepsilon_1}_{\mathcal{I}_2\to\mathcal{I}_1}\right)_{(\varepsilon_1,\mathcal{I}_1)\preccurlyeq(\varepsilon_2,\mathcal{I}_2)},\ \left(\delta^{\varepsilon}_{\mathcal{I}}\right)_{(\varepsilon,\mathcal{I})\in\mathcal{E}\mathcal{L}'}\right)$ is a regularized reduction of $(\mathcal{L} \sqcup \{\mathcal{H}\},\ \mathcal{M}^{\text{KIN}},\ \pi^{\text{KIN}})^{\downarrow}$ and we have a bijective map $\kappa^{\text{DYN}} : \mathcal{S}^{\downarrow}_{(\mathcal{E}\mathcal{L}',\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})} \to \mathcal{S}^{\downarrow}_{(\mathcal{L}_H,\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})}$.

Proof $\mathcal{E}$ is a directed set (prop. 16.5) and $\left(\mathcal{L}'^{\varepsilon}\right)_{\varepsilon\in\mathcal{E}}$ is a family of decreasing cofinal parts of $\mathcal{L}\sqcup\{\mathcal{H}\}$.

Next, we have $\forall (\varepsilon, \mathcal{I}) \in \mathcal{E}\mathcal{L}'$, $\mathcal{M}^{\text{DYN},\varepsilon}_{\mathcal{I}} = \mathcal{M}^{\text{DYN}}_{\ell^{\text{DYN}}(\varepsilon,\mathcal{I})}$ hence for $(\varepsilon_1, \mathcal{I}_1) \preccurlyeq (\varepsilon_2, \mathcal{I}_2) \in \mathcal{E}\mathcal{L}'$, $\pi^{\text{DYN},\varepsilon_2\to\varepsilon_1}_{\mathcal{I}_2\to\mathcal{I}_1}$ is well-defined as a surjective map $\mathcal{M}^{\text{DYN},\varepsilon_2}_{\mathcal{I}_2} \to \mathcal{M}^{\text{DYN},\varepsilon_1}_{\mathcal{I}_1}$ and it is compatible with the symplectic structures.

Moreover, for $\varepsilon_1 = \varepsilon_2 \in \mathcal{E}$, this definition coincides with the map $\pi^{\text{DYN},\varepsilon\to\varepsilon}_{\mathcal{I}_2\to\mathcal{I}_1}$ that has been introduced in prop. 16.6. Hence, for all $\varepsilon \in \mathcal{E}$, we have from prop. 16.6 that $(\mathcal{M}^{\text{DYN},\varepsilon},\ \mathcal{M}^{\text{SHELL},\varepsilon},\ \pi^{\text{DYN},\varepsilon\to\varepsilon},\ \delta^{\varepsilon})$ is an elementary reduction of $\left(\mathcal{L}'^{\varepsilon},\ \mathcal{M}^{\text{KIN}},\ \pi^{\text{KIN}}\right)^{\downarrow}$.

Lastly, using $\ell^{\text{DYN}} : \mathcal{E}\mathcal{L}' \to \mathcal{L}_H$ (with $\ell^{\text{DYN}}\left\langle \mathcal{E}\mathcal{L}' \right\rangle = \mathcal{L}_H$), we have by prop. 2.8 that $\left(\mathcal{E}\mathcal{L}', \mathcal{M}^{\text{DYN}}, \pi^{\text{DYN}}\right)^{\downarrow}$ is a projective system of phase spaces, thus:

$$\left(\mathcal{E},\ \left(\mathcal{L}'^{\varepsilon}\right)_{\varepsilon\in\mathcal{E}},\ \left(\mathcal{M}^{\text{DYN},\varepsilon}_{\mathcal{I}}\right)_{(\varepsilon,\mathcal{I})\in\mathcal{E}\mathcal{L}'},\ \left(\mathcal{M}^{\text{SHELL},\varepsilon}_{\mathcal{I}}\right)_{(\varepsilon,\mathcal{I})\in\mathcal{E}\mathcal{L}'},\ \left(\pi^{\text{DYN},\varepsilon_2\to\varepsilon_1}_{\mathcal{I}_2\to\mathcal{I}_1}\right)_{(\varepsilon_1,\mathcal{I}_1)\preccurlyeq(\varepsilon_2,\mathcal{I}_2)},\ \left(\delta^{\varepsilon}_{\mathcal{I}}\right)_{(\varepsilon,\mathcal{I})\in\mathcal{E}\mathcal{L}'}\right)$$

is a regularized reduction of $(\mathcal{L} \sqcup \{\mathcal{H}\},\ \mathcal{M}^{\text{KIN}},\ \pi^{\text{KIN}})^{\downarrow}$. And, in addition, there exists a bijective map $\kappa^{\text{DYN}} : \mathcal{S}^{\downarrow}_{(\mathcal{E}\mathcal{L}',\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})} \to \mathcal{S}^{\downarrow}_{(\mathcal{L}_H,\mathcal{M}^{\text{DYN}},\pi^{\text{DYN}})}$. $\square$

Proposition 16.9 We consider the same objects as in prop. 16.8 and we additionally define:

1. $\mathcal{M}^{\text{DYN},\mathcal{D}}_{\mathcal{H}} := \mathcal{M}^{\text{DYN}}_{\mathcal{D}} = \mathcal{D}$;

2. $\mathcal{M}^{\text{SHELL},\mathcal{D}}_{\mathcal{H}} := \{(\psi,\ t,\ E) \in \mathcal{M}^{\text{KIN}}_{\mathcal{H}} \mid \psi \in \mathcal{D} \quad \& \quad E = \langle \psi,\ H\psi \rangle\}$;

3. $\begin{array}{rcl} \delta^{\mathcal{D}}_{\mathcal{H}} : \quad \mathcal{M}^{\text{SHELL},\mathcal{D}}_{\mathcal{H}} & \to & \mathcal{M}^{\text{DYN},\mathcal{D}}_{\mathcal{H}} \\ (\psi,\ t,\ E) & \mapsto & \exp(i\,t\,H)\ \psi \end{array}$;

4. $\forall (\varepsilon,\ \mathcal{I}) \in \mathcal{E}\mathcal{L}',\ \pi^{\text{DYN},\mathcal{D}\to\varepsilon}_{\mathcal{H}\to\mathcal{I}} := \pi^{\text{DYN}}_{\mathcal{D}\to\ell^{\text{DYN}}(\varepsilon,\mathcal{I})}$.

Then, we have:

5. $\delta^{\mathcal{D}}_{\mathcal{H}}$ is surjective and, for all $\psi_o \in \mathcal{M}^{\text{DYN},\mathcal{D}}_{\mathcal{H}}$, $\left(\delta^{\mathcal{D}}_{\mathcal{H}}\right)^{-1}\langle \psi_o \rangle$ is connected;

6. for all $(\varepsilon,\ \mathcal{I}) \in \mathcal{E}\mathcal{L}'$, $\pi^{\text{DYN},\mathcal{D}\to\varepsilon}_{\mathcal{H}\to\mathcal{I}}$ is surjective and, for all $(\varepsilon_1,\ \mathcal{I}_1) \preccurlyeq (\varepsilon_2,\ \mathcal{I}_2) \in \mathcal{E}\mathcal{L}'$, $\pi^{\text{DYN},\varepsilon_2\to\varepsilon_1}_{\mathcal{I}_2\to\mathcal{I}_1} \circ \pi^{\text{DYN},\mathcal{D}\to\varepsilon_2}_{\mathcal{H}\to\mathcal{I}_2} = \pi^{\text{DYN},\mathcal{D}\to\varepsilon_1}_{\mathcal{H}\to\mathcal{I}_1}$.

Proof Since H is self-adjoint, $\exp(-i\,t\,H)$ defines a unitary operator on $\mathcal{H}$, and this operator stabilizes $\mathcal{D}$, for $\forall \psi \in \mathcal{D}$, $\|H \exp(-i\,t\,H)\ \psi\| = \|\exp(-i\,t\,H)\ H\,\psi\| = \|H\,\psi\| < \infty$. Hence, for $\psi_o \in \mathcal{M}^{\text{DYN},\mathcal{D}}_{\mathcal{H}}$:

$$\left(\delta^{\mathcal{D}}_{\mathcal{H}}\right)^{-1}\langle \psi_o \rangle = \{(\psi,\ t,\ E) \in \mathcal{M}^{\text{KIN}}_{\mathcal{H}} \mid t \in \mathbb{R},\ \psi = \exp(-i\,t\,H)\ \psi_o \in \mathcal{D} \quad \& \quad E = \langle \psi_o,\ H\psi_o \rangle < \infty\}.$$

Next, the statements 16.9.6 follows from the proof of prop. 16.7 (for $\forall(\varepsilon, \mathcal{I}) \in \mathcal{EL}'$, $\mathcal{M}^{\text{DYN},\mathcal{E}}_{\mathcal{I}} = \mathcal{M}^{\text{DYN}}_{\ell^{\text{DYN}}(\varepsilon,\mathcal{I})}$ and $\mathcal{M}^{\text{DYN},\mathcal{D}}_{\mathcal{H}} = \mathcal{M}^{\text{DYN}}_{\mathcal{D}}$). □

We close the discussion of the classical part of this toy model by proving that we indeed have suitable convergence at least for the dynamical states corresponding to vectors in $\mathcal{D}$, and, more precisely, that the successive projective families of orbits approximating such a state on the kinematical side correctly converge to the family arising from its associated orbit in the pseudo phase space reduction of $\mathcal{M}^{\text{KIN}}$ (that we introduced in prop. 16.9).

Theorem 16.10 We consider the same objects as in prop. 16.9. Let $\psi_o \in \mathcal{D}$. For $\varepsilon \in \mathcal{E}$, we define:

1. $\psi^\varepsilon := (\delta^{\varepsilon}_{\mathcal{H}})^{-1} \left\langle \pi^{\text{DYN},\mathcal{D}\to\varepsilon}_{\mathcal{H}\to\mathcal{H}}(\psi_o) \right\rangle \subset \mathcal{M}^{\text{SHELL},\varepsilon}_{\mathcal{H}} \subset \mathcal{M}^{\text{KIN}}_{\mathcal{H}} = \mathcal{M}^{\text{KIN}}$;
2. $\Psi^\varepsilon := \widehat{\sigma}^{\text{KIN}}_{\downarrow}(\psi^\varepsilon)$, where $\widehat{\sigma}^{\text{KIN}}_{\downarrow} : \mathcal{P}(\mathcal{M}^{\text{KIN}}) \to \widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ is defined as in prop. 3.23.

Then, the net $(\Psi^\varepsilon)_{\varepsilon\in\mathcal{E}}$ converges in $\widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ to $\widehat{\sigma}^{\text{KIN}}_{\downarrow}\left(\left(\delta^{\mathcal{D}}_{\mathcal{H}}\right)^{-1}\langle\psi_o\rangle\right)$ (def. 3.21).

Proof For $\varepsilon = (\mathcal{J}, \epsilon) \in \mathcal{E}$, we have, from the proof of prop. 16.6:

$$\psi^\varepsilon = \left\{ \left(e^{-\frac{i}{\epsilon}\sin\theta\,\Pi_{\mathcal{J}} H}\,\Pi_{\mathcal{J}}(\psi_o),\ \tfrac{1}{\epsilon}\sin\theta,\ E_{\mathcal{J}} + \epsilon\cos\theta \right) \,\middle|\, \theta \in [0, 2\pi[\right\},$$

where $E_{\mathcal{J}} := \langle \Pi_{\mathcal{J}}(\psi_o),\ H\,\Pi_{\mathcal{J}}(\psi_o)\rangle$. Hence, for all $\mathcal{I} \in \mathcal{L}$:

$$[\Psi^\varepsilon]_{\mathcal{I}} = \left\{ \left(\Pi_{\mathcal{I}}\, e^{-\frac{i}{\epsilon}\sin\theta\,\Pi_{\mathcal{J}} H}\,\Pi_{\mathcal{J}}(\psi_o),\ \tfrac{1}{\epsilon}\sin\theta,\ E_{\mathcal{J}} + \epsilon\cos\theta \right) \,\middle|\, \theta \in [0, 2\pi[\right\}.$$

And, from the proof of prop. 16.9:

$$\left[\widehat{\sigma}^{\text{KIN}}_{\downarrow}\left(\left(\delta^{\mathcal{D}}_{\mathcal{H}}\right)^{-1}\langle\psi_o\rangle\right)\right]_{\mathcal{I}} = \left\{ \left(\Pi_{\mathcal{I}}\, e^{-i\,t\,H}\,\psi_o,\ t,\ E_{\mathcal{D}}\right) \,\middle|\, t \in \mathbb{R} \right\} \text{ where } E_{\mathcal{D}} := \langle \psi_o,\ H\,\psi_o\rangle.$$

Let $\mathcal{I} \in \mathcal{L}$ and let U be an open set in $\mathcal{M}^{\text{KIN}}_{\mathcal{I}}$, such that $U \cap \left[\widehat{\sigma}^{\text{KIN}}_{\downarrow}\left(\left(\delta^{\mathcal{D}}_{\mathcal{H}}\right)^{-1}\langle\psi_o\rangle\right)\right]_{\mathcal{I}} \neq \varnothing$. Let $t \in \mathbb{R}$ such that:

$$\left(\Pi_{\mathcal{I}}\, e^{-i\,t\,H}\,\psi_o,\ t,\ E_{\mathcal{D}}\right) \in U,$$

and let $\epsilon_1 > 0$ such that:

$$\begin{aligned}&\forall\psi \in \mathcal{I},\ \forall E \in \mathbb{R},\\ &\qquad \left(\left\|\psi - \Pi_{\mathcal{I}}\, e^{-itH}\,\psi_o\right\| \leqslant \epsilon_1\ (\|\psi_o\| + 1) \quad \& \quad |E - E_{\mathcal{D}}| \leqslant \epsilon_1\right) \ \Rightarrow\ ((\psi,\ t,\ E) \in U).\end{aligned}$$

Let $\epsilon_2 = \frac{1}{1+|t|}\log\left(1 + \frac{\epsilon_1}{2}\right) > 0$ and $\epsilon_3 = \min\left(\epsilon_1,\ \frac{1}{1+|t|}\right) > 0$.

By spectral resolution, we can define $H^{\epsilon_2} := \epsilon_2 \left\lfloor \frac{1}{\epsilon_2} H \right\rfloor$ (where $\lfloor\cdot\rfloor$ denotes the floor function). Then, we have $[H, H^{\epsilon_2}] = 0$, $\|H - H^{\epsilon_2}\| \leqslant \epsilon_2$ and H^{ϵ_2} has discrete spectrum included in $\epsilon_2\,\mathbb{Z}$. Hence, there exists $N \in \mathbb{N}$ such that:

$$\left\|\psi_o - \sum_{k=-N}^{+N} \psi_o^{\epsilon_2,k}\right\| \leqslant \frac{\epsilon_1}{2},$$

where, for $k \in \mathbb{Z}$, $\psi_o^{\epsilon_2,k}$ is the projection of ψ_o on the eigenspace of H^{ϵ_2} with eigenvalue $\epsilon_2\, k$ (defining these eigenspace to be $\{0\}$ if $\epsilon_2\, k$ is not in the spectrum of H^{ϵ_2}).

We define $\mathcal{J}_2 = \text{Vect}\,\left\{\psi_o^{\epsilon_2,k} \;\middle|\; k \in \{-N,\ldots,N\}\right\}$, $\mathcal{J}_2$ is finite dimensional, and $\forall\psi \in \mathcal{J}_2$, $\|H\psi\| \leqslant \epsilon_2\,\|\psi\| + N\epsilon_2\,\|\psi\| < \infty$, hence $\mathcal{J}_2 \in \mathcal{L}_H$. Moreover, $\mathcal{J}_2$ is stabilized by H^{ϵ_2} and $\|\psi_o - \Pi_{\mathcal{J}_2}\psi_o\| \leqslant \frac{1}{2}\,\epsilon_1$. Next, we define $\mathcal{J}_3 = \mathcal{J}_2 + \text{Vect}\,\{\psi_o\} \in \mathcal{L}_H$ (because $\psi_o \in \mathcal{D}$).

Now, we consider $\varepsilon = (\mathcal{J},\,\epsilon) \in \mathcal{E}$, with $(\mathcal{J},\,\epsilon) \succcurlyeq (\mathcal{J}_3,\,\epsilon_3)$. We choose $\theta \in [0, 2\pi[$ such that $\sin\theta = \epsilon\, t$ ($|\epsilon t| \leqslant \epsilon_3\,|t| \leqslant 1$) and we define:

$$\psi := \Pi_{\mathfrak{J}}\, e^{-\frac{i}{\epsilon}\sin\theta\,\Pi_{\mathcal{J}}\,H}\,\Pi_{\mathcal{J}}(\psi_o)\,.$$

We have:

$$\begin{aligned}
\left\|\psi - \Pi_{\mathfrak{J}}\, e^{-it\,H}\,\psi_o\right\| &\leqslant \left\|e^{-it\,\Pi_{\mathcal{J}}\,H}\,\Pi_{\mathcal{J}}(\psi_o) - e^{-it\,H}\,\psi_o\right\| \\
&\leqslant \left\|e^{-it\,\Pi_{\mathcal{J}}\,(H-H^{\epsilon_2})} - \text{id}_{\mathcal{J}}\right\|\,\left\|e^{-it\,\Pi_{\mathcal{J}}\,H^{\epsilon_2}}\,\Pi_{\mathcal{J}}(\psi_o)\right\| + \left\|e^{-it\,\Pi_{\mathcal{J}}\,H^{\epsilon_2}}\,\Pi_{\mathcal{J}}(\psi_o) - e^{-it\,H^{\epsilon_2}}\,\psi_o\right\| + \\
&+ \left\|e^{-it\,(H-H^{\epsilon_2})} - \text{id}_{\mathcal{H}}\right\|\,\left\|e^{-it\,H^{\epsilon_2}}\,(\psi_o)\right\| \\
&\leqslant 2\,\left|e^{|t|\,\|H-H^{\epsilon_2}\|} - 1\right|\,\|\psi_o\| + \left\|e^{-it\,\Pi_{\mathcal{J}}\,H^{\epsilon_2}}\,\Pi_{\mathcal{J}_2}(\psi_o) - e^{-it\,H^{\epsilon_2}}\,\Pi_{\mathcal{J}_2}(\psi_o)\right\| + \\
&+ \left\|e^{-it\,\Pi_{\mathcal{J}}\,H^{\epsilon_2}}\,(\psi_o - \Pi_{\mathcal{J}_2}(\psi_o)) - e^{-it\,H^{\epsilon_2}}\,(\psi_o - \Pi_{\mathcal{J}_2}(\psi_o))\right\| \\
&\leqslant 2\,\left|e^{|t|\,\epsilon_2} - 1\right|\,\|\psi_o\| + \left\|e^{-it\,H^{\epsilon_2}}\,\Pi_{\mathcal{J}_2}(\psi_o) - e^{-it\,H^{\epsilon_2}}\,\Pi_{\mathcal{J}_2}(\psi_o)\right\| + 2\,\|\psi_o - \Pi_{\mathcal{J}_2}(\psi_o)\| \\
&\leqslant 2\,\frac{\epsilon_1}{2}\,\|\psi_o\| + 2\,\frac{\epsilon_1}{2} \leqslant \epsilon_1\,(1 + \|\psi_o\|)\,,
\end{aligned}$$

and:

$$|E_{\mathcal{J}} + \epsilon\,\cos\theta - E_{\mathcal{D}}| \leqslant |\langle\psi_o,\,(H - \Pi_{\mathcal{J}}\,H\,\Pi_{\mathcal{J}})\,\psi_o\rangle| + \epsilon_1 = \epsilon_1 \text{ (since } \psi_o \in \mathcal{J}_3 \subset \mathcal{J}).$$

Therefore, $\left(\psi,\,\frac{1}{\epsilon}\,\sin\theta,\,E_{\mathcal{J}} + \epsilon\cos\theta\right) \in U$, but since $\left(\psi,\,\frac{1}{\epsilon}\,\sin\theta,\,E_{\mathcal{J}} + \epsilon\cos\theta\right) \in [\psi^{\varepsilon}]_{\mathfrak{J}}$, we have $\forall\varepsilon = (\mathcal{J},\,\epsilon) \succcurlyeq (\mathcal{J}_3,\,\epsilon_3)\,,\ [\psi^{\varepsilon}]_{\mathfrak{J}} \cap U \neq \varnothing$.

Let K be a compact subset of $\mathcal{M}_{\mathfrak{J}}^{\text{KIN}}$, such that $K \cap \left[\widehat{\sigma}_{\downarrow}^{\text{KIN}}\left(\left(\delta_{\mathcal{H}}^{\mathcal{D}}\right)^{-1}\langle\psi_o\rangle\right)\right]_{\mathfrak{J}} = \varnothing$. Hence, there exist $T > 0$ and $\epsilon_1 > 0$ such that:

$$\forall\psi \in \mathfrak{J},\ \forall t \in \mathbb{R},\ \forall E \in \mathbb{R},$$

$$\left[\left(\left\|\psi - \Pi_{\mathfrak{J}}\, e^{-it\,H}\,\psi_o\right\| \leqslant \epsilon_1\,(\|\psi_o\| + 1) \quad \& \quad |E - E_{\mathcal{D}}| \leqslant \epsilon_1\right) \quad \text{or} \quad |t| \geqslant T\right] \;\Rightarrow\; ((\psi,\,t,\,E) \notin K)\,.$$

Following the same path as above, we can define:

$$\epsilon_2 = \tfrac{1}{1+T}\log\left(1 + \tfrac{\epsilon_1}{2}\right) > 0,$$

and construct a vector subspace $\mathcal{J}_2 \in \mathcal{L}_H$ such that:

$$\forall\mathcal{J} \in \mathcal{L}_H\,/\,\mathcal{J} \supset \mathcal{J}_2,\ \forall t \in \mathbb{R},\quad \left\|e^{-it\,\Pi_{\mathcal{J}}\,H}\,\Pi_{\mathcal{J}}(\psi_o) - e^{-it\,H}\,\psi_o\right\| \leqslant 2\,\left|e^{|t|\,\epsilon_2} - 1\right|\,\|\psi_o\| + \epsilon_1\,.$$

Analogously, we define $\epsilon_3 = \epsilon_1$ and $\mathcal{J}_3 = \mathcal{J}_2 + \text{Vect}\,\{\psi_o\} \in \mathcal{L}_H$.

Now, we consider $\varepsilon = (\mathcal{J},\,\epsilon) \in \mathcal{E}$, with $(\mathcal{J},\,\epsilon) \succcurlyeq (\mathcal{J}_3,\,\epsilon_3)$, and $\theta \in [0, 2\pi[$. If $\left|\frac{1}{\epsilon}\sin\theta\right| < T$, then we have, with $t = \frac{1}{\epsilon} \sin\theta$:

$$\left\| \Pi_{\mathcal{J}}\, e^{-\frac{i}{\epsilon}\, \sin\theta\, \Pi_{\mathcal{J}}\, H}\, \Pi_{\mathcal{J}}(\psi_o) - \Pi_{\mathcal{J}}\, e^{-it\,H}\, \psi_o \right\| \leqslant \epsilon_2\, (1 + \|\psi_o\|)\,,$$

and:

$$|E_{\mathcal{J}} + \epsilon\, \cos\theta - E_{\mathcal{D}}| \leqslant \epsilon_1\,.$$

Therefore, $\forall \varepsilon = (\mathcal{J},\, \epsilon) \succcurlyeq (\mathcal{J}_3,\, \epsilon_3)\,,\, [\psi^{\varepsilon}]_{\mathcal{J}} \cap K = \varnothing$.

So, for every $\mathcal{J} \in \mathcal{L}$, the net $\left([\Psi^{\varepsilon}]_{\mathcal{J}}\right)_{\varepsilon\in\mathcal{E}}$ converges in $\mathcal{P}\,(\mathcal{M}^{\text{KIN}}_{\mathcal{J}})$ to $\left[\widehat{\sigma}^{\text{KIN}}_{\downarrow}\left(\left(\delta^{\mathcal{D}}_{\mathcal{H}}\right)^{-1}\langle\psi_o\rangle\right)\right]_{\mathcal{J}}$, thus, the net $(\Psi^{\varepsilon})_{\varepsilon\in\mathcal{E}}$ converges in $\widehat{\mathcal{S}}^{\downarrow}_{(\mathcal{L},\mathcal{M}^{\text{KIN}},\pi^{\text{KIN}})}$ to $\widehat{\sigma}^{\text{KIN}}_{\downarrow}\left(\left(\delta^{\mathcal{D}}_{\mathcal{H}}\right)^{-1}\langle\psi_o\rangle\right)$. □

16.2 Quantum theory

We now want to implement this construction at the quantum level, with the aim of using this simple toy model to get a first hold on the implementation of constraints in projective systems of quantum state spaces.

To fix the notations, we begin by summarizing the main properties of (bosonic) Fock spaces [41, section I.3.4].

Definition 16.11 Let $\mathcal{H}$ be a separable Hilbert space. We define the Fock space $\widehat{\mathcal{H}}$ by:

$$\widehat{\mathcal{H}} := \overline{\bigoplus_{n\in\mathbb{N}} \mathcal{H}^{\otimes n,\,\text{sym}}} \text{ where } \mathcal{H}^{\otimes n,\,\text{sym}} \text{ is the symmetric vector subspace of } \mathcal{H}^{\otimes n}.$$

For $(e_i)_{i\in I}$ an orthonormal basis of $\mathcal{H}$ ($I \subset \mathbb{N}$), we define:

$$\Lambda_I := \left\{(n_i)_{i\in I} \mid \forall i \in I,\, n_i \in \mathbb{N} \quad \& \quad \textstyle\sum_i n_i < \infty\right\},$$

indexing the orthonormal basis $\left(\,|(n_i)_{i\in I}\,,\, (e_i)_{i\in I}\rangle\right)_{(n_i)_{i\in I}\in\Lambda_I}$ of $\widehat{\mathcal{H}}$:

$$\forall\, (n_i)_{i\in I} \in \Lambda_I,\, |(n_i)_{i\in I}\,,\, (e_i)_{i\in I}\rangle := \sqrt{\frac{\prod_{i\in I}(n_i!)}{N!}} \sum_{\substack{k_1,\dots,\,k_N \\ \forall i\in I,\, \#\{l \,\mid\, k_l = i\} = n_i}} |e_{k_1}\rangle \otimes \ldots \otimes |e_{k_N}\rangle\,,$$

where $N = \sum_{i\in I} n_i$.

If $\left(f_j\right)_{j\in I}$ is an other orthonormal basis of $\mathcal{H}$, we have:

$$\forall\, (n_i)_{i\in I}\,,\, \left(m_j\right)_{j\in I} \in \Lambda_I,\, \left\langle (n_i)_{i\in I}\,,\, (e_i)_{i\in I} \,\middle|\, \left(m_j\right)_{j\in I}\,,\, \left(f_j\right)_{j\in I} \right\rangle =$$

$$= \sum_{\substack{l_{i,j} \in \mathbb{N}^{I \times I} \\ \forall i,\, n_i = \sum_j l_{i,j} \\ \forall j,\, m_j = \sum_i l_{i,j}}} \prod_{i \in I} \left(\sqrt{\frac{n_i!}{\prod_j l_{i,j}!}} \right) \prod_{j \in I} \left(\sqrt{\frac{m_j!}{\prod_i l_{i,j}!}} \right) \prod_{i,j} \left\langle e_i,\, f_j \right\rangle^{l_{i,j}} . \tag{16.11.1}$$

Definition 16.12 We consider the same objects as in def. 16.11. Let $e \in \mathcal{H}$, $N \geqslant 1$ and $l \in \{1, \dots, N\}$. We define the operators $\widehat{a}_e^{N,l} : \mathcal{H}^{\otimes N} \to \mathcal{H}^{\otimes N-1}$ and $\left(\widehat{a}_e^{N,l}\right)^+ : \mathcal{H}^{\otimes N-1} \to \mathcal{H}^{\otimes N}$ by:

$$\forall \varphi_1, \dots, \varphi_N \in \mathcal{H},\;\; \widehat{a}_e^{N,\,l}\, \varphi_1^{(1)} \otimes \dots \otimes \varphi_N^{(N)} := \frac{\langle e,\, \varphi_l \rangle}{\sqrt{N}}\, \varphi_1^{(1)} \otimes \dots \otimes \cancel{\varphi_l^{(l)}} \otimes \dots \otimes \varphi_N^{(N-1)} ,$$

$$\text{and} \quad \forall \varphi_1, \dots, \varphi_{N-1} \in \mathcal{H},\;\; \left(\widehat{a}_e^{N,\,l}\right)^+ \varphi_1^{(1)} \otimes \dots \otimes \varphi_{N-1}^{(N-1)} := \frac{1}{\sqrt{N}} \varphi_1^{(1)} \otimes \dots \otimes e^{(l)} \otimes \dots \otimes \varphi_{N-1}^{(N)} .$$

Then, on $\widehat{\mathcal{H}}$ we can define (unbounded) operators $\widehat{a}_e$ and $\widehat{a}_e^+$, such that:

$$\forall \psi \in \mathcal{H}^{\otimes N,\,\text{sym}},\;\; \widehat{a}_e\, \psi = \sum_{l=1}^{N} \widehat{a}_e^{N,\,l}\, \psi \,\in \mathcal{H}^{\otimes N-1,\,\text{sym}} ,$$

$$\text{and} \quad \forall \psi \in \mathcal{H}^{\otimes N-1,\,\text{sym}},\;\; \widehat{a}_e^+\, \psi = \sum_{l=1}^{N} \left(\widehat{a}_e^{N,\,l}\right)^+ \psi \,\in \mathcal{H}^{\otimes N,\,\text{sym}} .$$

Let A be a bounded self-adjoint operator on $\mathcal{H}$. We can define an (unbounded) essentially self-adjoint operator $\widehat{A}$ on $\widehat{\mathcal{H}}$ such that:

$$\forall \psi \in \mathcal{H}^{\otimes N,\,\text{sym}},\;\; \widehat{A}\, \psi = \sum_{l=1}^{N} \text{id}_{\mathcal{H}}^{(1)} \otimes \dots \otimes A^{(l)} \otimes \dots \otimes \text{id}_{\mathcal{H}}^{(N)}\, \psi \,\in H^{\otimes N,\,\text{sym}} .$$

For $(e_i)_{i \in I}$ an orthonormal basis of $\mathcal{H}$, we have:

$$\widehat{A} = \sum_{i,j \in I} \left\langle e_i,\, A\, e_j \right\rangle \widehat{a}_{e_i}^+ \widehat{a}_{e_j} .$$

Lastly, let e, $f \in \mathcal{H}$ and let A, B be bounded self-adjoint operators on $\mathcal{H}$. The commutators between the operators defined above are given by:

$$[\widehat{a}_e,\, \widehat{a}_f] = 0,\; \left[\widehat{a}_e^+,\, \widehat{a}_f^+\right] = 0, \text{ and } \left[\widehat{a}_e,\, \widehat{a}_f^+\right] = \langle e,\, f \rangle\, \text{id}_{\widehat{\mathcal{H}}} ,$$

$$\left[\widehat{A},\, \widehat{B}\right] = \widehat{[A, B]_{\mathcal{H}}} ,\; \left[\widehat{a}_e,\, \widehat{A}\right] = \widehat{a}_{Ae} , \text{ and } \left[\widehat{a}_e^+,\, \widehat{A}\right] = -\widehat{a}_{Ae}^+ .$$

Before going on to the quantization using projective structures, we recall the more conventional quantization of $\mathcal{M}^{\text{DYN}}$, Ω_{DYN} (ie. a reduced phase space quantization for the theory we are considering) using Fock spaces techniques. The notable fact is that this direct quantization of the Schrödinger equation (considered as a classical field theory, aka. second quantization) can be identified with the (bosonic) Fock space describing an arbitrary number of independent, indistinguishable quantum particles of the corresponding first quantized theory [21]. This identification is not merely a naive matching of the Hilbert spaces: we can check that the quantized observables correspond in a natural way to the observables built on the Fock space (in fact, it can be understood as a holomorphic quantization, see [105, section 9.2]).

Proposition 16.13 We consider the objects introduced in prop. 16.2 and def. 16.12. We define the Fock quantization of $\mathcal{M}^{\text{DYN}}$ as $\widehat{\mathcal{M}}^{\text{DYN}}_{\text{Fock}} := \widehat{\mathcal{H}}$. For A a bounded self-adjoint operator on $\mathcal{H}$ and $e \in \mathcal{H}$ we define the following quantizations for the observables on $\mathcal{M}^{\text{DYN}}$:

$\widehat{\langle A\rangle}_{\text{Fock}} := \widehat{A}$, $\widehat{(\mathrm{a}_e)}_{\text{Fock}} := \widehat{\mathrm{a}}_e$, and $\widehat{(\mathrm{a}^*_e)}_{\text{Fock}} := \widehat{\mathrm{a}}^+_e$.

Then, we have:

$$\forall O,\, O' \in \{\mathrm{a}_e \mid e \in \mathcal{H}\} \cup \{\mathrm{a}^*_e \mid e \in \mathcal{H}\} \cup \{\langle A\rangle \mid A \text{ bounded, self-adj on } \mathcal{H}\}\,,$$

$$\left[\widehat{O}_{\text{Fock}},\, \widehat{O}'_{\text{Fock}}\right] = -i\left(\widehat{\{O,\, O'\}}_{\text{DYN}}\right)_{\text{Fock}}.$$

Proof This can be directly checked by comparing prop. 16.2 with def. 16.12. □

The key tool for constructing a projective system of quantum state spaces reproducing the classical structure from prop. 16.3 is the realization that the Fock space arising from a direct orthogonal sum of two Hilbert space can be naturally identified with the tensor product of the two corresponding Fock spaces. This is in fact a special case of the well-known property of quantization, that translates a Cartesian product of symplectic manifold into a tensor product of Hilbert spaces (for a direct sum is indeed a Cartesian product).

Proposition 16.14 Let $\mathcal{I}$ be a separable Hilbert space. Let $\mathcal{J}$ be a vector subspace of $\mathcal{I}$ and $\mathcal{J}^\perp$ the orthogonal complement of $\mathcal{J}$ in $\mathcal{I}$. Let $(e_i)_{i\in J}$ be an orthonormal basis of $\mathcal{J}$ and $(e_i)_{i\in I\setminus J}$ be an orthonormal basis of $\mathcal{J}^\perp$ (with $I \supset J$). Hence, $(e_i)_{i\in I}$ is an orthonormal basis of $\mathcal{I} = \mathcal{J} \oplus \mathcal{J}^\perp$.

We consider the corresponding Fock spaces $\widehat{\mathcal{I}}$, $\widehat{\mathcal{J}}$ & $\widehat{\mathcal{J}^\perp}$ (def. 16.11) and we define the linear application $\widehat{\varphi}_{\mathcal{I}\to\mathcal{J}} : \widehat{\mathcal{I}} \to \widehat{\mathcal{J}^\perp} \otimes \widehat{\mathcal{J}}$ by its action on the orthonormal basis $\left(\left|(n_i)_{i\in I},\, (e_i)_{i\in I}\right\rangle\right)_{(n_i)_{i\in I}\in\Lambda_I}$ of $\widehat{\mathcal{I}}$:

$$\widehat{\varphi}_{\mathcal{I}\to\mathcal{J}}\, \left|(n_i)_{i\in I},\, (e_i)_{i\in I}\right\rangle := \left|(n_i)_{i\in I\setminus J},\, (e_i)_{i\in I\setminus J}\right\rangle \otimes \left|(n_i)_{i\in J},\, (e_i)_{i\in J}\right\rangle. \tag{16.14.1}$$

Then, $\widehat{\varphi}_{\mathcal{I}\to\mathcal{J}}$ is an Hilbert space isomorphism. Moreover, $\widehat{\varphi}_{\mathcal{I}\to\mathcal{J}}$ does not depend on the choice of the bases $(e_i)_{i\in J}$ and $(e_i)_{i\in I\setminus J}$.

Proof $\widehat{\varphi}_{\mathcal{I}\to\mathcal{J}}$ sends an orthonormal basis to an orthonormal basis, since the map:

$$\begin{array}{rcl} \Lambda_I & \to & \Lambda_{J\setminus I} \times \Lambda_J \\ (n_i)_{i\in I} & \mapsto & (n_i)_{i\in I\setminus J},\, (n_i)_{i\in J} \end{array},$$

is bijective.

Then, if $(f_i)_{i\in J}$ is an other orthonormal basis of $\mathcal{J}$ and $(f_i)_{i\in I\setminus J}$ is an other orthonormal basis of $\mathcal{J}^\perp$, we have, using eq. (16.11.1) for $\left(m_j\right) j \in I \in \Lambda_I$:

$$\widehat{\varphi}_{\mathcal{I}\to\mathcal{J}} \left|\left(m_j\right)_{j\in I}, \left(f_j\right)_{j\in I}\right\rangle =$$

$$= \sum_{\substack{l_{i,j}\in\mathbb{N}^{I\times I} \\ \forall j\in I,\, m_j=\sum_i l_{i,j}}} \prod_{i\in I}\left(\sqrt{\left(\sum_{j\in I} l_{i,j}\right)!}\right) \prod_{j\in I}\left(\sqrt{m_j!}\right) \prod_{i,j\in I} \frac{\left\langle e_i,\, f_j\right\rangle^{l_{i,j}}}{l_{i,j}!}\, \widehat{\varphi}_{\mathcal{I}\to\mathcal{J}} \left|\left(\sum_{j\in I} l_{i,j}\right)_{i\in I}, \left(e_i\right)_{i\in I}\right\rangle .$$

Now, for $i, j \in J \times (I \setminus J)$ or $(I \setminus J) \times J$, $\left\langle e_i,\, f_j\right\rangle = 0$ since $\left\langle \mathcal{J}, \mathcal{J}^\perp\right\rangle = 0$. Therefore, the only non-vanishing terms in the sum above are such that $l_{i,j} = \mathbb{1}_{(i,j)\in(I\setminus J)^2}\, k_{i,j} + \mathbb{1}_{(i,j)\in J^2}\, p_{i,j}$ with $k_{i,j} \in \mathbb{N}^{(I\setminus J)\times(I\setminus J)}$ and $p_{i,j} \in \mathbb{N}^{J\times J}$. Hence, using eq. (16.14.1):

$$\widehat{\varphi}_{\mathcal{I}\to\mathcal{J}} \left|\left(m_j\right)_{j\in I}, \left(f_j\right)_{j\in I}\right\rangle =$$

$$= \sum_{\substack{k_{i,j}\in\mathbb{N}^{(I\setminus J)\times(I\setminus J)} \\ \forall j\in(I\setminus J),\, m_j=\sum_i k_{i,j}}} \sum_{\substack{p_{i,j}\in\mathbb{N}^{J\times J} \\ \forall j\in J,\, m_j=\sum_i p_{i,j}}} \prod_{i\in I\setminus J}\left(\sqrt{\left(\sum_{j\in I\setminus J} k_{i,j}\right)!}\right) \prod_{i\in J}\left(\sqrt{\left(\sum_{j\in J} p_{i,j}\right)!}\right) \prod_{j\in I\setminus J}\left(\sqrt{m_j!}\right) \times$$

$$\times \prod_{j\in J}\left(\sqrt{m_j!}\right) \prod_{i,j\in I\setminus J} \frac{\left\langle e_i,\, f_j\right\rangle^{k_{i,j}}}{k_{i,j}!} \prod_{i,j\in J} \frac{\left\langle e_i,\, f_j\right\rangle^{p_{i,j}}}{p_{i,j}!} \left|\left(\sum_{j\in I\setminus J} k_{i,j}\right)_{i\in I\setminus J}, \left(e_i\right)_{i\in I\setminus J}\right\rangle \otimes \left|\left(\sum_{j\in J} p_{i,j}\right)_{i\in J}, \left(e_i\right)_{i\in J}\right\rangle$$

$$= \left|\left(m_j\right)_{j\in I\setminus J}, \left(f_j\right)_{j\in I\setminus J}\right\rangle \otimes \left|\left(m_j\right)_{j\in J}, \left(f_j\right)_{j\in J}\right\rangle,$$

where we used again eq. (16.11.1), both in $\widehat{\mathcal{J}^\perp}$ and in $\widehat{\mathcal{J}}$. □

Proposition 16.15 We consider the objects introduced in props. 16.3 and 16.14. We define:

1. $\forall \mathcal{I} \in \mathcal{L}$, $\widehat{\mathcal{M}}^{\text{KIN}}_{\mathcal{I}} := \widehat{\mathcal{I}} \otimes \mathcal{T}$ where $\mathcal{T} := L_2(\mathbb{R}, d\mu)$ (μ being the Lebesgue measure on $\mathbb{R}$);
2. $\forall \mathcal{I} \subset \mathcal{I}' \in \mathcal{L}$, $\widehat{\mathcal{M}}^{\text{KIN}}_{\mathcal{I}'\to\mathcal{I}} := \widehat{\mathcal{I}^\perp \cap \mathcal{I}'}$ (with the convention that $\widehat{\mathcal{M}}^{\text{KIN}}_{\mathcal{I}'\to\mathcal{I}} = \mathbb{C}$ if $\mathcal{I}' = \mathcal{I}$);
3. $\forall \mathcal{I} \subset \mathcal{I}' \in \mathcal{L}$, $\widehat{\varphi}^{\text{KIN}}_{\mathcal{I}'\to\mathcal{I}} := \widehat{\varphi}_{\mathcal{I}'\to\mathcal{I}} \otimes \text{id}_{\mathcal{T}} : \widehat{\mathcal{I}'} \otimes \mathcal{T} \to \widehat{\mathcal{I}^\perp \cap \mathcal{I}'} \otimes \left(\widehat{\mathcal{I}} \otimes \mathcal{T}\right)$;
4. $\forall \mathcal{I} \subset \mathcal{I}' \subset \mathcal{I}'' \in \mathcal{L}$, $\widehat{\varphi}^{\text{KIN}}_{\mathcal{I}''\to\mathcal{I}'\to\mathcal{I}} := \widehat{\varphi}_{\left(\mathcal{I}^\perp\cap\mathcal{I}''\right)\to\left(\mathcal{I}^\perp\cap\mathcal{I}'\right)} : \widehat{\mathcal{I}^\perp \cap \mathcal{I}''} \to \widehat{\mathcal{I}'^\perp \cap \mathcal{I}''} \otimes \widehat{\mathcal{I}^\perp \cap \mathcal{I}'}$ (note that $\left(\mathcal{I}^\perp \cap \mathcal{I}'\right)^\perp \cap \left(\mathcal{I}^\perp \cap \mathcal{I}''\right) = \mathcal{I}'^\perp \cap \mathcal{I}''$ since $\mathcal{I} \subset \mathcal{I}'$).

$\left(\mathcal{L}, \widehat{\mathcal{M}}^{\text{KIN}}, \widehat{\varphi}^{\text{KIN}}\right)^\otimes$ is a projective system of quantum state spaces (def. 5.1).

Proof $\mathcal{L}$ is directed since for $\mathcal{I}, \mathcal{I}' \in \mathcal{L}$, $\mathcal{I} + \mathcal{I}' \in \mathcal{L}$. And, for $\mathcal{I} \subset \mathcal{I}' \subset \mathcal{I}'' \in \mathcal{L}$, $\widehat{\varphi}^{\text{KIN}}_{\mathcal{I}'\to\mathcal{I}}$ and $\widehat{\varphi}^{\text{KIN}}_{\mathcal{I}''\to\mathcal{I}'\to\mathcal{I}}$ are Hilbert space isomorphisms.

Let $\mathcal{I} \subset \mathcal{I}' \subset \mathcal{I}'' \in \mathcal{L}$. We choose an orthonormal basis $(e_i)_{i\in I}$ of $\mathcal{I}$, an orthonormal basis $(e_i)_{i\in I'\setminus I}$ of $\mathcal{I}'\cap\mathcal{I}^\perp$ (with $I' \supset I$) and an orthonormal basis $(e_i)_{i\in I''\setminus I'}$ of $\mathcal{I}''\cap\mathcal{I}'^\perp$ (with $I'' \supset I'$). Since eq. (16.14.1) is valid for any choice of orthonormal bases, we have for $(n_i)_{i\in I''} \in \Lambda_{I''}$:

$$\left(\text{id}_{\widehat{\mathcal{I}''\cap\mathcal{I}'^\perp}} \otimes \widehat{\varphi}_{\mathcal{I}'\to\mathcal{I}}\right) \circ \widehat{\varphi}_{\mathcal{I}''\to\mathcal{I}'} \left|(n_i)_{i\in I''}, (e_i)_{i\in I''}\right\rangle =$$

$$= \left|(n_i)_{i\in I''\setminus I'}\,,\,(e_i)_{i\in I''\setminus I'}\right\rangle \otimes \left|(n_i)_{i\in I'\setminus I}\,,\,(e_i)_{i\in I'\setminus I}\right\rangle \otimes \left|(n_i)_{i\in I}\,,\,(e_i)_{i\in I}\right\rangle$$

$$= \left(\widehat{\varphi}_{(\mathcal{J}''\cap\mathcal{J}^\perp)\to(\mathcal{J}'\cap\mathcal{J}^\perp)} \otimes \mathrm{id}_{\widehat{\mathcal{J}}}\right) \circ \widehat{\varphi}_{\mathcal{J}''\to\mathcal{J}} \left|(n_i)_{i\in I''}\,,\,(e_i)_{i\in I''}\right\rangle . \tag{16.15.1}$$

Hence, eq. (5.1.1) is fulfilled:

$$\left(\mathrm{id}_{\widehat{\mathcal{M}}^{\text{KIN}}_{\mathcal{J}''\to\mathcal{J}'}} \otimes \widehat{\varphi}^{\text{KIN}}_{\mathcal{J}'\to\mathcal{J}}\right) \circ \left(\widehat{\varphi}^{\text{KIN}}_{\mathcal{J}''\to\mathcal{J}'}\right) = \left(\widehat{\varphi}^{\text{KIN}}_{\mathcal{J}''\to\mathcal{J}'\to\mathcal{J}} \otimes \mathrm{id}_{\widehat{\mathcal{M}}^{\text{KIN}}_{\mathcal{J}}}\right) \circ \widehat{\varphi}^{\text{KIN}}_{\mathcal{J}''\to\mathcal{J}} .$$

□

Proposition 16.16 We consider the objects introduced in props. 16.7 and 16.14. We define:

1. $\forall \mathcal{J} \in \mathcal{L}_H$, $\widehat{\mathcal{M}}^{\text{DYN}}_{\mathcal{J}} := \widehat{\mathcal{J}}$;
2. $\forall \mathcal{J} \subset \mathcal{J}' \in \mathcal{L}_H$, $\widehat{\mathcal{M}}^{\text{DYN}}_{\mathcal{J}'\to\mathcal{J}} := \widehat{\mathcal{J}^\perp \cap \mathcal{J}'}$ & $\widehat{\varphi}^{\text{DYN}}_{\mathcal{J}'\to\mathcal{J}} := \widehat{\varphi}_{\mathcal{J}'\to\mathcal{J}}$;
3. $\forall \mathcal{J} \subset \mathcal{J}' \subset \mathcal{J}'' \in \mathcal{L}_H$, $\widehat{\varphi}^{\text{DYN}}_{\mathcal{J}''\to\mathcal{J}'\to\mathcal{J}} := \widehat{\varphi}_{(\mathcal{J}^\perp\cap\mathcal{J}'')\to(\mathcal{J}^\perp\cap\mathcal{J}')}$.

$\left(\mathcal{L}_H, \widehat{\mathcal{M}}^{\text{DYN}}, \widehat{\varphi}^{\text{DYN}}\right)^\otimes$ is a projective system of quantum state spaces.

Let A be a bounded self-adjoint operator on $\mathcal{H}$ and suppose that $(\mathrm{Ker}A)^\perp \in \mathcal{L}_H$. For $\mathcal{J} \in \mathcal{L}_H$ such that $(\mathrm{Ker}A)^\perp \subset \mathcal{J}$, we define $\widehat{A}_{\mathcal{J}} := \left(\widehat{A|_{\mathcal{J}\to\mathcal{J}}}\right)$ (def. 16.12). For $\mathcal{J}, \mathcal{J}' \in \mathcal{L}_H$ such that $(\mathrm{Ker}A)^\perp \subset \mathcal{J}, \mathcal{J}'$, we have:

$\widehat{A}_{\mathcal{J}} \sim \widehat{A}_{\mathcal{J}'}$ (with the equivalence relation $\sim$ defined in eq. (5.3.2)),

hence, we can define $\widehat{A}_{\mathcal{L}_H} := \left[\widehat{A}_{\mathcal{J}}\right]_\sim \in \mathcal{O}^\otimes_{(\mathcal{L}_H, \widehat{\mathcal{M}}^{\text{DYN}}, \widehat{\varphi}^{\text{DYN}})}$ (prop. 5.5).

Proof We know from prop. 16.7 that $\mathcal{L}_H$ is a directed set. Then, we can show that $\left(\mathcal{L}_H, \widehat{\mathcal{M}}^{\text{DYN}}, \widehat{\varphi}^{\text{DYN}}\right)^\otimes$ is a projective system of quantum state spaces exactly like in the proof of prop. 16.15.

Let $\mathcal{J} \subset \mathcal{J}'' \in \mathcal{L}_H$, $(e_i)_{i\in J}$ be an orthonormal basis of $\mathcal{J}$ and $(e_i)_{i\in J''\setminus J}$ (with $J'' \supset J$) an orthonormal basis of $\mathcal{J}'' \cap \mathcal{J}^\perp$. For $k, l \in J$ and $(n_i)_{i\in J''} \in \Lambda_{J''}$, we have:

$$\widehat{\varphi}^{-1}_{\mathcal{J}''\to\mathcal{J}} \left(\mathrm{id}_{\widehat{\mathcal{J}''\cap\mathcal{J}^\perp}} \otimes \widehat{a}^{\mathcal{J},+}_{e_k}\, \widehat{a}^{\mathcal{J}}_{e_l}\right) \widehat{\varphi}_{\mathcal{J}''\to\mathcal{J}} \left|(n_i)_{i\in J''}\,,\,(e_i)_{i\in J''}\right\rangle =$$

$$= \sqrt{n_l}\sqrt{n_k + 1 - \delta_{lk}}\; \left|(n_i - \delta_{li} + \delta_{ki})_{i\in J''}\,,\,(e_i)_{i\in J''}\right\rangle$$

$$= \widehat{a}^{\mathcal{J}'',+}_{e_k}\, \widehat{a}^{\mathcal{J}''}_{e_l} \left|(n_i)_{i\in J''}\,,\,(e_i)_{i\in J''}\right\rangle .$$

Let A be a bounded, self-adjoint operator on $\mathcal{H}$. We have $A\langle\mathcal{H}\rangle \subset (\mathrm{Ker}A)^\perp$. Now, if $(\mathrm{Ker}A)^\perp \subset \mathcal{J} \subset \mathcal{J}''$, both $\mathcal{J}$ and $\mathcal{J}''$ are in particular stabilised by A. Moreover, we have $\mathcal{J}'' \cap \mathcal{J}^\perp \subset \mathcal{J}^\perp \subset \mathrm{Ker}A$, therefore:

$$\left(\widehat{A|_{\mathcal{J}''\to\mathcal{J}''}}\right) = \sum_{k,l\in J''} \langle e_k\,,\,A\,e_l\rangle\, \widehat{a}^{\mathcal{J}'',+}_{e_k}\, \widehat{a}^{\mathcal{J}''}_{e_l}$$

$$= \sum_{k,l\in J} \langle e_k\,,\,A\,e_l\rangle\, \widehat{a}^{\mathcal{J}'',+}_{e_k}\, \widehat{a}^{\mathcal{J}''}_{e_l}$$

$$= \widehat{\varphi}^{-1}_{\mathcal{J}''\to\mathcal{J}} \left[\mathrm{id}_{\widehat{\mathcal{J}''\cap\mathcal{J}^\perp}} \otimes \left(\sum_{k,l\in J} \langle e_k\,,\, A\, e_l\rangle\, \widehat{a}^{\mathcal{J},+}_{e_k}\, \widehat{a}^{\mathcal{J}}_{e_l} \right) \right] \widehat{\varphi}_{\mathcal{J}''\to\mathcal{J}}$$

$$= \widehat{\varphi}^{-1}_{\mathcal{J}''\to\mathcal{J}} \left[\mathrm{id}_{\widehat{\mathcal{J}''\cap\mathcal{J}^\perp}} \otimes \left(\widehat{A|_{\mathcal{J}\to\mathcal{J}}} \right) \right] \widehat{\varphi}_{\mathcal{J}''\to\mathcal{J}} \tag{16.16.1}$$

(using the previous result, the third equality can be checked on finite linear combinations of the basis elements $|(n_i)_{i\in J''}\,,\,(e_i)_{i\in J''}\rangle$, $(n_i)_{i\in J''}\in\Lambda_{J''}$, which span a domain of essential self-adjointness for $\left(\widehat{A|_{\mathcal{J}''\to\mathcal{J}''}}\right)$, so this equality holds for the unique self-adjoint extensions of both sides, see also prop. 5.5). Hence, $\widehat{A}_{\mathcal{J}''} = \widehat{\varphi}^{\text{DYN},-1}_{\mathcal{J}''\to\mathcal{J}} \left[\mathrm{id}_{\widehat{\mathcal{M}}^{\text{DYN}}_{\mathcal{J}''\to\mathcal{J}}} \otimes \widehat{A}_{\mathcal{J}} \right] \widehat{\varphi}^{\text{DYN}}_{\mathcal{J}''\to\mathcal{J}}$.

Finally, if $\mathcal{J},\,\mathcal{J}'\in\mathcal{L}_H$ are such that $(\mathrm{Ker}A)^\perp\subset\mathcal{J},\mathcal{J}'$, we can find $\mathcal{J}''\in\mathcal{L}_H$ such that $\mathcal{J},\mathcal{J}'\subset\mathcal{J}''$ (because $\mathcal{L}_H$ is directed), so $\widehat{A}_{\mathcal{J}}\sim\widehat{A}_{\mathcal{J}'}$. □

Using the general result derived in theorem 5.9, we are able to embed the space of density matrices on the Fock space into the larger quantum state space constructed by projective techniques, and to precisely characterize the image of this embedding, by giving a condition for a projective state to be representable as a density matrix on $\widehat{\mathcal{H}}$.

Proposition 16.17 We consider the same objects as in prop. 16.16. There exists an injective map $\widehat{\sigma}_\downarrow : \overline{\mathcal{S}}_{\text{Fock}} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}_H,\widehat{\mathcal{M}}^{\text{DYN}},\widehat{\varphi}^{\text{DYN}})}$ (where $\overline{\mathcal{S}}_{\text{Fock}}$ is the space of (self-adjoint) positive semi-definite, traceclass operators over $\widehat{\mathcal{M}}^{\text{DYN}}_{\text{Fock}}$ and $\overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}_H,\widehat{\mathcal{M}}^{\text{DYN}},\widehat{\varphi}^{\text{DYN}})}$ was defined in def. 5.1) satisfying, for any bounded self-adjoint operator A on $\mathcal{H}$ with $(\mathrm{Ker}A)^\perp\in\mathcal{L}_H$, and any $\rho\in\overline{\mathcal{S}}_{\text{Fock}}$:

$$\mathrm{Tr}_{\widehat{\mathcal{M}}^{\text{DYN}}_{\text{Fock}}} \left[\rho\ \ \mathbb{I}_W\left(\widehat{\langle A\rangle}_{\text{Fock}}\right) \right] = \mathrm{Tr}\left[\widehat{\sigma}_\downarrow(\rho)\ \ \mathbb{I}_W\left(\widehat{A}_{\mathcal{L}_H}\right)\right], \tag{16.17.1}$$

where W is a measurable subset in the spectrum of $\widehat{\langle A\rangle}_{\text{Fock}}$, and $\mathbb{I}_W(\,\cdot\,)$ denotes the corresponding spectral projectors.

Moreover, we have:

$$\widehat{\sigma}_\downarrow\langle\mathcal{S}_{\text{Fock}}\rangle = \left\{ (\rho_{\mathcal{J}})_{\mathcal{J}\in\mathcal{L}_H} \,\middle|\, \sup_{\mathcal{J}\in\mathcal{L}_H}\inf_{\mathcal{J}'\supset\mathcal{J}} \mathrm{Tr}_{\widehat{\mathcal{M}}^{\text{DYN}}_{\mathcal{J}'}}\left(\rho_{\mathcal{J}'}\,\widehat{\Pi}_{\mathcal{J}'|\mathcal{J}}\right) = \mathrm{Tr}\,\rho = 1 \right\},$$

where $\mathcal{S}_{\text{Fock}}$ is the space of density matrices over $\widehat{\mathcal{M}}^{\text{DYN}}_{\text{Fock}}$ and:

$$\forall N\in\mathbb{N}, \forall\psi\in\mathcal{J}'^{\otimes N,\text{sym}},\ \widehat{\Pi}_{\mathcal{J}'|\mathcal{J}}\,\psi := (\Pi_{\mathcal{J}})^{\otimes N}\ \psi\in\mathcal{J}'^{\otimes N,\text{sym}},$$

$\Pi_{\mathcal{J}}$ being the orthogonal projection on $\mathcal{J}$.

Proof For $\mathcal{J}\subset\mathcal{J}'\in\mathcal{L}_H$, we define $\zeta_{\mathcal{J}'\to\mathcal{J}}\in\widehat{\mathcal{M}}^{\text{DYN}}_{\mathcal{J}'\to\mathcal{J}}$ as the vacuum state of $\widehat{\mathcal{M}}^{\text{DYN}}_{\mathcal{J}'\to\mathcal{J}} = \widehat{\mathcal{J}'\cap\mathcal{J}^\perp}$ (ie. $\zeta_{\mathcal{J}'\to\mathcal{J}} = |(O)_{i\in I}\,,\,(e_i)_{i\in I}\rangle$ for any basis $(e_i)_{i\in I}$ of $\mathcal{J}'\cap\mathcal{J}^\perp$). The family of vectors $(\zeta_{\mathcal{J}'\to\mathcal{J}})_{\mathcal{J}\subset\mathcal{J}'}$ fulfills the hypotheses of theorem 5.9.

Next, for all $\mathcal{J}\in\mathcal{L}_H$, we define an injection $\tau_{\text{Fock}\leftarrow\mathcal{J}}$ from $\widehat{\mathcal{M}}^{\text{DYN}}_{\mathcal{J}} = \widehat{\mathcal{J}}$ into $\widehat{\mathcal{M}}^{\text{DYN}}_{\text{Fock}} = \widehat{\mathcal{H}}$ by:

$$\tau_{\text{Fock}\leftarrow\mathcal{J}} = \widehat{\varphi}^{-1}_{\mathcal{H}\to\mathcal{J}}\left(\zeta_{\text{Fock}\to\mathcal{J}}\otimes(\,\cdot\,)\right),$$

where $\zeta_{\text{Fock}\to\mathcal{J}}$ is the vacuum state of $\widehat{\mathcal{J}^\perp}$. Using eq. (16.15.1), we can show that $\forall \mathcal{J} \subset \mathcal{J}' \in \mathcal{L}_H$, $\tau_{\text{Fock}\leftarrow\mathcal{J}'} \circ \tau_{\mathcal{J}'\leftarrow\mathcal{J}} = \tau_{\text{Fock}\leftarrow\mathcal{J}}$ (where $\tau_{\mathcal{J}'\leftarrow\mathcal{J}}$ is defined from $\zeta_{\mathcal{J}'\to\mathcal{J}}$ as in theorem 5.9).

Now, we can choose an orthonormal basis $(e_i)_{i\in\mathbb{N}}$ of $\mathcal{H}$ such that $\forall i \in \mathbb{N}$, $\|H\, e_i\| < \infty$ and consider for $N \geqslant 1$, $\mathcal{J}_N := \text{Vect}\left\{e_i \mid i \leqslant N\right\} \in \mathcal{L}_H$. Using eq. (16.14.1) with the orthonormal basis $(e_i)_{i\leqslant N}$ of $\mathcal{J}_N$ and the orthonormal basis $(e_i)_{i>N}$ of $\mathcal{J}_N^\perp$, we get:

$$\tau_{\text{Fock}\leftarrow\mathcal{J}_N}\left\langle \widehat{\mathcal{J}_N}\right\rangle = \overline{\text{Vect}\left\{\left|(n_i)_{i\in\mathbb{N}},\, (e_i)_{i\in\mathbb{N}}\right\rangle \,\middle|\, (n_i)_{i\in\mathbb{N}} \in \Lambda_{\mathbb{N}}^N\right\}},$$

where $\Lambda_{\mathbb{N}}^N := \left\{(n_i)_{i\in\mathbb{N}} \in \Lambda_{\mathbb{N}} \mid \forall i > N,\, n_i = 0\right\}$. Hence, from $\Lambda_{\mathbb{N}} = \bigcup_{N\geqslant 1}\Lambda_{\mathbb{N}}^N$, we have:

$$\widehat{\mathcal{M}}^{\text{DYN}}_{\text{Fock}} = \overline{\bigcup_{\mathcal{J}\in\mathcal{L}_H} \text{Im}\,\tau_{\text{Fock}\leftarrow\mathcal{J}}}\,.$$

Therefore, we can identify $\widehat{\mathcal{M}}^{\text{DYN}}_{\text{Fock}}$ with the inductive limit $\widehat{\mathcal{M}}^{\text{DYN}}_{\zeta}$ introduced in theorem 5.9, so we have an injection $\widehat{\sigma}_\downarrow : \overline{\mathcal{S}}_{\text{Fock}} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}_H, \widehat{\mathcal{H}}^{\text{DYN}}, \widehat{\varphi}^{\text{DYN}})}$, satisfying:

$$\widehat{\sigma}_\downarrow\left\langle \mathcal{S}_{\text{Fock}}\right\rangle = \left\{(\rho_{\mathcal{J}})_{\mathcal{J}\in\mathcal{L}_H} \,\middle|\, \sup_{\mathcal{J}\in\mathcal{L}_H}\inf_{\mathcal{J}'\supset\mathcal{J}} \text{Tr}_{\widehat{\mathcal{M}}^{\text{DYN}}_{\mathcal{J}'}}\left(\rho_{\mathcal{J}'}\,\widehat{\Pi}_{\mathcal{J}'|\mathcal{J}}\right) = 1\right\},$$

where:

$$\forall \mathcal{J} \subset \mathcal{J}' \in \mathcal{L}_H,\, \widehat{\Pi}_{\mathcal{J}'|\mathcal{J}} = \widehat{\varphi}^{-1}_{\mathcal{J}'\to\mathcal{J}} \circ \left(|\zeta_{\mathcal{J}'\to\mathcal{J}}\rangle\langle \zeta_{\mathcal{J}'\to\mathcal{J}}| \otimes \text{id}_{\widehat{\mathcal{J}}}\right) \circ \widehat{\varphi}_{\mathcal{J}'\to\mathcal{J}}\,.$$

Let $\mathcal{J} \subset \mathcal{J}' \in \mathcal{L}_H$ and let $(e_i)_{i\in J}$, resp. $(e_i)_{i\in J'\setminus J}$ be an orthonormal basis of $\mathcal{J}$, resp. $\mathcal{J}' \cap \mathcal{J}^\perp$. For $(n_i)_{i\in} J' \in \Lambda_{J'}$, we have:

$$\widehat{\Pi}_{\mathcal{J}'|\mathcal{J}}\left|(n_i)_{i\in J'},\, (e_i)_{i\in J'}\right\rangle = \begin{cases} \left|(n_i)_{i\in J'},\, (e_i)_{i\in J'}\right\rangle & \text{if } \forall i \in J' \setminus J,\, n_i = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \Pi_{\mathcal{J}}^{\otimes \sum_{i\in J'} n_i}\left|(n_i)_{i\in J'},\, (e_i)_{i\in J'}\right\rangle,$$

therefore $\forall N \in \mathbb{N}$, $\forall \psi \in \mathcal{J}'^{\otimes N,\text{sym}}$, $\widehat{\Pi}_{\mathcal{J}'|\mathcal{J}}\,\psi = (\Pi_{\mathcal{J}})^{\otimes N}\,\psi$.

Lastly, let A be a bounded self-adjoint operator on $\mathcal{H}$ such that $\mathcal{J} := (\text{Ker}A)^\perp \in \mathcal{L}_H$. Eq. (16.17.1) is then an application of prop. 5.5, using the definition of $\widehat{\sigma}_\downarrow$ (given in the proof of theorem 5.9) together with:

$$\widehat{\langle A\rangle}_{\text{Fock}} := \widehat{A} = \widehat{\varphi}^{-1}_{\mathcal{H}\to\mathcal{J}}\left[\text{id}_{\widehat{\mathcal{J}^\perp}} \otimes \widehat{A}_{\mathcal{J}}\right]\widehat{\varphi}_{\mathcal{H}\to\mathcal{J}}\,,$$

which can be shown like in the proof of prop. 16.16 (eq. (16.16.1)). □

We can now implement and solve in the quantum theory the approximated constraints we had on the classical side, and thus define a family of maps (indexed by the regularization parameter ε) from the dynamical projective system of quantum state spaces introduced above into the kinematical one.

Proposition 16.18 We consider the objects introduced in def. 16.4 and prop. 16.15. Let $\varepsilon = (\mathcal{J}, \epsilon) \in \mathcal{E}$ and let $\mathfrak{I} \in \mathcal{L}^\varepsilon$. We define the map:

$$\begin{array}{rcl} \widehat{\delta}^{\varepsilon}_{\mathcal{J}} : \widehat{\mathcal{J}} & \to & \widehat{\mathcal{I}} \otimes \mathcal{T} \\ \psi & \mapsto & \left(\widehat{\varphi}^{-1}_{\mathcal{I}\to\mathcal{J}} \otimes \mathrm{id}_{\mathcal{T}}\right) \left[\zeta_{\mathcal{I}\to\mathcal{J}} \otimes \exp\left(-i \left(\widehat{(\Pi_{\mathcal{J}} H \Pi_{\mathcal{J}})|_{\mathcal{J}\to\mathcal{J}}}\right) \otimes \widehat{T}\right) (\psi \otimes \delta_\epsilon)\right] \end{array},$$

where $\zeta_{\mathcal{I}\to\mathcal{J}}$ is the vacuum state in $\widehat{\mathcal{I} \cap \mathcal{J}^\perp}$, $\Pi_{\mathcal{J}}$ is the orthogonal projection on $\mathcal{J}$, $\widehat{T}$ is the position operator on $\mathcal{T} = L_2(\mathcal{R}, d\mu)$ and $\delta_\epsilon \in \mathcal{T}$ is defined by:

$$\forall t \in \mathbb{R},\ \delta_\epsilon(t) = \frac{\sqrt{\epsilon}}{\pi^{1/4}} \exp\left(-\frac{\epsilon^2\, t^2}{2}\right).$$

Then, we have:

$$\widehat{\delta}^{\varepsilon}_{\mathcal{J}} \left\langle \widehat{\mathcal{J}} \right\rangle = \left\{ \psi \in \mathcal{I} \,\middle|\, \left(\widehat{\Pi}_{\mathcal{I}|\mathcal{J}} \otimes \mathrm{id}_{\mathcal{T}}\right) \psi = \psi \quad \& \quad \widehat{C}^{\varepsilon}\, \psi = \psi \right\},$$

$$\text{with } \widehat{C}^{\varepsilon} = \frac{1}{\epsilon^2} \left(\mathrm{id}_{\widehat{\mathcal{J}}} \otimes \widehat{E} - \left(\widehat{(\Pi_{\mathcal{J}} H \Pi_{\mathcal{J}})|_{\mathcal{J}\to\mathcal{J}}}\right) \otimes \mathrm{id}_{\mathcal{T}}\right)^2 + \epsilon^2\, \mathrm{id}_{\widehat{\mathcal{J}}} \otimes \widehat{T}^2,$$

where $\widehat{\Pi}_{\mathcal{I}|\mathcal{J}}$ is defined as in prop. 16.17 and $\widehat{E}$ is the operator $i\,\partial_t$ on $\mathcal{T}$. Moreover, $\widehat{\delta}^{\varepsilon}_{\mathcal{J}}\Big|_{\widehat{\mathcal{J}} \to \widehat{\delta}^{\varepsilon}_{\mathcal{J}}\left\langle\widehat{\mathcal{J}}\right\rangle}$ is a unitary map.

Proof We define:

$$\begin{array}{rcl} \widehat{\delta}_1 : \widehat{\mathcal{J}} & \to & \widehat{\mathcal{J}} \otimes \mathcal{T} \\ \psi & \mapsto & \psi \otimes \delta_\epsilon \end{array}, \qquad \begin{array}{rcl} \widehat{\delta}_2 : \widehat{\mathcal{J}} \otimes \mathcal{T} & \to & \widehat{\mathcal{J}} \otimes \mathcal{T} \\ \psi & \mapsto & \exp\left(-i\widehat{H}^{\varepsilon}_{\mathcal{J}} \otimes \widehat{T}\right) \psi \end{array} \quad \text{with } H^{\varepsilon} := \Pi_{\mathcal{J}}\, H\, \Pi_{\mathcal{J}},$$

$$\begin{array}{rcl} \widehat{\delta}_3 : \widehat{\mathcal{J}} \otimes \mathcal{T} & \to & \widehat{\mathcal{I} \cap \mathcal{J}^\perp} \otimes \widehat{\mathcal{J}} \otimes \mathcal{T} \\ \psi & \mapsto & \zeta_{\mathcal{I}\to\mathcal{J}} \otimes \psi \end{array}, \qquad \begin{array}{rcl} \widehat{\delta}_4 : \widehat{\mathcal{I} \cap \mathcal{J}^\perp} \otimes \widehat{\mathcal{J}} \otimes \mathcal{T} & \to & \widehat{\mathcal{I}} \otimes \mathcal{T} \\ \psi & \mapsto & \left(\widehat{\varphi}^{-1}_{\mathcal{I}\to\mathcal{J}} \otimes \mathrm{id}_{\mathcal{T}}\right) \psi \end{array}.$$

We have $\widehat{\delta}_1 \left\langle \widehat{\mathcal{J}} \right\rangle = \widehat{\mathcal{J}} \otimes \mathrm{Vect}\{\delta_\epsilon\} = \widehat{\mathcal{J}} \otimes \left\{ \psi \in \mathcal{T} \,\middle|\, \widehat{C}_1\, \psi = \psi \right\}$, where $\widehat{C}_1 := \frac{1}{\epsilon^2}\, \widehat{E}^2 + \epsilon^2\, \widehat{T}^2$, and $\widehat{\delta}_1\Big|_{\widehat{\mathcal{J}} \to \widehat{\delta}_1\left\langle\widehat{\mathcal{J}}\right\rangle}$ is a unitary map.

$\widehat{\delta}_2$ is a unitary map, because $\widehat{H}^{\varepsilon}_{\mathcal{J}}$ and $\widehat{T}$ are essentially self-adjoint ($\forall N \in \mathbb{N}$, $\widehat{H}^{\varepsilon}_{\mathcal{J}}$ stabilize $\mathcal{H}^{\otimes N,\mathrm{sym}}$ and the restriction of $\widehat{H}^{\varepsilon}_{\mathcal{J}}$ to $\mathcal{H}^{\otimes N,\mathrm{sym}}$ is a bounded self-adjoint operator, for so is $H^{\varepsilon}|_{\mathcal{J}\to\mathcal{J}}$, by definition of $\mathcal{L}_H$). And we have:

$$\widehat{\delta}_2 \circ \widehat{\delta}_1 \left\langle \widehat{\mathcal{J}} \right\rangle = \left\{ \psi \in \widehat{\mathcal{J}} \otimes \mathcal{T} \,\middle|\, \widehat{C}_2\, \psi = \psi \right\},$$

with:

$$\begin{aligned} \widehat{C}_2 &:= \widehat{\delta}_2 \circ \left(\mathrm{id}_{\widehat{\mathcal{J}}} \otimes \widehat{C}_1\right) \circ \widehat{\delta}_2^{-1} \\ &= \exp\left(-i\left[\widehat{H}^{\varepsilon}_{\mathcal{J}} \otimes \widehat{T},\, \cdot\,\right]\right) \left(\mathrm{id}_{\widehat{\mathcal{J}}} \otimes \widehat{C}_1\right) \\ &= \frac{1}{\epsilon^2} \left(\mathrm{id}_{\widehat{\mathcal{J}}} \otimes \widehat{E} - \widehat{H}^{\varepsilon}_{\mathcal{J}} \otimes \mathrm{id}_{\mathcal{T}}\right)^2 + \epsilon^2\, \mathrm{id}_{\widehat{\mathcal{J}}} \otimes \widehat{T}^2. \end{aligned}$$

Next, we compute:

$$\widehat{\delta}_3 \circ \widehat{\delta}_2 \circ \widehat{\delta}_1 \left\langle \widehat{\mathcal{J}} \right\rangle = \left\{ \zeta_{\mathcal{I}\to\mathcal{J}} \otimes \psi \in \widehat{\mathcal{I} \cap \mathcal{J}^\perp} \otimes \widehat{\mathcal{J}} \otimes \mathcal{T} \;\middle|\; \widehat{C}_2\, \psi = \psi \right\}$$

$$= \left\{ \psi \in \widehat{\mathcal{I} \cap \mathcal{J}^\perp} \otimes \widehat{\mathcal{J}} \otimes \mathcal{T} \;\middle|\; \mathrm{id}_{\widehat{\mathcal{I}\cap\mathcal{J}^\perp}} \otimes \widehat{C}_2\, \psi = \psi \quad \& \quad \left(|\zeta_{\mathcal{I}\to\mathcal{J}} \,\rangle\langle\, \zeta_{\mathcal{I}\to\mathcal{J}}| \otimes \mathrm{id}_{\widehat{\mathcal{J}}\otimes\mathcal{T}} \right) \psi = \psi \right\},$$

and $\widehat{\delta}_3\Big|_{\widehat{\mathcal{J}}\otimes\mathcal{T}\to\widehat{\delta}_3\left\langle\widehat{\mathcal{J}}\otimes\mathcal{T}\right\rangle}$ is a unitary map.

Finally, $\widehat{\delta}_4$ is unitary (from prop. 16.14) and:

$$\widehat{\delta}^\varepsilon_{\mathcal{I}} \left\langle \widehat{\mathcal{J}} \right\rangle = \widehat{\delta}_4 \circ \widehat{\delta}_3 \circ \widehat{\delta}_2 \circ \widehat{\delta}_1 \left\langle \widehat{\mathcal{J}} \right\rangle = \left\{ \psi \in \widehat{\mathcal{I}} \otimes \mathcal{T} \;\middle|\; \widehat{C}_4\, \psi = \psi \quad \& \quad \widehat{D}_4\, \psi = \psi \right\},$$

with:

$$\widehat{C}_4 := \left(\widehat{\varphi}^{-1}_{\mathcal{I}\to\mathcal{J}} \otimes \mathrm{id}_{\mathcal{T}} \right) \left(\mathrm{id}_{\widehat{\mathcal{I}\cap\mathcal{J}^\perp}} \otimes \widehat{C}_2 \right) \left(\widehat{\varphi}_{\mathcal{I}\to\mathcal{J}} \otimes \mathrm{id}_{\mathcal{T}} \right)$$

$$= \frac{1}{\epsilon^2} \left(\mathrm{id}_{\widehat{\mathcal{I}}} \otimes \widehat{E} - \left[\widehat{\varphi}^{-1}_{\mathcal{I}\to\mathcal{J}} \left(\mathrm{id}_{\widehat{\mathcal{I}\cap\mathcal{J}^\perp}} \otimes \widehat{H}^\varepsilon_{\mathcal{J}} \right) \widehat{\varphi}_{\mathcal{I}\to\mathcal{J}} \right] \otimes \mathrm{id}_{\mathcal{T}} \right)^2 + \epsilon^2\, \mathrm{id}_{\widehat{\mathcal{I}}} \otimes \widehat{T}^2$$

$$= \frac{1}{\epsilon^2} \left(\mathrm{id}_{\widehat{\mathcal{I}}} \otimes \widehat{E} - \widehat{H}^\varepsilon_{\mathcal{I}} \otimes \mathrm{id}_{\mathcal{T}} \right)^2 + \epsilon^2\, \mathrm{id}_{\widehat{\mathcal{I}}} \otimes \widehat{T}^2 \text{ (using eq. (16.16.1)),}$$

and:

$$\widehat{D}_4 := \left(\widehat{\varphi}^{-1}_{\mathcal{I}\to\mathcal{J}} \otimes \mathrm{id}_{\mathcal{T}} \right) \left(|\zeta_{\mathcal{I}\to\mathcal{J}} \,\rangle\langle\, \zeta_{\mathcal{I}\to\mathcal{J}}| \otimes \mathrm{id}_{\widehat{\mathcal{J}}\otimes\mathcal{T}} \right) \left(\widehat{\varphi}_{\mathcal{I}\to\mathcal{J}} \otimes \mathrm{id}_{\mathcal{T}} \right)$$

$$= \widehat{\sqcap}_{\mathcal{I}|\mathcal{J}} \otimes \mathrm{id}_{\mathcal{T}} \text{ (as was shown in the proof of prop. 16.17).}$$

□

Proposition 16.19 We consider the same objects as in prop. 16.18. For $\varepsilon = (\mathcal{J}, \epsilon) \in \mathcal{E}$ and $\rho_{\mathcal{J}}$ a (self-adjoint) positive semi-definite, traceclass operator on $\widehat{\mathcal{J}}$, we define:

$$\forall \mathcal{I} \in \mathcal{L}^\varepsilon,\ \widehat{\Delta}^\varepsilon_{\mathcal{I}}\left(\rho_{\mathcal{J}}\right) := \widehat{\delta}^\varepsilon_{\mathcal{I}}\, \rho_{\mathcal{J}} \left(\widehat{\delta}^\varepsilon_{\mathcal{I}} \right)^+.$$

Then, $\left(\widehat{\Delta}^\varepsilon_{\mathcal{I}}\left(\rho_{\mathcal{J}}\right) \right)_{\mathcal{I}\in\mathcal{L}^\varepsilon} \in \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^\varepsilon, \widehat{\mathcal{M}}^{\text{KIN}}, \widehat{\varphi}^{\text{KIN}})}$.

Hence, for $\rho = (\rho_{\mathcal{J}})_{\mathcal{J}\in\mathcal{L}_H} \in \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}_H, \widehat{\mathcal{M}}^{\text{DYN}}, \widehat{\varphi}^{\text{DYN}})}$ (prop. 16.16), we can define:

$$\widehat{\Delta}^\varepsilon\left(\rho\right) = \widehat{\sigma}^{-1} \left(\left(\widehat{\Delta}^\varepsilon_{\mathcal{I}}\left(\rho_{\mathcal{J}}\right) \right)_{\mathcal{I}\in\mathcal{L}^\varepsilon} \right),$$

where the map $\widehat{\sigma} : \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}, \widehat{\mathcal{M}}^{\text{KIN}}, \widehat{\varphi}^{\text{KIN}})} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^\varepsilon, \widehat{\mathcal{M}}^{\text{KIN}}, \widehat{\varphi}^{\text{KIN}})}$ is defined as in prop. 5.6 (and is bijective, since $\mathcal{L}^\varepsilon$ is cofinal in $\mathcal{L}$).

Proof We need to prove that $\forall \mathcal{I}, \mathcal{I}' \in \mathcal{L}^\varepsilon$, with $\mathcal{I} \subset \mathcal{I}'$, $\mathrm{Tr}_{\mathcal{I}'\to\mathcal{I}} \widehat{\Delta}^\varepsilon_{\mathcal{I}'}\left(\rho_{\mathcal{J}}\right) = \widehat{\Delta}^\varepsilon_{\mathcal{I}}\left(\rho_{\mathcal{J}}\right)$. We have:

$$\forall \psi \in \widehat{\mathcal{J}},\ \widehat{\varphi}^{\text{KIN}}_{\mathcal{I}'\to\mathcal{I}} \circ \widehat{\delta}^\varepsilon_{\mathcal{I}'}(\psi) = \left(\left[\widehat{\varphi}_{\mathcal{I}'\to\mathcal{I}} \circ \widehat{\varphi}^{-1}_{\mathcal{I}'\to\mathcal{J}} \right] \otimes \mathrm{id}_{\mathcal{T}} \right) \left[\zeta_{\mathcal{I}'\to\mathcal{J}} \otimes e^{-i\,\widehat{H}^\varepsilon_{\mathcal{J}}\otimes\widehat{T}} \left(\psi \otimes \delta_\epsilon \right) \right]$$

$$= \left(\left[\left(\mathrm{id}_{\widehat{\mathcal{I}'\cap\mathcal{I}^\perp}} \otimes \widehat{\varphi}^{-1}_{\mathcal{I}\to\mathcal{J}} \right) \circ \left(\widehat{\varphi}_{\mathcal{I}'\to\mathcal{I}\to\mathcal{J}} \otimes \mathrm{id}_{\widehat{\mathcal{J}}} \right) \right] \otimes \mathrm{id}_{\mathcal{T}} \right) \left[\zeta_{\mathcal{I}'\to\mathcal{J}} \otimes e^{-i\,\widehat{H}^\varepsilon_{\mathcal{J}}\otimes\widehat{T}} \left(\psi \otimes \delta_\epsilon \right) \right]$$

$$= \left(\left(\mathrm{id}_{\widehat{\mathcal{J}' \cap \mathcal{J}^\perp}} \otimes \widehat{\varphi}^{-1}_{\mathcal{J} \to \mathcal{J}} \right) \otimes \mathrm{id}_{\mathcal{T}} \right) \left[\zeta_{\mathcal{J}' \to \mathcal{J}} \otimes \zeta_{\mathcal{J} \to \mathcal{J}} \otimes e^{-i \widehat{H}^\varepsilon_{\mathcal{J}} \otimes \widehat{T}} (\psi \otimes \delta_\epsilon) \right]$$

$$= \zeta_{\mathcal{J}' \to \mathcal{J}} \otimes \widehat{\delta}^\varepsilon_{\mathcal{J}} (\psi) \,,$$

hence $\widehat{\varphi}^{\text{KIN}}_{\mathcal{J}' \to \mathcal{J}} \circ \widehat{\Delta}^\varepsilon_{\mathcal{J}'} (\rho_{\mathcal{J}}) \circ \widehat{\varphi}^{\text{KIN},-1}_{\mathcal{J}' \to \mathcal{J}} = |\zeta_{\mathcal{J}' \to \mathcal{J}} \rangle\langle \zeta_{\mathcal{J}' \to \mathcal{J}}| \otimes \widehat{\Delta}^\varepsilon_{\mathcal{J}} (\rho_{\mathcal{J}})$, therefore:

$$\mathrm{Tr}_{\mathcal{J}' \to \mathcal{J}} \widehat{\Delta}^\varepsilon_{\mathcal{J}'} (\rho_{\mathcal{J}}) = \mathrm{Tr}_{\widehat{\mathcal{J}' \cap \mathcal{J}^\perp}} |\zeta_{\mathcal{J}' \to \mathcal{J}} \rangle\langle \zeta_{\mathcal{J}' \to \mathcal{J}}| \otimes \widehat{\Delta}^\varepsilon_{\mathcal{J}} (\rho_{\mathcal{J}}) = \widehat{\Delta}^\varepsilon_{\mathcal{J}} (\rho_{\mathcal{J}}) \,.$$

□

As a preparation for the study of convergence, we define a subset $\widehat{\mathcal{R}}$ of the space of states over the quantum projective structure. The motivation is to implement a quantum version of the regularity condition that was ensuring convergence on the classical side: at the classical level we have proved the convergence for normalized states, so in analogy we consider here states with a bounded expectation value for the total number of particles (which indeed corresponds to the quantization of the classical observable $\psi \mapsto \langle \psi, \psi \rangle$).

Note that, as we show in the following result, the regular states (the elements of $\widehat{\mathcal{R}}$) can be seen as states in the Fock space via the embedding of prop. 16.17. This is not really surprising, since we know that the Fock space quantization is appropriate for a basic non-interacting field theory like the Schrödinger equation.

Proposition 16.20 We consider the same objects as in prop. 16.17 and we define:

$$\widehat{\mathcal{R}} := \left\{ \rho \in \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}_H, \widehat{\mathcal{M}}^{\text{DYN}}, \widehat{\varphi}^{\text{DYN}})} \,\middle|\, \sup_{\mathcal{J} \in \mathcal{L}_H} \mathrm{Tr} \left(\rho \, (\widehat{\Pi_{\mathcal{J}}})_{\mathcal{L}_H} \right) < \infty \right\},$$

where $\mathrm{Tr} \left(\rho \, (\widehat{\Pi_{\mathcal{J}}})_{\mathcal{L}_H} \right) := \sum_{n \in \mathbb{N}} n \, \mathrm{Tr} \left(\rho \ \ \mathbb{I}_{\{n\}} \left((\widehat{\Pi_{\mathcal{J}}})_{\mathcal{L}_H} \right) \right)$ and $\mathbb{I}_{\{n\}} \left((\widehat{\Pi_{\mathcal{J}}})_{\mathcal{L}_H} \right)$ denotes the spectral projector as in prop. 5.5.

Then, $\widehat{\mathcal{R}} \subset \widehat{\sigma}_\downarrow \left\langle \overline{\mathcal{S}}_{\text{Fock}} \right\rangle$.

Proof Let $\rho \in \widehat{\mathcal{R}}$ and $N = \sup_{\mathcal{J} \in \mathcal{L}_H} \mathrm{Tr} \left(\rho \, (\widehat{\Pi_{\mathcal{J}}})_{\mathcal{L}_H} \right)$. If $\rho = 0$, then $\rho = \widehat{\sigma}_\downarrow(0)$. Otherwise, $\mathrm{Tr}\, \rho = r > 0$, hence $\rho = r \, \left(\frac{1}{r} \rho \right)$ with $\frac{1}{r} \rho \in \mathcal{S}^{\otimes}_{(\mathcal{L}_H, \widehat{\mathcal{M}}^{\text{DYN}}, \widehat{\varphi}^{\text{DYN}})}$. Let $\mathcal{J}, \mathcal{J}' \in \mathcal{L}_H$ with $\mathcal{J} \subset \mathcal{J}'$. Let $(e_i)_{i \in J}$ be an orthonormal basis of $\mathcal{J}$ and $(e_i)_{i \in J' \setminus J}$ ($J' \supset J$) be an orthonormal basis of $\mathcal{J}' \cap \mathcal{J}^\perp$. For $(n_i)_{i \in J'}$, $(m_i)_{i \in J'} \in \Lambda_{J'}$, we have:

$$\left\langle (n_i)_{i \in J'}, (e_i)_{i \in J'} \,\middle|\, \left(\widehat{\Pi_{\mathcal{J}' \cap \mathcal{J}^\perp}|_{\mathcal{J}' \to \mathcal{J}'}} \right) \,\middle|\, (m_i)_{i \in J'}, (e_i)_{i \in J'} \right\rangle = \begin{cases} \sum_{i \in J' \setminus J} n_i & \text{if } \forall i \in J', \, n_i = m_i \\ 0 & \text{else} \end{cases},$$

and:

$$\left\langle (n_i)_{i \in J'}, (e_i)_{i \in J'} \,\middle|\, \left(\mathrm{id}_{\widehat{\mathcal{J}'}} - \widehat{\Pi}_{\mathcal{J}'|\mathcal{J}} \right) \,\middle|\, (m_i)_{i \in J'}, (e_i)_{i \in J'} \right\rangle = \begin{cases} 1 & \text{if } \forall i \in J', \, n_i = m_i \quad \& \quad \sum_{i \in J' \setminus J} n_i \geqslant 1 \\ 0 & \text{else} \end{cases},$$

therefore:

$$\mathrm{Tr}_{\widehat{\mathcal{J}'}}\left(\rho_{\mathcal{J}'}\,\widehat{\Pi}_{\mathcal{J}'|\mathcal{J}}\right) \geqslant r - \sum_{n\in\mathbb{N}} n\,\mathrm{Tr}_{\widehat{\mathcal{J}'}}\left[\rho_{\mathcal{J}'}\;\; \mathbb{I}_{\{n\}}\left(\left(\widehat{\Pi_{\mathcal{J}'\cap\mathcal{J}^\perp}|_{\mathcal{J}'\to\mathcal{J}'}}\right)\right)\right] =: r - \mathrm{Tr}_{\widehat{\mathcal{J}'}}\left[\rho_{\mathcal{J}'}\left(\widehat{\Pi_{\mathcal{J}'\cap\mathcal{J}^\perp}|_{\mathcal{J}'\to\mathcal{J}'}}\right)\right].$$

Now, $\mathrm{Tr}_{\widehat{\mathcal{J}'}}\left[\rho_{\mathcal{J}'}\left(\widehat{\Pi_{\mathcal{J}'\cap\mathcal{J}^\perp}|_{\mathcal{J}'\to\mathcal{J}'}}\right)\right] = \sum_{n\in\mathbb{N}} n\,\mathrm{Tr}_{\widehat{\mathcal{J}'}}\left[\rho_{\mathcal{J}'}\;\; \mathbb{I}_{\{n\}}\left((\widehat{\Pi_{\mathcal{J}'}})_{\mathcal{J}'}\right)\right] - \sum_{n\in\mathbb{N}} n\,\mathrm{Tr}_{\widehat{\mathcal{J}'}}\left[\rho_{\mathcal{J}'}\;\; \mathbb{I}_{\{n\}}\left((\widehat{\Pi_{\mathcal{J}}})_{\mathcal{J}'}\right)\right] =$ $\mathrm{Tr}\left(\rho\,(\widehat{\Pi_{\mathcal{J}'}})_{\mathcal{L}_H}\right) - \mathrm{Tr}\left(\rho\,(\widehat{\Pi_{\mathcal{J}}})_{\mathcal{L}_H}\right)$, hence:

$$\inf_{\mathcal{J}'\supset\mathcal{J}} \mathrm{Tr}_{\widehat{\mathcal{J}'}}\left(\rho_{\mathcal{J}'}\,\widehat{\Pi}_{\mathcal{J}'|\mathcal{J}}\right) \geqslant r - \sup_{\mathcal{J}'\supset\mathcal{J}} \mathrm{Tr}\left(\rho\,(\widehat{\Pi_{\mathcal{J}'}})_{\mathcal{L}_H}\right) + \mathrm{Tr}\left(\rho\,(\widehat{\Pi_{\mathcal{J}}})_{\mathcal{L}_H}\right).$$

Finally, $\left(\mathrm{Tr}\left(\rho\,(\widehat{\Pi_{\mathcal{J}'}})_{\mathcal{L}_H}\right)\right)_{\mathcal{J}'\in\mathcal{L}_H}$ is increasing, so $\sup_{\mathcal{J}'\supset\mathcal{J}} \mathrm{Tr}\left(\rho\,(\widehat{\Pi_{\mathcal{J}'}})_{\mathcal{L}_H}\right) = N$ and:

$$\sup_{\mathcal{J}\in\mathcal{L}_H}\inf_{\mathcal{J}'\supset\mathcal{J}} \mathrm{Tr}_{\widehat{\mathcal{J}'}}\left(\rho_{\mathcal{J}'}\,\widehat{\Pi}_{\mathcal{J}'|\mathcal{J}}\right) \geqslant r - N + N = r\,.$$

On the other hand, $\forall \mathcal{J} \subset \mathcal{J}'$, $\mathrm{Tr}_{\widehat{\mathcal{J}'}}\left(\rho_{\mathcal{J}'}\,\widehat{\Pi}_{\mathcal{J}'|\mathcal{J}}\right) \leqslant r$, thus, using prop. 16.17, $\frac{1}{r}\rho \in \widehat{\sigma}_\downarrow\langle \mathcal{S}_{\mathrm{Fock}}\rangle$, and therefore $\rho \in \widehat{\sigma}_\downarrow\left\langle \overline{\mathcal{S}}_{\mathrm{Fock}}\right\rangle$. □

Finally, we prove a convergence result at the quantum level. We define here two different notions of convergence, one stronger than the other, in both cases requiring convergence of the expectation values for a certain class of observables. To assess how exactly the convergence should be adjusted would require a closer study of which observables are really measured in practice, for these constitute the class of kinematical observables that we want to be able to transport on the dynamical side.

In addition, we need to introduce an ϵ-dependent normalization parameter $\mathcal{N}$ that accounts for the fact that states solving the exact dynamics cannot be correctly normalized (they describe probability distributions invariant under a transformation running along the full time line from $t = -\infty$ to $t = +\infty$) so that it only makes sense to consider conditional probabilities, expressing the probability of measuring the system in a certain state, *knowing* that the measurement takes place at a certain time. So, as we lift the ϵ-regularization (that was making the gauge orbits compact and the solution of the quantum constraint normalizable), the probability of measuring the system in a certain time interval is dropping and needs to be accordingly compensated.

Theorem 16.21 We consider the same objects as in props. 16.19 and 16.20. Let $\mathcal{I} \in \mathcal{L}$, A be a bounded operator on $\widehat{\mathcal{I}}$ and $\varphi, \varphi' \in \mathcal{T}$. We additionally assume that φ, φ' have compact support. On $\widehat{\mathcal{M}}^{\mathrm{KIN}}_{\mathcal{I}}$, we define the operator:

$$R^{\mathcal{I}}_{A,\varphi,\varphi'} := A \otimes |\varphi\,\rangle\langle\,\varphi'|\,,$$

and, for $\varepsilon = (\mathcal{J}, \epsilon) \in \mathcal{E}$ and $\rho \in \widehat{\sigma}_\downarrow\left\langle \overline{\mathcal{S}}_{\mathrm{Fock}}\right\rangle$, we define:

$$R^{\mathcal{I},\varepsilon}_{A,\varphi,\varphi'}(\rho) := \frac{1}{\mathcal{N}(\epsilon,\varphi,\varphi')}\mathrm{Tr}_{\widehat{\mathcal{M}}^{\mathrm{KIN}}_{\mathcal{I}}}\left[\widehat{\Delta}^{\varepsilon}_{\mathcal{I}}(\rho)\,R^{\mathcal{I}}_{A,\varphi,\varphi'}\right],$$

where $\mathcal{N}(\epsilon,\varphi,\varphi') = \mathrm{Tr}_{\mathcal{T}}\,|\varphi\,\rangle\langle\,\varphi'|\;|\delta_\epsilon\,\rangle\langle\,\delta_\epsilon| = \langle\varphi',\delta_\epsilon\rangle\,\langle\delta_\epsilon,\varphi\rangle$.

Then, the net $\left(R^{\mathcal{I},\varepsilon}_{A,\varphi,\varphi'}(\rho)\right)_{\varepsilon\in\mathcal{E}}$ converges.

For $e, f \in \mathcal{I}$, we also define on $\widehat{\mathcal{M}}^{\text{KIN}}_{\mathcal{J}}$ the operator:

$$R^{\mathcal{I}}_{e,f,\varphi,\varphi'} := \widehat{a}^{\mathcal{I},+}_{e}\,\widehat{a}^{\mathcal{I}}_{f} \otimes |\varphi \rangle\langle \varphi'|,$$

and, for $\varepsilon = (\mathcal{J}, \epsilon) \in \mathcal{E}$ and $\rho \in \widehat{\mathcal{R}}$, we define:

$$R^{\mathcal{I},\varepsilon}_{e,f,\varphi,\varphi'}(\rho) := \frac{1}{\mathcal{N}(\epsilon,\,\varphi,\,\varphi')}\text{Tr}_{\widehat{\mathcal{M}}^{\text{KIN}}_{\mathcal{J}}}\left[\widehat{\Delta}^{\varepsilon}_{\mathcal{J}}(\rho)\,R^{\mathcal{I}}_{e,f,\varphi,\varphi'}\right].$$

Then, the net $\left(R^{\mathcal{I},\varepsilon}_{e,f,\varphi,\varphi'}(\rho)\right)_{\varepsilon\in\mathcal{E}}$ converges.

Proof *Bounded operator & Fock state.* Let $\rho \in \widehat{\sigma}_{\downarrow}\left\langle \overline{\mathcal{S}}_{\text{Fock}}\right\rangle$. For $\varepsilon = (\mathcal{J}, \epsilon) \in \mathcal{E}$ and $\mathcal{J}' \in \mathcal{L}^{\varepsilon}$, we have:

$$\begin{aligned}
\widehat{\Delta}^{\varepsilon}_{\mathcal{J}'}(\rho) &= \widehat{\varphi}^{\text{KIN},-1}_{\mathcal{J}'\to\mathcal{J}}\,\left(|\zeta_{\mathcal{J}'\to\mathcal{J}}\rangle\langle \zeta_{\mathcal{J}'\to\mathcal{J}}| \otimes \left[e^{-i\,\widehat{H}^{\varepsilon}_{\mathcal{J}}\otimes\widehat{T}}\,\left(\rho_{\mathcal{J}} \otimes |\delta_{\epsilon}\rangle\langle \delta_{\epsilon}|\right)\,e^{i\,\widehat{H}^{\varepsilon}_{\mathcal{J}}\otimes\widehat{T}}\right]\right)\,\widehat{\varphi}^{\text{KIN}}_{\mathcal{J}'\to\mathcal{J}}\\
&= \widehat{\varphi}^{\text{KIN},-1}_{\mathcal{J}'\to\mathcal{J}}\,e^{-i\,\text{id}_{\widehat{\mathcal{J}'\cap\mathcal{J}^{\perp}}}\otimes\widehat{H}^{\varepsilon}_{\mathcal{J}}\otimes\widehat{T}}\,\left(|\zeta_{\mathcal{J}'\to\mathcal{J}}\rangle\langle \zeta_{\mathcal{J}'\to\mathcal{J}}| \otimes \rho_{\mathcal{J}} \otimes |\delta_{\epsilon}\rangle\langle \delta_{\epsilon}|\right)\,e^{i\,\text{id}_{\widehat{\mathcal{J}'\cap\mathcal{J}^{\perp}}}\otimes\widehat{H}^{\varepsilon}_{\mathcal{J}}\otimes\widehat{T}}\,\widehat{\varphi}^{\text{KIN}}_{\mathcal{J}'\to\mathcal{J}}\\
&= e^{-i\,\widehat{H}^{\varepsilon}_{\mathcal{J}'}\otimes\widehat{T}}\,\widehat{\varphi}^{\text{KIN},-1}_{\mathcal{J}'\to\mathcal{J}}\,\left(|\zeta_{\mathcal{J}'\to\mathcal{J}}\rangle\langle \zeta_{\mathcal{J}'\to\mathcal{J}}| \otimes \rho_{\mathcal{J}} \otimes |\delta_{\epsilon}\rangle\langle \delta_{\epsilon}|\right)\,\widehat{\varphi}^{\text{KIN}}_{\mathcal{J}'\to\mathcal{J}}\,e^{i\,\widehat{H}^{\varepsilon}_{\mathcal{J}'}\otimes\widehat{T}}\ \text{(like in eq. (16.16.1))}\\
&= e^{-i\,\widehat{H}^{\varepsilon}_{\mathcal{J}'}\otimes\widehat{T}}\,\left(\left[\tau_{\mathcal{J}'\leftarrow\mathcal{J}}\,\rho_{\mathcal{J}}\,\tau^{+}_{\mathcal{J}'\leftarrow\mathcal{J}}\right] \otimes |\delta_{\epsilon}\rangle\langle \delta_{\epsilon}|\right)\,e^{i\,\widehat{H}^{\varepsilon}_{\mathcal{J}'}\otimes\widehat{T}},
\end{aligned}$$

where $\tau_{\mathcal{J}'\leftarrow\mathcal{J}} = \widehat{\varphi}^{-1}_{\mathcal{J}'\to\mathcal{J}}\,\left(\zeta_{\mathcal{J}'\to\mathcal{J}} \otimes (\,\cdot\,)\right)$. Hence, for $\mathcal{I} \subset \mathcal{J}'$:

$$\begin{aligned}
R^{\mathcal{I},\varepsilon}_{A,\varphi,\varphi'}(\rho) &= \frac{1}{\mathcal{N}(\epsilon,\,\varphi,\,\varphi')}\text{Tr}_{\widehat{\mathcal{M}}^{\text{KIN}}_{\mathcal{J}}}\left[\,\widehat{\Delta}^{\varepsilon}_{\mathcal{J}}(\rho)\,A \otimes |\varphi\rangle\langle \varphi'|\,\right] =\\
&= \frac{\text{Tr}_{\widehat{\mathcal{M}}^{\text{KIN}}_{\mathcal{J}'}}\left[\widehat{\Delta}^{\varepsilon}_{\mathcal{J}'}(\rho)\,\left(\widehat{\varphi}^{-1}_{\mathcal{J}'\to\mathcal{I}}\,\left(\text{id}_{\widehat{\mathcal{J}'\cap\mathcal{I}^{\perp}}} \otimes A\right)\,\widehat{\varphi}_{\mathcal{J}'\to\mathcal{I}}\right) \otimes |\varphi\rangle\langle \varphi'|\right]}{\mathcal{N}(\epsilon,\,\varphi,\,\varphi')}\\
&= \frac{\text{Tr}_{\widehat{\mathcal{M}}^{\text{KIN}}_{\mathcal{J}'}}\left[e^{-i\,\widehat{H}^{\varepsilon}_{\mathcal{J}'}\otimes\widehat{T}}\,\left(\left[\tau_{\mathcal{J}'\leftarrow\mathcal{J}}\,\rho_{\mathcal{J}}\,\tau^{+}_{\mathcal{J}'\leftarrow\mathcal{J}}\right] \otimes |\delta_{\epsilon}\rangle\langle \delta_{\epsilon}|\right)\,e^{i\,\widehat{H}^{\varepsilon}_{\mathcal{J}'}\otimes\widehat{T}}\,\left(\widehat{\varphi}^{-1}_{\mathcal{J}'\to\mathcal{I}}\,\left(\text{id}_{\widehat{\mathcal{J}'\cap\mathcal{I}^{\perp}}} \otimes A\right)\,\widehat{\varphi}_{\mathcal{J}'\to\mathcal{I}}\right) \otimes |\varphi\rangle\langle \varphi'|\right]}{\mathcal{N}(\epsilon,\,\varphi,\,\varphi')}\\
&= \frac{\displaystyle\int_{-T}^{T} dt\,dt'\;\;\varphi(t)\varphi'^{*}(t')\,\delta_{\epsilon}(t')\delta^{*}_{\epsilon}(t)\;\;Z^{\varepsilon}_{\mathcal{J}'}(t,t')}{\displaystyle\int_{-T}^{T} dt\,dt'\;\;\varphi(t)\varphi'^{*}(t')\,\delta_{\epsilon}(t')\delta^{*}_{\epsilon}(t)},
\end{aligned}$$

where $T > 0$ is such that the support of φ and φ' is included in $[-T,\,T]$, and Z^{ε} is defined as:

$$\begin{aligned}
Z^{\varepsilon}(t,t') &= \text{Tr}_{\widehat{\mathcal{J}'}}\left[e^{-it'\,\widehat{H}^{\varepsilon}_{\mathcal{J}'}}\,\left(\tau_{\mathcal{J}'\leftarrow\mathcal{J}}\,\rho_{\mathcal{J}}\,\tau^{+}_{\mathcal{J}'\leftarrow\mathcal{J}}\right)\,e^{it\,\widehat{H}^{\varepsilon}_{\mathcal{J}'}}\,\left(\widehat{\varphi}^{-1}_{\mathcal{J}'\to\mathcal{I}}\,\left(\text{id}_{\widehat{\mathcal{J}'\cap\mathcal{I}^{\perp}}} \otimes A\right)\,\widehat{\varphi}_{\mathcal{J}'\to\mathcal{I}}\right)\right]\\
&= \text{Tr}_{\widehat{\mathcal{J}}}\;\rho_{\mathcal{J}}\left[\tau^{+}_{\mathcal{J}'\leftarrow\mathcal{J}}\,e^{it\,\widehat{H}^{\varepsilon}_{\mathcal{J}'}}\,\left(\widehat{\varphi}^{-1}_{\mathcal{J}'\to\mathcal{I}}\,\left(\text{id}_{\widehat{\mathcal{J}'\cap\mathcal{I}^{\perp}}} \otimes A\right)\,\widehat{\varphi}_{\mathcal{J}'\to\mathcal{I}}\right)\,e^{-it'\,\widehat{H}^{\varepsilon}_{\mathcal{J}'}}\,\tau_{\mathcal{J}'\leftarrow\mathcal{J}}\right]\\
&= \text{Tr}_{\widehat{\mathcal{J}}}\;\rho_{\mathcal{J}}\left[e^{it\,\widehat{H}^{\varepsilon}_{\mathcal{J}}}\,\tau^{+}_{\mathcal{J}'\leftarrow\mathcal{J}}\,\left(\widehat{\varphi}^{-1}_{\mathcal{J}'\to\mathcal{I}}\,\left(\text{id}_{\widehat{\mathcal{J}'\cap\mathcal{I}^{\perp}}} \otimes A\right)\,\widehat{\varphi}_{\mathcal{J}'\to\mathcal{I}}\right)\,\tau_{\mathcal{J}'\leftarrow\mathcal{J}}\,e^{-it'\,\widehat{H}^{\varepsilon}_{\mathcal{J}}}\right],
\end{aligned}$$

for any $\mathcal{I}' \in \mathcal{L}$ such that $\mathcal{I}, \mathcal{J} \subset \mathcal{I}'$.

Next, $\frac{\sqrt{\pi}}{\epsilon}\, \delta_\epsilon(t')\delta_\epsilon^*(t)$ converges uniformly to 1 for $t, t' \in [-T, T]$, when $\epsilon \to 0$. Therefore, we need to show that the net $\left(Z^\varepsilon(t,t')\right)_{\varepsilon\in\mathcal{E}}$ converges uniformly for $t, t' \in [-T, T]$.

Let $\rho_{\text{Fock}} \in \overline{\mathcal{S}}_{\text{Fock}}$ such that $\rho = \widehat{\sigma}_{\downarrow}\left(\rho_{\text{Fock}}\right)$. Using the definition of $\widehat{\sigma}_{\downarrow}$, we can show:

$$Z^\varepsilon(t,t') = \text{Tr}_{\widehat{\mathcal{H}}}\ \rho_{\text{Fock}}\, \widehat{\varphi}^{-1}_{\mathcal{H}\to\mathcal{J}} \left[\mathbb{1}_{\widehat{\mathcal{J}^\perp}} \otimes e^{it\,\widehat{H}^\varepsilon_{\mathcal{J}}}\, \tau^+_{\mathcal{I}'\leftarrow\mathcal{J}} \left(\widehat{\varphi}^{-1}_{\mathcal{I}'\to\mathcal{I}} \left(\text{id}_{\widehat{\mathcal{I}'\cap\mathcal{I}^\perp}} \otimes A\right) \widehat{\varphi}_{\mathcal{I}'\to\mathcal{I}}\right)\, \tau_{\mathcal{I}'\leftarrow\mathcal{J}}\, e^{-it'\,\widehat{H}^\varepsilon_{\mathcal{J}}}\right] \widehat{\varphi}_{\mathcal{H}\to\mathcal{J}}$$

$$= \text{Tr}_{\widehat{\mathcal{H}}}\ \rho_{\text{Fock}}\, e^{it\,\widehat{H}^\varepsilon}\, \widehat{\varphi}^{-1}_{\mathcal{H}\to\mathcal{J}} \left[\mathbb{1}_{\widehat{\mathcal{J}^\perp}} \otimes \tau^+_{\mathcal{I}'\leftarrow\mathcal{J}} \left(\widehat{\varphi}^{-1}_{\mathcal{I}'\to\mathcal{I}} \left(\text{id}_{\widehat{\mathcal{I}'\cap\mathcal{I}^\perp}} \otimes A\right) \widehat{\varphi}_{\mathcal{I}'\to\mathcal{I}}\right)\, \tau_{\mathcal{I}'\leftarrow\mathcal{J}}\right] \widehat{\varphi}_{\mathcal{H}\to\mathcal{J}}\, e^{-it'\,\widehat{H}^\varepsilon}$$

(like in eq. (16.16.1))

$$= \text{Tr}_{\widehat{\mathcal{J}}}\ \left[\text{Tr}_{\mathcal{H}\to\mathcal{J}}\, e^{-it'\,\widehat{H}^\varepsilon}\, \rho_{\text{Fock}}\, e^{it\,\widehat{H}^\varepsilon}\right] \left[\tau^+_{\mathcal{I}'\leftarrow\mathcal{J}} \left(\widehat{\varphi}^{-1}_{\mathcal{I}'\to\mathcal{I}} \left(\text{id}_{\widehat{\mathcal{I}'\cap\mathcal{I}^\perp}} \otimes A\right) \widehat{\varphi}_{\mathcal{I}'\to\mathcal{I}}\right)\, \tau_{\mathcal{I}'\leftarrow\mathcal{J}}\right]$$

(by definition of $\text{Tr}_{\mathcal{H}\to\mathcal{J}}$)

And using twice eq. (5.1.1) (for $\mathcal{J}, \mathcal{I} \subset \mathcal{I}' \subset \mathcal{H}$), we have, for any $\psi \in \mathcal{J}$:

$$\widehat{\varphi}_{\mathcal{H}\to\mathcal{I}} \circ \tau_{\text{Fock}\leftarrow\mathcal{J}}(\psi) = \widehat{\varphi}_{\mathcal{H}\to\mathcal{I}} \circ \widehat{\varphi}^{-1}_{\mathcal{H}\to\mathcal{J}}\left(\zeta_{\text{Fock}\to\mathcal{J}} \otimes \psi\right)$$

$$= \widehat{\varphi}_{\mathcal{H}\to\mathcal{I}} \circ \widehat{\varphi}^{-1}_{\mathcal{H}\to\mathcal{J}} \circ \left(\widehat{\varphi}^{-1}_{\mathcal{J}^\perp\to\mathcal{I}'\cap\mathcal{J}^\perp} \otimes \mathbb{1}_{\mathcal{J}}\right)\left(\zeta_{\text{Fock}\to\mathcal{I}'} \otimes \zeta_{\mathcal{I}'\to\mathcal{J}} \otimes \psi\right)$$

$$= \widehat{\varphi}_{\mathcal{H}\to\mathcal{I}} \circ \widehat{\varphi}^{-1}_{\mathcal{H}\to\mathcal{I}'}\left(\zeta_{\text{Fock}\to\mathcal{I}'} \otimes \tau_{\mathcal{I}'\leftarrow\mathcal{J}}(\psi)\right)$$

$$= \left(\widehat{\varphi}^{-1}_{\mathcal{I}^\perp\to\mathcal{I}'\cap\mathcal{I}^\perp} \otimes \text{id}_{\mathcal{I}}\right)\left(\zeta_{\text{Fock}\to\mathcal{I}'} \otimes \widehat{\varphi}_{\mathcal{I}'\to\mathcal{I}} \circ \tau_{\mathcal{I}'\leftarrow\mathcal{J}}(\psi)\right).$$

Hence, we get:

$$Z^\varepsilon(t,t') = \text{Tr}_{\widehat{\mathcal{J}}}\ \left[\text{Tr}_{\mathcal{H}\to\mathcal{J}}\, e^{-it'\,\widehat{H}^\varepsilon}\, \rho_{\text{Fock}}\, e^{it\,\widehat{H}^\varepsilon}\right] \left[\tau^+_{\text{Fock}\leftarrow\mathcal{J}} \left(\widehat{\varphi}^{-1}_{\mathcal{H}\to\mathcal{I}} \left(\text{id}_{\widehat{\mathcal{I}^\perp}} \otimes A\right) \widehat{\varphi}_{\mathcal{H}\to\mathcal{I}}\right)\, \tau_{\text{Fock}\leftarrow\mathcal{J}}\right]$$

$$= \text{Tr}_{\widehat{\mathcal{H}}}\ \left[\tau_{\text{Fock}\leftarrow\mathcal{J}} \left(\text{Tr}_{\mathcal{H}\to\mathcal{J}}\, e^{-it'\,\widehat{H}^\varepsilon}\, \rho_{\text{Fock}}\, e^{it\,\widehat{H}^\varepsilon}\right)\, \tau^+_{\text{Fock}\leftarrow\mathcal{J}}\right] \left[\widehat{\varphi}^{-1}_{\mathcal{H}\to\mathcal{I}} \left(\text{id}_{\widehat{\mathcal{I}^\perp}} \otimes A\right) \widehat{\varphi}_{\mathcal{H}\to\mathcal{I}}\right]$$

But we have:

$$\left\|\widehat{\varphi}^{-1}_{\mathcal{H}\to\mathcal{I}} \left(\text{id}_{\widehat{\mathcal{I}^\perp}} \otimes A\right) \widehat{\varphi}_{\mathcal{H}\to\mathcal{I}}\right\| = \|A\| < \infty,$$

therefore, what remains to be shown is that the net:

$$\left(\tau_{\text{Fock}\leftarrow\mathcal{J}} \left(\text{Tr}_{\mathcal{H}\to\mathcal{J}}\, e^{-it'\,\widehat{H}^\varepsilon}\, \rho_{\text{Fock}}\, e^{it\,\widehat{H}^\varepsilon}\right)\, \tau^+_{\text{Fock}\leftarrow\mathcal{J}}\right)_{\varepsilon=(\mathcal{J},\epsilon)\in\mathcal{E}}$$

converges in trace norm (which was defined in lemma 5.10), uniformly for $t, t' \in [-T, T]$.

Let $\epsilon_o > 0$. We have:

$$\widehat{\mathcal{H}} = \overline{\bigoplus_{\mathcal{J}\in\mathcal{L}_H, N\geqslant 1} \widehat{\mathcal{J}}^N} \text{ where } \widehat{\mathcal{J}}^N = \bigoplus_{n\leqslant N} \widehat{\mathcal{J}}^{\otimes n,\,\text{sym}},$$

because $\mathcal{H} = \overline{\bigoplus_{\mathcal{J}\in\mathcal{L}_H} \mathcal{J}}$. Hence, we can prove, using the spectral decomposition of the self-adjoint traceclass operator ρ_{Fock} and the directed preorder on $\mathcal{L}_H$ and $\mathbb{N}$, that there exist $\mathcal{J}_o \in \mathcal{L}_H$ and $N_o \geqslant 1$ such that:

$$\|\rho_{\text{Fock}} - \rho^o_{\text{Fock}}\|_1 \leqslant \frac{\epsilon_o}{6},$$

where $\rho^o_{\text{Fock}} := \widehat{\Pi}_{\mathcal{J}_o,N_o}\,\rho_{\text{Fock}}\,\widehat{\Pi}_{\mathcal{J}_o,N_o}$ (with $\widehat{\Pi}_{\mathcal{J}_o,N_o}$ the orthogonal projection on $\widehat{\mathcal{J}}_o^{N_o}$) and $\|\cdot\|_1$ denotes the trace norm.

Since $\widehat{\mathcal{J}}_o^{N_o}$ is finite dimensional, there exist vectors $\psi_\alpha \in \widehat{\mathcal{J}}_o^{N_o}$, $\alpha \in \{1,\dots,K\}$ (with $K \in \mathbb{N}$) such that:

$$\rho^o_{\text{Fock}} = \sum_{\alpha=1}^{K} |\psi_\alpha \,\rangle\langle\, \psi_\alpha|.$$

We define:

$$\epsilon_1 := \frac{1}{N_o}\,\frac{1}{1+|T|}\log\left(1 + \frac{\epsilon_o}{12K\,\left(1+\max_\alpha \|\psi_\alpha\|^2\right)}\right) > 0,$$

and $H^{\epsilon_1} := \epsilon_1 \left\lfloor \frac{1}{\epsilon_1} H \right\rfloor$ (as in the proof of theorem 16.10). Then, since H^{ϵ_1} has discrete spectrum and K, $N_o < \infty$, we can construct $\mathcal{J}_1 \in \mathcal{L}_H$, such that $\mathcal{J}_1$ is stabilized by H^{ϵ_1} and:

$$\forall \alpha \leqslant K,\ \left\|\psi_\alpha - \widehat{\Pi}_{\mathcal{J}_1,N_o}\,\psi_\alpha\right\| \leqslant \frac{\epsilon_o}{12K\,(1+\max_\alpha \|\psi_\alpha\|)}.$$

Thus, we get:

$$\left\|\rho_{\text{Fock}} - \widehat{\Pi}_{\mathcal{J}_1,N_o}\,\rho^o_{\text{Fock}}\,\widehat{\Pi}_{\mathcal{J}_1,N_o}\right\|_1 \leqslant \frac{\epsilon_o}{6} + \sum_{\alpha=1}^{K}\left\||\psi_\alpha \,\rangle\langle\, \psi_\alpha| - \left|\widehat{\Pi}_{\mathcal{J}_1,N_o}\,\psi_\alpha \,\right\rangle\left\langle\, \widehat{\Pi}_{\mathcal{J}_1,N_o}\,\psi_\alpha\right|\right\|_1$$

$$\leqslant \frac{\epsilon_o}{6} + \sum_{\alpha=1}^{K}\left\|\psi_\alpha - \widehat{\Pi}_{\mathcal{J}_1,N_o}\,\psi_\alpha\right\|\,\|\psi_\alpha\| + \left\|\widehat{\Pi}_{\mathcal{J}_1,N_o}\,\psi_\alpha\right\|\,\left\|\psi_\alpha - \widehat{\Pi}_{\mathcal{J}_1,N_o}\,\psi_\alpha\right\| \leqslant \frac{\epsilon_o}{3}$$

Now, we consider $\mathcal{J}_2 \in \mathcal{L}_H$ such that $\mathcal{J}_o + \mathcal{J}_1 \subset \mathcal{J}_2$. For all $t, t' \in [-T,\,T]$, we have:

$$\left\|e^{-it'\left(\widehat{\Pi_{\mathcal{J}_2} H \Pi_{\mathcal{J}_2}}\right)}\,\rho_{\text{Fock}}\,e^{it\left(\widehat{\Pi_{\mathcal{J}_2} H \Pi_{\mathcal{J}_2}}\right)} - e^{-it'\widehat{H}^{\epsilon_1}}\,\widehat{\Pi}_{\mathcal{J}_1,N_o}\,\rho^o_{\text{Fock}}\,\widehat{\Pi}_{\mathcal{J}_1,N_o}\,e^{it\widehat{H}^{\epsilon_1}}\right\|_1 \leqslant$$

$$\leqslant \frac{\epsilon_o}{3} + \left\|e^{-it'\left(\widehat{\Pi_{\mathcal{J}_2} H \Pi_{\mathcal{J}_2}}\right)} - e^{-it'\left(\widehat{\Pi_{\mathcal{J}_2} H^{\epsilon_1} \Pi_{\mathcal{J}_2}}\right)}\right\|\,\|\rho^o_{\text{Fock}}\|_1 + \|\rho^o_{\text{Fock}}\|_1\,\left\|e^{it\left(\widehat{\Pi_{\mathcal{J}_2} H \Pi_{\mathcal{J}_2}}\right)} - e^{it\left(\widehat{\Pi_{\mathcal{J}_2} H^{\epsilon_1} \Pi_{\mathcal{J}_2}}\right)}\right\|$$

(since H^{ϵ_1} stabilizes $\mathcal{J}_1 \subset \mathcal{J}_2$)

$$\leqslant \frac{\epsilon_o}{3} + 2\left|e^{T\,N_o\,\epsilon_1} - 1\right|\,\|\rho^o_{\text{Fock}}\|_1 \leqslant \frac{\epsilon_o}{2},$$

and similarly:

$$\left\|e^{-it'\widehat{H}}\,\rho_{\text{Fock}}\,e^{it\widehat{H}} - e^{-it'\widehat{H}^{\epsilon_1}}\,\widehat{\Pi}_{\mathcal{J}_1,N_o}\,\rho^o_{\text{Fock}}\,\widehat{\Pi}_{\mathcal{J}_1,N_o}\,e^{it\widehat{H}^{\epsilon_1}}\right\|_1 \leqslant \frac{\epsilon_o}{2}.$$

Next, using again that H^{ϵ_1} stabilizes $\mathcal{J}_1 \subset \mathcal{J}_2$, we also have:

$$\tau_{\text{Fock}\leftarrow\mathcal{J}_2}\left[\text{Tr}_{\mathcal{H}\to\mathcal{J}_2}\ e^{-it'\widehat{H}^{\epsilon_1}}\,\widehat{\Pi}_{\mathcal{J}_1,N_o}\,\rho^o_{\text{Fock}}\,\widehat{\Pi}_{\mathcal{J}_1,N_o}\,e^{it\widehat{H}^{\epsilon_1}}\right]\tau^+_{\text{Fock}\leftarrow\mathcal{J}_2} = e^{-it'\widehat{H}^{\epsilon_1}}\,\widehat{\Pi}_{\mathcal{J}_1,N_o}\,\rho^o_{\text{Fock}}\,\widehat{\Pi}_{\mathcal{J}_1,N_o}\,e^{it\widehat{H}^{\epsilon_1}}.$$

Hence, for any $\varepsilon = (\mathcal{J}_2, \epsilon_2) \in \mathcal{E}$ such that $(\mathcal{J}_o + \mathcal{J}_1, 1) \preccurlyeq \varepsilon$, and any $t, t' \in [-T, T]$, we have:

$$\left\| \tau_{\text{Fock}\leftarrow\mathcal{J}_2} \left[\text{Tr}_{\mathcal{H}\to\mathcal{J}_2} e^{-it'\widehat{H}^\varepsilon} \rho_{\text{Fock}} e^{it\widehat{H}^\varepsilon} \right] \tau^+_{\text{Fock}\leftarrow\mathcal{J}_2} - e^{-it'\widehat{H}} \rho_{\text{Fock}} e^{it\widehat{H}} \right\|_1 \leqslant \epsilon_o ,$$

which provides the desired convergence.

Transition operator & regular state. Let $\rho \in \widehat{\mathcal{R}}$, $\mathcal{I} \in \mathcal{L}$ and $e, f \in \mathcal{I}$. Since $\widehat{\mathcal{R}} \subset \widehat{\sigma}_\downarrow \left\langle \overline{\mathcal{S}}_{\text{Fock}} \right\rangle$ (prop. 16.20), there exists $\rho_{\text{Fock}} \in \overline{\mathcal{S}}_{\text{Fock}}$ such that $\rho = \widehat{\sigma}_\downarrow(\rho_{\text{Fock}})$. Like above, a sufficient condition for the convergence of the net $\left(R^{\mathcal{I},\varepsilon}_{e, f, \varphi, \varphi'}(\rho) \right)_{\varepsilon\in\mathcal{E}}$ is uniform convergence for $t, t' \in [-T, T]$ of the net:

$$\left(\text{Tr}_{\widehat{\mathcal{H}}} \left[\tau_{\text{Fock}\leftarrow\mathcal{J}} \left(\text{Tr}_{\mathcal{H}\to\mathcal{J}} e^{-it'\widehat{H}^\varepsilon} \rho_{\text{Fock}} e^{it\widehat{H}^\varepsilon} \right) \tau^+_{\text{Fock}\leftarrow\mathcal{J}} \right] T_{e,f} \right)_{\varepsilon=(\mathcal{J},\epsilon)\in\mathcal{E}} ,$$

where we define:

$$T_{e,f} := \widehat{\varphi}^{-1}_{\mathcal{H}\to\mathcal{I}} \left(\text{id}_{\widehat{\mathcal{I}^\perp}} \otimes \widehat{\text{a}}^{\mathcal{I},+}_e \widehat{\text{a}}^{\mathcal{I}}_f \right) \widehat{\varphi}_{\mathcal{H}\to\mathcal{I}} .$$

Choosing an orthonormal basis $(e_i)_{i\in I}$ of $\mathcal{I}$ and completing it into an orthonormal basis $(e_i)_{i\in\mathbb{N}}$ of $\mathcal{I}'$ ($I \subset \mathbb{N}$), we get:

$$\begin{aligned} T_{e,f} &= \sum_{i,j\in I} \langle e_i, e\rangle \left\langle f, e_j\right\rangle \widehat{\varphi}^{-1}_{\mathcal{H}\to\mathcal{I}} \left(\text{id}_{\widehat{\mathcal{I}^\perp}} \otimes \widehat{\text{a}}^{\mathcal{I},+}_{e_i} \widehat{\text{a}}^{\mathcal{I}}_{e_j} \right) \widehat{\varphi}_{\mathcal{H}\to\mathcal{I}} \\ &= \sum_{i,j\in I} \langle e_i, e\rangle \left\langle f, e_j\right\rangle \left(\widehat{\text{a}}^{\text{Fock},+}_{e_i} \widehat{\text{a}}^{\text{Fock}}_{e_j} \right) \text{ (like in the proof of prop. 16.16)} \\ &= \widehat{\text{a}}^{\text{Fock},+}_e \widehat{\text{a}}^{\text{Fock}}_f . \end{aligned}$$

Now, from the definition of the creation and annihilation operators, we have:

$$T_{e,f} = \sum_{n=0}^{\infty} \widehat{\Pi}^{(n)} \widehat{\text{a}}^{\text{Fock},+}_e \widehat{\text{a}}^{\text{Fock}}_f \widehat{\Pi}^{(n)} ,$$

where, for all $n \in \mathbb{N}$, $\widehat{\Pi}^{(n)}$ is the orthogonal projection on the subspace $\mathcal{H}^{\otimes n,\,\text{sym}}$ of $\widehat{\mathcal{H}}$, and:

$$\left\| \widehat{\Pi}^{(n)} \widehat{\text{a}}^{\text{Fock},+}_e \widehat{\text{a}}^{\text{Fock}}_f \widehat{\Pi}^{(n)} \right\| \leqslant n \, \|e\| \, \|f\| .$$

On the other hand, we have, by definition of $\widehat{\mathcal{R}}$, $\sup_{\mathcal{J}\in\mathcal{L}_H} \text{Tr} \left(\rho \, \widehat{(\Pi_{\mathcal{J}})}_{\mathcal{L}_H} \right) =: N_{\text{tot}} < \infty$, so:

$$\begin{aligned} \sum_{n=0}^{\infty} n \, \text{Tr}_{\widehat{\mathcal{H}}} \, \rho_{\text{Fock}} \, \widehat{\Pi}^{(n)} &= \sum_{n\in\mathbb{N}} n \, \text{Tr}_{\widehat{\mathcal{H}}} \, \rho_{\text{Fock}} \, \mathbb{I}_{\{n\}} \left(\widehat{\langle \text{id}_{\mathcal{H}} \rangle}_{\text{Fock}} \right) \\ &= \sum_{n\in\mathbb{N}} n \sup_{\mathcal{J}\in\mathcal{L}_H} \text{Tr}_{\widehat{\mathcal{H}}} \, \rho_{\text{Fock}} \, \mathbb{I}_{\{n\}} \left(\widehat{\langle \Pi_{\mathcal{J}} \rangle}_{\text{Fock}} \right) \text{ (using lemma 5.10)} \\ &= \sum_{n\in\mathbb{N}} n \sup_{\mathcal{J}\in\mathcal{L}_H} \text{Tr}_{\widehat{\mathcal{J}}} \, \rho_{\mathcal{J}} \, \mathbb{I}_{\{n\}} \left(\widehat{(\Pi_{\mathcal{J}})}_{\mathcal{J}} \right) = N_{\text{tot}} \text{ (from eq. (16.17.1)).} \end{aligned}$$

Let $\epsilon_o > 0$. Then, there exists $N_o \geqslant 1$ such that:

$$\sum_{n>N_o} n\, \mathrm{Tr}_{\widehat{\mathcal{H}}}\, \rho_{\mathrm{Fock}}\, \widehat{\Pi}^{(n)} \leqslant \frac{\epsilon_o}{3}\,,$$

and therefore, for all $\varepsilon = (\mathcal{J},\, \epsilon) \in \mathcal{E}$:

$$\sum_{n>N_o} n \left\| \left[\tau_{\mathrm{Fock}\leftarrow\mathcal{J}} \left(\mathrm{Tr}_{\mathcal{H}\to\mathcal{J}}\, e^{-it'\,\widehat{H}^\varepsilon}\, \rho_{\mathrm{Fock}}\, e^{it\,\widehat{H}^\varepsilon} \right) \tau^+_{\mathrm{Fock}\leftarrow\mathcal{J}} \right] \widehat{\Pi}^{(n)} \right\|_1 =$$

$$= \sum_{n>N_o} n \left\| \tau_{\mathrm{Fock}\leftarrow\mathcal{J}} \left(\mathrm{Tr}_{\mathcal{H}\to\mathcal{J}}\, e^{-it'\,\widehat{H}^\varepsilon}\, \rho_{\mathrm{Fock}}\, e^{it\,\widehat{H}^\varepsilon} \right) \widehat{\Pi}^{(n)}_{\mathcal{J}}\, \tau^+_{\mathrm{Fock}\leftarrow\mathcal{J}} \right\|_1$$

(where for all $n \in \mathbb{N}$, $\widehat{\Pi}^{(n)}_{\mathcal{J}}$ is the orthogonal projection on the subspace $\mathcal{J}^{\otimes n,\,\mathrm{sym}}$ of $\widehat{\mathcal{J}}$)

$$= \sum_{n>N_o} n \left\| \left(\mathrm{Tr}_{\mathcal{H}\to\mathcal{J}}\, e^{-it'\,\widehat{H}^\varepsilon}\, \rho_{\mathrm{Fock}}\, e^{it\,\widehat{H}^\varepsilon} \right) \widehat{\Pi}^{(n)}_{\mathcal{J}} \right\|_1$$

$$= \sum_{n>N_o} n \left\| \mathrm{Tr}_{\mathcal{H}\to\mathcal{J}}\;\; e^{-it'\,\widehat{H}^\varepsilon}\, \rho_{\mathrm{Fock}}\, e^{it\,\widehat{H}^\varepsilon}\, \widehat{\varphi}^{-1}_{\mathcal{H}\to\mathcal{J}} \left(\mathrm{id}_{\widehat{\mathcal{J}^\perp}} \otimes \widehat{\Pi}^{(n)}_{\mathcal{J}} \right) \widehat{\varphi}_{\mathcal{H}\to\mathcal{J}} \right\|_1$$

$$\leqslant \sum_{n>N_o} n \left\| e^{-it'\,\widehat{H}^\varepsilon}\, \rho_{\mathrm{Fock}}\, e^{it\,\widehat{H}^\varepsilon}\, \widehat{\varphi}^{-1}_{\mathcal{H}\to\mathcal{J}} \left(\mathrm{id}_{\widehat{\mathcal{J}^\perp}} \otimes \widehat{\Pi}^{(n)}_{\mathcal{J}} \right) \widehat{\varphi}_{\mathcal{H}\to\mathcal{J}} \right\|_1$$

$$\leqslant \sum_{n>N_o} \sum_{n'\geqslant 0} n \left\| e^{-it'\,\widehat{H}^\varepsilon}\, \rho_{\mathrm{Fock}}\, e^{it\,\widehat{H}^\varepsilon}\, \widehat{\varphi}^{-1}_{\mathcal{H}\to\mathcal{J}} \left(\widehat{\Pi}^{(n')}_{\mathcal{J}^\perp} \otimes \widehat{\Pi}^{(n)}_{\mathcal{J}} \right) \widehat{\varphi}_{\mathcal{H}\to\mathcal{J}} \right\|_1$$

$$\leqslant \sum_{n>N_o} \sum_{n'\geqslant 0} (n+n') \left\| e^{-it'\,\widehat{H}^\varepsilon}\, \rho_{\mathrm{Fock}}\, e^{it\,\widehat{H}^\varepsilon}\, \widehat{\varphi}^{-1}_{\mathcal{H}\to\mathcal{J}} \left(\widehat{\Pi}^{(n')}_{\mathcal{J}^\perp} \otimes \widehat{\Pi}^{(n)}_{\mathcal{J}} \right) \widehat{\varphi}_{\mathcal{H}\to\mathcal{J}} \right\|_1$$

$$= \sum_{n>N_o} \sum_{n'\geqslant 0} (n+n') \left\| e^{-it'\,\widehat{H}^\varepsilon}\, \rho_{\mathrm{Fock}}\, \widehat{\varphi}^{-1}_{\mathcal{H}\to\mathcal{J}} \left(\widehat{\Pi}^{(n')}_{\mathcal{J}^\perp} \otimes \widehat{\Pi}^{(n)}_{\mathcal{J}} \right) \widehat{\varphi}_{\mathcal{H}\to\mathcal{J}}\, e^{it\,\widehat{H}^\varepsilon} \right\|_1$$

(for $\widehat{H}^\varepsilon$ stabilizes the subspaces $\left[\left(\mathcal{J}^\perp \right)^{\otimes n'} \otimes \mathcal{J}^{\otimes n} \right]^{\mathrm{sym}}$ for all n, n')

$$\leqslant \sum_{n>N_o} \sum_{n'\geqslant 0} (n+n') \left\| \rho_{\mathrm{Fock}}\, \widehat{\varphi}^{-1}_{\mathcal{H}\to\mathcal{J}} \left(\widehat{\Pi}^{(n')}_{\mathcal{J}^\perp} \otimes \widehat{\Pi}^{(n)}_{\mathcal{J}} \right) \widehat{\varphi}_{\mathcal{H}\to\mathcal{J}} \right\|_1$$

$$= \sum_{n>N_o} \sum_{n'\geqslant 0} (n+n')\, \mathrm{Tr}_{\widehat{\mathcal{H}}}\, \rho_{\mathrm{Fock}}\, \widehat{\varphi}^{-1}_{\mathcal{H}\to\mathcal{J}} \left(\widehat{\Pi}^{(n')}_{\mathcal{J}^\perp} \otimes \widehat{\Pi}^{(n)}_{\mathcal{J}} \right) \widehat{\varphi}_{\mathcal{H}\to\mathcal{J}}$$

$$\leqslant \sum_{n''>N_o} n''\, \mathrm{Tr}_{\widehat{\mathcal{H}}}\, \rho_{\mathrm{Fock}}\, \widehat{\Pi}^{(n'')} \leqslant \frac{\epsilon_o}{3}\,.$$

Next, from the previous point there exists $\varepsilon_o \in \mathcal{E}$ such that, for all $\varepsilon = (\mathcal{J},\, \epsilon) \succcurlyeq \varepsilon_o$:

$$\left\| \tau_{\mathrm{Fock}\leftarrow\mathcal{J}} \left[\mathrm{Tr}_{\mathcal{H}\to\mathcal{J}}\, e^{-it'\,\widehat{H}^\varepsilon}\, \rho_{\mathrm{Fock}}\, e^{it\,\widehat{H}^\varepsilon} \right] \tau^+_{\mathrm{Fock}\leftarrow\mathcal{J}} - e^{-it'\,\widehat{H}}\, \rho_{\mathrm{Fock}}\, e^{it\,\widehat{H}} \right\|_1 \leqslant \frac{\epsilon_o}{3\, N_o}\,,$$

thus:

$$\left| \mathrm{Tr}_{\widehat{\mathcal{H}}}\;\; \left[\tau_{\mathrm{Fock}\leftarrow\mathcal{J}} \left(\mathrm{Tr}_{\mathcal{H}\to\mathcal{J}}\, e^{-it'\,\widehat{H}^\varepsilon}\, \rho_{\mathrm{Fock}}\, e^{it\,\widehat{H}^\varepsilon} \right) \tau^+_{\mathrm{Fock}\leftarrow\mathcal{J}} \right] T_{e,f} - \mathrm{Tr}_{\widehat{\mathcal{H}}}\;\; e^{-it'\,\widehat{H}}\, \rho_{\mathrm{Fock}}\, e^{it\,\widehat{H}}\, T_{e,f} \right|$$

$$\leqslant \left\| \left[\tau_{\mathrm{Fock}\leftarrow\mathcal{J}} \left(\mathrm{Tr}_{\mathcal{H}\to\mathcal{J}}\, e^{-it'\widehat{H}^{\varepsilon}}\, \rho_{\mathrm{Fock}}\, e^{it\widehat{H}^{\varepsilon}} \right) \tau^{+}_{\mathrm{Fock}\leftarrow\mathcal{J}} \right] - e^{-it'\widehat{H}}\, \rho_{\mathrm{Fock}}\, e^{it\widehat{H}} \right\|_1 \times$$

$$\times \left\| \sum_{n\leqslant N_o} \widehat{\sqcap}^{(n)}\, \widehat{a}^{\mathrm{Fock},+}_{e}\, \widehat{a}^{\mathrm{Fock}}_{f}\, \widehat{\sqcap}^{(n)} \right\| + \sum_{n>N_o} \left\| \left[\tau_{\mathrm{Fock}\leftarrow\mathcal{J}} \left(\mathrm{Tr}_{\mathcal{H}\to\mathcal{J}}\, e^{-it'\widehat{H}^{\varepsilon}}\, \rho_{\mathrm{Fock}}\, e^{it\widehat{H}^{\varepsilon}} \right) \tau^{+}_{\mathrm{Fock}\leftarrow\mathcal{J}} \right] \widehat{\sqcap}^{(n)} \right\|_1 \times$$

$$\times \left\| \widehat{\sqcap}^{(n)}\, \widehat{a}^{\mathrm{Fock},+}_{e}\, \widehat{a}^{\mathrm{Fock}}_{f}\, \widehat{\sqcap}^{(n)} \right\| + \sum_{n>N_o} \left\| e^{-it'\widehat{H}}\, \rho_{\mathrm{Fock}}\, e^{it\widehat{H}}\, \widehat{\sqcap}^{(n)} \right\|_1 \left\| \widehat{\sqcap}^{(n)}\, \widehat{a}^{\mathrm{Fock},+}_{e}\, \widehat{a}^{\mathrm{Fock}}_{f}\, \widehat{\sqcap}^{(n)} \right\|$$

$$\leqslant \frac{\epsilon_o}{3\,N_o}\, N_o\, \|e\|\, \|f\| + \frac{\epsilon_o}{3}\, \|e\|\, \|f\| + \sum_{n>N_o} \left\| e^{-it'\widehat{H}}\, \rho_{\mathrm{Fock}}\, \widehat{\sqcap}^{(n)}\, e^{it\widehat{H}} \right\|_1 n\, \|e\|\, \|f\|$$

(for $e^{it\widehat{H}}$ stabilizes the subspaces $\mathcal{H}^{\otimes n,\,\mathrm{sym}}$ for all n)

$$\leqslant \epsilon_o\, \|e\|\, \|f\|\,,$$

which concludes the proof. □

While it will be essential to play with more toy models (and especially with more sophisticated ones), in order to sharpen our still rather crude proposal for dealing with constraints, we have at least ascertained that the program of subsection 3.2 can be applied to the most simple quantum field theory, where it satisfactorily reproduces established results. Indeed, we found that we can define a sensible convergence at the quantum level, on a subspace of states that can either be identified with the Fock space or with a subset of it. This is reassuring, for we know that the Fock space is the right arena to describe interaction-free theory (since such a theory preserves the subspaces of fixed particles number). It would be interesting to study whether more general quantum field theories can be translated in this language too.

In addition, we might be able to gain a deeper understanding of the formalism considered here by studying its relations to approaches that incorporate similar ingredients, like lattice quantum field theory. This could help shed light on issues that are shared with these approaches, notably the problem of 'universality': in other words, the concern about how to ensure that the results we are getting are robust, and do not depend critically on some arbitrary choices entering the definition of the regularization scheme.

We have displayed in section 15 a trick to circumvent this pitfall: by assembling all reasonable approximations into a huge label set $\mathcal{E}$, and ordering them by their respective quality, we can view a specific regularization prescription as simply selecting a cofinal subset in $\mathcal{E}$. However, it is not clear whether this could still be done for less trivial systems, because it could become difficult to arrange for $\mathcal{E}$ to be directed. It would be interesting to investigate whether this idea could be combined with the techniques we will develop in section 19, which aim precisely at extracting universal (countable, directed) subsets from (uncountable, possibly non-directed) label sets, although in a different context.

Chapter 6 – Searching for Semi-Classical States

17. Introduction

Recall that our motivation to set up a projective state space for the holonomy-flux algebra was to obtain better semi-classical states. As argued in the introduction of the present work, the Ashtekar-Lewandowski Hilbert space, being built out of a vacuum which is a momentum eigenstate (with vanishing fluxes), cannot accommodate states in which the quantum fluctuations would balanced between position and momentum variables. However, now that we could overcome this limitation, we uncover a deeper issue, which has its root not in the restriction to a particular representation of the algebra of observables, but in the algebra *itself*.

As underlined in subsection 12.2 the holonomy and flux variables do not come in independent, canonically conjugate pairs. In fact, any given flux observable has non trivial commutators with infinitely, uncountably many holonomies. As we will demonstrate below in the $G = \mathbb{R}$ case, the sheer amount of Heisenberg uncertainty relations arising from these non-vanishing commutators forces the quantum fluctuations to blow up uncontrollably as more and more degrees of freedom are taken into account: in the end, it is not even possible to keep the variance of all elementary variables *finite*, let alone small. Note that obstructions to the design of states with specific properties on the holonomy-flux algebra have been pointed out in earlier works [80]: in particular, the Ashtekar-Lewandowski vacuum turns out to be the only diffeomorphism-invariant state [55], which notably forbids the design of a diffeomorphism-invariant coherent state.

These issues can be traced back to the fact that holonomy-flux algebra along analytical (or semi-analytical) edges and surfaces is generated by uncountably many elementary observables, which motivates the construction we will present in section 19: we will explore a strategy to drastically reduce the algebra of observables, while keeping intact the physical content of the theory. Such a reduction allows the systematic construction of projective states, in particular semi-classical ones (subsection 19.1). After laying out a general framework in subsection 19.2, we will, in subsection 19.3, explain how it can be implemented in the case of a (slightly simplified) one-dimensional version of the holonomy-flux algebra. The generalization to higher dimensions, especially the physically relevant $d = 3$ case, is currently a work in progress.

Once the label set is trimmed down to countable cardinality, a corresponding inductive limit (constructed along the lines of theorem 5.9) will automatically be *separable* (assuming all 'building block' Hilbert spaces are), rather that non-separable like eg. the Ashtekar-Lewandowski Hilbert space. Hence, constructing states will get easier on the inductive limit side too: besides lifting the technical issues plaguing non-separable Hilbert spaces [28], it will allow states to include all basis vectors at once (by contrast, non-separable Hilbert spaces tend, paradoxically, to be 'too small', as their orthonormal basis are uncountable while linear combinations can only be at most countable). Nevertheless, the advantages of the projective formalism over an inductive limit Hilbert space remains: as we will check in prop. 19.2, the semi-classical quantum states that one can construct within projective state spaces on countable label sets typically do *not* belong to corresponding inductive limit Hilbert space arising from vacuum states that are far from semi-classical. In other words, the argument put forward in the main introduction, that discrete quantum excitations cannot mask the core properties of the vacuum, still holds in the case of a countable label set.

When the label set is countable, one can also, in the spirit of theorem 5.11, produce from the projective system associated infinite tensor products (ITP, see [99, 93]), and the states we will be considering often *do* belong to these ITP Hilbert spaces (see again prop. 19.2). In fact, the whole construction of section 19 is very closely connected with Algebraïc Quantum Gravity (AQG, see [36]): the idea of AQG is to choose an infinitely extended graph and to write a state space for quantum gravity as the infinite tensor product of the $L_2(G)$ Hilbert spaces carried by the individual edges. Like in the present development, this switch to discrete degrees of freedom in AQG is motivated by the search for better semi-classical states. We will comment where appropriate on the similarities and differences between the two approaches, and delineate some advantages of the projective formulation (namely a lesser dependence on arbitrary choices and an improved diffeomorphism invariance).

Finally, the benefits of simplifying the structure of the theory reach beyond the sole semi-classical analysis. In section 20, we will contemplate how it could help solving the constraints of LQG (continuing the discussion started in section 7).

18. Obstructions to the construction of narrow states

We will, in subsection 18.1, work in the context of a general linear projective system $(\mathcal{L}, \mathcal{M}, \pi)$: namely, a projective system such that all small phase spaces $\mathcal{M}_\eta$ are *linear* and across which the distinction between configuration and momentum variables can be defined consistently (ie. all projections preserve this distinction). Note that this is precisely the setup of [68] and we recalled in subsection 6.1 how to quantize such a classical projective limit into a projective system of quantum state space in the position representation.

In projective quantum state spaces of this kind, we can systematically study the construction of *narrow states*, namely states in which the measurement probabilities of *all* configuration and momentum variables have *finite variance*. In particular, we will spell out the conditions, arising from the Heisenberg uncertainty relations, for a certain assignment of variances to be realizable in a quantum state. Of special interest for the construction of semi-classical states are the *Gaussian states*, and, in fact, we will see that there exists, for any narrow state, a Gaussian state with the same variances.

In subsection 18.2, we will apply this study to the projective quantum state space set up in section 12, in the special case $G = \mathbb{R}$ (where it constitutes a linear projective system in the sense above, as will be shown in prop. 18.13). We will then be able to prove (prop. 18.14) that there are not *any* narrow states in this state space: in other words, all states therein have infinite variance for at least some of the variables, and thus are not good candidates as semi-classical states. We will also formulate some weaker notions of semi-classicality that are excluded as well (props. 18.15 and 18.16).

Note that, while the argument cannot be easily generalized beyond the $G = \mathbb{R}$ case (because the systematic study of narrow states in subsection 18.1 makes heavy use of the linearity, viz. the discussion preceding prop. 18.9), we will take it as a hint that the construction of semi-classical states should be altogether attacked differently. On the one hand, the negative result of subsection 18.2 casts serious doubts on the possibility to design admissible semi-classical states even in the compact group case: although diverging variances are of course excluded in this case (at least for the configuration variables), we expect that, if states very peaked around the group identity *could* be designed, the group structure should *not* matter much to them (heuristically, the portion of the configuration space seen by such states would be nearly linear). On the other hand, the approach we will develop in section 19 to go around the obstruction will simplify the construction of projective states to the point that keeping G arbitrary will become effortless.

18.1 Projective families of characteristic functions

The central tool of the present subsection will be Wigner characteristic functions (see [102, 47] and the review in appendix C): any quantum state can be represented by a function defined on the dual of the classical phase space, and this alternative representation is strictly equivalent to its representation as a density matrix ρ. This is the quantum version of characteristic distributions in classical statistical physics, in the sense that the moments of the probability distributions governing quantum measurements can be recovered from the derivatives of the characteristic function (prop. C.8). However, the positivity requirement for quantum characteristic functions (eq. (C.6.1), ensuring the positivity of ρ) differs from its classical counterpart (that would ensure positivity of the corresponding probability distribution), as it encodes the non-commutation of quantum observables (see the discussion preceding prop. C.6).

Unsurprisingly, the characteristic functions of the partial density matrices composing a projective state combine again into a projective family (the dual spaces $\mathcal{M}_\eta^*$ form naturally an inductive structure, with injections given by the pullback under the projections $\pi_{\eta'\to\eta}$, so functions on them arrange themselves in a projective structure, see also the discussion at the beginning of appendix C.2). This is the advantage of working with Wigner *characteristic* functions rather than Wigner quasi-probabilities, which would be in analogy to classical probability distributions (the computation of partial traces, or, at the classical level, of marginal probability distribution, is translated into a simple restriction when working with characteristic functions).

Proposition 18.1 Let $(\mathcal{L}, \mathcal{M}, \pi)^\downarrow$ be a projective system of phase spaces such that:

1. $\forall \eta \in \mathcal{L}$, there exists two finite-dimensional real vector spaces $\mathcal{C}_\eta$, $\mathcal{P}_\eta$, an invertible linear map $\Xi_\eta : \mathcal{P}_\eta \to \mathcal{C}_\eta^*$ (where $\mathcal{C}_\eta^*$ denotes the dual of $\mathcal{C}_\eta$) and a symplectomorphism $L_\eta : \mathcal{M}_\eta \to \mathcal{C}_\eta \times \mathcal{P}_\eta$, with respect to the symplectic structure Ω_η defined on $\mathcal{C}_\eta \times \mathcal{P}_\eta$ by:

$\forall (u, v),\ (u', v') \in \mathcal{C}_\eta \times \mathcal{P}_\eta\,,\ \Omega_\eta(u, v;\ u', v') := \Xi_\eta(v')(u) - \Xi_\eta(v)(u')\,;$

2. $\forall \eta \preccurlyeq \eta' \in \mathcal{L}\,,\ \pi_{\eta' \to \eta} = L_\eta^{-1} \circ (Q_{\eta' \to \eta} \times P_{\eta' \to \eta}) \circ L_{\eta'}$ with $Q_{\eta' \to \eta}$, resp. $P_{\eta' \to \eta}$, a linear map $\mathcal{C}_{\eta'} \to \mathcal{C}_\eta$, resp. $\mathcal{P}_{\eta'} \to \mathcal{P}_\eta$.

We choose, for any $\eta \in \mathcal{L}$, a Lebesgue measure μ_η on $\mathcal{C}_\eta$. For any $\eta \preccurlyeq \eta' \in \mathcal{L}$, we define:

3. $\mathcal{C}_{\eta' \to \eta} := \operatorname{Ker} Q_{\eta' \to \eta}\,;$

4. $\begin{array}{rccl} \varphi_{\eta' \to \eta} & : & \mathcal{C}_{\eta'} & \to & \mathcal{C}_{\eta' \to \eta} \times \mathcal{C}_\eta \\ & & x' & \mapsto & R_{\eta' \to \eta}(x'),\ Q_{\eta' \to \eta}(x') \end{array}$, with $R_{\eta' \to \eta} : \mathcal{C}_{\eta'} \to \mathcal{C}_{\eta' \to \eta}$ the projection on $\mathcal{C}_{\eta' \to \eta}$ parallel to $\left(\Xi_\eta \circ P_{\eta' \to \eta} \circ \Xi_{\eta'}^{-1}\right)^* \langle \mathcal{C}_\eta \rangle$ (where $(\,\cdot\,)^*$ denotes the dual map).

Then, we can complete these elements into a factorizing system of measured manifolds (def. 6.1) $(\mathcal{L},\ (\mathcal{C}, \mu),\ \varphi)^\times$, from which a projective system of quantum state spaces $(\mathcal{L},\ \mathcal{H},\ \Phi)^\otimes$ can be constructed as described in prop. 6.3.

Proof Since a finite-dimensional vector space is in particular an additive simply-connected Lie group, this is a special case of theorem 6.2. The explicit expressions for $\mathcal{C}_{\eta' \to \eta}$ and $\varphi_{\eta' \to \eta}$ are obtained by adapting the ones from the proof of theorem 6.2 to the slightly different notations we are using here. □

Definition 18.2 We consider the same objects as in prop. 18.1. A projective family of characteristic functions is a family $\left(W_\eta\right)_{\eta \in \mathcal{L}}$ such that:

1. for any $\eta \in \mathcal{L}$, W_η is a *continuous* function $\mathcal{C}_\eta^* \times \mathcal{P}_\eta^* \to \mathbb{C}$;
2. for any $\eta \in \mathcal{L}$, any $N \in \mathbb{N}$, any $(s_1, t_1), \ldots, (s_N, t_N) \in \mathcal{C}_\eta^* \times \mathcal{P}_\eta^*$ and any $z_1, \ldots, z_N \in \mathbb{C}$:

$$\sum_{i,j=1}^{N} \overline{z_j}\, z_i\, e^{i\, \xi_{ij}}\, W_\eta(s_i - s_j,\ t_i - t_j) \geqslant 0\,, \tag{18.2.1}$$

   where $\xi_{ij} := \frac{1}{2}\left(t_i\left(\Xi_\eta^{-1}(s_j)\right) - t_j\left(\Xi_\eta^{-1}(s_i)\right)\right);$

3. for any $\eta \preccurlyeq \eta' \in \mathcal{L}$, $W_\eta = W_{\eta'} \circ \left(Q^*_{\eta' \to \eta} \times P^*_{\eta' \to \eta}\right)$.

We denote by $\mathcal{W}^{\downarrow}_{(\mathcal{L},(\mathcal{C},\mathcal{P}),(Q,P))}$ the space of all projective families of characteristic functions.

Proposition 18.3 We consider the same objects as in prop. 18.1. Let $\eta \in \mathcal{L}$. On $\mathcal{H}_\eta = L_2(\mathcal{C}_\eta\,,\ d\mu_\eta)$ we define, for any $(s, t) \in \mathcal{C}_\eta^* \times \mathcal{P}_\eta^*$, a unitary operator $\mathrm{T}_\eta(s, t)$ on $\mathcal{H}_\eta$ by:

$$\forall \psi \in \mathcal{H}_\eta\,,\ \forall x \in \mathcal{C}_\eta\,,\ [\mathrm{T}_\eta(s, t)\, \psi](x) = \exp\left(i\, s(x) + \tfrac{i}{2}\, t\left(\Xi_\eta^{-1}(s)\right)\right)\, \psi\left(x + \Xi_\eta^{*,-1}(t)\right),$$

as well as a densely defined, essentially self-adjoint operator $\mathrm{X}_\eta(s, t)$ with dense domain $\mathcal{D}_\eta := C_o^\infty(\mathcal{C}_\eta,\ \mathbb{C})$ (the space of smooth, compactly supported, complex-valued functions on $\mathcal{C}_\eta$) by:

$$\forall \psi \in \mathcal{D}_\eta\,,\ \forall x \in \mathcal{C}_\eta\,,\ [\mathrm{X}_\eta(s, t)\, \psi](x) = s(x)\, \psi(x) - i\left[T_x \psi\right]\left(\Xi_\eta^{*,-1}(t)\right)$$

(where $T_x\psi$ denotes the differential of ψ at x and $\Xi_\eta^{*,-1} = \left(\Xi_\eta^*\right)^{-1} = \left(\Xi_\eta^{-1}\right)^*$).

For any $\eta \preccurlyeq \eta' \in \mathcal{L}$ and any $(s, t) \in \mathcal{C}_\eta^* \times \mathcal{P}_{\eta'}^*$, we have:

$$\mathrm{T}_{\eta'}(s', t') = \Phi^{-1}_{\eta'\to\eta} \circ \Big(\mathrm{id}_{\mathcal{H}_{\eta'\to\eta}} \otimes \mathrm{T}_\eta(s, t)\Big) \circ \Phi_{\eta'\to\eta}\,,$$

$$\& \quad \mathrm{X}_{\eta'}(s', t') = \Phi^{-1}_{\eta'\to\eta} \circ \Big(\mathrm{id}_{\mathcal{H}_{\eta'\to\eta}} \otimes \mathrm{X}_\eta(s, t)\Big) \circ \Phi_{\eta'\to\eta}\,, \tag{18.3.1}$$

where $(s', t') := \big(Q^*_{\eta'\to\eta}(s),\ P^*_{\eta'\to\eta}(t)\big)$.

Proof Let $\eta \in \mathcal{L}$. For any $(s, t) \in \mathcal{C}^*_\eta \times \mathcal{P}^*_\eta$, it has been proven in prop. C.5, that $\mathrm{T}_\eta(s, t)$, resp. $\mathrm{X}_\eta(s, t)$, is well-defined and is unitary, resp. essentially self-adjoint.

Let $\eta \preccurlyeq \eta' \in \mathcal{L}$ and $(s, t) \in \mathcal{C}^*_\eta \times \mathcal{P}^*_\eta$. Let $(s', t') := \big(Q^*_{\eta'\to\eta}(s),\ P^*_{\eta'\to\eta}(t)\big)$. For any $\psi_\eta \in \mathcal{H}_\eta$ and any $\psi_{\eta'\to\eta} \in \mathcal{H}_{\eta'\to\eta} = L_2(\mathcal{C}_{\eta'\to\eta}\,,\ d\mu_{\eta'\to\eta})$, we have, using the expression for $\Phi_{\eta'\to\eta}$ from prop. 6.3:

$$\forall (y, x) \in \mathcal{C}_{\eta'\to\eta} \times \mathcal{C}_\eta,\quad \Big[\Phi_{\eta'\to\eta}\, \mathrm{T}_{\eta'}(s', t')\, \Phi^{-1}_{\eta'\to\eta}\, (\psi_{\eta'\to\eta} \otimes \psi_\eta)\Big](y, x) =$$

$$= \Big[\mathrm{T}_{\eta'}(s', t')\, \Phi^{-1}_{\eta'\to\eta}\, (\psi_{\eta'\to\eta} \otimes \psi_\eta)\Big] \circ \varphi^{-1}_{\eta'\to\eta}(y, x)$$

$$= \exp\Big(i\, s' \circ \varphi^{-1}_{\eta'\to\eta}(y, x) + \tfrac{i}{2}\, t' \left(\Xi^{-1}_{\eta'}(s')\right)\Big)\ \ \Phi^{-1}_{\eta'\to\eta}(\psi_{\eta'\to\eta} \otimes \psi_\eta)\Big(\varphi^{-1}_{\eta'\to\eta}(y, x) + \Xi^{*,-1}_{\eta'}(t')\Big)$$

$$= \exp\Big(i\, s(x) + \tfrac{i}{2}\, t\left(\Xi^{-1}_\eta(s)\right)\Big)\ \psi_{\eta'\to\eta}(y)\, \psi_\eta\left(x + \Xi^{*,-1}_\eta(t)\right)$$

$$= \psi_{\eta'\to\eta}(y)\, \big[\mathrm{T}_\eta(s, t)\, \psi_\eta\big](x)\,,$$

where we have used:

$$\forall (y, x) \in \mathcal{C}_{\eta'\to\eta} \times \mathcal{C}_\eta,$$

$$Q_{\eta'\to\eta} \circ \varphi^{-1}_{\eta'\to\eta}(y, x) = x\,,$$

$$\& \quad \varphi_{\eta'\to\eta}\Big(\varphi^{-1}_{\eta'\to\eta}(y, x) + \Xi^{*,-1}_{\eta'} \circ P^*_{\eta'\to\eta}(t)\Big) = \Big(y,\ x + Q_{\eta'\to\eta} \circ \Xi^{*,-1}_{\eta'} \circ P^*_{\eta'\to\eta}(t)\Big)$$

(from the definition of $\varphi_{\eta'\to\eta}$ in prop. 18.1) as well as:

$$Q_{\eta'\to\eta} \circ \Xi^{*,-1}_{\eta'} \circ P^*_{\eta'\to\eta} = \Xi^{*,-1}_\eta\,,$$

as follows from the compatibility of $L^{-1}_\eta \circ \big(Q_{\eta'\to\eta} \times P_{\eta'\to\eta}\big) \circ L_{\eta'}$ with the symplectic structures $\Omega_{\eta'}$ and Ω_η (see def. 2.1 and eq. (6.2.1)). Thus, we get:

$$\forall \psi_\eta \in \mathcal{H}_\eta\,,\ \forall \psi_{\eta'\to\eta} \in \mathcal{H}_{\eta'\to\eta}\,,\quad \Phi_{\eta'\to\eta}\, \mathrm{T}_{\eta'}(s', t')\, \Phi^{-1}_{\eta'\to\eta}\, (\psi_{\eta'\to\eta} \otimes \psi_\eta) = \psi_{\eta'\to\eta} \otimes \big[\mathrm{T}_\eta(s, t)\, \psi_\eta\big]\,,$$

and therefore:

$$\mathrm{T}_{\eta'}(s', t') = \Phi^{-1}_{\eta'\to\eta} \circ \Big(\mathrm{id}_{\mathcal{H}_{\eta'\to\eta}} \otimes \mathrm{T}_\eta(s, t)\Big) \circ \Phi_{\eta'\to\eta}\,.$$

Similarly, we have, for any $\psi_\eta \in \mathcal{D}_\eta$ and any $\psi_{\eta'\to\eta} \in \mathcal{D}_{\eta'\to\eta} := C^\infty_o(\mathcal{C}_{\eta'\to\eta}, \mathbb{C})$, $\Phi^{-1}_{\eta'\to\eta}\, (\psi_{\eta'\to\eta} \otimes \psi_\eta) \in \mathcal{D}_{\eta'}$ and:

$$\Phi_{\eta'\to\eta}\, \mathrm{X}_{\eta'}(s', t')\, \Phi^{-1}_{\eta'\to\eta}\, (\psi_{\eta'\to\eta} \otimes \psi_\eta) = \psi_{\eta'\to\eta} \otimes \big[\mathrm{X}_\eta(s, t)\, \psi_\eta\big]\,,$$

using:

$$\forall (y, x) \in \mathcal{C}_{\eta'\to\eta} \times \mathcal{C}_\eta\,,$$

$$\Big[T_{\varphi^{-1}_{\eta'\to\eta}(y,x)}\, \Phi^{-1}_{\eta'\to\eta}\, \big(\psi_{\eta'\to\eta} \otimes \psi_\eta\big)\Big] = \psi_\eta(x)\ \big[T_y \psi_{\eta'\to\eta}\big] \circ R_{\eta'\to\eta} + \psi_{\eta'\to\eta}(y)\ [T_x\, \psi_\eta] \circ Q_{\eta'\to\eta}\,.$$

Moreover, $\mathcal{D}_{\eta'\to\eta}\otimes\mathcal{D}_\eta$ (understood as a tensor product of vector spaces, ie. *without* any completion) is a dense subspace of $\Phi_{\eta'\to\eta}\langle\mathcal{D}_{\eta'}\rangle$ and $\left(\Phi_{\eta'\to\eta}\,\mathrm{X}_{\eta'}(s',t')\,\Phi^{-1}_{\eta'\to\eta}\right)\Big|_{\mathcal{D}_{\eta'\to\eta}\otimes\mathcal{D}_\eta} = \mathrm{id}_{\mathcal{H}_{\eta'\to\eta}}\Big|_{\mathcal{D}_{\eta'\to\eta}}\otimes\mathrm{X}_\eta(s,t)$ is essentially self-adjoint (as a tensor product of essentially self-adjoint operators, see [73, theorem VIII.33]), hence the unique self-adjoint extensions of $\Phi_{\eta'\to\eta}\,\mathrm{X}_{\eta'}(s',t')\,\Phi^{-1}_{\eta'\to\eta}$ and $\mathrm{id}_{\mathcal{H}_{\eta'\to\eta}}\otimes\mathrm{X}_\eta(s,t)$ coincide. □

Proposition 18.4 We consider the same objects as in def. 18.2 and prop. 18.3. For any $\eta\in\mathcal{L}$ and any $\rho_\eta\in\overline{\mathcal{S}}_\eta$ (with $\overline{\mathcal{S}}_\eta$ the space of non-negative, traceclass operators on $\mathcal{H}_\eta$), we define W_{ρ_η} by:

$$\begin{array}{rccl} W_{\rho_\eta} & : & \mathcal{C}^*_\eta\times\mathcal{P}^*_\eta & \to \mathbb{C} \\ & & (s,t) & \mapsto \mathrm{Tr}_{\mathcal{H}_\eta}\,\rho_\eta\,\mathrm{T}_\eta(s,t) \end{array}.$$

The map $W:\left(\rho_\eta\right)_{\eta\in\mathcal{L}}\mapsto\left(W_{\rho_\eta}\right)_{\eta\in\mathcal{L}}$ is a bijection $\overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}\to\mathcal{W}^{\downarrow}_{(\mathcal{L},(\mathcal{C},\mathcal{P}),(Q,P))}$ (where $\overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ has been defined in def. 5.2).

Proof For any $\eta\in\mathcal{L}$, we denote by $\mathcal{W}_\eta$ the space of all continuous functions of positive type on $\mathcal{C}^*_\eta\times\mathcal{P}^*_\eta$ (prop. C.6). From props. C.7 and C.10, the map $W_\eta:\rho_\eta\mapsto W_{\rho_\eta}$ is a bijection $\overline{\mathcal{S}}_\eta\to\mathcal{W}_\eta$. Moreover, we have:

$$\overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} = \left\{\left(\rho_\eta\right)_{\eta\in\mathcal{L}}\,\middle|\,\forall\eta,\,\rho_\eta\in\overline{\mathcal{S}}_\eta \quad \& \quad \forall\eta\preccurlyeq\eta',\,\rho_\eta=\mathrm{Tr}_{\mathcal{H}_{\eta'\to\eta}}\left(\Phi_{\eta'\to\eta}\,\rho_{\eta'}\,\Phi^{-1}_{\eta'\to\eta}\right)\right\},$$

and:

$$\mathcal{W}^{\downarrow}_{(\mathcal{L},(\mathcal{C},\mathcal{P}),(Q,P))} = \left\{\left(W_\eta\right)_{\eta\in\mathcal{L}}\,\middle|\,\forall\eta,\,W_\eta\in\mathcal{W}_\eta \quad \& \quad \forall\eta\preccurlyeq\eta',\,W_\eta=W_{\eta'}\circ\left(Q^*_{\eta'\to\eta}\times P^*_{\eta'\to\eta}\right)\right\}.$$

Now, from eq. (18.3.1), we have, for any $\eta\preccurlyeq\eta'$ and any $\rho_{\eta'}\in\overline{\mathcal{S}}_{\eta'}$:

$$W_\eta\left(\mathrm{Tr}_{\mathcal{H}_{\eta'\to\eta}}\left(\Phi_{\eta'\to\eta}\,\rho_{\eta'}\,\Phi^{-1}_{\eta'\to\eta}\right)\right) = W_{\eta'}(\rho_{\eta'})\circ\left(Q^*_{\eta'\to\eta}\times P^*_{\eta'\to\eta}\right).$$

Thus, W is well-defined as a map $\overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}\to\mathcal{W}^{\downarrow}_{(\mathcal{L},(\mathcal{C},\mathcal{P}),(Q,P))}$ and is bijective. □

Asking for a state to have finite variances in all position and configuration variables is expressed at the level of its Wigner characteristic function by asking the latter to be twice differentiable at 0, and the corresponding covariance matrix can be obtained from the Hessian of the characteristic function (prop. C.8). In particular, the positivity requirement for the characteristic function (eq. (C.6.1)) gives rise to inequalities that have to be satisfied by the covariance matrix (def. 18.6.2): these are nothing but the Heisenberg uncertainty relations (as is manifest from the rewriting in prop. 18.7.2 or 18.7.3, since $V_\eta(s,s)=\Delta\mathrm{X}^2_\eta(s,0)$ and $U_\eta(t,t)=\Delta\mathrm{X}^2_\eta(0,t)$). These inequalities will play a crucial role for the result of the next subsection.

Moreover, the projective structure binding the partial characteristic functions of a projective state together goes down to its covariance matrices on the various labels η, which arrange naturally into their own projective structure.

Definition 18.5 Let $\rho=\left(\rho_\eta\right)_{\eta\in\mathcal{L}}\in\overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ and let $\left(W_\eta\right)_{\eta\in\mathcal{L}}:=W(\rho)$. We say that ρ is a narrow state if there exist, for any $\eta \in \mathcal{L}$, a linear form $W^{(1)}_\eta$ and a symmetric bilinear form $W^{(2)}_\eta$ on $\mathcal{C}^*_\eta \times \mathcal{P}^*_\eta$ such that:

$$\forall (s, t) \in \mathcal{C}^*_\eta \times \mathcal{P}^*_\eta\,,\ W_\eta(\tau s,\ \tau t) = 1 + i\,\tau\, W^{(1)}_\eta(s, t) - \frac{\tau^2}{2}\, W^{(2)}_\eta(s, t;\ s, t) + o(\tau^2)\,. \tag{18.5.1}$$

From props. C.5 and C.8, we then have, for any $\eta \in \mathcal{L}$:

1. $\mathrm{Tr}_{\mathcal{H}_\eta} \rho_\eta = 1$ (so that ρ_η is a density matrix on $\mathcal{H}_\eta$);
2. $\forall (s, t) \in \mathcal{C}^*_\eta \times \mathcal{P}^*_\eta$, $\mathrm{Tr}_{\mathcal{H}_\eta} \rho_\eta\, \mathrm{X}_\eta(s, t) = W^{(1)}_\eta(s, t)$;
3. $\forall (s, t), (s', t') \in \mathcal{C}^*_\eta \times \mathcal{P}^*_\eta$, $\mathrm{Tr}_{\mathcal{H}_\eta} \dfrac{\mathrm{X}_\eta(s, t)\, \rho_\eta\, \mathrm{X}_\eta(s', t') + \mathrm{X}_\eta(s', t')\, \rho_\eta\, \mathrm{X}_\eta(s, t)}{2} = W^{(2)}_\eta(s, t;\ s', t')$.

We denote the space of all narrow states by $\hat{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$.

Definition 18.6 We consider the same objects as in prop. 18.1. A projective family of variances is a family $\left(V_\eta\,,\ U_\eta\right)_{\eta \in \mathcal{L}}$ such that:

1. for any $\eta \in \mathcal{L}$, V_η , resp. U_η , is a strictly *positive* symmetric bilinear form on $\mathcal{C}^*_\eta$, resp. $\mathcal{P}^*_\eta$;
2. for any $\eta \in \mathcal{L}$ and any $(s, t) \in \mathcal{C}^*_\eta \times \mathcal{P}^*_\eta$, $V_\eta(s,\ s) + U_\eta(t,\ t) - t\big(\Xi^{-1}_\eta(s)\big) \geqslant 0$;
3. for any $\eta \preccurlyeq \eta' \in \mathcal{L}$, $V_\eta = V_{\eta'} \circ (Q^*_{\eta' \to \eta} \times Q^*_{\eta' \to \eta})$ & $U_\eta = U_{\eta'} \circ (P^*_{\eta' \to \eta} \times P^*_{\eta' \to \eta})$.

We denote by $\mathcal{V}^{\downarrow}_{(\mathcal{L},\,(\mathcal{C},\mathcal{P}),\,(Q,P))}$ the space of all projective families of variances.

Proposition 18.7 Let $\eta \in \mathcal{L}$ and let V_η , resp. U_η , be a strictly positive symmetric bilinear form on $\mathcal{C}^*_\eta$, resp. $\mathcal{P}^*_\eta$. Let V^{-1}_η , resp. U^{-1}_η , be the strictly positive symmetric bilinear form on $\mathcal{C}_\eta$, resp. $\mathcal{P}_\eta$, characterized by:

$$\forall s, s' \in \mathcal{C}^*_\eta\,,\ V^{-1}_\eta\big(V_\eta(s,\ \cdot\,),\ V_\eta(s',\ \cdot\,)\big) = V_\eta(s, s')\,,$$
$$\text{resp. } \forall t, t' \in \mathcal{P}^*_\eta\,,\ U^{-1}_\eta\big(U_\eta(t,\ \cdot\,),\ U_\eta(t',\ \cdot\,)\big) = U_\eta(t, t')\,,$$

with the canonical identification $\mathcal{C}^{**}_\eta \approx \mathcal{C}_\eta$, resp. $\mathcal{P}^{**}_\eta \approx \mathcal{P}_\eta$.

Then, the three following conditions are equivalent:

1. $\forall (s, t) \in \mathcal{C}^*_\eta \times \mathcal{P}^*_\eta$, $V_\eta(s,\ s) + U_\eta(t,\ t) - t\big(\Xi^{-1}_\eta(s)\big) \geqslant 0$;
2. $\forall s \in \mathcal{C}^*_\eta$, $2\, V_\eta(s,\ s) - \dfrac{1}{2}\, U^{-1}_\eta\big(\Xi^{-1}_\eta(s),\ \Xi^{-1}_\eta(s)\big) \geqslant 0$;
3. $\forall t \in \mathcal{P}^*_\eta$, $2\, U_\eta(t,\ t) - \dfrac{1}{2}\, V^{-1}_\eta\big(\Xi^{*,-1}_\eta(t),\ \Xi^{*,-1}_\eta(t)\big) \geqslant 0$.

Proof Let $\widetilde{V}_\eta$ be the linear map $\mathcal{C}^*_\eta \to \mathcal{C}_\eta$ defined by:

$$\forall s \in \mathcal{C}^*_\eta,\ \widetilde{V}_\eta(s) = V_\eta(s,\ \cdot\,) \in \mathcal{C}^{**}_\eta \approx \mathcal{C}_\eta\,.$$

For any $s \neq 0$, we have $s\big(\widetilde{V}_\eta(s)\big) = V_\eta(s,\ s) > 0$ (for V_η is strictly positive), hence $\widetilde{V}_\eta(s) \neq 0$. Thus, $\widetilde{V}_\eta$ is injective, and therefore bijective, since $\mathcal{C}^*_\eta$ and $\mathcal{C}_\eta$ are finite-dimensional vector spaces of the same dimension. V^{-1}_η is then defined by:

$$\forall x, x' \in \mathcal{C}_\eta\,,\; V_\eta^{-1}(x,\, x') := V_\eta\big(\widetilde{V}_\eta^{-1}(x),\, \widetilde{V}_\eta^{-1}(x')\big)\,,$$

and is therefore a strictly positive symmetric bilinear form on $\mathcal{C}_\eta$. We have similarly $\widetilde{U}_\eta : \mathcal{P}_\eta^* \to \mathcal{P}_\eta$ and $U_\eta^{-1} : \mathcal{P}_\eta \times \mathcal{P}_\eta \to \mathbb{R}$.

18.7.1 ⇒ 18.7.2 & 18.7.1 ⇒ 18.7.3. We assume that 18.7.1 holds. Applying with $(s, t) = \big(s,\, \frac{1}{2}\,\widetilde{U}_\eta^{-1} \circ \Xi_\eta^{-1}(s)\big)$ for some $s \in \mathcal{C}_\eta^*$ yields:

$$V_\eta(s,\, s) + \frac{1}{4} U_\eta^{-1}\big(\Xi_\eta^{-1}(s),\, \Xi_\eta^{-1}(s)\big) - \frac{1}{2} U_\eta^{-1}\big(\Xi_\eta^{-1}(s),\, \Xi_\eta^{-1}(s)\big) \geqslant 0\,,$$

where we have used:

$$\forall p, p' \in \mathcal{P}_\eta\,,\; \big(\widetilde{U}_\eta^{-1}(p)\big)(p') = U_\eta\big(\widetilde{U}_\eta^{-1}(p),\, \widetilde{U}_\eta^{-1}(p')\big) = U_\eta^{-1}(p,\, p')\,.$$

Thus, we obtain the condition 18.7.2. Similarly, we can prove that 18.7.1 implies 18.7.3.

18.7.2 ⇒ 18.7.1 & 18.7.3 ⇒ 18.7.1. We assume that 18.7.2 holds. V_η provides a scalar product on $\mathcal{C}_\eta^*$, so the strictly positive symmetric bilinear form $U_\eta^{-1}\big(\Xi_\eta^{-1}(\,\cdot\,),\, \Xi_\eta^{-1}(\,\cdot\,)\big)$ can be diagonalized in a V_η-orthonormal basis $(f_i)_{i\in\{1,\dots,n\}}$ (with $n := \dim \mathcal{C}_\eta$), ie. we have:

$$\forall i, j \in \{1, \dots, n\}\,,\; V_\eta(f_i\,,\, f_j) = \delta_{ij} \quad \& \quad U_\eta^{-1}\big(\Xi_\eta^{-1}(f_i),\, \Xi_\eta^{-1}(f_j)\big) = \lambda_{(i)}\, \delta_{ij}\,,$$

with $\forall i \in \{1, \dots, n\}\,,\; \lambda_{(i)} > 0$. Next, the condition 18.7.2 can be rewritten:

$$\forall i \in \{1, \dots, n\}\,,\; \lambda_{(i)} \leqslant 4\,.$$

Moreover, defining, for any $i \in \{1, \dots, n\}$, $g_i = \Xi_\eta^*(e_i)$ with $(e_i)_{i\in\{1,\dots,n\}}$ the basis in $\mathcal{C}_\eta$ dual to $(f_i)_{i\in\{1,\dots,n\}}$, we have:

$$\forall i, j \in \{1, \dots, n\}\,,\; \lambda_{(i)}\, \delta_{ij} = U_\eta^{-1}\big(\Xi_\eta^{-1}(f_i),\, \Xi_\eta^{-1}(f_j)\big) = U_\eta\big(\lambda_{(i)}\, g_i,\, \lambda_{(j)}\, g_j\big)$$

where we have used that, for any $i \in \{1, \dots, n\}$, $\widetilde{U}_\eta^{-1} \circ \Xi_\eta^{-1}(f_i) = \lambda_{(i)}\, g_i$. Hence, we get:

$$\forall i, j \in \{1, \dots, n\}\,,\; U_\eta\big(g_i,\, g_j\big) = \frac{\delta_{ij}}{\lambda_{(j)}}\,.$$

Let $s = s^i\, f_i \in \mathcal{C}_\eta^*$ and $t = t^j\, g_j \in \mathcal{P}_\eta^*$ (with implicit summation). We have:

$$V_\eta(s,\, s) + U_\eta(t,\, t) - t\big(\Xi_\eta^{-1}(s)\big) = s^i s^i + \frac{t^j t^j}{\lambda_{(j)}} - s^i\, t^i\,.$$

Now, for any $\sigma, \tau \in \mathbb{R}$, and any $\lambda \in\,]0,\, 4]$, we have:

$$\sigma^2 + \frac{\tau^2}{\lambda} - \sigma\tau \geqslant 0\,,$$

so 18.7.1 is fulfilled. Similarly, we can prove that 18.7.3 implies 18.7.1. □

Proposition 18.8 Let $\rho = \big(\rho_\eta\big)_{\eta\in\mathcal{L}} \in \hat{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ and let $\big(W_\eta\big)_{\eta\in\mathcal{L}} := W(\rho)$. For any $\eta \in \mathcal{L}$, we define:

$$\forall (s, s') \in \mathcal{C}_\eta^*\,,\; V_\eta(s,\, s') := W_\eta^{(2)}(s, 0;\, s', 0) - W_\eta^{(1)}(s, 0)\, W_\eta^{(1)}(s', 0)\,,$$

$$\&\quad \forall (t, t') \in \mathcal{P}_\eta^*\,,\; U_\eta(t,\, t') := W_\eta^{(2)}(0, t;\, 0, t') - W_\eta^{(1)}(0, t)\, W_\eta^{(1)}(0, t')\,,$$

with $W^{(1)}_\eta$ and $W^{(2)}_\eta$ as in def. 18.5.

Then, $\left(V_\eta\,,\,U_\eta\right)_{\eta\in\mathcal{L}} \in \mathcal{V}^{\downarrow}_{(\mathcal{L},\,(\mathcal{C},\mathcal{P}),\,(Q,P))}$. Accordingly, we define the map V as:

$$\begin{array}{rccl} V & : & \hat{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} & \rightarrow & \mathcal{V}^{\downarrow}_{(\mathcal{L},\,(\mathcal{C},\mathcal{P}),\,(Q,P))} \\ & & \rho & \mapsto & \left(V_\eta\,,\,U_\eta\right)_{\eta\in\mathcal{L}} \end{array}.$$

Proof Let $\eta \in \mathcal{L}$, $(s,t) \in \mathcal{C}^*_\eta \times \mathcal{P}^*_\eta$, and $\tau \in \mathbb{R}$. Applying eq. (18.2.1) for the points $(0,0)$, $(\tau s,0)$, $(0,\tau t)$ with respective coefficients $-(1+i)$, $e^{-i\tau\,W^{(1)}_\eta(s,0)}$, $i\,e^{-i\tau\,W^{(1)}_\eta(0,t)}$ yields:

$$4 + 2\,\mathrm{Re}\Big[(i-1)\,e^{-i\tau\,W^{(1)}_\eta(s,0)}\,W_\eta(\tau s,0) - (1+i)\,e^{-i\tau\,W^{(1)}_\eta(0,t)}\,W_\eta(0,\tau t) + \\ + i\,e^{-i\tau\,W^{(1)}_\eta(-s,t)}\,e^{i\frac{\tau^2}{2}\,t\circ\Xi^{-1}_\eta(s)}\,W_\eta(-\tau s,\tau t)\Big] \geqslant 0\,,$$

where we have used that $W_\eta(0,0) = 1$ and that, for any $(s',t') \in \mathcal{C}^*_\eta \times \mathcal{P}^*_\eta$, $W_\eta(-s',-t') = \overline{W(s',t')}$ (prop. C.6; note that this in particular accounts for the reality of $W^{(1)}_\eta$ and $W^{(2)}_\eta$). Now, for any $(s',t') \in \mathcal{C}^*_\eta \times \mathcal{P}^*_\eta$, we have the expansion:

$$e^{-i\tau\,W^{(1)}_\eta(s',t')}\,W_\eta(\tau s',\tau t') = 1 - \frac{\tau^2}{2}W^{(2)}_\eta(s',t';\,s',t') + \frac{\tau^2}{2}\,\left[W^{(1)}_\eta(s',t')\right]^2 + o(\tau^2)\,.$$

Inserting in the previous inequality, we get:

$$4 + 2\left[-2 + \frac{\tau^2}{2}V_\eta(s,\,s) + \frac{\tau^2}{2}U_\eta(t,\,t) - \frac{\tau^2}{2}\,t\circ\Xi^{-1}_\eta(s) + o(\tau^2)\right] \geqslant 0\,.$$

Hence, we have, for any $(s,t) \in \mathcal{C}^*_\eta \times \mathcal{P}^*_\eta$:

$$V_\eta(s,\,s) + U_\eta(t,\,t) - t\circ\Xi^{-1}_\eta(s) \geqslant 0\,. \tag{18.8.1}$$

Let $s \in \mathcal{C}^*_\eta$ with $s \neq 0$. We have $\Xi^{-1}_\eta(s) \neq 0$, so there exists $t_s \in \mathcal{P}^*_\eta$ such that $t_s\circ\Xi^{-1}_\eta(s) = 1$. Applying eq. (18.8.1) with $(0,t_s)$ yields $u := U_\eta(t_s,\,t_s) \geqslant 0$, allowing us to define $t := \frac{1}{1+u}\,t_s$. Applying again eq. (18.8.1), now with (s,t), we get:

$$V_\eta(s,\,s) - \frac{1}{(1+u)^2} \geqslant 0\,,$$

so $V_\eta(s,\,s) > 0$. Similarly, we have, for any $t \in \mathcal{P}^*_\eta$, $t \neq 0 \Rightarrow U_\eta(t,\,t) > 0$. Thus, def. 18.6.1 and 18.6.2 are fulfilled (actually, we have shown that 18.6.1 is implied by 18.6.2).

Let $\eta \preccurlyeq \eta' \in \mathcal{L}$. From $W_\eta = W_{\eta'}\circ(Q^*_{\eta'\to\eta}\times P^*_{\eta'\to\eta})$ together with eq. (18.5.1) implies, for any $(s,t) \in \mathcal{C}^*_\eta \times \mathcal{P}^*_\eta$:

$$W^{(1)}_\eta(s,t) = W^{(1)}_{\eta'}(s',t') \quad \& \quad W^{(2)}_\eta(s,t;\,s,t) = W^{(2)}_{\eta'}(s',t';\,s',t')\,,$$

with $(s',\,t') := \left(Q^*_{\eta'\to\eta}(s),\,P^*_{\eta'\to\eta}(t)\right)$. Hence, def. 18.6.3 holds. □

The Heisenberg uncertainty relations that a projective family of covariance matrices need to satisfy if there exists a narrow quantum state with these covariances turn out to also be a *sufficient* condition for the existence of such a state. Indeed, we can construct a Gaussian operator combining the covariance matrix for the configuration variables with the one for the momentum variables, and the uncertainty relations are precisely what is required for this operator to be a *positive semi-definite* operator of unit trace, ie. a density matrix. Moreover, the projective conditions between the covariance matrices ensure that these Gaussian operators assemble into a projective quantum state. Although this is remarkably easy to check at the level of the characteristic functions (where the projections are simply restrictions so that the compatibility of the Gaussian states on various labels can be directly read out from the expression of their characteristic functions), it is instructive to understand how taking the partial trace of Gaussian states works, in particular how the form of factorization $\mathcal{C}_{\eta'} \approx \mathcal{C}_{\eta' \to \eta} \times \mathcal{C}_\eta$ conspires with the expression for the Gaussian states so that the correct projections get respectively applied on the covariance matrices for the positions and momenta (in accordance with the projection $\mathcal{M}_{\eta'} \to \mathcal{M}_\eta$).

Note that this is the point where it is critical to be working on *linear* configuration spaces. While the Wigner transform machinery could be adapted, for example, to the case of a compact group G [1], the nice projection property of Gaussian distributions is what makes the construction of projective coherent states in the linear case much easier than in the $L_2\left(G^N\right)$ case: even in the easiest – $G = \mathcal{U}(1)$ – case, the equivalent of Gaussian states, namely Hall states [42], do not have such a nice behavior under partial trace.

Proposition 18.9 Let $\left(V_\eta\,,\,U_\eta\right)_{\eta\in\mathcal{L}} \in \mathcal{V}^{\downarrow}_{(\mathcal{L},\,(\mathcal{C},\mathcal{P}),\,(Q,P))}$. Then, there exists $\rho \in \hat{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ such that $V(\rho) = \left(V_\eta\,,\,U_\eta\right)_{\eta\in\mathcal{L}}$, with V the map introduced in prop. 18.8. In other words, the map V is surjective.

Proof Let $\eta \in \mathcal{L}$. Let $(e_i)_{i\in\{1,\dots,n\}}$, $(f_i)_{i\in\{1,\dots,n\}}$ and $\left(\lambda_{(i)}\right)_{i\in\{1,\dots,n\}}$ be as in the proof of prop. 18.7. For any $k_1, \dots, k_n \in \mathbb{N}$, we define $\psi_{k_1,\dots,k_n}$ as:

$$\forall x = x^i\, e_i \in \mathcal{C}_\eta\,,\ \psi_{k_1,\dots,k_n}(x) := \frac{1}{\sqrt{\alpha_\eta}} \prod_{i=1}^{n} \frac{1}{\pi^{1/4}\,\lambda_{(i)}^{1/8}\,\sqrt{k_i!\,2^{k_i}}}\, e^{-\frac{(x^i)^2}{2\sqrt{\lambda_{(i)}}}}\ H_{k_i}\left(\frac{x^i}{\lambda_{(i)}^{1/4}}\right),$$

where for any $k \in \mathbb{N}$, H_k is the k-th Hermite polynomial:

$$\forall t \in \mathbb{R},\ H_k(t) := (-1)^k\, e^{t^2}\, \frac{d^k}{dt^k} e^{-t^2}\,,$$

and α_η is such that $d\mu_\eta(x^i\,e_i) = \alpha_\eta\, dx^1 \dots dx^n$. Then, for any $k_1, \dots, k_n \in \mathbb{N}$, $\psi_{k_1,\dots,k_n} \in \mathcal{H}_\eta = L_2(\mathcal{C}_\eta\,,\,d\mu_\eta)$ with $\|\psi_{k_1,\dots,k_n}\|_{\mathcal{H}_\eta} = 1$. Next, we define:

$$\rho_\eta := \sum_{k_1,\dots,k_n=0}^{\infty} \left[\prod_{i=1}^{n} \frac{2\sqrt{\lambda_{(i)}}}{2+\sqrt{\lambda_{(i)}}} \left(\frac{2-\sqrt{\lambda_{(i)}}}{2+\sqrt{\lambda_{(i)}}}\right)^{k_i}\right] |\psi_{k_1,\dots,k_n}\,\rangle\ \langle\,\psi_{k_1,\dots,k_n}|\,.$$

Using that $\forall i \in \{1,\dots,n\}\,,\ \lambda_{(i)} \in\,]0,\,4]$ (from the proof of prop. 18.7), ρ_η is a non-negative operator on $\mathcal{H}_\eta$ and we have:

$$\|\rho_\eta\|_1 \leqslant \sum_{k_1,\dots,k_n=0}^{\infty} \left|\prod_{i=1}^{n} \frac{2\sqrt{\lambda_{(i)}}}{2+\sqrt{\lambda_{(i)}}} \left(\frac{2-\sqrt{\lambda_{(i)}}}{2+\sqrt{\lambda_{(i)}}}\right)^{k_i}\right| \|\psi_{k_1,\dots,k_n}\|^2_{\mathcal{H}_\eta} = 1\,,$$

where $\|\cdot\|_1$ denotes the trace norm [73, theorem VI.20]. Hence, ρ is a (self-adjoint), positive semi-definite, traceclass operator on $\mathcal{H}_\eta$. Using Mehler formula [61], its kernel is given by:

$$\forall x = x^i\, e_i\,,\ y = y^i\, e_i \in \mathcal{C}_\eta\,,$$

$$\begin{aligned}
\rho_\eta(x;\,y) &:= \sum_{k_1,\dots,k_n=0}^{\infty} \left[\prod_{i=1}^{n} \frac{2\sqrt{\lambda_{(i)}}}{2+\sqrt{\lambda_{(i)}}}\ \left(\frac{2-\sqrt{\lambda_{(i)}}}{2+\sqrt{\lambda_{(i)}}}\right)^{k_i}\right] \psi_{k_1,\dots,k_n}(x)\,\overline{\psi_{k_1,\dots,k_n}(y)} \\
&= \frac{1}{\alpha_\eta\,(2\pi)^{n/2}}\ \exp\left[-\sum_{i=1}^{n}\frac{1}{2}\ \left(\frac{x^i+y^i}{2}\right)^2 + \frac{1}{2\,\lambda_{(i)}}(x^i-y^i)^2\right] \\
&= \frac{1}{\alpha_\eta\,(2\pi)^{n/2}}\ \exp\left[-\frac{1}{2}\,V_\eta^{-1}\left(\frac{x+y}{2}\,,\ \frac{x+y}{2}\right) - \frac{1}{2}\,U_\eta\left(\Xi^*_\eta(x-y)\,,\ \Xi^*_\eta(x-y)\right)\right].
\end{aligned}$$

Thus, for any $(s,t) \in \mathcal{C}^*_\eta \times \mathcal{P}^*_\eta$, we get, using the expression for $\mathrm{T}_\eta(s,t)$ from prop. 18.3:

$$\begin{aligned}
W_{\rho_\eta}(s,t) &:= \mathrm{Tr}_{\mathcal{H}_\eta}\,\rho_\eta\,\mathrm{T}_\eta(s,t) \\
&= \frac{1}{\alpha_\eta\,(2\pi)^{n/2}} \int d\mu_\eta(x)\ \exp\left[i\,s(x) + \frac{i}{2}\,t\left(\Xi_\eta^{-1}(s)\right) + \right. \\
&\qquad \left. -\frac{1}{2}\,V_\eta^{-1}\left(x + \frac{\Xi_\eta^{*,-1}(t)}{2}\,,\ x + \frac{\Xi_\eta^{*,-1}(t)}{2}\right) - \frac{1}{2}\,U_\eta\,(t,\ t)\right] \\
&= \exp\left[-\frac{1}{2}\,V_\eta(s,\ s) - \frac{1}{2}\,U_\eta(t,\ t)\right].
\end{aligned}$$

From 18.6.3, we have, for any $\eta \preccurlyeq \eta' \in \mathcal{L}$, $W_{\rho_\eta} = W_{\rho_{\eta'}} \circ \left(Q^*_{\eta'\to\eta} \times P^*_{\eta'\to\eta}\right)$, hence, from the proof of prop. 18.4, $\rho_\eta = \mathrm{Tr}_{\mathcal{H}_{\eta'\to\eta}}\left(\Phi_{\eta'\to\eta}\,\rho_{\eta'}\,\Phi^{-1}_{\eta'\to\eta}\right)$. So $\rho := \left(\rho_\eta\right)_{\eta\in\mathcal{L}} \in \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$, and we have $\left(W_{\rho_\eta}\right)_{\eta\in\mathcal{L}} = W(\rho)$. Moreover, for any $\eta \in \mathcal{L}$, we get the expansion:

$$\forall (s,t) \in \mathcal{C}^*_\eta \times \mathcal{P}^*_\eta,\ W_{\rho_\eta}(\tau s,\ \tau t) = 1 - \frac{\tau^2}{2}\,V_\eta(s,\ s) - \frac{\tau^2}{2}\,U_\eta(t,\ t) + o(\tau^2)\,.$$

Therefore, $\rho \in \hat{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ with:

$$\forall \eta \in \mathcal{L},\ W^{(1)}_\eta(s,t) = 0 \quad \& \quad W^{(2)}_\eta(s,t;\,s,t) = V_\eta(s,\ s) + U_\eta(t,\ t)\,.$$

In particular, this implies $V(\rho) = \left(V_\eta\,,\ U_\eta\right)_{\eta\in\mathcal{L}}$.

Note. One can also check directly that, for any $\eta \preccurlyeq \eta' \in \mathcal{L}$, $\rho_\eta = \mathrm{Tr}_{\mathcal{H}_{\eta'\to\eta}}\left(\Phi_{\eta'\to\eta}\,\rho_{\eta'}\,\Phi^{-1}_{\eta'\to\eta}\right)$. Indeed, denoting by $\rho^{(\eta')}_\eta$ the integral kernel of $\mathrm{Tr}_{\mathcal{H}_{\eta'\to\eta}}\left(\Phi_{\eta'\to\eta}\,\rho_{\eta'}\,\Phi^{-1}_{\eta'\to\eta}\right)$, we have:

$$\forall x, y \in \mathcal{C}_\eta\,,\ \rho^{(\eta')}_\eta(x;\,y) = \int d\mu_{\eta'\to\eta}(z)\,\rho_{\eta'}\left(\varphi^{-1}_{\eta'\to\eta}(z,x);\ \varphi^{-1}_{\eta'\to\eta}(z,y)\right).$$

Now, for any $x, y \in \mathcal{C}_\eta$ and any $z \in \mathcal{C}_{\eta'\to\eta}$, the definition of $\varphi_{\eta'\to\eta}$ together with $Q_{\eta'\to\eta} \circ \Xi^{*,-1}_{\eta'} \circ P^*_{\eta'\to\eta} = \Xi^{*,-1}_\eta$ (from the proof of prop. 18.3) yields:

$$\varphi^{-1}_{\eta'\to\eta}(z,x) = K_{\eta'\to\eta}(z) + J_{\eta'\to\eta}(x)\,,$$

with $J_{\eta'\to\eta} := \Xi^{*,-1}_{\eta'} \circ P^*_{\eta'\to\eta} \circ \Xi^*_\eta$ and $K_{\eta'\to\eta}$ the canonical injection of $\mathcal{C}_{\eta'\to\eta} = \mathrm{Ker}\,Q_{\eta'\to\eta}$ in $\mathcal{C}_{\eta'}$. So we get:

$$U_{\eta'}\left(\Xi^*_{\eta'}\left[\varphi^{-1}_{\eta'\to\eta}(z,x)-\varphi^{-1}_{\eta'\to\eta}(z,y)\right],\ \Xi^*_{\eta'}\left[\varphi^{-1}_{\eta'\to\eta}(z,x)-\varphi^{-1}_{\eta'\to\eta}(z,y)\right]\right)=$$
$$=U_{\eta'}\left(P^*_{\eta'\to\eta}\circ\Xi^*_\eta(x-y),\ P^*_{\eta'\to\eta}\circ\Xi^*_\eta(x-y)\right)$$
$$=U_\eta\left(\Xi^*_\eta(x-y),\ \Xi^*_\eta(x-y)\right),$$

as well as:

$$V^{-1}_{\eta'}\left(\frac{\varphi^{-1}_{\eta'\to\eta}(z,x)+\varphi^{-1}_{\eta'\to\eta}(z,y)}{2},\ \frac{\varphi^{-1}_{\eta'\to\eta}(z,x)+\varphi^{-1}_{\eta'\to\eta}(z,y)}{2}\right)=$$
$$=\left[\widetilde{Z}^{(\eta')}_\eta(z)\right](z)+2\left[\widetilde{X}^{(\eta')}_\eta\left(\frac{x+y}{2}\right)\right](z)+\left[\widetilde{Y}^{(\eta')}_\eta\left(\frac{x+y}{2}\right)\right]\left(\frac{x+y}{2}\right),$$

with:

$$\widetilde{X}^{(\eta')}_\eta:=K^*_{\eta'\to\eta}\circ\widetilde{V}^{-1}_{\eta'}\circ J_{\eta'\to\eta},\quad \widetilde{Y}^{(\eta')}_\eta:=J^*_{\eta'\to\eta}\circ\widetilde{V}^{-1}_{\eta'}\circ J_{\eta'\to\eta}\quad \&\quad \widetilde{Z}^{(\eta')}_\eta:=K^*_{\eta'\to\eta}\circ\widetilde{V}^{-1}_{\eta'}\circ K_{\eta'\to\eta},$$

where $\widetilde{V}_{\eta'}$ is defined as in the proof of prop. 18.7. For any $z\in\mathcal{C}_{\eta'\to\eta}\setminus\{0\}$, def. 18.6.1 implies:

$$\left[\widetilde{Z}^{(\eta')}_\eta(z)\right](z)=V_{\eta'}\left(\widetilde{V}^{-1}_{\eta'}\circ K_{\eta'\to\eta}(z),\ \widetilde{V}^{-1}_{\eta'}\circ K_{\eta'\to\eta}(z)\right)>0,$$

hence $\widetilde{Z}^{(\eta')}_\eta:\mathcal{C}_{\eta'\to\eta}\to\mathcal{C}^*_{\eta'\to\eta}$ is invertible, and the above equation becomes:

$$V^{-1}_{\eta'}\left(\frac{\varphi^{-1}_{\eta'\to\eta}(z,x)+\varphi^{-1}_{\eta'\to\eta}(z,y)}{2},\ \frac{\varphi^{-1}_{\eta'\to\eta}(z,x)+\varphi^{-1}_{\eta'\to\eta}(z,y)}{2}\right)=$$
$$=\left[\widetilde{Z}^{(\eta')}_\eta\left(z+(\widetilde{Z}^{(\eta')}_\eta)^{-1}\circ\widetilde{X}^{(\eta')}_\eta\left(\frac{x+y}{2}\right)\right)\right]\left(z+(\widetilde{Z}^{(\eta')}_\eta)^{-1}\circ\widetilde{X}^{(\eta')}_\eta\left(\frac{x+y}{2}\right)\right)+$$
$$+\left[\left(\widetilde{Y}^{(\eta')}_\eta-(\widetilde{X}^{(\eta')}_\eta)^*\circ(\widetilde{Z}^{(\eta')}_\eta)^{-1}\circ\widetilde{X}^{(\eta')}_\eta\right)\left(\frac{x+y}{2}\right)\right]\left(\frac{x+y}{2}\right).$$

On the other hand, using:

$$K_{\eta'\to\eta}\circ R_{\eta'\to\eta}+J_{\eta'\to\eta}\circ Q_{\eta'\to\eta}=\mathrm{id}_{\mathcal{C}_{\eta'}},\quad Q_{\eta'\to\eta}\circ K_{\eta'\to\eta}=0\quad \&\quad Q_{\eta'\to\eta}\circ J_{\eta'\to\eta}=\mathrm{id}_{\mathcal{C}_\eta}$$

(as follows from the expressions for $\varphi_{\eta'\to\eta}$ and $\varphi^{-1}_{\eta'\to\eta}$), together with $\widetilde{V}_\eta=Q_{\eta'\to\eta}\circ\widetilde{V}_{\eta'}\circ Q^*_{\eta'\to\eta}$, we can check that:

$$\widetilde{V}_\eta\left(\widetilde{Y}^{(\eta')}_\eta-(\widetilde{X}^{(\eta')}_\eta)^*\circ(\widetilde{Z}^{(\eta')}_\eta)^{-1}\circ\widetilde{X}^{(\eta')}_\eta\right)=\mathrm{id}_{\mathcal{C}_\eta}\tag{18.9.1}$$

(noting that $\widetilde{V}_\eta$ is the lower right block of $\left(R_{\eta'\to\eta},\ Q_{\eta'\to\eta}\right)\circ\widetilde{V}_{\eta'}\circ\left(R^*_{\eta'\to\eta},\ Q^*_{\eta'\to\eta}\right)$, which is the inverse of $\left(K^*_{\eta'\to\eta},\ J^*_{\eta'\to\eta}\right)\circ\widetilde{V}^{-1}_{\eta'}\circ\left(K_{\eta'\to\eta},\ J_{\eta'\to\eta}\right)$, eq. (18.9.1) can be visualized by performing a blockwise matrix inversion of the latter). Putting everything together and carrying out the Gaussian integration over z, we get:

$$\forall x,y\in\mathcal{C}_\eta,\ \rho^{(\eta')}_\eta(x;\,y)\propto\exp\left[-\frac{1}{2}V^{-1}_\eta\left(\frac{x+y}{2},\ \frac{x+y}{2}\right)-\frac{1}{2}U_\eta\left(\Xi^*_\eta(x-y),\ \Xi^*_\eta(x-y)\right)\right],$$

so $\mathrm{Tr}_{\mathcal{H}_{\eta'\to\eta}}\left(\Phi_{\eta'\to\eta}\,\rho_{\eta'}\,\Phi^{-1}_{\eta'\to\eta}\right)\propto\rho_\eta$, the proportionality factor being determined to be 1 from:

$$\mathrm{Tr}_{\mathcal{H}_\eta}\, \mathrm{Tr}_{\mathcal{H}_{\eta'\to\eta}} \left(\Phi_{\eta'\to\eta}\, \rho_{\eta'}\, \Phi^{-1}_{\eta'\to\eta}\right) = \mathrm{Tr}_{\mathcal{H}_{\eta'}}\, \rho_{\eta'} = 1 = \mathrm{Tr}_{\mathcal{H}_\eta}\, \rho_\eta\,.$$

□

18.2 A no-go result in the $G = \mathbb{R}$ case

To prove the advertised no-go result in the case of the holonomy-flux algebra with $G = \mathbb{R}$ (using the projective system set up in sections 10 and 12), it will be enough to concentrate on a certain (uncountable) subset of observables out of this algebra. So, we will choose an edge, that we will identify with the line segment $[0, 1]$, and a continuous stack of surfaces intersecting this edge, one for each point in $]0, 1]$ (see fig. 18.1). We will keep, for each surface, only its face looking at 0. Moreover, for all holonomies included in the selected edge to have finite variance, it would be sufficient that all holonomies *ending at* 0 would have finite variance, since the holonomy between e and e' can be expressed as a composition of the one between e and 0 and the one between e' and 0 (hence would have finite variance if those two had).

Thus, we will attach, to each point e in $]0, 1]$, the holonomy starting at e (and ending at 0), as well as the flux that acts at the onset of this holonomy. It is manifest from the symplectic structure in prop. 18.10 that these pairs of variables attached to the various points in $]0, 1]$ are *not independent* canonically conjugate pairs: instead, the flux at some point e acts on *all* holonomies that start *at or above* e. As announced in section 17, these uncountably many non-zero commutators will play a decisive role in the proof.

Now, suppose that it would be possible to construct a quantum state in which all those holonomies and fluxes would have *finite* variances. Then, the points in $]0, 1]$ could be organized into countably many (overlapping) classes: a point would belong to the class indexed by some $A \in \mathbb{N}$ if the variance of both the holonomy and flux attached to this point would be less than A. The assumption of all those variances being finite would ensure that each point belongs at least to one of those classes, so, as there are *uncountably many* points in $]0, 1]$, there should exist some A whose class $\mathcal{K}$ would be uncountable, hence *infinite*.

Finally given n points in this infinite set $\mathcal{K}$, we will, in the proof of lemma 18.12, combine the variances of the holonomies and fluxes attached to these points into a quadratic expression, that should, if all these variances were bounded by a constant A, be bounded *independently* of n. However, we will show that, by virtue of the Heisenberg uncertainty relations, this quadratic expression can be bounded *below* by a *diverging* expression of n.

Proposition 18.10 We define the label set $\mathcal{L}^{(\mathrm{aux})}$ as the set of all *finite* subsets of points in $]0, 1]$:

$$\mathcal{L}^{(\mathrm{aux})} := \{\kappa \subset\,]0, 1] \mid \#\kappa < \infty\}\,.$$

We equip $\mathcal{L}^{(\mathrm{aux})}$ with the partial order $\subset$ (set inclusion). For any $\kappa = (e_1, \dots, e_n) \in \mathcal{L}^{(\mathrm{aux})}$, with $e_1 < \dots < e_n$, we define:

$$\mathcal{C}^{(\mathrm{aux})}_\kappa := \{h : \kappa \to \mathbb{R}\} \quad \& \quad \mathcal{P}^{(\mathrm{aux})}_\kappa := \{P : \kappa \to \mathbb{R}\}\,,$$

as well as the linear map $\Xi^{(\mathrm{aux})}_\kappa : \mathcal{P}^{(\mathrm{aux})}_\kappa \to \mathcal{C}^{(\mathrm{aux}),*}_\kappa$ given by:

$$\forall P \in \mathcal{P}^{(\text{aux})}_{\kappa}\,,\ \forall h \in \mathcal{C}^{(\text{aux})}_{\kappa}\,,\ \ \Xi^{(\text{aux})}_{\kappa}(P)(h) := \sum_{k=1}^{n} P(e_k)\,h(e_k) - \sum_{k=1}^{n-1} P(e_{k+1})\,h(e_k)\,.$$

$\Xi^{(\text{aux})}_{\kappa}$ is invertible, with $\Xi^{(\text{aux}),-1}_{\kappa}$ such that:

$$\forall s \in \mathcal{C}^{(\text{aux}),*}_{\kappa}\,,\ \forall e \in \kappa,\ \left[\Xi^{(\text{aux}),-1}_{\kappa}(s)\right](e) = s\left(e' \mapsto \begin{cases} 1 & \text{if } e \leqslant e' \\ 0 & \text{else} \end{cases}\right).$$

We equip $\mathcal{M}^{(\text{aux})}_{\kappa} := \mathcal{C}^{(\text{aux})}_{\kappa} \times \mathcal{P}^{(\text{aux})}_{\kappa}$ with the symplectic structure $\Omega^{(\text{aux})}_{\kappa}$ defined from $\Xi^{(\text{aux})}_{\kappa}$ as in prop. 18.1.

For any $\kappa \subset \kappa' \in \mathcal{L}^{(\text{aux})}$, we define:

$$\begin{array}{rccc} Q^{(\text{aux})}_{\kappa'\to\kappa} : & \mathcal{C}^{(\text{aux})}_{\kappa'} & \to & \mathcal{C}^{(\text{aux})}_{\kappa} \\ & h & \mapsto & h\big|_{\kappa} \end{array} \quad \& \quad \begin{array}{rccc} P^{(\text{aux})}_{\kappa'\to\kappa} : & \mathcal{P}^{(\text{aux})}_{\kappa'} & \to & \mathcal{P}^{(\text{aux})}_{\kappa} \\ & P & \mapsto & P\big|_{\kappa} \end{array},$$

and $\pi^{(\text{aux})}_{\kappa'\to\kappa} = Q^{(\text{aux})}_{\kappa'\to\kappa} \times P^{(\text{aux})}_{\kappa'\to\kappa}$. Then, $\left(\mathcal{L}^{(\text{aux})}, \mathcal{M}^{(\text{aux})}, \pi^{(\text{aux})}\right)^{\downarrow}$ is a projective system of phase spaces and fulfills prop. 18.1.1 and 18.1.2. We denote by $\left(\mathcal{L}^{(\text{aux})}, \mathcal{H}^{(\text{aux})}, \Phi^{(\text{aux})}\right)^{\otimes}$ the corresponding projective system of quantum state spaces.

Proof $\mathcal{L}^{(\text{aux})}$ is a directed set, since any two finite subsets of $]0, 1]$ are included in their union and this union is finite. Let $\kappa = (e_1, \ldots, e_n) \in \mathcal{L}^{(\text{aux})}$ with $e_1 < \ldots < e_n$. $\Xi^{(\text{aux})}_{\kappa}$ being invertible can be checked from the expression given for $\Xi^{(\text{aux}),-1}_{\kappa}$ and this ensures that $\Omega^{(\text{aux})}_{\kappa}$ is indeed a symplectic form (aka. an anti-symmetric, non-degenerate form).

For any $\kappa \subset \kappa' \in \mathcal{L}^{(\text{aux})}$, $\pi^{(\text{aux})}_{\kappa'\to\kappa}$ is a surjective map $\mathcal{M}_{\kappa'} \to \mathcal{M}_{\kappa}$ and is compatible with the symplectic structures, for we have:

$$\Xi^{(\text{aux}),-1}_{\kappa} = P^{(\text{aux})}_{\kappa'\to\kappa} \circ \Xi^{(\text{aux}),-1}_{\kappa'} \circ Q^{(\text{aux}),*}_{\kappa'\to\kappa}\,.$$

Moreover, for any $\kappa \subset \kappa' \subset \kappa'' \in \mathcal{L}^{(\text{aux})}$, $\pi^{(\text{aux})}_{\kappa''\to\kappa} = \pi^{(\text{aux})}_{\kappa'\to\kappa} \circ \pi^{(\text{aux})}_{\kappa''\to\kappa'}$. Therefore, $\left(\mathcal{L}^{(\text{aux})}, \mathcal{M}^{(\text{aux})}, \pi^{(\text{aux})}\right)$ is a projective system of phase spaces. Prop. 18.1.1 and 18.1.2 are fulfilled by construction with $\forall \kappa \in \mathcal{L}^{(\text{aux})}$, $L_{\kappa} = \text{id}_{\mathcal{M}_{\kappa}}$. □

Theorem 18.11 With the notations of def. 18.6 and prop. 18.10, $\mathcal{V}^{\downarrow}_{(\text{aux})} := \mathcal{V}^{\downarrow}_{\left(\mathcal{L}^{(\text{aux})}, (\mathcal{C}^{(\text{aux})},\mathcal{P}^{(\text{aux})}), (Q^{(\text{aux})},P^{(\text{aux})})\right)} = \varnothing$. Hence, prop. 18.8 implies that $\hat{\mathcal{S}}^{\otimes}_{(\text{aux})} := \hat{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{(\text{aux})},\mathcal{H}^{(\text{aux})},\Phi^{(\text{aux})})} = \varnothing$.

Lemma 18.12 Let $\mathcal{K}$ be a countably *infinite* subset of $]0, 1]$ and let $A > 0$. We define:

$$\mathcal{V}^{\downarrow}_{(\mathcal{K},A)} := \left\{ (V_{\kappa}, U_{\kappa})_{\kappa} \in \mathcal{V}^{\downarrow}_{(\text{aux})} \;\middle|\; \forall \kappa \subset \mathcal{K},\ \forall e \in \kappa,\ V_{\kappa}(\mathsf{h}^{[e]}_{\kappa}, \mathsf{h}^{[e]}_{\kappa}) \leqslant A \quad \& \quad U_{\kappa}(\mathsf{P}^{[e]}_{\kappa}, \mathsf{P}^{[e]}_{\kappa}) \leqslant A \right\},$$

with $\forall \kappa \in \mathcal{L}^{(\text{aux})}, \forall e \in \kappa,\ \ \mathsf{h}^{[e]}_{\kappa} := \big(h \mapsto h(e)\big) \quad \& \quad \mathsf{P}^{[e]}_{\kappa} := \big(P \mapsto P(e)\big)$. Then, $\mathcal{V}^{\downarrow}_{(\mathcal{K},A)} = \varnothing$.

Proof Proceeding by contradiction, we suppose that there exists $(V_{\kappa}, U_{\kappa})_{\kappa} \in \mathcal{V}^{\downarrow}_{(\mathcal{K},A)}$. Let $n \in \mathbb{N}$. Since $\mathcal{K}$ is infinite, there exist $\kappa = (e_1, \ldots, e_n) \subset \mathcal{K}$ with $e_1 < \ldots < e_n$. Then, we have:

$$\frac{1}{n}\sum_{k=1}^{n} V_{\kappa}\left(\mathsf{h}^{[e_k]}_{\kappa}, \mathsf{h}^{[e_k]}_{\kappa}\right) + U_{\kappa}\left(\mathsf{P}^{[e_k]}_{\kappa}, \mathsf{P}^{[e_k]}_{\kappa}\right) \leqslant 2A.$$

We define the n by n matrix $M^{(n)}$ by:

$$\forall k, l \in \{1, \ldots, n\}\,,\; M^{(n)}_{kl} := \mathrm{P}^{[e_k]}_{\kappa}\left(\Xi^{(\mathrm{aux}),-1}_{\kappa}\big(\mathrm{h}^{[e_l]}_{\kappa}\big)\right) = \begin{cases} 1 & \text{if } k \leqslant l \\ 0 & \text{else} \end{cases}\,.$$

Performing a singular value decomposition of $M^{(n)}$ there exist two n by n orthogonal matrices $O^{(n)}$ and $O'^{(n)}$, and a diagonal matrix $\Delta^{(n)}$ with non-negative real entries, such that:

$$M^{(n)} = O'^{(n)\top}\,\Delta^{(n)}\,O^{(n)}\,,$$

where $(\,\cdot\,)^{\top}$ denotes the transpose matrix. Defining, for any $k \in \{1, \ldots, n\}$, $s_k \in \mathcal{C}^{(\mathrm{aux}),*}_{\kappa}$ and $t_k \in \mathcal{P}^{(\mathrm{aux}),*}_{\kappa}$ by:

$$s_k := \sum_{l=1}^{n} O^{(n)}_{kl}\,\mathrm{h}^{[e_l]}_{\kappa} \quad \& \quad t_k := \sum_{l=1}^{n} O'^{(n)}_{kl}\,\mathrm{P}^{[e_l]}_{\kappa}\,,$$

we get:

$$\frac{1}{n}\sum_{k=1}^{n} V_{\kappa}(s_k\,,\,s_k) + U_{\kappa}(t_k\,,\,t_k) = \frac{1}{n}\sum_{l=1}^{n} V_{\kappa}\left(\mathrm{h}^{[e_l]}_{\kappa}\,,\,\mathrm{h}^{[e_l]}_{\kappa}\right) + U_{\kappa}\left(\mathrm{P}^{[e_l]}_{\kappa}\,,\,\mathrm{P}^{[e_l]}_{\kappa}\right) \leqslant 2A\,.$$

On the other hand, from def. 18.6.2, we have:

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} V_{\kappa}(s_k\,,\,s_k) + U_{\kappa}(t_k\,,\,t_k) &\geqslant \frac{1}{n}\sum_{k=1}^{n} t_k\left(\Xi^{(\mathrm{aux}),-1}_{\kappa}(s_k)\right) \\ &= \frac{1}{n}\mathrm{Tr}\left[O'^{(n)}\,M^{(n)}\,O^{(n)\top}\right] = \frac{1}{n}\mathrm{Tr}\,\Delta^{(n)} = \frac{1}{n}\mathrm{Tr}\,\sqrt{M^{(n)\top}\,M^{(n)}}\,.\end{aligned}$$

Now, $N^{(n)} := \dfrac{1}{n^2}\,M^{(n)\top}\,M^{(n)}$ is given by:

$$\forall k, l \in \{1, \ldots, n\}\,,\; N^{(n)}_{kl} = \frac{1}{n^2}\sum_{m=1}^{n} M^{(n)}_{mk}\,M^{(n)}_{ml} = \frac{1}{n^2}\sum_{m=1}^{\min(k,l)} 1 = \frac{1}{n}\min\left(\frac{k}{n},\,\frac{l}{n}\right),$$

and the previous inequalities requires $\mathrm{Tr}\,\sqrt{N^{(n)}} \leqslant 2\,A$.

On the complex Hilbert space $\mathcal{J} := L_2\left([0,\,1]\,,\,d\mu_{[0,1]}\right)$ (with $d\mu_{[0,1]}$ the usual, normalized measure on $[0,\,1]$), we define, for any $n \in \mathbb{N}$, a bounded operator $N^{[n]}$ by:

$$\begin{aligned}\forall \zeta \in \mathcal{J},\, \forall x \in [0,\,1]\,,\; \left[N^{[n]}\zeta\right](x) &:= \sum_{l=1}^{n} \min\left([x]_n\,,\,\frac{l}{n}\right)\int_{\frac{l-1}{n}}^{\frac{l}{n}} dy\;\zeta(y) \\ &= \int_0^1 dy\;\min\left([x]_n\,,\,[y]_n\right)\;\zeta(y)\,,\end{aligned}$$

with $[x]_n := \dfrac{1}{n}\lceil n\,x\rceil$ and $\lceil\,\cdot\,\rceil$ the ceiling function. For any $m \in \{1, \ldots, n\}$, we define $u^{[n]}_m \in \mathcal{J}$ by:

$$\forall x \in [0,\,1]\,,\; u^{[n]}_m(x) := \sqrt{n}\,O^{(n)}_{m\,\lceil nx\rceil}\,.$$

We have $\left\langle u^{[n]}_m \,\middle|\, u^{[n]}_{m'}\right\rangle_{\mathcal{J}} = \delta_{mm'}$ and, using $N^{(n)} = O^{(n)\top}\left(\dfrac{\Delta^{(n)}}{n}\right)^2 O^{(n)}$:

$$N^{[n]} = \sum_{m=1}^{n} \left(\frac{\Delta^{(n)}_{mm}}{n} \right)^2 \left| u^{[n]}_m \right\rangle \left\langle u^{[n]}_m \right| .$$

Now, we define a bounded operator $N^{[\infty]}$ on $\mathcal{J}$ by:

$$\forall \zeta \in \mathcal{J},\ \forall x \in [0,\, 1]\,,\ \left[N^{[\infty]} \zeta \right](x) = \int_0^1 dy\ \min(x,\ y)\ \zeta(y)$$
$$= \int_0^x dy\ y\,\zeta(y) + x \int_x^1 dy\ \zeta(y)\,,$$

and, for any $m \in \mathbb{N} \setminus \{0\}$, we define $u^{[\infty]}_m \in \mathcal{J}$ by:

$$\forall x \in [0,\, 1]\,,\ u^{[\infty]}_m(x) := \sqrt{2}\, \sin \left[\left(m - \frac{1}{2} \right) \pi\, x \right].$$

We have $\left\langle u^{[\infty]}_m \,\middle|\, u^{[\infty]}_{m'} \right\rangle_{\mathcal{J}} = \delta_{mm'}$ and:

$$N^{[\infty]} u^{[\infty]}_m = \frac{4}{\pi^2 \,(2m-1)^2}\, u^{[\infty]}_m\,.$$

Since $\displaystyle\sum_{m=1}^{\infty} \frac{2}{\pi\,(2m-1)} = \infty$, there exists $M \in \mathbb{N} \setminus \{0\}$ such that:

$$\sum_{m=1}^{M} \frac{2}{\pi\,(2m-1)} > 2A + 1\,.$$

Let $n \in \mathbb{N}$ such that $n > 8\,\pi^2\, M^3$ and complete $\left(u^{[n]}_m \right)_{m \in \{1,\ldots,n\}}$ into an orthonormal basis $\left(u^{[n]}_m \right)_{m \in \mathbb{N} \setminus \{0\}}$ of $\mathcal{J}$. For any $m \in \{1, \ldots, M\}$, we have:

$$\left\| N^{[\infty]}\, u^{[\infty]}_m - N^{[n]}\, u^{[\infty]}_m \right\| \leqslant \left\| N^{[\infty]} - N^{[n]} \right\| \, \left\| u^{[\infty]}_m \right\| \leqslant \frac{1}{n}\,.$$

But we also have:

$$\left\| N^{[\infty]}\, u^{[\infty]}_m - N^{[n]}\, u^{[\infty]}_m \right\|^2 = \frac{1}{n^4} \sum_{m'=1}^{\infty} \left| \left(\Delta^{(\infty)}_{mm} \right)^2 - \left(\Delta^{(n)}_{m'm'} \right)^2 \right|^2 \, \left| \left\langle u^{[n]}_{m'} \,\middle|\, u^{[\infty]}_m \right\rangle_{\mathcal{J}} \right|^2$$

where $\Delta^{(\infty)}_{mm} := \dfrac{2\,n}{\pi\,(2m-1)}$ and, for any $m' > n$, $\Delta^{(n)}_{m'm'} := 0$. Thus, we get:

$$\frac{1}{n^4} \left[\inf_{m' \in \mathbb{N} \setminus \{0\}} \left| \left(\Delta^{(\infty)}_{mm} \right)^2 - \left(\Delta^{(n)}_{m'm'} \right)^2 \right| \right]^2 \sum_{m'=1}^{\infty} \left| \left\langle u^{[n]}_{m'} \,\middle|\, u^{[\infty]}_m \right\rangle_{\mathcal{J}} \right|^2 \leqslant \frac{1}{n^2}\,,$$

and therefore:

$$\inf_{m' \in \mathbb{N} \setminus \{0\}} \left| \left(\Delta^{(\infty)}_{mm} \right)^2 - \left(\Delta^{(n)}_{m'm'} \right)^2 \right| \leqslant n\,.$$

Hence, for any $m \in \{1, \ldots, M\}$, there exists $\widetilde{m} \in \mathbb{N} \setminus \{0\}$, such that:

$$\left| \frac{4}{\pi^2 (2m-1)^2} - \left(\frac{\Delta^{(n)}_{\widetilde{m}\widetilde{m}}}{n} \right)^2 \right| \leqslant \frac{2}{n} \,.$$

Using $\Delta^{(n)}_{\widetilde{m}\widetilde{m}} \geqslant 0$ and $n > 8\,\pi^2 M^3$, this implies:

$$\left| \frac{2}{\pi\,(2m-1)} - \frac{\Delta^{(n)}_{\widetilde{m}\widetilde{m}}}{n} \right| = \left| \frac{4}{\pi^2 (2m-1)^2} - \left(\frac{\Delta^{(n)}_{\widetilde{m}\widetilde{m}}}{n} \right)^2 \right| \left| \frac{2}{\pi\,(2m-1)} + \frac{\Delta^{(n)}_{\widetilde{m}\widetilde{m}}}{n} \right|^{-1} \leqslant \frac{1}{4\,\pi\,M^2} \,.$$

On the other hand, for any $m' > n$, we have:

$$\left| \frac{2}{\pi\,(2m-1)} - \frac{\Delta^{(n)}_{m'm'}}{n} \right| \geqslant \frac{1}{\pi\,M} \geqslant \frac{1}{\pi\,M^2} \,,$$

so $\widetilde{m} \leqslant n$. Next, for any $m \neq m' \in \{1, \dots, M\}$, the inequality:

$$\left| \frac{2}{\pi\,(2m-1)} - \frac{2}{\pi\,(2m'-1)} \right| \geqslant \frac{1}{\pi\,M^2}$$

holds, so $\widetilde{m} \neq \widetilde{m}'$. Therefore, we obtain:

$$\sum_{\widetilde{m}=1}^{n} \frac{\Delta^{(n)}_{\widetilde{m}\widetilde{m}}}{n} \geqslant \sum_{m=1}^{M} \frac{\Delta^{(n)}_{\widetilde{m}\widetilde{m}}}{n} \geqslant \sum_{m=1}^{M} \frac{2}{\pi\,(2m-1)} - \frac{1}{4\,\pi\,M} > 2A \,,$$

in contradiction with $\frac{1}{n} \operatorname{Tr} \Delta^{(n)} \leqslant 2A$, whence the initial assumption on the existence of $(V_\kappa,\, U_\kappa)_\kappa \in \mathcal{V}^{\downarrow}_{(\mathcal{K},A)}$ is proven wrong. □

Proof of theorem 18.11 Again, we proceed by contradiction and we suppose that there exists $(V_\kappa\,,\, U_\kappa)_\kappa \in \mathcal{V}^{\downarrow}_{(\text{aux})}$. For any $e \in\,]0,\, 1]$, we define $V^{[e]} := V_\kappa \left(\mathsf{h}^{[e]}_\kappa,\, \mathsf{h}^{[e]}_\kappa \right)$ and $U^{[e]} := U_\kappa \left(\mathsf{P}^{[e]}_\kappa,\, \mathsf{P}^{[e]}_\kappa \right)$ for some $\kappa \in \mathcal{L}^{(\text{aux})}$ such that $e \in \kappa$. If $\kappa' \in \mathcal{L}^{(\text{aux})}$ is such that $e \in \kappa'$, then $\kappa, \kappa' \subset \kappa \cup \kappa' \in \mathcal{L}^{(\text{aux})}$ and:

$$\begin{aligned} V_{\kappa'} \left(\mathsf{h}^{[e]}_{\kappa'},\, \mathsf{h}^{[e]}_{\kappa'} \right) &= V_{\kappa\cup\kappa'} \left(Q^{(\text{aux}),*}_{\kappa\cup\kappa'\to\kappa'} \left(\mathsf{h}^{[e]}_{\kappa'} \right),\, Q^{(\text{aux}),*}_{\kappa\cup\kappa'\to\kappa'} \left(\mathsf{h}^{[e]}_{\kappa'} \right) \right) = V_{\kappa\cup\kappa'} \left(\mathsf{h}^{[e]}_{\kappa\cup\kappa'},\, \mathsf{h}^{[e]}_{\kappa\cup\kappa'} \right) \\ &= V_\kappa \left(\mathsf{h}^{[e]}_\kappa,\, \mathsf{h}^{[e]}_\kappa \right) = V^{[e]} . \end{aligned}$$

Similarly, $U_{\kappa'} \left(\mathsf{P}^{[e]}_{\kappa'},\, \mathsf{P}^{[e]}_{\kappa'} \right) = U^{[e]}$. Now, we have:

$$]0,\, 1] = \bigcup_{A=1}^{\infty} \left\{ e \in\,]0,\, 1] \,\middle|\, V^{[e]} \leqslant A \quad \& \quad U^{[e]} \leqslant A \right\} ,$$

and since $]0,\, 1]$ is *uncountably* infinite, there exists $A \in \mathbb{N} \setminus \{0\}$ such that:

$$\left\{ e \in\,]0,\, 1] \,\middle|\, V^{[e]} \leqslant A \quad \& \quad U^{[e]} \leqslant A \right\}$$

is infinite, hence contains some countably infinite subset $\mathcal{K}$. Then, we have $(V_\kappa\,,\, U_\kappa)_\kappa \in \mathcal{V}^{\downarrow}_{(\mathcal{K},A)}$, but this is impossible since $\mathcal{V}^{\downarrow}_{(\mathcal{K},A)} = \varnothing$ from the previous lemma. □

Proposition 18.13 Let $\mathcal{L}_{\text{HF}}$ be the directed label set defined in def. 10.12. For any $\eta \in \mathcal{L}_{\text{HF}}$, let

$\mathcal{C}^{\mathbb{R}}_\eta := \{h : \gamma(\eta) \to \mathbb{R}\}$, $\mathcal{P}^{\mathbb{R}}_\eta := \{P : \mathcal{F}(\eta) \to \mathbb{R}\}$ (with $\gamma(\eta)$ and $\mathcal{F}(\eta)$ as in def. 10.12), $\mathcal{M}^{\mathbb{R}}_\eta := T^*(\mathcal{C}^{\mathbb{R}}_\eta)$ and, for any $\eta \preccurlyeq \eta' \in \mathcal{L}_{\text{HF}}$, let $\pi^{\mathbb{R}}_{\eta' \to \eta} : \mathcal{M}^{\mathbb{R}}_{\eta'} \to \mathcal{M}^{\mathbb{R}}_\eta$ be defined as in prop. 10.26 in the special case $G = (\mathbb{R}, +)$. Then, $\left(\mathcal{L}_{\text{HF}}, \mathcal{M}^{\mathbb{R}}, \pi^{\mathbb{R}}_{\eta' \to \eta}\right)^{\downarrow}$ is a projective system of phase spaces fulfilling the hypotheses of prop. 18.1, with:

1. $\forall \eta \in \mathcal{L}_{\text{HF}}, \forall P \in \mathcal{P}^{\mathbb{R}}_\eta, \ \Xi^{\mathbb{R}}_\eta(P) : h \mapsto \sum_{e \in \gamma(\eta)} P \circ \chi_\eta(e)\ h(e)$ (with χ_η as in def. 10.12);

2. $\forall \eta \in \mathcal{L}_{\text{HF}}, \forall (h, p) \in \mathcal{M}^{\mathbb{R}}_\eta, \ L^{\mathbb{R}}_\eta(h, p) := \left(h,\ F \mapsto p\left(\delta_{\chi_\eta^{-1}(F)}\right)\right)$ (with, for any $F \in \mathcal{F}(\eta)$ and any $e \in \gamma(\eta)$, $\delta_{\chi_\eta^{-1}(F)}(e) = 1$ if $\chi_\eta(e) = F$, and 0 otherwise);

3. $\forall \eta \preccurlyeq \eta' \in \mathcal{L}_{\text{HF}}, \forall h_{\eta'} \in \mathcal{C}^{\mathbb{R}}_{\eta'}, \ Q^{\mathbb{R}}_{\eta' \to \eta}(h_{\eta'}) : e \mapsto \sum_{k=1}^{n_{\eta' \to \eta, e}} \epsilon_{\eta' \to \eta, e}(k)\ h_{\eta'} \circ a_{\eta' \to \eta, e}(k)$ (with the notations of prop. 10.23);

4. $\forall \eta \preccurlyeq \eta' \in \mathcal{L}_{\text{HF}}, \forall P_{\eta'} \in \mathcal{P}^{\mathbb{R}}_{\eta'}, \ P^{\mathbb{R}}_{\eta' \to \eta}(P_{\eta'}) : F \mapsto \sum_{F' \in H^{(1,3)}_{\eta' \to \eta, F}} P_{\eta'}(F')$ (with $H^{(1,3)}_{\eta' \to \eta, F}$ also from prop. 10.23).

The projective system of quantum state spaces $\left(\mathcal{L}_{\text{HF}}, \mathcal{H}^{\mathbb{R}}, \Phi^{\mathbb{R}}\right)^{\otimes}$ provided by prop. 12.1 (in the special case $G = (\mathbb{R}, +)$) can be identified with the one provided by prop. 18.1. Moreover, for any $\eta \in \mathcal{L}_{\text{HF}}$ and any $(s, t) \in \mathcal{C}^{\mathbb{R},*}_\eta \times \mathcal{P}^{\mathbb{R},*}_\eta$,

$$\mathsf{X}^{\mathbb{R}}_\eta(s, t) = \sum_{e \in \gamma(\eta)} s(\delta_e)\ \widehat{\mathsf{h}^{(e, \text{id}_{\mathbb{R}})}_\eta} - \sum_{F \in \mathcal{F}(\eta)} t(\delta_F)\ \widehat{\mathsf{P}^{(\overline{F}, 1)}_\eta},$$

using the densely defined, essentially self-adjoint operators introduced in eqs. (12.2.1) and (12.2.2) (the minus sign is the result of conflicting conventions in props. B.14 and C.5).

Proof *Assertions 18.1.1 and 18.1.2.* Let $\eta \in \mathcal{L}_{\text{HF}}$. $\Xi^{\mathbb{R}}_\eta$ is an invertible linear map $\mathcal{P}_\eta \to \mathcal{C}^*_\eta$ with:

$$\forall p \in \mathcal{C}^*_\eta,\ \Xi^{\mathbb{R},-1}_\eta(p) : F \mapsto p\left(\delta_{\chi_\eta^{-1}(F)}\right).$$

Next, $L^{\mathbb{R}}_\eta$ is an invertible linear map $\mathcal{M}^{\mathbb{R}}_\eta \to \mathcal{C}^{\mathbb{R}}_\eta \times \mathcal{P}^{\mathbb{R}}_\eta$, with:

$$\forall (h, P) \in \mathcal{C}^{\mathbb{R}}_\eta \times \mathcal{P}^{\mathbb{R}}_\eta,\ L^{\mathbb{R},-1}_\eta(h, P) = \left(h,\ h' \mapsto \sum_{e \in \gamma(\eta)} P \circ \chi_\eta(e)\ h'(e)\right).$$

Let $\Omega^{\mathbb{R}}_\eta$ be the symplectic structure on $\mathcal{C}^{\mathbb{R}}_\eta \times \mathcal{P}^{\mathbb{R}}_\eta$ defined by:

$$\forall h, h' \in \mathcal{C}_\eta,\ \forall P, P' \in \mathcal{P}_\eta,\ \ \Omega^{\mathbb{R}}_\eta(h, P;\ h', P') := \Xi^{\mathbb{R}}_\eta(P')(h) - \Xi^{\mathbb{R}}_\eta(P)(h').$$

We have:

$$\forall (h, p), (h', p') \in \mathcal{M}^{\mathbb{R}}_\eta,\ \left(L^{\mathbb{R},*}_\eta \Omega^{\mathbb{R}}_\eta\right)(h, p;\ h', p') = p'(h) - p(h').$$

Hence, $L^{\mathbb{R}}_\eta$ is a symplectomorphism (with respect to the canonical symplectic structure on $\mathcal{M}^{\mathbb{R}}_\eta = T^*(\mathcal{C}^{\mathbb{R}}_\eta)$, see eq. (2.16.1)). Moreover, it coincides with the map defined in prop. 10.26 in the case $G = (\mathbb{R}, +)$.

Let $\eta \preccurlyeq \eta' \in \mathcal{L}_{\text{HF}}$. Using prop. 10.26 we have, for any $h_{\eta'}, P_{\eta'} \in \mathcal{C}^{\mathbb{R}}_{\eta'} \times \mathcal{P}^{\mathbb{R}}_{\eta'}$:

$$L^{\mathbb{R}}_{\eta} \circ \pi^{\mathbb{R}}_{\eta' \to \eta} \circ L^{\mathbb{R},-1}_{\eta'}(h_{\eta'}, P_{\eta'}) = \left(Q^{\mathbb{R}}_{\eta' \to \eta}(h_{\eta'}),\, P^{\mathbb{R}}_{\eta' \to \eta}(P_{\eta'})\right).$$

Projective of quantum state spaces. For any $\eta \preccurlyeq \eta' \in \mathcal{L}_{\text{HF}}$, let $\mathcal{C}^{\mathbb{R}}_{\eta' \to \eta}$ and $\varphi^{\mathbb{R}}_{\eta' \to \eta}$, resp. $\mathcal{C}_{\eta' \to \eta}$ and $\varphi_{\eta' \to \eta}$, be defined as in prop. 10.24 (with $G = (\mathbb{R}, +)$), resp. as in prop. 18.1. It follows from the uniqueness part of prop. 2.10 and from the connectedness of $\mathcal{M}_{\eta'}$ that the tangential lifts of the maps $\varphi^{\mathbb{R}}_{\eta' \to \eta}$ and $\varphi_{\eta' \to \eta}$ must coincide modulo a suitable symplectomorphic identification of the cotangent bundles $T^*(\mathcal{C}^{\mathbb{R}}_{\eta' \to \eta})$ and $T^*(\mathcal{C}_{\eta' \to \eta})$. Hence, the maps $\varphi^{\mathbb{R}}_{\eta' \to \eta}$ and $\varphi_{\eta' \to \eta}$ themselves coincide modulo a diffeomorphic identification of the manifolds $\mathcal{C}^{\mathbb{R}}_{\eta' \to \eta}$ and $\mathcal{C}_{\eta' \to \eta}$. Therefore, the associated projective systems of quantum state spaces coincide modulo a unitary identification of the Hilbert spaces $\mathcal{H}^{\mathbb{R}}_{\eta' \to \eta} := L_2(\mathcal{C}^{\mathbb{R}}_{\eta' \to \eta})$ and $\mathcal{H}_{\eta' \to \eta} := L_2(\mathcal{C}_{\eta' \to \eta})$.

Observables. Let $(s, t) \in \mathcal{C}^{\mathbb{R},*}_{\eta} \times \mathcal{P}^{\mathbb{R},*}_{\eta}$ and $\psi \in \mathcal{D}^{\mathbb{R}}_{\eta} := C^{\infty}_{o}(\mathcal{C}^{\mathbb{R}}_{\eta}, \mathbb{C})$. For any $h \in \mathcal{C}^{\mathbb{R}}_{\eta}$, we have:

$$\begin{aligned}\left[\mathsf{X}^{\mathbb{R}}_{\eta}(s, t)\, \psi\right](h) &:= s(h)\, \psi(h) - i\, [T_h \psi]\left(\Xi^{\mathbb{R},*,-1}_{\eta}(t)\right) \\ &= \sum_{e \in \gamma(\eta)} s(\delta_e)\, h(e)\, \psi(h) - \sum_{F \in \mathcal{F}(\eta)} i\, t(\delta_F)\, [T_h \psi](\delta_{\chi_\eta^{-1}(F)}),\end{aligned}$$

where we have used that, for any $F \in \mathcal{F}(\eta)$, $\Xi^{\mathbb{R},*,-1}_{\eta}(t)(\chi_\eta^{-1}(F)) = t(\delta_F)$. Now, specializing the definitions from prop. 12.2 to the case $G = (\mathbb{R}, +)$, this can be rewritten as:

$$\left[\mathsf{X}^{\mathbb{R}}_{\eta}(s, t)\, \psi\right](h) = \sum_{e \in \gamma(\eta)} s(\delta_e) \left[\widehat{\mathsf{h}^{(e, \mathrm{id}_{\mathbb{R}})}_{\eta}}\, \psi\right](h) - \sum_{F \in \mathcal{F}(\eta)} t(\delta_F) \left[\widehat{\mathsf{P}^{(\overline{F}, 1)}_{\eta}}\, \psi\right](h).$$

□

Proposition 18.14 With the notations of def. 18.5 and prop. 18.13, $\hat{\mathcal{S}}^{\otimes}_{(\mathcal{L}_{\text{HF}}, \mathcal{H}^{\mathbb{R}}, \Phi^{\mathbb{R}})} = \varnothing$.

Proof *Directed pre-order on $\mathcal{L}_{\text{HF}} \sqcup \mathcal{L}^{(\text{aux})}$.* Let $\Psi : U \to V$ be an analytical coordinate patch on Σ, with U an open neighborhood of 0 in $\mathbb{R}^d$ (recall that $d := \dim(\Sigma) \geqslant 1$). Let ϵ such that $B^{(d)}_{\epsilon} \subset U$. For any $\tau \neq \tau' \in [0, 2]$ we define:

$$\begin{array}{rrcl} \breve{e}_{\tau,\tau'} : & U_{\tau,\tau'} \subset \mathbb{R}^d \approx \mathbb{R} \times \mathbb{R}^{d-1} & \to & V \\ & (x, y) & \mapsto & \Psi\left(\frac{\epsilon}{2}\left(\tau + x\,(\tau' - \tau)\right), \frac{\epsilon}{2}\, y\right) \end{array},$$

where $U_{\tau,\tau'} = \left\{ (x, y) \in \mathbb{R}^d \approx \mathbb{R} \times \mathbb{R}^{d-1} \,\middle|\, \left(\tau + x\,(\tau' - \tau),\, y\right) \in \dfrac{2}{\epsilon}\, U \right\}$ is an open neighborhood of $[0, 1] \times \{0\}^{d-1}$ in $\mathbb{R}^d$. We denote by $e_{\tau,\tau'}$ be the corresponding edge. Next, for any $\tau \in [0, 1]$ we define $\breve{S}_\tau \equiv \breve{e}_{\tau,2}$ and we note that $U_{\tau,2}$ is an open neighborhood of $\{0\} \times B^{(d-1)}$ in $\mathbb{R}^d$. We denote by S_τ the corresponding surface.

Let $\kappa = (\tau_1, \ldots, \tau_n) \in \mathcal{L}^{(\text{aux})}$, with $0 < \tau_1 < \ldots < \tau_n \leqslant 1$. We define (fig. 18.1):

$$\gamma_\kappa := \left\{ e_{\tau_i, \tau_{i-1/2}} \,\middle|\, i \in \{1, \ldots, n\} \right\} \cup \left\{ e_{\tau_i, \tau_{i+1/2}} \,\middle|\, i \in \{1, \ldots, n\} \right\} \in \mathcal{L}_{\text{graphs}},$$

where $\tau_{1/2} := 0$, $\tau_{n+1/2} := 2$ and, for any $i \in \{1, \ldots, n-1\}$, $\tau_{i+1/2} := \frac{1}{2}(\tau_i + \tau_{i+1})$. We also define:

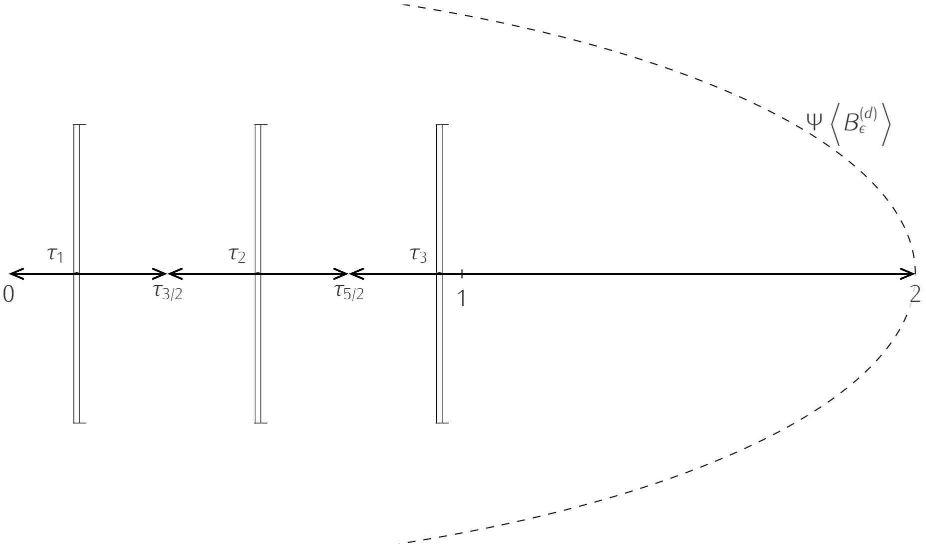


Figure 18.1 – Constructing η_κ

$$\lambda_\kappa := \Big[\,\{S_\tau \mid \tau \in \kappa\}\,\Big]_\sim \in \mathcal{L}_{\text{profls}}\,,$$

where $\sim$ denotes the equivalence relation from def. 10.10. Since, for any $\tau \neq \tau'$, $r(S_\tau) \cap r(S_{\tau'}) = \varnothing$, we have:

$$\mathcal{F}(\lambda_\kappa) = \left\{F^\tau_\diamond(\kappa) \,\middle|\, \tau \in \kappa,\ \diamond \in \{\uparrow, \downarrow\}\right\},$$

where, for any $\tau \in \kappa$ and any $\diamond \in \{\uparrow, \downarrow\}$:

$$F^\tau_\diamond(\kappa) := \{e \in \mathcal{L}_{\text{edges}} \mid e \diamond S_\tau \ \ \&\ \ \forall \tau' \in \kappa \setminus \{\tau\}\,,\ e \looparrowright S_{\tau'}\}\,.$$

Thus, $\eta_\kappa := (\gamma_\kappa\,,\,\lambda_\kappa) \in \mathcal{L}_{\text{HF}}$ with:

$$\forall i \in \{1, \dots, n\}\ ,\ \chi_{\eta_\kappa}(e_{\tau_i\,,\,\tau_{i-1/2}}) = F^{\tau_i}_\downarrow(\kappa) \quad \& \quad \chi_{\eta_\kappa}(e_{\tau_i\,,\,\tau_{i+1/2}}) = F^{\tau_i}_\uparrow(\kappa)\,.$$

Moreover, for any $i \in \{1, \dots, n\}$, we have:

$$e_{\tau_i\,,0} = e_{\tau_1\,,\,\tau_{1/2}} \circ e^{-1}_{\tau_1\,,\,\tau_{3/2}} \circ \dots \circ e_{\tau_i\,,\,\tau_{i-1/2}}\,,$$

$$\text{and}\ \ \overline{F^{\tau_i}_\downarrow(\kappa)} = \overline{F^{\tau_i}} \in \mathcal{L}_{\text{faces}}\,,\ \text{with}\ \ F^{\tau_i} := \left\{e \in \mathcal{L}_{\text{edges}} \,\middle|\, e \downarrow S_{\tau_i}\right\},$$

where, for $F \subset \mathcal{L}_{\text{edges}}$, $F^\perp$ and $\overline{F} := F^\perp \circ F$ are defined as in prop. 10.21, and the equality $\overline{F^{\tau_i}_\downarrow(\kappa)} = \overline{F^{\tau_i}}$ can be proved using prop. 10.9.4.

Now, let $\mathcal{L} := \mathcal{L}_{\text{HF}} \sqcup \mathcal{L}^{(\text{aux})}$ and extend the pre-orders on $\mathcal{L}_{\text{HF}}$ and $\mathcal{L}^{(\text{aux})}$ with:

$$\forall \eta = (\gamma, \lambda) \in \mathcal{L}_{\text{HF}},\ \forall \kappa \in \mathcal{L}^{(\text{aux})},$$
$$\kappa \preccurlyeq \eta \Leftrightarrow \left(\forall \tau \in \kappa,\ \gamma \in \mathcal{L}_{\text{graphs}/e_{\tau,0}} \ \ \&\ \ \lambda \in \mathcal{L}_{\text{profls}/\overline{F^\tau}}\right),$$

where $\mathcal{L}_{\text{graphs}/e}$ for $e \in \mathcal{L}_{\text{edges}}$, resp. $\mathcal{L}_{\text{profls}/\overline{F}}$ for $\overline{F} \in \mathcal{L}_{\text{faces}}$, has been defined in prop. 10.20, resp. in prop. 10.21. To check that this defines a pre-order $\mathcal{L}$, we note that:

$$\forall e \in \mathcal{L}_{\text{edges}},\ \forall \gamma \preccurlyeq \gamma' \in \mathcal{L}_{\text{graphs}}\,,\ \ \gamma \in \mathcal{L}_{\text{graphs}/e} \Rightarrow \gamma' \in \mathcal{L}_{\text{graphs}/e}\,,$$
$$\forall \overline{F} \in \mathcal{L}_{\text{faces}},\ \forall \lambda \preccurlyeq \lambda' \in \mathcal{L}_{\text{profls}}\,,\ \ \lambda \in \mathcal{L}_{\text{profls}/\overline{F}} \Rightarrow \lambda' \in \mathcal{L}_{\text{profls}/\overline{F}}\,.$$

Since, for any $\kappa \in \mathcal{L}^{(\text{aux})}$, $\kappa \preccurlyeq \eta_\kappa$ with η_κ the label constructed above, $\mathcal{L}_{\text{HF}}$ is cofinal in $\mathcal{L}$ and, in particular, $\mathcal{L}$ is directed.

Projective system on $\mathcal{L}$. For any $\kappa \in \mathcal{L}^{(\text{aux})}$ and any $\eta = (\gamma, \lambda) \in \mathcal{L}_{\text{HF}}$ such that $\kappa \preccurlyeq \eta$, we define:

$$\begin{array}{rcrcl} Q_{\eta\to\kappa} & : & \mathcal{C}^{\mathbb{R}}_\eta & \to & \mathcal{C}^{(\text{aux})}_\kappa \\ & & h & \mapsto & \left[\tau \mapsto \sum_{k=1}^{n_{\gamma\to e_{\tau,0}}} \epsilon_{\gamma\to e_{\tau,0}}(k)\; h \circ a_{\gamma\to e_{\tau,0}}(k)\right] \end{array},$$

as well as:

$$\begin{array}{rcrcl} P_{\eta\to\kappa} & : & \mathcal{P}^{\mathbb{R}}_\eta & \to & \mathcal{P}^{(\text{aux})}_\kappa \\ & & P & \mapsto & \left[\tau \mapsto \sum_{F' \in H_{\lambda\to\overline{F^\tau}}} P(F')\right] \end{array},$$

where $n_{\gamma\to e_{\tau,0}}$, $\epsilon_{\gamma\to e_{\tau,0}}$ and $a_{\gamma\to e_{\tau,0}}$ have been defined in prop. 10.20, and $H_{\lambda\to\overline{F^\tau}}$ in prop. 10.21.

Defining, for any $\kappa \in \mathcal{L}^{(\text{aux})}$ and any $\eta \in \mathcal{L}_{\text{HF}}$, $\pi_{\eta\to\kappa} : \mathcal{M}^{\mathbb{R}}_\eta \to \mathcal{M}^{(\text{aux})}_\kappa$ as:

$$\pi_{\eta\to\kappa} = \left(Q_{\eta\to\kappa} \times P_{\eta\to\kappa}\right) \circ L^{\mathbb{R}}_\eta\,,$$

and specializing the definitions from prop. 10.27 to the case $G = (\mathbb{R}, +)$, we have:

$$\forall (h,\, p) \in \mathcal{M}^{\mathbb{R}}_\eta\,,\; \pi_{\eta\to\kappa}(h,\, p) = \left(\tau \mapsto \mathrm{h}_\eta^{(e_{\tau,0}\,,\, \mathrm{id}_{\mathbb{R}})}(h,\, p),\; \tau \mapsto \mathrm{P}_\eta^{(\overline{F^\tau},1)}(h,\, p)\right).$$

For any $\kappa \preccurlyeq \kappa' \in \mathcal{L}^{(\text{aux})}$ and any $\eta \in \mathcal{L}_{\text{HF}}$ with $\kappa' \preccurlyeq \eta$, we thus have, using the definition of $\pi^{(\text{aux})}_{\kappa'\to\kappa}$ from prop. 18.10:

$$\pi_{\eta\to\kappa} = \pi^{(\text{aux})}_{\kappa'\to\kappa} \circ \pi_{\eta\to\kappa'}\,.$$

Moreover, for any $\kappa \in \mathcal{L}^{(\text{aux})}$ and any $\eta \preccurlyeq \eta' \in \mathcal{L}_{\text{HF}}$, with $\kappa \preccurlyeq \eta$, we have shown in prop. 10.27 that:

$$\forall \tau \in \kappa,\; \mathrm{h}_\eta^{(e_{\tau,0}\,,\,\mathrm{id}_{\mathbb{R}})} \circ \pi^{\mathbb{R}}_{\eta'\to\eta} = \mathrm{h}_{\eta'}^{(e_{\tau,0}\,,\,\mathrm{id}_{\mathbb{R}})} \quad \& \quad \mathrm{P}_\eta^{(\overline{F^\tau},1)} \circ \pi^{\mathbb{R}}_{\eta'\to\eta} = \mathrm{P}_{\eta'}^{(\overline{F^\tau},1)}\,,$$

therefore, we also have:

$$\pi_{\eta'\to\kappa} = \pi_{\eta\to\kappa} \circ \pi^{\mathbb{R}}_{\eta'\to\eta}\,.$$

Let $\kappa = (\tau_1\,, \ldots, \tau_n) \in \mathcal{L}^{(\text{aux})}$ with $0 < \tau_1 < \ldots < \tau_n \leqslant 1$ and let $\eta_\kappa = (\gamma_\kappa\,, \lambda_\kappa)$ be constructed as above. We have:

$$\forall h \in \mathcal{C}^{\mathbb{R}}_{\eta_\kappa}\,,\, \forall \tau_j \in \kappa,\;\; \left[Q_{\eta_\kappa\to\kappa}(h)\right](\tau_j) = \sum_{i=1}^{j} h(e_{\tau_i,\tau_{i-1/2}}) - \sum_{i=1}^{j-1} h(e_{\tau_i,\tau_{i+1/2}})\,,$$

as well as:

$$\forall P \in \mathcal{P}^{\mathbb{R}}_{\eta_\kappa}\,,\, \forall \tau_i \in \kappa,\;\; \left[P_{\eta_\kappa\to\kappa}(P)\right](\tau_i) = P\left(F^{\tau_i}_{\downarrow}(\kappa)\right).$$

Inserting the expression for $\Xi^{\mathbb{R},-1}_{\eta_\kappa}$ from the proof of prop. 18.13, we get, for any $s \in \mathcal{C}^{(\text{aux}),*}_\kappa$:

$$P_{\eta_\kappa\to\kappa} \circ \Xi^{\mathbb{R},-1}_{\eta_\kappa} \circ Q^*_{\eta_\kappa\to\kappa}(s) : \tau_i \mapsto s \circ Q_{\eta_\kappa\to\kappa}\left(\delta_{\chi^{-1}_{\eta_\kappa}\left(F^{\tau_i}_{\downarrow}(\kappa)\right)}\right) = s\left(\tau_j \mapsto \sum_{k=1}^{j} \delta_{ik}\right).$$

Thus, $P_{\eta_\kappa\to\kappa} \circ \Xi^{\mathbb{R},-1}_{\eta_\kappa} \circ Q^*_{\eta_\kappa\to\kappa} = \Xi^{(\text{aux}),-1}_\kappa$, in other words $\pi_{\eta_\kappa\to\kappa}$ is a projection compatible with the symplectic structures. Then, for any $\eta \in \mathcal{L}_{\text{HF}}$ such that $\eta \succcurlyeq \kappa$, there exists $\eta' \succcurlyeq \eta, \eta_\kappa$ such that

$\pi_{\eta\to\kappa} \circ \pi^{\mathbb{R}}_{\eta'\to\eta} = \pi_{\eta'\to\kappa} = \pi_{\eta_\kappa\to\kappa} \circ \pi^{\mathbb{R}}_{\eta'\to\eta_\kappa}$, so $\pi_{\eta\to\kappa}$ is a projection compatible with the symplectic structures.

Thus, we can combine $\left(\mathcal{L}^{(\text{aux})}, \mathcal{M}^{(\text{aux})}, \pi^{(\text{aux})}\right)^{\downarrow}$ and $\left(\mathcal{L}_{\text{HF}}, \mathcal{M}^{\mathbb{R}}, \pi^{\mathbb{R}}\right)^{\downarrow}$ into a projective system of phase spaces on $\mathcal{L}$, fulfilling prop. 18.1.1 and 18.1.2. We denote by $(\mathcal{L}, \mathcal{H}, \Phi)^{\otimes}$ the corresponding projective system of quantum state spaces.

Mapping narrow states on $\mathcal{L}_{\text{HF}}$ to narrow states on $\mathcal{L}^{(\text{aux})}$. Applying prop. 5.6 twice, to go from $\mathcal{L}_{\text{HF}}$ to $\mathcal{L}$ (using that $\mathcal{L}_{\text{HF}}$ is cofinal in $\mathcal{L}$) and from $\mathcal{L}$ to $\mathcal{L}^{(\text{aux})}$, there exist a map $\sigma : \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}_{\text{HF}}, \mathcal{H}^{\mathbb{R}}, \Phi^{\mathbb{R}})} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{(\text{aux})}, \mathcal{H}^{(\text{aux})}, \Phi^{(\text{aux})})}$ and a map $\alpha : \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{(\text{aux})}, \mathcal{H}^{(\text{aux})}, \Phi^{(\text{aux})})} \to \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}_{\text{HF}}, \mathcal{H}^{\mathbb{R}}, \Phi^{\mathbb{R}})}$ such that:

$$\forall \rho \in \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}_{\text{HF}}, \mathcal{H}^{\mathbb{R}}, \Phi^{\mathbb{R}})},\ \forall A \in \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{(\text{aux})}, \mathcal{H}^{(\text{aux})}, \Phi^{(\text{aux})})},\quad \text{Tr}\big(\rho\, \alpha(A)\big) = \text{Tr}\big(\sigma(\rho)\, A\big)\,.$$

Moreover, for any $\kappa \in \mathcal{L}^{(\text{aux})}$ and any $(s, t) \in \mathcal{C}^{(\text{aux}),*}_{\kappa} \times \mathcal{P}^{(\text{aux}),*}_{\kappa}$, we have from eq. (18.3.1):

$$\begin{aligned}\alpha\left(\left[\text{T}^{(\text{aux})}_{\kappa}(s, t)\right]_{\sim,\mathcal{L}^{(\text{aux})}}\right) &= \left[\Phi^{-1}_{\eta_\kappa\to\kappa} \circ \left(\text{id}_{\mathcal{H}_{\eta_\kappa\to\kappa}} \otimes \text{T}^{(\text{aux})}_{\kappa}(s, t)\right) \circ \Phi_{\eta_\kappa\to\kappa}\right]_{\sim,\mathcal{L}_{\text{HF}}} \\ &= \left[\text{T}^{\mathbb{R}}_{\eta_\kappa}\left(s \circ Q_{\eta_\kappa\to\kappa}\,,\ t \circ P_{\eta_\kappa\to\kappa}\right)\right]_{\sim,\mathcal{L}_{\text{HF}}}\,.\end{aligned}$$

Thus, for any $\rho \in \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}_{\text{HF}}, \mathcal{H}^{\mathbb{R}}, \Phi^{\mathbb{R}})}$ and any $\kappa \in \mathcal{L}^{(\text{aux})}$, we get:

$$\forall (s, t) \in \mathcal{C}^{(\text{aux}),*}_{\kappa} \times \mathcal{P}^{(\text{aux}),*}_{\kappa}\,,\quad W_{\sigma(\rho)_\kappa}(s, t) = W_{\rho_{\eta_\kappa}}\left(s \circ Q_{\eta_\kappa\to\kappa}\,,\ t \circ P_{\eta_\kappa\to\kappa}\right),$$

hence $\sigma\left\langle \hat{\mathcal{S}}^{\otimes}_{(\mathcal{L}_{\text{HF}},\mathcal{H}^{\mathbb{R}},\Phi^{\mathbb{R}})}\right\rangle \subset \hat{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{(\text{aux})},\mathcal{H}^{(\text{aux})},\Phi^{(\text{aux})})} = \varnothing$. □

If we cannot have Gaussian states, the next best thing would be to have some quantum analogue of the classical probability distributions whose characteristic function takes the form:

$$\left\langle e^{-is(X)}\right\rangle_X = \exp\left[ia(s) - \big(b(s)\big)^{\alpha/2}\right],$$

with $\alpha \in \,]0, 2]$, a a linear form and b a (symmetric) non-negative bilinear form on the dual space. These distributions have the nice property that their marginal probabilities (the classical equivalent of the partial trace of a quantum state) are again of the same form, with the same α (this is manifest by recalling that the characteristic function for a marginal probability is the restriction of the full characteristic function to the corresponding vector subspace of the dual space, in complete analogy to def. 18.2.3). For $\alpha = 2$ the distribution is Gaussian, while for $\alpha < 2$ it is heavy-tailed (aka. has infinite variance), and these distributions generalize the Gaussian distribution in the sense that they appear as attractors in heavy-tailed generalizations of the central limit theorem [38].

However, as prop. 18.15 below shows, the previous arguments excludes in the same stroke a large class of states.

Proposition 18.15 We consider the same objects as in prop. 18.4 and we suppose that there exists a state $\rho \in \mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$, integers $m, n \in \mathbb{N}$, and reals $\epsilon_1, \ldots, \epsilon_{m+n} > 0$ such that:

1. for any $\eta \in \mathcal{L}$ there exists m linear forms $a^{(1)}_\eta, \ldots, a^{(m)}_\eta$ and n (symmetric) non-negative bilinear forms $b^{(1)}_\eta, \ldots, b^{(n)}_\eta$ on $\mathcal{C}^*_\eta \times \mathcal{P}^*_\eta$;

2. for any $\eta \preccurlyeq \eta' \in \mathcal{L}$:

$$\forall k \in \{1, \ldots, m\}\,,\ a^{(k)}_\eta = a^{(k)}_{\eta'} \circ (Q^*_{\eta'\to\eta} \times P^*_{\eta'\to\eta})$$
$$\&\quad \forall l \in \{1, \ldots, n\}\,,\ b^{(l)}_\eta = b^{(l)}_{\eta'} \circ \big((Q^*_{\eta'\to\eta} \times P^*_{\eta'\to\eta}) \times (Q^*_{\eta'\to\eta} \times P^*_{\eta'\to\eta})\big)\,;$$

3. for any $\eta \in \mathcal{L}$ and any $(s, t) \in \mathcal{C}^*_\eta \times \mathcal{P}^*_\eta$:

$$\Big(\forall k \in \{1, \ldots, m\}\,,\ \big|a^{(k)}_\eta(s, t)\big|^2 < \epsilon_k \quad \& \quad \forall l \in \{1, \ldots, n\}\,,\ b^{(l)}_\eta(s, t;\, s, t) < \epsilon_{m+l}\Big)$$
$$\Rightarrow\ \big|W_{\rho_\eta}(s, t) - 1\big| < \frac{1}{2}\,.$$

Then, $\hat{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \neq \varnothing$.

Proof For any $\eta \in \mathcal{L}$ we define:

$$\forall (s, t), (s', t') \in \mathcal{C}^*_\eta \times \mathcal{P}^*_\eta\,,$$
$$B_\eta(s, t;\, s', t') := \frac{1}{4\pi\ \min(\epsilon_1\,,\ \ldots\,,\ \epsilon_{m+n})} \left(\sum_{k=1}^{m} a^{(k)}_\eta(s, t)\, a^{(k)}_\eta(s', t') + \sum_{l=1}^{n} b^{(l)}_\eta(s, t;\, s', t')\right),$$

as well as:

$$\forall s, s' \in \mathcal{C}^*_\eta\,,\ V_\eta(s, s') := B_\eta(s, 0;\, s', 0) \quad \& \quad \forall t, t' \in \mathcal{P}^*_\eta\,,\ U_\eta(t, t') := B_\eta(0, t;\, 0, t')\,.$$

V_η, resp. U_η, is a (symmetric) non-negative bilinear form on $\mathcal{C}^*_\eta$, resp. $\mathcal{P}^*_\eta$, and for any $\eta \preccurlyeq \eta' \in \mathcal{L}$, we have:

$$V_\eta = V_{\eta'} \circ (Q^*_{\eta'\to\eta} \times Q^*_{\eta'\to\eta}) \quad \& \quad U_\eta = U_{\eta'} \circ (P^*_{\eta'\to\eta} \times P^*_{\eta'\to\eta})\,.$$

Let $\eta \in \mathcal{L}$. Reasoning by contradiction, we suppose that there exists $(s', t') \in \mathcal{C}^*_\eta \times \mathcal{P}^*_\eta$ such that:

$$V_\eta(s', s') + U_\eta(t', t') < t'\big(\Xi^{-1}_\eta(s')\big)\,.$$

In particular, this implies $\epsilon := t'\big(\Xi^{-1}_\eta(s')\big) > 0$. We define $s := \sqrt{\frac{2\pi}{\epsilon}}s'$ and $t := \sqrt{\frac{2\pi}{\epsilon}}t'$, so that:

$$V_\eta(s, s) + U_\eta(t, t) < 2\pi \quad \& \quad t\big(\Xi^{-1}_\eta(s)\big) = 2\pi\,.$$

Then, we have:

$$B_\eta(s, 0;\, s, 0) = V_\eta(s, s) < 4\pi \quad \& \quad B_\eta(0, t;\, 0, t) = U_\eta(t, t) < 4\pi\,,$$

as well as:

$$B_\eta(-s, t;\, -s, t) \leqslant B_\eta(-s, t;\, -s, t) + B_\eta(s, t;\, s, t) = 2\,\big(V_\eta(s, s) + U_\eta(t, t)\big) < 4\pi\,.$$

Thus, we get:

$$\big|W_{\rho_\eta}(s, 0) - 1\big| < \frac{1}{2}\,,\quad \big|W_{\rho_\eta}(0, t) - 1\big| < \frac{1}{2} \quad \& \quad \big|W_{\rho_\eta}(-s, t) - 1\big| < \frac{1}{2}\,. \tag{18.15.1}$$

Now, the positivity condition eq. (18.2.1) applied to the points $(0, 0)$, $(s, 0)$ and $(0, t)$ can be rewritten as:

$$\forall z_o\,,\ z_1\,,\ z_2 \in \mathbb{C},$$

$$|z_o|^2 + |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}\left(\overline{z_o}\, z_1\, W_{\rho_\eta}(s,0) + \overline{z_o}\, z_2\, W_{\rho_\eta}(0,t) - \overline{z_1} z_2\, W_{\rho_\eta}(-s,t)\right) \geqslant 0 \quad (18.15.2)$$

where we have used $t\big(\Xi_\eta^{-1}(s)\big) = 2\pi$ and $W_{\rho_\eta}(0,0) = 1$ (for $\rho \in \mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$, hence $\mathrm{Tr}_{\mathcal{H}_\eta} \rho_\eta = 1$). Applying eq. (18.15.2) with $z_o = 1$ and $z_1 = z_2 = -1$ yields:

$$\begin{aligned}
0 &\leqslant 3 - \mathrm{Re}\left(2\, W_{\rho_\eta}(s,0) - 2\, W_{\rho_\eta}(0,t) - 2\, W_{\rho_\eta}(-s,t)\right) \\
&\leqslant 2\,\left|1 - W_{\rho_\eta}(s,0)\right| + 2\,\left|1 - W_{\rho_\eta}(0,t)\right| - 1 - 2\,\mathrm{Re}\left(W_{\rho_\eta}(-s,t)\right) \\
&< \mathrm{Re}\left(1 - W_{\rho_\eta}(-s,t)\right) - \mathrm{Re}\left(W_{\rho_\eta}(-s,t)\right).
\end{aligned}$$

Thus, $-\mathrm{Re}\big(1 - W_{\rho_\eta}(-s,t)\big) < \mathrm{Re}\big(W_{\rho_\eta}(-s,t)\big) < \mathrm{Re}\big(1 - W_{\rho_\eta}(-s,t)\big)$, and since we also have $\mathrm{Im}\big(W_{\rho_\eta}(-s,t)\big) = -\mathrm{Im}\big(1 - W_{\rho_\eta}(-s,t)\big)$, this implies $\left|W_{\rho_\eta}(-s,t)\right| < \left|1 - W_{\rho_\eta}(-s,t)\right| < \frac{1}{2} < 1$.

Next, we apply eq. (18.15.2) with:

$$\begin{aligned}
&z_o = 1 - \left|W_{\rho_\eta}(-s,t)\right|^2, \\
&\qquad z_1 = -\overline{W_{\rho_\eta}(s,0)} - \overline{W_{\rho_\eta}(0,t)}\, W_{\rho_\eta}(-s,t) \quad \& \quad z_2 = -\overline{W_{\rho_\eta}(0,t)} - \overline{W_{\rho_\eta}(s,0)}\, \overline{W_{\rho_\eta}(-s,t)}
\end{aligned}$$

(these values arise by optimizing on z_1, z_2, keeping z_o fixed). After simplifications, we get:

$$0 \leqslant z_o^2 - z_o \left[\left|W_{\rho_\eta}(s,0)\right|^2 + \left|W_{\rho_\eta}(0,t)\right|^2\right] - 2\, z_o\, \mathrm{Re}\left(W_{\rho_\eta}(s,0)\, \overline{W_{\rho_\eta}(0,t)}\, W_{\rho_\eta}(-s,t)\right).$$

Since $\left|W_{\rho_\eta}(-s,t)\right| < 1$, $z_o > 0$, so this leads to:

$$\left|W_{\rho_\eta}(-s,t)\right|^2 + \left|W_{\rho_\eta}(s,0)\right|^2 + \left|W_{\rho_\eta}(0,t)\right|^2 + 2\,\mathrm{Re}\left(W_{\rho_\eta}(s,0)\, \overline{W_{\rho_\eta}(0,t)}\, W_{\rho_\eta}(-s,t)\right) \leqslant 1.$$

Finally, we rewrite this inequality in terms of $w_o := W_{\rho_\eta}(-s,t) - 1$, $w_1 := W_{\rho_\eta}(s,0) - 1$ and $w_2 := W_{\rho_\eta}(0,t) - 1$:

$$\begin{aligned}
0 &\geqslant 4 + 4\mathrm{Re}\,(w_o + w_1 + \overline{w_2}) + |w_o + w_1 + \overline{w_2}|^2 + 2\,\mathrm{Re}\,(w_o w_1 \overline{w_2}) \\
&\geqslant 4 - 4\,|w_o + w_1 + \overline{w_2}| + |w_o + w_1 + \overline{w_2}|^2 - 2\,|w_o w_1 \overline{w_2}|.
\end{aligned}$$

However, this contradicts eq. (18.15.1) which requires $|w_o + w_1 + \overline{w_2}| < 3/2$ and $|w_o w_1 \overline{w_2}| < 1/8$, since $(x \mapsto x^2 - 4\,x)\left\langle\, [0,\, 3/2[\, \right\rangle = \,]-15/4,\, 0]$.

Thus, we have proven that:

$$\forall (s,t) \in \mathcal{C}^*_\eta \times \mathcal{P}^*_\eta,\ V_\eta(s,s) + U_\eta(t,t) - t\left(\Xi_\eta^{-1}(s)\right) \geqslant 0.$$

In particular, for any $s \neq 0 \in \mathcal{C}^*_\eta$, there exists $x \in \mathcal{C}_\eta$ such that $s(x) \neq 0$, so defining $t := \Xi^*_\eta \left(\dfrac{x}{s(x)}\right) \in \mathcal{P}^*_\eta$, we have:

$$\forall \lambda \in \mathbb{R},\ \lambda^2\, V_\eta(s,s) + U_\eta(t,t) - \lambda \geqslant 0,$$

hence the symmetric bilinear form V_η is strictly positive. Similarly, U_η is also strictly positive. Therefore, $\left(V_\eta\,,\, U_\eta\right)_{\eta \in \mathcal{L}} \in \mathcal{V}^{\downarrow}_{(\mathcal{L},(\mathcal{C},\mathcal{P}),(Q,P))}$, so by prop. 18.9, $\hat{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \neq \varnothing$. □

The previous result can also be seen as excluding a different notion of state confinement, focusing on *quantiles* rather than *moments* (in other words, working with the cumulative distribution function rather than with the characteristic or moment-generating one).

Proposition 18.16 We consider the same objects a in prop. 18.4. Let $\mathcal{C}^*_{\text{cyl}}$, resp. $\mathcal{P}^*_{\text{cyl}}$, be the inductive limit (*without any completion*) of the inductive system $\left(\left(\mathcal{C}^*_\eta\right)_{\eta\in\mathcal{L}}, \left(Q^*_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'} \right)$, resp. $\left(\left(\mathcal{P}^*_\eta\right)_{\eta\in\mathcal{L}}, \left(P^*_{\eta'\to\eta}\right)_{\eta\preccurlyeq\eta'} \right)$. For any $\eta \in \mathcal{L}$, let $Q^*_{\text{cyl}\to\eta}$, resp. $P^*_{\text{cyl}\to\eta}$, be the canonical injection of $\mathcal{C}^*_\eta$ in $\mathcal{C}^*_{\text{cyl}}$, resp. of $\mathcal{P}^*_\eta$ in $\mathcal{P}^*_{\text{cyl}}$.

We suppose that there exists a state $\rho \in \mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$, a real $p > 3/4$ and a non-negative (symmetric) bilinear form B_{cyl} on $\mathcal{C}^*_{\text{cyl}} \times \mathcal{P}^*_{\text{cyl}}$ such that, for any $\eta \in \mathcal{L}$ and any $(s, t) \in \mathcal{C}^*_\eta \times \mathcal{P}^*_\eta$:

$$\text{Tr}_{\mathcal{H}_\eta}\left[\rho_\eta\, \Pi_\eta(s, t)\right] \geqslant p\,,$$

where $\Pi_\eta(s, t)$, is the spectral projector of $\mathrm{X}_\eta(s, t)$ on $\left[-A, A\right]$ with:

$$A = \sqrt{B_{\text{cyl}}\left(Q^*_{\text{cyl}\to\eta}(s), P^*_{\text{cyl}\to\eta}(t);\ Q^*_{\text{cyl}\to\eta}(s), P^*_{\text{cyl}\to\eta}(t)\right)}\,.$$

Then, $\hat{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \neq \varnothing$.

Proof For any $\eta \in \mathcal{L}$, we define:

$$\forall (s, t), (s', t') \in \mathcal{L},\ b^{(1)}_\eta(s, t;\ s', t') := B_{\text{cyl}}\left(Q^*_{\text{cyl}\to\eta}(s), P^*_{\text{cyl}\to\eta}(t);\ Q^*_{\text{cyl}\to\eta}(s'), P^*_{\text{cyl}\to\eta}(t')\right).$$

By construction, $\left(b^{(1)}_\eta\right)_{\eta\in\mathcal{L}}$ fulfills prop. 18.15.1 and 18.15.2.

Let $\epsilon_1 := \left(2p - \frac{3}{2}\right)^2 > 0$. For any $\eta \in \mathcal{L}$ and any $(s, t) \in \mathcal{C}^*_\eta \times \mathcal{P}^*_\eta$ such that $b^{(1)}_\eta(s, t;\ s, t) < \epsilon_1$, we have:

$$\begin{aligned}\left\|\Pi_\eta(s, t)\ \left(\mathrm{T}_\eta(s, t) - \text{id}_{\mathcal{H}_\eta}\right)\right\| &\leqslant \sup_{x\in\left[-\sqrt{b^{(1)}_\eta(s,t;\,s,t)},\ \sqrt{b^{(1)}_\eta(s,t;\,s,t)}\right]} \left|e^{i\,x} - 1\right| \\ &= 2\sin\left(\frac{\sqrt{b^{(1)}_\eta(s, t;\ s, t)}}{2}\right) < 2p - \tfrac{3}{2}\,,\end{aligned}$$

as well as:

$$\left\|\mathrm{T}_\eta(s, t) - \text{id}_{\mathcal{H}_\eta}\right\| \leqslant \left\|\mathrm{T}_\eta(s, t)\right\| + \left\|\text{id}_{\mathcal{H}_\eta}\right\| = 2\,.$$

Thus, we get:

$$\begin{aligned}\left|W_{\rho_\eta}(s, t) - 1\right| &= \left|Tr_{\mathcal{H}_\eta}\,\rho_\eta\ \left(\mathrm{T}_\eta(s, t) - \text{id}_{\mathcal{H}_\eta}\right)\right| \\ &< 2p - \tfrac{3}{2} + 2\,Tr_{\mathcal{H}_\eta}\,\rho_\eta\ \left(\text{id}_{\mathcal{H}_\eta} - \Pi_\eta(s, t)\right) < \frac{1}{2}\,.\end{aligned}$$

Hence, prop. 18.15 implies $\hat{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \neq \varnothing$. □

Any state on the algebra $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ (aka. positive linear form of norm 1, see [41, part III, def. 2.2.8] and prop. 5.4) yields a $\mathbb{C}$-valued function on the inductive limit $\mathcal{C}^*_{\text{cyl}} \times \mathcal{P}^*_{\text{cyl}}$, which satisfies suitable positivity requirements (in accordance with def. 18.2.2). Since the space of states on a C^*-algebra is never empty [87, theorem I.9.18], we are assured that there do exist such functions of positive type on $\mathcal{C}^*_{\text{cyl}} \times \mathcal{P}^*_{\text{cyl}}$. However, a function of positive type comes from a projective state in $\mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ if and only if it is *continuous* with respect to the inductive limit topology on $\mathcal{C}^*_{\text{cyl}} \times \mathcal{P}^*_{\text{cyl}}$ (this reflects the characterization from prop. 5.12). What we have established above is that, in the case of the projective structure constructed as in sections 10 and 12 for the holonomy-flux algebra with $G = \mathbb{R}$:

- there do not exist any function of positive type *twice-differentiable* at 0 with respect to the *inductive limit topology*;
- given some scalar product on $\mathcal{C}^*_{\text{cyl}} \times \mathcal{P}^*_{\text{cyl}}$, there do not exist any function of positive type *continuous* at 0 with respect to the *corresponding (strong) topology*.

This still leaves a fairly large window for projective quantum states to be found, however, it would be reassuring to have an actual proof of their existence, and, even better, constructive techniques to obtain them.

On the other hand, the first of these two points demonstrates that the failure to find admissible semi-classical states would *not* be solved by looking for an even larger state space: if there would exist such states on the algebra they would be twice-differentiable, hence continuous, at 0, and they would automatically belong to the projective state space (thanks to the positivity condition, continuity at zero implies continuity everywhere). As announced in section 17, the problem we uncovered here has its roots in the holonomy-flux algebra *itself*. Therefore, the resolution we will propose in the next section will act directly on the structure of this algebra. At the same time, it will bypass the concerns just raised about the projective state space being possibly empty, since it will provide us with a systematic procedure to construct arbitrary projective quantum states, no matter whether the gauge group is compact or not (subsection 19.1).

19. Quasi-cofinal sequences

The fact that $\mathcal{L}_{\text{HF}}$ is uncountable plays a crucial role in the negative result of prop. 18.14. As we will see in subsection 19.1, constructing projective quantum states is significantly easier when the label set is countable, since, in this case, all projective states can be constructed recursively in a systematic way. In particular, there is no risk for a projective limit on a countable label set to be empty, and, more generally, for any label η, and any density matrix ρ_η on $\mathcal{H}_\eta$, it will always be possible to find a projective state whose restriction to the label η coincides with ρ_η.

To benefit of these advantages, our goal will therefore be to restrict $\mathcal{L}_{\text{HF}}$ to a carefully chosen countable *subset*. However, we have to be cautious of the dangerous side-effects such an endeavor could have:

- we do not want to introduce any objectionable arbitrariness in the theory: recall that we put forward in the introduction of the present work the improved universality of the projective approach compared to the choice of a particular representation of the algebra of observables;
- in addition, going over to discrete structures carries a serious risk of breaking diffeomorphism invariance [74], something we want to avoid at any cost in view of applications to *background independant* quantum gravity [75, 77].

In subsection 19.2, we will therefore spell out, in a general setting, the properties that a label subset should satisfy to ensure that restricting the projective system to this subset will preserve suitable notions of both universality and diffeomorphism invariance (or, more generally, invariance under whatever the group of symmetries is for the particular theory under consideration). This strategy will be put to practice on a simple 1-dimensional toy model in subsection 19.3, while the proof that there indeed exist countable subsets in $\mathcal{L}_{\text{HF}}$ satisfying the requirements of subsection 19.2 is currently under progress for $d > 1$.

Note that, from a *physical* point of view, it seems in fact very reasonable to expect the *elementary* observables of a theory (in the sense discussed in section 1) to form a *countable* set: these observables are meant to be in one-to-one correspondence with the experimental protocols describing their measurement, and such protocols should indeed form a countable set (since they can be encoded eg. as finite sequences of chars). To say it differently, if there would be uncountably many elementary observables, we would not even be able to accurately *tell* which one we are measuring in a given experiment.

One could at first think that such an argument should be made at the level of the *physical* observables, since those are often thought of as the only 'real' ones. However, in the spirit of appendix A (viz. the extended discussions in section 3), we adopt the interpretation that the kinematical observables are not just byproducts of the construction of the final, physical theory, to be discarded as soon as the latter has been obtained, but that they instead play a prominent role to formulate the interface between the mathematical theory and the experimental reality: they are used to label with physical meaning the dynamical observables to which they give rise (as stressed in the discussion preceding def. A.2, the redundancy of this labeling is deliberate: it reflects the predictive power of the theory). So, in this perspective, we indeed expect countability already at the level of the *kinematical* elementary observables.

19.1 Factorized states on cofinal sequences

As underlined in prop. 5.6, restricting the label set to a cofinal part does not affect the projective quantum state space, so, rather than considering a countable label set, it is sufficient to look at a label set admitting a *countable cofinal subset*. This weaker condition is of course equivalent if the part of the label set that is below any given label is countable, like in $\mathcal{L}_{\text{HF}}$ (as the labels in $\mathcal{L}_{\text{HF}}$ are finite collections of edges and faces, they actually have only finitely many sublabels, see prop. 19.3). But, for example, in the label set considered in prop. 16.3, which consists of finite dimensional vector subspaces, most labels (namely the vector subspaces of dimension greater than 2) are above uncountably many others (while the label set $\mathcal{L}$ of prop. 16.3 itself does not admit a countable cofinal subset, one could easily construct an uncountable part of $\mathcal{L}$ that does).

If we do have a countable cofinal part, then we can construct recursively an *increasing* cofinal sequence from it, and, along this sequence, we can use the fact that the partial traces $\text{Tr}_{\eta_{n+1} \to \eta_n}$ are surjective to construct a projective quantum state, by recursively choosing a density matrix $\rho_{\eta_{n+1}}$ in the preimage of ρ_{η_n}. Clearly, all projective states can be constructed in this way. The 'factorized pure states', satisfying $\rho_{\eta_{n+1}} \approx |\psi_{n+1} \rangle\langle \psi_{n+1}| \otimes \rho_{\eta_n}$ for some vector $\psi_{n+1} \in \mathcal{H}_{\eta_{n+1} \to \eta_n}$, are particularly simple, and their convex closure is dense in the projective state space (with respect to a topology defined like in prop. 12.12).

Proposition 19.1 Let $(\mathcal{L}, \mathcal{H}, \Phi)^{\otimes}$ be a projective system of quantum state spaces (def. 5.1) and suppose that $\mathcal{L}$ admits a *countable* cofinal subset $\widetilde{\mathcal{L}}_{\text{seq}}$. Then, there exists an increasing sequence $(\eta_n)_{n\in\mathbb{N}}$ such that $\{\eta_n \mid n \in \mathbb{N}\}$ is cofinal in $\widetilde{\mathcal{L}}_{\text{seq}}$, hence in $\mathcal{L}$. We *choose* such an increasing sequence, and we define $\mathcal{L}_{\text{seq}} := \{\eta_n \mid n \in \mathbb{N}\}$ as well as:

$$\mathcal{J}_o := \mathcal{H}_{\eta_o} \quad \& \quad \forall n > 0,\, \mathcal{J}_n := \mathcal{H}_{\eta_n \to \eta_{n-1}}\,.$$

Then, for any sequence $\psi = (\psi_n)_{n\in\mathbb{N}}$ such that:

$$\forall n \in \mathbb{N},\ \ \psi_n \in \mathcal{J}_n \quad \& \quad \|\psi_n\|_{\mathcal{J}_n} = 1\,, \tag{19.1.1}$$

there exists a *unique* state $\rho[\psi] \in \mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ such that:

$$\rho_{\eta_o}[\psi] = |\psi_o\,\rangle\langle\,\psi_o| \quad \& \quad \forall n > 0,\, \rho_{\eta_n}[\psi] = \Phi^{-1}_{\eta_n\to\eta_{n-1}} \circ \Big(\,|\psi_n\,\rangle\langle\,\psi_n| \otimes \rho_{\eta_{n-1}}[\psi]\Big) \circ \Phi_{\eta_n\to\eta_{n-1}}\,. \tag{19.1.2}$$

Proof *Auxiliary projective system on* $\mathbb{N}$. Since $\widetilde{\mathcal{L}}_{\text{seq}}$ is countable, there exists a sequence $\left(\widetilde{\eta}_n\right)_{n\in\mathbb{N}}$ such that $\widetilde{\mathcal{L}}_{\text{seq}} = \left\{\widetilde{\eta}_n \middle| n \in \mathbb{N}\right\}$ (if $\widetilde{\mathcal{L}}_{\text{seq}}$ happens to be finite, we can simply choose the sequence to be eventually constant). Next, $\mathcal{L}$ being directed and $\widetilde{\mathcal{L}}_{\text{seq}}$ being cofinal in $\mathcal{L}$, there exists, for any $n, n' \in \mathbb{N}$, $N \in \mathbb{N}$ such that:

$$\widetilde{\eta}_n\,,\, \widetilde{\eta}_{n'} \preccurlyeq \widetilde{\eta}_N\,.$$

Hence, we can define recursively a sequence $(N_n)_{n\in\mathbb{N}}$ via:

$$N_o = 0 \quad \& \quad \forall n > 0,\, N_n := \min\left\{N \in \mathbb{N} \mid \widetilde{\eta}_n\,,\, \widetilde{\eta}_{N_{n-1}} \preccurlyeq \widetilde{\eta}_N\right\}.$$

By construction, the sequence $\left(\eta_n := \widetilde{\eta}_{N_n}\right)_{n\in\mathbb{N}}$ is increasing and $\mathcal{L}_{\text{seq}} := \{\eta_n \mid n \in \mathbb{N}\}$ is cofinal in $\widetilde{\mathcal{L}}_{\text{seq}}$.

We define recursively a family of unitary isomorphisms $\left(\widetilde{\Phi}_n : \mathcal{H}_{\eta_n} \to \mathcal{J}_n \otimes \ldots \otimes \mathcal{J}_o\right)_{n\in\mathbb{N}}$ via:

$$\widetilde{\Phi}_o := \mathrm{id}_{\mathcal{J}_o} \quad \& \quad \forall n > 0,\, \widetilde{\Phi}_n := \left(\mathrm{id}_{\mathcal{J}_n} \otimes \widetilde{\Phi}_{n-1}\right) \circ \Phi_{\eta_n\to\eta_{n-1}}\,.$$

Next, for any $n \in \mathbb{N}$, we define recursively a family of unitary isomorphisms $\Big(\widetilde{\Phi}_{n'\to n} : \mathcal{H}_{\eta_{n'}\to\eta_n} \to \mathcal{J}_{n'} \otimes \ldots \otimes \mathcal{J}_{n+1}\Big)_{n'>n}$ via:

$$\widetilde{\Phi}_{(n+1)\to n} := \mathrm{id}_{\mathcal{J}_{n+1}} \quad \& \quad \forall n' > n+1,\, \widetilde{\Phi}_{n'\to n} := \left(\mathrm{id}_{\mathcal{J}_{n'}} \otimes \widetilde{\Phi}_{(n'-1)\to n}\right) \circ \Phi_{\eta_{n'}\to\eta_{n'-1}\to\eta_n}\,.$$

Thus, we get, for any $n < n' \in \mathbb{N}$:

$$\left(\widetilde{\Phi}_{n'\to n} \otimes \widetilde{\Phi}_n\right) \circ \Phi_{\eta_{n'}\to\eta_n} \circ \widetilde{\Phi}^{-1}_{n'} = \mathrm{id}_{\mathcal{J}_{n'}\otimes\ldots\otimes\mathcal{J}_{n+1}\otimes\mathcal{J}_n\otimes\ldots\otimes\mathcal{J}_o}\,, \tag{19.1.3}$$

as can be shown by recursion over $n' - n$, using the definitions above together with eq. (5.1.1).

Let $\sigma_{\mathcal{L}\to\mathcal{L}_{\text{seq}}} : \overline{\mathcal{S}}^{\otimes}_{\mathcal{L},\mathcal{H},\Phi} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}_{\text{seq}},\mathcal{H},\Phi)}$ and $\alpha_{\mathcal{L}\leftarrow\mathcal{L}_{\text{seq}}} : \overline{\mathcal{A}}^{\otimes}_{\mathcal{L}_{\text{seq}},\mathcal{H},\Phi} \to \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ be the bijective maps constructed as in prop. 5.6. We define, for any $n \in \mathbb{N}$:

$$\mathcal{K}_n := \mathcal{J}_n \otimes \ldots \otimes \mathcal{J}_o \quad \& \quad \mathcal{K}_{n\to n} := \mathbb{C}\,,$$

and for any $n < n' \in \mathbb{N}$:

$$\mathcal{K}_{n'\to n} := \mathcal{J}_{n'} \otimes \ldots \otimes \mathcal{J}_{n+1} .$$

Moreover, we define, for any $n \leqslant n' \in \mathbb{N}$, $\Psi_{n'\to n}$ to be the natural identification $\mathcal{K}_{n'} \approx \mathcal{K}_{n'\to n} \otimes \mathcal{K}_n$ and, for any $n \leqslant n' \leqslant n'' \in \mathbb{N}$, $\Psi_{n''\to n'\to n}$ to be the natural identification $\mathcal{K}_{n''\to n} \approx \mathcal{K}_{n''\to n'} \otimes \mathcal{K}_{n'\to n}$. Thus, $(\mathbb{N}, \mathcal{K}, \Psi)^{\otimes}$ is a projective system of quantum state spaces. Now, we define:

$$\begin{array}{rccc} \sigma_{\mathcal{L}_{\text{seq}}\to\mathbb{N}} : & \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}_{\text{seq}},\mathcal{H},\Phi)} & \to & \overline{\mathcal{S}}^{\otimes}_{(\mathbb{N},\mathcal{K},\Psi)} \\ & \left(\rho_\eta\right)_{\eta\in\mathcal{L}_{\text{seq}}} & \mapsto & \left(\widetilde{\Phi}_n \circ \rho_{\eta_n} \circ \widetilde{\Phi}_n^{-1}\right)_{n\in\mathbb{N}} \end{array} .$$

Let $\rho \in \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}_{\text{seq}},\mathcal{H},\Phi)}$. For any $n \in \mathbb{N}$, we have:

$$\mathrm{Tr}_{n\to n}\widetilde{\Phi}_n \circ \rho_{\eta_n} \circ \widetilde{\Phi}_n^{-1} = \widetilde{\Phi}_n \circ \rho_{\eta_n} \circ \widetilde{\Phi}_n^{-1} ,$$

and for any $n < n' \in \mathbb{N}$, eq. (19.1.3) yields:

$$\begin{aligned} \mathrm{Tr}_{n'\to n}\widetilde{\Phi}_{n'} \circ \rho_{\eta_{n'}} \circ \widetilde{\Phi}_{n'}^{-1} &= \widetilde{\Phi}_n \circ \left[\mathrm{Tr}_{\mathcal{H}_{\eta_{n'}\to\eta_n}} \Phi_{\eta_{n'}\to\eta_n} \circ \rho_{\eta_{n'}} \circ \Phi^{-1}_{\eta_{n'}\to\eta_n}\right] \circ \widetilde{\Phi}_n^{-1} \\ &= \widetilde{\Phi}_n \circ \rho_{\eta_n} \circ \widetilde{\Phi}_n^{-1} . \end{aligned}$$

Therefore, $\sigma_{\mathcal{L}_{\text{seq}}\to\mathbb{N}}$ is well-defined as a map $\overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}_{\text{seq}},\mathcal{H},\Phi)} \to \overline{\mathcal{S}}^{\otimes}_{(\mathbb{N},\mathcal{K},\Psi)}$. In addition, $\sigma_{\mathcal{L}_{\text{seq}}\to\mathbb{N}}$ is injective, since $\mathcal{L}_{\text{seq}} = \{\eta_n \mid n \in \mathbb{N}\}$.

We now want to prove that $\sigma_{\mathcal{L}_{\text{seq}}\to\mathbb{N}}$ is surjective as well. Let $\widetilde{\rho} \in \overline{\mathcal{S}}^{\otimes}_{(\mathbb{N},\mathcal{K},\Psi)}$. Let $n, n' \in \mathbb{N}$ such that $\eta_n \preccurlyeq \eta_{n'}$. If $n < n'$, we have, in a way similar to above:

$$\mathrm{Tr}_{\eta_{n'}\to\eta_n}\widetilde{\Phi}_{n'}^{-1} \circ \widetilde{\rho}_{n'} \circ \widetilde{\Phi}_{n'} = \widetilde{\Phi}_n^{-1} \circ \widetilde{\rho}_n \circ \widetilde{\Phi}_n . \tag{19.1.4}$$

Clearly, eq. (19.1.4) also holds if $n = n'$. Finally, if $n > n'$, $\eta_n \succcurlyeq \eta_{n'} \succcurlyeq \eta_n$, hence applying eq. (5.1.1) implies $\mathrm{Tr}_{\eta_{n'}\to\eta_n} = \left(\mathrm{Tr}_{\eta_n\to\eta_{n'}}\right)^{-1}$. Making use of the first case for n', n then yields eq. (19.1.4) in this case too. In particular, if $\eta_n = \eta_{n'}$, we get:

$$\widetilde{\Phi}_{n'}^{-1} \circ \widetilde{\rho}_{n'} \circ \widetilde{\Phi}_{n'} = \widetilde{\Phi}_n^{-1} \circ \widetilde{\rho}_n \circ \widetilde{\Phi}_n .$$

Therefore, there exists $\rho = \left(\rho_\eta\right)_{\eta\in\mathcal{L}_{\text{seq}}} \in \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}_{\text{seq}},\mathcal{H},\Phi)}$ such that $\forall n \in \mathbb{N}$, $\rho_{\eta_n} = \widetilde{\Phi}_n^{-1} \circ \widetilde{\rho}_n \circ \widetilde{\Phi}_n$, ie. $\sigma_{\mathcal{L}_{\text{seq}}\to\mathbb{N}}(\rho) = \widetilde{\rho}$.

Next, for any $n, n' \in \mathbb{N}$, and any $A_{\eta_n} \in \mathcal{A}_{\eta_n}$, $A_{\eta_{n'}} \in \mathcal{A}_{\eta_{n'}}$, we have:

$$\begin{aligned} &\Big(\exists\, \eta'' \in \mathcal{L}_{\text{seq}} \,/\, \eta'' \succcurlyeq \eta_n\,,\, \eta_{n'} \quad \& \\ &\qquad \Phi^{-1}_{\eta''\to\eta_n} \circ (\mathrm{id}_{\mathcal{H}_{\eta''\to\eta_n}} \otimes A_{\eta_n}) \circ \Phi_{\eta''\to\eta_n} = \Phi^{-1}_{\eta''\to\eta_{n'}} \circ (\mathrm{id}_{\mathcal{H}_{\eta''\to\eta_{n'}}} \otimes A_{\eta_{n'}}) \circ \Phi_{\eta''\to\eta_{n'}}\Big) \Leftrightarrow \\ &\Leftrightarrow \Big(\exists\, n'' \geqslant n, n' \,/\, \mathrm{id}_{\mathcal{K}_{n''\to n}} \otimes (\widetilde{\Phi}_n \circ A_{\eta_n} \circ \widetilde{\Phi}_n^{-1}) = \mathrm{id}_{\mathcal{K}_{n''\to n'}} \otimes (\widetilde{\Phi}_{n'} \circ A_{\eta_{n'}} \circ \widetilde{\Phi}_{n'}^{-1})\Big) \end{aligned}$$

(the direction '$\Rightarrow$' can be shown by choosing $\widetilde{n}$ such that $\eta'' = \eta_{\widetilde{n}}$ and $n'' > n, n', \widetilde{n}$). Thus, we can define the algebra isomorphism:

$$\begin{array}{rccc} \alpha_{\mathcal{L}_{\text{seq}}\leftarrow\mathbb{N}} : & \mathcal{A}^{\otimes}_{\mathbb{N},\mathcal{K},\Psi} & \to & \mathcal{A}^{\otimes}_{\mathcal{L}_{\text{seq}},\mathcal{H},\Phi} \\ & \left[A_n\right]_\sim & \mapsto & \left[\widetilde{\Phi}_n^{-1} \circ A_n \circ \widetilde{\Phi}_n\right]_\sim \end{array} ,$$

and extends it by continuity into a C^*-algebra isomorphism $\alpha_{\mathcal{L}_{\text{seq}}\leftarrow\mathbb{N}} : \overline{\mathcal{A}}^{\otimes}_{\mathbb{N},\mathcal{K},\Psi} \to \overline{\mathcal{A}}^{\otimes}_{\mathcal{L}_{\text{seq}},\mathcal{H},\Phi}$. Finally, we define the bijective maps $\sigma := \sigma_{\mathcal{L}_{\text{seq}}\to\mathbb{N}} \circ \sigma_{\mathcal{L}\to\mathcal{L}_{\text{seq}}} : \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \to \overline{\mathcal{S}}^{\otimes}_{(\mathbb{N},\mathcal{K},\Psi)}$ and $\alpha := \alpha_{\mathcal{L}\leftarrow\mathcal{L}_{\text{seq}}} \circ \alpha_{\mathcal{L}_{\text{seq}}\leftarrow\mathbb{N}} : \overline{\mathcal{A}}^{\otimes}_{(\mathbb{N},\mathcal{K},\Psi)} \to \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$.

Existence and uniqueness of $\rho[\psi]$. Let $\psi = (\psi_n)_{n\in\mathbb{N}}$ be a sequence satisfying eq. (19.1.1). We define $\widetilde{\rho}[\psi] \in \mathcal{S}^{\otimes}_{(\mathbb{N},\mathcal{K},\Psi)}$ via:

$$\forall n \in \mathbb{N},\ \widetilde{\rho}_n := |\psi_n \otimes \ldots \otimes \psi_o\,\rangle\langle\, \psi_n \otimes \ldots \otimes \psi_o| \,.$$

Then, $\rho[\psi] := \sigma^{-1}\big(\widetilde{\rho}[\psi]\big) \in \mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ fulfills eq. (19.1.2). Reciprocally, if $\rho \in \mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ fulfills eq. (19.1.2), then $\sigma(\rho) = \widetilde{\rho}[\psi]$ (this can be checked recursively on n), hence $\rho = \rho[\psi]$. □

Supposing that the projective system of quantum state spaces under consideration has been obtained through the quantization of a factorizing system of symplectic manifolds (eg. along the lines of section 6), and that the latter forms a rendering (def. 2.6) of some classical, continuum phase space $\mathcal{M}_\infty$, we can use this technique to construct a semi-classical state centered on a classical point $x_\infty \in \mathcal{M}_\infty$. To this intend, the vector $\psi_{n+1} \in \mathcal{H}_{\eta_{n+1}\to\eta_n}$ should be chosen as a semi-classical state centered around the point $x_{\eta_{n+1}\to\eta_n} \in \mathcal{M}_{\eta_{n+1}\to\eta_n}$, computed from x_∞ via $\left(x_{\eta_{n+1}\to\eta_n}, x_{\eta_n}\right) := \varphi_{\eta_{n+1}\to\eta_n}\left(x_{\eta_{n+1}}\right) := \varphi_{\eta_{n+1}\to\eta_n} \circ \pi_{\infty\to\eta_{n+1}}(x_\infty)$.

For small n, we can think of the coarse labels η_n as describing some *collective*, *macroscopic* degrees of freedom, so that the prescription above offers a concrete implementation of the approach advocated in [71]: namely, we start by forming states having good peaking properties at macroscopic scales, and, going down step by step toward smaller and smaller scales, we impose, at each step, as much semi-classicality as the Heisenberg uncertainty relations will allow (taking heed of the already fixed behavior at larger scales). This is readily achieved here because the largest part of the work was done beforehand while setting up the factorizing system, by identifying the degrees of freedom in η_{n+1} that commute with the ones from η_n (recall the discussion before prop. 2.10): those are precisely the variables on which semi-classicality can be imposed independently of the already chosen state on η_n.

We can then ask whether a semi-classical state constructed this way would belong to the inductive limit Hilbert space arising from a choice of vacuum state (prop. 5.8). Assuming this vacuum is itself a factorized pure state, the characterization given in theorem 5.9 can be reformulated into the condition 19.2.3 below. In particular, if the vacuum state is a momentum eigenstate, like the Ashtekar-Lewandowski vacuum, a factorized semi-classical state could only be made an element of the corresponding inductive limit by deteriorating the semi-classicality of ψ_n: eg. if ψ_n is taken as a coherent state, controlled by a semi-classicality parameter that determines the repartition of the quantum uncertainties between position and momentum variables, this parameter will have to be shifted fast enough, as n grows, toward maximally peaked momenta and maximally spread positions. By contrast, if the vacuum state is itself a coherent state, like the Fock vacuum, the condition 19.2.3 can be interpreted as delimiting a domain in the classical projective limit (def. 2.3) such that, for x_∞ belonging to this domain, the factorized semi-classical state centered around x_∞ will belong to the corresponding inductive limit: assuming the vacuum is centered around 0, this requires that $x_{\eta_{n+1}\to\eta_n}$ tends to 0 fast enough. The question will then be whether the image of $\mathcal{M}_\infty$ in the projective limit (prop. 2.7) happens to be contained in this admissible domain.

Finally, we also notice that the tensor product factors $\mathcal{H}_{\eta_{n+1}\to\eta_n}$ can be arranged into an infinite tensor product (ITP, see [99, 93] and theorem 5.11), and, not surprisingly, all factorized states do belong to this ITP. Still, as we will argue below, working with a projective state space instead of an ITP Hilbert space allows to overcome certain limitations of the ITP construction, in particular with respect to universality (prop. 19.13).

To comment on the relation with the Algebraic Quantum Gravity framework (AQG, see [36] and the brief explanation in section 17), note that, while a primary motivation for introducing an ITP Hilbert space in AQG was the availability of factorized coherent states very similar to the one discussed above, an important difference lies in the type of tensor product factors we are using: the building blocs of the ITP in AQG describe individual, *microscopic* degrees of freedom, meant to represent the smallest atoms of a quantum geometry (presumably at Plank scale), instead of holding complementary degrees of freedom added step by step as we refine our description from macroscopic to microscopic scales.

Proposition 19.2 We consider the same objects as in prop. 19.1. We denote by $\mathcal{H}_{\text{seq}}$ the infinite tensor product of $(\mathcal{J}_n)_{n\in\mathbb{N}}$ (see [99] and theorem 5.11). There exist a map $\sigma_{\text{seq}} : \overline{\mathcal{S}}_{\text{seq}} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ and an algebra morphism $\alpha_{\text{seq}} : \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \to \mathcal{A}_{\text{seq}}$ ($\overline{\mathcal{S}}_{\text{seq}}$, resp. $\mathcal{A}_{\text{seq}}$, being the space of non-negative traceclass operators, resp. the algebra of bounded operators, on $\mathcal{H}_{\text{seq}}$) such that:

$$\forall \rho \in \overline{\mathcal{S}}_{\text{seq}}\,,\ \forall A \in \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}\,,\quad \text{Tr}_{\mathcal{H}_{\text{seq}}} \rho\, \alpha_{\text{seq}}(A) = \text{Tr}\, \sigma_{\text{seq}}(\rho)\, A\,.$$

Similarly, for any sequence ψ satisfying eq. (19.1.1), we denote by $\mathcal{H}_{[\psi]}$ the GNS representation of $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ arising from the state $\rho[\psi]$ (see props. 5.4 and 5.8). There exist an *injective* map $\sigma_{[\psi]} : \overline{\mathcal{S}}_{[\psi]} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ and an algebra morphism $\alpha_{[\psi]} : \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \to \mathcal{A}_{[\psi]}$ ($\overline{\mathcal{S}}_{[\psi]}$, resp. $\mathcal{A}_{[\psi]}$, being the space of non-negative traceclass operators, resp. the algebra of bounded operators, on $\mathcal{H}_{[\psi]}$) such that:

$$\forall \rho \in \overline{\mathcal{S}}_{[\psi]}\,,\ \forall A \in \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}\,,\quad \text{Tr}_{\mathcal{H}_{[\psi]}} \rho\, \alpha_{[\psi]}(A) = \text{Tr}\, \sigma_{[\psi]}(\rho)\, A\,.$$

Moreover, $\rho[\psi] \in \sigma_{[\psi]}\left\langle \mathcal{S}_{[\psi]} \right\rangle$ and $\sigma_{[\psi]}\left\langle \mathcal{S}_{[\psi]} \right\rangle \subset \sigma_{\text{seq}}\left\langle \mathcal{S}_{\text{seq}} \right\rangle$ (with $\mathcal{S}_{[\psi]}$, resp. $\mathcal{S}_{\text{seq}}$, the space of density matrices on $\mathcal{H}_{[\psi]}$, resp. $\mathcal{H}_{\text{seq}}$).

Let ψ, ψ' be two sequences satisfying eq. (19.1.1). The following statements are equivalents:

1. $\rho[\psi'] \in \sigma_{[\psi]}\left\langle \mathcal{S}_{[\psi]} \right\rangle$;
2. $\sigma_{[\psi']}\left\langle \mathcal{S}_{[\psi']} \right\rangle = \sigma_{[\psi]}\left\langle \mathcal{S}_{[\psi]} \right\rangle$;
3. $\displaystyle\sum_{n=0}^{\infty} \Big(1 - \big|\, \langle \psi_n \mid \psi'_n \rangle_{\mathcal{J}_n} \big|\Big) < \infty\,.$

Proof *Construction of $\mathcal{H}_{[\psi]}$, $\sigma_{[\psi]}$ and $\alpha_{[\psi]}$.* We define:

$$\mathcal{Z}^{\otimes}_{(\mathbb{N},\mathcal{J})} := \Big\{ (\psi_n)_{n\in\mathbb{N}} \;\Big|\; \forall n \in \mathbb{N},\ \psi_n \in \mathcal{J}_n \ \ \&\ \ \|\psi_n\|_{\mathcal{J}_n} = 1 \Big\}\,.$$

Let $\psi \in \mathcal{Z}^{\otimes}_{(\mathbb{N},\mathcal{J})}$ and let $\widetilde{\rho}[\psi] \in \mathcal{S}^{\otimes}_{(\mathbb{N},\mathcal{K},\Psi)}$ be defined as in the proof of prop. 19.1. Let $\mathcal{H}_{[\psi]}$ be the GNS representation of $\overline{\mathcal{A}}^{\otimes}_{(\mathbb{N},\mathcal{K},\Psi)}$ arising from the state $\widetilde{\rho}[\psi]$. From prop. 5.8 and theorem 5.9, there exist an injective map $\widetilde{\sigma}_{[\psi]} : \overline{\mathcal{S}}_{[\psi]} \to \overline{\mathcal{S}}^{\otimes}_{(\mathbb{N},\mathcal{K},\Psi)}$ and a C^*-algebra morphism $\widetilde{\alpha}_{[\psi]} : \overline{\mathcal{A}}^{\otimes}_{(\mathbb{N},\mathcal{K},\Psi)} \to \mathcal{A}_{[\psi]}$, such that

$\mathrm{Tr}_{\mathcal{H}_{[\psi]}}\left(\,\cdot\,\widetilde{\alpha}_{[\psi]}(\,\cdot\,)\right) = \mathrm{Tr}\left(\widetilde{\sigma}_{[\psi]}(\,\cdot\,)\,\cdot\,\right)$. Moreover, we have:

$$\widetilde{\sigma}_{[\psi]}\left\langle \mathcal{S}_{[\psi]}\right\rangle := \left\{ \left(\widetilde{\rho}_n\right)_{n\in\mathbb{N}} \in \mathcal{S}^{\otimes}_{(\mathbb{N},\mathcal{K},\Psi)} \,\middle|\, \sup_{n\in\mathbb{N}} \inf_{n'>n} \left\langle \zeta_{n'\to n}\middle| \left(\mathrm{Tr}_{\mathcal{K}_n} \widetilde{\rho}_{n'}\right) \zeta_{n'\to n}\right\rangle = \mathrm{Tr}\,\widetilde{\rho} = 1 \right\},$$

where $\forall n < n' \in \mathbb{N}$, $\zeta_{n'\to n} := \psi_{n'} \otimes \ldots \otimes \psi_{n+1} \in \mathcal{K}_{n'\to n}$.

Using the Tr-intertwined bijective maps σ and α defined in the proof of prop. 19.1, we can identify $\mathcal{H}_{[\psi]}$ with the GNS representation of $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$ arising from the state $\rho[\psi]$, and define the injective map $\sigma_{[\psi]} := \sigma^{-1} \circ \widetilde{\sigma}_{[\psi]} : \overline{\mathcal{S}}_{[\psi]} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$, as well as the algebra morphism $\alpha_{[\psi]} := \widetilde{\alpha}_{[\psi]} \circ \alpha^{-1} : \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \to \mathcal{A}_{[\psi]}$. We then have $\mathrm{Tr}_{\mathcal{H}_{[\psi]}}\left(\,\cdot\,\alpha_{[\psi]}(\,\cdot\,)\right) = \mathrm{Tr}\left(\sigma_{[\psi]}(\,\cdot\,)\,\cdot\,\right)$, and:

$$\sigma_{[\psi]}\left\langle \mathcal{S}_{[\psi]}\right\rangle := \left\{ \left(\rho_\eta\right)_{\eta\in\mathcal{L}} \in \mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \,\middle|\, \sup_{n\in\mathbb{N}} \inf_{n'>n} \left\langle \zeta_{n'\to n}\middle| \left(\mathrm{Tr}_{\mathcal{K}_n} \widetilde{\Phi}_{n'} \circ \rho_{\eta_{n'}} \circ \widetilde{\Phi}^{-1}_{n'}\right) \zeta_{n'\to n}\right\rangle = \mathrm{Tr}\,\rho = 1 \right\}. \tag{19.2.1}$$

In particular, $\rho[\psi] \in \sigma_{[\psi]}\left\langle \mathcal{S}_{[\psi]}\right\rangle$.

Comparing $\sigma_{[\psi]}\left\langle \mathcal{S}_{[\psi]}\right\rangle$ *and* $\sigma_{[\psi']}\left\langle \mathcal{S}_{[\psi']}\right\rangle$. Let $\psi, \psi' \in \mathcal{Z}^{\otimes}_{(\mathbb{N},\mathcal{J})}$. Since $\rho[\psi'] \in \sigma_{[\psi']}\left\langle \mathcal{S}_{[\psi']}\right\rangle$, statement 19.2.2 implies statement 19.2.1. We suppose that statement 19.2.1 holds. Then, the characterization above implies:

$$\begin{aligned} 1 &= \sup_{n\in\mathbb{N}} \inf_{n'>n} \left|\left\langle \psi_{n'} \otimes \ldots \otimes \psi_{n+1} \middle| \psi'_{n'} \otimes \ldots \otimes \psi'_{n+1}\right\rangle\right|^2 \\ &= \sup_{n\in\mathbb{N}} \inf_{n'>n} \prod_{k=n+1}^{n'} \left|\left\langle \psi_k \mid \psi'_k\right\rangle_{\mathcal{J}_k}\right|^2. \end{aligned}$$

This can only holds if there exists $N \in \mathbb{N}$ such that $\forall k \geqslant N$, $\langle \psi_k \mid \psi'_k\rangle_{\mathcal{J}_k} \neq 0$. Then, for any $k \geqslant N$, $0 < \left|\langle \psi_k \mid \psi'_k\rangle_{\mathcal{J}_k}\right| \leqslant \|\psi_k\|_{\mathcal{J}_k}\, \|\psi'_k\|_{\mathcal{J}_k} = 1$, so that $0 \leqslant -\log\left|\langle \psi_k \mid \psi'_k\rangle_{\mathcal{J}_k}\right| < \infty$. Hence, we get:

$$0 = \inf_{n\geqslant N} \sum_{k=n+1}^{\infty} \left(-\log\left|\langle \psi_k \mid \psi'_k\rangle_{\mathcal{J}_k}\right|\right),$$

in other words $\sum_{n=N+1}^{\infty} \left(-\log\left|\langle \psi_n \mid \psi'_n\rangle_{\mathcal{J}_n}\right|\right)$ converges. Then, using that $\log x \leqslant x - 1$ for $x \in \,]0, \infty[$, this implies that:

$$\sum_{n=N+1}^{\infty} \left(1 - \left|\langle \psi_n \mid \psi'_n\rangle_{\mathcal{J}_n}\right|\right)$$

converges, hence statement 19.2.3 holds.

Reciprocally, we now suppose that 19.2.3 holds. Let $\rho \in \sigma_{[\psi]}\left\langle \mathcal{S}_{[\psi]}\right\rangle$ and let $\epsilon > 0$. Then, there exists $N \in \mathbb{N}$ such that:

$$\forall n' > N,\quad 1 - \epsilon \leqslant \left\langle \psi_{n'} \otimes \ldots \otimes \psi_{N+1}\middle| \left(\mathrm{Tr}_{\mathcal{K}_N} \widetilde{\Phi}_{n'} \circ \rho_{\eta_{n'}} \circ \widetilde{\Phi}^{-1}_{n'}\right) \psi_{n'} \otimes \ldots \otimes \psi_{N+1}\right\rangle.$$

Thus, for any $n \geqslant N$ and any $n' > n$, we have:

$$1-\epsilon \leqslant \left\langle \psi_{n'} \otimes \ldots \otimes \psi_{N+1} \middle| \left(\mathrm{Tr}_{\mathcal{K}_N} \widetilde{\Phi}_{n'} \circ \rho_{\eta_{n'}} \circ \widetilde{\Phi}_{n'}^{-1}\right) \psi_{n'} \otimes \ldots \otimes \psi_{N+1} \right\rangle$$

$$= \left\langle \psi_{n'} \otimes \ldots \otimes \psi_{n+1} \otimes \ldots \otimes \psi_{N+1} \middle| \left(\mathrm{Tr}_{\mathcal{K}_N} \widetilde{\Phi}_{n'} \circ \rho_{\eta_{n'}} \circ \widetilde{\Phi}_{n'}^{-1}\right) \psi_{n'} \otimes \ldots \otimes \psi_{n+1} \otimes \ldots \otimes \psi_{N+1} \right\rangle$$

$$\leqslant \left\langle \psi_{n'} \otimes \ldots \otimes \psi_{n+1} \middle| \mathrm{Tr}_{\mathcal{K}_{n\to N}} \left(\mathrm{Tr}_{\mathcal{K}_N} \widetilde{\Phi}_{n'} \circ \rho_{\eta_{n'}} \circ \widetilde{\Phi}_{n'}^{-1}\right) \psi_{n'} \otimes \ldots \otimes \psi_{n+1} \right\rangle$$

$$= \left\langle \psi_{n'} \otimes \ldots \otimes \psi_{n+1} \middle| \left(\mathrm{Tr}_{\mathcal{K}_n} \widetilde{\Phi}_{n'} \circ \rho_{\eta_{n'}} \circ \widetilde{\Phi}_{n'}^{-1}\right) \psi_{n'} \otimes \ldots \otimes \psi_{n+1} \right\rangle.$$

Let $n \geqslant N$, $n' > n$, $\zeta_{n'\to n} := \psi_{n'} \otimes \ldots \otimes \psi_{n+1}$, $\zeta'_{n'\to n} := \psi'_{n'} \otimes \ldots \otimes \psi'_{n+1}$ and $\rho_{n'\to n} := \mathrm{Tr}_{\mathcal{K}_n} \left(\widetilde{\Phi}_{n'} \circ \rho_{\eta_{n'}} \circ \widetilde{\Phi}_{n'}^{-1}\right)$. We have:

$$1 - \left\langle \zeta'_{n'\to n} \middle| \rho_{n'\to n}\, \zeta'_{n'\to n} \right\rangle =$$

$$= 1 - \left\langle \zeta_{n'\to n} \middle| \rho_{n'\to n}\, \zeta_{n'\to n} \right\rangle + \mathrm{Tr}_{\mathcal{K}_{n'\to n}} \rho_{n'\to n} \left(\left|\zeta_{n'\to n}\right\rangle\!\left\langle \zeta_{n'\to n}\right| - \left|\zeta'_{n'\to n}\right\rangle\!\left\langle \zeta'_{n'\to n}\right| \right)$$

$$\leqslant \epsilon + \left\| \left|\zeta_{n'\to n}\right\rangle\!\left\langle \zeta_{n'\to n}\right| - \left|\zeta'_{n'\to n}\right\rangle\!\left\langle \zeta'_{n'\to n}\right| \right\|_{\mathcal{A}_{n'\to n}}, \tag{19.2.2}$$

where $\|\cdot\|_{\mathcal{A}_{n'\to n}}$ denotes the operator norm on $\mathcal{K}_{n'\to n}$ and we have used that $\rho_{n'\to n}$ is a density matrix on $\mathcal{K}_{n'\to n}$ (as $\rho \in \sigma_{[\psi]} \left\langle \mathcal{S}_{[\psi]} \right\rangle \subset \mathcal{S}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$). Now, $\|\zeta_{n'\to n}\|_{\mathcal{K}_{n'\to n}} = \|\zeta'_{n'\to n}\|_{\mathcal{K}_{n'\to n}} = 1$, so:

$$\left\| \left|\zeta_{n'\to n}\right\rangle\!\left\langle \zeta_{n'\to n}\right| - \left|\zeta'_{n'\to n}\right\rangle\!\left\langle \zeta'_{n'\to n}\right| \right\|_{\mathcal{A}_{n'\to n}} = \inf_{\theta\in[0,2\pi[} \left\| \left|\zeta_{n'\to n}\right\rangle\!\left\langle \zeta_{n'\to n}\right| - \left|e^{i\theta}\, \zeta'_{n'\to n}\right\rangle\!\left\langle e^{i\theta}\, \zeta'_{n'\to n}\right| \right\|_{\mathcal{A}_{n'\to n}}$$

$$\leqslant 2 \inf_{\theta\in[0,2\pi[} \left\| \zeta_{n'\to n} - e^{i\theta}\, \zeta'_{n'\to n} \right\|_{\mathcal{K}_{n'\to n}}$$

$$= 2\sqrt{2} \sqrt{1 - \left| \langle \zeta_{n'\to n} \mid \zeta'_{n'\to n} \rangle_{\mathcal{K}_{n'\to n}} \right|}. \tag{19.2.3}$$

Let $\epsilon_1 := \min(\epsilon, 2) > 0$ and $\epsilon_2 := -\log\left(1 - \frac{\epsilon_1^2}{8}\right) > 0$. Making use of statement 19.2.3, let $N' \geqslant N$ such that:

$$\sum_{k=} N' + 1^{\infty} \left(1 - \left| \langle \psi_k \mid \psi'_k \rangle_{\mathcal{J}_k} \right|\right) \leqslant \frac{\epsilon_2}{2}.$$

In particular, this implies:

$$\forall k > N',\ \left| \langle \psi_k \mid \psi'_k \rangle_{\mathcal{J}_k} \right| \in \left[1 - \frac{\log 2}{2}, 1\right] \subset \left[\frac{1}{2}, 1\right].$$

Using that $\log' x = 1/x \leqslant 2$ for any $x \in [1/2, 1]$, we thus get:

$$\forall k > N',\ -\log \left| \langle \psi_k \mid \psi'_k \rangle_{\mathcal{J}_k} \right| \leqslant 2 \left(1 - \left| \langle \psi_k \mid \psi'_k \rangle_{\mathcal{J}_k} \right|\right).$$

Therefore, we have, for any $n' > N'$:

$$-\log \left| \langle \zeta_{n'\to N'} \mid \zeta'_{n'\to N'} \rangle_{\mathcal{K}_{n'\to N'}} \right| \leqslant -\sum_{k=} N' + 1^{\infty} \log \left| \langle \psi_k \mid \psi'_k \rangle_{\mathcal{J}_k} \right| \leqslant \epsilon_2,$$

and, using the definition of ϵ_2:

$$1 - \left| \langle \zeta_{n'\to N'} \mid \zeta'_{n'\to N'} \rangle_{\mathcal{K}_{n'\to N'}} \right| \leqslant \frac{\epsilon^2}{8} . \tag{19.2.4}$$

Finally, for $n' > N'$, combining eqs. (19.2.2), (19.2.3) and (19.2.4) yields:

$$1 - \left\langle \zeta'_{n'\to N'} \middle| \rho_{n'\to N'}\, \zeta'_{n'\to N'} \right\rangle \leqslant \epsilon + 2\sqrt{2}\sqrt{1 - \left| \langle \zeta_{n'\to N'} \mid \zeta'_{n'\to N'} \rangle_{\mathcal{K}_{n'\to N'}} \right|} \leqslant 2\,\epsilon .$$

Hence, we get:

$$\sup_{n\in\mathbb{N}} \inf_{n'>n} \left\langle \zeta'_{n'\to n} \middle| \rho_{n'\to n}\, \zeta'_{n'\to n} \right\rangle \geqslant 1 - 2\,\epsilon .$$

Since this holds for any $\epsilon > 0$ and the right hand side is bounded above by 1 (for $\operatorname{Tr}\rho = 1$), $\rho \in \sigma_{[\psi']} \left\langle \mathcal{S}_{[\psi']} \right\rangle$. Thus, we have proved that $\sigma_{[\psi]} \left\langle \mathcal{S}_{[\psi]} \right\rangle \subset \sigma_{[\psi']} \left\langle \mathcal{S}_{[\psi']} \right\rangle$, and, statement 19.2.3 being symmetric in ψ, ψ', we can prove as well $\sigma_{[\psi']} \left\langle \mathcal{S}_{[\psi']} \right\rangle \subset \sigma_{[\psi]} \left\langle \mathcal{S}_{[\psi]} \right\rangle$, ie. statement 19.2.2 holds.

Construction of $\mathcal{H}_{\text{seq}}$, σ_{seq} *and* α_{seq}. Like in the proof of theorem 5.11, the ITP $\mathcal{H}_{\text{seq}}$ of $(\mathcal{J}_n)_{n\in\mathbb{N}}$ can be written as:

$$\mathcal{H}_{\text{seq}} = \overline{\bigoplus_{[\|\psi\|]} \mathcal{H}_{[\|\psi\|]}} ,$$

where the $[\|\psi\|]$ are the equivalence classes in $\mathcal{Z}^{\otimes}_{(\mathbb{N},\mathcal{J})}$ for the equivalence relation:

$$(\psi_n)_{n\in\mathbb{N}} \simeq (\psi'_n)_{n\in\mathbb{N}} \Leftrightarrow \sum_{n\in\mathbb{N}} \left| 1 - \langle \psi'_n , \psi_n \rangle_{\mathcal{J}_n} \right| < \infty ,$$

and the Hilbert space $\mathcal{H}_{[\|\psi\|]}$ can be identified with $\mathcal{H}_{[\psi]}$ for some representative ψ of $[\|\psi\|]$ (to check that the inductive limit mentioned in the proof of theorem 5.11 coincides with the one defining $\mathcal{H}_{[\psi]}$, as described in prop. 5.8, we notice that the subsets of $\mathbb{N}$ of the form $\{0, \ldots, n\}$ for some $n \in \mathbb{N}$ constitute a cofinal part, with respect to the inclusion order, in the set of all *finite* subsets of $\mathbb{N}$). We choose such a representative ψ for each equivalence class $[\|\psi\|]$ and we define:

$$\begin{array}{rcl} \sigma_{\text{seq}} : \overline{\mathcal{S}}_{\text{seq}} & \to & \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \\ \rho & \mapsto & \sum_{[\|\psi\|]} \sigma_{[\psi]}\big(\Pi_{[\|\psi\|]}\, \rho\, \Pi_{[\|\psi\|]}\big) \end{array} ,$$

where, for any equivalence class $[\|\psi\|]$, $\Pi_{[\|\psi\|]}$ denotes the orthogonal projection on $\mathcal{H}_{[\|\psi\|]} \approx \mathcal{H}_{[\psi]}$. Note that the sum over $[\|\psi\|]$ is absolutely convergent in trace-norm, since, for each $[\|\psi\|]$, $\Pi_{[\|\psi\|]}\, \rho\, \Pi_{[\|\psi\|]}$ is a non-negative traceclass operator and we have $\sum_{[\|\psi\|]} \operatorname{Tr}_{\mathcal{H}_{[\psi]}}\big(\Pi_{[\|\psi\|]}\, \rho\, \Pi_{[\|\psi\|]}\big) = \operatorname{Tr}_{\mathcal{H}_{\text{seq}}} \rho < \infty$. We also define:

$$\begin{array}{rcl} \alpha_{\text{seq}} : \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} & \to & \mathcal{A}_{\text{seq}} \\ A & \mapsto & \sum_{[\|\psi\|]} \Pi_{[\|\psi\|]}\, \alpha_{[\psi]}(A)\, \Pi_{[\|\psi\|]} \end{array} .$$

Again, the sum involved converges, because the projections $\Pi_{[\|\psi\|]}$ are mutually orthogonal. We have, for any $\rho \in \overline{\mathcal{S}}_{\text{seq}}$ and any $A \in \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$:

$$\operatorname{Tr}_{\mathcal{H}_{\text{seq}}} \big(\rho\, \alpha_{\text{seq}}(A)\big) = \operatorname{Tr} \big(\sigma_{\text{seq}}(\rho)\, A\big) ,$$

as follows from the corresponding property fulfilled by each pair $\sigma_{[\psi]}$, $\alpha_{[\psi]}$.

Now, let $\psi' \in \mathcal{Z}^{\otimes}_{(\mathbb{N},\mathcal{J})}$ and let ψ be the representative chosen in $[\|\psi'\|]$. The definition of $\simeq$ implies that statement 19.2.3 holds for ψ, ψ', hence:

$$\sigma_{[\psi']} \left\langle \mathcal{S}_{[\psi']} \right\rangle = \sigma_{[\psi]} \left\langle \mathcal{S}_{[\psi]} \right\rangle \subset \sigma_{\text{seq}} \left\langle \mathcal{S}_{\text{seq}} \right\rangle .$$

Note. The detailed description of the mapping from the state space of the ITP into the projective state space obtained for this proof, in particular the characterization of which super-selection sectors of the ITP are sent onto identical images (owing to the equivalence relation from statement 19.2.3 being strictly coarser than the relation $\simeq$, as the latter is sensible to the relative *phase* of the factors ψ_n, while the former is not), could be easily generalized to the situation considered in theorem 5.11 (with possibly uncountably many tensor product factors). □

The non-existence of narrow states in the $G = \mathbb{R}$ case (prop. 18.14) indirectly proves that $\mathcal{L}_{\text{HF}}$ does not admit a countable cofinal subset (otherwise, such states could be constructed as described above): indeed, this can be checked directly.

Proposition 19.3 Let $\mathcal{L}_{\text{HF}}$ be the directed label set defined in def. 10.12. $\mathcal{L}_{\text{HF}}$ is *uncountable* and for any $\eta' \in \mathcal{L}_{\text{HF}}$:

$$\mathcal{L}_{\text{HF}}[\eta'] := \{\eta \in \mathcal{L}_{\text{HF}} \mid \eta \preccurlyeq \eta'\}$$

is *finite*. Hence, $\mathcal{L}_{\text{HF}}$ does not admit any countable cofinal subset.

Proof $\mathcal{L}_{\text{HF}}$ *is uncountable.* Let $\Psi : U \to V$ be an analytical coordinate patch on Σ, with U an open neighborhood of 0 in $\mathbb{R}^d$, $\epsilon > 0$ such that $B^{(d)}_\epsilon \subset U$, and, for any $\tau \neq \tau' \in [0, 2]$, $\breve{e}_{\tau,\tau'}$ be defined as in the proof of prop. 18.14. For any $\tau \in \,]0, 1[$, let e^-_τ, resp. e^+_τ, be the edge corresponding to $\breve{e}_{\tau,0}$, resp. $\breve{e}_{\tau,1}$, and S_τ be the surface corresponding to $\breve{S}_\tau \equiv \breve{e}_{\tau,2}$. We have, for any $\tau \in \,]0, 1[$:

$$r(e^-_\tau) \cap r(e^+_\tau) = \left\{\Psi(\tfrac{\epsilon\tau}{2}, 0)\right\} = \{b(e^-_\tau)\} = \{b(e^+_\tau)\} \;, \quad e^-_\tau \downarrow S_\tau \quad \& \quad e^+_\tau \uparrow S_\tau \,,$$

hence $\eta_\tau := (\gamma_\tau\,, \lambda_\tau) \in \mathcal{L}_{\text{HF}}$, with:

$$\gamma_\tau := \{e^-_\tau\,, e^+_\tau\} \in \mathcal{L}_{\text{graphs}} \quad \& \quad \lambda_\tau := \left[\,\{S_\tau\}\,\right]_\sim \in \mathcal{L}_{\text{profls}} \,.$$

Moreover, for any $(\tau, s), (\tau', s') \in \,]0, 1[\, \times \{0, 1\}$, we have:

$$\breve{e}_{\tau,s} \sim \breve{e}_{\tau',s'} \;\Rightarrow\; r(\breve{e}_{\tau,s}) = r(\breve{e}_{\tau',s'}) \;\Rightarrow\; \left[\tfrac{\epsilon\tau}{2}, \tfrac{\epsilon s}{2}\right] = \left[\tfrac{\epsilon\tau'}{2}, \tfrac{\epsilon s'}{2}\right] \;\Rightarrow\; (\tau, s) = (\tau', s')\,,$$

hence $\eta_\tau = \eta_{\tau'} \Leftrightarrow \tau = \tau'$. Since $]0, 1[$ is uncountable, so is $\mathcal{L}_{\text{HF}}$.

The part below any label η' is finite. Let $\gamma' \in \mathcal{L}_{\text{graphs}}$ and let $N = \#\gamma'$. For any $\gamma \preccurlyeq \gamma'$, we define:

$$\mathrm{A}_{\gamma'}(\gamma) \;:=\; \left\{ \Big(\big(a_{\gamma'\to e}(1), \epsilon_{\gamma'\to e}(1)\big), \,\ldots\,, \big(a_{\gamma'\to e}(n_{\gamma'\to e}), \epsilon_{\gamma'\to e}(n_{\gamma'\to e})\big) \Big) \;\Big|\; e \in \gamma \right\} \;\subset\; \bigcup_{1\leqslant n\leqslant N} \big(\gamma' \times \{\pm 1\}\big)^n \,,$$

where, for any $e \in \gamma$, $a_{\gamma'\to e}$, $\epsilon_{\gamma'\to e}$ and $n_{\gamma'\to e}$ have been defined in prop. 10.20, and $n_{\gamma'\to e} \leqslant N$ for $a_{\gamma'\to e}$ is injective $\{1, \ldots, n_{\gamma'\to e}\} \to \gamma'$. We have:

$$\gamma = \left\{ (e'_n)^{\epsilon'}_n \circ \ldots \circ (e'_1)^{\epsilon'}_1 \;\Big|\; \big((e'_1\,, \epsilon'_1), \,\ldots\,, (e'_n\,, \epsilon'_n)\big) \in \mathrm{A}_{\gamma'}(\gamma) \right\},$$

hence $\mathsf{A}_{\gamma'}$ is injective from $\mathcal{L}_{\text{graphs}}[\gamma'] := \{\gamma \in \mathcal{L}_{\text{graphs}} \mid \gamma \preccurlyeq \gamma'\}$ into the set of parts of $\bigcup_{1\leqslant n\leqslant N}(\gamma' \times \{\pm 1\})^n$, so:

$$\#\mathcal{L}_{\text{graphs}}[\gamma'] \leqslant 2^{\frac{(2N)^{N+1}-2N}{2N-1}} < \infty\,.$$

Next, let $\lambda' \in \mathcal{L}_{\text{profls}}$ and let $N' = \#\mathcal{F}(\lambda')$. For any $\lambda \preccurlyeq \lambda'$, we define:

$$\mathsf{H}_{\lambda'}(\lambda) := \left\{ H_{\lambda'\to\overline{F}} \;\middle|\; \overline{F} = F^{\perp} \circ F,\ F \in \mathcal{F}(\lambda)\right\} \subset \mathcal{P}(\mathcal{F}(\lambda'))\,,$$

where, for any $F \in \mathcal{F}(\lambda)$, $F^{\perp}$, $\overline{F}$ and $H_{\lambda'\to\overline{F}}$ have been defined in prop. 10.21, and $\mathcal{P}(\mathcal{F}(\lambda'))$ denotes the set of parts of $\mathcal{F}(\lambda')$. Let $F \in \mathcal{F}(\lambda)$. Since $\lambda \preccurlyeq \lambda'$, there exist $F'_1, \ldots, F'_m \in \mathcal{F}(\lambda')$ ($m \geqslant 1$) such that:

$$F = F_{\urcorner}(\lambda) \circ \bigcup_{i=1}^{m} F'_i\,,$$

and for any $i \in \{1, \ldots, m\}$, $F'_i \in H_{\lambda'\to\overline{F}}$ (see the proof of prop. 10.21). Moreover, for any $F' \in H_{\lambda'\to\overline{F}} \subset \mathcal{F}(\lambda')$, there exists, by definition of $\mathcal{F}(\lambda')$ (prop. 10.9), $e \in F'$, and there exists, by definition of $H_{\lambda'\to\overline{F}}$, $i \in \{1, \ldots, m\}$ such that $e = e'' \circ e'$ with $e' \in F'_i$. Thus, using props. 10.6.6 and 10.9.1, $F'_i = F'$. Hence, $\{F'_1, \ldots, F'_m\} = H_{\lambda'\to\overline{F}}$, in other words:

$$F = F_{\urcorner}(\lambda) \circ \bigcup_{F' \in H_{\lambda'\to\overline{F}}} F'\,. \tag{19.3.1}$$

Now, using props. 10.9.2 and 10.9.3, we get:

$$F_{\urcorner}(\lambda) = \left(F_{\urcorner}(\lambda) \circ \bigcup_{H \in \mathsf{H}_{\lambda'}(\lambda)} \bigcup_{F' \in H} F'\right)^{\perp} = \left(\bigcup_{H \in \mathsf{H}_{\lambda'}(\lambda)} \bigcup_{F' \in H} F'\right)^{\perp}.$$

Together with eq. (19.3.1) and def. 10.10, this ensures that $\mathsf{H}_{\lambda'}$ is injective from $\mathsf{L}_{\text{profls}}[\lambda'] := \{\lambda \in \mathcal{L}_{\text{profls}} \,|\, \lambda \preccurlyeq \lambda'\}$ into the set of parts of $\mathcal{P}(\mathcal{F}(\lambda'))$, hence:

$$\#\mathcal{L}_{\text{profls}}[\lambda'] \leqslant 2^{2^{N'}} < \infty\,.$$

Finally, for any $\eta' = (\gamma', \lambda') \in \mathcal{L}_{\text{HF}}$, $\mathcal{L}_{\text{HF}}[\eta'] \subset \mathcal{L}_{\text{graphs}}[\gamma'] \times \mathcal{L}_{\text{profls}}[\lambda']$, so:

$$\#\mathcal{L}_{\text{HF}}[\eta'] \leqslant 2^{2^{N} + \frac{(2N)^{N+1}-2N}{2N-1}} < \infty,$$

where $N := \#\gamma' = \#\mathcal{F}(\lambda)$. In particular, if $\mathcal{L}'$ is a cofinal subset of $\mathcal{L}_{\text{HF}}$, we have:

$$\mathcal{L}_{\text{HF}} = \bigcup_{\eta' \in \mathcal{L}'} \mathcal{L}_{\text{HF}}[\eta'],$$

so $\mathcal{L}'$ must be uncountable. □

The problem with the label set $\mathcal{L}_{\text{HF}}$ is basically that the slightest deformation or displacement of an edge (or surface) yields an observable, which, according to the structure set up in section 10, is completely independent of the original one. As argued at the beginning of the present section, this is physically not justifiable and the task of the next subsection will therefore be to formalize the idea that, whenever two edges (or surfaces) are related to each other by an infinitesimal deformation, they should be considered indistinguishable. This will allow us to cut down the algebra to a countable cardinal, while preserving both universality and diffeomorphism-invariance.

Note that the kind of result we are aiming at should not be confused with various results in the context of LQG displaying how countable cardinality or universality can be obtained or restored *after* quotienting out the diffeomorphisms [10, 28]: since we do not yet fully understand how this quotienting should be done in the projective formalism, we have to simplify the algebra of observables already at the diffeomorphism-*covariant* level, rather than at the diffeomorphism-*invariant* one (where those designations refer to the individual quantum states, not to the invariance of the overall state space). In fact, such an upfront simplification of algebra could make the strategy exposed in section 13 easier to implement in practice, thus helping to *solve* the diffeomorphism constraints.

19.2 Quasi-cofinal sequences: definition and properties

This subsection intends to clarify, in a general setting, which requirements should an increasing sequence of labels satisfy to ensure that it captures the whole algebra of observables, up to small deformations. To give a precise meaning to this notion of 'small deformations', closeness of observables will be defined with respect to a (topological) group a transformations acting on the algebra. Our definition for the action of a group on a projective system (def. 19.5) is inspired by [66, section 3.5].

We call such sequences *quasi-cofinal*, to underline their affinity with cofinal sequences, which, as recalled in the previous subsection, capture the whole algebra *exactly*. Indeed, as we will show below, a rather innocent-looking condition (def. 19.7.3), which can be understood as 'cofinality up to small deformations', is sufficient to prove two strong results:

- the projective system of quantum state spaces obtained by restricting the label set to a quasi-cofinal sequence is universal: it only depends on the original projective system and on the action of the group of transformations (theorem 19.8);
- and any transformation in this group can be approximated by a transformation acting on the restricted projective system (prop. 19.9).

Since the initial label set will eventually be restricted to a part admitting an *increasing cofinal sequence* (and thus automatically directed, for $\mathbb{N}, \leqslant$ is directed), we can afford to start from a very large, 'extended' label set $\mathcal{L}^{(\text{ext})}$, which will not even be required to be directed: as stressed in section 7, there can sometimes be a tension between ensuring the pivotal three-spaces consistency condition (eq. (5.1.1) and fig. 5.1), and preserving the directedness of the label set, so that it might prove convenient to initially relax this requirement (we will discuss how this added flexibility could be exploited at the end of the present section and in section 20).

To make the abstract construction of the present subsection clearer, it will be sufficient for now to imagine $\mathcal{L}^{(\text{ext})}$ to be the semi-analytical version of $\mathcal{L}_{\text{HF}}$ and the group of transformations $\mathcal{T}$ to consist of all semi-analytic diffeomorphisms (see [92, section IV.20], as well as the beginning of subsection 12.2). In contrast to *fully* analytic diffeomorphisms, semi-analytic ones can be local, so are usable as small deformations, and while the group of semi-analytic diffeomorphisms *do* act on the algebra generated by $\mathcal{L}_{\text{HF}}$ (since, as underlined many times above, this algebra is *identical* to the one generated by semi-analytical labels), its action is easier to write down if we use semi-analytical labels: in particular, it can then be put in the convenient form described in def. 19.5.

Definition 19.4 A projective pre-system of quantum state spaces is a quintuple:

$$\Big(\mathcal{L}^{(\text{ext})}, \big(\mathcal{H}_\eta\big)_{\eta\in\mathcal{L}^{(\text{ext})}}, \big(\mathcal{H}_{\eta'\to\eta}\big)_{\eta\preccurlyeq\eta'}, \big(\Phi_{\eta'\to\eta}\big)_{\eta\preccurlyeq\eta'}, \big(\Phi_{\eta''\to\eta'\to\eta}\big)_{\eta\preccurlyeq\eta'\preccurlyeq\eta''}\Big)$$

where $\mathcal{L}^{(\text{ext})}$ is a pre-ordered set (*not* necessarily directed) and defs. 5.1.2 to 5.1.5 hold. Whenever possible, we will use the shortened notation $\big/\mathcal{L}^{(\text{ext})}, \mathcal{H}, \Phi\big/^{\otimes}$ instead of $\Big(\mathcal{L}^{(\text{ext})}, \big(\mathcal{H}_\eta\big)_{\eta\in\mathcal{L}^{(\text{ext})}}, \big(\mathcal{H}_{\eta'\to\eta}\big)_{\eta\preccurlyeq\eta'}, \big(\Phi_{\eta'\to\eta}\big)_{\eta\preccurlyeq\eta'}, \big(\Phi_{\eta''\to\eta'\to\eta}\big)_{\eta\preccurlyeq\eta'\preccurlyeq\eta''}\Big)$.

For any $\eta \preccurlyeq \eta' \in \mathcal{L}^{(\text{ext})}$, we define $\text{Tr}_{\eta'\to\eta} : \overline{\mathcal{S}}_{\eta'} \to \overline{\mathcal{S}}_\eta$ and $\iota_{\eta'\leftarrow\eta} : \mathcal{A}_\eta \to \mathcal{A}_{\eta'}$ as in defs. 5.2 and 5.3. From eq. (5.1.1), we have, for any $\eta \preccurlyeq \eta' \preccurlyeq \eta''$:

$$\text{Tr}_{\eta''\to\eta} = \text{Tr}_{\eta'\to\eta} \circ \text{Tr}_{\eta''\to\eta'} \quad \& \quad \iota_{\eta''\leftarrow\eta} = \iota_{\eta''\leftarrow\eta'} \circ \iota_{\eta'\leftarrow\eta}\,.$$

Definition 19.5 Let $\big/\mathcal{L}^{(\text{ext})}, \mathcal{H}, \Phi\big/^{\otimes}$ be a projective pre-system of quantum state spaces and let $\mathcal{T}$ be a group. An action of $\mathcal{T}$ on $\big/\mathcal{L}^{(\text{ext})}, \mathcal{H}, \Phi\big/^{\otimes}$ is an action $T, \eta \mapsto T\eta$ of $\mathcal{T}$ on $\mathcal{L}^{(\text{ext})}$ together with families $\big(\text{U}_\eta\big)_{\eta\in\mathcal{L}^{(\text{ext})}}$ and $\big(\text{U}_{\eta'\to\eta}\big)_{\eta\preccurlyeq\eta'}$ such that:

1. $\forall T \in \mathcal{T}$, $\forall \eta \in \mathcal{L}^{(\text{ext})}$, $\text{U}_\eta(T)$ is an isomorphism of Hilbert spaces $\mathcal{H}_\eta \to \mathcal{H}_{T\eta}$;
2. $\forall T \in \mathcal{T}$, $\forall \eta \preccurlyeq \eta' \in \mathcal{L}^{(\text{ext})}$, $T\eta \preccurlyeq T\eta'$ and $\text{U}_{\eta'\to\eta}(T)$ is an isomorphism of Hilbert spaces $\mathcal{H}_{\eta'\to\eta} \to \mathcal{H}_{T\eta'\to T\eta}$, such that:
$$\Phi_{T\eta'\to T\eta} \circ \text{U}_{\eta'}(T) = \big(\text{U}_{\eta'\to\eta}(T) \otimes \text{U}_\eta(T)\big) \circ \Phi_{\eta'\to\eta}\,;$$
3. for any $\eta \in \mathcal{L}^{(\text{ext})}$,
$$\forall T, T' \in \mathcal{T},\ \ \text{U}_{T\eta}(T^{-1}) = \text{U}_\eta(T)^{-1} \quad \& \quad \text{U}_{T\eta}(T') \circ \text{U}_\eta(T) = \text{U}_\eta(T'.T)\,,$$
and for any $\eta \preccurlyeq \eta' \in \mathcal{L}^{(\text{ext})}$:
$$\forall T, T' \in \mathcal{T},\ \ \text{U}_{T\eta'\to T\eta}(T^{-1}) = \text{U}_{\eta'\to\eta}(T)^{-1} \quad \& \quad \text{U}_{T\eta'\to T\eta}(T') \circ \text{U}_{\eta'\to\eta}(T) = \text{U}_{\eta'\to\eta}(T'.T)\,.$$

In particular, for any $\eta \in \mathcal{L}^{(\text{ext})}$, $\text{U}_\eta(\mathbf{1}) = \text{id}_{\mathcal{H}_\eta}$ and, for any $\eta \preccurlyeq \eta' \in \mathcal{L}^{(\text{ext})}$, $\text{U}_{\eta'\to\eta}(\mathbf{1}) = \text{id}_{\mathcal{H}_{\eta'\to\eta}}$.

For any $\eta \in \mathcal{L}^{(\text{ext})}$ and any $T \in \mathcal{T}$, we define $T\triangleright : \overline{\mathcal{S}}_\eta \to \overline{\mathcal{S}}_{T\eta}$, resp. $\mathcal{A}_\eta \to \mathcal{A}_{T\eta}$, via:

$$\forall \rho_\eta \in \overline{\mathcal{S}}_\eta\,,\ T\triangleright\rho_\eta := \text{U}_\eta(T)\,\rho_\eta\,\text{U}_\eta(T)^{-1}\,, \quad \text{resp. } \forall A_\eta \in \mathcal{A}_\eta\,,\ T\triangleright A_\eta := \text{U}_\eta(T)\,A_\eta\,\text{U}_\eta(T)^{-1}\,.$$

From assumption 19.5.3, $\triangleright$ is a group action of $\mathcal{T}$ on $\bigsqcup_{\eta\in\mathcal{L}^{(\text{ext})}} \mathcal{A}_\eta$, and, from assumption 19.5.2, we have, for any $\eta \preccurlyeq \eta' \in \mathcal{L}^{(\text{ext})}$ and any $T \in \mathcal{T}$:

$$\text{Tr}_{T\eta'\to T\eta}\big(T\triangleright\,\cdot\,\big) = T\triangleright\big(\text{Tr}_{\eta'\to\eta}(\,\cdot\,)\big) \quad \& \quad \iota_{T\eta'\leftarrow T\eta}\big(T\triangleright\,\cdot\,\big) = T\triangleright\big(\iota_{\eta'\leftarrow\eta}(\,\cdot\,)\big)\,. \tag{19.5.1}$$

Proposition 19.6 Let $\big/\mathcal{L}^{(\text{ext})}, \mathcal{H}, \Phi\big/^{\otimes}$ be a projective pre-system of quantum state spaces and let

$\mathcal{T}$ be a *topological* group acting on $/\mathcal{L}^{(\mathrm{ext})}, \mathcal{H}, \Phi/^{\otimes}$. We denote by $\mathcal{A}^{\sqcup}_{/\mathcal{L}^{(\mathrm{ext})},\mathcal{H},\Phi/}$ the set of all subsets in $\bigsqcup_{\eta\in\mathcal{L}^{(\mathrm{ext})}} \mathcal{A}_\eta$, and we define, for any open neighborhood V of $\mathbf{1}$ in $\mathcal{T}$:

$$\mathcal{U}_V := \left\{(Y,\ Y') \in \mathcal{A}^{\sqcup}_{/\mathcal{L}^{(\mathrm{ext})},\mathcal{H},\Phi/} \times \mathcal{A}^{\sqcup}_{/\mathcal{L}^{(\mathrm{ext})},\mathcal{H},\Phi/} \,\middle|\, Y' \subset V \triangleright Y \quad \& \quad Y \subset V^{-1} \triangleright Y'\right\},$$

where $V^{-1} := \left\{T^{-1} \,\middle|\, T \in V\right\}$ and, for any $Y \in \mathcal{A}^{\sqcup}_{/\mathcal{L}^{(\mathrm{ext})},\mathcal{H},\Phi/}$, $V \triangleright Y := \{T \triangleright A \mid T \in V,\, A \in Y\}$.

The set $\left\{\mathcal{U} \,\middle|\, \exists\, V \text{ open neighborhood of } \mathbf{1} \text{ in } \mathcal{T} \,/\, \mathcal{U}_V \subset \mathcal{U}\right\}$ is a uniform structure on $\sqcup_{\eta\in\mathcal{L}^{(\mathrm{ext})}}\mathcal{A}_\eta$ [25, def. IX.11.1]. For any $Y, Y' \in \mathcal{A}^{\sqcup}_{/\mathcal{L}^{(\mathrm{ext})},\mathcal{H},\Phi/}$, we say that Y' is V-close to Y if $(Y, Y') \in \mathcal{U}_V$.

Proof To prove that $\left\{\mathcal{U} \,\middle|\, \exists\, V \text{ open neighborhood of } \mathbf{1} \text{ in } \mathcal{T} \,/\, \mathcal{U}_V \subset \mathcal{U}\right\}$ is a uniform structure, we need to prove that:

1. for any open neighborhood V of $\mathbf{1}$ in $\mathcal{T}$, $\left\{(Y, Y) \,\middle|\, Y \in \mathcal{A}^{\sqcup}_{/\mathcal{L}^{(\mathrm{ext})},\mathcal{H},\Phi/}\right\} \subset \mathcal{U}_V$;
2. for any open neighborhoods V_1, V_2 of $\mathbf{1}$ in $\mathcal{T}$, there exists an open neighborhood W of $\mathbf{1}$ in $\mathcal{T}$ such that $\mathcal{U}_W \subset \mathcal{U}_{V_1} \cap \mathcal{U}_{V_2}$;
3. for any open neighborhood V of $\mathbf{1}$ in $\mathcal{T}$, there exists an open neighborhood W of $\mathbf{1}$ in $\mathcal{T}$ such that:

   $$\left\{(Y,\ Y') \in \mathcal{A}^{\sqcup}_{/\mathcal{L}^{(\mathrm{ext})},\mathcal{H},\Phi/} \times \mathcal{A}^{\sqcup}_{/\mathcal{L}^{(\mathrm{ext})},\mathcal{H},\Phi/} \,\middle|\, \exists Y'' \in \mathcal{A}^{\sqcup}_{/\mathcal{L}^{(\mathrm{ext})},\mathcal{H},\Phi/} \,/\, (Y,\ Y''),\, (Y',\ Y'') \in \mathcal{U}_W\right\} \subset \mathcal{U}_V,$$

   which can be rewritten as:

   $$\left\{(Y,\ Y') \in \mathcal{A}^{\sqcup}_{/\mathcal{L}^{(\mathrm{ext})},\mathcal{H},\Phi/} \times \mathcal{A}^{\sqcup}_{/\mathcal{L}^{(\mathrm{ext})},\mathcal{H},\Phi/} \,\middle|\, (Y \cup Y') \subset W^{-1} \triangleright (W \triangleright Y \,\cap\, W \triangleright Y')\right\} \subset \mathcal{U}_V.$$

Def. 19.5.3 ensures that for any $A \in \bigsqcup_{\eta\in\mathcal{L}^{(\mathrm{ext})}} \mathcal{A}_\eta$, $\mathbf{1} \triangleright A = A$, so statement 19.6.1 holds. Next, for any open neighborhoods V_1, V_2 of $\mathbf{1}$ in $\mathcal{T}$, $W := V_1 \cap V_2$ is an open neighborhood of $\mathbf{1}$ in $\mathcal{T}$ and we have $W^{-1} = V_1^{-1} \cap V_2^{-1}$, so $\mathcal{U}_W \subset \mathcal{U}_{V_1} \cap \mathcal{U}_{V_2}$, hence statement 19.6.2 holds.

Let V be an open neighborhood of $\mathbf{1}$ in $\mathcal{T}$. Since V is a topological group $T, T' \mapsto (T')^{-1} . T$ is continuous, hence:

$$\widetilde{W} := \left\{(T, T') \in \mathcal{T} \times \mathcal{T} \,\middle|\, (T')^{-1} . T \in V\right\}$$

is an open neighborhood of $(\mathbf{1}, \mathbf{1})$ in $\mathcal{T} \times \mathcal{T}$. Then, there exists open neighborhoods W', W'' of $\mathbf{1}$ in $\mathcal{T}$ such that $W' \times W'' \subset \widetilde{W}$. Defining $W := W' \cap W''$, W is also an open neighborhood of $\mathbf{1}$ in $\mathcal{T}$, and we have:

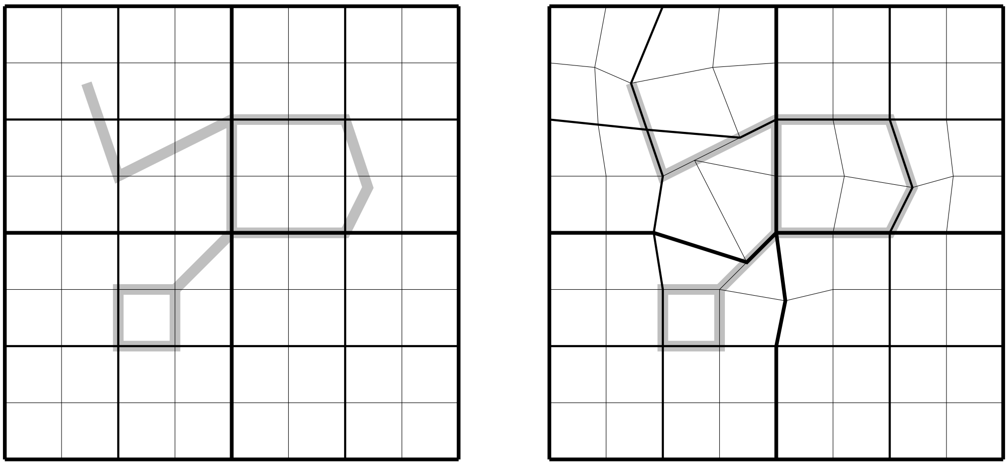
$$\forall Y \in \mathcal{A}^{\sqcup}_{/\mathcal{L}^{(\mathrm{ext})},\mathcal{H},\Phi/}, \quad W^{-1} \triangleright (W \triangleright Y) \subset (V \cap V^{-1}) \triangleright Y.$$

Thus, for any Y, $Y' \in \mathcal{A}^{\sqcup}_{/\mathcal{L}^{(\mathrm{ext})},\mathcal{H},\Phi/}$, we get:

$$W^{-1} \triangleright (W \triangleright Y \,\cap\, W \triangleright Y') \subset \left(W^{-1} \triangleright (W \triangleright Y)\right) \cap \left(W^{-1} \triangleright (W \triangleright Y')\right) \subset (V \triangleright Y) \cap (V^{-1} \triangleright Y'),$$

which proves statement 19.6.3. □

A first idea to express the notion of an increasing sequence $(\kappa_n)_n$ being 'cofinal up to small



We symbolically represent the quasi-cofinal sequence by finer and finer grids (in black)
and the label to be approximated by thick line segments (in gray)

Figure 19.1 – Deforming the quasi-cofinal sequence to adapt it to an arbitrary label, while preserving all parts that are already in place

deformations' would be to require that, for any label $\eta' \in \mathcal{L}^{(\text{ext})}$, an arbitrarily small deformation of the sequence $(\kappa_n)_n$ should be sufficient to make η' a sublabel of some sufficiently fine κ_n. However, it turns out that a slighly stronger 'quasi-cofinality' condition (def. 19.7.3) can be much more powerful, leading to the advertised results regarding universality of the restricted projective system and approximation of the transformations in $\mathcal{T}$. The key adjustment, that will be crucial to prove these results, is to require that, whenever η' have some parts that are *already adapted* to the quasi-cofinal sequence, the small deformation mentioned above should *leave these parts untouched* (fig. 19.1).

Definition 19.7 Let $\left/\mathcal{L}^{(\text{ext})}, \mathcal{H}, \Phi\right/^{\otimes}$ be a projective pre-system of quantum state spaces and let $\mathcal{T}$ be a topological group acting on $\left/\mathcal{L}^{(\text{ext})}, \mathcal{H}, \Phi\right/^{\otimes}$. A quasi-cofinal sequence in $\mathcal{L}^{(\text{ext})}$ with respect to this action is a sequence $(\kappa_n)_{n\in\mathbb{N}}$ in $\mathcal{L}^{(\text{ext})}$ such that:

1. κ_o is a least element in $\mathcal{L}^{(\text{ext})}$, ie. $\forall \eta \in \mathcal{L}^{(\text{ext})},\ \kappa_o \preccurlyeq \eta$;
2. $(\kappa_n)_{n\in\mathbb{N}}$ is increasing, ie. $\forall n \leqslant n' \in \mathbb{N},\ \kappa_n \preccurlyeq \kappa_{n'}$;
3. for any open neighborhood V of $\mathbf{1}$ in $\mathcal{T}$, any $n \in \mathbb{N}$ and any $\eta' \succcurlyeq \eta \in \mathcal{L}^{(\text{ext})}$ such that $\eta \preccurlyeq \kappa_n$, there exists $n' \geqslant n$ and $T \in V$ such that:

   $$T\eta = \eta\,,\quad \mathrm{U}_\eta(T) = \mathrm{id}_{\mathcal{H}_\eta} \quad \& \quad \eta' \preccurlyeq T\kappa_{n'}\,.$$

For any quasi-cofinal sequence $\kappa = (\kappa_n)_{n\in\mathbb{N}}$, we define:

$$\mathcal{L}^{[\kappa]} := \left\{\eta \in \mathcal{L}^{(\text{ext})} \;\middle|\; \exists n \in \mathbb{N} \,/\, \eta \preccurlyeq \kappa_n\right\}.$$

$\mathcal{L}^{[\kappa]}$ is a directed set (for $\mathbb{N}$ is), so that $\left(\mathcal{L}^{[\kappa]}, \mathcal{H}, \Phi\right)$ is a projective system of quantum state spaces. By construction, $\{\kappa_n \mid n \in \mathbb{N}\}$ is cofinal in $\mathcal{L}^{[\kappa]}$.

While the choice of a quasi-cofinal sequence is far from unique (if only because omitting terms will not void the requirements of def. 19.7), we now want to show that the resulting projective system does *not* depend on this choice. More precisely, the projective systems defined from two different quasi-cofinal sequences $(\kappa_n)_n$ and $(\lambda_m)_m$ can be matched through an arbitrarily small deformation. The idea of the proof is to *interlace* the two sequences, by applying small deformations to both $(\kappa_n)_n$ and $(\lambda_m)_m$: we will then be able to identify their associated projective systems using the same extension/restriction routine that we used repeatedly, eg. in subsection 5.2. Here is the reason why we insisted, in the formulation of the quasi-cofinality property 19.7.3, to protect against deformation any part of the quasi-cofinal sequence that happens to be already adapted to the target label: this allows us to *recursively* construct the required deformations of $(\kappa_n)_n$ and $(\lambda_m)_m$, by alternately adapting $(\kappa_n)_n$ to a certain λ_m , and, in the next step, $(\lambda_m)_m$ to a certain κ_n .

Theorem 19.8 Let $\left/\mathcal{L}^{(\text{ext})}, \mathcal{H}, \Phi\right/^{\otimes}$ be a projective pre-system of quantum state spaces and let $\mathcal{T}$ be a topological group acting on $\left/\mathcal{L}^{(\text{ext})}, \mathcal{H}, \Phi\right/^{\otimes}$. Let $\kappa = (\kappa_n)_{n\in\mathbb{N}}$ and $\lambda = (\lambda_m)_{m\in\mathbb{N}}$ be two quasi-cofinal sequences in $\mathcal{L}^{(\text{ext})}$ with respect to this action. Then, there exist, for any open neighborhood V of $\mathbf{1}$ in $\mathcal{T}$, a bijective map $\sigma : \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)}$ and a C^*-algebra isomorphism $\alpha : \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)} \to \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$ such that:

1. for any $\rho \in \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$ and any $A \in \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)}$, $\operatorname{Tr} \rho\, \alpha(A) = \operatorname{Tr} \sigma(\rho)\, A$;
2. $\alpha\left\langle \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)} \right\rangle = \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$ and for any $A \in \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)}$, $\alpha(A)$ is V-close to A (note that $\mathcal{A}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$, $\mathcal{A}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)} \subset \mathcal{A}^{\sqcup}_{/\mathcal{L}^{(\text{ext})},\mathcal{H},\Phi/}$).

Proof *Sequences of deformations* $(T_k)_{k\in\mathbb{N}}$, $(S_k)_{k\in\mathbb{N}}$. We define recursively families $(V_k)_{k\in\mathbb{N}}$, $(W_k)_{k\in\mathbb{N}}$ of open neighborhoods of $\mathbf{1}$ in $\mathcal{T}$ as follows:

3. V_o and W_o are chosen (in a way similar to the proof of prop. 19.6) so that:
$$\forall (T, S) \in V_o \times W_o\,,\ T^{-1} . S \in V\,;$$
4. for any $k \geqslant 1$, V_k and W_k are chosen (again in a similar way) so that:
$$\forall T, T' \in V_k\,,\ T . T' \in V_{k-1} \quad \& \quad \forall S, S' \in W_k\,,\ S . S' \in W_{k-1}\,.$$

Next, we define recursively families $(n_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}}$ of integers, and families $(T_k)_{k\in\mathbb{N}}$, $(S_k)_{k\in\mathbb{N}}$ of elements in $\mathcal{T}$, in the following way:

5. $n_o := 0$, $m_o := 0$, $T_o := \mathbf{1}$ and $S_o := \mathbf{1}$.
6. For any $k \geqslant 1$, we have $\kappa_{n_{k-1}} \preccurlyeq \left(T^{-1}_{k-1} . S_{k-1}\right)\lambda_{m_{k-1}}$ (as follows from either point 19.8.5 if $k = 1$ or from point 19.8.7 for the step $k-1$ if $k > 1$) and $\kappa_{n_{k-1}} \preccurlyeq \kappa_{n_{k-1}+1}$ (from def. 19.7.2). Using def. 19.7.3 for the cofinal family $(\kappa_n)_{n\in\mathbb{N}}$, we choose $n_k \geqslant n_{k-1} + 1$ and $\widetilde{T}_k \in V_k$ such that:
$$\widetilde{T}_k \kappa_{n_{k-1}} = \kappa_{n_{k-1}}\,,\ \ \mathrm{U}_{\kappa_{n_{k-1}}}(\widetilde{T}_k) = \mathrm{id}_{\mathcal{H}_{\kappa_{n_{k-1}}}} \quad \& \quad \left(T^{-1}_{k-1} . S_{k-1}\right)\lambda_{m_{k-1}} \preccurlyeq \widetilde{T}_k \kappa_{n_k}\,,$$
and, defining $T_k := T_{k-1} . \widetilde{T}_k$, we have:
$$T_k \kappa_{n_{k-1}} = T_{k-1}\kappa_{n_{k-1}}\,,\ \ \mathrm{U}_{\kappa_{n_{k-1}}}(T_k) = \mathrm{U}_{\kappa_{n_{k-1}}}(T_{k-1}) \quad \& \quad S_{k-1}\lambda_{m_{k-1}} \preccurlyeq T_k \kappa_{n_k}\,.$$
7. For any $k \geqslant 1$, we have $\lambda_{m_{k-1}} \preccurlyeq \left(S^{-1}_{k-1} . T_k\right)\kappa_{n_k}$ (as follows from point 19.8.6) and $\lambda_{m_{k-1}} \preccurlyeq \lambda_{m_{k-1}+1}$

(from def. 19.7.2). Using def. 19.7.3 for the cofinal family $(\lambda_m)_{m\in\mathbb{N}}$, we choose $m_k \geqslant m_{k-1}+1$ and $\widetilde{S}_k \in W_k$ such that:

$$\widetilde{S}_k\lambda_{m_{k-1}} = \lambda_{m_{k-1}}\,,\quad \mathrm{U}_{\lambda_{m_{k-1}}}(\widetilde{S}_k) = \mathrm{id}_{\mathcal{H}_{\lambda_{m_{k-1}}}} \quad \& \quad \big(S_{k-1}^{-1}\,.\,T_k\big)\kappa_{n_k} \preccurlyeq \widetilde{S}_k\lambda_{m_k}\,,$$

and, defining $S_k := S_{k-1}\,.\,\widetilde{S}_k$, we have:

$$S_k\lambda_{m_{k-1}} = S_{k-1}\lambda_{m_{k-1}}\,,\quad \mathrm{U}_{\lambda_{m_{k-1}}}(S_k) = \mathrm{U}_{\lambda_{m_{k-1}}}(S_{k-1}) \quad \& \quad T_k\kappa_{n_k} \preccurlyeq S_k\lambda_{m_k}\,.$$

For any $k \in \mathbb{N}$, we introduce the notations $K_k := \kappa_{n_k}$ & $L_k := \lambda_{m_k}$, so that we have:

$$T_kK_k \preccurlyeq S_kL_k \preccurlyeq T_{k+1}K_{k+1}\,,$$

as well as:

$$T_{k+1}K_k = T_kK_k\,,\quad \mathrm{U}_{K_k}(T_{k+1}) = \mathrm{U}_{K_k}(T_k) \quad \& \quad S_{k+1}L_k = S_kL_k\,,\quad \mathrm{U}_{L_k}(S_{k+1}) = \mathrm{U}_{L_k}(S_k)\,.$$

We also define, for any $k \in \mathbb{N}$, $R_k := S_k^{-1}\,.\,T_k$.

Using the definitions of the sequences $(V_k)_{k\in\mathbb{N}}$, $(W_k)_{k\in\mathbb{N}}$ and $(T_k)_{k\in\mathbb{N}}$, $(S_k)_{k\in\mathbb{N}}$, we can prove recursively that:

$$\forall k \in \mathbb{N},\quad \forall T \in V_k\,,\ T_k\,.\,T \in V_o \quad \& \quad \forall S \in W_k\,,\ S_k\,.\,S \in W_o\,.$$

Thus, for any $k \in \mathbb{N}$, $(T_k\,,\,S_k) \in V_o \times W_o$ (since $\mathbf{1} \in V_k \cap W_k$), so $R_k^{-1} \in V$ and $R_k \in V^{-1}$.

In addition, for any $k \geqslant 1$, we have $n_k > n_{k-1}$, resp. $m_k > m_{k-1}$, so the sequence $(n_k)_{k\in\mathbb{N}}$, resp. $(m_k)_{k\in\mathbb{N}}$, is strictly increasing, and $\mathcal{K} := \{K_k \mid k \in \mathbb{N}\}$, resp. $\mathcal{L} := \{L_k \mid k \in \mathbb{N}\}$, is cofinal in $\mathcal{L}^{[\kappa]}$, resp. $\mathcal{L}^{[\lambda]}$. Thus, from prop. 5.6, there exist a bijective map $\sigma_{\mathcal{K}} : \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{K},\mathcal{H},\Phi)}$, resp. $\sigma_{\mathcal{L}} : \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$, and a C^*-algebra isomorphism $\alpha_{\mathcal{K}} : \overline{\mathcal{A}}^{\otimes}_{(\mathcal{K},\mathcal{H},\Phi)} \to \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$, resp. $\alpha_{\mathcal{L}} : \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \to \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)}$, such that $\sigma_{\mathcal{K}}$ and $\alpha_{\mathcal{K}}$, resp. $\sigma_{\mathcal{L}}$ and $\alpha_{\mathcal{L}}$, are Tr-intertwined.

Mapping states. Let $\rho \in \overline{\mathcal{S}}^{\otimes}_{(\mathcal{K},\mathcal{H},\Phi)}$. For any $k \in \mathbb{N}$, we have $S_kL_k \preccurlyeq T_{k+1}K_{k+1}$, so we can define a non-negative traceclass operator $\widetilde{\rho}_k$ on $\mathcal{H}_{L_k}$ via:

$$\widetilde{\rho}_k := S_k^{-1}\triangleright\big(\mathrm{Tr}_{T_{k+1}K_{k+1}\to S_kL_k}\,(T_{k+1}\triangleright\rho_{K_{k+1}})\big)\,. \tag{19.8.1}$$

Using $S_kL_k = S_{k+1}L_k$ and $\mathrm{U}_{L_k}(S_k) = \mathrm{U}_{L_k}(S_{k+1})$ together with eq. (19.5.1) yields:

$$\widetilde{\rho}_k = \mathrm{Tr}_{R_{k+1}K_{k+1}\to L_k}\,\big(R_{k+1}\triangleright\rho_{K_{k+1}}\big)\,. \tag{19.8.2}$$

Moreover, from eqs. (19.5.1) and (19.8.1), we get:

$$\begin{aligned}
\mathrm{Tr}_{L_{k+1}\to L_k}\,\widetilde{\rho}_{k+1} &= S_k^{-1}\triangleright\big(\mathrm{Tr}_{T_{k+2}K_{k+2}\to S_kL_k}\;T_{k+2}\triangleright\rho_{K_{k+2}}\big)\\
&= S_k^{-1}\triangleright\Big[\mathrm{Tr}_{T_{k+1}K_{k+1}\to S_kL_k}\,\big(\mathrm{Tr}_{T_{k+2}K_{k+2}\to T_{k+1}K_{k+1}}\;T_{k+2}\triangleright\rho_{K_{k+2}}\big)\Big]\\
&= S_k^{-1}\triangleright\Big(\mathrm{Tr}_{T_{k+1}K_{k+1}\to S_kL_k}\,(T_{k+1}\triangleright\rho_{K_{k+1}})\Big) = \widetilde{\rho}_k\,,
\end{aligned}$$

where we have used $L_k \preccurlyeq L_{k+1}$, $S_{k+1}L_k = S_kL_k$ and $\mathrm{U}_{L_k}(S_{k+1}) = \mathrm{U}_{L_k}(S_k)$ in the first line, $S_kL_k \preccurlyeq T_{k+1}K_{k+1} = T_{k+2}K_{k+1} \preccurlyeq T_{k+2}K_{k+2}$ in the second, and $\mathrm{U}_{K_{k+1}}(T_{k+2}) = \mathrm{U}_{K_{k+1}}(T_{k+1})$ and $\mathrm{Tr}_{K_{k+2}\to K_{k+1}}\,\rho_{K_{k+2}} = \rho_{K_{k+1}}$ in the third.

Now, for any $k \leqslant k' \in \mathbb{N}$, this allows to prove recursively:

$$\operatorname{Tr}_{L_{k'}\to L_k} \widetilde{\rho}_{k'} = \widetilde{\rho}_k\,, \tag{19.8.3}$$

so for any $k, k' \in \mathbb{N}$, such that $L_k \preccurlyeq L_{k'}$, either $k \leqslant k'$, in which case eq. (19.8.3) holds, or $k > k'$, in which case $L_k \preccurlyeq L_{k'} \preccurlyeq L_k$, so $\operatorname{Tr}_{L_{k'}\to L_k} = \left(\operatorname{Tr}_{L_k\to L_{k'}}\right)^{-1}$ and eq. (19.8.3) follows from the equality for $k' \leqslant k$. In particular, if $L_k = L_{k'}$, $\widetilde{\rho}_{k'} = \widetilde{\rho}_k$. Thus, the map:

$$\begin{array}{rccc} \sigma_{\mathcal{K}\to\mathcal{L}} : & \overline{\mathcal{S}}^{\otimes}_{(\mathcal{K},\mathcal{H},\Phi)} & \to & \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \\ & \left(\rho_{K_k}\right)_{K_k\in\mathcal{K}} & \mapsto & \left(\operatorname{Tr}_{R_{k+1}K_{k+1}\to L_k}\left(R_{k+1}\triangleright\rho_{K_{k+1}}\right)\right)_{L_k\in\mathcal{L}} \end{array},$$

is well-defined as a map $\overline{\mathcal{S}}^{\otimes}_{(\mathcal{K},\mathcal{H},\Phi)} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}$.

Mapping observables. Let $A_{L_k} \in \mathcal{A}_{L_k}$. Since $S_{k+1}L_k = S_kL_k \preccurlyeq T_{k+1}K_{k+1}$ and $\mathrm{U}_{L_k}(S_{k+1}) = \mathrm{U}_{L_k}(S_k)$, we have, using eq. (19.5.1):

$$\widetilde{(A_{L_k})}_{k+1} := \iota_{K_{k+1}\leftarrow R^{-1}_{k+1}L_k}\left(R^{-1}_{k+1}\triangleright A_{L_k}\right) = T^{-1}_{k+1}\triangleright\left(\iota_{T_{k+1}K_{k+1}\leftarrow S_kL_k}(S_k\triangleright A_{L_k})\right) \in \mathcal{A}_{K_{k+1}}.$$

Next, let $A_{L_{k+1}} = \iota_{L_{k+1}\leftarrow L_k}(A_{L_k}) \in \mathcal{A}_{L_{k+1}}$. In a way similar to the computation of $\operatorname{Tr}_{L_{k+1}\to L_k}\widetilde{\rho}_{k+1}$ above, we have:

$$\begin{aligned} \widetilde{(A_{L_{k+1}})}_{k+2} &= T^{-1}_{k+2}\triangleright\left(\iota_{T_{k+2}K_{k+2}\leftarrow T_{k+1}K_{k+1}} \circ \iota_{T_{k+1}K_{k+1}\leftarrow S_kL_k}(S_k\triangleright A_{L_k})\right) \\ &\sim_{\mathcal{K}} T^{-1}_{k+1}\triangleright\left(\iota_{T_{k+1}K_{k+1}\leftarrow S_kL_k}(S_k\triangleright A_{L_k})\right) = \widetilde{(A_{L_k})}_{k+1}\,, \end{aligned}$$

with $\sim_{\mathcal{K}}$ defined as in eq. (5.3.2) for the projective system $(\mathcal{K}, \mathcal{H}, \Phi)^{\otimes}$.

We can then prove recursively that for any $k \leqslant k'$ and any $A_{L_k} \in \mathcal{A}_{L_k}$, $A_{L_{k'}} := \iota_{L_{k'}\leftarrow L_k}(A_{L_k}) \in \mathcal{A}_{L_{k'}}$:

$$\widetilde{(A_{L_k})}_{k+1} \sim_{\mathcal{K}} \widetilde{(A_{L_{k'}})}_{k'+1}\,.$$

Now, for any $k, k' \in \mathbb{N}$ and any $A_{L_k} \in \mathcal{A}_{L_k}$, $A_{L_{k'}} \in \mathcal{A}_{L_{k'}}$ such that $A_{L_k} \sim_{\mathcal{L}} A_{L_{k'}}$ (with $\sim_{\mathcal{L}}$ defined as in eq. (5.3.2) for the projective system $(\mathcal{L}, \mathcal{H}, \Phi)^{\otimes}$), there exists $\widetilde{k} \in \mathbb{N}$ such that $L_{\widetilde{k}} \succcurlyeq L_k$, $L_{k'}$ and $\iota_{L_{\widetilde{k}}\leftarrow L_k}(A_{L_k}) = \iota_{L_{\widetilde{k}}\leftarrow L_{k'}}(A_{L_{k'}})$. Hence, there exists $k'' \geqslant k, k', \widetilde{k}$, such that $\iota_{L_{k''}\leftarrow L_k}(A_{L_k}) = \iota_{L_{k''}\leftarrow L_{k'}}(A_{L_{k'}}) =: A_{L_{k''}}$, and:

$$\widetilde{(A_{L_k})}_{k+1} \sim_{\mathcal{K}} \widetilde{(A_{L_{k''}})}_{k''+1} \sim_{\mathcal{K}} \widetilde{(A_{L_{k'}})}_{k'+1}\,.$$

Thus, the map:

$$\begin{array}{rccc} \alpha_{\mathcal{L}\to\mathcal{K}} : & \mathcal{A}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} & \to & \mathcal{A}^{\otimes}_{(\mathcal{K},\mathcal{H},\Phi)} \\ & \left[A_{L_k}\right]_{\sim_{\mathcal{L}}} & \mapsto & \left[\iota_{K_{k+1}\leftarrow R^{-1}_{k+1}L_k}(R^{-1}_{k+1}\triangleright A_{L_k})\right]_{\sim_{\mathcal{K}}} \end{array},$$

is well-defined as an isometric $*$-algebra morphism $\mathcal{A}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \to \mathcal{A}^{\otimes}_{(\mathcal{K},\mathcal{H},\Phi)}$, and it can be extended by continuity into a C^*-algebra morphism $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \to \overline{\mathcal{A}}^{\otimes}_{(\mathcal{K},\mathcal{H},\Phi)}$. Moreover, $\alpha_{\mathcal{L}\to\mathcal{K}}$ and the previously defined map $\sigma_{\mathcal{K}\to\mathcal{L}}$ are Tr-intertwined.

Inverse mapping. In a similar fashion, we can define the maps:

$$\begin{array}{rccc} \sigma_{\mathcal{L}\to\mathcal{K}} : & \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} & \to & \overline{\mathcal{S}}^{\otimes}_{(\mathcal{K},\mathcal{H},\Phi)} \\ & \left(\rho_{L_k}\right)_{L_k\in\mathcal{L}} & \mapsto & \left(\operatorname{Tr}_{R^{-1}_kL_k\to K_k}\left(R^{-1}_k\triangleright\rho_{L_k}\right)\right)_{K_k\in\mathcal{K}} \end{array},$$

and:

$$\begin{array}{rcl} \alpha_{\mathcal{K}\to\mathcal{L}} & : & \mathcal{A}^{\otimes}_{(\mathcal{K},\mathcal{H},\Phi)} \to \mathcal{A}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)} \\ & & \left[A_{K_k}\right]_{\sim_{\mathcal{K}}} \mapsto \left[\iota_{L_k\leftarrow R_k K_k}(R_k \triangleright A_{K_k})\right]_{\sim_{\mathcal{L}}} \end{array}.$$

We have $\sigma_{\mathcal{K}\to\mathcal{L}} \circ \sigma_{\mathcal{L}\to\mathcal{K}} = \mathrm{id}_{\overline{\mathcal{S}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}}$ and $\sigma_{\mathcal{L}\to\mathcal{K}} \circ \sigma_{\mathcal{K}\to\mathcal{L}} = \mathrm{id}_{\overline{\mathcal{S}}^{\otimes}_{(\mathcal{K},\mathcal{H},\Phi)}}$, hence $\sigma_{\mathcal{K}\to\mathcal{L}}$ is bijective with $\sigma^{-1}_{\mathcal{K}\to\mathcal{L}} = \sigma_{\mathcal{L}\to\mathcal{K}}$. Similarly, $\alpha_{\mathcal{L}\to\mathcal{K}} \circ \alpha_{\mathcal{K}\to\mathcal{L}} = \mathrm{id}_{\overline{\mathcal{A}}^{\otimes}_{(\mathcal{K},\mathcal{H},\Phi)}}$ and $\alpha_{\mathcal{K}\to\mathcal{L}} \circ \alpha_{\mathcal{L}\to\mathcal{K}} = \mathrm{id}_{\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L},\mathcal{H},\Phi)}}$, hence $\alpha_{\mathcal{L}\to\mathcal{K}}$ is bijective with $\alpha^{-1}_{\mathcal{L}\to\mathcal{K}} = \alpha_{\mathcal{K}\to\mathcal{L}}$.

Closeness. We define the bijective map $\sigma := \sigma^{-1}_{\mathcal{L}} \circ \sigma_{\mathcal{K}\to\mathcal{L}} \circ \sigma_{\mathcal{K}} : \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)}$, and the C^*-algebra isomorphism $\alpha := \alpha_{\mathcal{K}} \circ \alpha_{\mathcal{L}\to\mathcal{K}} \circ \alpha^{-1}_{\mathcal{L}} : \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)} \to \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$. Let $A \in \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)}$ and $\widetilde{A} := \alpha(A) \in \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$.

Let $\eta \in \mathcal{L}^{[\lambda]}$ and $A_\eta \in \mathcal{A}_\eta$, such that $A_\eta \in A$. Let $n \in \mathbb{N}$ such that $\eta \preccurlyeq \lambda_n$ and $k \in \mathbb{N}$ such that $n \leqslant m_k$. Let $A_k := \iota_{L_k\leftarrow\eta}(A_\eta) \in \mathcal{A}_{L_k}$ and $\widetilde{A}_{k+1} := \iota_{K_{k+1}\leftarrow R^{-1}_{k+1}L_k}(R^{-1}_{k+1}\triangleright A_{L_k}) \in \mathcal{A}_{K_{k+1}}$. We have $\widetilde{A} = \left[\widetilde{A}_{k+1}\right]_{\sim_{\mathcal{L}^{[\kappa]}}}$. In particular, $\widetilde{A} \in \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$. Moreover, $\widetilde{A}_{k+1} = \iota_{K_{k+1}\leftarrow R^{-1}_{k+1}\eta}(R^{-1}_{k+1}\triangleright A_\eta)$ and $R^{-1}_{k+1}\eta \preccurlyeq R^{-1}_{k+1}L_k \preccurlyeq K_{k+1}$, so $R^{-1}_{k+1}\eta \in \mathcal{L}^{[\kappa]}$ and $R^{-1}_{k+1}\triangleright A_\eta \in \widetilde{A}$. Since $R_{k+1} \in V^{-1}$, $A_\eta \in V^{-1}\triangleright\widetilde{A}$. Therefore, $A \subset V^{-1}\triangleright\widetilde{A}$.

Similarly, for any $\widetilde{A} \in \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$, $A := \alpha^{-1}(\widetilde{A}) \in \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)}$ and $\widetilde{A} \subset V\triangleright A$, which proves statement 19.8.2. □

It is an immediate corollary of the just proven universality result that any transformation $T \in \mathcal{T}$ can be approximated, at an arbitrary precision, by a transformation that *stabilizes* the restricted projective system over a quasi-cofinal sequence $(\kappa_n)_n$: indeed, T maps $(\kappa_n)_n$ to a new quasi-cofinal sequence $(\lambda_m)_m$, and the projective system over $(\lambda_m)_m$ can then, by universality, be deformed back into the one over $(\kappa_n)_n$.

Proposition 19.9 Let $\big/\mathcal{L}^{(\text{ext})}, \mathcal{H}, \Phi\big/^{\otimes}$ be a projective pre-system of quantum state spaces and let $\mathcal{T}$ be a topological group acting on $\big/\mathcal{L}^{(\text{ext})}, \mathcal{H}, \Phi\big/^{\otimes}$. Let $\kappa = (\kappa_n)_{n\in\mathbb{N}}$ be a quasi-cofinal sequence in $\mathcal{L}^{(\text{ext})}$ with respect to this action. Then, there exist, for any $T \in \mathcal{T}$ and any open neighborhood V of $\mathbf{1}$ in $\mathcal{T}$, a bijective map $\sigma : \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$ and a C^*-algebra isomorphism $\alpha : \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \to \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$ such that:

1. for any $\rho \in \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$ and any $A \in \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$, $\operatorname{Tr}\rho\, A = \operatorname{Tr}\sigma(\rho)\, \alpha(A)$;
2. $\alpha\left\langle \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \right\rangle = \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$ and for any $A \in \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$, $\alpha(A)$ is V-close to $T\triangleright A := \{T\triangleright A_\bullet \mid A_\bullet \in A\}$.

Proof Let $T \in \mathcal{T}$ and let V be an open neighborhood of $\mathbf{1}$ in $\mathcal{T}$. For any $n \in \mathbb{N}$, we define $\underline{\kappa}_n := T\kappa_n$. For any $\underline{\eta} \in \mathcal{L}^{(\text{ext})}$, $\kappa_o \preccurlyeq T^{-1}\underline{\eta}$, hence $\underline{\kappa}_o \preccurlyeq \underline{\eta}$, and, for any $n \leqslant n' \in \mathbb{N}$, $\kappa_n \preccurlyeq \kappa_{n'}$, hence $\underline{\kappa}_n \preccurlyeq \underline{\kappa}_{n'}$. Let $\underline{W}$ be an open neighborhood of $\mathbf{1}$ in $\mathcal{T}$, $n \in \mathbb{N}$ and $\underline{\eta}' \succcurlyeq \underline{\eta} \in \mathcal{L}^{(\text{ext})}$ such that $\underline{\eta} \preccurlyeq \underline{\kappa}_n$. Then, $W := \left\{S \in \mathcal{T} \middle| T \,.\, S \,.\, T^{-1} \in \underline{W}\right\}$ is an open neighborhood of $\mathbf{1}$ in $\mathcal{T}$, $T^{-1}\underline{\eta}' \succcurlyeq T^{-1}\underline{\eta}$ and $T^{-1}\underline{\eta} \preccurlyeq \kappa_n$. Thus, there exists $n' \geqslant n$ and $S \in W$ such that:

$$S(T^{-1}\underline{\eta}) = T^{-1}\underline{\eta}\,,\quad \mathrm{U}_{T^{-1}\underline{\eta}}(S) = \mathrm{id}_{\mathcal{H}_{T^{-1}\underline{\eta}}} \quad \& \quad T^{-1}\underline{\eta}' \preccurlyeq S\kappa_{n'}\,.$$

Defining $\underline{S} := T\,.\,S\,.\,T^{-1} \in \underline{W}$, this can be rewritten as:

$$\underline{S}\underline{\eta} = \underline{\eta}\,,\quad \mathrm{U}_{\underline{\eta}}(\underline{S}) = \mathrm{id}_{\mathcal{H}_{\underline{\eta}}} \quad \& \quad \underline{\eta}' \preccurlyeq \underline{S}\,\underline{\kappa}_{n'}\,.$$

Therefore, $(\underline{\kappa}_n)_{n\in\mathbb{N}}$ is a quasi-cofinal sequence in $\mathcal{L}^{(\mathrm{ext})}$. Moreover, $\mathcal{L}^{[\underline{\kappa}]} = \left\{\underline{\eta} \,\middle|\, T^{-1}\underline{\eta} \in \mathcal{L}^{[\kappa]}\right\}$.

Now, we define:

$$\begin{array}{rccl} \underline{\sigma} & : & \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} & \to & \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\underline{\kappa}]},\mathcal{H},\Phi)} \\ & & \left(\rho_\eta\right)_{\eta\in\mathcal{L}^{[\kappa]}} & \mapsto & \left(T \triangleright \rho_{T^{-1}\underline{\eta}}\right)_{\underline{\eta}\in\mathcal{L}^{[\underline{\kappa}]}} \end{array},$$

as well as:

$$\begin{array}{rccl} \underline{\alpha} & : & \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\underline{\kappa}]},\mathcal{H},\Phi)} & \to & \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \\ & & \left[\underline{A}_{\underline{\eta}}\right]_{\sim_{\mathcal{L}^{[\underline{\kappa}]}}} & \mapsto & \left[T^{-1} \triangleright \underline{A}_{\underline{\eta}}\right]_{\sim_{\mathcal{L}^{[\kappa]}}} \end{array}.$$

Def. 19.5 ensures that $\underline{\sigma}$ is well-defined as a bijective map $\overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\underline{\kappa}]},\mathcal{H},\Phi)}$, that $\underline{\alpha}$ is well-defined as an isometric $*$-algebra isomorphism $\mathcal{A}^{\otimes}_{(\mathcal{L}^{[\underline{\kappa}]},\mathcal{H},\Phi)} \to \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$ and can be extended by continuity into a C^*-algebra isomorphism $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\underline{\kappa}]},\mathcal{H},\Phi)} \to \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$, that $\underline{\sigma}$ and $\underline{\alpha}$ are Tr-intertwined, and that, for any $\underline{A} \in \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\underline{\kappa}]},\mathcal{H},\Phi)}$, $\underline{\alpha}(\underline{A}) = T^{-1} \triangleright \underline{A}$.

Next, V^{-1} is an open neighborhood of $\mathbf{1}$ in $\mathcal{T}$ and, from theorem 19.8, there exists a bijective map $\widetilde{\sigma} : \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\underline{\kappa}]},\mathcal{H},\Phi)} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$ and a C^*-algebra isomorphism $\widetilde{\alpha} : \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \to \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\underline{\kappa}]},\mathcal{H},\Phi)}$, such that $\widetilde{\sigma}$ and $\widetilde{\alpha}$ are Tr-intertwined, $\widetilde{\alpha}\left\langle \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \right\rangle = \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\underline{\kappa}]},\mathcal{H},\Phi)}$ and:

$$\forall A \in \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}\,,\quad \widetilde{\alpha}(A) \text{ is } V^{-1}\text{-close to } A\,.$$

Thus, $\sigma := \widetilde{\sigma} \circ \underline{\sigma}$ is a bijective map $\overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$, $\alpha := \widetilde{\alpha}^{-1} \circ \underline{\alpha}^{-1}$ is a C^*-algebra isomorphism $\overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \to \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$ and statements 19.9.1 and 19.9.2 are fulfilled. □

Taking $\mathcal{T}$ to be the group of diffeomorphisms, and assuming the *existence* of a quasi-cofinal sequence for the projective system under consideration (see the next subsection for a $d = 1$ example; the existence proof in higher dimensions is currently under study), the previous result would allow to define a *discretized* theory, while preserving a notion of *diffeomorphism invariance*: such a theory would only have *countably* many observables, instead of a *continuum* thereof, but it would have *enough* automorphisms to approximate the full group of diffeomorphisms.

Restoring diffeomorphism invariance is indeed a serious concern when discretizing a background-independent theory [59, 74]: for example, a *fixed* lattice does *not* have enough automorphisms to appropriately account for diffeomorphism invariance. It would be tempting to bypass this issue altogether, by declaring such a lattice to be 'non-embedded', in the hope that one would thus quotient out any coordinate dependency. This strategy is however known to give the wrong answer, as it fails to remove enough degrees of freedom from the theory. The intuitive reason for this failure is that the lattice *itself* effectively provides a coordinate system: the disposition of the fields with respect to the lattice should therefore *also* be quotiented out when going diffeomorphism-invariant.

In the context of AQG, these difficulties are in particular the reason why diffeomorphisms have to be treated through the so-called 'Extended Master Constraint' approach [91], which can accommodate the absence of an action of the diffeomorphism group on the ITP Hilbert space of AQG.

19.3 One-dimensional toy-model

To illustrate the abstract framework laid in the previous subsection, we now want to work out a concrete example. The projective system we are considering here can be though as a one-dimensional version of $\mathcal{L}_{\text{HF}}$. To further simplify the argument below, we take Σ to be the line segment $]0, 1]$, and for each surface (which, in dimension $d = 1$, is pointlike), we only keep its downward face (ie. the face oriented toward 0): these additional simplifications are purely for convenience, and could be easily lifted. The resulting projective system is precisely the one we set up in subsection 18.2 on the label set $\mathcal{L}^{(\text{aux})}$ (see in particular prop. 18.10 and the proof of prop. 18.14), except that we will, in the following, keep the gauge group G arbitrary. As group of transformations $\mathfrak{T}$ we will use the homeomorphisms of $]0, 1]$, which act on this projective system in a transparent way, mapping a label (which is a finite set of points in $]0, 1]$) to its image, and identifying the associated Hilbert spaces accordingly (this is similar to [66, section 3.5]).

Proposition 19.10 Let $\mathcal{L}^{(\text{aux})}$ be defined as in prop. 18.10, G be a finite-dimensional Lie-group and μ be a right-invariant Haar measure on G. For any $\kappa \in \mathcal{L}^{(\text{aux})} = (e_1, \dots, e_n)$ with $0 < e_1 < \dots < e_n \leqslant 1$, we define $\mathcal{C}_\kappa := \{h : \kappa \to G\}$ and:

$$\begin{array}{rccl} E_\kappa & : & \mathcal{C}_\kappa & \to & G^{\#\kappa} \\ & & h & \mapsto & \left(h_{(-1)}(e_k)^{-1}.h(e_k)\right)_{k\in\{1,\dots,n\}} \end{array},$$

where for any $h \in \mathcal{C}_\kappa$, $h_{(-1)}$ is given by:

$$h_{(-1)}(e_1) = \mathbf{1} \quad \& \quad \forall k \in \{2, \dots, n\}\,,\ h_{(-1)}(e_k) = h(e_{k-1})\,.$$

E_κ is a diffeomorphism $\mathcal{C}_\kappa \to G^{\#\kappa}$ and we equip $\mathcal{C}_\kappa$ with the push-forward measure $\mu_\kappa := E^{-1}_{\kappa,*}\,\mu^n$. Next, for any $\kappa \subset \kappa'$, we define $\mathcal{C}_{\kappa'\to\kappa} := \mathcal{C}_{\kappa'\setminus\kappa}$ as well as:

$$\begin{array}{rccl} \varphi_{\kappa'\to\kappa} & : & \mathcal{C}_{\kappa'} & \to & \mathcal{C}_{\kappa'\to\kappa} \times \mathcal{C}_\kappa \\ & & h' & \mapsto & \left((h'_{(-1)})^{-1}.h'\right)\Big|_{\kappa'\setminus\kappa}\,,\ h'\big|_\kappa \end{array},$$

Then, these objects can be completed into a factorizing system of measured manifolds $\left(\mathcal{L}^{(\text{aux})},\, (\mathcal{C}, \mu),\, \varphi\right)^{\times}$ (def. 6.1) and we denote by $\left(\mathcal{L}^{(\text{aux})},\, \mathcal{H},\, \Phi\right)^{\otimes}$ the corresponding projective system of quantum state spaces (prop. 6.3).

Let $\mathfrak{T}$ be the group of homeomorphism $]0, 1] \to]0, 1]$ equipped with the metric:

$$\forall T, T' \in \mathfrak{T},\quad d(T,\, T') := \sup_{x\in]0,1]} |T(x) - T'(x)|\,.$$

For any $T \in \mathfrak{T}$ and any $\kappa \in \mathcal{L}^{(\text{aux})}$, we define $T\kappa := T\langle\kappa\rangle$ and:

$$\begin{array}{rcl} \mathrm{U}_\kappa(T) \ : \ \mathcal{H}_\kappa & \to & \mathcal{H}_{T\kappa} \\ \psi & \mapsto & \psi\big(\,\cdot\circ T|_\kappa\,\big) \end{array} .$$

Then, these objects can be completed into an action of $\mathcal{T}$ on $\big(\mathcal{L}^{(\text{aux})}, \mathcal{H}, \Phi\big)^{\otimes}$ (which, being a projective system of quantum state spaces, is *a fortiori* a projective pre-system of quantum state spaces).

Proof Let $\kappa \subset \kappa' \in \mathcal{L}^{(\text{aux})}$ with $\kappa' = (e'_1, \ldots, e'_{n'})$ $(0 < e'_1 < \ldots < e'_{n'} \leqslant 1)$. G being a Lie group, $\varphi_{\kappa'\to\kappa}$ is smooth $\mathcal{C}_{\kappa'} \to \mathcal{C}_{\kappa'\to\kappa} \times \mathcal{C}_\kappa$. Next, for any $(j, h) \in \mathcal{C}_{\kappa'\to\kappa} \times \mathcal{C}_\kappa$, we define $\widetilde{h}' \in \mathcal{C}_{\kappa'}$ recursively via:

$$\widetilde{h}'(e'_1) = \begin{cases} h(e'_1) & \text{if } e'_1 \in \kappa \\ j(e'_1) & \text{if } e'_1 \notin \kappa \end{cases} \quad \& \quad \forall k \in \{2, \ldots, n'\}\ ,\ \widetilde{h}'(e'_k) = \begin{cases} h(e'_k) & \text{if } e'_k \in \kappa \\ \widetilde{h}'(e'_{k-1}).j(e'_k) & \text{if } e'_k \notin \kappa \end{cases} ,$$

and we define $\widetilde{\varphi}_{\kappa'\to\kappa} : \mathcal{C}_{\kappa'\to\kappa} \times \mathcal{C}_\kappa \to \mathcal{C}_{\kappa'}$, $(j, h) \to \widetilde{h}'$. $\widetilde{\varphi}_{\kappa'\to\kappa}$ is smooth and we have $\widetilde{\varphi}_{\kappa'\to\kappa} \circ \varphi_{\kappa'\to\kappa} = \mathrm{id}_{\mathcal{C}_{\kappa'}}$ as well as $\varphi_{\kappa'\to\kappa} \circ \widetilde{\varphi}_{\kappa'\to\kappa} = \mathrm{id}_{\mathcal{C}_{\kappa'\to\kappa}} \times \mathrm{id}_{\mathcal{C}_\kappa}$, so $\varphi_{\kappa'\to\kappa}$ is a diffeomorphism. In particular, for any $\kappa \in \mathcal{L}^{(\text{aux})}$, $E_\kappa = \varphi_{\kappa\to\varnothing} : \mathcal{C}_\kappa \to \mathcal{C}_{\kappa'\to\varnothing} \times \mathcal{C}_\varnothing \approx G^{\#\kappa}$ is a diffeomorphism.

Next, for any $\kappa \subset \kappa' \in \mathcal{L}^{(\text{aux})}$ with $\kappa = (e_1, \ldots, e_n)$ $(0 < e_1 < \ldots < e_n \leqslant 1)$ and $\kappa' = (e'_1, \ldots, e'_{n'})$ $(0 < e'_1 < \ldots < e'_{n'} \leqslant 1)$, we define an action of G^n (equipped with a Lie group structure using pointwise operations) on $\mathcal{C}_{\kappa'}$ as:

$$\forall g \in G^n, \forall h' \in \mathcal{C}_{\kappa'}\ ,\ \ R^{(\kappa,g)}_{\kappa'}(h') := E^{-1}_{\kappa'}\big(E_{\kappa'}(h') \,.\, g'\big)\,,$$

where:

$$\forall g \in G^n, \forall l \in \{1, \ldots, n'\}\ ,\ \ g'_l := \begin{cases} g_k & \text{if } e'_l = e_k \\ \mathbf{1} & \text{if } e'_l \notin \kappa \end{cases} .$$

Also, for any $\kappa' \subset \kappa''$, with $\kappa' = (e'_1, \ldots, e'_{n'})$ $(0 < e'_1 < \ldots < e'_{n'} \leqslant 1)$ and $\kappa'' = (e''_1, \ldots, e''_{n''})$ $(0 < e''_1 < \ldots < e''_{n''} \leqslant 1)$, we have:

$$\big(\mathrm{id}_{\mathcal{C}_{\kappa''\to\kappa'}} \times E_{\kappa'}\big) \circ \varphi_{\kappa''\to\kappa'} \circ E^{-1}_{\kappa''}(j'') = \big(j''_m\big)_{m\in\{1,\ldots,n''\}\setminus\{m_l \,|\, 1\leqslant l\leqslant n'\}}\,, \big(j''_{m_{l-1}+1}\cdot\ \ldots\ \cdot j''_{m_l}\big)_{l\in\{1,\ldots,n'\}}\,, \tag{19.10.1}$$

where $(m_l)_{l\in\{0,\ldots,n'\}}$ is defined via:

$$m_o := 0 \quad \& \quad \forall l \in \{1, \ldots, n'\}\ ,\ e''_{m_l} = e'_l\,.$$

and we have identified $\mathcal{C}_{\kappa''\to\kappa'} \approx G^{n''-n'}$. Hence, for any $\kappa \subset \kappa' \subset \kappa''$, we obtain:

$$\forall g \in G^{\#\kappa},\ \varphi_{\kappa''\to\kappa'} \circ R^{(\kappa,g)}_{\kappa''} = (\mathrm{id}_{\mathcal{C}_{\kappa''\to\kappa'}} \times R^{(\kappa,g)}_{\kappa'}) \circ \varphi_{\kappa''\to\kappa'}\,. \tag{19.10.2}$$

Applying eq. (19.10.2) repeatedly yields, for any $\kappa \subset \kappa' \subset \kappa''$ and any $g \in G^{\#\kappa}$:

$$\begin{aligned} &\big(\mathrm{id}_{\mathcal{C}_{\kappa''\to\kappa'}} \times \varphi_{\kappa'\to\kappa}\big) \circ \varphi_{\kappa''\to\kappa'} \circ \varphi^{-1}_{\kappa''\to\kappa} \circ \big(\mathrm{id}_{\mathcal{C}_{\kappa''\to\kappa}} \times R^{(\kappa,g)}_\kappa\big) = \\ &\qquad = \big(\mathrm{id}_{\mathcal{C}_{\kappa''\to\kappa'}} \times \mathrm{id}_{\mathcal{C}_{\kappa'\to\kappa}} \times R^{(\kappa,g)}_\kappa\big) \circ \big(\mathrm{id}_{\mathcal{C}_{\kappa''\to\kappa'}} \times \varphi_{\kappa'\to\kappa}\big) \circ \varphi_{\kappa''\to\kappa'} \circ \varphi^{-1}_{\kappa''\to\kappa}\,. \end{aligned}$$

Now, there exists a map $\varphi_{\kappa''\to\kappa'\to\kappa} : \mathcal{C}_{\kappa''\to\kappa} \to \mathcal{C}_{\kappa''\to\kappa'} \times \mathcal{C}_{\kappa'\to\kappa}$ such that:

$$\forall j'' \in \mathcal{C}_{\kappa''\to\kappa}\ ,\ \ \big(\mathrm{id}_{\mathcal{C}_{\kappa''\to\kappa'}} \times \varphi_{\kappa'\to\kappa}\big) \circ \varphi_{\kappa''\to\kappa'} \circ \varphi^{-1}_{\kappa''\to\kappa}(j'', \mathbf{1}_\kappa) = \big(\varphi_{\kappa''\to\kappa'\to\kappa}(j''), \mathbf{1}_\kappa\big)\,,$$

with $\mathbf{1}_\kappa : \kappa \to G,\ e \mapsto \mathbf{1}$, so using:

$$\forall h \in \mathcal{C}_\kappa\,,\quad h = R_\kappa^{(\kappa, E_\kappa(h))}\left(\mathbf{1}_\kappa\right),$$

we get:

$$\left(\mathrm{id}_{\mathcal{C}_{\kappa''\to\kappa'}} \times \varphi_{\kappa'\to\kappa}\right) \circ \varphi_{\kappa''\to\kappa'} = \left(\varphi_{\kappa''\to\kappa'\to\kappa} \times \mathrm{id}_{\mathcal{C}_\kappa}\right) \circ \varphi_{\kappa''\to\kappa}\,. \tag{19.10.3}$$

In particular, this requires that $\varphi_{\kappa''\to\kappa'\to\kappa}$ is a diffeomorphism $\mathcal{C}_{\kappa''\to\kappa} \to \mathcal{C}_{\kappa''\to\kappa'} \times \mathcal{C}_{\kappa'\to\kappa}$. Therefore, $\left(\mathcal{L}^{(\text{aux})}, \mathcal{C}, \varphi\right)$ is a factorizing system of smooth, finite dimensional manifolds.

Moreover, for any $\kappa \subset \kappa' \in \mathcal{L}^{(\text{aux})}$ and any $g \in G^{\#\kappa}$, we have:

$$R_{\kappa',*}^{(\kappa,g)}\, \mu_{\kappa'} = \mu_{\kappa'}\,,$$

for μ is invariant under right-translations on G. Eq. (19.10.2) then yields, for any $g \in G^{\#\kappa}$:

$$\varphi_{\kappa'\to\kappa,*}\, \mu_{\kappa'} = (\mathrm{id}_{\mathcal{C}_{\kappa'\to\kappa}} \times R_\kappa^{(\kappa,g)})_* \left[\varphi_{\kappa'\to\kappa,*}\, \mu_{\kappa'}\right]$$

Using the uniqueness up to a global positive factor of the right-invariant Haar measure on $G^{\#\kappa}$, we conclude that there exists a smooth measure $\mu_{\kappa'\to\kappa}$ on $\mathcal{C}_{\kappa'\to\kappa}$ such that:

$$\varphi_{\kappa'\to\kappa,*}\mu_{\kappa'} = \mu_{\kappa'\to\kappa} \times \mu_\kappa\,.$$

Finally, from eq. (19.10.3), this also implies:

$$\varphi_{\kappa''\to\kappa'\to\kappa,*}\mu_{\kappa''\to\kappa} = \mu_{\kappa''\to\kappa'} \times \mu_{\kappa'\to\kappa}\,.$$

Action of $\mathfrak{T}$. Let T be an homeomorphism $]0, 1] \to]0, 1]$. From the intermediate value theorem, T is strictly monotonous. If T would be decreasing, we would have $T\left\langle]0, 1]\right\rangle \subset [T(1), 1]$ with $T(1) > 0$, which would contradict the surjectivity of T, so T has to be strictly increasing. T can then be extended into an homeomorphism $\widetilde{T} : [0, 1] \to [0, 1]$ with $\widetilde{T}(0) := 0$, and, in particular, T is uniformly continuous. Therefore, for any $\epsilon > 0$ there exists $\tau > 0$ such that:

$$\forall x, y \in]0, 1]\,,\quad |x - y| \leqslant \tau \Rightarrow |T(x) - T(y)| \leqslant \epsilon/2\,,$$

so, for any $T' \in \mathfrak{T}$:

$$\forall S, S' \in \mathfrak{T},\ \left(d(S, T) \leqslant \epsilon/2 \quad \& \quad d(S', T') \leqslant \tau\right)$$
$$\Rightarrow\ d(S \circ S',\ T \circ T') \leqslant d(S \circ S',\ T \circ S') + d(T \circ S',\ T \circ T') \leqslant \epsilon\,.$$

Similarly, there exists $\tau' > 0$ such that:

$$\forall S \in \mathfrak{T},\ d(S,\ T) \leqslant \tau' \ \Rightarrow\ d\left(T^{-1} \circ S,\ \mathrm{id}_{]0,1]}\right) = d\left(T^{-1},\ S^{-1}\right) \leqslant \epsilon\,.$$

Thus, the metric d makes $\mathfrak{T}$ into a topological group.

For any $\kappa \in \mathcal{L}^{(\text{aux})}$, we have $\mathrm{id}_{]0,1]}\left\langle\kappa\right\rangle = \kappa$, $\forall T,\ T\left\langle\kappa\right\rangle \in \mathcal{L}^{(\text{aux})}$ and $\forall T, T' \in \mathfrak{T}$, $(T \circ T')\left\langle\kappa\right\rangle = T\left\langle T'\left\langle\kappa\right\rangle\right\rangle$, hence $\kappa \mapsto T\kappa$ is a group action of $\mathfrak{T}$ on $\mathcal{L}^{(\text{aux})}$. Next, for any $T \in \mathfrak{T}$, T is strictly increasing, as mentioned above, so:

$$\forall h \in \mathcal{C}_{T\kappa}\,,\quad E_\kappa\left(h \circ T\big|_\kappa\right) = E_{T\kappa}(h)\,.$$

Therefore, $h \mapsto h \circ T\big|_\kappa$ is a volume-preserving diffeomorphism $(\mathcal{C}_{T\kappa}\,, \mu_{T\kappa}) \to (\mathcal{C}_\kappa\,, \mu_\kappa)$, so $\mathrm{U}_\kappa(T)$ is a unitary isomorphism from $\mathcal{H}_\kappa = L_2\left(\mathcal{C}_\kappa\,, d\mu_\kappa\right)$ into $\mathcal{H}_{T\kappa} = L_2\left(\mathcal{C}_{T\kappa}\,, d\mu_{T\kappa}\right)$. Moreover, we have:

$$\forall h \in \mathcal{C}_\kappa\,,\; h \circ \mathrm{id}_{]0,1]}\big|_\kappa = h \quad \& \quad \forall T, T' \in \mathfrak{T},\, \forall h \in \mathcal{C}_{(T'\circ T)\kappa}\,,\; h \circ (T' \circ T)\big|_\kappa = \big(h \circ T'|_{T\kappa}\big) \circ T|_\kappa\,,$$

so that:

$$\mathrm{U}_\kappa\left(\mathrm{id}_{]0,1]}\right) = \mathrm{id}_{\mathcal{H}_\kappa} \quad \& \quad \forall T, T' \in \mathfrak{T},\, \mathrm{U}_{T\kappa}(T') \circ \mathrm{U}_\kappa(T) = \mathrm{U}_\kappa(T' \circ T)\,, \tag{19.10.4}$$

which, in addition, implies:

$$\forall T \in \mathfrak{T},\, \mathrm{U}_{T\kappa}(T^{-1}) = \mathrm{U}_\kappa(T)^{-1}\,. \tag{19.10.5}$$

Let $\kappa \subset \kappa' \in \mathcal{L}^{(\mathrm{aux})}$ and $T \in \mathfrak{T}$. We then have $T\langle\kappa\rangle \subset T\langle\kappa'\rangle$ as well as $T\langle\kappa' \setminus \kappa\rangle = T\langle\kappa'\rangle \setminus T\langle\kappa\rangle$ (for T is bijective), and, T being strictly increasing:

$$\forall h' \in \mathcal{C}_{T\kappa'}\,,\quad \big(h' \circ T|_{\kappa'}\big)_{(-1)} = h'_{(-1)} \circ T|_{\kappa'}\,.$$

Thus, defining:

$$\begin{array}{rccl} \mathrm{U}_{\kappa'\to\kappa}(T) \;:\; & \mathcal{H}_{\kappa'\to\kappa} & \to & \mathcal{H}_{T\kappa'\to T\kappa} \\ & \psi & \mapsto & \psi\left(\,\cdot\, \circ T|_{\kappa'\setminus\kappa}\right) \end{array},$$

we get:

$$\mathrm{U}_{\kappa'}(T) \circ \Phi^{-1}_{\kappa'\to\kappa} = \Phi^{-1}_{T\kappa'\to T\kappa} \circ \left(\mathrm{U}_{\kappa'\to\kappa}(T) \otimes \mathrm{U}_\kappa(T)\right). \tag{19.10.6}$$

Therefore, $\mathrm{U}_{\kappa'\to\kappa}(T)$ is a unitary isomorphism $\mathcal{H}_{\kappa'\to\kappa} \to \mathcal{H}_{T\kappa'\to T\kappa}$ satisfying def. 19.5.2.

Finally, let $\kappa \subset \kappa' \in \mathcal{L}^{(\mathrm{aux})}$ and let $T, T' \in \mathfrak{T}$. Using eqs. (19.10.5) and (19.10.6), we have:

$$\begin{aligned} \mathrm{U}_{T\kappa'\to T\kappa}(T^{-1}) \otimes \mathrm{U}_{T\kappa}(T^{-1}) &= \Phi_{\kappa'\to\kappa} \circ \mathrm{U}_{T\kappa'}(T^{-1}) \circ \Phi^{-1}_{T\kappa'\to T\kappa} \\ &= \left[\Phi_{T\kappa'\to T\kappa} \circ \mathrm{U}_{\kappa'}(T) \circ \Phi^{-1}_{\kappa'\to\kappa}\right]^{-1} \\ &= \mathrm{U}_{\kappa'\to\kappa}(T)^{-1} \otimes \mathrm{U}_\kappa(T)^{-1}\,, \end{aligned}$$

hence $\mathrm{U}_{T\kappa'\to T\kappa}(T^{-1}) = \mathrm{U}_{\kappa'\to\kappa}(T)^{-1}$, and, similarly, using eqs. (19.10.4) and (19.10.6), $\mathrm{U}_{T\kappa'\to T\kappa}(T') \circ \mathrm{U}_{\kappa'\to\kappa}(T) = \mathrm{U}_{\kappa'\to\kappa}(T' \circ T)$. □

A simple quasi-cofinal sequence for this system is obtained by taking the set K of all points of the form $k/2^n$ (as will become clear in the course of the present subsection, it in fact does not really matter how these points are layered on finer and finer levels of the quasi-cofinal sequence). To prove that the quasi-cofinality property 19.7.3 holds, we will consider a set of points κ' (to be approximated), of which a subset κ already belong to K. For any two *successive* points e, e' in κ, we simply need to approximate the points in $\kappa' \cap \left]e, e'\right[$ by points in $K \cap \left]e, e'\right[$, and to find a deformation of $\left]e, e'\right[$ mapping one set of points into the other: we can, for example, use the corresponding piecewise-linear mapping.

Proposition 19.11 We consider the same objects as in prop. 19.10. Then, there exists a quasi-cofinal sequence in $\mathcal{L}^{(\mathrm{aux})}$ with respect to the action of $\mathfrak{T}$ on $\left(\mathcal{L}^{(\mathrm{aux})}, \mathcal{H}, \Phi\right)^{\otimes}$.

Proof We define $\kappa_o := \varnothing \in \mathcal{L}^{(\mathrm{aux})}$ and, for any $n \geqslant 1$:

$$\kappa_n := \left\{ \frac{k}{2^n} \,\middle|\, k \in \{1, \dots, 2^n\} \right\} \in \mathcal{L}^{(\mathrm{aux})}\,.$$

We have, for any $\kappa \in \mathcal{L}^{(\text{aux})}$, $\kappa_o \subset \kappa$, and, for any $n \leqslant n'$, $\kappa_n \subset \kappa_{n'}$.

Let V be an open neighborhood of $\mathbf{1}$ in $\mathfrak{T}$, $n \in \mathbb{N}$ and $\kappa \subset \kappa' \in \mathcal{L}^{(\text{aux})}$ with $\kappa \subset \kappa_n$. Let $\epsilon > 0$ such that $B_\epsilon \subset V$ (where B_ϵ denotes the closed ball of radius ϵ and center $\text{id}_{]0,1]}$ in $\mathfrak{T}$) and let $\epsilon' > 0$ be given by:

$$\epsilon' = \min_{\substack{e,e' \in \kappa' \\ e \neq e'}} |e - e'| .$$

Let $n' \geqslant n$ such that $\frac{1}{2^{n'}} < \frac{1}{2} \min(\epsilon, \epsilon')$. For any $k' \in \left\{1, \dots, 2^{n'} - 1\right\}$, we define $I_{k'} \subset]0, 1]$ via:

$$I_1 := \left]0, \frac{3}{2^{n'+1}}\right], \quad I_{2^{n'}-1} := \left]1 - \frac{3}{2^{n'+1}}, 1\right[$$
$$\& \quad \forall k' \in \left\{2, \dots, 2^{n'} - 2\right\}, \; I_{k'} := \left] \frac{2k'-1}{2^{n'+1}}, \frac{2k'+1}{2^{n'+1}} \right].$$

The family $(I_{k'})_{k' \in \left\{1,\dots,2^{n'}-1\right\}}$ is then a partition of $]0, 1[$, and, for each $k' \in \left\{1, \dots, 2^{n'} - 1\right\}$, $I_{k'}$ contains at most one element of κ'. For $k' \in \left\{0, \dots, 2^{n'}\right\}$, we define $e'_{k'}$ via:

$$e'_o := 0, \quad e'_{2^{n'}} := 1 \quad \& \quad \forall k' \in \left\{1, \dots, 2^{n'} - 1\right\}, \; e'_{k'} = \begin{cases} e' & \text{if } I_{k'} \cap \kappa' = \{e'\} \\ \frac{k'}{2^{n'}} & \text{else} \end{cases} .$$

For any $k' \in \left\{1, \dots, 2^{n'} - 1\right\}$, $e'_{k'} \in I_{k'}$, thus the family $(e_{k'})_{k' \in \left\{0,\dots,2^{n'}\right\}}$ is strictly increasing. Next, we define a piecewise linear homeomorphism $T :]0, 1] \to]0, 1]$, via:

$$\forall k' \in \left\{1, \dots, 2^{n'}\right\}, \forall x \in \left] \frac{k'-1}{2^{n'}}, \frac{k'}{2^{n'}} \right], \quad T(x) := (k' - 2^{n'} x)\, e_{k'-1} + (1 - k' + 2^{n'} x)\, e_{k'} .$$

For any $x \in]0, 1]$, we have $|T(x) - x| \leqslant \frac{2}{2^{n'}} < \epsilon$, so $T \in V$.

Let $e' \in \kappa'$. If $e' = 1$, then $e' = T(1) = T(\frac{2^{n'}}{2^{n'}})$. Otherwise, $k' \in]0, 1[$, hence there exists $k' \in \left\{1, \dots, 2^{n'} - 1\right\}$ such that $e' \in I_{k'}$, so $e' = e'_{k'} = T(\frac{k'}{2^{n'}})$. Therefore, $\kappa' \subset T\kappa_{n'}$. Finally, for any $k' \in \left\{1, \dots, 2^{n'}\right\}$ such that $\frac{k'}{2^{n'}} \in \kappa'$, we have $e'_{k'} = \frac{k'}{2^{n'}}$, so $T(\frac{k'}{2^{n'}}) = \frac{k'}{2^{n'}}$. Thus, $T|_{\kappa' \cap \kappa_{n'}} = \text{id}_{\kappa' \cap \kappa_{n'}}$ and, in particular, $T|_\kappa = \text{id}_\kappa$ (for $\kappa \subset \kappa' \cap \kappa_n \subset \kappa' \cap \kappa_{n'}$), yielding $T\kappa = \kappa$ and $\text{U}_\kappa(T) = \text{id}_{\mathcal{H}_\kappa}$. $\square$

In theorem 19.8, we stated that, given two quasi-cofinal sequences $(\kappa_n)_n$ and $(\lambda_m)_m$, the restricted projective system over $(\kappa_n)_n$ can be deformed into the one over $(\lambda_m)_m$. However, the deformation maps, acting at the level of the quantum states and observables, that we constructed when proving this result *a priori* do *not* arise from an element of $\mathfrak{T}$: instead, while their action on any *given* label κ_n coincides with the action of an element $T_n \in \mathfrak{T}$, this element could be n-dependent. In prop. 19.12 below, we show that *in the particular example* we are now considering, the deformation actually *does arise* from an homeomorphism of $]0, 1]$, in other words that T_n can be made independent of n. An important ingredient of the proof is to realize that a bijection from $]0, 1]$ into itself is an homeomorphism if, and only if, it is strictly increasing (basically because the topology of $\mathbb{R}$ is closely related to its order).

In particular, this result allows to derive a stronger version of prop. 19.9: not only can any element in $\mathfrak{T}$ be approximated by an automorphism of the restricted projective system, but the set of automorphisms that arise in this way moreover forms a *group* (in fact, it is a subgroup of $\mathfrak{T}$). This property could be relevant when turning to the imposition of the 'diffeomorphism' constraints (aka. the homeomorphism invariance in the present context).

Proposition 19.12 We consider the same objects as in prop. 19.10 and we assume that G is non-trivial (ie. it is not reduced to $\{1\}$). Let κ and λ be quasi-cofinal sequences in $\mathcal{L}^{(\text{aux})}$ and let $\epsilon > 0$. For any C^*-algebra isomorphism $\alpha : \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)} \to \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$ fulfilling statement 19.8.2 with respect to $V := B_\epsilon$ (the closed ball of radius ϵ and center $\text{id}_{]0,1]}$ in $\mathfrak{T}$) and any bijective map $\sigma : \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)}$ such that σ, α are Tr-intertwined, there exists $T \in B_\epsilon$ such that $\mathcal{L}^{[\lambda]} = \left\{ T\eta \;\middle|\; \eta \in \mathcal{L}^{[\kappa]} \right\}$ and:

$$\forall A \in \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)}\,,\ \alpha(A) = T^{-1} \triangleright A\,,$$

$$\forall \rho \in \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}\,,\ \sigma(\rho) = \left(T \triangleright \rho_{T^{-1}\underline{\eta}} \right)_{\underline{\eta} \in \mathcal{L}^{[\lambda]}}\,.$$

In particular, for any quasi-cofinal sequence $\kappa = (\kappa_n)_{n\in\mathbb{N}}$, the group:

$$\mathfrak{T}^{[\kappa]} := \{ T \in \mathfrak{T} \mid T\langle K \rangle = K \}\,, \text{ with } K := \bigcup_{n\in\mathbb{N}} \kappa_n\,,$$

acts on the projective system $\left(\mathcal{L}^{[\kappa]}, \mathcal{H}, \Phi \right)^{\otimes}$ and is dense in $\mathfrak{T}$.

Proof *Determination of $T \in B_\epsilon$.* Let $\kappa = (\kappa_n)_{n\in\mathbb{N}}$, $\lambda = (\lambda_m)_{m\in\mathbb{N}}$ be two quasi-cofinal sequences in $\mathcal{L}^{(\text{aux})}$ with respect to the action of $\mathfrak{T}$ on $\left(\mathcal{L}^{(\text{aux})}, \mathcal{H}, \Phi \right)^{\otimes}$, and let $V := B_\epsilon$ for some $\epsilon > 0$. Let $\alpha : \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)} \to \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$ be a C^*-algebra isomorphism fulfilling statement 19.8.2 with respect to V and let $\sigma : \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)}$ be a bijective map such that σ, α are Tr-intertwined.

For any bounded measurable function $m : G \to \mathbb{C}$, any $e \in]0, 1]$ and any $\eta \in \mathcal{L}^{(\text{aux})}$ such that $e \in \eta$, we define an operator $\widehat{\mathsf{h}_\eta^{(e,m)}} \in \mathcal{A}_\eta$ via:

$$\forall \psi \in \mathcal{H}_\eta\,, \forall h \in \mathcal{C}_\eta\,,\ \left[\widehat{\mathsf{h}_\eta^{(e,m)}}\, \psi \right](h) := m\big(h(e)\big)\, \psi(h)\,.$$

For any $\eta, \eta' \in \mathcal{L}^{(\text{aux})}$ such that $e \in \eta \cap \eta'$, we have, from the definition of $\varphi_{\eta'\to\eta}$ in prop. 19.10:

$$\widehat{\mathsf{h}_\eta^{(e,m)}} \sim \widehat{\mathsf{h}_{\eta'}^{(e,m)}}\,.$$

We denote by $\widehat{\mathsf{h}^{(e,m)}}$ the corresponding element of $\mathcal{A}^{\otimes}_{(\mathcal{L}^{(\text{aux})},\mathcal{H},\Phi)}$. Next, for any $e \in L := \bigcup_{m\in\mathbb{N}} \lambda_m$, we define $\widehat{\mathsf{h}_{[\lambda]}^{(e,m)}} := \widehat{\mathsf{h}^{(e,m)}} \cap \bigsqcup_{\eta \in \mathcal{L}^{[\lambda]}} \mathcal{A}_\eta \neq \varnothing$. Since $\mathcal{L}^{[\lambda]}$ is cofinal in $\mathcal{L}^{(\text{aux})}$, $\widehat{\mathsf{h}_{[\lambda]}^{(e,m)}} \in \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)}$. By assumption, $\alpha\big(\widehat{\mathsf{h}_{[\lambda]}^{(e,m)}}\big)$ is then V-close to $\widehat{\mathsf{h}_{[\lambda]}^{(e,m)}}$, so there exists $S_{(e,m)} \in V$ such that:

$$S_{(e,m)}^{-1} \triangleright \widehat{\mathsf{h}_{\{e\}}^{(e,m)}} \in \alpha\big(\widehat{\mathsf{h}_{[\lambda]}^{(e,m)}}\big)\,.$$

This implies $S^{-1}_{(e,m)}\ \{e\} = \left\{S^{-1}_{(e,m)}(e)\right\} \in \mathcal{L}^{[\kappa]}$, ie. $S^{-1}_{(e,m)}(e) \in K := \bigcup_{n\in\mathbb{N}} \kappa_n$, and:

$$\alpha\left(\widehat{\mathsf{h}^{(e,m)}_{[\lambda]}}\right) = \left[S^{-1}_{(e,m)} \triangleright \widehat{\mathsf{h}^{(e,m)}_{\{e\}}}\right]_{\sim_{\mathcal{L}^{[\kappa]}}} = \left[\mathsf{h}^{\widehat{\left(S^{-1}_{(e,m)}(e),m\right)}}_{\left\{S^{-1}_{(e,m)}(e)\right\}}\right]_{\sim_{\mathcal{L}^{[\kappa]}}} = \mathsf{h}^{\widehat{\left(S^{-1}_{(e,m)}(e),m\right)}}_{[\kappa]}\,,$$

where the second equality comes from the definition of the action of $\mathcal{T}$ on $\left(\mathcal{L}^{(\mathrm{aux})},\ \mathcal{H},\ \Phi\right)$ (prop. 19.10). We *choose* a *non-constant*, smooth, compactly supported map $m_o : G \to \mathbb{C}$ (thanks to G being non-trivial), and for any $e \in L$, we define $\tilde{t}(e) := S^{-1}_{(e,m_o)}(e)$. Since $S_{(e,m_o)} \in V = B_\epsilon$, we have $\left|\tilde{t}(e) - e\right| \leqslant \epsilon$.

Next, we consider $e, e' \in L$ such that $e < e'$. From the assumptions on α, we get that:

$$\alpha\left(\widehat{\mathsf{h}^{(e,m_o)}_{[\lambda]}} - \widehat{\mathsf{h}^{(e',m_o)}_{[\lambda]}}\right) = \widehat{\mathsf{h}^{(\tilde{t}(e),m_o)}_{[\kappa]}} - \widehat{\mathsf{h}^{(\tilde{t}(e'),m_o)}_{[\kappa]}}$$

is V-close to $\widehat{\mathsf{h}^{(e,m_o)}_{[\lambda]}} - \widehat{\mathsf{h}^{(e',m_o)}_{[\lambda]}}$. Hence, there exists $S \in V$ such that:

$$S^{-1} \triangleright \left(\widehat{\mathsf{h}^{(e,m_o)}_{\{e,e'\}}} - \widehat{\mathsf{h}^{(e',m_o)}_{\{e,e'\}}}\right) \in \widehat{\mathsf{h}^{(\tilde{t}(e),m_o)}_{[\kappa]}} - \widehat{\mathsf{h}^{(\tilde{t}(e'),m_o)}_{[\kappa]}}\,,$$

which can be rewritten, in a way similar to above, as:

$$\widehat{\mathsf{h}^{(S^{-1}(e),m_o)}_{\eta'}} - \widehat{\mathsf{h}^{(S^{-1}(e'),m_o)}_{\eta'}} \sim \widehat{\mathsf{h}^{(\tilde{t}(e),m_o)}_{\eta}} - \widehat{\mathsf{h}^{(\tilde{t}(e'),m_o)}_{\eta}}\,,$$

where $\eta := \left\{\tilde{t}(e),\ \tilde{t}(e')\right\}$ and $\eta' := \left\{S^{-1}(e),\ S^{-1}(e')\right\}$. Let $\eta'' := \eta \cup \eta'$. For any $\psi \in \mathcal{H}_{\eta''}$, we then have:

$$\forall h \in \mathcal{C}_{\eta''}\,,\ \ \left[m_o \circ h\left(S^{-1}(e)\right) - m_o \circ h\left(S^{-1}(e')\right)\right]\psi(h) = \left[m_o \circ h\left(\tilde{t}(e)\right) - m_o \circ h\left(\tilde{t}(e')\right)\right]\psi(h)\,.$$

Since this holds for any ψ, this implies:

$$\forall h \in \mathcal{C}_{\eta''}\,,\ \ m_o \circ h\left(S^{-1}(e)\right) - m_o \circ h\left(S^{-1}(e')\right) = m_o \circ h\left(\tilde{t}(e)\right) - m_o \circ h\left(\tilde{t}(e')\right),$$

and therefore, m_o being non-constant, $S^{-1}(e) = \tilde{t}(e)$ and $S^{-1}(e') = \tilde{t}(e')$ (this can be check by distinguishing cases and plugging specific values of h in the equality above). Now, $S^{-1} \in \mathcal{T}$, ie. it is an homeomorphism $]0,\ 1] \to]0,\ 1]$, so, by the intermediate value theorem, it is *strictly increasing*. Thus, $\tilde{t}(e) < \tilde{t}(e')$.

To summarize, we have proved that there exists a strictly increasing map $\tilde{t} : L \to K$ such that:

$$\forall e \in L,\ \ \alpha\left(\widehat{\mathsf{h}^{(e,m_o)}_{[\lambda]}}\right) = \widehat{\mathsf{h}^{(\tilde{t}(e),m_o)}_{[\kappa]}} \quad \& \quad \left|\tilde{t}(e) - e\right| \leqslant \epsilon\,.$$

Applying the same reasoning to the C^*-algebra isomorphism $\alpha^{-1} : \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \to \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)}$, which satisfies statement 19.8.2 with respect to $V^{-1} = B_\epsilon = V$, yields that $\tilde{t}$ is actually *bijective* $L \to K$ (for, m_o being non-constant, we have $\widehat{\mathsf{h}^{(e,m_o)}} = \widehat{\mathsf{h}^{(e',m_o)}} \Rightarrow e = e'$). Now, let $x \in]0,\ 1]$ and let $\epsilon' > 0$. We have $\varnothing \subset \{x\} \in \mathcal{L}^{(\mathrm{aux})}$ and $\varnothing \subset \lambda_o$, so, from 19.7.3, there exists $S \in B_{\epsilon'}$ and $m \in \mathbb{N}$ such that $\{x\} \subset S\lambda_m$, ie. there exists $y := S^{-1}(x) \in L$ such that $|x - y| \leqslant \epsilon'$. In other words, L is dense in $]0,\ 1]$, and, similarly, so is K. Then the topology on L (as a subspace of $]0,\ 1]$), coincides with its order topology, ie. it is generated by the base:

$$\left\{\,]e,\,e'[\,\cap L \middle| e,\,e' \in L\right\} \,\cup\, \left\{\,]0,\,e[\,\cap L \middle| e \in L\right\} \,\cup\, \left\{\,]e,\,1]\cap L \middle| e \in L\right\}.$$

The same holds for K and $\tilde{t}$, being bijective and strictly increasing, is thus an homeomorphism $L \to K$. Then, we can extend $\tilde{t}$ into a continuous function $\widetilde{T} :]0, 1] \to [0, 1]$. Let $x \in]0, 1]$. L being dense, there exists $e \in]0, x[\, \cap L$, and, for any $e' \in]e, x] \cap L$, $\widetilde{T}(e') = \tilde{t}(e') > \tilde{t}(e)$. By continuity, this implies $\widetilde{T}(x) \geqslant \tilde{t}(e) > 0$. So $\widetilde{T}$ is actually valued in $]0, 1]$. Similarly, $\tilde{t}^{-1}$ can be extended into a continuous function $T :]0, 1] \to [0, 1]$, and we have $T \circ \widetilde{T}\Big|_{L\to L} = \mathrm{id}_L$, as well as $\widetilde{T} \circ T\Big|_{K\to K} = \mathrm{id}_K$. Therefore, T is an homeomorphism $]0, 1] \to]0, 1]$ with $T^{-1} = \widetilde{T}$. Moreover, for any $e \in K$, $|T(e) - e| = \left|\tilde{t}^{-1}(e) - \tilde{t}\big(\tilde{t}^{-1}(e)\big)\right| \leqslant \epsilon$, so by continuity $T \in B_\epsilon$.

Since $\mathcal{L}^{(\mathrm{aux})}$ consists of *finite* subsets of $]0, 1]$ and the sequence $(\lambda_m)_{m\in\mathbb{N}}$ is increasing, we have:

$$\mathcal{L}^{[\lambda]} := \left\{\eta \in \mathcal{L}^{(\mathrm{aux})} \,\middle|\, \exists m \in \mathbb{N} \,/\, \eta \subset \lambda_m\right\} = \left\{\eta \in \mathcal{L}^{(\mathrm{aux})} \,\middle|\, \eta \subset L\right\},$$

and, similarly, $\mathcal{L}^{[\kappa]} = \left\{\eta \in \mathcal{L}^{(\mathrm{aux})} \,\middle|\, \eta \subset K\right\}$. Thus, $T\langle K\rangle = L$ yields $\mathcal{L}^{[\lambda]} = \left\{T\eta \,\middle|\, \eta \in \mathcal{L}^{[\kappa]}\right\}$. Like in the proof of prop. 19.9, we can then define an isometric $*$-algebra isomorphism $\widetilde{\alpha}$ via:

$$\begin{array}{rccc} \widetilde{\alpha} \;:\; & \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)} & \to & \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \\ & A & \mapsto & T^{-1}\triangleright A \end{array},$$

and extend it into a C^*-algebra isomorphism $\widetilde{\alpha} : \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)} \to \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$. We now want to prove that $\alpha = \widetilde{\alpha}$.

α and σ come from T. Let $e \in L$ and let $m : G \to \mathbb{C}$ be a bounded measurable function. Like above, there exists $S \in V$ such that:

$$S^{-1}\triangleright\Big(\widehat{\mathrm{h}^{(e,m)}_{\{e\}}} - \widehat{\mathrm{h}^{(e,m_o)}_{\{e\}}}\Big) \in \widehat{\mathrm{h}^{(S^{-1}_{(e,m)}(e),m_o)}_{[\kappa]}} - \widehat{\mathrm{h}^{(T^{-1}(e),m_o)}_{[\kappa]}},$$

and, defining $\eta' := \left\{S^{-1}(e),\, S^{-1}_{(e,m)}(e),\, T^{-1}(e)\right\}$, we get:

$$\forall h \in \mathcal{C}_{\eta'},\quad m \circ h\big(S^{-1}(e)\big) - m_o \circ h\big(S^{-1}(e)\big) = m \circ h\big(S^{-1}_{(e,m)}(e)\big) - m_o \circ h\big(T^{-1}(e)\big).$$

This requires either $S^{-1}_{(e,m)}(e) = T^{-1}(e)$, or m being constant. In the latter case, $\widehat{\mathrm{h}^{(e',m)}_{[\kappa]}} = m(\mathbf{1})\,\left[\mathrm{id}_{\mathcal{H}_\varnothing}\right]_{\sim_{\mathcal{L}^{[\kappa]}}}$ for any $e' \in K$, so in both cases:

$$\alpha\Big(\widehat{\mathrm{h}^{(e,m)}_{[\lambda]}}\Big) = \widehat{\mathrm{h}^{(T^{-1}(e),m)}_{[\kappa]}} = \widetilde{\alpha}\Big(\widehat{\mathrm{h}^{(e,m)}_{[\lambda]}}\Big).$$

Next, let $e \in L$ and $A_{\{e\}} \in \mathcal{A}_{\{e\}}$. Again, there exists $S \in V$ such that:

$$S^{-1}\triangleright A_{\{e\}} \in \alpha\left(\left[A_{\{e\}}\right]_{\mathcal{L}^{[\lambda]}}\right).$$

Defining $\eta := \left\{S^{-1}(e)\right\}$ and $\eta' := \left\{T^{-1}(e),\, S^{-1}(e)\right\}$, we get, for any bounded measurable function $m : G \to \mathbb{C}$:

$$\left[\iota_{\eta'\leftarrow\eta}\big(S^{-1}\triangleright A_{\{e\}}\big),\, \widehat{\mathrm{h}^{(T^{-1}(e),m)}_{\eta'}}\right] \in \alpha\left(\left[A^{(e,m)}_{\{e\}}\right]_{\mathcal{L}^{[\lambda]}}\right),$$

where $A^{(e,m)}_{\{e\}} := \left[A_{\{e\}},\, \widehat{\mathrm{h}^{(e,m)}_{\{e\}}}\right]$.

If we suppose $T^{-1}(e) < S^{-1}(e)$, we have, from the definition of $\varphi_{\eta' \to \left\{S^{-1}(e)\right\}}$:

$$\widehat{\mathrm{h}_{\eta'}^{(T^{-1}(e),m)}} = \Phi^{-1}_{\eta' \to \eta} \circ \left(\widehat{\mathrm{h}_{\eta' \to \eta}^{(T^{-1}(e),m)}} \otimes \mathrm{id}_{\mathcal{H}_\eta} \right) \circ \Phi_{\eta' \to \eta} ,$$

where:

$$\forall \psi \in \mathcal{H}_{\eta' \to \eta} \,,\, \forall j \in \mathcal{C}_{\eta' \to \eta} \,, \quad \left[\widehat{\mathrm{h}_{\eta' \to \eta}^{(T^{-1}(e),m)}} \, \psi \right](j) = m \circ j\big(T^{-1}(e)\big) \, \psi(j) \,.$$

Thus, we get $\alpha \left(\left[A^{(e,m)}_{\{e\}} \right]_{\mathcal{L}^{[\lambda]}} \right) = 0$. α being a C^*-algebra isomorphism, this implies that, for any bounded measurable function m, $A_{\{e\}}$ commutes with $\widehat{\mathrm{h}^{(e,m)}_{\{e\}}}$. Now, let ψ_o be a smooth, nowhere-vanishing, square-integrable function on $\mathcal{C}_{\{e\}}$ (eg. using a partition of unity [54, lemma 2.16 and theorem 2.18] with suitable dumping factors). For any smooth, compactly supported function ψ on $\mathcal{C}_{\{e\}}$, $m_\psi := {}^{\psi}\!/\!{}_{\psi_o}$ is a bounded measurable function on $\mathcal{C}_{\{e\}} \approx G$ and we have:

$$\widehat{\mathrm{h}^{(e,m_\psi)}_{\{e\}}} \, \psi_o = \psi \,.$$

We then get:

$$\forall h \in \mathcal{C}_{\{e\}} \,, \quad \left[A_{\{e\}} \, \psi \right](h) = \left[\widehat{\mathrm{h}^{(e,m_\psi)}_{\{e\}}} \, A_{\{e\}} \, \psi_o \right](h) = \frac{\left[A_{\{e\}} \, \psi_o \right](h)}{\psi_o(h)} \, \psi(h) \,.$$

Since this holds for any smooth compactly supported ψ, $m := \left[A_{\{e\}} \, \psi_o \right] / \psi_o$ is almost everywhere bounded by the operator norm $\left\| A_{\{e\}} \right\|_{\mathcal{A}_{\{e\}}}$ of $A_{\{e\}}$, and, by density, $A_{\{e\}} = \widehat{\mathrm{h}^{(e,m)}_{\{e\}}}$.

Therefore, we either have $\alpha \left(\left[A_{\{e\}} \right]_{\mathcal{L}^{[\lambda]}} \right) = \widetilde{\alpha} \left(\left[A_{\{e\}} \right]_{\mathcal{L}^{[\lambda]}} \right)$ or $S^{-1}(e) \leqslant T^{-1}(e)$. In the latter case, the same reasoning applied to α^{-1} yields $S^{-1}(e) = T^{-1}(e)$, thus $S^{-1}\big|_{\{e\}} = T^{-1}\big|_{\{e\}}$, so that, in this case too, $\alpha \left(\left[A_{\{e\}} \right]_{\mathcal{L}^{[\lambda]}} \right) = \widetilde{\alpha} \left(\left[A_{\{e\}} \right]_{\mathcal{L}^{[\lambda]}} \right)$.

We want to prove:

$$\forall \eta \in \mathcal{L}^{[\lambda]},\ \forall A_\eta \in \mathcal{A}_\eta \,,\ \alpha \left([A_\eta]_{\mathcal{L}^{[\lambda]}} \right) = \widetilde{\alpha} \left([A_\eta]_{\mathcal{L}^{[\lambda]}} \right) . \tag{19.12.1}$$

We proceed by recursion on $\#\eta$. The case $\#\eta = 1$ has been treated above and it implies the case $\#\eta = 0$. We now suppose that eq. (19.12.1) holds up to $\#\eta = N \geqslant 1$. Let $\eta = (e_1 \,, \ldots, e_{N+1}) \in \mathcal{L}^{[\lambda]}$ (with $0 < e_1 < \ldots < e_{N+1} \leqslant 1$). Let $\eta_1 := \{e_1\}$ and $\eta_2 := \eta \setminus \{e_1\}$. Using eq. (19.10.1), we have:

$$\forall j \in G^{N+1}, \quad \left(\mathrm{id}_{\mathcal{C}_{\eta \to \eta_1}} \times E_{\eta_1} \right) \circ \varphi_{\eta \to \eta_1} \circ E^{-1}_\eta(j) = \big(j_2 \,, \ldots, j_{N+1} \,;\, j_1 \big) \,,$$

$$\& \quad \left(\mathrm{id}_{\mathcal{C}_{\eta \to \eta_2}} \times E_{\eta_2} \right) \circ \varphi_{\eta \to \eta_2} \circ E^{-1}_\eta(j) = \big(j_1 \,;\, (j_1 \, . \, j_2) \,, j_3 \,, \ldots, j_{N+1} \big) \,.$$

We define, for any $\eta' \in \mathcal{L}^{(\mathrm{aux})}$, the unitary isomorphism $\Phi_{\eta'} : \mathcal{H}_{\eta'} \to \mathcal{H}^{\otimes \# \eta'}$, $\psi \mapsto \psi \circ E^{-1}_\eta$, where $\mathcal{H} := L_2(G, d\mu)$. We then have, for any bounded operator A_1 on $\mathcal{H}$:

$$\forall \psi \in \mathcal{H}_\eta \,,\ \forall j \in G^{N+1}, \quad \left[\Phi_\eta \circ \iota_{\eta \leftarrow \eta_1} \big(\Phi^{-1}_{\eta_1} A_1 \, \Phi_{\eta_1} \big)(\psi) \right](j) =$$

$$= \left[A_1 \Big(\Phi_\eta(\psi) \big(\,\cdot\, , j_2 \,, \ldots, j_{N+1} \big) \Big) \right](j_1) = \big(A_1 \otimes \mathrm{id}^{\otimes N}_{\mathcal{H}} \big) \, \big(\Phi_\eta(\psi) \big)(j) \,,$$

and, for any bounded operator A_2 on $\mathcal{H}^{\otimes N}$:

$$\forall \psi \in \mathcal{H}_\eta\,,\ \forall j \in G^{N+1},\quad \Big[\Phi_\eta \,\circ\, \iota_{\eta\leftarrow\eta_2}\big(\Phi^{-1}_{\eta_2} A_2\, \Phi_{\eta_2}\big)(\psi)\Big](j) =$$

$$= \left[A_2\left(\Phi_\eta(\psi)\big(j_1\,,\, (j_1^{-1}\,.\,\cdot\,),\, \cdot\,,\, \ldots\,,\, \cdot\,\big)\right)\right]\big((j_1\,.\,j_2),\, j_3\,,\, \ldots\,,\, j_{N+1}\big)$$

$$= \left(\mathrm{id}_{\mathcal{H}} \otimes \left(\mathsf{L}^{-1}_{(j_1)} A_2\, \mathsf{L}_{(j_1)}\right)\right) \big(\Phi_\eta(\psi)\big)(j)\,,$$

where, for any $u \in G$:

$$\begin{array}{rcl} \mathsf{L}_{(u)}\ :\ \mathcal{H}^{\otimes N} & \to & \mathcal{H}^{\otimes N} \\ \psi & \mapsto & \Big[(j_2\,,\, \ldots\,,\, j_{N+1}) \mapsto \psi\big((u^{-1}\,.\,j_2),\, j_3\,,\, \ldots\,,\, j_{N+1}\big)\Big] \end{array}.$$

Chaining these expressions we get:

$$\forall \psi \in \mathcal{H}_\eta\,,\ \forall j \in G^{N+1},\quad \Big[\Phi_\eta \,\circ\, \iota_{\eta\leftarrow\eta_2}\big(\Phi^{-1}_{\eta_2} A_2\, \Phi_{\eta_2}\big) \,\circ\, \iota_{\eta\leftarrow\eta_1}\big(\Phi^{-1}_{\eta_1} A_1\, \Phi_{\eta_1}\big)(\psi)\Big](j) =$$

$$= \left(A_1 \otimes \left(\mathsf{L}^{-1}_{(j_1)} A_2\, \mathsf{L}_{(j_1)}\right)\right)\big(\Phi_\eta(\psi)\big)(j)\,. \tag{19.12.2}$$

For any bounded operator A_1 on $\mathcal{H}$, resp. A_2 on $\mathcal{H}^{\otimes N}$, we define a bounded operator $I(A_1\,,\, A_2)$ on $\mathcal{H}^{\otimes (N+1)}$ via:

$$\forall \psi \in \mathcal{H}^{\otimes (N+1)},\ \forall j \in G^{N+1},\quad \big(I(A_1\,,\, A_2)\,\psi\big)(j) := \left[\left(A_1 \otimes \left(\mathsf{L}^{-1}_{(j_1)} A_2\, \mathsf{L}_{(j_1)}\right)\right)(\psi)\right](j)$$

(note that it follows in particular from the previous expression that this indeed defines a bounded operator on $\mathcal{H}^{\otimes (N+1)}$). We denote by $\mathcal{I}$ the vector subspace spanned by these operators in the space of bounded operators on $\mathcal{H}^{\otimes (N+1)}$:

$$\mathcal{I} := \mathrm{Vect}\,\left\{I(A_1\,,\, A_2)\ \middle|\ A_1 \text{ bounded operator on } \mathcal{H},\ A_2 \text{ on } \mathcal{H}^{\otimes N}\right\}$$

(*without* any completion, ie. considering only *finite* linear combinations). The recursion hypothesis, together with eq. (19.12.2) and the fact that both α and $\widetilde{\alpha}$ are C^*-algebra isomorphisms, then ensures:

$$\forall A \in \mathcal{I},\ \alpha\left(\left[\Phi^{-1}_\eta \,\circ\, A \,\circ\, \Phi_\eta\right]_{\sim_{\mathcal{L}^{[\lambda]}}}\right) = \widetilde{\alpha}\left(\left[\Phi^{-1}_\eta \,\circ\, A \,\circ\, \Phi_\eta\right]_{\sim_{\mathcal{L}^{[\lambda]}}}\right).$$

Let and $\epsilon > 0$. For any $u \in G$, $(u\,\cdot)_*\,\mu$ is a right-invariant Haar measure, so there exists $\Delta_u < \infty$ such that $(u\,.\,\cdot)_*\,\mu = \Delta_u\,\mu$. Defining, for any $\psi \in \mathcal{H}$ and any $u \in G$, $\psi^{(u)} \in \mathcal{H}$ via:

$$\forall j \in G,\ \psi^{(u)}(j) := \psi(u\,.\,j)\,,$$

we then have $\left\|\psi^{(u)}\right\|_{\mathcal{H}} = \sqrt{\Delta_u}\ \|\psi\|_{\mathcal{H}}$ (which in particular ensures that $\psi^{(u)}$ is indeed in $\mathcal{H}$). Moreover, $u \mapsto \Delta_u$ is a smooth character on G (aka. modular function of G, see [43, appendix C.4]). Hence, there exists an open neighborhood D of $\mathbf{1}$ in G such that:

$$\forall u \in D,\ |1 - \Delta_u| \leqslant \frac{\min(1,\ \epsilon)}{2}\,.$$

We now choose an orthonormal basis $(\phi_i)_{i\in\mathbb{N}}$ in $\mathcal{H}$ and an integer $M \in \mathbb{N}$. For any $i \leqslant M$, there exists a smooth, compactly supported function $\widetilde{\phi}_i$ on G (with support $C_i \subset G$) such that:

$$\left\|\phi_i - \widetilde{\phi}_i\right\|_{\mathcal{H}} \leqslant \frac{\epsilon}{2}.$$

We define the compact $C := \bigcup_{i\leqslant M} C_i \subset G$. For any $i \leqslant M$, $\widetilde{\phi}_i$ is in particular uniformly continuous, so there exists an open neighborhood V_i of $\mathbf{1}$ in G such that:

$$\forall u \in V_i\,, \quad \|\widetilde{\phi}_i - \widetilde{\phi}_i^{(u)}\|_\infty \leqslant \frac{\epsilon}{4\sqrt{\mu(C)}}.$$

In particular, we thus have:

$$\begin{aligned}\forall u \in V_i\,, \quad \|\phi_i - \phi_i^{(u)}\|_{\mathcal{H}} &\leqslant \|\phi_i - \widetilde{\phi}_i\|_{\mathcal{H}} + \|\widetilde{\phi}_i - \widetilde{\phi}_i^{(u)}\|_{\mathcal{H}} + \|\widetilde{\phi}_i^{(u)} - \phi_i^{(u)}\|_{\mathcal{H}} \\ &\leqslant \frac{\epsilon}{2}\left(1 + \sqrt{\Delta_u}\right) + \frac{\epsilon}{4}\sqrt{\frac{\mu(C_i \,\cup\, u^{-1}\,.\,C_i)}{\mu(C)}} \\ &\leqslant \frac{\epsilon}{2}\left(1 + \sqrt{\Delta_u} + \frac{\sqrt{1+\Delta_u}}{2}\right).\end{aligned}$$

We define $V := D \cap \bigcap_{i\leqslant M} V_i$, so that V is an open neighborhood of $\mathbf{1}$ in G. Let $(u_k)_{k\leqslant r}$ with $r < \infty$ be a *finite* family of points in C such that $C \subset \bigcup_{k\leqslant r} V_{[u_k]}$, where $\forall u \in G,\ V_{[u]} := \{v\,.\,u \mid v \in V\} \cap \{u\,.\,v' \mid v' \in V\}$. For any $k \leqslant r$, we define:

$$F_k := \left(V_{[u_k]} \cap C\right) \setminus \left(\textstyle\bigcup_{l<k} V_{[u_k]}\right).$$

Since $F_k \subset C$, $\mu(F_k) < \infty$, and we define $R := \{k \mid k \leqslant r \ \ \&\ \ \mu(F_k) > 0\}$. For any $k \in R$, we define $\chi_k \in \mathcal{H}$ as $\chi_k := {}^1\!/\!\sqrt{\mu(F_k)}\ \mathbb{1}_{F_k}$, where $\mathbb{1}_{F_k}$ denotes the indicator function of F_k. $(\chi_k)_{k\in R}$ is then an orthonormal family and we have, for any $i \leqslant M$:

$$\begin{aligned}\left\|\widetilde{\phi}_i - \sum_{k\in R}\left\langle \chi_k \;\middle|\; \widetilde{\phi}_i\right\rangle \chi_k\right\|^2_{\mathcal{H}} &= \sum_{l\in R}\int_{F_l} d\mu(u)\,\left|\widetilde{\phi}_i(u) - \frac{1}{\mu(F_l)}\int_{F_l} d\mu(v)\,\widetilde{\phi}_i(v)\right|^2 \\ &\leqslant \sum_{l\in R}\int_{F_l} d\mu(u)\,\left(\left|\widetilde{\phi}_i(u) - \widetilde{\phi}_i(u_l)\right| + \frac{1}{\mu(F_l)}\int_{F_l} d\mu(v)\,\left|\widetilde{\phi}_i(u_l) - \widetilde{\phi}_i(v)\right|\right)^2 \leqslant \frac{\epsilon^2}{4},\end{aligned}$$

so that $\left\|\phi_i - \sum_{k\in R}\left\langle \chi_k \;\middle|\; \widetilde{\phi}_i\right\rangle \chi_k\right\|_{\mathcal{H}} \leqslant \epsilon$. Since $\sum_{k\in R}|\chi_k\,\rangle\langle\,\chi_k|$ is an orthogonal projection, this implies $\left\|\phi_i - \sum_{k\in R}\langle \chi_k \mid \phi_i\rangle\,\chi_k\right\|_{\mathcal{H}} \leqslant \epsilon$.

Let $(i_p)_p\,,\ (j_p)_p \in \{0,\dots,M\}^{N+1}$. For any $k \in R$, we define:

$$A_1^k := \langle \chi_k \mid \phi_{i_1}\rangle\ \left|\chi_k\,\right\rangle\left\langle\,\phi_{j_1}\right|$$

$$\&\quad A_2^k := \Delta_{u_k}\left|\phi_{i_2}^{(u_k^{-1})} \otimes \phi_{i_3} \otimes \dots \otimes \phi_{i_{N+1}}\right\rangle\left\langle\,\phi_{j_2}^{(u_k^{-1})} \otimes \phi_{j_3} \otimes \dots \otimes \phi_{j_{N+1}}\right|,$$

so that $A^{(i_p)_p\,(j_p)_p} := \sum_{k\in R} I(A_1^k\,,\,A_2^k) \in \mathcal{I}$. Then, for any $\psi \in \mathcal{H}^{\otimes(N+1)}$, we have:

$$\begin{aligned}\forall h \in G^{N+1},\ \left[A^{(i_p)_p\,(j_p)_p}\,\psi\right](h) = \sum_{k\in R}\langle \chi_k \mid \phi_{i_1}\rangle\,\chi_k(h_1)\,\phi_{i_2}^{(u_k^{-1}\,.\,h_1)}(h_2)\,\times \\ \times\phi^{(3\dots)}_{(i_p)_p}(h_3\,,\,\dots\,,\,h_{N+1})\,\Delta_{h_1^{-1}\,.\,u_k}\left\langle \phi_{j_1} \otimes \phi_{j_2}^{(u_k^{-1}\,.\,h_1)} \otimes \phi^{(3\dots)}_{(j_p)_p}\;\middle|\;\psi\right\rangle,\end{aligned}$$

where, for any $(i'_p)_p \in \{0,\dots,M\}^{N+1}$, $\phi^{(3\dots)}_{(i'_p)_p} \in \mathcal{H}^{\otimes(N-1)}$ is defined by:

$$\phi^{(3\dots)}_{(i'_p)_p} := \phi_{i'_3} \otimes \dots \otimes \phi_{i'_{N+1}} .$$

Using $\left\| \phi_{i_1} - \sum_{k\in R} \langle \chi_k \mid \phi_{i_1} \rangle \, \chi_k \right\|_{\mathcal{H}} \leqslant \epsilon$, this yields:

$$\left\| A^{(i_p)_p \, (j_p)_p} \, \psi - \left| \phi_{i_1} \otimes \phi_{i_2} \otimes \phi^{(3\dots)}_{(i_p)_p} \right\rangle \left\langle \phi_{j_1} \otimes \phi_{j_2} \otimes \phi^{(3\dots)}_{(j_p)_p} \;\middle|\; \psi \right\rangle \right\|_{\mathcal{H}^{\otimes(N+1)}}$$

$$\leqslant \epsilon \, \|\psi\|_{\mathcal{H}^{\otimes(N+1)}} + \left\| \sum_{k\in R} \langle \chi_k \mid \phi_{i_1} \rangle \, \beta_k \otimes \phi^{(3\dots)}_{(i_p)_p} \right\|_{\mathcal{H}^{\otimes(N+1)}} ,$$

where, for any $k \in R$, $\beta_k \in \mathcal{H}^{\otimes 2}$ is defined by:

$$\forall h_1 , \, h_2 \in G, \quad \beta_k(h_1, h_2) := \chi_k(h_1) \left[\phi_{i_2}^{(u_k^{-1} \,.\, h_1)}(h_2) \, \Delta_{h_1^{-1} \,.\, u_k} \left\langle \phi_{j_1} \otimes \phi_{j_2}^{(u_k^{-1} \,.\, h_1)} \otimes \phi^{(3\dots)}_{(j_p)_p} \;\middle|\; \psi \right\rangle - \right.$$

$$\left. - \phi_{i_2}(h_2) \left\langle \phi_{j_1} \otimes \phi_{j_2} \otimes \phi^{(3\dots)}_{(j_p)_p} \;\middle|\; \psi \right\rangle \right] .$$

Next, for any $k \in R$, and any $h_1 \in F_k$, we have $u_k^{-1} \,.\, h_1 \in V$, so that:

$$\|\beta_k\|^2_{\mathcal{H}^{\otimes 2}} = \int_{F_k} \frac{d\mu(h_1)}{\mu(F_k)} \left\| \phi_{i_2}^{(u_k^{-1} \,.\, h_1)} \Delta_{h_1^{-1} \,.\, u_k} \left\langle \phi_{j_1} \otimes \phi_{j_2}^{(u_k^{-1} \,.\, h_1)} \otimes \phi^{(3\dots)}_{(j_p)_p} \;\middle|\; \psi \right\rangle - \right.$$

$$\left. - \phi_{i_2} \left\langle \phi_{j_1} \otimes \phi_{j_2} \otimes \phi^{(3\dots)}_{(j_p)_p} \;\middle|\; \psi \right\rangle \right\|^2_{\mathcal{H}}$$

$$\leqslant \int_{F_k} \frac{d\mu(h_1)}{\mu(F_k)} \left(\Delta_{u_k^{-1} \,.\, h_1} \left| \frac{1}{\Delta_{u_k^{-1} \,.\, h_1}} - 1 \right| + \sqrt{\Delta_{u_k^{-1} \,.\, h_1}} \left\| \phi_{i_2}^{(u_k^{-1} \,.\, h_1)} - \phi_{i_2} \right\|_{\mathcal{H}} + \right.$$

$$\left. + \left\| \phi_{j_2}^{(u_k^{-1} \,.\, h_1)} - \phi_{j_2} \right\|_{\mathcal{H}} \right)^2 \|\psi\|^2_{\mathcal{H}^{\otimes(N+1)}}$$

$$\leqslant \left(5 \, \epsilon \, \|\psi\|_{\mathcal{H}^{\otimes(N+1)}} \right)^2 .$$

Moreover, for any $k \neq l$, we have $\langle \beta_k \mid \beta_l \rangle_{\mathcal{H}^{\otimes 2}} = 0$, so we get:

$$\left\| A^{(i_p)_p \, (j_p)_p} - \left| \phi_{i_1} \otimes \phi_{i_2} \otimes \phi^{(3\dots)}_{(i_p)_p} \right\rangle \left\langle \phi_{j_1} \otimes \phi_{j_2} \otimes \phi^{(3\dots)}_{(j_p)_p} \right| \right\|_{\mathcal{A}^{\otimes(N+1)}} \leqslant 6 \, \epsilon ,$$

where $\| \cdot \|_{\mathcal{A}^{\otimes(N+1)}}$ denotes the operator norm on $\mathcal{H}^{\otimes(N+1)}$.

Let $A_\eta \in \mathcal{A}_\eta$ and let $\eta' \in \mathcal{L}^{[\kappa]}$ such that $\alpha \left([A_\eta]_{\sim_{\mathcal{L}^{[\lambda]}}} \right) = \left[A'_{\eta'} \right]_{\sim_{\mathcal{L}^{[\kappa]}}}$ for some $A'_{\eta'} \in \mathcal{A}_{\eta'}$ (thanks to statement 19.8.2). Let $\eta'' := \eta' \cup T^{-1}\eta \in \mathcal{L}^{[\kappa]}$ and let $\rho_{\eta''}$ be a non-negative traceclass operator on $\mathcal{H}_{\eta''}$. Let $n'' \in \mathbb{N}$ such that $\eta'' \subset \kappa_{n''}$, choose $\tau_{n''} \in \mathcal{H}_{\kappa_{n''} \to \eta''}$ and, for any $n > n''$, choose $\tau_n \in \mathcal{H}_{\kappa_n \to \kappa_{n-1}}$. In a way similar to prop. 19.1, there exists $\rho' \in \mathcal{S}^{\otimes}_{(\mathcal{L}^{[\kappa]}, \mathcal{H}, \Phi)}$ such that:

$$\Phi_{\kappa_{n''} \to \eta''} \, \rho'_{\kappa_{n''}} \, \Phi^{-1}_{\kappa_{n''} \to \eta''} = \left| \tau_{n''} \right\rangle \! \left\langle \tau_{n''} \right| \otimes \rho_{\eta''} \quad \& \quad \forall n > n'', \, \Phi_{\kappa_n \to \kappa_{n-1}} \, \rho'_{\kappa_n} \, \Phi^{-1}_{\kappa_n \to \kappa_{n-1}} = \left| \tau_n \right\rangle \! \left\langle \tau_n \right| \otimes \rho'_{\kappa_{n-1}} .$$

In particular, we then have $\rho'_{\eta''} = \rho_{\eta''}$. Let $\rho := \left[\sigma(\rho') \right]_\eta$ and $\widetilde{\rho} := \mathrm{U}_{T^{-1}\eta}(T) \left(\mathrm{Tr}_{\eta'' \to T^{-1}\eta} \, \rho_{\eta''} \right) \mathrm{U}_\eta(T^{-1})$. Next, for any $M \in \mathbb{N}$, we define a finite-dimensional vector subspace $\mathcal{J}_M \subset \mathcal{H}_\eta$ as:

$$\mathcal{J}_M := \mathrm{Vect} \left\{ \Phi_\eta^{-1} \left| \phi_{i_1} \otimes \dots \otimes \phi_{i_{N+1}} \right\rangle \;\middle|\; \left(i_p \right)_p \in \{0, \dots, M\}^{N+1} \right\} .$$

Let $\epsilon > 0$. Applying lemma 5.10.1 to the family $(\mathcal{J}_M)_{M\in\mathbb{N}}$ and the non-negative traceclass operators ρ and $\widetilde{\rho}$, there exists $M \in \mathbb{N}$ such that:

$$\left\| \rho - \Pi_M \rho \Pi_M \right\|_1 \leqslant \frac{\epsilon}{4} \quad \& \quad \left\| \widetilde{\rho} - \Pi_M \widetilde{\rho} \Pi_M \right\|_1 \leqslant \frac{\epsilon}{4},$$

where $\|\cdot\|_1$ denotes the trace-norm and Π_M is the orthogonal projection on $\mathcal{J}_M$. Now, from the previous step (with $M \in \mathbb{N}$ and $\epsilon/24\,(M+1)^2 > 0$), there exists, for any $\left(i_p\right)_p, \left(j_p\right)_p \in \{0, \ldots, M\}^{N+1}$, $A^{\left(i_p\right)_p \left(j_p\right)_p} \in \mathfrak{I}$ such that:

$$\left\|\Phi_\eta^{-1} \left|\phi_{i_1} \otimes \ldots \otimes \phi_{i_{N+1}} \right\rangle \left\langle \phi_{j_1} \otimes \ldots \otimes \phi_{j_{N+1}}\right| \Phi_\eta - \Phi_\eta^{-1} A^{\left(i_p\right)_p \left(j_p\right)_p} \Phi_\eta\right\|_{\mathcal{A}_\eta} \leqslant \frac{\epsilon}{4\,(M+1)^2},$$

where $\|\cdot\|_{\mathcal{A}_\eta}$ denotes the operator norm on $\mathcal{H}_\eta$. Thus, defining a bounded operator $\widetilde{A}$ on $\mathcal{H}_\eta$ as:

$$\widetilde{A} := \sum_{\left(i_p\right)_p, \left(j_p\right)_p \in\{0,\ldots,M\}^{N+1}} \left\langle \phi_{i_1} \otimes \ldots \otimes \phi_{i_{N+1}} \,\middle|\, \Phi_\eta A_\eta \Phi_\eta^{-1} \phi_{j_1} \otimes \ldots \otimes \phi_{j_{N+1}} \right\rangle \Phi_\eta^{-1} A^{\left(i_p\right)_p \left(j_p\right)_p} \Phi_\eta,$$

we have $\Phi_\eta \widetilde{A} \Phi_\eta^{-1} \in \mathfrak{I}$ and $\left\|\widetilde{A} - \Pi_M A_\eta \Pi_M\right\| \leqslant \frac{\epsilon}{4} \|A_\eta\|_{\mathcal{A}_\eta}$. Putting everything together, we obtain:

$$\begin{aligned}
\left|\mathrm{Tr}_{\mathcal{H}_{\eta''}} \rho_{\eta''} \left(\iota_{\eta''\leftarrow\eta'}(A'_{\eta'}) - \iota_{\eta''\leftarrow T^{-1}\eta}(T^{-1}\triangleright A_\eta)\right)\right| &= \left|\mathrm{Tr}_{\mathcal{H}_\eta} \left(\rho - \widetilde{\rho}\right) A_\eta\right| \\
&\leqslant \epsilon \|A_\eta\|_{\mathcal{A}_\eta} + \left|\mathrm{Tr}_{\mathcal{H}_\eta} \left(\rho - \widetilde{\rho}\right) \widetilde{A}\right| \\
&= \epsilon \|A_\eta\|_{\mathcal{A}_\eta} + \left|\mathrm{Tr}\, \rho' \left(\alpha\left(\left[\widetilde{A}\right]_{\sim_{\mathcal{L}^{[\lambda]}}}\right) - \widetilde{\alpha}\left(\left[\widetilde{A}\right]_{\sim_{\mathcal{L}^{[\lambda]}}}\right)\right)\right| = \epsilon \|A_\eta\|_{\mathcal{A}_\eta}.
\end{aligned}$$

Since this holds for any density matrix $\rho_{\eta''}$ on $\mathcal{H}_{\eta''}$ and any $\epsilon > 0$, it follows that $\iota_{\eta''\leftarrow\eta'}(A'_{\eta'}) = \iota_{\eta''\leftarrow T^{-1}\eta}(T^{-1}\triangleright A_\eta)$, ie. $\alpha\left([A_\eta]_{\sim_{\mathcal{L}^{[\lambda]}}}\right) = \widetilde{\alpha}\left([A_\eta]_{\sim_{\mathcal{L}^{[\lambda]}}}\right)$, which concludes the recursive proof of eq. (19.12.1).

From eq. (19.12.1), we have:

$$\forall A \in \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)}, \quad \alpha(A) = \widetilde{\alpha}(A),$$

hence, by continuity, $\alpha = \widetilde{\alpha}$. Finally, like in the proof of prop. 19.9, we can then define a bijective map $\widetilde{\sigma} : \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)}$ via:

$$\begin{array}{rccc}
\widetilde{\sigma} : & \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} & \to & \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\lambda]},\mathcal{H},\Phi)} \\
& \left(\rho_\eta\right)_{\eta\in\mathcal{L}^{[\kappa]}} & \mapsto & \left(T\triangleright\rho_{T^{-1}\underline{\eta}}\right)_{\underline{\eta}\in\mathcal{L}^{[\lambda]}}
\end{array},$$

$\widetilde{\sigma}$ and $\widetilde{\alpha} = \alpha$ are Tr-intertwined by construction, as are σ and α. Thus, for any $\rho \in \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$, any $\underline{\eta} \in \mathcal{L}^{[\lambda]}$ and any $A_{\underline{\eta}} \in \mathcal{A}_{\underline{\eta}}$, we have:

$$\mathrm{Tr}_{\mathcal{H}_{\underline{\eta}}} \left[\sigma(\rho)\right]_{\underline{\eta}} A_{\underline{\eta}} = \mathrm{Tr}\, \rho\, \alpha\left(\left[A_{\underline{\eta}}\right]_{\sim_{\mathcal{L}^{[\lambda]}}}\right) = \mathrm{Tr}_{\mathcal{H}_{\underline{\eta}}} \left[\widetilde{\sigma}(\rho)\right]_{\underline{\eta}} A_{\underline{\eta}},$$

which ensures that $\sigma = \widetilde{\sigma}$.

Approximation of $\mathfrak{T}$ by $\mathfrak{T}^{[\kappa]}$. $\mathfrak{T}^{[\kappa]}$ is stable under composition and inverse, hence it forms a subgroup of $\mathfrak{T}$. Moreover, using the characterization $\mathcal{L}^{[\kappa]} = \left\{\eta \in \mathcal{L}^{(\mathrm{aux})} \,\middle|\, \eta \subset K\right\}$, we have, for any $T \in \mathfrak{T}^{[\kappa]}$

and any $\eta \in \mathcal{L}^{(\text{aux})}$, $\eta \in \mathcal{L}^{[\kappa]} \Leftrightarrow T\eta \in \mathcal{L}^{[\kappa]}$. Thus, the group action of $\mathcal{T}$ on the projective system $\left(\mathcal{L}^{(\text{aux})}, \mathcal{H}, \Phi\right)^{\otimes}$ induces a group action of $\mathcal{T}^{[\kappa]}$ on the projective system $\left(\mathcal{L}^{[\kappa]}, \mathcal{H}, \Phi\right)^{\otimes}$.

Next, let $T \in \mathcal{T}$ and let V be an open neighborhood of T in $\mathcal{T}$. Let $\epsilon > 0$ such that $B_\epsilon \,.\, T := \{T'.T \mid T' \in B_\epsilon\} \subset V$. Let $\sigma : \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$ and $\alpha : \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \to \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$ be as in prop. 19.9 with respect to the transformation $T \in \mathcal{T}$ and the open neighborhood B_ϵ of $\text{id}_{]0,1]}$ in $\mathcal{T}$. Define $\underline{\sigma} : \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\underline{\kappa}]},\mathcal{H},\Phi)}$ and $\underline{\alpha} : \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\underline{\kappa}]},\mathcal{H},\Phi)} \to \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$ like in the proof of prop. 19.9, with $\underline{\kappa}$ the quasi-cofinal sequence $\underline{\kappa} := (T\kappa_n)_{n\in\mathbb{N}}$. Then, $\underline{\sigma} \circ \sigma^{-1} : \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)} \to \overline{\mathcal{S}}^{\otimes}_{(\mathcal{L}^{[\underline{\kappa}]},\mathcal{H},\Phi)}$ and $\alpha \circ \underline{\alpha} : \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\underline{\kappa}]},\mathcal{H},\Phi)} \to \overline{\mathcal{A}}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}$ fulfills the hypotheses of the first part of the present proof, hence there exists $T' \in B_\epsilon$ such that:

$$\left\{T\eta \;\middle|\; \eta \in \mathcal{L}^{[\kappa]}\right\} = \mathcal{L}^{[\underline{\kappa}]} = \left\{T'\eta \;\middle|\; \eta \in \mathcal{L}^{[\kappa]}\right\}$$
$$\& \quad \forall A \in \mathcal{A}^{\otimes}_{(\mathcal{L}^{[\kappa]},\mathcal{H},\Phi)}\,, \quad \alpha(A) = T'^{-1} \triangleright \underline{\alpha}^{-1}(A) = (T'^{-1}\,.\,T) \triangleright A\,.$$

In particular, $\widetilde{T} := T'^{-1} \,.\, T \in \mathcal{T}^{\kappa} \cap V$. Therefore, $\mathcal{T}^{\kappa}$ is dense in $\mathcal{T}$. □

As announced in the discussion preceding prop. 19.2, neither the universality of the restricted projective systems built from quasi-cofinal sequences, nor the possibility of approximating transformations (that directly follows from it), extend to the infinite tensor products that one can assemble from these sequences. In prop. 19.13 below, we construct an example of a deformation that provides an identification at the level of the projective systems, but fails to provide a unitary mapping of the corresponding ITP's. The proof is somewhat similar to the one of theorem 5.11, and relies on the fact that grouping the tensor product factors pairwise yields an *inequivalent* ITP [99, section 4.2].

Proposition 19.13 We consider the same objects as in prop. 19.10 and we assume that G is non-trivial. For any quasi-cofinal sequence $\kappa = (\kappa_n)_{n\in\mathbb{N}}$ in $\mathcal{L}^{(\text{aux})}$, we define $\mathcal{H}^{[\kappa]}_{\text{seq}}$ and $\sigma^{[\kappa]}_{\text{seq}}$ as in prop. 19.2 with respect to the projective system $\left(\mathcal{L}^{[\kappa]}, \mathcal{H}, \Phi\right)$ and the increasing sequence $(\kappa_n)_{n\in\mathbb{N}}$. There exists quasi-cofinal sequences κ, λ and an homeomorphism $T \in \mathcal{T}$ such that:

$$\mathcal{L}^{[\lambda]} = \left\{T\eta \;\middle|\; \eta \in \mathcal{L}^{[\kappa]}\right\} \quad \& \quad \sigma^{[\lambda]}_{\text{seq}}\left\langle \overline{\mathcal{S}}^{[\lambda]}_{\text{seq}} \right\rangle \neq \left\{ \left(T \triangleright \rho_{T^{-1}\underline{\eta}} \right)_{\underline{\eta}\in\mathcal{L}^{[\lambda]}} \;\middle|\; \rho \in \sigma^{[\kappa]}_{\text{seq}}\left\langle \overline{\mathcal{S}}^{[\kappa]}_{\text{seq}} \right\rangle \right\}$$

(where $\overline{\mathcal{S}}^{[\kappa]}_{\text{seq}}$, resp. $\overline{\mathcal{S}}^{[\lambda]}_{\text{seq}}$, denotes the space of non-negative traceclass operators on $\mathcal{H}^{[\kappa]}_{\text{seq}}$, resp. $\mathcal{H}^{[\lambda]}_{\text{seq}}$). In particular, this means that there does not exist any isomorphism of Hilbert spaces $\text{U} : \mathcal{H}^{[\kappa]}_{\text{seq}} \to \mathcal{H}^{[\lambda]}_{\text{seq}}$ such that:

$$\forall \rho \in \overline{\mathcal{S}}^{[\kappa]}_{\text{seq}}\,, \; \sigma^{[\lambda]}_{\text{seq}}\big(\text{U}\,\rho\,\text{U}^{-1}\big) = \left(T \triangleright \left[\sigma^{[\kappa]}_{\text{seq}}(\rho)\right]_{T^{-1}\underline{\eta}} \right)_{\underline{\eta}\in\mathcal{L}^{[\lambda]}}. \tag{19.13.1}$$

Proof Let $\kappa = (\kappa_n)_{n\in\mathbb{N}}$ be a *strictly* increasing quasi-cofinal sequence (eg. the one we constructed in the proof of prop. 19.11) and let $\lambda := (\lambda_m)_{m\in\mathbb{N}}$, with $\forall m \in \mathbb{N}$, $\lambda_m := \kappa_{2m}$. Then, one can check that λ is also a quasi-cofinal sequence and, setting $T := \text{id}_{]0,1]} \in \mathcal{T}$, we have $\mathcal{L}^{[\lambda]} = \mathcal{L}^{[\kappa]} = \left\{T\eta \;\middle|\; \eta \in \mathcal{L}^{[\kappa]}\right\}$.

We define $\mathcal{J}_o = \mathcal{J}'_o := \mathcal{H}_{\kappa_o} = \mathcal{H}_{\lambda_o} = \mathcal{H}_\varnothing$, as well as, for any $n > 0$, $\mathcal{J}_n := \mathcal{H}_{\kappa_n \to \kappa_{n-1}}$, and for any $m > 0$, $\mathcal{J}'_m := \mathcal{H}_{\lambda_m \to \lambda_{m-1}}$. Then, for any $m > 0$, $\Phi_{\kappa_{2m} \to \kappa_{2m-1} \to \kappa_{2m-2}}$ is an isomorphism $\mathcal{J}'_m \to \mathcal{J}_{2m} \otimes \mathcal{J}_{2m-1}$. We choose a normalized vector $\psi_o \in \mathcal{J}_o$. For any $n > 0$, we have $\kappa_{n-1} \subsetneq \kappa_n$, hence, G being non-trivial, $\dim \mathcal{J}_n \geqslant 2$, so we can choose two normalized, mutually orthogonal vectors $\psi_n^{(1)}, \psi_n^{(2)} \in \mathcal{J}_n$. Then, we define:

$$\phi_o := \psi_o \in \mathcal{J}'_o \quad \& \quad \forall m > 0,\ \phi_m := \Phi^{-1}_{\kappa_{2m} \to \kappa_{2m-1} \to \kappa_{2m-2}} \left(\frac{\psi_{2m}^{(1)} \otimes \psi_{2m-1}^{(1)} + \psi_{2m}^{(2)} \otimes \psi_{2m-1}^{(2)}}{\sqrt{2}} \right) \in \mathcal{J}'_m,$$

and, by prop. 19.1, we construct $\rho[\phi] \in \mathcal{S}^{\otimes}_{(\mathcal{L}^{[\lambda]}, \mathcal{H}, \Phi)}$ such that:

$$\rho_{\lambda_o}[\phi] = |\phi_o\rangle\langle\phi_o| \quad \& \quad \forall m > 0,\ \rho_{\lambda_m}[\phi] = \Phi^{-1}_{\lambda_m \to \lambda_{m-1}} \circ \big(|\phi_m\rangle\langle\phi_m| \otimes \rho_{\lambda_{m-1}}[\phi] \big) \circ \Phi_{\lambda_m \to \lambda_{m-1}}.$$

From prop. 19.2, we have $\rho[\phi] \in \sigma^{[\lambda]}_{\text{seq}} \left\langle \mathcal{S}^{[\lambda]}_{\text{seq}} \right\rangle \subset \sigma^{[\lambda]}_{\text{seq}} \left\langle \overline{\mathcal{S}}^{[\lambda]}_{\text{seq}} \right\rangle$.

Reasoning by contradiction we suppose that there exists a non-negative traceclass operator $\widetilde{\rho}$ on $\mathcal{H}^{[\kappa]}_{\text{seq}}$ such that:

$$\forall \eta \in \mathcal{L}^{[\lambda]} = \mathcal{L}^{[\kappa]},\ \rho_\eta[\phi] = T \triangleright \left[\sigma^{[\kappa]}_{\text{seq}}\big(\widetilde{\rho}\big) \right]_{T^{-1}\eta},$$

ie. $\rho[\phi] = \sigma^{[\kappa]}_{\text{seq}}\big(\widetilde{\rho}\big)$. Using the same notations as in the proofs of props. 19.1 and 19.2, we have:

$$\rho[\phi] = \sigma^{[\kappa]}_{\text{seq}}\big(\widetilde{\rho}\big) = \sum_{[\![\psi']\!]} \sigma^{[\kappa]}_{[\psi']}\big(\Pi_{[\![\psi']\!]}\, \widetilde{\rho}\, \Pi_{[\![\psi']\!]}\big),$$

as well as:

$$1 = \operatorname{Tr} \rho[\phi] = \operatorname{Tr}_{\mathcal{H}^{[\kappa]}_{\text{seq}}} \left[\widetilde{\rho} \right] = \sum_{[\![\psi']\!]} \operatorname{Tr}_{\mathcal{H}^{[\kappa]}_{[\psi']}} \left[\Pi_{[\![\psi']\!]}\, \widetilde{\rho}\, \Pi_{[\![\psi']\!]} \right].$$

Hence, there should exist $\psi' = \big(\psi'_n\big)_{n \in \mathbb{N}} \in \mathcal{Z}^{\otimes}_{(\mathbb{N}, \mathcal{J})}$ such that $\operatorname{Tr}_{\mathcal{H}^{[\kappa]}_{[\psi']}} \left[\Pi_{[\![\psi']\!]}\, \widetilde{\rho}\, \Pi_{[\![\psi']\!]} \right] > 0$, and eq. (19.2.1) then yields:

$$\begin{aligned}
&\sup_{n \in \mathbb{N}} \inf_{n' > n} \left\langle \zeta'_{n' \to n} \,\middle|\, \left(\operatorname{Tr}_{\mathcal{K}_n} \widetilde{\Phi}_{n'}\, \rho[\phi]_{\kappa_{n'}}\, \widetilde{\Phi}^{-1}_{n'} \right) \zeta'_{n' \to n} \right\rangle = \\
&\quad = \sum_{[\![\psi'']\!]} \sup_{n \in \mathbb{N}} \inf_{n' > n} \left\langle \zeta'_{n' \to n} \,\middle|\, \left(\operatorname{Tr}_{\mathcal{K}_n} \widetilde{\Phi}_{n'} \left[\sigma^{[\kappa]}_{[\psi'']}\big(\Pi_{[\![\psi'']\!]}\, \widetilde{\rho}\, \Pi_{[\![\psi'']\!]}\big) \right]_{\kappa_{n'}} \widetilde{\Phi}^{-1}_{n'} \right) \zeta'_{n' \to n} \right\rangle \\
&\quad \geqslant \sup_{n \in \mathbb{N}} \inf_{n' > n} \left\langle \zeta'_{n' \to n} \,\middle|\, \left(\operatorname{Tr}_{\mathcal{K}_n} \widetilde{\Phi}_{n'} \left[\sigma^{[\kappa]}_{[\psi']}\big(\Pi_{[\![\psi']\!]}\, \widetilde{\rho}\, \Pi_{[\![\psi']\!]}\big) \right]_{\kappa_{n'}} \widetilde{\Phi}^{-1}_{n'} \right) \zeta'_{n' \to n} \right\rangle \\
&\quad = \operatorname{Tr}_{\mathcal{H}^{[\kappa]}_{[\psi']}} \left[\Pi_{[\![\psi']\!]}\, \widetilde{\rho}\, \Pi_{[\![\psi']\!]} \right] > 0,
\end{aligned}$$

with $\forall n < n' \in \mathbb{N}$, $\zeta'_{n' \to n} := \psi'_{n'} \otimes \ldots \otimes \psi'_{n+1} \in \mathcal{K}_{n' \to n}$ (we have used that the sum $\sum_{[\![\psi'']\!]}$ is absolutely convergent in trace norm, and that the argument of $\inf_{n' > n}$, resp. of $\sup_{n \in \mathbb{N}}$, is positive and decreasing with n', resp. increasing with n).

From the definition of $\widetilde{\Phi}_n$ and $\rho[\phi]$, we have $\widetilde{\Phi}_o\, \rho[\phi]_{\kappa_o}\, \widetilde{\Phi}^{-1}_o = |\phi_o\rangle\langle\phi_o|$, as well as, for any $m > 0$:

$$\widetilde{\Phi}_{2m}\, \rho[\phi]_{\kappa_{2m}}\, \widetilde{\Phi}^{-1}_{2m} = \frac{1}{2} \left| \psi_{2m}^{(1)} \otimes \psi_{2m-1}^{(1)} + \psi_{2m}^{(2)} \otimes \psi_{2m-1}^{(2)} \right\rangle \left\langle \psi_{2m}^{(1)} \otimes \psi_{2m-1}^{(1)} + \psi_{2m}^{(2)} \otimes \psi_{2m-1}^{(2)} \right| \otimes$$

$$\otimes \left(\widetilde{\Phi}_{2m-2}\, \rho_{\kappa_{2m-2}}[\phi]\, \widetilde{\Phi}^{-1}_{2m-2}\right).$$

Hence, we get, for any $m > 0$:

$$\widetilde{\Phi}_{2m}\, \rho[\phi]_{\kappa_{2m}}\, \widetilde{\Phi}^{-1}_{2m} = \frac{1}{2^m} \left|\psi^{(1)}_{2m} \otimes \psi^{(1)}_{2m-1} + \psi^{(2)}_{2m} \otimes \psi^{(2)}_{2m-1} \right\rangle \left\langle \psi^{(1)}_{2m} \otimes \psi^{(1)}_{2m-1} + \psi^{(2)}_{2m} \otimes \psi^{(2)}_{2m-1}\right| \otimes$$

$$\otimes \ldots \otimes \left|\psi^{(1)}_{2} \otimes \psi^{(1)}_{1} + \psi^{(2)}_{2} \otimes \psi^{(2)}_{1} \right\rangle \left\langle \psi^{(1)}_{2} \otimes \psi^{(1)}_{1} + \psi^{(2)}_{2} \otimes \psi^{(2)}_{1}\right| \otimes |\phi_o\rangle\langle \phi_o|.$$

Let $n \in \mathbb{N}$ and let $m > n/2$. If $n = 2p$, we have $p < m$ and:

$$\left\langle \zeta'_{2m\to n} \,\middle|\, \left(\mathrm{Tr}_{\mathcal{K}_n}\, \widetilde{\Phi}_{2m}\, \rho[\phi]_{\kappa_{2m}}\, \widetilde{\Phi}^{-1}_{2m}\right)\, \zeta'_{2m\to n} \right\rangle = \prod_{k=p+1}^{m} \xi_k\,,$$

where, for any $k > 0$, $\xi_k := \frac{1}{2}\left|\left\langle \psi'_{2k} \otimes \psi'_{2k-1} \,\middle|\, \psi^{(1)}_{2k} \otimes \psi^{(1)}_{2k-1} + \psi^{(2)}_{2k} \otimes \psi^{(2)}_{2k-1}\right\rangle\right|^2$. If $n = 2p+1$, we also have $p < m$ and:

$$\left\langle \zeta'_{2m\to n} \,\middle|\, \left(\mathrm{Tr}_{\mathcal{K}_n}\, \widetilde{\Phi}_{2m}\, \rho[\phi]_{\kappa_{2m}}\, \widetilde{\Phi}^{-1}_{2m}\right)\, \zeta'_{2m\to n} \right\rangle = \widetilde{\xi}_{p+1} \prod_{k=p+2}^{m} \xi_k\,,$$

where $\widetilde{\xi}_{p+1} := \frac{1}{2}\left|\left\langle \psi'_{2p+2} \,\middle|\, \psi^{(1)}_{2p+2}\right\rangle\right|^2 + \frac{1}{2}\left|\left\langle \psi'_{2p+2} \,\middle|\, \psi^{(2)}_{2p+2}\right\rangle\right|^2$. Now, for any $k > 0$, we have:

$$\begin{aligned}
\xi_k &= \frac{1}{2}\left|\sum_{\epsilon=1}^{2} \left\langle \psi'_{2k} \,\middle|\, \psi^{(\epsilon)}_{2k}\right\rangle \left\langle \psi'_{2k-1} \,\middle|\, \psi^{(\epsilon)}_{2k-1}\right\rangle\right|^2 \\
&\leqslant \frac{1}{2}\left(\sum_{\epsilon=1}^{2} \left|\left\langle \psi'_{2k} \,\middle|\, \psi^{(\epsilon)}_{2k}\right\rangle\right|^2\right) \left(\sum_{\epsilon=1}^{2} \left|\left\langle \psi'_{2k-1} \,\middle|\, \psi^{(\epsilon)}_{2k-1}\right\rangle\right|^2\right) \\
&\leqslant \frac{1}{2}\, \|\psi'_{2k}\|^2\, \|\psi'_{2k-1}\|^2 = \frac{1}{2}.
\end{aligned}$$

Thus, we get:

$$\left\langle \zeta'_{2m\to n} \,\middle|\, \left(\mathrm{Tr}_{\mathcal{K}_n}\, \widetilde{\Phi}_{2m}\, \rho[\phi]_{\kappa_{2m}}\, \widetilde{\Phi}^{-1}_{2m}\right)\, \zeta'_{2m\to n} \right\rangle \leqslant \frac{1}{2^m - \lfloor (n+1)/2 \rfloor}\,,$$

where $\lfloor \cdot \rfloor$ denotes the floor function. This yields, for any $n \in \mathbb{N}$:

$$\inf_{n'>n} \left\langle \zeta'_{n'\to n} \,\middle|\, \left(\mathrm{Tr}_{\mathcal{K}_n}\, \widetilde{\Phi}_{n'}\, \rho[\phi]_{\kappa_{n'}}\, \widetilde{\Phi}^{-1}_{n'}\right)\, \zeta'_{n'\to n} \right\rangle = 0,$$

and therefore:

$$\sup_{n\in\mathbb{N}} \inf_{n'>n} \left\langle \zeta'_{n'\to n} \,\middle|\, \left(\mathrm{Tr}_{\mathcal{K}_n}\, \widetilde{\Phi}_{n'}\, \rho[\phi]_{\kappa_{n'}}\, \widetilde{\Phi}^{-1}_{n'}\right)\, \zeta'_{n'\to n} \right\rangle = 0,$$

which provides the desired contradiction. Hence, $\rho[\phi] \notin \left\{ \left(T \triangleright \rho_{T^{-1}\eta}\right)_{\eta\in\mathcal{L}^{[\lambda]}} \,\middle|\, \rho \in \sigma^{[\kappa]}_{\mathrm{seq}} \left\langle \overline{\mathcal{S}}^{[\kappa]}_{\mathrm{seq}} \right\rangle \right\}$, so:

$$\sigma_{\text{seq}}^{[\lambda]} \left\langle \overline{\mathcal{S}}_{\text{seq}}^{[\lambda]} \right\rangle \neq \left\{ \left(T \triangleright \rho_{T^{-1}\eta} \right)_{\eta \in \mathcal{L}^{[\lambda]}} \,\middle|\, \rho \in \sigma_{\text{seq}}^{[\kappa]} \left\langle \overline{\mathcal{S}}_{\text{seq}}^{[\kappa]} \right\rangle \right\}.$$

□

To exhibit a quasi-cofinal sequence for $\mathcal{L}_{\text{HF}}$ in the physically more interesting $d = 3$ case, the idea would be to construct a discrete but dense, fractal-like structure made of edges and surfaces: note that the sequence constructed above for our one-dimensional toy-model can also be seen as a fractal in $]0, 1]$. The proof that such a structure can be designed that satisfies the quasi-cofinality property 19.7.3 is however significantly more involved than in the one-dimensional case, and is not yet finished.

An interesting use of the possibility to start from a *non-directed*, extended label set $\mathcal{L}^{(\text{ext})}$ would be that we could drop the somewhat ad-oc (semi-)analyticity requirement for edges and surfaces: recall that analyticity was used solely in lemmas 10.5 and 10.8, with the aim of proving the directedness of $\mathcal{L}_{\text{HF}}$. Also, we could take advantage of this possibility to eliminate the unphysical (in the sense of having no equivalent in the classical continuum phase space from subsection 9.1), *degenerated* fluxes. These fluxes supported on geometrical objects of dimension strictly less that $d - 1$ had to be included in $\mathcal{L}_{\text{HF}}$ because they arose from the commutators of non-degenerate, $(d - 1)$-supported fluxes: we could not exclude *a priori* that an edge, hitting the intersection, would see the non-vanishing commutator. In contrast, in the context of a fractal-structure, where a discrete set of admissible edges and surfaces is chosen *beforehand*, we can effectively forbid that an edge will ever go through the intersection of any two surfaces, and simply *set* the commutator of *degenerately* intersecting fluxes to zero (this would not, however completely solve the problem of the 'non-commuting fluxes', since, as stressed at the end of section 10, the core of this problem concerns the *non-degenerately* intersecting fluxes).

Another consideration that will enter the definition of $\mathcal{L}^{(\text{ext})}$ is that it seems necessary to allow for *piecewise* smooth edges and surfaces, and for the group of transformations to consist of *piecewise diffeomorphisms*. This is the price we pay to satisfy an improved quasi-cofinality condition, that respects any already adapted sublabel, and it is already apparent in fig. 19.1: to adapt the grid to the target label in this schematic example, we had to deform it in a way that *flattened* some right angles (see eg. the diagonal edge in the bottom left quadrant). This complication can for example occur if there are two vertices b and f already at their final position (ie. they belong to the sublabel η of η' that we are no longer allowed to deform): there might be, in η', a straight edge e joining b to f, while there is no straight line available in the quasi-cofinal sequence to connect these two particular vertices: we will then have to approximate e by a zig-zag line along the fractal fabric of our quasi-cofinal sequence.

Switching to piecewise diffeomorphisms is presumably harmless anyway. In the context of quantum gravity, in particular, it has been argued that geometrical knowledge, including angles, should ultimately come from the gravitational degrees of freedom themselves (see eg. [28]). And while admitting only strict diffeomorphisms is enforced in LQG for the benefit of the regularization of the volume operator [7], and therefore of the Hamiltonian constraint that depends on it [88], the additional, fractal-like structure provided by a quasi-cofinal sequence could probably serve as a drop-in replacement for the differentiable structure used in these calculations.

20. Discussion: imposing the constraints (2/2)

Recall that we put forward in section 13 the need to 'anchor' the fluxes, in order to improve their transformation properties under gauge transformations. The problem was then to build a *directed* label set while keeping track of the auxiliary systems of paths entering the definition of these anchored fluxes. Remarkably, a fractal-like structure as sketched above could provide these anchoring paths automatically. Indeed, considering a given face, with a conjugate edge starting from it, the refinement taking place as we go deeper and deeper in the quasi-cofinal sequence would require to simultaneously subdivide the original face and ramify the original edge. In this way, we would recursively construct a dense tree reaching this face, that one could use for anchoring the corresponding flux.

Going over to the diffeomorphism constraints, one could imagine to simply declare the labels of the quasi-cofinal sequence to be non-embedded, and to keep the relations between them as they are: this would be a way to ignore the difficulty described in section 13 regarding the proper way of relating non-embedded labels. However, as we explained at the end of subsection 19.2, this would not actually make the states diffeomorphism invariant, as the fractal structure would just take over the role of a coordinate system. Instead, we should follow the strategy outlined in section 13 and, for every single state at a certain level κ_n, we should reposition κ_{n-1} within κ_n to capture the best picture of this particular state.

Note that obtaining the diffeomorphism-invariant state space in this way could, incidentally, be beneficial for the semi-classical limit. This is because the factorized coherent states constructed along the lines of subsection 19.1 are not immune against the so-called 'staircase problem' (see [92, subsection 11.2.5]). The variables added at each step of the quasi-cofinal sequence (aka. the variables from the spaces $\mathcal{H}_{\kappa_{n+1} \to \kappa_n}$) are the most semi-classical ones in such a state. Thus, while the elementary variables of the first few elements of the sequence will have very peaked distributions, other variables only slightly different may be very spread, due to their complicated dependence in terms of these preferred directions of the factorized state. The problem is less severe than in the case of fixed-graph coherent states [90] (in which the variables on the graph are perfectly semi-classical, while variables only slightly different, that are not supported by the graph, are not semi-classical at all), but it would be further mitigated by adapting the imposition of the diffeomorphism constraints to the quasi-cofinal sequence. Indeed, the collective, large-scale variables that compose the coarsest elements of the quasi-cofinal sequence, and are thus prioritized in the semi-classical approximation, would then be interpreted as reflecting the most prominent features of the state.

Finally, it would also be natural to perform the regularization of the Hamiltonian constraints *along* the quasi-cofinal sequence, eg. by adapting the approximation scheme developed in [88] , and doing so could potentially help the dynamical stability of the factorized semi-classical states (recall the discussion in the general introduction: an important limitation of fixed-graph coherent states is that they are not well adapted to graph-changing Hamiltonian constraints). Interestingly, the need for fractal constructions also emerges from this perspective [92, subsection II.12.2.5].

Conclusion

Summary

We have taken the first steps toward a reformulation of Loop Quantum Gravity withing the projective framework proposed by Kijowski [48] , with the hope of shedding new light on its *semi-classical regime*, and perhaps also on its *dynamics*. Along the way, we obtained the following main results:

Extension of the projective formalism. We have investigated the formalism in detail, on the classical side (section 2), as well as on the quantum side (section 5). This allowed us to formulate fairly systematic *quantization prescriptions*, along the lines of geometric quantization (position quantization, in particular on simply connected Lie groups, in subsection 6.1; holomorphic quantization in subsection 6.2).

Application to the holonomy-flux algebra. We were able to show that the construction developed in [68] can be generalized from the linear case to an arbitrary gauge group G. The key ingredient is still the same, namely the use of labels defined as collections of *edges* and *surfaces*. However, the somewhat involved algebra built by the holonomies and fluxes in the case of a non-Abelian group requires to be more restrictive as to which such collections qualify as labels (subsections 10.1 and 10.2). The *factorization maps*, which are the central objects of the formalism, can then be expressed *explicitly* in terms of the group operations (subsections 10.3 and 12.1), so that no further restrictions on the Lie group G are needed. In particular, the case $G = \mathcal{SL}(2, \mathbb{C})$ could have application to the treatment of the complex Ashtekar variables (see section 11), which cannot be handled by the standard LQG methods.

Relation with standard constructions. We exhibited mappings from the spaces of density matrices on Hilbert spaces constructed as *inductive limits* or *infinite tensor products*, into corresponding projective state spaces, and we were able to precisely characterize the *image* of these mappings (subsection 5.2). This allowed us to prove, in subsection 12.2, that the projective state space we had set up for the holonomy-flux algebra *extends* the well-established Ashtekar-Lewandowski one (in the *compact* group case, where the latter is defined).

Proposal for dealing with constraints. We identified the need for *regularizing* the dynamics in order to implement it in the projective framework, and proposed to describe dynamical states as (projective) families of kinematical ones, that *converge* to *exact* solutions of the constraints (section 3). It is via this connection with the kinematical state space that the degrees of freedom on the dynamical side acquire their *physical meaning*. This proposal was put to the proof on simple toy-models in sections 15 and 16. In particular, the study of the second quantization of the Schrödinger equation (aka. non-relativistic quantum field theory) provides interesting insight on how considerations of convergence when implementing the dynamics could help *select* a subset of well-behaved states in the (typically very large) projective state space: this is how the *Fock space* reemerges in this approach.

Procedure for simplifying the algebra of observables. We have analyzed the remaining *ob-*

structions to the existence of satisfactory *semi-classical states* (section 18), and devised a strategy to go around this problem, by trimming the label set down to *countable cardinality*, without denaturing the *physical content* of the algebra of observables, its *universality*, nor its *symmetries* (subsection 19.2, with a one-dimensional proof of principle in subsection 19.3). This paves the way for the construction of states whose semi-classicality is enforced step by step, starting from *collective*, *macroscopic* degrees of freedom and going down progressively toward smaller and smaller scales (subsection 19.1).

Outlook

The program started in the present work is still far from complete and has uncovered many open questions. Directions for further research include:

Systematic quantization prescriptions. As discussed in section 7, it should be possible to generalize the quantization procedure outlined in section 6. Given an arbitrary projective system of classical phase spaces, one would like to immediately obtain a corresponding projective quantum state space from a consistent choice of *polarizations* (in the sense of geometric quantization [105]) on the small, partial phase spaces. As these partial phase spaces are meant to be *finite dimensional*, different choices of polarization can often be related through *unitary identification* of the resulting quantized partial theories and this could allow to weaken the consistency conditions (by demanding consistency only up to unitary equivalence). This could have applications to the description of quantum fields on *arbitrary spacetimes* (eg. in the spirit of [64]).

Proof of non-emptyness. Projective systems built on *countable* label sets are guaranteed to yield non-empty quantum state spaces, as projective states are easily constructed in this case (subsection 19.1). The situation is less clear in the case of *uncountable* labels sets (see prop. 5.12 and the discussion at the end of section 18) and it would be helpful to delineate *sufficient* conditions for projective state spaces to be non-empty. In particular, we would like to construct projective states on the holonomy-flux algebra in the *non-compact* group case.

Refining the implementation of the dynamics. We need to develop the tools to deal with constraints in the projective framework, expanding the strategy proposed in section 3. On the classical side, we would like to formulate *systematic* recipes to generate the input needed for the *regularization* of the constraints. On the quantum side, we still have to provide a rigorous procedure, including rules for defining an effective and physically meaningful notion of *convergence* (see the discussion surrounding theorem 16.21). As a general guiding principle, we should strive to reflect the concrete *experimental* implementation of the observables. In particular, when considering a theory of gravity, it might prove legitimate to define the convergence in a way that completely ignores the gravitational degrees of freedom: indeed, geometry is only *probed by matter*, and never measured directly.

Solving the constraints of LQG. The implementation of the Gauss and diffeomorphism constraints on the projective quantum state space set up in section 12 would be a good opportunity to further refine these tools, before eventually turning to the Hamiltonian constraints [88, 91]. In sections 13 and 20, we explored how these constraints could be regularized, notably by making use of the *fractal structure* introduced in section 19. Note that the analysis of this regularization at the *classical* level could already be interesting on its own, to investigate how the correct dynamics can be recovered in a formulation build on the holonomy and flux variables (instead of a more conventional formulation using a Sobolev-like space of regular connections and electrical fields, see subsection 9.1): in particular, it could clarify the impact of having *non-commuting fluxes* (see [92, section II.6.1] and the discussion at the end of section 10).

Relation with covariant approaches. By giving a central role to truncations and refinement in the construction of the quantum state space, the projective formalism appears well suited to import into the canonical setting some strengths of the *path-integral* formulation of quantum field theory, such as *approximation schemes* and *renormalization*. In the context of quantum gravity, taking advantage of the techniques developed in covariant approaches (eg. Spin Foams [9, 76] or Causal Dynamical Triangulations [2]) could in particular help *computing solutions* of the Hamiltonian constraints of LQG [94].

Semi-classical states and applications. Finally, we shall complete the construction of semi-classical states begun in section 19, in particular by extending the one-dimensional results of subsection 19.3 to the $d = 3$ case. The thus obtained states could then be applied to the study of the *semi-classical limit* [90, 36] and to the derivation of *symmetry reduced models* [16, 30, 27, 18] (as hinted in the introduction). A first target in this direction would be to recover the *photon states* of usual quantum electrodynamics out of a theory quantized in terms of holonomies and fluxes (with gauge group $G = \mathcal{U}(1)$, see [97]).

Appendix

A. Classical constrained systems

To fix the notations and definitions, we summarize here some facts about constrained classical systems. We recall how a reduced phase space arises from a constraint surface in a symplectic manifold [105, section 1.7], we introduce a notion of transport of observables to translate kinematical observables into dynamical ones (this facility is the main object of the physical discussion in section 3), and give a very brief account of partial gauge fixing [62].

When considering a constraint surface $\mathcal{M}^{\text{SHELL}}$ in a symplectic manifold $\mathcal{M}^{\text{KIN}}$ the pullback of the symplectic structure Ω_{KIN} does not, in general, define a symplectic structure on $\mathcal{M}^{\text{SHELL}}$: there might be directions in the tangent space of $\mathcal{M}^{\text{SHELL}}$ on which this pullback vanishes. These directions correspond to the gauge flow generated by first class constraints, and the gauge orbits need to be quotiented out in order to get a reduced phase space $\mathcal{M}^{\text{DYN}}$ with a non-degenerate symplectic structure Ω_{DYN}.

Except for the first few definitions (which are tailored to match the needs of some results in section 3), this appendix focuses on finite dimensional manifolds: this is anyway the point of the formalism presented in the main text that we aim at describing a field theory in such a way that we can work mostly within the context of finite dimensional manifolds.

Definition A.1 Let $\mathcal{M}^{\text{KIN}}$ be a (possibly infinite dimensional) smooth symplectic manifold (with symplectic structure Ω_{KIN}). A phase space reduction of $\mathcal{M}^{\text{KIN}}$ is a triple $(\mathcal{M}^{\text{DYN}}, \mathcal{M}^{\text{SHELL}}, \delta)$ such that:

1. $\mathcal{M}^{\text{SHELL}}$ is a submanifold of $\mathcal{M}^{\text{KIN}}$ and $\mathcal{M}^{\text{DYN}}$ is a symplectic manifold (with symplectic structure Ω_{DYN});
2. $\delta : \mathcal{M}^{\text{SHELL}} \to \mathcal{M}^{\text{DYN}}$ is a surjective map and, for all $y \in \mathcal{M}^{\text{DYN}}$, $\delta^{-1}\langle y \rangle$ is connected;
3. for all $x \in \mathcal{M}^{\text{SHELL}}$, $\text{Im}(T_x\delta) = T_{\delta(x)}(\mathcal{M}^{\text{DYN}}) \quad \& \quad \Omega_{\text{KIN},x}\big|_{T_x(\mathcal{M}^{\text{SHELL}})} = [\delta^* \Omega_{\text{DYN}}]_x$.

For any bounded real-valued function f on $\mathcal{M}^{\text{KIN}}$, we define a corresponding dynamical observable on $\mathcal{M}^{\text{DYN}}$ by mapping to a point y in the reduced phase space the supremum of f on the corresponding orbit $\delta^{-1}\langle y \rangle$. The motivation for this definition is that we regard indicator functions as the most fundamental observables: with the transport of observables defined this way, the indicator function of some region in $\mathcal{M}^{\text{KIN}}$ is mapped into the indicator function on the space of orbits that characterize whether a given orbit crosses this region or not. In other words, the dynamical observable related to the indicator function of some region of $\mathcal{M}^{\text{KIN}}$ will tell us whether the dynamical state of the system allows it to be measured in that region.

Note that there can be relations between the dynamical observables $f_1^{\text{DYN}}, \dots, f_k^{\text{DYN}}$ arising from functionally independent kinematical observables $f_1, \dots, f_k$, or to state this more precisely we can have dependencies:

$$\mathrm{Im}\,(f_1^{\text{DYN}} \times \ldots \times f_k^{\text{DYN}}) \neq (\mathrm{Im}\, f_1^{\text{DYN}}) \times \ldots \times (\mathrm{Im}\, f_k^{\text{DYN}}),$$

although the corresponding kinematical observables were independent:

$$\mathrm{Im}\,(f_1 \times \ldots \times f_k) = (\mathrm{Im}\, f_1) \times \ldots \times (\mathrm{Im}\, f_k).$$

This is a crucial observation, since, indeed, the dynamical content of theory lies in such functional relations emerging between observables that were kinematically independent.

Definition A.2 Let $(\mathcal{M}^{\text{DYN}}, \mathcal{M}^{\text{SHELL}}, \delta)$ be a phase space reduction of $\mathcal{M}^{\text{KIN}}$. We denote by $B(\mathcal{M}^{\text{KIN}})$ the space of bounded, real-valued, functions on $\mathcal{M}^{\text{KIN}}$. For all $f \in B(\mathcal{M}^{\text{KIN}})$, we define $f^{\text{DYN}} \in B(\mathcal{M}^{\text{DYN}})$ by:

$$\forall y \in \mathcal{M}^{\text{DYN}},\ f^{\text{DYN}}(y) := \sup \left\{ f(x) \,\middle|\, x \in \delta^{-1} \langle y \rangle \right\}. \tag{A.2.1}$$

As explained above, a phase space reduction is entirely specified by its constraint surface and, provided we exclude any pathology that could occur regarding the rank of pullback (pointA.3.2) or the quotienting under gauge (pointsA.3.3 and A.3.4), we can reconstruct the elements listed in def. A.1 from the constraint surface $\mathcal{M}^{\text{SHELL}}$. The degenerate directions of the pullback of the symplectic form on $\mathcal{M}^{\text{SHELL}}$ naturally generate a foliation [54, chapter 14] (the required integrability condition follows from Ω_{KIN} being closed, see [105, section 1.7]), and quotienting along this foliation ensures that we recover a non-degenerate symplectic structure on $\mathcal{M}^{\text{DYN}}$.

For the rest of this appendix all manifolds will be *finite dimensional* manifolds.

Definition A.3 Let $\mathcal{M}^{\text{KIN}}$ be a smooth, finite dimensional, symplectic manifold (with symplectic structure Ω_{KIN}). A pre-reduction of $\mathcal{M}^{\text{KIN}}$ is a triple $(\mathcal{M}^{\text{DYN}}, \mathcal{M}^{\text{SHELL}}, \delta)$ such that:

1. $\mathcal{M}^{\text{SHELL}}$ is a submanifold of $\mathcal{M}^{\text{KIN}}$ and $\mathcal{M}^{\text{DYN}}$ is a manifold;
2. the restriction of Ω_{KIN} to $T(\mathcal{M}^{\text{SHELL}})$ is of constant rank, thus defining a foliation $K(\mathcal{M}^{\text{SHELL}})$ by $\forall x \in \mathcal{M}^{\text{SHELL}},\ K_x(\mathcal{M}^{\text{SHELL}}) := \left\{ v \in T_x(\mathcal{M}^{\text{SHELL}}) \,\middle|\, \Omega_{\text{KIN},x}(v,\,\cdot)|_{T_x(\mathcal{M}^{\text{SHELL}})} = 0 \right\} \subset T_x(\mathcal{M}^{\text{SHELL}})$;
3. $\delta : \mathcal{M}^{\text{SHELL}} \to \mathcal{M}^{\text{DYN}}$ is a surjective map and $\forall x \in \mathcal{M}^{\text{SHELL}},\ \mathrm{Im}(T_x\delta) = T_{\delta(x)}(\mathcal{M}^{\text{DYN}})$;
4. $\forall y \in \mathcal{M}^{\text{DYN}},\ \delta^{-1} \langle y \rangle$ is a leaf of the foliation $K(\mathcal{M}^{\text{SHELL}})$.

Proposition A.4 Let $\mathcal{M}^{\text{KIN}}$ be a smooth, finite dimensional, symplectic manifold (with symplectic structure Ω_{KIN}) and let $(\mathcal{M}^{\text{DYN}}, \mathcal{M}^{\text{SHELL}}, \delta)$ be a phase space reduction of $\mathcal{M}^{\text{KIN}}$. Then, $(\mathcal{M}^{\text{DYN}}, \mathcal{M}^{\text{SHELL}}, \delta)$ is a pre-reduction of $\mathcal{M}^{\text{KIN}}$ and we have:

$$\forall x \in \mathcal{M}^{\text{SHELL}},\quad K_x(\mathcal{M}^{\text{SHELL}}) = \mathrm{Ker}\, T_x\delta = T_x\left(\delta^{-1}\langle \delta(x) \rangle\right). \tag{A.4.1}$$

Proof Defs. A.3.1 and A.3.3 are directly implied by def. A.1.

Let $y \in \mathcal{M}^{\text{DYN}}$ and $x \in \delta^{-1}\langle y \rangle$. Since δ has surjective derivative at each point, we have as an implication of the rank theorem [54, theorem 5.22] that $\delta^{-1}\langle y \rangle$ is a submanifold of $\mathcal{M}^{\text{SHELL}}$ with tangent space $\mathrm{Ker}\, T_x\delta \subset T_x(\mathcal{M}^{\text{SHELL}})$ at x. Now, from def. A.1.3, together with the non-degeneracy of Ω_{DYN} (for $\mathcal{M}^{\text{DYN}}$ is a symplectic manifold), we have:

$$\forall x' \in \delta^{-1}\langle y \rangle,\quad \mathrm{Ker}\, T_{x'}\delta = K_{x'}(\mathcal{M}^{\text{SHELL}}).$$

Hence, $K(\mathcal{M}^{\text{SHELL}})$ has constant dimension, so def. A.3.2 is fulfilled.

Additionaly, by maximality of the leaves, the connected submanifold $\delta^{-1}\langle y\rangle$ is included in the leaf of the foliation $K(\mathcal{M}^{\text{SHELL}})$ that goes through x. Reciprocally, since the leaf that goes through x is connected, and has tangent space $K_{x'}(\mathcal{M}^{\text{SHELL}}) = \operatorname{Ker} T_{x'}\delta$ at any point, δ is constant on it, hence it is included in $\delta^{-1}\langle y\rangle$. Thus, def. A.3.4 is fulfilled. □

Proposition A.5 Let $\mathcal{M}^{\text{KIN}}$ be a smooth, finite dimensional, symplectic manifold and let $(\mathcal{M}^{\text{DYN}}, \mathcal{M}^{\text{SHELL}}, \delta)$ be a pre-reduction of $\mathcal{M}^{\text{KIN}}$. Then, there exists a symplectic structure Ω_{DYN} on $\mathcal{M}^{\text{DYN}}$ such that $(\mathcal{M}^{\text{DYN}}, \mathcal{M}^{\text{SHELL}}, \delta)$ is a phase space reduction of $\mathcal{M}^{\text{KIN}}$.

Proof What we need to prove is that there exists a symplectic structure Ω_{DYN} on $\mathcal{M}^{\text{DYN}}$ such that:

$$\forall x \in \mathcal{M}^{\text{SHELL}},\ \Omega_{\text{KIN},x}\big|_{T_x(\mathcal{M}^{\text{SHELL}})} = [\delta^*\,\Omega_{\text{DYN}}]_x\,.$$

The others points in def. A.1 are immediately fulfilled (in particular, for any $y \in \mathcal{M}^{\text{DYN}}$, $\delta^{-1}\langle y\rangle$ is connected as a leaf of a foliation).

Let $x \in \mathcal{M}^{\text{SHELL}}$ and let $y := \delta(x)$. Since δ has surjective derivative at each point, there exist by the rank theorem [54, theorem 5.13] open neighborhoods U of x in $\mathcal{M}^{\text{SHELL}}$, V of y in $\mathcal{M}^{\text{DYN}}$ and W of 0 in $\mathbb{R}^{s-d}$ (with $s := \dim \mathcal{M}^{\text{SHELL}}$ and $d := \dim \mathcal{M}^{\text{DYN}}$), and a diffeomorphism $\varphi : V \times W \to U$ such that:

$$\forall y' \in V,\ \forall z' \in W,\ \delta \circ \varphi(y', z') = y'\,.$$

For any $y', z' \in V \times W$, we define $\Omega^{\varphi,z'}_{\text{DYN},y'}$ by:

$$\forall v, v' \in T_{y'}(\mathcal{M}^{\text{DYN}}),\ \Omega^{\varphi,z'}_{\text{DYN},y'}\left(v,\ v'\right) = \Omega_{\text{KIN},\varphi(y',z')}\left(T_{y',z'}\,\varphi(v,0),\ T_{y',z'}\,\varphi(v',0)\right).$$

Then, setting $x' := \varphi(y', z')$, $\Omega^{\varphi,z'}_{\text{DYN},y'}$ satisfies:

$$\forall u, u' \in T_{x'}(\mathcal{M}^{\text{SHELL}}),\quad \Omega_{\text{KIN},x'}\left(u,\ u'\right) = \Omega^{\varphi,z'}_{\text{DYN},y'}\left(T_{x'}\delta(u),\ T_{x'}\delta(u')\right),$$

for we have from def. A.3.4:

$$\left\{T_{y',z'}\,\varphi(0,w) \,\middle|\, w \in T_{z'}(W)\right\} = T_{x'}\left(\delta^{-1}\langle y'\rangle\right) = K_{x'}\left(\mathcal{M}^{\text{SHELL}}\right). \tag{A.5.1}$$

Let $\widetilde{Y}, \widetilde{Y}'$ be vector fields on V and $\widetilde{Z}$ be a vector field on W. Defining $Y := \varphi_*\left(\widetilde{Y}, 0\right)$, $Y' := \varphi_*\left(\widetilde{Y}', 0\right)$, and $Z := \varphi_*\left(0, \widetilde{Z}\right)$, we have $[Y,\ Z] = [Y',\ Z] = 0$ and, from eq. (A.5.1):

$$\Omega_{\text{KIN}}(Z,\ \cdot)\big|_{T(\mathcal{M}^{\text{SHELL}})} = 0\,.$$

Hence, we get, for any $y', z' \in V \times W$:

$$\begin{aligned} d\Omega_{\text{KIN}}\left(Y,\ Y',\ Z\right)_{\varphi(y',z')} &= 0 \quad \text{(by definition of a symplectic form)} \\ &= Z\left(\Omega_{\text{KIN}}(Y,\ Y')\right)_{\varphi(y',z')} \\ &= \widetilde{Z}_{z'}\left(z'' \mapsto \Omega^{\varphi,z''}_{\text{DYN},y'}\left(\widetilde{Y}_{y'},\ \widetilde{Y}'_{y'}\right)\right). \end{aligned}$$

Now, for any $x \in \mathcal{M}^{\text{SHELL}}$, we define $\Omega^{x}_{\text{DYN}} : T_{\delta(x)}(\mathcal{M}^{\text{DYN}}) \times T_{\delta(x)}(\mathcal{M}^{\text{DYN}}) \to \mathbb{R}$ by:

$$\forall v, w \in T_x(\mathcal{M}^{\text{SHELL}}),\ \Omega^{x}_{\text{DYN}}\left(T_x\delta(v),\ T_x\delta(w)\right) = \Omega_{\text{KIN},x}(v, w)\,.$$

That such an Ω^x_{DYN} exists is established by the previous discussion and, since $\operatorname{Im} T_x\delta = T_{\delta(x)}(\mathcal{M}^{\text{DYN}})$, it is moreover unique. Thus, Ω^x_{DYN} is well-defined.

The previous argument also shows that, for any vector fields Y, Y' on $\mathcal{M}^{\text{DYN}}$, $x \mapsto \Omega^x_{\text{DYN}}\left(Y_{\delta(x)},\, Y'_{\delta(x)}\right)$ is smooth and satisfies:

$$\forall x \in \mathcal{M}^{\text{SHELL}},\ \forall w \in T_x\left(\delta^{-1}\left\langle\delta(x)\right\rangle\right),\quad T_x\left(x' \mapsto \Omega^{x'}_{\text{DYN}}\left(Y_{\delta(x')},\, Y'_{\delta(x')}\right)\right)(w) = 0\,.$$

The level sets of δ being connected, as underlined above, this allows us to define a smooth differential 2-form Ω_{DYN} satisfying:

$$\forall x \in \mathcal{M}^{\text{SHELL}},\ \Omega_{\text{KIN},x}\big|_{T_x(\mathcal{M}^{\text{SHELL}})} = \left[\delta^*\,\Omega_{\text{DYN}}\right]_x .$$

Lastly, for any $x \in \mathcal{M}^{\text{SHELL}}$, we also have:

$$\left[\delta^*\, d\Omega_{\text{DYN}}\right]_x = d\Omega_{\text{KIN},x}\big|_{T_x(\mathcal{M}^{\text{SHELL}})} = 0\,,$$

and, from eq. (A.5.1):

$$\operatorname{Ker} T_x\delta = K_x(\mathcal{M}^{\text{SHELL}})\,.$$

Thus, $T_x\delta$ being surjective, Ω_{DYN} is closed and non-degenerate, so it is indeed a symplectic form on $\mathcal{M}^{\text{DYN}}$. □

Proposition A.6 Let $\mathcal{M}^{\text{KIN}}$ be a smooth, finite dimensional, symplectic manifold and let $(\mathcal{M}^{\text{DYN},1}, \mathcal{M}^{\text{SHELL}}, \delta_1)$ and $(\mathcal{M}^{\text{DYN},2}, \mathcal{M}^{\text{SHELL}}, \delta_2)$ be two phase space reductions of $\mathcal{M}^{\text{KIN}}$ arising from the same submanifold $\mathcal{M}^{\text{SHELL}}$ of $\mathcal{M}^{\text{KIN}}$. Then there exists a unique map $\psi : \mathcal{M}^{\text{DYN},1} \to \mathcal{M}^{\text{DYN},2}$ such that $\delta_2 = \psi \circ \delta_1$. Moreover, ψ is a symplectomorphism.

Proof From def. A.3.4 δ_2 is constant on the level sets of δ_1 and reciprocally, hence, as a consequence [54, prop. 5.21] of the rank theorem (using that both δ_1 and δ_2 are surjective and have surjective derivative at each point, from def. A.3.3), there exists a unique diffeomorphism $\psi : \mathcal{M}^{\text{DYN},1} \to \mathcal{M}^{\text{DYN},2}$ such that $\delta_2 = \psi \circ \delta_1$.

In particular, for $x \in \mathcal{M}^{\text{SHELL}}$ (with $y := \delta_1(x)$), we have $T_x\delta_2 = T_y\psi \circ T_x\delta_1$, so that, using def. A.1.3:

$$\left[\delta_1^*\,\Omega_{\text{DYN},1}\right]_x = \Omega_{\text{KIN},x}\big|_{T_x(\mathcal{M}^{\text{SHELL}})} = \left[\delta_1^*\,\psi^*\,\Omega_{\text{DYN},2}\right]_x.$$

Since $T_x\delta_1$ and δ_1 are surjective, ψ is a symplectomorphism. □

The subalgebra of kinematical observables whose Hamiltonian flow stabilizes the constraint surface (and are therefore constant on the gauge orbits) is mapped into the algebra of dynamical observables in a way that preserves the Poisson brackets. Equivalently, these compatible kinematical observables can be characterized as the one having vanishing commutators with the constraints (at least on shell). This property is closely related to the possibility of solving mutually commuting sets of constraints *successively*, rather than simultaneously.

Proposition A.7 Let $\mathcal{M}^{\text{KIN}}$ be a smooth, finite dimensional, symplectic manifold and let $(\mathcal{M}^{\text{DYN}}, \mathcal{M}^{\text{SHELL}}, \delta)$ be a phase space reduction of $\mathcal{M}^{\text{KIN}}$. Let f, g and $\{f, g\}_{\text{KIN}} \in C^\infty(\mathcal{M}^{\text{KIN}}, \mathbb{R}) \cap B(\mathcal{M}^{\text{KIN}})$, and assume that:

$$\forall x \in \mathcal{M}^{\text{SHELL}},\ X_{f,x}, X_{g,x} \in T_x(\mathcal{M}^{\text{SHELL}})\,,$$

where the Hamiltonian vector field $X_f := \underline{df}$ is defined by $\Omega_{\text{KIN}}(X_f,\, \cdot\,) = df$.

Then $f^{\text{DYN}}, g^{\text{DYN}} \in C^\infty(\mathcal{M}^{\text{DYN}}, \mathbb{R})$ and $\{f^{\text{DYN}}, g^{\text{DYN}}\}_{\text{DYN}} = \left(\{f, g\}_{\text{KIN}}\right)^{\text{DYN}}$.

Proof For all $x \in \mathcal{M}^{\text{SHELL}}$, $X_{f,x} \in T_x(\mathcal{M}^{\text{SHELL}})$, hence $df_x \langle K_x(\mathcal{M}^{\text{SHELL}})\rangle = \Omega_{\text{KIN},x}\left(X_{f,x}, K_x(\mathcal{M}^{\text{SHELL}})\right) \subset \Omega_{\text{KIN},x}(T_x(\mathcal{M}^{\text{SHELL}}), K_x(\mathcal{M}^{\text{SHELL}})) = \{0\}$. The same holds for g. Therefore, f and g are constant on the leaves of the foliation $K(\mathcal{M}^{\text{SHELL}})$ on $\mathcal{M}^{\text{SHELL}}$. As a consequence [54, prop. 5.20] of the rank theorem (using defs. A.3.3 and A.3.4), there exist smooth maps $\widetilde{f}$ and $\widetilde{g} : \mathcal{M}^{\text{DYN}} \to \mathbb{R}$ such that $f|_{\mathcal{M}^{\text{SHELL}}} = \widetilde{f} \circ \delta$ and $g|_{\mathcal{M}^{\text{SHELL}}} = \widetilde{g} \circ \delta$. Hence, $f^{\text{DYN}} = \widetilde{f}$ and $g^{\text{DYN}} = \widetilde{g}$.

In addition, we have for any $x \in \mathcal{M}^{\text{SHELL}}$ (with $y := \delta(x)$):

$$\begin{aligned}
[\delta^* df^{\text{DYN}}]_x &= df_x|_{T_x(\mathcal{M}^{\text{SHELL}})} \\
&= \Omega_{\text{KIN},x}|_{T_x(\mathcal{M}^{\text{SHELL}})}\left(X_{f,x};\, \cdot\,\right) \ \text{(using } X_{f,x} \in T_x(\mathcal{M}^{\text{SHELL}})\text{)} \\
&= \Omega_{\text{DYN},y}\left(T_x\delta\left(X_{f,x}\right);\, T_x\delta\left(\,\cdot\,\right)\right) \ \text{(using def. A.1.3).}
\end{aligned}$$

Thus, $T_x\delta$ being surjective, $X_{f^{\text{DYN}},y} = T_x\delta\left(X_{f,x}\right)$ and, similarly $X_{g^{\text{DYN}},y} = T_x\delta\left(X_{g,x}\right)$.

Hence, we have:

$$\begin{aligned}
\{f^{\text{DYN}}, g^{\text{DYN}}\}_{\text{DYN},y} &= \Omega_{\text{DYN},y}\left(X_{g^{\text{DYN}},y}, X_{f^{\text{DYN}},y}\right) \\
&= \Omega_{\text{DYN},y}\left(T_x\delta\left(X_{g,x}\right), T_x\delta\left(X_{f,x}\right)\right) \\
&= \Omega_{\text{KIN},x}\left(X_{g,x}, X_{f,x}\right) \ \text{(using def. A.1.3 and } X_{f,x}, X_{g,x} \in T_x(\mathcal{M}^{\text{SHELL}})\text{)} \\
&= \{f, g\}_{\text{KIN},x}.
\end{aligned}$$

Since this holds for all $x \in \delta^{-1}\langle y\rangle$, this implies in particular that $\{f, g\}_{\text{KIN}}$ is constant on $\delta^{-1}\langle y\rangle$. Therefore $\left(\{f, g\}_{\text{KIN}}\right)^{\text{DYN}}(y) = \{f, g\}_{\text{KIN},x} = \{f^{\text{DYN}}, g^{\text{DYN}}\}_{\text{DYN},y}$. □

Finally, we briefly recall the idea of partial gauge fixing: imposing additional constraints, in such a way that the resulting constraint surface cuts transversally through each gauge orbit, yields the same reduced phase space. However, the dynamical observable associated to a given kinematical observable according to the *gauge-fixed* dynamics does *not* coincides with the one associated according to the original dynamics: while the reduced phase space is not affected by gauge fixing, its *relation* to the kinematical phase space *is* (and, as stressed above, this relation is where the *physical* interpretation of the reduced phase space comes from). At least, the gauge-fixed dynamics does reproduce the correct dynamical observable whenever the kinematical observable under consideration is compatible with original set of constraints (because it is then constant on gauge orbits).

Proposition A.8 Let $(\mathcal{M}^{\text{DYN}}, \mathcal{M}^{\text{SHELL}}, \delta)$ be a phase space reduction of $\mathcal{M}^{\text{KIN}}$ and $\mathcal{M}^{\text{FIX}}$ a submanifold of $\mathcal{M}^{\text{SHELL}}$ such that:

1. for all $x \in \mathcal{M}^{\text{FIX}}$, $T_x(\mathcal{M}^{\text{SHELL}}) = T_x(\mathcal{M}^{\text{FIX}}) + K_x(\mathcal{M}^{\text{SHELL}})$;
2. the intersection of a leaf of the foliation $K(\mathcal{M}^{\text{SHELL}})$ with $\mathcal{M}^{\text{FIX}}$ is not void and is connected;

We define $\delta^{\text{FIX}} : \mathcal{M}^{\text{FIX}} \to \mathcal{M}^{\text{DYN}}$ by $\delta^{\text{FIX}} := \delta|_{\mathcal{M}^{\text{FIX}}}$. Then, $(\mathcal{M}^{\text{DYN}}, \mathcal{M}^{\text{FIX}}, \delta^{\text{FIX}})$ is a phase space reduction of $\mathcal{M}^{\text{KIN}}$.

Moreover, if $f \in C^\infty(\mathcal{M}^{\text{KIN}}, \mathbb{R}) \cap B(\mathcal{M}^{\text{KIN}})$ and $\forall x \in \mathcal{M}^{\text{SHELL}}$, $X_{f,x} \in T_x(\mathcal{M}^{\text{SHELL}})$, we have $f^{\text{DYN}} = f^{\text{FIX}}$ where:

$$\forall y \in \mathcal{M}^{\text{DYN}},\ f^{\text{DYN}}(y) := \sup\left\{ f(x) \;\middle|\; x \in \delta^{-1}\langle y \rangle \right\} \quad \text{(def. A.2)}$$

$$\text{and } f^{\text{FIX}}(y) := \sup\left\{ f(x) \;\middle|\; x \in \delta^{\text{FIX},-1}\langle y \rangle \right\}.$$

Proof *Statements A.1.1 & A.1.2.* $\mathcal{M}^{\text{FIX}}$ is a submanifold of $\mathcal{M}^{\text{SHELL}}$, and $\mathcal{M}^{\text{SHELL}}$ is a submanifold of $\mathcal{M}^{\text{KIN}}$, hence $\mathcal{M}^{\text{FIX}}$ is a submanifold of $\mathcal{M}^{\text{KIN}}$. $\mathcal{M}^{\text{DYN}}$ is a symplectic manifold.

The level sets of δ^{FIX} are the intersection with $\mathcal{M}^{\text{FIX}}$ of the leaves of the foliation $K(\mathcal{M}^{\text{SHELL}})$ (using def. A.3.4), hence from assumption A.8.2, δ^{FIX} is surjective and its level sets are connected.

Statement A.1.3. Let $x \in \mathcal{M}^{\text{FIX}}$. We have $T_x\delta^{\text{FIX}} = T_x\delta|_{T_x(\mathcal{M}^{\text{FIX}})}$, hence:

$$T_x\delta^{\text{FIX}}\langle T_x(\mathcal{M}^{\text{FIX}})\rangle = T_x\delta\langle T_x(\mathcal{M}^{\text{FIX}})\rangle = T_x\delta\langle T_x(\mathcal{M}^{\text{FIX}}) + \operatorname{Ker} T_x\delta\rangle$$

$$= T_x\delta\langle T_x(\mathcal{M}^{\text{FIX}}) + K_x(\mathcal{M}^{\text{SHELL}})\rangle = T_x\delta\langle T_x(\mathcal{M}^{\text{SHELL}})\rangle = T_{\delta^{\text{FIX}}(x)}(\mathcal{M}^{\text{DYN}})$$ (using assumption A.8.1, eq. (A.4.1) and def. A.1.3 for the phase space reduction $(\mathcal{M}^{\text{DYN}}, \mathcal{M}^{\text{SHELL}}, \delta)$).

Next, we have:

$\Omega_{\text{KIN},x}|_{T_x(\mathcal{M}^{\text{FIX}})} = [\delta^*\,\Omega_{\text{DYN}}]_x|_{T_x(\mathcal{M}^{\text{FIX}})}$ (using def. A.1.3 for the phase space reduction $(\mathcal{M}^{\text{DYN}}, \mathcal{M}^{\text{SHELL}}, \delta)$ and $T_x(\mathcal{M}^{\text{FIX}}) \subset T_x(\mathcal{M}^{\text{SHELL}})$)

$$= \left[\left(\delta|_{\mathcal{M}^{\text{FIX}}}\right)^*\,\Omega_{\text{DYN}}\right]_x$$

$$= [\delta^{\text{FIX},*}\,\Omega_{\text{DYN}}]_x\,.$$

Observables. Let $f \in C^\infty(\mathcal{M}^{\text{KIN}}, \mathbb{R}) \cap B(\mathcal{M}^{\text{KIN}})$ with $\forall x \in \mathcal{M}^{\text{SHELL}}$, $X_{f,x} \in T_x(\mathcal{M}^{\text{SHELL}})$. From the proof of prop. A.7, f is then constant on the leaves of the foliation $K(\mathcal{M}^{\text{SHELL}})$ on $\mathcal{M}^{\text{SHELL}}$. Therefore, $f^{\text{DYN}} = f^{\text{FIX}}$.

□

B. Geometric quantization

The aim of this appendix is to import a few definitions and properties from geometric quantization, that are needed in particular for section 6. We try here to give a short self-contained introduction, leading rapidly to the definition of the holomorphic representation on a Kähler manifold [105, sections 8.4 & 9.2], and of the position representation arising from a choice of configuration variables on a symplectic manifold [105, sections 4.5 & 9.3]. Accordingly, we skip advanced aspects, including underlying insights and technical subtleties.

In this appendix all manifolds are assumed to be smooth, *finite dimensional* manifolds.

B.1 Prequantization

Definition B.1 An hermitian line bundle is a vector bundle ($\mathcal{B}$, $\pi_{\mathcal{B}}$, $\mathcal{M}$) associated to a $\mathcal{U}(1)$-principal bundle on a smooth manifold $\mathcal{M}$ via the standard action of $\mathcal{U}(1)$ on $\mathbb{C}$. Since the $\mathbb{C}$-linear structure and the Hermitian product $\langle \cdot , \cdot \rangle : z, z' \mapsto z^{*z'}$ on $\mathbb{C}$ is preserved under the action of $\mathcal{U}(1)$, each fiber of $\mathcal{B}$ can be equipped with a natural hermitian structure.

Any connection in the $\mathcal{U}(1)$-principal bundle defines a covariant derivative ∇ on $\mathcal{B}$, and we can associate to its curvature a Lie($\mathcal{U}(1)$) $\approx \mathbb{R}$-valued 2-form $D\nabla$ on $\mathcal{M}$, such that for any cross-section s of $\mathcal{B}$ and any vector fields X, Y on $\mathcal{M}$:

$$[\nabla_X, \nabla_Y](s) = \nabla_{[X,Y]}(s) + i\, D\nabla(X, Y)\, s\,. \tag{B.1.1}$$

Definition B.2 Let $\mathcal{M}$ be a symplectic manifold (with symplectic structure Ω). A prequantum bundle ($\mathcal{B}$, ∇) on $\mathcal{M}$ is an hermitian line bundle $\mathcal{B}$, with base $\mathcal{M}$, equipped with a connection, with corresponding covariant derivative ∇ and such that:

$$D\nabla = -\Omega\,.$$

On $\mathcal{M}$, we define the symplectic volume form $\omega := \frac{1}{p!}\,\Omega\wedge \ldots \wedge\Omega = \frac{1}{p!}\,\Omega^{\wedge p}$ (where $p := \dim \mathcal{M}/2$) and the corresponding measure μ_ω.

Definition B.3 Considering the same objects as in def. B.2, we define the prequantum Hilbert space $\mathcal{H}_{\text{preQ}}$ as the space $L_2(\mathcal{M} \to \mathcal{B},\, d\mu_\omega)$ of (equivalence classes up to almost-everywhere equality of) cross-sections of $\mathcal{B}$ whose norm, defined using the hermitian structure on $\mathcal{B}$, is square-integrable with respect to μ_ω.

For $f \in C^\infty(\mathcal{M}, \mathbb{C})$, we define the prequantization $\widehat{f}$ of f as a (densely defined) operator on $\mathcal{H}_{\text{preQ}}$ by:

$$\forall s \in \mathcal{D}_f \subset \mathcal{H}_{\text{preQ}},\quad \widehat{f}s := f s + i\, \nabla_{X_f}\, s\,,$$

where $\nabla_{X_f} := \nabla_{X_{\text{Re}(f)}} + i\, \nabla_{X_{\text{Im}(f)}}$.

Proposition B.4 Let $f, g \in C^\infty(\mathcal{M}, \mathbb{C})$. Then:

$$\left[\widehat{f}, \widehat{g}\right] = i\, \widehat{\{f, g\}}\,.$$

and:

$$\forall s, s' \in \mathcal{D}_f,\ \left\langle \widehat{f^*}(s), s' \right\rangle = \left\langle s, \widehat{f}(s') \right\rangle.$$

If f is *real*-valued and if, moreover, X_f is a *complete* vector field on $\mathcal{M}$ [54, chap. 12], then $\widehat{f}$ is an essentially self-adjoint operator on $\mathcal{H}_{\text{preQ}}$.

Proof Let $s \in \mathcal{D}_{f,g}$ (defining the common domain $\mathcal{D}_{f,g}$ such that both $\widehat{f}\,\widehat{g}\,s$ and $\widehat{g}\,\widehat{f}\,s$ are well-defined; since $\mathcal{D}_{f,g}$ contains at least the compactly supported smooth cross-sections, it is dense in $\mathcal{H}_{\text{preQ}}$). Using eq. (B.1.1) we have:

$$\left[\widehat{f}, \widehat{g}\right](s) = -\left[\nabla_{X_f}, \nabla_{X_g}\right](s) + i\, d_{X_f}(g)\, s - i\, d_{X_g}(f)\, s$$

$$= -\nabla_{[X_f, X_g]}(s) + i\,\Omega\left(X_f, X_g\right) s + i\,\Omega\left(X_g, X_f\right) s - i\,\Omega\left(X_f, X_g\right) s$$

$$= -\nabla_{X_{\{f,g\}}}(s) + i\,\{f, g\}\, s = i\,\widehat{\{f, g\}}\,(s)\,.$$

Let $s, s' \in \mathcal{D}_f$. We have for all X vector field on $\mathcal{M}$:

$$\forall x \in \mathcal{M},\ d_X \langle s, s'\rangle (x) = \langle \nabla_X(s)(x), s'(x)\rangle + \langle s(x), \nabla_X(s')(x)\rangle\,, \tag{B.4.1}$$

for ∇ comes from a $\mathcal{U}(1)$-connection. Hence, we get:

$$\left\langle \widehat{f^*}(s), s' \right\rangle = \int d\mu_\omega(x)\ \ \left\langle i\,\nabla_{X^*_f}(s)(x), s'(x)\right\rangle + \int d\mu_\omega(x)\ \ \langle f^*\,s(x), s'(x)\rangle$$

$$= -\int d\mu_\omega(x)\ \ i\,d_{X_f}\langle s, s'\rangle(x) + \int d\mu_\omega(x)\ \ \langle s(x), i\,\nabla_{X_f}(s')(x)\rangle + \int d\mu_\omega(x)\ \ \langle s(x), f\,s'(x)\rangle$$

$$= \int i\,\langle s, s'\rangle\ (\mathfrak{L}_{X_f}\omega) + \int d\mu_\omega(x)\ \ \left\langle s(x), \widehat{f}(s')(x)\right\rangle$$

(using Stokes theorem [54, theorem 10.23]; $\mathcal{M}$ is assumed to be without boundary, and $\mathcal{D}_f$ is required to ensure suitable fall-off conditions)

$$= \int d\mu_\omega(x)\ \ \left\langle s(x), \widehat{f}(s')(x)\right\rangle = \left\langle s, \widehat{f}(s')\right\rangle$$

(for X_f generates symplectomorphisms, thus preserving the symplectic volume form ω).

We now assume that f is real-valued. From the previous point, $\widehat{f}$ is a symmetric operator on a dense domain containing at least the compactly supported smooth cross-sections. We moreover assume that the Hamiltonian vector field X_f of f is a complete vector field on $\mathcal{M}$, ie. there exists a smooth map $\theta : \mathbb{R}\times\mathcal{M} \to \mathcal{M}$ such that [54, lemma 12.7]:

1. $\theta(0, \cdot\,) = \mathrm{id}_{\mathcal{M}}\quad \&\quad \forall t, t' \in \mathbb{R},\ \theta(t, \cdot\,)\circ\theta(t', \cdot\,) = \theta(t + t', \cdot\,)\,;$
2. $\forall x \in \mathcal{M},\quad X_{f,x} = \left.\dfrac{d}{d\tau}\theta(\tau, x)\right|_{\tau=0}\,;$
3. and, for any $t \in \mathbb{R}$, $\theta(t, \cdot\,)$ is a symplectomorphism of $\mathcal{M}$ [54, prop. 13.23].

Using ∇-parallel transport [50, section II.3], we can lift θ to a flow Θ on $\boldsymbol{\mathcal{B}}$, such that, for any $t \in \mathbb{R}$, $\Theta(t, \cdot\,)$ is a bundle automorphism of $\boldsymbol{\mathcal{B}}$. We then have, for any smooth cross-section s:

$$\left.\frac{d}{d\tau}s_{(\tau)}(x)\right|_{\tau=t} = -\left(\nabla_{X_f} s_{(t)}\right)(x)\,,$$

where $\forall t \in \mathbb{R},\ s_{(t)} := \Theta(t, \cdot\,)\circ s\circ\theta(-t, \cdot\,)$.

Next, let $s \in \mathrm{Ker}\left(\widehat{f}^\dagger \pm i\right) \subset \mathcal{H}_{\text{preQ}}$. This requires:

$$\forall s' \in \mathcal{D}_f\,,\ \left\langle \widehat{f}(s') \,\middle|\, s\right\rangle = \mp i\,\langle s' \mid s\rangle\,.$$

For any smooth, compactly supported cross-section $s' \in \mathcal{D}_f$, and any $t \in \mathbb{R}$, we have $s'_{(t)} \in \mathcal{D}_f$, so we get, using the dominated convergence theorem:

$$0 = \int d\mu_\omega(x) \;\; \left\langle (f \circ \theta(t,x) \mp i) \; s'(x) \middle| s_{(-t)}(x) \right\rangle - i \; \frac{d}{d\tau} \int d\mu_\omega(x) \;\; \left\langle s'(x) \middle| s_{(-\tau)}(x) \right\rangle \Big|_{\tau = t} ,$$

where we have used that for any $\tau \in \mathbb{R}$, $\theta(\tau, \cdot)$ preserves μ_ω (as a symplectomorphism on $\mathcal{M}$) and $\Theta(\tau, \cdot)$ preserves $\langle \cdot , \cdot \rangle$ (as a line bundle automorphism). Defining, for any smooth, compactly supported cross-section $s' \in \mathcal{D}_f$, and any $t \in \mathbb{R}$:

$$I(s', t) := \int d\mu_\omega(x) \;\; \left\langle s'_{(t)}(x) \;\middle|\; s(x) \right\rangle = \int d\mu_\omega(x) \;\; \left\langle s'(x) \mid s_{(-t)}(x) \right\rangle ,$$

this can be rewritten as:

$$\frac{d}{d\tau} I(s', \tau) \Big|_{\tau = t} = I\Big(\big(i\, f \circ \theta(t, \cdot) \pm 1 \big) \, s', \; t \Big) . \tag{B.4.2}$$

Defining, for any $t \in \mathbb{R}$ and any $x \in \mathcal{M}$:

$$F(t, x) := \exp \left(\mp t - i \int_0^t d\tau \;\; f \circ \theta(\tau, x) \right) ,$$

we obtain, for any smooth, compactly supported cross-section $s' \in \mathcal{D}_f$:

$$\forall t \in \mathbb{R}, \; \frac{d}{d\tau} I\big(F(\tau, \cdot) \, s', \; \tau \big) \Big|_{\tau = t} = 0 .$$

hence, for any $t \in \mathbb{R}$, and almost everywhere in $(\mathcal{M}, \mu_\omega)$, $s = \overline{F}(t, \cdot) \, s_{(-t)}$. In particular, we then have:

$$\forall t \in \mathbb{R}, \;\; \big| s \big| = e^{\mp t} \left| s \circ \theta(t, \cdot) \right| \;\; \text{a.e. in } (\mathcal{M}, \mu_\omega) .$$

Since $s \in L_2(\mathcal{M} \to \boldsymbol{\mathcal{B}}, \, d\mu_\omega)$ and $\theta(t, \cdot)$ preserves μ_ω for any $t \in \mathbb{R}$, this requires $s = 0$. Thus, we obtain $\operatorname{Ker} \left(\widehat{f}^\dagger \pm i \right) = \{0\}$, which ensures that $\widehat{f}$ is essentially self-adjoint [73, theorem VIII.3]. □

The prequantization of $\mathcal{M}$ leads to a faithful representation of the full Poisson-algebra $C^\infty(\mathcal{M}, \mathbb{C})$. However, this representation is typically much too big (as is to be expected from the Groenewold-Van-Hove theorem [40] and generalizations thereof [39]), so the next step will be to implement additional prescriptions yielding a physically admissible Hilbert space (at the cost of restricting which observables can be quantized).

B.2 Holomorphic representation

To discuss holomorphic quantization we need to equip $\mathcal{M}$ with an almost complex structure J (def. B.5.1), which is required to be integrable (def. B.5.2, ensuring the existence of local holomorphic coordinates, thus making J into a complex structure for $\mathcal{M}$) and compatible with the symplectic structure Ω (def. B.5.3). An additional positivity requirement (def. B.5.4) allows to define from Ω and J a Riemannian metric on $\mathcal{M}$ (the so-called Kähler metric) and makes $\mathcal{M}$ into a Kähler manifold [50, section IX.4].

Definition B.5 A Kähler manifold $(\mathcal{M}, \Omega, J)$ is a symplectic manifold $(\mathcal{M}, \Omega)$ equipped with a smooth field J satisfying:

1. $\forall x \in \mathcal{M}$, J_x is an endomorphism of $T_x(\mathcal{M})$ such that $J_x^2 = -\mathrm{id}_{T_x(\mathcal{M})}$;
2. $\forall X, Y \in \mathcal{T}^\infty(\mathcal{M})$, $[JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] = 0$ (where $\mathcal{T}^\infty(\mathcal{M})$ is the space of smooth vector fields on $\mathcal{M}$);
3. $\forall x \in \mathcal{M}$, $\forall v, w \in T_x(\mathcal{M})$, $\Omega_x(J_x v, J_x w) = \Omega(v, w)$;
4. $\forall x \in \mathcal{M}$, $\forall v \neq 0 \in T_x(\mathcal{M})$, $\Omega_x(v, J_x v) > 0$.

Proposition B.6 Let $\mathcal{M}$ be a Kähler manifold and $(\boldsymbol{\mathcal{B}}, \nabla)$ be a prequantum bundle on $\mathcal{M}$. We define the holomorphic quantization $\mathcal{H}_{\text{Holo}}$ of $\mathcal{M}$ to be $\mathcal{H}_{\text{Holo}} := \mathcal{H}_{\text{preQ}} \cap \text{Holo}(\mathcal{M} \to \boldsymbol{\mathcal{B}})$, where $\text{Holo}(\mathcal{M} \to \boldsymbol{\mathcal{B}})$ is the space of holomorphic cross-sections of $\boldsymbol{\mathcal{B}}$:

$$\text{Holo}(\mathcal{M} \to \boldsymbol{\mathcal{B}}) := \{s \in C^\infty(\mathcal{M} \to \boldsymbol{\mathcal{B}}) \mid \forall x \in \mathcal{M}, \forall v \in T_x(\mathcal{M}), \ \nabla_{Jv} s = i \nabla_v s\}.$$

$\mathcal{H}_{\text{Holo}}$ is a closed vector subspace of $\mathcal{H}_{\text{preQ}}$, hence is itself an Hilbert space.

Proof Let $(s_\alpha)_{\alpha \in A}$ be a net in $\mathcal{H}_{\text{Holo}}$ that converges (for the norm $\| \cdot \|_{\text{preQ}}$) to $s \in \mathcal{H}_{\text{preQ}}$.

Let $x \in \mathcal{M}$. There exist a neighborhood U of x in $\mathcal{M}$, holomorphic coordinates $z^1, \dots, z^p$ ($2p := \dim\mathcal{M}$) on U and a real valued function $K(z, z^*)$ on U such that [105, section 5.4]:

$$\forall x' \in U, \ \Omega_{x'} = i \frac{\partial^2 K}{\partial z^j \, \partial z^{l,*}}(x') \, dz^j \wedge dz^{l,*},$$

and from B.5.4 $\dfrac{\partial^2 K}{\partial z^j \, \partial z^{l,*}}$ has to be a positive definite hermitian matrix at every point in U. The symplectic measure is then given over U by:

$$\mu_\omega := \beta \, \mu_{\mathbb{C}}^{(p)} = 2^p \det\left(\frac{\partial^2 K}{\partial z \, \partial z^*}\right) \mu_{\mathbb{C}}^{(p)} \text{ (where } \mu_{\mathbb{C}}^{(p)} \text{ is the standard measure on } \mathbb{C}^p).$$

We choose $t \in \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}}^{-1} \langle x \rangle$ (the fiber of $\boldsymbol{\mathcal{B}}$ above x), with $|t| = e^{-K(x)/2}$, and we define the cross-section r of $\boldsymbol{\mathcal{B}}|_U$ by:

$$r(x) = t \quad \& \quad \forall x' \in U, \forall v \in T_{x'}(\mathcal{M}), \ \nabla_v r(x') = -\left(\frac{\partial K}{\partial z^j}(x') \left[dz^j\right]_{x'}(v)\right) r(x').$$

We can check using eq. (B.1.1) that this characterizes a well-defined cross-section of $\boldsymbol{\mathcal{B}}|_U$ and we have moreover:

$$\forall x' \in U, \forall v \in T_{x'}(\mathcal{M}), \ \nabla_{Jv} r(x') = i \nabla_v r(x'),$$

$$\text{and} \quad [d \langle r, r \rangle]_{x'}(v) = -dK_{x'}(v) \, \langle r, r \rangle (x'),$$

so r is an holomorphic cross-section of $\boldsymbol{\mathcal{B}}|_U$ and $\forall x' \in U$, $|r(x')| = e^{-K(x')/2}$.

Next, for all $\alpha \in A$ we can define f_α as the holomorphic function $f_\alpha : U \to \mathbb{C}$ such that $\forall x' \in U$, $s_\alpha(x') = f_\alpha(x') \, r(x')$. Similarly, we define $f : U \to \mathbb{C}$ such that $\forall x' \in U$, $s(x') = f(x') \, r(x')$.

Let $\epsilon > 0$ and let U_1 be a closed ball (with respect to the coordinates $z^1, \dots, z^p$) of center x and radius $r > 0$ such that $U_1 \subset U$ and $\forall x' \in U_1$, $\beta(x') \, e^{-K(x')} > \epsilon$. Let U_2 be the closed ball of center x and radius $r/2$. For all $x' \in U_2$ we call $U_2^{x'}$ the closed ball of center x' and radius $r/2$. Hence,

$U_2^{x'} \subset U_1$. For g an holomorphic function on $U_1 \to \mathbb{C}$, we have:

$$\forall x' \in U_2,\ g(x') = \frac{8^p}{r^{2p}} \left[\prod_{k=1}^{p} \int_0^{r/2} q^k \, dq^k\right] g(x')$$

$$= \frac{4^p}{\pi^p \, r^{2p}} \left[\prod_{k=1}^{p} \int_0^{r/2} dq^k \int_0^{2\pi} q^k \, d\theta^k\right] \ g\left(x' + \left(q^1 \, e^{i\theta^1}, \ldots, q^p \, e^{i\theta^p}\right)\right),$$

hence:

$$\forall x' \in U_2,\ \left|g(x')\right|^2 \leqslant \left(\frac{4}{\pi \, r^2}\right)^{2p} \left[\int_{U_2^{x'}} d\mu_{\mathbb{C}}^{(p)}(z) \ \ |g(z)|\right]^2$$

$$\leqslant \left(\frac{4}{\pi \, r^2}\right)^{2p} \frac{\pi^p \, r^{2p}}{4^p} \int_{U_2^{x'}} d\mu_{\mathbb{C}}^{(p)}(z) \ \ |g(z)|^2 \text{ (by convexity of } x \mapsto x^2\text{)}$$

$$\leqslant \frac{4}{\pi \, r^2} \int_{U_1} d\mu_{\mathbb{C}}^{(p)}(z) \ \ |g(z)|^2 .$$

Therefore, for all $\alpha, \alpha' \in A$:

$$\forall x' \in U_2,\ \left|f_\alpha(x') - f_{\alpha'}(x')\right|^2 \leqslant \frac{4}{\pi \, r^2} \int_{U_1} d\mu_{\mathbb{C}}^{(p)}(z) \ \ \left|f_\alpha(z) - f_{\alpha'}(z)\right|^2$$

$$\leqslant \frac{1}{\epsilon} \frac{4}{\pi \, r^2} \int_{U_1} d\mu_\omega(z) \ \ \left|f_\alpha(z) - f_{\alpha'}(z)\right|^2 \, |r|^2$$

$$\leqslant \frac{4}{\epsilon \, \pi \, r^2} \left\|s_\alpha - s_{\alpha'}\right\|^2_{\text{preQ}},$$

hence the net $\left(f_\alpha\big|_{U_2}\right)_{\alpha \in A}$ converges uniformly to a function $f' : U_2 \to \mathbb{C}$. Cauchy's integral formula implies that f' is holomorphic on the interior of U_2. On the other hand, the net $\left(f_\alpha\big|_{U_2}\right)_{\alpha \in A}$ converges in L_2-norm to $f\big|_{U_2}$ (for $\forall \alpha \in A$, $\left\|f_\alpha\big|_{U_2} - f\big|_{U_2}\right\|_2 \leqslant \frac{1}{\sqrt{\epsilon}} \left\|s_\alpha - s\right\|_{\text{preQ}}$), hence $f' = f\big|_{U_2}$ (μ_ω-almost-everywhere). Therefore, $s \in \mathcal{H}_{\text{Holo}}$. □

Since we restrict the quantum Hilbert space to $\mathcal{H}_{\text{Holo}}$, we should also restrict the admissible observables to be the ones that stabilize $\mathcal{H}_{\text{Holo}}$ (note that we do not discuss here whether the intersection of $\mathcal{H}_{\text{Holo}}$ with the dense domain of such an observable will also be dense in $\mathcal{H}_{\text{Holo}}$; this is a non-trivial question, for the usual tools based on bump functions are not available in the holomorphic class).

Proposition B.7 We consider the same objects as in prop. B.6. We define:

$$\mathcal{O}_{\text{Holo},\mathbb{C}} := \left\{f \in C^\infty(\mathcal{M}, \mathbb{C}) \mid \forall Y \in \mathcal{T}^\infty(\mathcal{M}), \exists Z \in \mathcal{T}^\infty(\mathcal{M}) \ / \ \left[Y + i\,JY, X_f\right] = Z + i\,JZ\right\},$$

where $\mathcal{T}^\infty(\mathcal{M})$ is the space of smooth vector fields on $\mathcal{M}$.

Then, for all $f \in \mathcal{O}_{\text{Holo},\mathbb{C}}$, $\widehat{f}$ stabilizes $\mathcal{H}_{\text{Holo}}$.

Proof Let $f \in \mathcal{O}_{\text{Holo},\mathbb{C}}$ and $s \in \mathcal{D}_f \cap \mathcal{H}_{\text{Holo}}$. Let $x \in \mathcal{M}$ and $v \in T_x(\mathcal{M})$. Then, there exists $Y \in \mathcal{T}^\infty(\mathcal{M})$ such that $Y_x = v$ (we can construct such a vector field using local smooth coordinates around x and an appropriate bump function). Since $f \in \mathcal{O}_{\text{Holo},\mathbb{C}}$, there also exists $Z \in \mathcal{T}^\infty(\mathcal{M})$ such that $[Y + i\,JY,\, X_f] = Z + i\,JZ$. Hence:

$$
\begin{aligned}
\nabla_{Jv,\,x} \widehat{f}\, s &= \left(\nabla_{JY} \widehat{f}\, s\right)(x) \\
&= \left(f\ (\nabla_{JY}\, s)\right)(x) + \left(\Omega\,(X_f,\, JY)\ s\right)(x) + i\ (\nabla_{JY} \nabla_{X_f}\ s)\,(x) \\
&= \left(f\ (\nabla_{JY}\, s)\right)(x) - i\ \left(\nabla_{[X_f,\, J\,Y]}\ s\right)(x) + i\ (\nabla_{X_f} \nabla_{JY}\ s)\,(x) \quad \text{(using eq. (B.1.1))} \\
&= \left(f\ (\nabla_{JY}\, s)\right)(x) + (\nabla_{Z+i\,JZ}\, s)\,(x) + \left(\nabla_{[X_f,\, Y]}\ s\right)(x) + i\ (\nabla_{X_f} \nabla_{JY}\ s)\,(x) \\
&= i\ \left(f\ (\nabla_Y\, s)\right)(x) + \left(\nabla_{[X_f,\, Y]}\ s\right)(x) - (\nabla_{X_f} \nabla_Y\ s)\,(x) \quad (\text{using } s \in \mathcal{H}_{\text{Holo}}) \\
&= i\,\nabla_Y\ \left(\widehat{f}\, s\right)(x) = i\,\nabla_{v,\,x} \widehat{f}\, s\,,
\end{aligned}
$$

therefore $\widehat{f}\, s \in \mathcal{H}_{\text{Holo}}$. □

Proposition B.8 We consider the same objects as in prop. B.7. Let r be a nowhere-vanishing cross-section of $\boldsymbol{\mathcal{B}}$ such that $r \in \mathcal{H}_{\text{Holo}}$ and let μ_r be the measure on $\mathcal{M}$ defined by $\mu_r = \langle r,\, r\rangle\ \mu_\omega$. Then, the map:

$$
\begin{array}{rcrcl}
\Phi_r & : & L_2(\mathcal{M},\, d\mu_r) \cap \text{Holo}(\mathcal{M}) & \to & \mathcal{H}_{\text{Holo}} \\
 & & \psi & \mapsto & \psi\, r
\end{array}\,,
$$

is an Hilbert space isomorphism.

If $f \in \mathcal{O}_{\text{Holo},\mathbb{C}}$ and $\psi \in \Phi_r^{-1} \langle \mathcal{D}_f \rangle$, we have:

$$
\widehat{f}^r\ \psi := \left(\Phi_r^{-1}\, \widehat{f}\, \Phi_r\right)\ \psi = f\, \psi + i\ (d_{X_f} \psi) + \frac{i}{2}\, X_f^r\, \psi\,, \tag{B.8.1}
$$

where X_f^r is defined by $2\, \nabla_{X_f}\, r = X_f^r\, r$.

Proof Let $s \in \mathcal{H}_{\text{Holo}}$. Since r is a nowhere-vanishing holomorphic cross-section there exists a unique smooth function $\psi : \mathcal{M} \to \mathbb{C}$ such that $s = \psi\, r$. Moreover, for all $x \in \mathcal{M}$ and all $v \in T_x(\mathcal{M})$:

$$
i\ (d_v \psi)\ r = i\, \nabla_v\, s - i\, \psi\, \nabla_v\, r = \nabla_{Jv}\, s - \psi\, \nabla_{Jv}\, r = (d_{Jv} \psi)\ r\,,
$$

hence $\psi \in \text{Holo}(\mathcal{M})$. Moreover:

$$
\|s\|_{\text{Holo}}^2 = \int_{\mathcal{M}} d\mu_\omega(x)\ \langle \psi(x)\, r(x),\ \psi(x)\, r(x)\rangle = \int_{\mathcal{M}} d\mu_r(x)\, \psi^*(x)\, \psi(x) = \|\psi\|_{2,\mu_r}^2.
$$

Therefore, $\psi \in L_2(\mathcal{M},\, d\mu_r)$. But since we have $\Phi_r(\psi) = s$ and $\|s\|_{\text{Holo}} = \|\psi\|_{2,\mu_r}$, Φ_r is an Hilbert space isomorphism.

Eq. (B.8.1) can be checked from the definition of $\widehat{(\cdot)}$ (def. B.3). □

B.3 Position representation

We now turn to the position representation. We describe a choice of configuration variables as a map γ from the phase space into the configuration space. The typical example occurs when $\mathcal{M}$ is given as a cotangent bundle (with its canonical symplectic structure) in which case γ is simply the projection on the base manifold.

Definition B.9 Let $\mathcal{M}$ be a symplectic manifold. A configuration space for $\mathcal{M}$ is a manifold $\mathcal{C}$ and a surjective map $\gamma : \mathcal{M} \to \mathcal{C}$ such that:

1. $\forall x \in \mathcal{M}$, $\mathrm{Im}(T_x\gamma) = T_{\gamma(x)}(\mathcal{C})$;
2. $\forall x \in \mathcal{M}$, $\mathrm{Ker}(T_x\gamma) = \left(\mathrm{Ker}(T_x\gamma)\right)^{\perp} := \{v \in T_x(\mathcal{M}) \mid \forall w \in \mathrm{Ker}(T_x\gamma),\ \Omega_{\mathcal{M},x}(v,\, w) = 0\}$.
3. $\forall y \in \mathcal{C}$, $\gamma^{-1}\langle\{y\}\rangle$ is connected.

Definition B.10 Let $\mathcal{M}$ be a symplectic manifold, $(\mathcal{C},\ \gamma)$ be a configuration space for $\mathcal{M}$ and $(\boldsymbol{\mathcal{B}}, \nabla)$ be a prequantum bundle on $\mathcal{M}$. A configuration quantum bundle on $\mathcal{C}$ is an hermitian line bundle $\boldsymbol{\mathcal{B}}_{\mathcal{C}}$, with base $\mathcal{C}$, and a smooth map $\Gamma : \boldsymbol{\mathcal{B}} \to \boldsymbol{\mathcal{B}}_{\mathcal{C}}$ such that:

1. $\forall z \in \boldsymbol{\mathcal{B}}$, $\boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}_{\mathcal{C}}} \circ \Gamma(z) = \gamma \circ \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}}(z)$ (where $\boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}}$ and $\boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}_{\mathcal{C}}}$ are the bundle projections);
2. $\forall z \in \boldsymbol{\mathcal{B}}$, $\forall \lambda \in \mathbb{C}$, $\Gamma(\lambda\, z) = \lambda\, \Gamma(z)$;
3. $\forall z \in \boldsymbol{\mathcal{B}}$, $|\Gamma(z)| = |z|$;
4. $\forall z \in \boldsymbol{\mathcal{B}}$, $\mathrm{Ker}\, T_z\Gamma \subset \mathrm{Hor}_z(\boldsymbol{\mathcal{B}},\ \nabla)$ (where $\mathrm{Hor}_z(\boldsymbol{\mathcal{B}},\ \nabla)$ is defined as the ∇-horizontal subspace of $T_z(\boldsymbol{\mathcal{B}})$).

Proposition B.11 Let $\mathcal{M}$ be a symplectic manifold, $(\mathcal{C},\ \gamma)$ be a configuration space for $\mathcal{M}$ and $(\boldsymbol{\mathcal{B}}, \nabla)$ be a prequantum bundle on $\mathcal{M}$. If, for all $y \in \mathcal{C}$, $\gamma^{-1}\langle\{y\}\rangle$ is simply-connected, then there exists a configuration quantum bundle $(\boldsymbol{\mathcal{B}}_{\mathcal{C}},\ \Gamma)$ on $\mathcal{C}$.

Proof *Definition of* $\boldsymbol{\mathcal{B}}_{\mathcal{C}}$. Let $y \in \mathcal{C}$ and let $x \in \gamma^{-1}\langle y\rangle$. Since the derivative of γ is surjective at any point in $\mathcal{M}$ (def. B.9.1), we have, by the rank theorem [54, theorem 5.13], $T_x\left(\gamma^{-1}\langle y\rangle\right) = \mathrm{Ker}\, T_x\gamma$. So, using def. B.9.2, $T_x\left(\gamma^{-1}\langle y\rangle\right) = T_x\left(\gamma^{-1}\langle y\rangle\right)^{\perp}$, hence $\Omega_x|_{T_x\left(\gamma^{-1}\langle y\rangle\right)} = 0$. Therefore, if we call $\left(\boldsymbol{\mathcal{B}}_y,\ \nabla_y\right)$ the restriction of $(\boldsymbol{\mathcal{B}}, \nabla)$ over $\gamma^{-1}\langle y\rangle$, the connection ∇_y is flat.

Therefore, $z \mapsto \mathrm{Hor}_z(\boldsymbol{\mathcal{B}}, \nabla) \cap \mathrm{Ker}\left[T_z(\gamma \circ \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}})\right]$ is a smooth involutive tangent distribution on $\boldsymbol{\mathcal{B}}$, so by the global Frobenius theorem [54, theorem 14.13], it defines a foliation of $\boldsymbol{\mathcal{B}}$. Moreover, if Λ is a leaf of this foliation, there exists $y \in \mathcal{C}$ such that $\boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}}\langle\Lambda\rangle = \gamma^{-1}\langle y\rangle$ and $\boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}}|_{\Lambda\to\gamma^{-1}\langle y\rangle}$ is a diffeomorphism. Indeed, the leaf Λ being connected by definition, $\gamma \circ \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}}$ is constant on Λ, so there exists $y \in \mathcal{C}$ such that $\boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}}\langle\Lambda\rangle \subset \gamma^{-1}\langle y\rangle$, ie. $\Lambda \subset \boldsymbol{\mathcal{B}}_y$, and, $\gamma^{-1}\langle y\rangle$ being simply-connected, Λ is just a global horizontal cross-section of $\left(\boldsymbol{\mathcal{B}}_y,\ \nabla_y\right)$ [50, corollary II.9.2].

We define $\boldsymbol{\mathcal{B}}_{\mathcal{C}}$ as the set of all leaves and $\Gamma : \boldsymbol{\mathcal{B}} \to \boldsymbol{\mathcal{B}}_{\mathcal{C}}$ as the quotient map. Since $\gamma \circ \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}}$ is constant on a leaf, we can define a map $\boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}_{\mathcal{C}}}$ on $\boldsymbol{\mathcal{B}}_{\mathcal{C}}$ such that $\gamma \circ \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}} = \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}_{\mathcal{C}}} \circ \Gamma$. Moreover, for any leaf Λ and any $\lambda \in \mathbb{C}$, $\lambda.\Lambda$ is also a leaf, therefore, we can define an action of $\mathbb{C}$ on $\boldsymbol{\mathcal{B}}_{\mathcal{C}}$ such that $\forall z \in \boldsymbol{\mathcal{B}}$, $\forall \lambda \in \mathbb{C}$, $\Gamma(\lambda z) = \lambda \Gamma(z)$. And since ∇ is a $\mathcal{U}(1)$ connection, the norm $|\cdot|$ on $\boldsymbol{\mathcal{B}}$ is constant on each leaf, so we also can define a norm on $\boldsymbol{\mathcal{B}}_{\mathcal{C}}$ such that $\forall z \in \boldsymbol{\mathcal{B}}$, $|\Gamma(z)| = |z|$.

Local description of the quotient. Let $x \in \mathcal{M}$ and let $y = \gamma(x)$. Let U_1 be an open neighborhood of x in $\mathcal{M}$ and $\varphi_1 : U_1 \times \mathbb{C} \to \boldsymbol{\mathcal{B}}$ a local trivialization of the bundle $\boldsymbol{\mathcal{B}}$. Since $T\gamma$ is surjective at any point in $\mathcal{M}$, there exist, by the rank theorem, open neighborhoods V_2 of y in $\mathcal{C}$, W_2 of 0 in $\mathbb{R}^p$ ($p := \dim \mathcal{M} / 2 = \dim \mathcal{C}$), and U_2 of x in U_1, and a diffeomorphism $\varphi_2 : V_2 \times W_2 \to U_2$ such that $\gamma \circ \varphi_2$ is the first projection map $V_2 \times W_2 \to V_2$.

Next, by definition of a foliation, there exist a neighborhood T_2 of $\varphi_1(x, 0)$ in $\varphi_1 \langle U_2 \times \mathbb{C} \rangle$, neighborhoods Q_2 and R_2 of 0 in $\mathbb{R}^{p+2}$ and $\mathbb{R}^p$ respectively and a diffeomorphism $\psi_1 : Q_2 \times R_2 \to T_2$ such that for any $u \in \boldsymbol{\mathcal{B}}_{\mathcal{C}}$ there exists a (possibly empty) countable subset $Q_u \subset Q_2$ with:

$$\Gamma^{-1} \langle u \rangle \cap T_2 = \psi_1 \langle Q_u \times R_2 \rangle \ . \tag{B.11.1}$$

Let V_3, W_3 and S_3 be neighborhoods of y in V_2, 0 in W_2 and 0 in $\mathbb{C}$ respectively, such that $\varphi_1 \langle \varphi_2 \langle V_3 \times W_3 \rangle \times S_3 \rangle \subset T_2$, and define:

$$\begin{array}{rccl} \psi_2 & : & V_3 \times S_3 & \to & Q_2 \\ & & v, \lambda & \mapsto & \pi_{Q_2} \circ \psi_1^{-1} \circ \varphi_1(\varphi_2(v, 0), \lambda) \end{array} ,$$

where $\pi_{Q_2} : Q_2 \times R_2 \to Q_2$ is the first projection map. Since we have:

$$[T_{(0,0)} \psi_1] \langle \{0\} \times T_0(R_2) \rangle = \mathrm{Hor}_{\varphi_1(x,0)} (\boldsymbol{\mathcal{B}}, \nabla) \cap \mathrm{Ker}\, [T_{\varphi_1(x,0)} (\gamma \circ \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}})]$$

$$\text{and } [T_{(x,0)} \varphi_1] \left\langle \left[T_{y,0}\, d\varphi_2\right] \left\langle T_y(V_3) \times \{0\} \right\rangle \times \mathbb{C} \right\rangle \cap \mathrm{Hor}_{\varphi_1(x,0)} (\boldsymbol{\mathcal{B}}, \nabla) \cap \mathrm{Ker}\, [T_{\varphi_1(x,0)} (\gamma \circ \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}})] = \{0\} ,$$

$\left[T_{y,0}\, \psi_2\right]$ is surjective, thus invertible, so by the inverse function theorem [54, theorem 5.11], we can narrow W_3 and S_3 so that there exists a neighborhood Q_3 of 0 in Q_2 with ψ_2 inducing a diffeomorphism $V_3 \times S_3 \to Q_3$.

Now, we define $T_3 := \psi_1 \langle Q_3 \times R_2 \rangle \cap \varphi_1 \langle \varphi_2 \langle V_3 \times W_3 \rangle \times \mathbb{C} \rangle$ and:

$$\begin{array}{rccl} \varphi_3 & : & T_3 & \to & V_3 \times S_3 \times W_3 \\ & & z & \mapsto & \psi_2^{-1} \circ \pi_{Q_3} \circ \psi_1^{-1}(z),\ \pi_{W_3} \circ \varphi_2^{-1} \circ \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}}(z) \end{array} .$$

Precomposing φ_3 by $\varphi_1 \circ (\varphi_2 \times \mathrm{id}_{\mathbb{C}})$, we can check that $[T_{\varphi_1(x,0)}\, \varphi_3]$ is injective, thus invertible, so using again the inverse function theorem, there exist neighborhoods T, V, W and S of $\varphi_1(x, 0)$ in $\boldsymbol{\mathcal{B}}$, y in $\mathcal{C}$, 0 in $\mathbb{R}^p$ and 0 in $\mathbb{C}$ respectively, and a diffeomorphism $\varphi : V \times W \times S \to T$ satisfying, for all $v, \lambda \in V \times S$:

$$\exists u \in \boldsymbol{\mathcal{B}}_{\mathcal{C}} \,/\, \Gamma^{-1} \langle u \rangle \cap \mathrm{Im}\, \varphi = \varphi \langle \{v\} \times W \times \{\lambda\} \rangle \quad \& \quad \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}_{\mathcal{C}}}(u) = v \tag{B.11.2}$$

(using eq. (B.11.1) and the injectivity of the restriction of $\boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}}$ to the leaf $\Gamma^{-1} \langle u \rangle$, together with $\varphi \langle \{v\} \times W \times \{\lambda\} \rangle = \psi_1 \langle \{\psi_2(v, \lambda)\} \times R_2 \rangle \cap \mathrm{Im}\, \varphi$) and for all $v, w, \lambda \in V \times W \times S$:

$$\forall \mu \in \mathbb{C} \,/\, \mu\lambda \in S, \quad \varphi(v, w, \mu\lambda) = \mu\, \varphi(v, w, \lambda) \tag{B.11.3}$$

(we can first check this for $w = 0$, and then use the previous point, since for all $u \in \boldsymbol{\mathcal{B}}_{\mathcal{C}}$ and all $\mu \in \mathbb{C}$, $\Gamma^{-1} \langle \mu.u \rangle = \mu \,.\, \Gamma^{-1} \langle u \rangle$).

Finally, using eq. (B.11.3), we can extend S to be all $\mathbb{C}$ while still satisfying eqs. (B.11.2) and (B.11.3).

Compatibility of the local descriptions. Let $x_0, x_1 \in \mathcal{M}$ such that $\gamma(x_0) = \gamma(x_1) =: y$. There exists a path $\kappa : [0,1] \to \gamma^{-1}\langle y\rangle$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$ (simple connectedness implies path connectedness).

Using the preceding point, for all $t \in [0,1]$ there exist open neighborhoods V_t of y in $\mathcal{C}$, W_t of 0 in $\mathbb{R}^p$ and U_t of $\kappa(t)$ in $\mathcal{M}$, and a diffeomorphism $\varphi_t : V_t \times W_t \times \mathbb{C} \to \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}}^{-1}\langle U_t\rangle$ satisfying eq. (B.11.2) and eq. (B.11.3). For any $t \in [0,1]$, we call π_t the projection map $\pi_t : V_t \times W_t \times \mathbb{C} \to V_t \times \mathbb{C}$ and we define the smooth map $\Gamma_t := \pi_t \circ \varphi_t^{-1} : \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}}^{-1}\langle U_t\rangle \to V_t \times \mathbb{C}$.

Next, there exist $t_1, \ldots, t_{N-1}$ ($1 \leqslant N < \infty$) such that $(U_{t_i})_{0\leqslant i\leqslant N}$ is an open cover of $\kappa\langle[0,1]\rangle$ (where we set $t_0 = 0$ and $t_N = 1$), and $\forall i \leqslant N-1$, $\kappa^{-1}\langle U_{t_i} \cap U_{t_{i+1}}\rangle \neq \varnothing$.

We define $V = \bigcap_{0\leqslant i\leqslant N-1} \gamma\langle U_{t_i} \cap U_{t_{i+1}}\rangle$. γ is an open map (for $T\gamma$ is surjective at any point), therefore V is an open subset of $\mathcal{C}$, and for any $i \leqslant N-1$, there exists $t \in [0,1]$ such that $\kappa(t) \in U_{t_i} \cap U_{t_{i+1}}$, hence $y = \gamma \circ \kappa(t) \in U_{t_i} \cap U_{t_{i+1}}$. Thus, V is an open neighborhood of y in $\mathcal{C}$.

Let $i \leqslant N-1$. The maps $\Gamma_{t_i}\big|_{\boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}}^{-1}\langle U_{t_i}\cap U_{t_{i+1}}\cap\gamma^{-1}\langle V\rangle\rangle \to V\times\mathbb{C}}$ and $\Gamma_{t_{i+1}}\big|_{\boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}}^{-1}\langle U_{t_i}\cap U_{t_{i+1}}\cap\gamma^{-1}\langle V\rangle\rangle \to V\times\mathbb{C}}$ are smooth, surjective, their derivatives are surjective at each point and they are constant on each other level sets (using eq. (B.11.2)), therefore the rank theorem implies [54, prop. 5.21] the existence of a diffeomorphism $\Phi_i : V \times \mathbb{C} \to V \times \mathbb{C}$ such that:

$$\forall x \in \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}}^{-1}\left\langle U_{t_i} \cap U_{t_{i+1}} \cap \gamma^{-1}\langle V\rangle\right\rangle,\ \Gamma_{t_i}(x) = \Phi_i \circ \Gamma_{t_{i+1}}(x)\,.$$

Thus, eq. (B.11.2) leads to:

$$\forall v \in V,\ \forall\lambda \in \mathbb{C},\ \exists u \in \boldsymbol{\mathcal{B}}_{\mathcal{C}} /$$

$$\Gamma^{-1}\langle u\rangle \cap \operatorname{Im}\varphi_{t_i} = \varphi_{t_i}\langle\{v\} \times W_{t_i} \times \{\lambda\}\rangle$$

$$\&\quad \Gamma^{-1}\langle u\rangle \cap \operatorname{Im}\varphi_{t_{i+1}} = \varphi_{t_{i+1}} \circ \widetilde{\Phi}_i^{-1}\langle\{v\} \times W_{t_{i+1}} \times \{\lambda\}\rangle\,,$$

where $\widetilde{\Phi}_i$ is defined naturally from Φ_i as a map $\widetilde{\Phi}_i : V \times W_{t_{i+1}} \times \mathbb{C} \to V \times W_{t_{i+1}} \times \mathbb{C}$.

Defining $\Phi := \Phi_0 \circ \ldots \circ \Phi_{N-1} : V \times \mathbb{C} \to V \times \mathbb{C}$ and $\widetilde{\Phi} : V \times W_{t_N} \times \mathbb{C} \to V \times W_{t_N} \times \mathbb{C}$, we have:

$$\forall v \in V,\ \forall\lambda \in \mathbb{C},\ \forall w_0 \in W_0,\ \forall w_1 \in W_1, \Gamma \circ \varphi_0(v, w_0, \lambda) = \Gamma \circ \varphi_1 \circ \widetilde{\Phi}^{-1}(v, w_1, \lambda)\,.$$

This way we have proved that for any $x_0, x_1 \in \mathcal{M}$ such that $\gamma(x_0) = \gamma(x_1) =: y$, there exist open neighborhoods V of y in $\mathcal{C}$, W_0 and W_1 of 0 in $\mathbb{R}^p$, $\widetilde{U}_0$ of x_0 in $\mathcal{M}$ and $\widetilde{U}_1$ of x_1 in $\mathcal{M}$, diffeomorphisms $\widetilde{\varphi}_0 : V \times W_0 \times \mathbb{C} \to \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}}^{-1}\left\langle\widetilde{U}_0\right\rangle$ and $\widetilde{\varphi}_1 : V \times W_1 \times \mathbb{C} \to \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}}^{-1}\left\langle\widetilde{U}_1\right\rangle$, and an injective map $\psi : V \times \mathbb{C} \to \boldsymbol{\mathcal{B}}_{\mathcal{C}}$, such that:

$$\forall v \in V,\ \forall\lambda \in \mathbb{C}, \forall w \in W_{0/1},\ \psi(v,\lambda) = \Gamma \circ \widetilde{\varphi}_{0/1}(v, w, \lambda\rangle$$

$$\forall v \in V,\ \forall\lambda \in \mathbb{C},\ \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}_{\mathcal{C}}} \circ \psi(v,\lambda) = v$$

$$\text{and}\quad \forall v, w \in V \times W_{0/1},\ \forall\lambda \in \mathbb{C},\ \widetilde{\varphi}_{0/1}(v, w, \lambda\,\cdot) = \lambda\,\widetilde{\varphi}_{0/1}(v, w, \cdot)\,.$$

Topological, differentiable and bundle structures on $\boldsymbol{\mathcal{B}}_{\mathcal{C}}$. We equip $\boldsymbol{\mathcal{B}}_{\mathcal{C}}$ with the final topology induced by Γ (so that $U \subset \boldsymbol{\mathcal{B}}_{\mathcal{C}}$ is open iff $\Gamma^{-1}\langle U\rangle$ is open in $\boldsymbol{\mathcal{B}}$). The previous point, together with $\gamma \circ \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}} = \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}_{\mathcal{C}}} \circ \Gamma$, ensures that Γ is an open map for this topology (because the preimage of the image of an open subset of $\boldsymbol{\mathcal{B}}$ is an open subset of $\boldsymbol{\mathcal{B}}$), and that we can use the local descriptions of the quotient to define a bundle structure on $\boldsymbol{\mathcal{B}}_{\mathcal{C}}$, with respect to which Γ will be a smooth surjective map with surjective derivative at each point. We can check that this structure is then compatible with the projection $\boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}_{\mathcal{C}}}$ and the action of $\mathbb{C}$ on $\boldsymbol{\mathcal{B}}_{\mathcal{C}}$ defined above. □

Since the cross-sections of $\boldsymbol{\mathcal{B}}$ that are ∇-horizontal over the level sets of γ are typically non-normalizable, we need to introduce a measure on $\mathcal{C}$. In general, there is however no preferred choice for this measure, hence we will associate to any smooth measure [31, section 11.4] on $\mathcal{C}$ a corresponding Hilbert space and we will restore the independence with respect to the choice of measure by providing identifications between these different Hilbert spaces.

Definition B.12 Let $\mathcal{C}$ be a smooth manifold. A smooth measure μ on $\mathcal{C}$ is a Borel measure on $\mathcal{C}$ such that, for any smooth coordinate chart $\varphi : U \to \mathbb{R}^p$ ($p := \dim\mathcal{C}$) on an open subset U of $\mathcal{C}$, there exists a smooth, nowhere vanishing, strictly positive function $\alpha_\varphi : U \to \mathbb{R}$ satisfying:

$$\mu\big|_U = \alpha_\varphi \left[\varphi_*^{-1}\mu_{\mathbb{R}}^{(p)}\right],$$

where $\mu_{\mathbb{R}}^{(p)}$ is the Lebesgue measure on $\mathbb{R}^p$.

In particular, the measure μ_ω associated to a nowhere vanishing volume form ω on $\mathcal{C}$ is a smooth measure on $\mathcal{C}$.

For any smooth vector field X on $\mathcal{C}$ we define its divergence with respect to μ as the smooth function $\operatorname{div}_\mu X$ on $\mathcal{C}$ satisfying:

$$\mathfrak{L}_X\mu = (\operatorname{div}_\mu X)\ \mu\,.$$

Finally, for any two smooth measures μ, μ' on $\mathcal{C}$ there exists a unique strictly positive smooth function α on $\mathcal{C}$ such that $\mu' = \alpha\,\mu$.

Definition B.13 We consider the same objects as in def. B.10. Let μ be a smooth measure on $\mathcal{C}$. The position representation with measure μ is the Hilbert space $\mathcal{H}^\mu_{\text{Pos}} := L_2(\mathcal{C} \to \boldsymbol{\mathcal{B}}_{\mathcal{C}},\, d\mu)$ of cross-sections of $\boldsymbol{\mathcal{B}}_{\mathcal{C}}$ with square-integrable norm with respect to μ.

As underlined above, trimming the prequantum representation down to a physically pertinent size comes at the price of restricting the algebra of observables that can be quantized. In the position representation, this quantization condition requires that the Hamiltonian flow of an admissible observable should send level sets of γ onto level sets of γ. In the case of a cotangent bundle, the quantizable functions are therefore the ones that depend at most linearly on the momentum variables.

Proposition B.14 We consider the same objects as in def. B.13. We define:

$$\mathcal{O}_{\text{Pos}} := \left\{ f \in C^\infty(\mathcal{M},\, \mathbb{R}) \;\middle|\; \exists \overline{X}_f \in \mathcal{T}^\infty(\mathcal{C}),\, \forall x \in \mathcal{M},\ T_x\gamma\left(X_{f,x}\right) = \overline{X}_{f,\gamma(x)} \right\},$$

where $\mathcal{T}^\infty(\mathcal{C})$ is the space of smooth vector fields on $\mathcal{C}$, and:

$$\mathcal{O}_{\text{Pos},\mathbb{C}} := \{f \in C^\infty(\mathcal{M},\, \mathbb{C}) \mid \operatorname{Re} f,\, \operatorname{Im} f \in \mathcal{O}_{\text{Pos}}\}.$$

Then, for $f \in \mathcal{O}_{\text{Pos}}$, we can define the quantization $\widehat{f}^\mu$ of f as a densely defined operator on $\mathcal{H}^\mu_{\text{Pos}}$ by:

$$\forall s \in \mathcal{D}^\mu_f,\, \forall y \in \mathcal{C},\, \left(\widehat{f}^\mu\, s\right)(y) := f(x)\, s(y) + i\, \Gamma\left(\nabla_{X_{f,x}} \widetilde{s}(x)\right) + \frac{i}{2}\left(\operatorname{div}_\mu \overline{X}_f\right)(y)\, s(y)\,,$$

where x is any point in $\gamma^{-1}\langle y\rangle$, $\widetilde{s}$ is the cross-section of $\boldsymbol{\mathcal{B}}$ such that $\Gamma \circ \widetilde{s} = s \circ \gamma$, and $\operatorname{div}_\mu \overline{X}_f \in C^\infty(\mathcal{C},\, \mathbb{R})$ is such that $\mathfrak{L}_{\overline{X}_f}\mu = \left(\operatorname{div}_\mu \overline{X}_f\right)\mu$. For $f \in \mathcal{O}_{\text{Pos},\mathbb{C}}$, we define $\widehat{f}^\mu := \widehat{\operatorname{Re}(f)}^\mu + \widehat{\operatorname{Im}(f)}^\mu$.

Moreover, we have for all $f, g \in \mathcal{O}_{\text{Pos},\mathbb{C}}$:

$$\{f,\, g\}_{\mathcal{M}} \in \mathcal{O}_{\text{Pos},\mathbb{C}}\,, \quad \left[\widehat{f}^\mu,\, \widehat{g}^\mu\right] = i\, \widehat{\{f,\, g\}}^\mu\,,$$

$$\text{and} \quad \forall s, s' \in \mathcal{D}^\mu_f,\, \left\langle s',\, \widehat{f}^\mu(s)\right\rangle = \left\langle \widehat{f^*}^\mu(s'),\, s\right\rangle.$$

If $f \in \mathcal{O}_{\text{Pos}}$ and $\overline{X}_f$ is a *complete* vector field on $\mathcal{C}$ [54, chap. 12], $\widehat{f}^\mu$ is essentially self-adjoint.

Proof Let $f \in \mathcal{O}_{\text{Pos}}$ and $s \in \mathcal{D}^\mu_f$. The cross-section $\widetilde{s}$ such that $\Gamma \circ \widetilde{s} = s \circ \gamma$ is well-defined, since for any $y \in \mathcal{C}$, $x \in \gamma^{-1}\langle y\rangle$ and $w \in \boldsymbol{\pi}^{-1}_{\boldsymbol{\mathcal{B}}_\mathcal{C}}\langle y\rangle$, there is a unique $z \in \boldsymbol{\pi}^{-1}_{\boldsymbol{\mathcal{B}}}\langle x\rangle$ such that $\Gamma(z) = w$ (this follows from def. B.10.2). We now want to prove that $f(x)\, s(y) + i\, \Gamma\left(\nabla_{X_{f,x}} \widetilde{s}(x)\right)$ does not depend on the choice of x in $\gamma^{-1}\langle y\rangle$.

Let V be any smooth vector field on $\mathcal{M}$ such that $\forall x \in \mathcal{M}$, $V_x \in \operatorname{Ker} T_x\gamma$. We have $\nabla_V \widetilde{s} = 0$ (since $\forall x \in \mathcal{M}$, $d_V\widetilde{s}(x) \in \operatorname{Ker} T_{s(x)}\Gamma \subset \operatorname{Hor}_{s(x)}(\boldsymbol{\mathcal{B}},\, \nabla)$ using def. B.10.4), therefore:

$$\begin{aligned}\forall x \in \mathcal{M},\, \nabla_{V_x} \nabla_{X_{f,x}} \widetilde{s}(x) &= \left[\nabla_{V_x},\, \nabla_{X_{f,x}}\right]\widetilde{s}(x)\\ &= \nabla_{\left[V_x,\, X_{f,x}\right]}\widetilde{s}(x) - i\, \Omega_{\mathcal{M},x}\left(V_x,\, X_{f,x}\right)\widetilde{s}(x)\,,\end{aligned}$$

and $T\gamma\left(\,[V,\, X_f]\right) = 0$ (using $\forall x \in \mathcal{M}$, $V_x \in \operatorname{Ker} T_x\gamma$ and $T_x\gamma\left(X_{f,x}\right) = \overline{X}_{f,\gamma(x)}$), hence $\forall x \in \mathcal{M}$, $\nabla_{V_x} \nabla_{X_{f,x}} \widetilde{s}(x) = i\, (d_{V_x} f)\, \widetilde{s}(x)$. Therefore:

$$\forall x \in \mathcal{M},\, \nabla_{V_x}\left[f(x)\widetilde{s}(x) + i\, \nabla_{X_{f,x}} \widetilde{s}(x)\right] = 0\,. \tag{B.14.1}$$

Let $z \in \boldsymbol{\mathcal{B}}$ and let V^{HOR} be the ∇-horizontal lift on $\boldsymbol{\mathcal{B}}$ of the vector field V on $\mathcal{M}$. Using $T\gamma(V) = 0$ together with def. B.10.1, we have $[T_{\Gamma(z)}\, \boldsymbol{\pi}_{\boldsymbol{\mathcal{B}}_\mathcal{C}}] \circ [T_z\Gamma](V^{\text{HOR}}_z) = 0$, so there exists $u \in \mathbb{C}$ such that $T_z\Gamma\,(V^{\text{HOR}}_z) = [T_1\,(\cdot\,\Gamma(z))](u) = T_z\Gamma \circ [T_1\,(\cdot\, z)](u)$ (where we used def. B.10.2 to get the second equality). Thus, using def. B.10.4, $V^{\text{HOR}}_z - [T_1\,(\cdot\, z)](u) \in \operatorname{Hor}_z(\boldsymbol{\mathcal{B}},\, \nabla)$, therefore $u = 0$, and $T_z\Gamma\,(V^{\text{HOR}}_z) = 0$.

Hence, eq. (B.14.1) becomes:

$$\left[T\left(x \mapsto \Gamma\left[f(x)\widetilde{s}(x) + i\, \nabla_{X_{f,x}} \widetilde{s}(x)\right]\right)\right](V) = 0\,,$$

so $\Gamma\,[f\, \widetilde{s} + i\, \nabla_{X_f} \widetilde{s}] = f\,(s \circ \gamma) + i\, \Gamma\,(\nabla_{X_f} \widetilde{s})$ is constant on the level sets of γ. This ensures that $\widehat{f}^\mu\, s$ is well-defined as a cross-section of $\boldsymbol{\mathcal{B}}_\mathcal{C}$.

Let $f, g \in \mathcal{O}_{\text{Pos}}$ (since $\widehat{\cdot}^\mu$ is $\mathbb{C}$-linear, and $[\,\cdot\,,\,\cdot\,]$, $\{\,\cdot\,,\,\cdot\,\}$ are $\mathbb{C}$-bilinear, it is enough to consider $\mathbb{R}$-valued functions to prove the commutator relations). Using the characterization of $\mathcal{O}_{\text{Pos}}$, we have:

$$\forall x \in \mathcal{M},\ T_x\gamma\left(X_{\{f,g\},x}\right) = T_x\gamma\left(\left[X_f, X_g\right]_x\right) = \left[\overline{X}_f, \overline{X}_g\right]_{\gamma(x)},$$

hence $\{f, g\} \in \mathcal{O}_{\text{Pos}}$ with $\overline{X}_{\{f,g\}} = \left[\overline{X}_f, \overline{X}_g\right]$.

Let $s \in \mathcal{D}^{\mu}_{f,g}$ (where, as in the proof of prop. B.4, the common domain $\mathcal{D}^{\mu}_{f,g}$, such that both $\widehat{f}^{\mu}\,\widehat{g}^{\mu}\,s$ and $\widehat{g}^{\mu}\,\widehat{f}^{\mu}\,s$ are well-defined, is dense in $\mathcal{H}^{\mu}_{\text{Pos}}$). Like in the proof of prop. B.4, we have:

$$\left[f + i\,\nabla_{X_f}, g + i\,\nabla_{X_g}\right]\widetilde{s} = i\,\{f, g\}\,\widetilde{s} - \nabla_{X_{\{f,g\}}}\widetilde{s}\,.$$

On the other hand, we can rewrite the definition of $\widehat{f}^{\mu}$ as:

$$\widetilde{\left(\widehat{f}^{\mu}\,s\right)} = \left(f + i\,\nabla_{X_f} + \frac{i}{2}\left(\operatorname{div}_{\mu}\overline{X}_f\right)\circ\gamma\right)\widetilde{s} \tag{B.14.2}$$

thus:

$$\widetilde{\left(\left[\widehat{f}^{\mu}, \widehat{g}^{\mu}\right]s\right)} = \left(i\,\{f, g\} - \nabla_{X_{\{f,g\}}} + \frac{1}{2}\left(d_{X_g}\left(\operatorname{div}_{\mu}\overline{X}_f\right)\circ\gamma - d_{X_f}\left(\operatorname{div}_{\mu}\overline{X}_g\right)\circ\gamma\right)\right)\widetilde{s}\,.$$

Next, we have:

$$d_{X_g}\left(\operatorname{div}_{\mu}\overline{X}_f\right)\circ\gamma - d_{X_f}\left(\operatorname{div}_{\mu}\overline{X}_g\right)\circ\gamma = \left(d_{\overline{X}_g}\left(\operatorname{div}_{\mu}\overline{X}_f\right) - d_{\overline{X}_f}\left(\operatorname{div}_{\mu}\overline{X}_g\right)\right)\circ\gamma\,,$$

and $\mathfrak{L}_{\overline{X}_g}\left(\mathfrak{L}_{\overline{X}_f}\mu\right) = \left(d_{\overline{X}_g}\left(\operatorname{div}_{\mu}\overline{X}_f\right)\right)\mu + \left(\operatorname{div}_{\mu}\overline{X}_g\right)\left(\operatorname{div}_{\mu}\overline{X}_f\right)\mu$, therefore, using $\overline{X}_{\{f,g\}} = \left[\overline{X}_f, \overline{X}_g\right]$:

$$d_{X_g}\left(\operatorname{div}_{\mu}\overline{X}_f\right)\circ\gamma - d_{X_f}\left(\operatorname{div}_{\mu}\overline{X}_g\right)\circ\gamma = -\left(\operatorname{div}_{\mu}\overline{X}_{\{f,g\}}\right)\circ\gamma\,.$$

Hence, using eq. (B.14.2) for $\widehat{\{f,g\}}^{\mu}$:

$$\left[\widehat{f}^{\mu}, \widehat{g}^{\mu}\right]s = i\,\widehat{\{f,g\}}^{\mu}\,s\,.$$

Lastly, let $f \in \mathcal{O}_{\text{Pos},\mathbb{C}}$ and $s, s' \in \mathcal{D}^{\mu}_{f}$. Using def. B.10.3, we have $\forall x \in \mathcal{M}$, $\langle\widetilde{s}'(x), \widetilde{s}(x)\rangle = \langle s' \circ \gamma(x), s \circ \gamma(x)\rangle$ and combining eq. (B.14.2) with eq. (B.4.1):

$$\forall x \in \mathcal{M},\ \left\langle \widetilde{s}', \widetilde{\widehat{f}^{\mu}s}\right\rangle(x) = i\,\left(\operatorname{div}_{\mu}\overline{X}_f\right)\circ\gamma(x)\,\langle\widetilde{s}', \widetilde{s}\rangle(x) + i\,d_{X_{f,x}}\langle\widetilde{s}', \widetilde{s}\rangle(x) + \left\langle \widetilde{\widehat{f^*}^{\mu}s'}, \widetilde{s}\right\rangle(x)\,,$$

therefore:

$$\forall y \in \mathcal{C},\ \left\langle s', \widehat{f}^{\mu}s\right\rangle(y) = i\,\left(\operatorname{div}_{\mu}\overline{X}_f\right)(y)\,\langle s', s\rangle(y) + i\,d_{\overline{X}_{f,y}}\langle s', s\rangle(y) + \left\langle \widehat{f^*}^{\mu}s', s\right\rangle(y)\,.$$

Now, using Stokes theorem [17, theorem 7.7] ($\mathcal{C}$ is assumed to be without boundary, and $\mathcal{D}^{\mu}_{f}$ is required to ensure suitable fall-off conditions) and the definition of $\overline{X}_f$, we have:

$$\int_{\mathcal{C}} d\mu(y)\quad\left(\operatorname{div}_{\mu}\overline{X}_f\right)(y)\,\langle s', s\rangle(y) + d_{\overline{X}_{f,y}}\langle s', s\rangle(y) = \int_{\mathcal{C}}\mathfrak{L}_X\left[\langle s', s\rangle\,d\mu\right] = 0\,,$$

thus $\left\langle s', \widehat{f}^{\mu}s\right\rangle = \left\langle \widehat{f^*}^{\mu}s', s\right\rangle$.

Checking that $\widehat{f}^{\mu}$ is essentially self-adjoint, as soon as f is real-valued and $\overline{X}_f$ is complete, can be done in a way similar to the proof of prop. B.4. □

Proposition B.15 We consider the same objects as in def. B.10. Let μ and μ' be two smooth measures on $\mathcal{C}$ and let $\mathcal{H}^{\mu}_{\text{Pos}}$ and $\mathcal{H}^{\mu'}_{\text{Pos}}$ be the corresponding position representations. Then there exists a Hilbert space isomorphism $\Phi_{\mu\to\mu'} : \mathcal{H}^{\mu}_{\text{Pos}} \to \mathcal{H}^{\mu'}_{\text{Pos}}$ such that:

$$\forall f \in \mathcal{O}_{\text{Pos},\mathbb{C}},\ \widehat{f}^{\mu'} = \Phi_{\mu\to\mu'} \circ \widehat{f}^{\mu} \circ \Phi^{-1}_{\mu\to\mu'}\,. \tag{B.15.1}$$

Moreover, we can define these maps in such a way that for any three smooth measures μ, μ', and μ'', on $\mathcal{C}$, $\Phi_{\mu\to\mu''} = \Phi_{\mu'\to\mu''} \circ \Phi_{\mu\to\mu'}$. Thus, this family of maps provides a position representation $\mathcal{H}_{\text{Pos}}$, that can be consistently identified with $\mathcal{H}^{\mu}_{\text{Pos}}$ for any μ.

Proof Let μ and μ' be two smooth measures on $\mathcal{C}$. Then, there exists a unique $\alpha \in C^{\infty}(\mathcal{C}, \mathbb{R}^*_+)$ such that $\mu' = \alpha\,\mu$ (def. B.12). We define $\Phi_{\mu\to\mu'}$ by:

$$\begin{array}{rccc} \Phi_{\mu\to\mu'} & : & \mathcal{H}^{\mu}_{\text{Pos}} & \to & \mathcal{H}^{\mu'}_{\text{Pos}} \\ & & s & \mapsto & \frac{1}{\sqrt{\alpha}}\, s \end{array}.$$

The factor $\dfrac{1}{\sqrt{\alpha}}$ ensures that $\Phi_{\mu\to\mu'}$ is a unitary map and we can check that for any three smooth measures μ, μ', and μ'', $\Phi_{\mu\to\mu''} = \Phi_{\mu'\to\mu''} \circ \Phi_{\mu\to\mu'}$. In particular, $\Phi_{\mu\to\mu'}$ is then invertible, hence it is a Hilbert space isomorphism.

Lastly, eq. (B.15.1) follows from:

$$\forall f \in \mathcal{O}_{\text{Pos}},\ 2\sqrt{\alpha}\,\left(d_{\overline{X}_f}\frac{1}{\sqrt{\alpha}}\right) + \left(\text{div}_{\mu'}\overline{X}_f\right) = \left(\text{div}_{\mu}\overline{X}_f\right).$$

□

C. Wigner characteristic functions

In this appendix, we give a brief overview of the Wigner-Weyl transform [102], that allows to represent quantum density matrices as functions on the phase space, mimicking the probability distributions of classical statistical physics. The presentation below follows very closely [47], merely adapting of the notations and definitions to the ones we will be using in section 18.

Note that, in view of applying this tool to projective systems of quantum state spaces (subsection 18.1), we will be doing only 'half' of the Wigner transform: going from the density matrix to the Wigner characteristic function. The second half would be to perform a Fourier transform yielding the Wigner quasi-probability distribution, in analogy to the reconstruction of a classical probability distribution from its characteristic function. However, Wigner characteristic functions are more convenient when working with partial traces of the underlying density matrices, exactly like classical characteristic functions are convenient for computing marginal probabilities.

C.1 Weakly continuous representations of the Weyl algebra

We begin by recalling basic facts regarding the Weyl algebra and weakly continuous representations thereof. These representation-theoretic considerations will come into play to prove that the Wigner transform is surjective onto a suitably defined space of functions.

Definition C.1 Let $\mathcal{C}$, $\mathcal{P}$ be two finite-dimensional real vector spaces and let $\Xi : \mathcal{P} \to \mathcal{C}^*$ be an invertible linear map (with $\mathcal{C}^*$ the dual of $\mathcal{C}$). We equip $\mathcal{C} \times \mathcal{P}$ with the symplectic form Ω given by:

$$\forall u, u' \in \mathcal{C},\ \forall v, v' \in \mathcal{P},\ \Omega(u, v;\ u', v') := \Xi(v')(u) - \Xi(v)(u')\,.$$

The map $M := \mathrm{id}_{\mathcal{C}} \times \Xi$ is then a symplectomorphism $\mathcal{C} \times \mathcal{P} \to T^*(\mathcal{C})$ ($T^*(\mathcal{C})$ being equipped with its canonical symplectic structure, see eg. eq. (2.16.1)).

Proposition C.2 Let $\mathcal{I}$ be the complex vector space of functions $\mathcal{C}^* \times \mathcal{P}^* \to \mathbb{C}$ with support on finitely many points, ie:

$$\forall \iota \in \mathcal{I},\quad \#\left\{(s, t) \in \mathcal{C}^* \times \mathcal{P}^* \mid \iota(s, t) \neq 0\right\} < \infty.$$

In particular, for any $(s, t) \in \mathcal{C}^* \times \mathcal{P}^*$, we define $\delta_{(s,t)} \in \mathcal{I}$ by:

$$\forall (s', t') \in \mathcal{C}^* \times \mathcal{P}^*,\ \delta_{(s,t)}(s', t') := \begin{cases} 1 \text{ if } (s', t') = (s, t) \\ 0 \text{ else} \end{cases}.$$

For ι_1 , $\iota_2 \in \mathcal{I}$, we define their product $\iota_1 \star \iota_2 \in \mathcal{I}$ by:

$$\forall (s, t) \in \mathcal{C}^* \times \mathcal{P}^*,\ \left(\iota_1 \star \iota_2\right)(s, t) \ := \sum_{\substack{(s_1,t_1),(s_2,t_2) \in \mathcal{C}^* \times \mathcal{P}^* \\ s = s_1 + s_2,\, t = t_1 + t_2}} e^{i\xi(s_1,t_1;s_2,t_2)}\, \iota_1(s_1, t_1)\, \iota_2(s_2, t_2)\,,$$

where, for any $(s_1, t_1), (s_2, t_2) \in \mathcal{C}^* \times \mathcal{P}^*$:

$$\xi(s_1, t_1; s_2, t_2) := \frac{t_1\left(\Xi^{-1}(s_2)\right) - t_2\left(\Xi^{-1}(s_1)\right)}{2}\,.$$

We also define, for any $\iota \in \mathcal{I}$, its conjugate ι^* by:

$$\forall (s, t) \in \mathcal{C}^* \times \mathcal{P}^*,\ \iota^*(s, t) := \overline{\iota(-s, -t)}\,,$$

where $\overline{\cdot}$ stands for complex conjugation.

$\mathcal{I}, \star, ^*$ is a *-algebra [41, section III.2.2], with unit $\delta_{(0,0)}$.

Proof Let ι_1 , $\iota_2 \in \mathcal{I}$. Since ι_1 and ι_2 are supported on finitely many points, the sum defining $\iota_1 \star \iota_2$ is finite and $\iota_1 \star \iota_2$ is again supported on finitely many points, ie. it is in $\mathcal{I}$. The operation $\star$ is bilinear, and, for any ι_1 , ι_2 , $\iota_3 \in \mathcal{I}$, we have:

$$\forall (s, t) \in \mathcal{C}^* \times \mathcal{P}^*,\ \left(\iota_1 \star (\iota_2 \star \iota_3)\right)(s,\ t) =$$

$$= \sum_{\substack{s = s_1 + s'_1 \\ t = t_1 + t'_1}} e^{i\,\xi(s_1,t_1;s'_1,t'_1)}\, \iota_1(s_1\,,\ t_1) \sum_{\substack{s'_1 = s_2 + s_3 \\ t'_1 = t_2 + t_3}} e^{i\,\xi(s_2,t_2;s_3,t_3)}\, \iota_2(s_2\,,\ t_2)\, \iota_3(s_3\,,\ t_3)$$

$$= \sum_{\substack{s=s_1+s_2+s_3 \\ t=t_1+t_2+t_3}} e^{i\,\xi(s_1,t_1;s_2,t_2)+i\,\xi(s_1,t_1;s_3,t_3)+i\,\xi(s_2,t_2;s_3,t_3)}\, \iota_1(s_1\,,\ t_1)\, \iota_2(s_2\,,\ t_2)\, \iota_3(s_3\,,\ t_3)$$

$$= \big((\iota_1 \star \iota_2) \star \iota_3\big)(s,\ t)\,,$$

so it is associative as well. We can check that $\delta_{(0,0)}$ is a unit for $\mathcal{I}, \star$. Finally, the operation $(\,\cdot\,)^*$ is an antilinear involution by construction, and, for any $\iota_1\,,\ \iota_2 \in \mathcal{I}$, we have:

$$\forall (s,t) \in \mathcal{C}^* \times \mathcal{P}^*,\ (\iota_1 \star \iota_2)^*(s,\ t) = \sum_{\substack{s=s_1+s_2 \\ t=t_1+t_2}} e^{-i\,\xi(s_1,t_1;s_2,t_2)}\, \iota_1^*(s_1\,,\ t_1)\, \iota_2^*(s_2\,,\ t_2) = (\iota_2^* \star \iota_1^*)(s,\ t)\,.$$

□

Note that the elementary observables $\mathrm{X}_\kappa(s,t)$ in a representation κ are labeled by elements of the dual space $\mathcal{C}^* \times \mathcal{P}^*$ of the phase space $\mathcal{M} \approx \mathcal{C} \times \mathcal{P}$, because they are actually labeled by the linear observables of which they are the quantization (in the spirit of def. B.3, up to a difference in sign convention). Therefore, the unitary operators $\mathrm{T}_\kappa(s,t)$, which are obtained by exponentiating these observables, are labeled by elements of the dual as well. This observation will be important for the development of subsection 18.1.

Definition C.3 Let $\mathcal{H}_\kappa$ be a complex Hilbert space. A weakly continuous, unitary representation of $\mathcal{C}^* \times \mathcal{P}^*$ on $\mathcal{H}_\kappa$ is a map $\mathrm{T}_\kappa : \mathcal{C}^* \times \mathcal{P}^* \to \mathcal{A}_\kappa$ (where $\mathcal{A}_\kappa$ is the algebra of bounded linear operators over $\mathcal{H}_\kappa$), satisfying the following properties:

1. $\forall (s,t) \in \mathcal{C}^* \times \mathcal{P}^*$, $\mathrm{T}_\kappa(s,t)$ is a unitary operator on $\mathcal{H}_\kappa$;
2. $\forall (s_1,t_1), (s_2,t_2) \in \mathcal{C}^* \times \mathcal{P}^*$, $\mathrm{T}_\kappa(s_1,t_1)\, \mathrm{T}_\kappa(s_2,t_2) = e^{i\xi(s_1,t_1;s_2,t_2)}\, \mathrm{T}_\kappa(s_1+s_2, t_1+t_2)$;
3. $\mathrm{T}_\kappa(0,0) = \mathrm{id}_{\mathcal{H}_\kappa}$;
4. $\forall \varphi, \varphi' \in \mathcal{H}_\kappa$, the map $(s,t) \mapsto \langle \varphi' \mid \mathrm{T}_\kappa(s,t)\, \varphi \rangle$ is a continuous function $\mathcal{C}^* \times \mathcal{P}^* \to \mathbb{C}$.

This provides a representation of the *-algebra $\mathcal{I}$ on $\mathcal{H}_\kappa$:

$$\forall \iota \in \mathcal{I},\ \mathrm{T}_\kappa\,\{\iota\} := \sum_{(s,t)\in\mathcal{C}^*\times\mathcal{P}^*} \iota(s,t)\, \mathrm{T}_\kappa(s,t)$$

(using in particular that points C.3.1 to C.3.3 imply $\forall (s,t) \in \mathcal{C}^* \times \mathcal{P}^*$, $\mathrm{T}_\kappa(-s,-t) = \mathrm{T}_\kappa(s,t)^{-1} = \mathrm{T}_\kappa(s,t)^\dagger$).

Proposition C.4 Let $\mathcal{H}_\kappa$, T_κ be as in def. C.3. Then, there exists, for any $(s,t) \in \mathcal{C}^* \times \mathcal{P}^*$ a densely defined (possibly unbounded), self-adjoint operator $\mathrm{X}_\kappa(s,t)$ on $\mathcal{H}_\kappa$ such that:

$$\forall \tau \in \mathbb{R},\ \mathrm{T}_\kappa(\tau s, \tau t) = \exp(i\,\tau\, \mathrm{X}_\kappa(s,t))$$

(the exponential being defined via spectral resolution [73, theorem VIII.5]).

Moreover, there exists a dense vector subspace $\mathcal{D}_\kappa \subset \mathcal{H}_\kappa$ such that:

1. for each $(s,t) \in \mathcal{C}^* \times \mathcal{P}^*$, $\mathcal{D}_\kappa \subset \mathrm{Dom}\big(\mathrm{X}_\kappa(s,t)\big)$ (where $\mathrm{Dom}\big(\mathrm{X}_\kappa(s,t)\big)$ denotes the dense domain of $\mathrm{X}_\kappa(s,t)$) and $\mathrm{X}_\kappa(s,t)\big|_{\mathcal{D}_\kappa}$ is essentially self-adjoint;
2. $\forall (s,t) \in \mathcal{C}^* \times \mathcal{P}^*$, $\mathrm{X}_\kappa(s,t)\,\langle \mathcal{D}_\kappa \rangle \subset \mathcal{D}_\kappa$;

3. $\forall (s,t), (s',t') \in \mathcal{C}^* \times \mathcal{P}^*$, $\forall \tau, \tau' \in \mathbb{R}$, $X_\kappa(\tau s + \tau' s',\, \tau t + \tau' t')\big|_{\mathcal{D}_\kappa} = \tau\ X_\kappa(s,t)\big|_{\mathcal{D}_\kappa} + \tau'\ X_\kappa(s',t')\big|_{\mathcal{D}_\kappa}$.

Proof Let $(s,t) \in \mathcal{C}^* \times \mathcal{P}^*$. For any $\tau \in \mathbb{R}$, $T_\kappa(\tau s, \tau t)$ is unitary (def. C.3.1), and $T_\kappa(0,0) = \mathrm{id}_{\mathcal{H}_\kappa}$ (def. C.3.3). Moreover, using def. C.3.2, we have, for any $\tau_1, \tau_2 \in \mathbb{R}$:

$$T_\kappa(\tau_1 s, \tau_1 t)\, T_\kappa(\tau_2 s, \tau_2 t) = T_\kappa\big((\tau_1 + \tau_2)s,\ (\tau_1 + \tau_2)t\big)\,, \tag{C.4.1}$$

since $\xi(\tau_1 s, \tau_1 t;\ \tau_2 s, \tau_2 t) = \tau_1\, \tau_2\, \xi(s,t;\ s,t) = 0$.

Let $\varphi \in \mathcal{H}_\kappa$, $(s',t') \in \mathcal{C}^* \times \mathcal{P}^*$ and $\epsilon > 0$. Let $\varphi' := T_\kappa(s',t')\,\varphi$. From def. C.3.4, there exists an open neighborhood U of (s',t') in $\mathcal{C}^* \times \mathcal{P}^*$, such that:

$$\forall (s'',t'') \in U,\ \left|\langle \varphi' \mid T_\kappa(s'',t'')\,\varphi\rangle - \|\varphi'\|^2\right| < \frac{\epsilon}{2}\,.$$

Hence, thanks to the unitarity of $T_\kappa(s'',t'')$ for any $(s'',t'') \in \mathcal{C}^* \times \mathcal{P}^*$, we get:

$$\forall (s'',t'') \in U,\ \|T_\kappa(s'',t'')\,\varphi - T_\kappa(s',t')\,\varphi\|^2 \leqslant 2\left|\langle \varphi' \mid T_\kappa(s'',t'')\,\varphi\rangle - \|\varphi'\|^2\right| < \epsilon\,. \tag{C.4.2}$$

Let $\varphi \in \mathcal{H}_\kappa$, $\tau_o \in \mathbb{R}$ and $\epsilon > 0$. Applying the previous point with $(s',t') = (\tau_o s, \tau_o t)$, there exists $\epsilon' > 0$ such that:

$$\forall \tau \in \left]\tau_{o-\epsilon'}, \tau_{o+\epsilon'}\right[\,,\ \|T_\kappa(\tau s, \tau t)\,\varphi - T_\kappa(\tau_o s, \tau_o t)\,\varphi\|^2 < \epsilon\,.$$

Thus, $\tau \mapsto T_\kappa(\tau s, \tau t)$ is a strongly continuous one-parameter unitary group, so, by Stone's theorem [73, theorem VIII.8], there exists a densely defined, self-adjoint operator $X_\kappa(s,t)$ on $\mathcal{H}_\kappa$ such that, for any $\tau \in \mathbb{R}$, $T_\kappa(\tau s, \tau t) = \exp\big(i\,\tau\, X_\kappa(s,t)\big)$.

Let $\widetilde{\mu}$ be a Lebesgue measure on the finite-dimensional real vector space $\mathcal{C}^* \times \mathcal{P}^*$. For any $f \in C^\infty_o(\mathcal{C}^* \times \mathcal{P}^*, \mathbb{C})$ (where $C^\infty_o(\mathcal{C}^* \times \mathcal{P}^*, \mathbb{C})$ denotes the space of smooth, complex valued, compactly supported functions on $\mathcal{C}^* \times \mathcal{P}^*$), and any $\varphi \in \mathcal{H}_\kappa$, we define $\varphi\,\{f\} \in \mathcal{H}_\kappa$ as:

$$\varphi\,\{f\} := \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s',t')\, f(s',t')\, T_\kappa(s',t')\,\varphi\,.$$

As shown above $(s',t') \mapsto T_\kappa(s',t')\,\varphi$ is a continuous map $\mathcal{C}^* \times \mathcal{P}^* \to \mathcal{H}_\kappa$, so $(s',t') \mapsto f(s',t')\,T_\kappa(s',t')\,\varphi$ is a continuous, compactly supported map $\mathcal{C}^* \times \mathcal{P}^* \to \mathcal{H}_\kappa$, hence it is integrable on $\mathcal{C}^* \times \mathcal{P}^*$, $\widetilde{\mu}$ [45, section III.1]. Therefore, $\varphi\,\{f\}$ is well-defined as an element of $\mathcal{H}_\kappa$.

We define the vector subspace $\mathcal{D}_\kappa \subset \mathcal{H}_\kappa$ as:

$$\mathcal{D}_\kappa := \mathrm{Vect}\left\{\varphi\,\{f\} \;\middle|\; f \in C^\infty_o(\mathcal{C}^* \times \mathcal{P}^*, \mathbb{C}),\ \varphi \in \mathcal{H}_\kappa\right\}.$$

Let $\varphi \in \mathcal{H}_\kappa$ and $\epsilon > 0$. From def. C.3.3 and eq. (C.4.2), there exists an open neighborhood U of $(0,0)$ in $\mathcal{C}^* \times \mathcal{P}^*$ such that:

$$\forall (s',t') \in U,\ \|T_\kappa(s',t')\,\varphi - \varphi\|^2 < \epsilon\,.$$

Let $f \in C^\infty_o(\mathcal{C}^* \times \mathcal{P}^*, \mathbb{C})$ be a bump function with:

$$f \geqslant 0,\quad \mathrm{supp}(f) \subset U \quad \& \quad \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}\ \ f = 1\,,$$

where $\mathrm{supp}(f)$ denotes the support of f. Then, we have:

$$\|\varphi - \varphi\,\{f\}\| \leqslant \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s',t')\, f(s',t')\ \|\varphi - T_\kappa(s',t')\,\varphi\| < \epsilon\,.$$

Thus, $\mathcal{D}_\kappa$ is a dense subspace of $\mathcal{H}_\kappa$.

Let $\varphi \in \mathcal{H}_\kappa$, $f \in C_o^\infty(\mathcal{C}^* \times \mathcal{P}^*, \mathbb{C})$ and $(s,t) \in \mathcal{C}^* \times \mathcal{P}^*$. We have, for any $\tau \in \mathbb{R}$:

$$\begin{aligned}\mathrm{T}_\kappa(\tau s, \tau t)\,\varphi\,\{f\} &= \int_{\mathcal{C}^*\times\mathcal{P}^*} d\widetilde{\mu}(s',t')\, f(s',t')\,\mathrm{T}_\kappa(\tau s,\tau t)\,\mathrm{T}_\kappa(s',t')\,\varphi \\ &= \int_{\mathcal{C}^*\times\mathcal{P}^*} d\widetilde{\mu}(s'',t'')\, f(s''-\tau s, t''-\tau t)\, e^{i\tau\,\xi(s,t;s'',t'')}\,\mathrm{T}_\kappa(s'',t'')\,\varphi\,,\end{aligned}$$

where we have used the translation invariance of the Lebesgue measure and the fact that $\forall (s'',t'') \in \mathcal{C}^* \times \mathcal{P}^*$, $\xi(\tau s, \tau t; s''-\tau s, t''-\tau t) = \tau\,\xi(s,t;s'',t'')$. Defining $f^{(\tau)}_{(s,t)} \in C_o^\infty(\mathcal{C}^* \times \mathcal{P}^*, \mathbb{C})$ by:

$$\forall (s',t') \in \mathcal{C}^* \times \mathcal{P}^*,\ f^{(\tau)}_{(s,t)}(s',t') := e^{i\tau\,\xi(s,t;s',t')}\, f(s'-\tau s, t'-\tau t)\,,$$

we thus get $\mathrm{T}_\kappa(\tau s, \tau t)\,\varphi\,\{f\} = \varphi\left\{f^{(\tau)}_{(s,t)}\right\}$.

We now define, for any $\tau \in \mathbb{R}$, $f^{1,(\tau)}_{(s,t)}, f^{2,(\tau)}_{(s,t)} \in C_o^\infty(\mathcal{C}^* \times \mathcal{P}^*, \mathbb{C})$ by:

$$\forall (s',t') \in \mathcal{C}^* \times \mathcal{P}^*,\ f^{1,(\tau)}_{(s,t)}(s',t') := \left.\frac{d}{d\tau'} f^{(\tau')}_{(s,t)}(s',t')\right|_{\tau'=\tau} \quad \& \quad f^{2,(\tau)}_{(s,t)}(s',t') := \left.\frac{d}{d\tau'} f^{1,(\tau')}_{(s,t)}(s',t')\right|_{\tau'=\tau}.$$

Then, we have:

$$\forall \tau \in \mathbb{R},\ \forall (s',t') \in \mathcal{C}^* \times \mathcal{P}^*,$$

$$g^{(\tau)}_{(s,t)}(s',t') := \frac{f^{(\tau)}_{(s,t)}(s',t') - f(s',t')}{\tau} - f^{1,(0)}_{(s,t)}(s',t') = \frac{1}{\tau}\int_0^\tau d\tau' \int_0^{\tau'} d\tau''\ f^{2,(\tau'')}_{(s,t)}(s',t')\,.$$

Let $K := \{(s'+\tau s, t'+\tau t) \mid (s',t') \in \mathrm{supp}(f),\ \tau \in [-1,1]\}$. K is compact as the continuous image of the compact $\mathrm{supp}(f) \times [-1,1]$ and we have:

$$\forall \tau \in [-1,1],\ \forall (s',t') \notin K,\ g^{(\tau)}_{(s,t)}(s',t') = 0\,.$$

Moreover, the map $s',t';\tau'' \mapsto f^{2,(\tau'')}_{(s,t)}(s',t')$ is continuous on $\mathcal{C}^* \times \mathcal{P}^* \times \mathbb{R}$, hence bounded on $K \times [-1,1]$, so there exists $M > 0$ such that:

$$\forall \tau \in [-1,1],\ \forall (s',t') \in \mathcal{C}^* \times \mathcal{P}^*,\ \left|g^{(\tau)}_{(s,t)}(s',t')\right| \leqslant M\,\mathbf{1}_K(s',t')\,,$$

where $\mathbf{1}_K$ denotes the indicator function of K. So, the dominated convergence theorem yields:

$$\lim_{\tau\to 0} \int_{\mathcal{C}^*\times\mathcal{P}^*} d\widetilde{\mu}(s',t')\ \left\|g^{(\tau)}_{(s,t)}(s',t')\,\mathrm{T}_\kappa(s',t')\,\varphi\right\| = 0\,,$$

and therefore:

$$\lim_{\tau\to 0} \left\| \frac{\mathrm{T}_\kappa(\tau s,\tau t)\,\varphi\,\{f\} - \varphi\,\{f\}}{\tau} - \varphi\,\{f_{(s,t)}\} \right\| = 0\,,$$

where:

$$\forall (s',t') \in \mathcal{C}^* \times \mathcal{P}^*,\ f_{(s,t)}(s',t') := f^{1,(0)}_{(s,t)}(s',t') = i\,\xi(s,t;s',t')\,f(s',t') - \left[T_{(s',t')}f\right](s,t)$$

(with $T_{(s',t')}f$ the differential of f at (s',t')).

By definition of $\mathrm{X}_\kappa(s,t)$, this implies [73, theorem VIII.7]:

$$\varphi\{f\} \in \mathrm{Dom}\big(\mathrm{X}_\kappa(s,t)\big) \quad \& \quad \mathrm{X}_\kappa(s,t)\,\varphi\{f\} = -i\,\varphi\{f_{(s,t)}\}\,.$$

Hence, $\mathcal{D}_\kappa \subset \mathrm{Dom}\big(\mathrm{X}_\kappa(s,t)\big)$, and, since $f_{(s,t)} \in C_o^\infty(\mathcal{C}^* \times \mathcal{P}^*, \mathbb{C})$ for any $f \in C_o^\infty(\mathcal{C}^* \times \mathcal{P}^*, \mathbb{C})$, $\mathrm{X}_\kappa(s,t)\,\langle\mathcal{D}_\kappa\rangle \subset \mathcal{D}_\kappa$. Next, $\mathrm{X}_\kappa(s,t)|_{\mathcal{D}_\kappa}$ is symmetric (as the restriction of a self-adjoint operator), and we have, for $\varphi' \in \mathcal{D}_\kappa$:

$$\mathrm{X}_\kappa(s,t)|_{\mathcal{D}_\kappa}\,\varphi' = -i\,\lim_{\tau\to 0}\frac{\mathrm{T}_\kappa(\tau s,\tau t)\,\varphi' - \varphi'}{\tau}\,,$$

so:

$$\varphi \in \mathrm{Ker}\left(\big[\mathrm{X}_\kappa(s,t)|_{\mathcal{D}_\kappa}\big]^\dagger \pm i\right) \Leftrightarrow$$

$$\Leftrightarrow \forall\varphi' \in \mathcal{D}_\kappa\,,\ -i\,\lim_{\tau\to 0}\frac{\langle\varphi \mid \mathrm{T}_\kappa(\tau s,\tau t)\,\varphi'\rangle - \langle\varphi \mid \varphi'\rangle}{\tau} \mp i\,\langle\varphi \mid \varphi'\rangle = 0$$

$$\Leftrightarrow \forall\varphi' \in \mathcal{D}_\kappa\,,\ \frac{d}{d\tau}\langle\varphi \mid \mathrm{T}_\kappa(\tau s,\tau t)\,\varphi'\rangle\bigg|_{\tau=0} = \mp\langle\varphi \mid \varphi'\rangle$$

$$\Leftrightarrow \forall\varphi' \in \mathcal{D}_\kappa\,,\ \forall\tau \in \mathbb{R}\,,\ \frac{d}{d\tau'}\langle\varphi \mid \mathrm{T}_\kappa(\tau' s,\tau' t)\,\varphi'\rangle\bigg|_{\tau'=\tau} = \mp\langle\varphi \mid \mathrm{T}_\kappa(\tau s,\tau t)\,\varphi'\rangle$$

(using $\mathrm{T}_\kappa(\tau s,\tau t)\,\langle\mathcal{D}_\kappa\rangle \subset \mathcal{D}_\kappa$ and eq. (C.4.1))

$$\Leftrightarrow \forall\varphi' \in \mathcal{D}_\kappa\,,\ \forall\tau \in \mathbb{R}\,,\ \langle\varphi \mid \mathrm{T}_\kappa(\tau s,\tau t)\,\varphi'\rangle = e^{\mp\tau}\,\langle\varphi \mid \varphi'\rangle$$

$$\Leftrightarrow \forall\varphi' \in \mathcal{D}_\kappa\,,\ \langle\varphi \mid \varphi'\rangle = 0$$

(since, for each $\varphi' \in \mathcal{D}_\kappa$, $\tau \mapsto \langle\varphi \mid \mathrm{T}_\kappa(\tau s,\tau t)\,\varphi'\rangle$ is bounded, $\mathrm{T}_\kappa(\tau s,\tau t)$ being unitary for any $\tau \in \mathbb{R}$).

Thus, $\mathrm{Ker}\left(\big[\mathrm{X}_\kappa(s,t)|_{\mathcal{D}_\kappa}\big]^\dagger \pm i\right) = \mathcal{D}_\kappa^\perp = \{0\}$ (for $\mathcal{D}_\kappa$ is dense in $\mathcal{H}_\kappa$), and therefore $\mathrm{X}_\kappa(s,t)|_{\mathcal{D}_\kappa}$ is essentially self-adjoint [73, theorem VIII.3].

Finally, for any $f \in C_o^\infty(\mathcal{C}^* \times \mathcal{P}^*, \mathbb{C})$, $(s,t), (s',t') \in \mathcal{C}^* \times \mathcal{P}^*$ and $\tau, \tau' \in \mathbb{R}$, we have:

$$f_{(\tau s+\tau' s',\tau t+\tau' t')} = \tau\,f_{(s,t)} + \tau'\,f_{(s',t')}\,,$$

which proves point C.4.3. □

Of course, the usual Schrödinger representation is such a weakly continuous representation of the Weyl algebra, and, in fact, the Stone-von Neumann theorem [98], which we will recall below (prop. C.10, following the proof given in [47, theorem 15a]), tells us that it is the only one (more precisely, it is the only *irreducible* one: an arbitrary weakly continuous representation thus decomposes as a direct sum of independent copies of the Schrödinger representation).

Proposition C.5 Let $\mathcal{H}_o := L_2(\mathcal{C}, d\mu)$ with μ a Lebesgue measure on the finite-dimensional real vector space $\mathcal{C}$. For any $(s,t) \in \mathcal{C}^* \times \mathcal{P}^*$, we define a unitary operator $\mathrm{T}_o(s,t)$ by:

$$\forall\varphi \in \mathcal{H}_o,\ \forall x \in \mathcal{C},\ [\mathrm{T}_o(s,t)\,\varphi](x) := e^{i\,s(x)+\frac{i}{2}\,t\left(\Xi^{-1}(s)\right)}\,\varphi\left(x + \Xi^{*,-1}(t)\right),$$

where $\Xi^* : \mathcal{C} \to \mathcal{P}^*$ is the dual of the map Ξ and $\Xi^{*,-1} = (\Xi^*)^{-1} = \left(\Xi^{-1}\right)^*$.

T_o is a weakly continuous, unitary representation of $\mathcal{C}^* \times \mathcal{P}^*$.

Let $\mathcal{D}^{\cdot}_o$ be the space of smooth, compactly supported, complex-valued functions on $\mathcal{C}$. $\mathcal{D}^{\cdot}_o$ is a dense vector subspace in $\mathcal{H}_o$, and, for any $(s, t) \in \mathcal{C}^* \times \mathcal{P}^*$, the linear operator $\mathrm{X}^{\cdot}_o(s, t)$ defined on $\mathcal{D}^{\cdot}_o$ by:

$$\forall \varphi \in \mathcal{D}^{\cdot}_o,\ \forall x \in \mathcal{C},\ \left[\mathrm{X}^{\cdot}_o(s, t)\, \varphi\right](x) := s(x)\, \varphi(x) - i\, [T_x \varphi] \left(\Xi^{*,-1}(t)\right),$$

is essentially self-adjoint (with $T_x\varphi$ the differential of φ at x). Moreover, we have:

$$\forall (s, t) \in \mathcal{C}^* \times \mathcal{P}^*,\ \mathcal{D}^{\cdot}_o \subset \mathrm{Dom}\big(\mathrm{X}_o(s, t)\big) \quad \& \quad \mathrm{X}^{\cdot}_o(s, t) = \mathrm{X}_o(s, t)|_{\mathcal{D}^{\cdot}_o},$$

where, for any $(s, t) \in \mathcal{C}^* \times \mathcal{P}^*$, $\mathrm{X}_o(s, t)$ is the self-adjoint generator introduced in prop. C.4.

Proof Let $\varphi \in \mathcal{H}_o$ and $(s, t) \in \mathcal{C}^* \times \mathcal{P}^*$. Let φ' be given by:

$$\forall x \in \mathcal{C},\ \varphi'(x) := e^{i\, s(x) + \frac{i}{2}\, t\left(\Xi^{-1}(s)\right)}\, \varphi\left(x + \Xi^{*,-1}(t)\right).$$

Using the translation invariance of the Lebesgue measure, we have:

$$\int_{\mathcal{C}} d\mu(x)\ \left|\varphi'(x)\right|^2 = \int_{\mathcal{C}} d\mu(x')\ \left|\varphi\left(x'\right)\right|^2 = \left\|\varphi\right\|^2,$$

hence $\mathrm{T}_o(s, t)$ is well-defined as a linear operator $\mathcal{H}_o \to \mathcal{H}_o$ and is isometric.

Moreover, we have $\mathrm{T}_o(0, 0) = \mathrm{id}_{\mathcal{H}_o}$ and, for any $(s_1, t_1), (s_2, t_2) \in \mathcal{C}^* \times \mathcal{P}^*$:

$$\begin{aligned}
&\forall \varphi \in \mathcal{H}_o,\ \forall x \in \mathcal{C},\\
&\quad [\mathrm{T}_o(s_1, t_1)\, \mathrm{T}_o(s_2, t_2)\, \varphi](x) = e^{i\, s_1(x) + \frac{i}{2}\, t_1\left(\Xi^{-1}(s_1)\right)}\, [\mathrm{T}_o(s_2, t_2)\, \varphi]\left(x + \Xi^{*,-1}(t_1)\right)\\
&\qquad = e^{i\,(s_1+s_2)(x) + \frac{i}{2}\left[(t_1+t_2)\left(\Xi^{-1}(s_1+s_2)\right) - t_2\left(\Xi^{-1}(s_1)\right) + t_1\left(\Xi^{-1}(s_2)\right)\right]}\, \varphi\left(x + \Xi^{*,-1}(t_1 + t_2)\right)\\
&\qquad = e^{i\,\xi(s_1, t_1;\, s_2, t_2)}\, [\mathrm{T}_o(s_1 + s_2, t_1 + t_2)\, \varphi](x),
\end{aligned}$$

where we have used $s_2\left(\Xi^{*,-1}(t_1)\right) = t_1\left(\Xi^{-1}(s_2)\right)$. In particular, we thus have, for any $(s, t) \in \mathcal{C}^* \times \mathcal{P}^*$, $\mathrm{T}_o(s, t)\, \mathrm{T}_o(-s, -t) = \mathrm{T}_o(-s, -t)\, \mathrm{T}_o(s, t) = \mathrm{id}_{\mathcal{H}_o}$, so $\mathrm{T}_o(s, t)$ is invertible, and, being isometric, it is a unitary operator on $\mathcal{H}_o$.

Let $\varphi \in \mathcal{D}^{\cdot}_o$, $(s, t) \in \mathcal{C}^* \times \mathcal{P}^*$ and $\epsilon > 0$. In particular, φ is bounded and has compact support, so:

$$M := \int_{\mathcal{C}} d\mu(x)\ |\varphi(x)| < \infty.$$

Moreover, φ is also absolutely continuous, so there exists an open neighborhood U_1 of 0 in $\mathcal{C}$ such that:

$$\forall x \in \mathcal{C},\ \forall x' \in U_1,\ |\varphi(x + x') - \varphi(x)| < \frac{\epsilon}{4M + 1}.$$

The map $\Xi^{*,-1}$ being linear on the finite-dimensional vector space $\mathcal{P}^*$, it is continuous, therefore $U_2 := \Xi^* \left\langle U_1 \right\rangle$ is an open neighborhood of 0 in $\mathcal{P}^*$.

Next, the map $z : \mathcal{C} \times \mathcal{C}^* \times \mathcal{P}^* \to \mathbb{C}$, defined by:

$$\forall x \in \mathcal{C},\ \forall (s', t') \in \mathcal{C}^* \times \mathcal{P}^*,\ z(x;\, s', t') := e^{-i\xi(s, t;\, s', t') + i\, s'(x) + \frac{i}{2}\, t'\left(\Xi^{-1}(s')\right)},$$

is continuous, so, for any $x \in \mathcal{C}$, there exists an open neighborhood V_x of x in $\mathcal{C}$ and an open neighborhood $U_3^{(x)}$ of $(0, 0)$ in $\mathcal{C}^* \times \mathcal{P}^*$ such that:

$$\forall x \in V_x,\ \forall (s',t') \in U_3^{(x)},\ |z(x;\,s',t') - 1| < \frac{\epsilon}{4\,\|\varphi\|^2 + 1}\,.$$

Since the support $\operatorname{supp}(\varphi)$ of φ is compact, there exist $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathcal{C}$ such that $\operatorname{supp}(\varphi) \subset \bigcup_{i=1}^{n} V_{x_i}$. Thus, defining $U_3 := \bigcap_{i=1}^{n} U_3^{(x_i)}$, U_3 is an open neighborhood of $(0,0)$ in $\mathcal{C}^* \times \mathcal{P}^*$, and:

$$\forall x \in \operatorname{supp}(\varphi),\ \forall (s',t') \in U_3,\ |z(x;\,s',t') - 1| < \frac{\epsilon}{4\,\|\varphi\|^2 + 1}\,.$$

Then, $U_4 := (\mathcal{C}^* \times U_2) \cap U_3$ is an open neighborhood of $(0,0)$ in $\mathcal{C}^* \times \mathcal{P}^*$, and, for any $(s',t') \in U_4$:

$$\|\mathrm{T}_o(s+s',t+t')\,\varphi - \mathrm{T}_o(s,t)\,\varphi\|^2$$
$$= 2\,\|\varphi\|^2 - 2\,\mathrm{Re}\left\langle \varphi \,\middle|\, e^{-i\xi(s,t;\,s',t')}\,\mathrm{T}_o(s',t')\,\varphi \right\rangle$$

(where we have used $\mathrm{T}_o(s+s',t+t') = e^{-i\xi(s,t;\,s',t')}\,\mathrm{T}_o(s,t)\,\mathrm{T}_o(s',t')$ and the fact that both $\mathrm{T}_o(s,t)$ and $\mathrm{T}_o(s',t')$ are unitary)

$$\leqslant 2\left|\left\langle \varphi \,\middle|\, e^{-i\xi(s,t;\,s',t')}\,\mathrm{T}_o(s',t')\,\varphi \right\rangle - \|\varphi\|^2\right|$$
$$\leqslant 2\int_{\mathcal{C}} d\mu(x)\ |\varphi(x)|\ \left|z(x;\,s',t')\,\varphi\left(x + \Xi^{*,-1}(t')\right) - \varphi(x)\right|$$
$$\leqslant 2\int_{\mathcal{C}} d\mu(x)\ |\varphi(x)|\ \left|\varphi\left(x + \Xi^{*,-1}(t')\right) - \varphi(x)\right| + 2\int_{\operatorname{supp}(\varphi)} d\mu(x)\ |\varphi(x)|^2\ |z(x;\,s',t') - 1|$$

(for $\forall x \in \mathcal{C},\ |z(x;\,s',t')| = 1$)

$$< \epsilon\,.$$

Hence, for any $\varphi \in \mathcal{D}_o^{\cdot}$, the map $(s,t) \mapsto \mathrm{T}_o(s,t)\,\varphi$ is continuous $\mathcal{C}^* \times \mathcal{P}^* \to \mathcal{H}_o$.

Now, the smooth, compactly supported functions are dense in $\mathcal{H}_o$, so for any $\varphi, \varphi' \in \mathcal{H}_o$, and any $\epsilon > 0$, there exist $\widetilde{\varphi} \in \mathcal{D}_o^{\cdot}$ such that:

$$\|\varphi - \widetilde{\varphi}\| < \frac{\epsilon}{3\ \|\varphi'\| + 1}\,.$$

Next, for any $(s,t) \in \mathcal{C}^* x \mathcal{P}^*$, there exists an open neighborhood U_5 of (s,t) in $\mathcal{C}^* x \mathcal{P}^*$ such that:

$$\forall s'', t'' \in U_5,\ \|\mathrm{T}_o(s'',t'')\,\widetilde{\varphi} - \mathrm{T}_o(s,t)\,\widetilde{\varphi}\| < \frac{\epsilon}{3\ \|\varphi'\| + 1}\,.$$

Thus, for any $s'', t'' \in U_5$,

$$|\langle \varphi' \mid \mathrm{T}_o(s'',t'')\,\varphi\rangle - \langle \varphi' \mid \mathrm{T}_o(s,t)\,\varphi\rangle| \leqslant \|\varphi'\|\ \left[2\ \|\varphi - \widetilde{\varphi}\| + \|\mathrm{T}_o(s'',t'')\,\widetilde{\varphi} - \mathrm{T}_o(s,t)\,\widetilde{\varphi}\|\ \right] < \epsilon$$

(since $\mathrm{T}_o(s,t)$ and $\mathrm{T}_o(s'',t'')$ are both unitary), which proves def. C.3.4.

Let $\varphi \in \mathcal{D}_o^{\cdot}$ and $(s,t) \in \mathcal{C}^* \times \mathcal{P}^*$. For any $\tau \in \mathbb{R}$, we have $[\mathrm{T}_o(\tau s, \tau t)\,\varphi] = \varphi_{(s,t)}^{(\tau)} \in \mathcal{D}_o^{\cdot}$, where:

$$\forall x \in \mathcal{C},\ \varphi_{(s,t)}^{(\tau)}(x) := e^{i\,\tau\,s(x) + \frac{i}{2}\,\tau^2\,t\left(\Xi^{-1}(s)\right)}\,\varphi\left(x + \tau\,\Xi^{*,-1}(t)\right),$$

and $\mathrm{X}_o^{\cdot}(s,t)\,\varphi = -i\,\varphi_{(s,t)} \in \mathcal{D}_o^{\cdot}$, where:

$$\forall x \in \mathcal{C},\ \varphi_{(s,t)}(x) := \left.\frac{d}{d\tau}\varphi_{(s,t)}^{(\tau)}(x)\right|_{\tau=0}\,.$$

In particular, $X^{\bullet}_o(s,t)\,\varphi$ belongs to $\mathcal{H}_o$, thus $X^{\bullet}_o(s,t)$ is well-defined as a linear operator on $\mathcal{D}^{\bullet}_o$.

Moreover, like in the proof of prop. C.4, the dominated convergence theorem yields:

$$\lim_{\tau\to 0}\int_{\mathcal{C}} d\mu(x)\,\left|\frac{\varphi^{(\tau)}_{(s,t)}(x)-\varphi(x)}{\tau}-\varphi_{(s,t)}(x)\right|^2 = 0\,,$$

and therefore:

$$\lim_{\tau\to 0}\left\|\frac{T_o(\tau s,\tau t)\,\varphi-\varphi}{\tau}-i\,X^{\bullet}_o(s,t)\,\varphi\right\| = 0\,.$$

Hence, we have [73, theorem VIII.7]:

$$\varphi\in \mathrm{Dom}\,(X_o(s,t)) \quad \& \quad X^{\bullet}_o(s,t)\,\varphi = X_o(s,t)\,\varphi\,,$$

so $\mathcal{D}^{\bullet}_o \subset \mathrm{Dom}\,(X_o(s,t))$ and $X^{\bullet}_o(s,t) = X_o(s,t)|_{\mathcal{D}^{\bullet}_o}$.

Finally, using: □

$$\forall\varphi\in\mathcal{D}^{\bullet}_o,\; X^{\bullet}_o(s,t)\,\varphi = -i\lim_{\tau\to 0}\frac{T_o(\tau s,\tau t)\,\varphi-\varphi}{\tau}\,,$$

together with $\forall\tau\in\mathbb{R}$, $T_o(\tau s,\tau t)\left\langle\mathcal{D}^{\bullet}_o\right\rangle\subset\mathcal{D}^{\bullet}_o$ and the density of $\mathcal{D}^{\bullet}_o$ in $\mathcal{H}_o$, we get, via the same argument as in the proof of prop. C.4, that $X^{\bullet}_o$ is essentially self-adjoint.

C.2 The Wigner-Weyl transform

We now come to the Wigner transform itself. The Wigner characteristic function associated to a quantum state maps to a point (s,t) the expectation value of $T_\kappa(s,t)$ in this state. In particular, it is defined as a function on the *dual* space $\mathcal{C}^*\times\mathcal{P}^*$, since, as underlined above, the operators $T_\kappa(s,t)$ are labeled by elements of the dual. Note that if we would complete the Wigner transform, by Fourier-transforming this characteristic function, we would pass to the dual again and obtain a quasi-probability defined on the *phase space*, in correct analogy with the probability distributions considered in classical statistical physics.

Insisting on a strict distinction between $\mathcal{M}$ and its dual, may seems an unnecessary care in the case of a linear space, since we could simply choose some identification between the two. However, it allows us to keep track of the nature of the objects we are considering, and in particular of the natural direction of their transformations under morphisms. Probability measures are naturally push-forwarded along a projection, while functions on $\mathcal{M}$ should be pull-backed, but functions on the dual, being pull-backed by the dual map, effectively flow in the same direction as probability measures (viz. def. 18.2): thus, defining characteristic functions as a functions on the dual is consistent with the fact that they encode the same information as probability measures.

We begin by characterizing the image of the (first half of) the Wigner transform. The positivity condition eq. (C.6.1) reflects the requirement for density matrices to be non-negative operators and it manifests the truly quantum nature of the Wigner quasi-probability: a similar equation for a classical characteristic function, reflecting the positivity of its associated probability distribution, would not have the twisting phase $e^{i\xi(s,t;s',t')}$. In other words, eq. (C.6.1) captures the non-commutation of the position and momentum variables (viz. prop. C.2), and it will play a central role in the derivation of the negative result in subsection 18.2. Finally, the continuity of the Wigner characteristic function expresses the fact that it comes from a density matrix on a *weakly continuous* representation (this is closely related to the characterization of normal states on a W^*-algebra, see [87, corollary III.3.11] and prop. 5.12)

Proposition C.6 A continuous function $W : \mathcal{C}^* \times \mathcal{P}^* \to \mathbb{C}$ is said of positive type if, for any $\iota \in \mathcal{I}$:

$$\sum_{(s,t),(s',t') \in \mathcal{C}^* \times \mathcal{P}^*} \overline{\iota(s',t')}\, \iota(s,t)\, e^{i\xi(s,t;s',t')}\, W(s-s', t-t') \geqslant 0. \tag{C.6.1}$$

In particular, this implies:

$$\forall (s,t) \in \mathcal{C}^* \times \mathcal{P}^*,\ W(-s,-t) = \overline{W(s,t)} \quad \& \quad |W(s,t)| \leqslant W(0,0)\,.$$

We denote by $\mathcal{W}$ the space of all continuous functions of positive type.

Proof Let $(s,t) \in \mathcal{C}^* \times \mathcal{P}^*$ and let $\lambda \in \mathbb{C}$. Applying eq. (C.6.1) to $\delta_{(0,0)} + \lambda\, \delta_{(s,t)} \in \mathcal{I}$ yields:

$$\left(1 + |\lambda|^2\right) W(0,0) + \lambda\, W(s,t) + \overline{\lambda}\, W(-s,-t) \geqslant 0\,,$$

where we have used $\xi(0,0;0,0) = \xi(s,t;0,0) = \xi(0,0;s,t) = \xi(s,t;s,t) = 0$. In particular, $W(0,0)$ is real positive (as follows from setting $\lambda = 0$). Thus, $\lambda\, W(s,t) + \overline{\lambda}\, W(-s,-t)$ is real for any $\lambda \in \mathbb{C}$, so $\overline{W(s,t)} = W(-s,-t)$. Now, $\left(1 + |\lambda|^2\right) W(0,0) + 2\,\mathrm{Re}\,[\lambda\, W(s,t)]$ is positive for any $\lambda \in \mathbb{C}$, hence $|W(s,t)| \leqslant W(0,0)$. □

Proposition C.7 Let $\mathcal{H}_\kappa$, T_κ be as in def. C.3 and let ρ be a traceclass, (self-adjoint) positive semi-definite operator on $\mathcal{H}_\kappa$. The map $W_\rho : \mathcal{C}^* \times \mathcal{P}^* \to \mathbb{C}$, defined by:

$$\forall (s,t) \in \mathcal{C}^* \times \mathcal{P}^*,\ W_\rho(s,t) := \mathrm{Tr}_{\mathcal{H}_\kappa}\left[\rho\, \mathrm{T}_\kappa(s,t)\right],$$

is a continuous function of positive type.

Proof For any $(s,t) \in \mathcal{C}^* \times \mathcal{P}^*$, $\mathrm{T}_\kappa(s,t)$ is a bounded operator, hence $\rho\, \mathrm{T}_\kappa(s,t)$ is traceclass, therefore W_ρ is well-defined. From the spectral theorem, there exist $N \in \mathbb{N} \cup \{\infty\}$, an orthonormal family $(\varphi_k)_{k \leqslant N}$ in $\mathcal{H}_\kappa$ and a family of non-negative reals $(p_k)_{k \leqslant N}$ such that:

$$\rho = \sum_{k \leqslant N} p_k\, |\varphi_k\,\rangle\langle\,\varphi_k| \quad \& \quad \sum_{k \leqslant N} p_k = \mathrm{Tr}_{\mathcal{H}_\kappa}\, \rho.$$

Let $(s,t) \in \mathcal{C}^* \times \mathcal{P}^*$ and $\epsilon > 0$. Let $K \in \mathbb{N}$ with $K \leqslant N$ such that:

$$\sum_{K < k \leqslant N} p_k < \frac{\epsilon}{3}\,.$$

From def. C.3.4, the map $(s,t) \mapsto \langle \varphi_k \mid \mathrm{T}_\kappa(s,t)\, \varphi_k \rangle$ is continuous for every $k \leqslant K$. Hence, for each $k \leqslant K$, there exists an open neighborhood U_k of (s,t) in $\mathcal{C}^* \times \mathcal{P}^*$ such that:

$$\forall (s',t') \in U_k\,,\ \left| \langle \varphi_k \mid \mathrm{T}_\kappa(s',t')\, \varphi_k \rangle - \langle \varphi_k \mid \mathrm{T}_\kappa(s,t)\, \varphi_k \rangle \right| < \frac{\epsilon}{3\, \mathrm{Tr}_{\mathcal{H}_\kappa}\, \rho + 1}\,.$$

Since K is finite, $U := \bigcap_{k \leqslant K} U_k$ is an open neighborhood of (s,t) in $\mathcal{C}^* \times \mathcal{P}^*$ and, for any $(s',t') \in U$, we have:

$$\begin{aligned} |W_\rho(s',t') - W_\rho(s,t)| &\leqslant \sum_{k \leqslant N} p_k \left| \langle \varphi_k \mid \mathsf{T}_\kappa(s',t')\, \varphi_k \rangle - \langle \varphi_k \mid \mathsf{T}_\kappa(s,t)\, \varphi_k \rangle \right| \\ &\leqslant 2 \sum_{K < k \leqslant N} p_k + \sum_{k \leqslant K} p_k \left| \langle \varphi_k \mid \mathsf{T}_\kappa(s',t')\, \varphi_k \rangle - \langle \varphi_k \mid \mathsf{T}_\kappa(s,t)\, \varphi_k \rangle \right| \\ &< 2\,\frac{\epsilon}{3} + \sum_{k \leqslant K} p_k\, \frac{\epsilon}{3\,\mathrm{Tr}_{\mathcal{H}_\kappa} \rho + 1} \leqslant \epsilon\,, \end{aligned}$$

where we have used that $\mathsf{T}_\kappa(s,t)$ and $\mathsf{T}_\kappa(s',t')$ are unitary operators and each φ_k for $k \leqslant N$ is a normalized vector, hence $|\langle \varphi_k \mid \mathsf{T}_\kappa(s,t)\, \varphi_k \rangle| \leqslant 1$ and $|\langle \varphi_k \mid \mathsf{T}_\kappa(s',t')\, \varphi_k \rangle| \leqslant 1$. Thus, W_ρ is a continuous function $\mathcal{C}^* \times \mathcal{P}^* \to \mathbb{C}$.

Let $\iota \in \mathcal{I}$. We have:

$$\begin{aligned} \sum_{(s,t),(s',t') \in \mathcal{C}^* \times \mathcal{P}^*} \overline{\iota(s',t')}\, \iota(s,t)\, e^{i\xi(s,t;s',t')}\, W_\rho(s-s', t-t') &= \sum_{(s'',t'') \in \mathcal{C}^* \times \mathcal{P}^*} [\iota^* \star \iota](s'',t'')\, W_\rho(s'',t'') \\ &= \mathrm{Tr}_{\mathcal{H}_\kappa} [\rho\, \mathsf{T}_\kappa \{\iota^* \star \iota\}] \end{aligned}$$

(where $\mathsf{T}_\kappa \{\iota'\}$ has been defined for $\iota' \in \mathcal{I}$ in def. C.3, as a *finite* linear combination of unitary operators belonging to $\mathsf{T}_\kappa \langle \mathcal{C}^* \times \mathcal{P}^* \rangle$)

$$= \sum_{k \leqslant N} p_k \left\langle \varphi_k \,\middle|\, \mathsf{T}_\kappa \{\iota\}^\dagger\, \mathsf{T}_\kappa \{\iota\}\, \varphi_k \right\rangle \geqslant 0.$$

Therefore, W_ρ fulfills eq. (C.6.1). □

Like its classical analogue, the Wigner characteristic function is related to the moment-generating function (precisely, the latter, when it exists, is the restriction to the complex axis of the analytical expansion of the former). In particular, a quantum state exhibits finite variances for the elementary observables $\mathsf{X}_\kappa(s,t)$ if its characteristic function is twice-differentiable at 0, and the covariance matrix can then be recovered from the corresponding Hessian.

Proposition C.8 Let $\mathcal{H}_\kappa$, T_κ, ρ and W_ρ be as in prop. C.7. We moreover assume that there exist a linear form $W_\rho^{(1)}$ and a symmetric bilinear form $W_\rho^{(2)}$ on $\mathcal{C}^* \times \mathcal{P}^*$ such that:

$$\forall (s,t) \in \mathcal{C}^* \times \mathcal{P}^*,\ W_\rho(\tau s, \tau t) = W_\rho(0,0) + i\,\tau\, W_\rho^{(1)}(s,t) - \frac{\tau^2}{2}\, W_\rho^{(2)}(s,t;s,t) + o(\tau^2)\,. \qquad \text{(C.8.1)}$$

Then, for any (s,t), $(s',t') \in \mathcal{C}^* \times \mathcal{P}^*$, the operators $\rho\, \mathsf{X}_\kappa(s,t)$ (densely defined on $\mathrm{Dom}\big(\mathsf{X}_\kappa(s,t)\big)$) and $\frac{\mathsf{X}_\kappa(s,t)\,\rho\,\mathsf{X}_\kappa(s',t') + \mathsf{X}_\kappa(s',t')\,\rho\,\mathsf{X}_\kappa(s,t)}{2}$ (which can be defined at least as a sesquilinear form on the dense subset $\mathcal{D}_\kappa \subset \mathcal{H}_\kappa$) admit a traceclass extension on $\mathcal{H}_\kappa$, with:

$$\mathrm{Tr}_{\mathcal{H}_\kappa} \rho\, \mathsf{X}_\kappa(s,t) = W_\rho^{(1)}(s,t) \quad \& \quad \mathrm{Tr}_{\mathcal{H}_\kappa} \frac{\mathsf{X}_\kappa(s,t)\,\rho\,\mathsf{X}_\kappa(s',t') + \mathsf{X}_\kappa(s',t')\,\rho\,\mathsf{X}_\kappa(s,t)}{2} = W_\rho^{(2)}(s,t;s',t')\,.$$

Proof Let $(s,t) \in \mathcal{C}^* \times \mathcal{P}^*$. For any $\tau \neq 0$, we define:

$$\mathrm{Y}_\kappa^{(\tau)}(s,t) := \frac{2\,\mathrm{id}_{\mathcal{H}_\kappa} - \mathrm{T}_\kappa(\tau s, \tau t) - \mathrm{T}_\kappa(\tau s, \tau t)^\dagger}{\tau^2}\,.$$

Since $\mathrm{T}_\kappa(\tau s, \tau t)$ is a unitary operator, $\mathrm{Y}_\kappa^{(\tau)}(s,t)$ is a bounded, (self-adjoint) positive semi-definite operator on $\mathcal{H}_\kappa$. Using $\mathrm{T}_\kappa(0,0) = \mathrm{id}_{\mathcal{H}_\kappa}$ and $\mathrm{T}_\kappa(\tau s, \tau t)^\dagger = \mathrm{T}_\kappa(-\tau s, -\tau t)$, we get:

$$\mathrm{Tr}_{\mathcal{H}_\kappa}\, \rho\, \mathrm{Y}_\kappa^{(\tau)}(s,t) = \frac{2\,W_\rho(0,0) - W_\rho(\tau s, \tau t) - W_\rho(-\tau s, -\tau t)}{\tau^2}\,.$$

Hence, eq. (C.8.1) implies:

$$\lim_{\tau\to 0} \mathrm{Tr}_{\mathcal{H}_\kappa}\, \rho\, \mathrm{Y}_\kappa^{(\tau)}(s,t) = W_\rho^{(2)}(s,t;s,t)\,.$$

Let $\tau_1 > 0$ such that:

$$\forall \tau \in \left]0,\ \tau_1\right[\ ,\ \mathrm{Tr}_{\mathcal{H}_\kappa}\, \rho\, \mathrm{Y}_\kappa^{(\tau)}(s,t) \leqslant W_\rho^{(2)}(s,t;s,t) + 1 =: A\,.$$

Applying the spectral theorem to the self-adjoint operator $\mathrm{X}_\kappa(s,t)$, we denote by $d\Pi_\kappa(s,t)$ its projection-valued measure, with:

$$\mathrm{X}_\kappa(s,t) = \int_{-\infty}^{+\infty} \sigma\, d\Pi_\kappa(s,t)[\sigma]\,.$$

Then, for any $\tau \neq 0$, we have, using $\mathrm{T}_\kappa(\tau s, \tau t) = \exp\left(i\,\tau\,\mathrm{X}_\kappa(s,t)\right)$:

$$\mathrm{Y}_\kappa^{(\tau)}(s,t) := \frac{2}{\tau^2}\int_{-\infty}^{+\infty} \left[1 - \cos(\tau\sigma)\right]\, d\Pi_\kappa(s,t)[\sigma]\,.$$

There exists $\epsilon > 0$ such that:

$$\forall x \in \left]-\epsilon, \epsilon\right[\ ,\ 1 - \cos(x) > \frac{x^2}{4}\,,$$

hence, for any $\tau \neq 0$, the operator:

$$\mathrm{Y}_\kappa^{(\tau)}(s,t) - \frac{2}{\tau^2}\int_{-\epsilon/\tau}^{\epsilon/\tau} \frac{(\tau\sigma)^2}{4}\, d\Pi_\kappa(s,t)[\sigma]\,,$$

is a bounded, (self-adjoint) positive semi-definite operator on $\mathcal{H}_\kappa$. Defining, for any $B \leqslant B' \in \mathbb{R}$, the spectral projector $\Pi_\kappa^{[B,B']}(s,t) := \int_B^{B'} d\Pi_\kappa(s,t)[\sigma]$, we thus have:

$$\forall B > \frac{\epsilon}{\tau_1}\ ,\ \mathrm{Tr}_{\mathcal{H}_\kappa}\, \rho\, \Pi_\kappa^{[-B,B]}(s,t)\, \mathrm{X}_\kappa(s,t)^2\, \Pi_\kappa^{[-B,B]}(s,t) \leqslant 2\,A\,. \tag{C.8.2}$$

Now, from the spectral theorem, there exist $N \in \mathbb{N} \cup \{\infty\}$, an orthonormal family $(\varphi_k)_{k\leqslant N}$ in $\mathcal{H}_\kappa$ and a family of strictly positive reals $(p_k)_{k\leqslant N}$ such that:

$$\rho = \sum_{k\leqslant N} p_k \left|\varphi_k\right\rangle\left\langle\, \varphi_k\right| \quad \& \quad \sum_{k\leqslant N} p_k = \mathrm{Tr}_{\mathcal{H}_\kappa}\, \rho\,.$$

For each $k \leqslant N$ and each $B \geqslant 0$, we have:

$$\left\langle \varphi_k \,\middle|\, \Pi_\kappa^{[-B,B]}(s,t)\, \mathrm{X}_\kappa(s,t)^2\, \Pi_\kappa^{[-B,B]}(s,t)\, \varphi_k \right\rangle = \left\| \mathrm{X}_\kappa(s,t)\, \Pi_\kappa^{[-B,B]}(s,t)\, \varphi_k \right\|^2 \geqslant 0,$$

so, for any $B > \epsilon/\tau_1$ and any $k \leqslant N$,

$$\left\| \mathrm{X}_\kappa(s,t)\, \Pi_\kappa^{[-B,B]}(s,t)\, \varphi_k \right\|^2 \leqslant \frac{2A}{p_k}\,.$$

Therefore, $\varphi_k \in \mathrm{Dom}\big(\mathrm{X}_\kappa(s,t)\big)$. Moreover, we have [73, theorem VIII.7]:

$$\lim_{\tau\to 0} \frac{\mathrm{T}_\kappa(\tau s, \tau t)\, \varphi_k - \varphi_k}{\tau} = i\, \mathrm{X}_\kappa(s,t)\, \varphi_k\,,$$

hence, we get:

$$\lim_{\tau\to 0} \left\langle \varphi_k \;\middle|\; \mathrm{Y}_\kappa^{(\tau)}(s,t)\, \varphi_k \right\rangle = \lim_{\tau\to 0} \left\| \frac{\mathrm{T}_\kappa(\tau s, \tau t)\, \varphi_k - \varphi_k}{\tau} \right\|^2 = \|\mathrm{X}_\kappa(s,t)\, \varphi_k\|^2\,.$$

Fatou's lemma then yields:

$$\sum_{k\leqslant N} p_k\ \|\mathrm{X}_\kappa(s,t)\, \varphi_k\|^2 \leqslant \liminf_{\tau\to 0} \sum_{k\leqslant N} p_k\ \left\langle \varphi_k \;\middle|\; \mathrm{Y}_\kappa^{(\tau)}(s,t)\, \varphi_k \right\rangle = W_\rho^{(2)}(s,t;\, s,t)\,.$$

Next, this provides the bound:

$$\sum_{k\leqslant N} p_k\ \|\varphi_k\|\ \|\mathrm{X}_\kappa(s,t)\, \varphi_k\| \leqslant \sqrt{W_\rho^{(2)}(s,t;\, s,t)\, \mathrm{Tr}_{\mathcal{H}_\kappa} \rho}\,,$$

where we have used $\|\varphi_k\| = 1$ and the Cauchy–Schwarz inequality (with $\sum_{k\leqslant N} p_k = \mathrm{Tr}_{\mathcal{H}_\kappa} \rho$). Hence, we can define a bounded operator $\{\rho\, \mathrm{X}_\kappa(s,t)\}$ on $\mathcal{H}_\kappa$ by:

$$\{\rho\, \mathrm{X}_\kappa(s,t)\} := \sum_{k\leqslant N} p_k\ |\varphi_k\ \rangle\langle\ \mathrm{X}_\kappa(s,t)\, \varphi_k|\,,$$

and this operator coincides with $\rho\, \mathrm{X}_\kappa(s,t)$ on $\mathrm{Dom}\big(\mathrm{X}_\kappa(s,t)\big)$, since $\mathrm{X}_\kappa(s,t)$ is self-adjoint. In addition, we have:

$$\big\|\ \{\rho\, \mathrm{X}_\kappa(s,t)\}\ \big\|_1 \leqslant \sum_{k\leqslant N} p_k\ \big\|\ |\varphi_k\ \rangle\langle\ \mathrm{X}_\kappa(s,t)\, \varphi_k|\ \big\|_1 = \sum_{k\leqslant N} p_k\ \|\varphi_k\|\ \|\mathrm{X}_\kappa(s,t)\, \varphi_k\| < \infty\,,$$

where $\|\cdot\|_1$ denotes the trace norm [73, theorem VI.20], so $\{\rho\, \mathrm{X}_\kappa(s,t)\}$ is traceclass. Its trace is given by:

$$\mathrm{Tr}_{\mathcal{H}_\kappa}\ \{\rho\, \mathrm{X}_\kappa(s,t)\} = \sum_{k\leqslant N} p_k\ \langle \mathrm{X}_\kappa(s,t)\, \varphi_k \mid \varphi_k\rangle = \sum_{k\leqslant N} p_k\ \langle \varphi_k \mid \mathrm{X}_\kappa(s,t)\, \varphi_k\rangle\,.$$

Now, from:

$$\forall x \in \mathbb{R},\ \left|e^{ix} - 1\right| \leqslant |x|\,,$$

together with the spectral decomposition of $\mathrm{X}_\kappa(s,t)$, we get, for any $\tau \neq 0$:

$$\left\| \frac{\mathrm{T}_\kappa(\tau s, \tau t)\, \varphi_k - \varphi_k}{\tau} \right\| \leqslant \|\mathrm{X}_\kappa(s,t)\, \varphi_k\|\,.$$

Thus, the dominated convergence theorem yields:

$$i \sum_{k\leqslant N} p_k\ \langle \varphi_k \mid \mathrm{X}_\kappa(s,t)\, \varphi_k\rangle = \sum_{k\leqslant N} p_k\ \lim_{\tau\to 0} \left\langle \varphi_k \;\middle|\; \frac{\mathrm{T}_\kappa(\tau s, \tau t)\, \varphi_k - \varphi_k}{\tau} \right\rangle$$

$$= \lim_{\tau \to 0} \sum_{k \leqslant N} p_k \left\langle \varphi_k \;\middle|\; \frac{\mathrm{T}_\kappa(\tau s, \tau t)\, \varphi_k - \varphi_k}{\tau} \right\rangle,$$

which can be rewritten, using eq. (C.8.1), as:

$$\mathrm{Tr}_{\mathcal{H}_\kappa} \left\{\rho\, \mathrm{X}_\kappa(s, t)\right\} = W_\rho^{(1)}(s, t)\,.$$

(note that $W_\rho(0, 0) = \mathrm{Tr}_{\mathcal{H}_\kappa} \rho$ for $\mathrm{T}_\kappa(0, 0) = \mathrm{id}_{\mathcal{H}_\kappa}$).

Similarily, the inequality:

$$\sum_{k \leqslant N} p_k \left\| \mathrm{X}_\kappa(s, t)\, \varphi_k \right\|^2 \leqslant W_\rho^{(2)}(s, t;\, s, t) < \infty\,,$$

ensures that we can define a traceclass operator $\{\mathrm{X}_\kappa(s, t)\, \rho\, \mathrm{X}_\kappa(s, t)\}$ on $\mathcal{H}_\kappa$ by:

$$\{\mathrm{X}_\kappa(s, t)\, \rho\, \mathrm{X}_\kappa(s, t)\} := \sum_{k \leqslant N} p_k \left| \mathrm{X}_\kappa(s, t)\, \varphi_k \right\rangle \left\langle \mathrm{X}_\kappa(s, t)\, \varphi_k \right|,$$

and we have:

$$\forall \varphi', \varphi'' \in \mathrm{Dom}\big(\mathrm{X}_\kappa(s, t)\big),\ \langle \varphi'' \mid \{\mathrm{X}_\kappa(s, t)\, \rho\, \mathrm{X}_\kappa(s, t)\}\ \varphi' \rangle = \langle \mathrm{X}_\kappa(s, t)\, \varphi'' \mid \rho\, \mathrm{X}_\kappa(s, t)\, \varphi' \rangle\,.$$

Moreover, using again the dominated convergence theorem, with:

$$\forall \tau \neq 0,\ \left\langle \varphi_k \;\middle|\; \mathrm{Y}_\kappa^{(\tau)}(s, t)\, \varphi_k \right\rangle = \left\| \frac{\mathrm{T}_\kappa(\tau s, \tau t)\, \varphi_k - \varphi_k}{\tau} \right\|^2 \leqslant \left\| \mathrm{X}_\kappa(s, t)\, \varphi_k \right\|^2,$$

we get:

$$\mathrm{Tr}_{\mathcal{H}_\kappa} \left\{\mathrm{X}_\kappa(s, t)\, \rho\, \mathrm{X}_\kappa(s, t)\right\} = \lim_{\tau \to 0} \mathrm{Tr}_{\mathcal{H}_\kappa} \rho\, \mathrm{Y}_\kappa^{(\tau)}(s, t) = W_\rho^{(2)}(s, t;\, s, t)\,.$$

Finally, for any (s, t), $(s', t') \in \mathcal{C}^* \times \mathcal{P}^*$, we have, thanks to prop. C.4.3:

$$\begin{aligned} \forall \varphi, \varphi' \in \mathcal{D}_\kappa\,,\ &\langle \mathrm{X}_\kappa(s, t)\, \varphi'' \mid \rho\, \mathrm{X}_\kappa(s', t')\, \varphi' \rangle + \langle \mathrm{X}_\kappa(s', t')\, \varphi'' \mid \rho\, \mathrm{X}_\kappa(s, t)\, \varphi' \rangle = \\ &= \langle \varphi'' \mid \{\mathrm{X}_\kappa(s + s', t + t')\, \rho\, \mathrm{X}_\kappa(s + s', t + t')\}\ \varphi' \rangle + \\ &\quad - \langle \varphi'' \mid \{\mathrm{X}_\kappa(s, t)\, \rho\, \mathrm{X}_\kappa(s, t)\}\ \varphi' \rangle - \langle \varphi'' \mid \{\mathrm{X}_\kappa(s', t')\, \rho\, \mathrm{X}_\kappa(s', t')\}\ \varphi' \rangle\,. \end{aligned}$$

Hence, $\frac{\{\mathrm{X}_\kappa(s+s', t+t')\, \rho\, \mathrm{X}_\kappa(s+s', t+t')\} - \{\mathrm{X}_\kappa(s, t)\, \rho\, \mathrm{X}_\kappa(s, t)\} - \{\mathrm{X}_\kappa(s', t')\, \rho\, \mathrm{X}_\kappa(s', t')\}}{2}$ provides a traceclass extension of the sesquilinear form $\frac{\mathrm{X}_\kappa(s, t)\, \rho\, \mathrm{X}_\kappa(s', t') + \mathrm{X}_\kappa(s', t')\, \rho\, \mathrm{X}_\kappa(s, t)}{2}$ on $\mathcal{D}_\kappa$ and its trace is given by:

$$\mathrm{Tr}_{\mathcal{H}_\kappa} \frac{\{\mathrm{X}_\kappa(s+s', t+t')\, \rho\, \mathrm{X}_\kappa(s+s', t+t')\} - \{\mathrm{X}_\kappa(s, t)\, \rho\, \mathrm{X}_\kappa(s, t)\} - \{\mathrm{X}_\kappa(s', t')\, \rho\, \mathrm{X}_\kappa(s', t')\}}{2} = W_\rho^{(2)}(s, t;\, s', t')\,.$$

□

As a first step toward recovering the density matrix from its Wigner characteristic function, we observe that the positivity condition eq. (C.6.1) is exactly what we need to turn a function on $\mathcal{C}^* \times \mathcal{P}^*$ into a state (aka. a normalized positive linear functional, see [41, part III, def. 2.2.8]) on the Weyl algebra. Then, from any state we can reconstruct a representation of the algebra, via the GNS construction [34], and the continuity of the characteristic function ensures that this representation will be weakly continuous.

Proposition C.9 Let $W \in \mathcal{W}$. Then, there exist a weakly continuous, unitary representation

$\mathcal{H}_W$, T_W of $\mathcal{C}^* \times \mathcal{P}^*$ and a vector $\zeta_W \in \mathcal{H}_W$ such that:

1. $\mathcal{H}_W = \overline{\mathrm{Vect}\,\{\mathrm{T}_W(s,t)\,\zeta_W \mid (s,t) \in \mathcal{C}^* \times \mathcal{P}^*\}}$ (ie. ζ_W is a cyclic vector for $\mathcal{H}_W$, T_W);
2. $\forall (s,t) \in \mathcal{C}^* \times \mathcal{P}^*$, $W(s,t) = \langle \zeta_W \mid \mathrm{T}_W(s,t)\,\zeta_W \rangle_{\mathcal{H}_W}$.

Proof Let $\iota_1, \iota_2 \in \mathcal{I}$. We define:

$$\langle \iota_1 \mid \iota_2 \rangle_W := \sum_{(s,t) \in \mathcal{C}^* \times \mathcal{P}^*} (\iota_1^* \star \iota_2)(s,t)\, W(s,t)\,.$$

This is well defined, for the sum has only finitely many non-zero contributions ($\iota_1^* \star \iota_2 \in \mathcal{I}$), and eq. (C.6.1) ensures that $\langle \cdot \mid \cdot \rangle_W$ provides an Hermitian, positive semi-definite sesquilinear form on the complex vector space $\mathcal{I}$ (the hermiticiy comes from $(\iota_1^* \star \iota_2)^* = \iota_2^* \star \iota_1$ together with $\forall (s,t) \in \mathcal{C}^* \times \mathcal{P}^*$, $W(-s,-t) = \overline{W(s,t)}$, as was proven in prop. C.6).

Let $\mathcal{N}_W := \{\upsilon \in \mathcal{I} \mid \langle \upsilon \mid \upsilon \rangle_W = 0\}$. For any $\iota \in \mathcal{I}$ and any $\upsilon \in \mathcal{N}_W$, we have:

$$\forall \lambda \in \mathbb{C},\ \langle \iota \mid \iota \rangle + 2\,\mathrm{Re}\big[\lambda\, \langle \iota \mid \upsilon \rangle\big] = \langle \iota + \lambda\, \upsilon \mid \iota + \lambda\, \upsilon \rangle_W \geqslant 0\,,$$

therefore $\langle \iota \mid \upsilon \rangle = 0$. Hence, for any $\iota_1, \iota_2 \in \mathcal{I}$ and any $\upsilon_1, \upsilon_2 \in \mathcal{N}_W$:

$$\langle \iota_1 + \upsilon_1 \mid \iota_2 + \upsilon_2 \rangle_W = \langle \iota_1 \mid \iota_2 \rangle_W\,,$$

so $\langle \cdot \mid \cdot \rangle_W$ induces an inner product on the quotient $\mathcal{I}/\mathcal{N}_W$. We denote by $\mathcal{H}_W$ the completion of $\mathcal{I}/\mathcal{N}_W$ with respect to the corresponding norm.

Let $(s,t) \in \mathcal{C}^* \times \mathcal{P}^*$. For any $\iota_1, \iota_2 \in \mathcal{I}$, we have:

$$\langle \delta_{(s,t)} \star \iota_1 \mid \delta_{(s,t)} \star \iota_2 \rangle_W = \langle \iota_1 \mid \iota_2 \rangle_W$$

(for $(\delta_{(s,t)} \star \iota_1)^* \star (\delta_{(s,t)} \star \iota_2) = \iota_1^* \star (\delta_{(-s,-t)} \star \delta_{(s,t)}) \star \iota_2 = \iota_1^* \star \iota_2$), so in particular $[\delta_{(s,t)} \star \cdot\,]\langle \mathcal{N}_W \rangle \subset \mathcal{N}_W$, and $\delta_{(s,t)} \star \cdot$ induces a unitary operator $\mathrm{T}_W(s,t)$ on $\mathcal{H}_W$. Then, def. C.3.2 and C.3.3 come from:

$$\forall (s_1,t_1), (s_2,t_2) \in \mathcal{C}^* \times \mathcal{P}^*,\ \delta_{(s_1,t_1)} \star \delta_{(s_2,t_2)} = e^{i\xi(s_1,t_1;s_2,t_2)}\, \delta_{(s_1+s_2,\, t_1+t_2)}\,,$$

and from the fact that $\delta_{(0,0)}$ is the unit of the *-algebra $\mathcal{I}$.

Let ι_1, $\iota_2 \in \mathcal{I}$. For any $(s,t) \in \mathcal{C}^* \times \mathcal{P}^*$, we have:

$$\langle \iota_1 \mid \delta_{(s,t)} \star \iota_2 \rangle_W = \sum_{\substack{(s_1,t_1),(s_2,t_2) \\ \in \mathcal{C}^* \times \mathcal{P}^*}} e^{i\xi(s_2,t_2;s_1,t_1) + i\xi(s,t;s_1+s_2,t_1+t_2)}\, \overline{\iota_1(s_1,t_1)}\, \iota_2(s_2,t_2)\, W(s+s_2-s_1, t+t_2-t_1)\,.$$

Thus, the map $(s,t) \mapsto \langle \iota_1 \mid \delta_{(s,t)} \star \iota_2 \rangle_W$ is a finite linear combination of translations of W, and those are continuous by definition of $\mathcal{W}$. So, for any $\widetilde{\iota_1}, \widetilde{\iota_2} \in \mathcal{I}/\mathcal{N}_W$, the map $(s,t) \mapsto \langle \widetilde{\iota_1} \mid \mathrm{T}_W(s,t)\,\widetilde{\iota_2} \rangle_{\mathcal{H}_W}$ is continuous. Now, $\mathcal{I}/\mathcal{N}_W$ being dense in $\mathcal{H}_W$, there exist, for any $\varphi_1, \varphi_2 \in \mathcal{H}_W$ and any $\epsilon > 0$, $\widetilde{\iota_1}, \widetilde{\iota_2} \in \mathcal{I}/\mathcal{N}_W$ such that:

$$\|\varphi_1 - \widetilde{\iota_1}\|_{\mathcal{H}_W} < \frac{\epsilon}{6\, \|\varphi_2\|_{\mathcal{H}_W} + \sqrt{6\epsilon}} \quad \& \quad \|\varphi_2 - \widetilde{\iota_2}\|_{\mathcal{H}_W} < \frac{\epsilon}{6\, \|\varphi_1\|_{\mathcal{H}_W} + \sqrt{6\epsilon}}\,.$$

Next, for any $(s,t) \in \mathcal{C}^* \times \mathcal{P}^*$, there exists an open neighborhood U of (s,t) in $\mathcal{C}^* \times \mathcal{P}^*$ such that:

$$\forall (s',t') \in U,\ \left|\langle \widetilde{\iota_1} \mid \mathrm{T}_W(s',t')\,\widetilde{\iota_2} \rangle_{\mathcal{H}_W} - \langle \widetilde{\iota_1} \mid \mathrm{T}_W(s,t)\,\widetilde{\iota_2} \rangle_{\mathcal{H}_W}\right| < \frac{\epsilon}{3}\,.$$

Hence, for any $(s',t') \in U$, we have:

$$|\langle \varphi_1 \mid \mathrm{T}_W(s', t')\, \varphi_2 \rangle - \langle \varphi_1 \mid \mathrm{T}_W(s, t)\, \varphi_2 \rangle| < \epsilon \,,$$

where we have used that $\mathrm{T}_W(s, t)$ and $\mathrm{T}_W(s', t')$ are unitary. This proves def. C.3.4.

Finally, we define ζ_W to be the equivalence class of $\delta_{(0,0)}$ in $\mathcal{I}/\mathcal{N}_W$. For any $(s, t) \in \mathcal{C}^* \times \mathcal{P}^*$, we have $\delta_{(s,t)} \star \delta_{(0,0)} = \delta_{(s,t)}$, so $\mathrm{T}_W(s, t)\, \zeta_W$ is the equivalence class of $\delta_{(s,t)}$ in $\mathcal{I}/\mathcal{N}_W$. But since $\mathcal{I} = \mathrm{Vect}\left\{\delta_{(s,t)} \mid (s, t) \in \mathcal{C}^* \times \mathcal{P}^*\right\}$, we have:

$$\mathcal{I}/\mathcal{N}_W = \mathrm{Vect}\left\{\mathrm{T}_W(s, t)\, \zeta_W \mid (s, t) \in \mathcal{C}^* \times \mathcal{P}^*\right\},$$

which proves point C.9.1. Moreover, we also have:

$$\langle \zeta_W \mid \mathrm{T}_W(s, t)\, \zeta_W \rangle_{\mathcal{H}_W} = \langle \delta_{(0,0)} \mid \delta_{(s,t)} \rangle_W = W(s, t)\,,$$

which proves point C.9.2. □

Finally, the invertibility of Wigner transform can be seen as a consequence of the Stone-von Neumann theorem [98] (which we implicitly prove below, following closely [47, theorem 15a]). Indeed, the Schrödinger representation being the unique irreducible, weakly continuous representation of the Weyl algebra, the Hilbert space $\mathcal{H}_W$ constructed from W above can be written as a direct sum of independent copies of $\mathcal{H}_o$. Projecting the vector ζ_W representing W in $\mathcal{H}_W$ on each of these copies, we obtain a collection of vectors in $\mathcal{H}_o$, and we can reconstruct a density matrix on $\mathcal{H}_o$ as the statistical superposition of the corresponding pure states (this is consistent with the general result that a state is pure if and only if its GNS representation is irreducible, see [41, part III, theorem 2.2.17]).

Proposition C.10 Let $W \in \mathcal{W}$. Then, there exists a *unique* traceclass, (self-adjoint) positive semi-definite operator ρ on $\mathcal{H}_o$ such that $W = W_\rho$ (with W_ρ defined as in prop. C.7).

Lemma C.11 Let $(\cdot \mid \cdot)$ be a real inner product on $\mathcal{C}$. Using the resulting identification of $\mathcal{C}^*$ with $\mathcal{C}$ and identifying $\mathcal{P}^*$ with $\mathcal{C}$ through the dual map Ξ^*, we are provided with a corresponding real inner product on $\mathcal{C}^* \times \mathcal{P}^*$, which we will denote by $(\cdot, \cdot \mid \cdot, \cdot)$. Let $\widetilde{\mu}$ be the Lebesgue measure on $\mathcal{C}^* \times \mathcal{P}^*$ normalized with respect to this Euclidean structure. We define a map Ψ on $\mathcal{C}^* \times \mathcal{P}^*$ through:

$$\forall (s, t) \in \mathcal{C}^* \times \mathcal{P}^*,\ \Psi(s, t) := \frac{1}{\alpha} \exp\left(-\frac{(s, t \mid s, t)}{4}\right),$$

where $\alpha := \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s, t) \exp\left(-\frac{(s,t \mid s,t)}{2}\right)$.

Let W be a bounded, continuous function on $\mathcal{C}^* \times \mathcal{P}^*$ such that:

$$\forall (s_o, t_o), (s_1, t_1) \in \mathcal{C}^* \times \mathcal{P}^*,\ \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s, t)\, \Psi(s, t)\, e^{i\, \xi(s,t;s_1,t_1)}\, W(s_o + s,\ t_o + t) = 0\,. \qquad \text{(C.11.1)}$$

Then, $W \equiv 0$.

Proof Let $(e_i)_{i \in \{1,\dots,p\}}$ be an orthonormal basis of $\mathcal{C}$, $(\cdot, \cdot)$, let $(f_i)_{i \in \{1,\dots,p\}}$ be the corresponding dual basis in $\mathcal{C}^*$ and let $(g_i)_{i \in \{1,\dots,p\}}$ be the basis in $\mathcal{P}^*$ given by:

$$\forall i \in \{1, \dots, p\}\,,\ g_i := \Xi^*(e_i)\,.$$

For any $(s, t) \in \mathcal{C}^* \times \mathcal{P}^*$, we will write $s =: s^i f_i$ and $t =: t^i g_i$ (with implicit summation). In particular, we then have:

$$\forall (s,t),\, (s',t') \in \mathcal{C}^* \times \mathcal{P}^*,\; \xi(s,t;s',t') = \frac{t^i s'^i - s^i t'^i}{2} \quad \& \quad \big(s,t \mid s',t'\big) = s^i s'^i + t^i t'^i ,$$

as well as $d\widetilde{\mu}(s,t) = ds^1 \dots ds^p \, dt^1 \dots dt^p$. Therefore, $\alpha = (2\pi)^p$.

Now, for any $\beta > 0$ and any $(s_o, t_o) \in \mathcal{C}^* \times \mathcal{P}^*$, we have:

$$\int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s_1,t_1) \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s,t) \; \left| \Psi(\beta s_1, \beta t_1)\, \Psi(s,t)\, e^{i\xi(s,t;s_1,t_1)}\, W(s_o + s,\; t_o + t) \right|$$

$$\leqslant \left[\frac{2}{\beta}\right]^{2p} \|W\|_\infty < \infty ,$$

where $\|W\|_\infty$ is the sup norm of the bounded function W. Thus, Fubini's theorem, together with eq. (C.11.1) yields:

$$0 = \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s,t)\, \Psi(s,t)\, W(s_o + s,\; t_o + t) \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s_1,t_1) \Psi(\beta s_1, \beta t_1)\, e^{i\xi(s,t;s_1,t_1)} .$$

Performing the Gaussian integration, we get:

$$0 = \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s,t) \left[\frac{\gamma^2 - 1}{\pi}\right]^p \exp\left(-\frac{\gamma^2}{4}\, (s,t \mid s,t)\right)\; W(s_o + s,\; t_o + t) ,$$

where $\gamma := \sqrt{1 + {}^1\!/\!_{\beta^2}}$.

Let $\epsilon > 0$. There exists $A > 0$ such that:

$$\int_A^\infty du\, u^{2p-1}\, e^{-\frac{u^2}{4}} < \frac{\epsilon}{4\, \|W\|_\infty + 1} \int_0^\infty du\, u^{2p-1}\, e^{-\frac{u^2}{4}} .$$

W being continuous, there also exists $\delta \in \,]0, A[$ such that:

$$\forall (s,t) \in \mathcal{C}^* \times \mathcal{P}^* \big/ \, (s,t \mid s,t) \leqslant \delta^2,\; |W(s_o + s,\; t_o + t) - W(s_o,\; t_o)| < \frac{\epsilon}{2} .$$

Hence, applying the previous equality with $\beta = \dfrac{\delta}{\sqrt{A^2 - \delta^2}}$, we have:

$$\begin{aligned}
|W(s_o,\; t_o)| \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s,t)\, e^{-\frac{\gamma^2}{4}(s,t \mid s,t)} &= \\
= \left| \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s,t)\, e^{-\frac{\gamma^2}{4}(s,t \mid s,t)} \left[W(s_o + s,\; t_o + t) - W(s_o,\; t_o)\right] \right| & \\
< \frac{\epsilon}{2} \int_{(s,t \mid s,t) \leqslant \delta^2} d\widetilde{\mu}(s,t)\, e^{-\frac{\gamma^2}{4}(s,t \mid s,t)} \; + \; 2\, \|W\|_\infty \int_{(s,t \mid s,t) \geqslant \delta^2} d\widetilde{\mu}(s,t)\, e^{-\frac{\gamma^2}{4}(s,t \mid s,t)} . &
\end{aligned}$$

Changing to spherical coordinates and dropping an overall constant, this can be rewritten as:

$$|W(s_o,\; t_o)| \int_0^\infty du\, u^{2p-1}\, e^{-\frac{u^2}{4}} < \frac{\epsilon}{2} \int_0^{\gamma\delta} du\, u^{2p-1}\, e^{-\frac{u^2}{4}} \; + \; 2\, \|W\|_\infty \int_{\gamma\delta}^\infty du\, u^{2p-1}\, e^{-\frac{u^2}{4}}$$

$$< \epsilon \int_0^\infty du\, u^{2p-1}\, e^{-\frac{u^2}{4}} .$$

where we have used that $\gamma\delta = A$. Thus, we have, for any $(s_o, t_o) \in \mathcal{C}^* \times \mathcal{P}^*$ and any $\epsilon > 0$, $|W(s_o, t_o)| < \epsilon$, and therefore $W \equiv 0$. □

Proof of prop. C.10 *Existence.* Let $W \in \mathcal{W}$ and let $\mathcal{H}_W$, T_W be the weakly continuous, unitary representation introduced in prop. C.9, with cyclic vector ζ_W. For any $\varphi, \varphi' \in \mathcal{H}_W$, the map $(s, t) \mapsto \Psi(s, t)\, \langle \varphi' \mid \mathrm{T}_W(s, t)\, \varphi\rangle_{\mathcal{H}_W}$ is continuous, hence measurable, and we have:

$$\int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s, t)\, \left|\Psi(s, t)\, \langle \varphi' \mid \mathrm{T}_W(s, t)\, \varphi\rangle_{\mathcal{H}_W}\right| \leqslant 2^p\, \|\varphi'\|_{\mathcal{H}_W}\, \|\varphi\|_{\mathcal{H}_W} < \infty\,.$$

Hence, there exists a bounded operator $\mathrm{T}_W\{\Psi\}$ on $\mathcal{H}_W$ such that, for any $\varphi, \varphi' \in \mathcal{H}_W$:

$$\langle \varphi' \mid \mathrm{T}_W\{\Psi\}\, \varphi\rangle_{\mathcal{H}_W} := \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s, t)\, \Psi(s, t)\, \langle \varphi' \mid \mathrm{T}_W(s, t)\, \varphi\rangle_{\mathcal{H}_W}\,.$$

In particular, $\mathrm{T}_W\{\Psi\}$ is symmetric (since, for any $(s, t) \in \mathcal{C}^* \times \mathcal{P}^*$, $\overline{\Psi(s, t)} = \Psi(s, t) = \Psi(-s, -t)$ and $\mathrm{T}_W(s, t)^\dagger = \mathrm{T}_W(-s, -t)$). Being bounded, it is therefore self-adjoint.

Now, for any $(s_1, t_1) \in \mathcal{C}^* \times \mathcal{P}^*$ and any $\varphi, \varphi' \in \mathcal{H}_W$, we have:

$$\begin{aligned}
\left\langle \varphi' \;\middle|\; \left[\mathrm{T}_W\{\Psi\}\, \mathrm{T}_W(s_1, t_1)\right] \varphi\right\rangle_{\mathcal{H}_W} &= \left\langle \varphi' \;\middle|\; \mathrm{T}_W\{\Psi\}\, \left[\mathrm{T}_W(s_1, t_1)\, \varphi\right]\right\rangle_{\mathcal{H}_W} \\
&= \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s, t)\, \Psi(s, t)\, \langle \varphi' \mid \mathrm{T}_W(s, t)\, \mathrm{T}_W(s_1, t_1)\, \varphi\rangle_{\mathcal{H}_W} \\
&= \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s, t)\, \Psi(s, t)\, e^{i\xi(s,t;s_1,t_1)}\, \langle \varphi' \mid \mathrm{T}_W(s + s_1, t + t_1)\, \varphi\rangle_{\mathcal{H}_W} \\
&= \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s, t)\, \Psi(s - s_1, t - t_1)\, e^{i\xi(s,t;s_1,t_1)}\, \langle \varphi' \mid \mathrm{T}_W(s, t)\, \varphi\rangle_{\mathcal{H}_W}\,,
\end{aligned}$$

where we have used def. C.3.2 and $\forall (s, t) \in \mathcal{C}^* \times \mathcal{P}^*$, $\xi(s - s_1, t - t_1; s_1, t_1) = \xi(s, t; s_1, t_1)$. Since $\mathrm{T}_W\{\Psi\}^\dagger = \mathrm{T}_W\{\Psi\}$ and $\mathrm{T}_W(s_1, t_1)^\dagger = \mathrm{T}_W(-s_1, -t_1)$, we also have:

$$\left\langle \varphi' \;\middle|\; \left[\mathrm{T}_W(s_1, t_1)\, \mathrm{T}_W\{\Psi\}\right] \varphi\right\rangle_{\mathcal{H}_W} = \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s, t)\, \Psi(s - s_1, t - t_1)\, e^{-i\xi(s,t;s_1,t_1)}\, \langle \varphi' \mid \mathrm{T}_W(s, t)\, \varphi\rangle_{\mathcal{H}_W}\,.$$

Applying these two formulas successively, we get, for any $(s_1, t_1) \in \mathcal{C}^* \times \mathcal{P}^*$ and any $\varphi, \varphi' \in \mathcal{H}_W$:

$$\begin{aligned}
&\left\langle \varphi' \;\middle|\; \left[\mathrm{T}_W\{\Psi\}\, \mathrm{T}_W(s_1, t_1)\, \mathrm{T}_W\{\Psi\}\right] \varphi\right\rangle = \\
&\quad = \left\langle \varphi' \;\middle|\; \mathrm{T}_W\{\Psi\}\, \mathrm{T}_W(s_1, t_1) \left[\mathrm{T}_W\{\Psi\}\, \varphi\right]\right\rangle \\
&\quad = \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s, t)\, \Psi(s - s_1, t - t_1)\, e^{i\xi(s,t;s_1,t_1)}\, \langle \varphi' \mid \mathrm{T}_W(s, t)\, \mathrm{T}_W\{\Psi\}\, \varphi\rangle_{\mathcal{H}_W} \\
&\quad = \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s, t) \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s', t')\, \Psi(s - s_1, t - t_1)\, \Psi(s' - s, t' - t)\, e^{i\xi(s,t;s_1+s',t_1+t')}\, \langle \varphi' \mid \mathrm{T}_W(s', t')\, \varphi\rangle_{\mathcal{H}_W}\,.
\end{aligned}$$

Invoking Fubini's theorem and performing the Gaussian integration, like in the proof of lemma C.11, we obtain:

$$\begin{aligned}
&\left\langle \varphi' \;\middle|\; \left[\mathrm{T}_W\{\Psi\}\, \mathrm{T}_W(s_1, t_1)\, \mathrm{T}_W\{\Psi\}\right] \varphi\right\rangle = \\
&\qquad = \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s', t')\, \exp\left(-\frac{(s_1, t_1 \mid s_1, t_1)}{4}\right)\, \Psi(s', t')\, \langle \varphi' \mid \mathrm{T}_W(s', t')\, \varphi\rangle_{\mathcal{H}_W}
\end{aligned}$$

$$= \exp\left(-\frac{(s_1, t_1 \mid s_1, t_1)}{4}\right) \langle \varphi' \mid \mathrm{T}_W \{\Psi\}\ \varphi\rangle_{\mathcal{H}_W}.$$

Since this holds for any $\varphi, \varphi' \in \mathcal{H}_W$, we have:

$$\forall (s,t) \in \mathcal{C}^* \times \mathcal{P}^*,\ \mathrm{T}_W \{\Psi\}\ \mathrm{T}_W(s,t)\, \mathrm{T}_W \{\Psi\} = \exp\left(-\frac{(s, t \mid s, t)}{4}\right) \mathrm{T}_W \{\Psi\}. \tag{C.10.1}$$

In particular, applying with $(s,t) = (0,0)$, and using that $\mathrm{T}_W(0,0) = \mathrm{id}_{\mathcal{H}_W}$, this implies:

$$\mathrm{T}_W \{\Psi\}\ \mathrm{T}_W \{\Psi\} = \mathrm{T}_W \{\Psi\}.$$

$\mathrm{T}_W \{\Psi\}$ being idempotent and self-adjoint, it is an orthogonal projection.

Next, we define $\psi_o \in \mathcal{H}_o$ by:

$$\forall x \in \mathcal{C},\ \psi_o(x) = \frac{1}{\pi^{p/4}} \exp\left(-\frac{(x \mid x)}{2}\right)$$

(choosing the Lebesgue measure μ, entering the definition of $\mathcal{H}_o$ in prop. C.5, to be normalized with respect to the scalar product $(\cdot \mid \cdot)$ on $\mathcal{C}$; this normalization can be chosen without loss of generality since the Hilbert spaces obtained using different normalizations of μ are unitarily identified in a natural way through a corresponding renormalization of the wave-functions). We can check that, for any $(s,t) \in \mathcal{C}^* \times \mathcal{P}^*$:

$$\langle \psi_o \mid \mathrm{T}_o(s,t)\, \psi_o\rangle_{\mathcal{H}_o} = \exp\left(-\frac{(s, t \mid s, t)}{4}\right). \tag{C.10.2}$$

Let $\left(\psi_W^{(a)}\right)_a$ be an orthonormal basis of the image of $\mathrm{T}_W \{\Psi\}$. Then, for any a, b and any (s,t), $(s',t') \in \mathcal{C}^* \times \mathcal{P}^*$, we have, using the hermiticity and idempotence of $\mathrm{T}_W \{\Psi\}$ together with the properties of the representations T_W and T_o:

$$\left\langle \mathrm{T}_W(s',t')\, \psi_W^{(b)} \,\middle|\, \mathrm{T}_W(s,t)\, \psi_W^{(a)} \right\rangle_{\mathcal{H}_W} =$$

$$= \left\langle \mathrm{T}_W \{\Psi\}\ \psi_W^{(b)} \,\middle|\, \mathrm{T}_W(-s',-t')\, \mathrm{T}_W(s,t)\, \mathrm{T}_W \{\Psi\}\ \psi_W^{(a)} \right\rangle_{\mathcal{H}_W}$$

$$= e^{i\xi(s,t;s',t')} \left\langle \psi_W^{(b)} \,\middle|\, \mathrm{T}_W \{\Psi\}\ \mathrm{T}_W(s-s', t-t')\, \mathrm{T}_W \{\Psi\}\ \psi_W^{(a)} \right\rangle_{\mathcal{H}_W}$$

$$= e^{i\xi(s,t;s',t')} \langle \psi_o \mid \mathrm{T}_o(s-s', t-t')\, \psi_o\rangle_{\mathcal{H}_o} \left\langle \psi_W^{(b)} \,\middle|\, \mathrm{T}_W \{\Psi\}\ \psi_W^{(a)} \right\rangle_{\mathcal{H}_W}$$

(combining eqs. (C.10.1) and (C.10.2))

$$= \delta_{ab} \langle \mathrm{T}_o(s',t')\, \psi_o \mid \mathrm{T}_o(s,t)\, \psi_o\rangle_{\mathcal{H}_o}.$$

For any a, we define $\mathcal{H}_W^{(a)} := \overline{\mathrm{Vect}\left\{\mathrm{T}_W(s,t)\, \psi_W^{(a)} \,\middle|\, (s,t) \in \mathcal{C}^* \times \mathcal{P}^*\right\}}$. Each $\mathcal{H}_W^{(a)}$ is thus stable under $\mathrm{T}_W \langle \mathcal{C}^* \times \mathcal{P}^* \rangle$. The previous computation shows that the $\mathcal{H}_W^{(a)}$ are mutually orthogonal and that there exists, for each a, an isometric injection $I^{(a)} : \mathcal{H}_W^{(a)} \to \mathcal{H}_o$ such that:

$$\forall \iota \in \mathcal{I},\ I^{(a)} \left[\mathrm{T}_W \{\iota\}\ \psi_W^{(a)}\right] = \mathrm{T}_o \{\iota\}\ \psi_o.$$

Then, using that:

$$\forall \iota_1, \iota_2 \in \mathcal{I},\ \mathrm{T}_W\{\iota_1\}\,\mathrm{T}_W\{\iota_2\} = \mathrm{T}_W\{\iota_1 \star \iota_2\} \quad \& \quad \mathrm{T}_o\{\iota_1\}\,\mathrm{T}_o\{\iota_2\} = \mathrm{T}_o\{\iota_1 \star \iota_2\},$$

together with the density of $\left\{\mathrm{T}_W\{\iota\}\,\psi_W^{(a)} \,\middle|\, \iota \in \mathcal{I}\right\} = \mathrm{Vect}\left\{\mathrm{T}_W(s,t)\,\psi_W^{(a)} \,\middle|\, (s,t) \in \mathcal{C}^* \times \mathcal{P}^*\right\}$ in $\mathcal{H}_W^{(a)}$, we get:

$$\forall \iota \in \mathcal{I},\ I^{(a)}\,\mathrm{T}_W\{\iota\} = \mathrm{T}_o\{\iota\}\,I^{(a)}.$$

Now, we define:

$$\mathcal{H}_W^{\mathrm{rest}} := \left(\bigoplus_a \mathcal{H}_W^{(a)}\right)^{\perp},$$

so that:

$$\mathcal{H}_W = \mathcal{H}_W^{\mathrm{rest}} \oplus \overline{\bigoplus_a \mathcal{H}_W^{(a)}}$$

(as can be seen as a consequence of Riesz lemma). Since each $\mathcal{H}_W^{(a)}$ is stable under $\mathrm{T}_W\langle \mathcal{C}^* \times \mathcal{P}^* \rangle$, so is $\mathcal{H}_W^{\mathrm{rest}}$. Next, we write:

$$\zeta_W = \zeta_W^{\mathrm{rest}} + \left(\sum_a \zeta_W^{(a)}\right),$$

with $\zeta_W^{\mathrm{rest}} \in \mathcal{H}_W^{\mathrm{rest}}$ and $\forall a,\ \zeta_W^{(a)} \in \mathcal{H}_W^{(a)}$. In particular, we have:

$$\|\zeta_W\|^2 = \left\|\zeta_W^{\mathrm{rest}}\right\|^2 + \sum_a \left\|\zeta_W^{(a)}\right\|^2,$$

so there can be only countably many non-zero $\zeta_W^{(a)}$ (actually, the cyclicity of ζ_W together with the stability of each $\mathcal{H}_W^{(a)}$ requires all $\zeta_W^{(a)}$ to be non-zero, so this even implies that a takes value in a countable set). Using prop. C.9.2, we have:

$$\begin{aligned}\forall (s,t) \in \mathcal{C}^* \times \mathcal{P}^*,\ W(s,t) &= \langle \zeta_W \mid \mathrm{T}_W(s,t)\,\zeta_W \rangle_{\mathcal{H}_W} \\ &= \left\langle \zeta_W^{\mathrm{rest}} \,\middle|\, \mathrm{T}_W(s,t)\,\zeta_W^{\mathrm{rest}} \right\rangle_{\mathcal{H}_W} + \sum_a \left\langle \zeta_W^{(a)} \,\middle|\, \mathrm{T}_W(s,t)\,\zeta_W^{(a)} \right\rangle_{\mathcal{H}_W}.\end{aligned}$$

Thus, defining $\zeta_o^{(a)} := I^{(a)}\,\zeta_W^{(a)} \in \mathcal{H}_o$ for each a, the isometric and intertwining properties of $I^{(a)}$ yield:

$$\forall (s,t) \in \mathcal{C}^* \times \mathcal{P}^*,\ W(s,t) = \left\langle \zeta_W^{\mathrm{rest}} \,\middle|\, \mathrm{T}_W(s,t)\,\zeta_W^{\mathrm{rest}} \right\rangle_{\mathcal{H}_W} + \sum_a \left\langle \zeta_o^{(a)} \,\middle|\, \mathrm{T}_o(s,t)\,\zeta_o^{(a)} \right\rangle_{\mathcal{H}_o}.$$

The bound:

$$\sum_a \left\|\zeta_o^{(a)}\right\|_{\mathcal{H}_o}^2 = \sum_a \left\|\zeta_W^{(a)}\right\|_{\mathcal{H}_W}^2 \leqslant \|\zeta_W\|^2 < \infty,$$

allows us to define a traceclass, (self-adjoint) positive semi-definite operator ρ on $\mathcal{H}_o$ by:

$$\rho := \sum_a \left|\zeta_o^{(a)}\right\rangle \left\langle \zeta_o^{(a)}\right|.$$

Then, we get:

$$\forall (s,t) \in \mathcal{C}^* \times \mathcal{P}^*,\ W(s,t) = W^{\text{rest}}(s,t) + \text{Tr}_{\mathcal{H}_o}\left[\rho \, \text{T}_o(s,t)\right] = W^{\text{rest}}(s,t) + W_\rho(s,t)\,,$$

where, for any $(s,t) \in \mathcal{C}^* \times \mathcal{P}^*$, $W^{\text{rest}}(s,t) := \left\langle \zeta_W^{\text{rest}} \,\middle|\, \text{T}_W(s,t)\, \zeta_W^{\text{rest}} \right\rangle_{\mathcal{H}_W}$.

T_W being a weakly continuous, unitary representation of $\mathcal{C}^* \times \mathcal{P}^*$ on $\mathcal{H}_W$, the function W^{rest} is continuous $\mathcal{C}^* \times \mathcal{P}^* \to \mathbb{C}$ and it is bounded by $\left\| \zeta_W^{\text{rest}} \right\|^2_{\mathcal{H}_W}$. Now, for any $(s',t') \in \mathcal{C}^* \times \mathcal{P}^*$, $\text{T}_W(s',t')\, \zeta_W^{\text{rest}} \in \mathcal{H}_W^{\text{rest}}$, so it is in particular orthogonal to each $\psi_W^{(a)}$, as $\psi_W^{(a)} \in \mathcal{H}_W^{(a)}$. But since $\left(\psi_W^{(a)}\right)_a$ is an orthonormal basis of the image of the orthogonal projection $\text{T}_W\left\{\Psi\right\}$, this implies:

$$\forall (s',t') \in \mathcal{C}^* \times \mathcal{P}^*,\ \text{T}_W\left\{\Psi\right\}\ \text{T}_W(s',t')\, \zeta_W^{\text{rest}} = 0\,,$$

and therefore:

$$\forall (s',t'),\ (s'',t'') \in \mathcal{C}^* \times \mathcal{P}^*,\ \left\langle \text{T}_W(s'',t'')\, \zeta_W^{\text{rest}} \,\middle|\, \text{T}_W\left\{\Psi\right\}\ \text{T}_W(s',t')\, \zeta_W^{\text{rest}} \right\rangle_{\mathcal{H}_W} = 0\,.$$

On the other hand, we have, for any $\varphi', \varphi'' \in \mathcal{H}_W$:

$$\begin{aligned}
&\forall (s',t'),\ (s'',t'') \in \mathcal{C}^* \times \mathcal{P}^*,\ \left\langle \text{T}_W(s'',t'')\, \varphi'' \mid \text{T}_W\left\{\Psi\right\}\ \text{T}_W(s',t')\, \varphi' \right\rangle_{\mathcal{H}_W} = \\
&\quad = \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s,t)\, \Psi(s,t)\, \left\langle \text{T}_W(s'',t'')\, \varphi'' \mid \text{T}_W(s,t)\, \text{T}_W(s',t')\, \varphi' \right\rangle_{\mathcal{H}_W} \\
&\quad = \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s,t)\, \Psi(s,t)\, e^{-i\xi(s'',t'';s,t)}\, e^{i\xi(s-s'',t-t'';s',t')}\, \left\langle \varphi'' \mid \text{T}_W(s+s'-s'', t+t'-t'')\, \varphi' \right\rangle_{\mathcal{H}_W} \\
&\quad = e^{-i\xi(s'',t'';s',t')} \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s,t)\, \Psi(s,t)\, e^{i\xi(s,t;s'+s'',t'+t'')}\, \left\langle \varphi'' \mid \text{T}_W(s+s'-s'', t+t'-t'')\, \varphi' \right\rangle_{\mathcal{H}_W}.
\end{aligned}$$

Thus, we have:

$$\forall (s_o,t_o),\ (s_1,t_1) \in \mathcal{C}^* \times \mathcal{P}^*,\ \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s,t)\, \Psi(s,t)\, e^{i\xi(s,t;s_1,t_1)}\, W^{\text{rest}}(s+s_o, t+t_o) = 0\,,$$

so, from lemma C.11, $W^{\text{rest}} \equiv 0$, hence $W = W_\rho$.

Uniqueness. Let ρ_1, ρ_2 be two traceclass, (self-adjoint) positive semi-definite operators on $\mathcal{H}_o$ such that $W_{\rho_1} = W_{\rho_2}$. Like above, we define a bounded linear operator $\text{T}_o\left\{\Psi\right\}$ on $\mathcal{H}_o$ satisfying, for any $\varphi_1, \varphi_2 \in \mathcal{H}_o$:

$$\left\langle \varphi_1 \mid \text{T}_o\left\{\Psi\right\}\ \varphi_2 \right\rangle_{\mathcal{H}_o} := \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s,t)\, \Psi(s,t)\, \left\langle \varphi_1 \mid \text{T}_o(s,t)\, \varphi_2 \right\rangle_{\mathcal{H}_o}.$$

Using the bases introduced in the proof of lemma C.11 together with the expression for $\text{T}_o(s,t)$ from prop. C.5, this can be rewritten as:

$$\begin{aligned}
&\left\langle \varphi_1 \mid \text{T}_o\left\{\Psi\right\}\ \varphi_2 \right\rangle_{\mathcal{H}_o} = \\
&\quad = \frac{1}{(2\pi)^p} \int ds^1 \ldots ds^p\, dt^1 \ldots dt^p \int dx_1^1 \ldots dx_1^p\ \overline{\varphi_1(x_1^i\, e_i)}\, \varphi_2\big((x_1^i + t^i)\, e_i\big) \times \\
&\qquad\quad \times \exp\left(-\frac{s^i\, s^i + t^i\, t^i}{4}\right)\ \exp\left(i\, s^i\, x_1^i + \frac{i}{2} t^i\, s^i\right) \\
&\quad = \frac{1}{(2\pi)^p} \int dx_1^1 \ldots dx_1^p\, dt^1 \ldots dt^p\ \overline{\varphi_1(x_1^i\, e_i)}\, \varphi_2\big((x_1^i + t^i)\, e_i\big)\ \exp\left(-\frac{t^i\, t^i}{4}\right) \times
\end{aligned}$$

$$\times \int ds^1 \dots ds^p \quad \exp\left(-\frac{s^i\, s^i}{4} + i\, s^i\, x_1^i + \frac{i}{2} t^i\, s^i\right)$$

(using Fubini's theorem as the integral is absolutely convergent)

$$= \frac{1}{\sqrt{\pi}^p} \int dx_1^1 \dots dx_1^p\, dt^1 \dots dt^p \;\; \overline{\varphi_1(x_1^i\, e_i)}\, \varphi_2\big((x_1^i + t^i)\, e_i\big) \;\exp\left(-\frac{t^i\, t^i}{4}\right) \times$$

$$\times \exp\left(-\frac{(2x_1^i + t^i)\,(2x_1^i + t^i)}{4}\right)$$

$$= \frac{1}{\sqrt{\pi}^p} \int dx_1^1 \dots dx_1^p\, dx_2^1 \dots dx_2^p \;\; \overline{\varphi_1(x_1^i\, e_i)}\, \varphi_2(x_2^i\, e_i) \;\exp\left(-\frac{x_1^i\, x_1^i + x_2^i\, x_2^i}{2}\right)$$

(with the change of variables $x_2^i := x_1^i + t^i$)

$$= \langle \varphi_1 \mid \psi_o \rangle_{\mathcal{H}_o} \; \langle \psi_o \mid \varphi_2 \rangle_{\mathcal{H}_o} .$$

Since this holds for any $\varphi_1, \varphi_2 \in \mathcal{H}_o$, we have $\mathrm{T}_o\,\{\Psi\} = |\psi_o\,\rangle\,\langle\,\psi_o|$.

Now, from the spectral theorem, there exist $N_1 \in \mathbb{N} \cup \{\infty\}$, an orthonormal family $\left(\varphi_k^{(1)}\right)_{k<N_1}$ in $\mathcal{H}_o$ and a family of strictly positive reals $\left(p_k^{(1)}\right)_{k<N_1}$ such that:

$$\rho_1 = \sum_{k<N_1} p_k^{(1)} \left|\varphi_k^{(1)}\,\right\rangle \left\langle\, \varphi_k^{(1)}\right| \quad \& \quad \sum_{k<N_1} p_k^{(1)} = \mathrm{Tr}_{\mathcal{H}_o}\, \rho_1 .$$

Thus, we get, for any (s_1, t_1), $(s_2, t_2) \in \mathcal{C}^* \times \mathcal{P}^*$:

$$\langle \mathrm{T}_o(s_1, t_1)\, \psi_o \mid \rho_1\, \mathrm{T}_o(s_2, t_2)\, \psi_o \rangle_{\mathcal{H}_o} =$$

$$= \sum_{k<N_1} p_k^{(1)} \left\langle \mathrm{T}_o(s_1, t_1)\, \psi_o \,\middle|\, \varphi_k^{(1)} \right\rangle \left\langle \varphi_k^{(1)} \,\middle|\, \mathrm{T}_o(s_2, t_2)\, \psi_o \right\rangle_{\mathcal{H}_o}$$

$$= \sum_{k<N_1} p_k^{(1)} \left\langle \varphi_k^{(1)} \,\middle|\, \mathrm{T}_o(s_2, t_2)\, \mathrm{T}_o\,\{\Psi\}\; \mathrm{T}_o(-s_1, -t_1)\, \varphi_k^{(1)} \right\rangle_{\mathcal{H}_o}$$

$$= \sum_{k<N_1} p_k^{(1)} \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s,t)\, \Psi(s,t)\, e^{-i\xi(s,t;\, s_1+s_2, t_1+t_2) + i\xi(s_1,t_1;\, s_2,t_2)} \left\langle \varphi_k^{(1)} \,\middle|\, \mathrm{T}_o(s + s_{2-s_1}, t + t_{2-t_1})\, \varphi_k^{(1)} \right\rangle_{\mathcal{H}_o}$$

$$= \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s,t)\, \Psi(s,t)\, e^{-i\xi(s,t;\, s_1+s_2, t_1+t_2) + i\xi(s_1,t_1;\, s_2,t_2)} \sum_{k<N_1} p_k^{(1)} \left\langle \varphi_k^{(1)} \,\middle|\, \mathrm{T}_o(s + s_{2-s_1}, t + t_{2-t_1})\, \varphi_k^{(1)} \right\rangle_{\mathcal{H}_o}$$

(using Fubini's theorem with the discrete measure on $\{k < N\}$)

$$= \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s,t)\, \Psi(s,t)\, e^{-i\xi(s,t;\, s_1+s_2, t_1+t_2) + i\xi(s_1,t_1;\, s_2,t_2)}\, \mathrm{Tr}_{\mathcal{H}_o} \left[\rho_1\, \mathrm{T}_o(s + s_{2-s_1}, t + t_{2-t_1})\right]$$

$$= e^{i\xi(s_1,t_1;\, s_2,t_2)} \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s,t)\, \Psi(s,t)\, e^{-i\xi(s,t;\, s_1+s_2, t_1+t_2)}\, W_{\rho_1}(s + s_{2-s_1}, t + t_{2-t_1}) .$$

As the same holds for ρ_2, $W_{\rho_1} = W_{\rho_2}$ implies:

$$\forall \varphi, \varphi' \in \mathcal{H}_o^{(0)},\; \langle \varphi' \mid \rho_1\, \varphi \rangle_{\mathcal{H}_o} = \langle \varphi' \mid \rho_2\, \varphi \rangle_{\mathcal{H}_o} ,$$

where $\mathcal{H}_o^{(0)} := \overline{\mathrm{Vect}\, \{\mathrm{T}_o(s,t)\, \psi_o \mid (s,t) \in \mathcal{C}^* \times \mathcal{P}^*\}}$.

Finally, let $\varphi \in \left[\mathcal{H}_o^{(0)}\right]^{\perp}$. Like above, we then have, for any (s', t'), $(s'', t'') \in \mathcal{C}^* \times \mathcal{P}^*$:

$$0 = \langle \mathsf{T}_o(s'', t'')\, \psi_o \mid \varphi \rangle_{\mathcal{H}_o} \, \langle \varphi \mid \mathsf{T}_o(s', t')\, \psi_o \rangle_{\mathcal{H}_o}$$
$$= e^{-i\xi(s',t';\, s'',t'')} \int_{\mathcal{C}^* \times \mathcal{P}^*} d\widetilde{\mu}(s, t)\, \Psi(s, t)\, e^{-i\xi(s,t;\, s'+s'',t'+t'')} \, \langle \varphi \mid \mathsf{T}_o(s + s' - s'', t + t' - t'')\, \varphi \rangle_{\mathcal{H}_o} \,,$$

so applying lemma C.11 to the bounded, continuous function $(s, t) \mapsto \langle \varphi \mid \mathsf{T}_o(s, t)\, \varphi \rangle_{\mathcal{H}_o}$ yields:

$$\forall (s, t) \in \mathcal{C}^* \times \mathcal{P}^*,\ \langle \varphi \mid \mathsf{T}_o(s, t)\, \varphi \rangle_{\mathcal{H}_o} = 0,$$

and in particular $\|\varphi\|^2 = \langle \varphi \mid \mathsf{T}_o(0, 0)\, \varphi \rangle_{\mathcal{H}_o} = 0$, so $\varphi = 0$. Hence, $\mathcal{H}_o^{(0)} = \mathcal{H}_o$ and, therefore, $\rho_1 = \rho_2$. □

References

[1] S. Twareque Ali, Natig M. Atakishiyev, Sergey M. Chumakov, and Kurt B. Wolf. The Wigner Function for General Lie Groups and the Wavelet Transform. *Annales Henri Poincaré*, **1**(4):685–714, 2000.

[2] Jan Ambjorn, Andrzej Görlich, Jerzy Jurkiewicz, and Renate Loll. Nonperturbative Quantum Gravity. *Physics Reports*, **519**:127–210, 2012. [arXiv: 1203.3591]

[3] Abhay Ashtekar. New Variables for Classical and Quantum Gravity. *Phys. Rev. Lett.*, **57**:2244–2247, November 1986.
Abhay Ashtekar. New Hamiltonian Formulation of General Relativity. *Phys. Rev. D*, **36**:1587–1602, 1987.

[4] Abhay Ashtekar. *Lectures on Non-perturbative Canonical Gravity*. Number 6 in Advanced Series in Astrophysics and Cosmology. World Scientific, 1991. Notes prepared in collaboration with Ranjeet S. Tate.

[5] Abhay Ashtekar and Martin Bojowald. Quantum geometry and the Schwarzschild Singularity. *Class. Quant. Grav.*, **23**:391–411, 2006. [arXiv: gr-qc/0509075]

[6] Abhay Ashtekar and Chris Isham. Representations of the Holonomy Algebras of Gravity and non-Abelian Gauge Theories. *Class. Quant. Grav.*, **9**:1433, 1992. [arXiv: hep-th/9202053]

[7] Abhay Ashtekar and Jerzy Lewandowski. Quantum Theory of Gravity I: Area Operators. *Class. Quant. Grav.*, **14**:A55–A82, 1997. [arXiv: gr-qc/9602046]
Abhay Ashtekar and Jerzy Lewandowski. Quantum Theory of Geometry II: Volume Operators. *Adv. Theor. Math. Phys.*, **1**:388–429, 1998. [arXiv: gr-qc/9711031]

[8] Abhay Ashtekar, Jerzy Lewandowski, Donald Marolf, Jose Mourao, and Thomas Thiemann. Quantization of Diffeomorphism Invariant Theories of Connections with Local Degrees of Freedom. *J. Math. Phys.*, **36**:6456–6493, 1995. [arXiv: gr-qc/9504018]

[9] John C. Baez. An Introduction to Spin Foam Models of Quantum Gravity and BF Theory. *Lect. Notes Phys.*, **543**:25–94, 2000. [arXiv: gr-qc/9905087]
John C. Baez. Spin Foam Models. *Class. Quant. Grav.*, **15**:1827–1858, 1998. [arXiv: gr-qc/9709052]

[10] John C. Baez and Stephen Sawin. Functional Integration on Spaces of Connections. *J. Funct. Anal.*, **150**:1–26, 1997. [arXiv: q-alg/9507023]
Jose A. Zapata. Combinatorial Space from Loop Quantum Gravity. *Gen. Rel. Grav.*, **30**:1229–1245, 1998. [arXiv: gr-qc/9703038]

[11] Aristide Baratin, Bianca Dittrich, Daniele Oriti, and Johannes Tambornino. Non-Commutative Flux Representation for Loop Quantum Gravity. *Class. Quant. Grav.*, **28**:175011, 2011. [arXiv: 1004.3450]
Bianca Dittrich, Carlos Guedes, and Daniele Oriti. On the Space of Generalized Fluxes for Loop Quantum Gravity. *Class. Quant. Grav.*, **30**:055008, 2013. [arXiv: 1205.6166]

[12] Fernando Barbero. Real Ashtekar Variables for Lorentzian Signature Space-times. *Phys.*

Rev. D, **51**:5507–5510, 1996. [arXiv: gr-qc/9410014]

[13] Norbert Bodendorfer and Yasha Neiman. Imaginary Action, Spinfoam Asymptotics and the 'Transplanckian' Regime of Loop Quantum Gravity. *Class. Quant. Grav.*, **30**:195018, 2013.
Muxin Han. Black Hole Entropy in Loop Quantum Gravity, Analytic Continuation, and Dual Holography. Preprint, 2014. [arXiv: 1402.2084]
Jibril Ben Achour and Karim Noui. Analytic Continuation of Real Loop Quantum Gravity : Lessons from Black Hole Thermodynamics. Preprint, 2015. [arXiv: 1501.05523]

[14] Norbert Bodendorfer, Alexander Stottmeister, and Andreas Thurn. On a Partially Reduced Phase Space Quantisation of General Relativity Conformally Coupled to a Scalar Field. *Class. Quant. Grav.*, **30**:115017, 2013. [arXiv: 1203.6526]

[15] Norbert Bodendorfer, Thomas Thiemann, and Andreas Thurn. New Variables for Classical and Quantum Gravity in all Dimensions I. Hamiltonian Analysis. *Class. Quant. Grav.*, **30**:045001, 2013. [arXiv: 1105.3703]
Norbert Bodendorfer, Thomas Thiemann, and Andreas Thurn. New Variables for Classical and Quantum Gravity in all Dimensions II. Lagrangian Analysis. *Class. Quant. Grav.*, **30**:045002, 2013. [arXiv: 1105.3704]

[16] Martin Bojowald. Loop Quantum Cosmology I: Kinematics. *Class. Quant. Grav.*, **17**:1489–1508, 2000. [arXiv: gr-qc/9910103]
Martin Bojowald. Loop Quantum Cosmology IV: Discrete Time Evolution. *Class. Quant. Grav.*, **18**:1071–1088, 2001. [arXiv: gr-qc/0008053]
Abhay Ashtekar, Martin Bojowald, and Jerzy Lewandowski. Mathematical Structure of Loop Quantum Cosmology. *Adv. Theor. Math. Phys.*, **7**:233–268, 2003. [arXiv: gr-qc/0304074]
Abhay Ashtekar, Tomasz Pawlowski, and Parampreet Singh. Quantum nature of the big bang. *Phys. Rev. Lett.*, **96**:141301, 2006. [arXiv: gr-qc/0602086]

[17] Raoul Bott and Loring W. Tu. *Differential Forms in Algebraic Topology*. Number 82 in Graduate texts in mathematics. Springer, first edition, 1982.

[18] Miguel Campiglia, Rodolfo Gambini, and Jorge Pullin. Loop Quantization of Spherically Symmetric Midi-Superspaces. *Class. Quant. Grav.*, **24**:3649–3672, 2007. [arXiv: gr-qc/0703135]
Rodolfo Gambini, Javier Olmedo, and Jorge Pullin. Quantum Black Holes in Loop Quantum Gravity. Preprint, 2013. [arXiv: 1310.5996]

[19] Abhay Ashtekar and Jerzy Lewandowski. Representation Theory of Analytic Holonomy C^*-Algebras. In John Baez, editor, *Knots and Quantum Gravity*. Oxford University Press, 1994. [arXiv: gr-qc/9311010]

[20] Yvonne Choquet-Bruhat and Cecile DeWitt-Morette. *Analysis, Manifolds and Physics*, volume 1. Elsevier, second edition, 1982.

[21] Paul A. M. Dirac. The Quantum Theory of the Emission and Absorption of Radiation. *Proc. Roy. Soc.*, **A114**:243, 1927.

[22] Paul A. M. Dirac. *Lectures on quantum mechanics*. Number 2 in Belfer Graduate School of Science Monographs. Academic Press, 1964.

[23] Bianca Dittrich and Marc Geiller. Flux Formulation of Loop Quantum Gravity: Classical Framework. Preprint, 2014. [arXiv: 1412.3752]

[24] Fay Dowker and Adrian Kent. Properties of Consistent Histories. *Phys. Rev. Lett.*, **75**:3038–3041, 1995. [arXiv: gr-qc/9409037]

[25] James Dugundji. *Topology*. Allyn and Bacon Series in Advanced Mathematics. Allyn and Bacon, 1966.

[26] Nelson Dunford and Jacob T. Schwartz. *Linear Operators – Spectral Theory*, volume 2. Wiley, 1988.

[27] Jonathan Engle. Relating Loop Quantum Cosmology to Loop Quantum Gravity: Symmetric Sectors and Embeddings. *Class. Quant. Grav.*, **24**:5777–5802, 2007. [arXiv: gr-qc/0701132]

[28] Winston Fairbairn and Carlo Rovelli. Separable Hilbert space in Loop Quantum Gravity. *J. Math. Phys.*, **45**:2802–2814, 2004. [arXiv: gr-qc/0403047]
J. Fernando Barbero G., Tomasz Pawłowski, and Eduardo J. S. Villaseñor. Separable Hilbert Space for Loop Quantization. *Phys. Rev. D*, **90**:067505, 2014. [arXiv: 1403.2974]

[29] James M. G. Fell. The Dual Spaces of C^*-Algebras. *Trans. Amer. Math. Soc.*, **94**(3):365–403, 1960.

[30] Christian Fleischhack. Loop Quantization and Symmetry: Configuration Spaces. Preprint, 2010. [arXiv: 1010.0449]
Maximilian Hanusch. Invariant Connections in Loop Quantum Gravity. Preprint, 2013. [arXiv: 1307.5303]
Maximilian Hanusch. Projective Structures in Loop Quantum Cosmology. Preprint, 2013. [arXiv: 1309.0713]
Maximilian Hanusch. A Characterization of Invariant Connections. Preprint, 2013. [arXiv: 1310.0318]

[31] Gerald B. Folland. *Real Analysis – Modern Techniques and Their Applications*. Pure and Applied Mathematics. Wiley-Interscience, second edition, 1999.

[32] Laurent Freidel and Etera R. Livine. Ponzano-Regge Model Revisited III: Feynman Diagrams and Effective Field Theory. *Class. Quant. Grav.*, **23**:2021–2062, 2006. [arXiv: hep-th/0502106]

[33] Laurent Freidel and Jonathan Ziprick. Spinning Geometry = Twisted Geometry. *Class. Quant. Grav.*, **31**:045007, 2014. [arXiv: 1308.0040]

[34] Israel Gelfand and Mark Naimark. On the Imbedding of Normed Rings into the Ring of Operators in Hilbert Space. *Mat. Sb., Nov. Ser.*, **12**:197–213, 1943.
Irving Segal. Irreducible Representations of Operator Algebras. *Bull. Amer. Math. Soc.*, **53**:73–88, 1947.

[35] Kristina Giesel, Stefan Hofmann, Thomas Thiemann, and Olivier Winkler. Manifestly Gauge-Invariant General Relativistic Perturbation Theory: I. Foundations. *Class. Quant. Grav.*, **27**:055005, 2010. [arXiv: 0711.0115]
Kristina Giesel and Thomas Thiemann. Algebraic Quantum Gravity (AQG). IV. Reduced Phase Space Quantisation of Loop Quantum Gravity. *Class. Quant. Grav.*, **27**:175009, 2010. [arXiv: 0711.0119]
Kristina Giesel and Thomas Thiemann. Scalar Material Reference Systems and Loop Quantum Gravity. Preprint, 2012. [arXiv: 1206.3807]

[36] Kristina Giesel and Thomas Thiemann. Algebraic Quantum Gravity (AQG) I. Conceptual Setup. *Class. Quant. Grav.*, **24**:2465–2498, 2007. [arXiv: gr-qc/0607099]

Kristina Giesel and Thomas Thiemann. Algebraic Quantum Gravity (AQG) II. Semiclassical Analysis. *Class. Quant. Grav.*, **24**:2499–2564, 2007. [arXiv: gr-qc/0607100]

Kristina Giesel and Thomas Thiemann. Algebraic Quantum Gravity (AQG) III. Semiclassical Perturbation Theory. *Class. Quant. Grav.*, **24**:2565–2588, 2007. [arXiv: gr-qc/0607101]

[37] Domenico Giulini and Donald Marolf. On the Generality of Refined Algebraic Quantization. *Class. Quant. Grav.*, **16**:2479–2488, 1999. [arXiv: gr-qc/9812024]

Domenico Giulini and Donald Marolf. A Uniqueness Theorem for Constraint Quantization. *Class. Quant. Grav.*, **16**:2489–2505, 1999. [arXiv: gr-qc/9902045]

[38] Boris W. Gnedenko and Andrei N. Kolmogorow. *Limit Distributions for Sums of Independent Random Variables*. Addison-Wesley Mathematics Series. Addison-Wesley, 1954.

[39] Mark J. Gotay, Hendrik Grundling, and C. A. Hurst. A Groenewold-Van Hove Theorem for S^2. *Trans. A.M.S.*, **348**:1579–1597, 1996. [arXiv: dg-ga/9502008]

Mark J. Gotay. On a Full Quantization of the Torus. In J.-P. Antoine et al., editor, *Quantization, Coherent States, and Complex Structures*, pages 55–62. Plenum Press, 1995. [arXiv: dg-ga/9507005]

Mark J. Gotay. Obstructions to Quantization. In *Mechanics: From Theory to Computation (Essays in Honor of Juan-Carlos Simo)*, pages 171–216. Springer, 2000. [arXiv: math-ph/9809011]

[40] Hilbrand J. Groenewold. On the Principles of Elementary Quantum Mechanics. *Physica*, **12**(7):405–460, 1946.

Léon Van Hove. Sur le Problème des Relations entre les Transformations Unitaires de la Mécanique Quantique et les Transformations Canoniques de la Mécanique Classique. *Acad. Roy. Belgique Bull. Cl. Sci.*, **37**:610–620, 1951.

Léon Van Hove. Sur Certaines Représentations Unitaires d'un Groupe Infini de Transformations. *Mem. Acad. Roy. Belgique Cl. Sci.*, **26**:61–102, 1951.

[41] Rudolf Haag. *Local Quantum Physics – Fields, Particles, Algebras*. Texts and Monographs in Physics. Springer, second edition, 1996.

[42] Brian C. Hall. The Segal-Bargmann "Coherent State" Transform for Compact Lie Groups. *J. Funct. Anal.*, **122**:103–151, 1994.

[43] Brian C. Hall. *Lie Groups, Lie Algebras, and Representations – An Elementary Introduction*. Number 222 in Graduate Texts in Mathematics. Springer, first edition, 2003.

[44] Hakan Hedenmal, Boris Korenblum, and Kehe Zhu. *Theory of Bergman Spaces*. Number 199 in Graduate Texts in Mathematics. Springer, first edition, 2000.

[45] Einar Hille and Ralph S. Phillips. *Functional Analysis and Semi-groups*. Number 31 in American Mathematical Society: Colloquium publications. American Mathematical Society, 1996.

[46] Chris J. Isham. *Modern Differential Geometry for Physicists*. Number 61 in Lecture Notes in Physics. World Scientific, second edition, 1999.

[47] Daniel Kastler. The C^*-Algebras of a Free Boson Field. I. Discussion of the Basic Facts. *Comm. Math. Phys.*, **1**(1):14–48, 1965.

[48] Jerzy Kijowski. Symplectic Geometry and Second Quantization. *Rep. Math. Phys.*, **11**:97–109,

1977.

[49] William D. Kirwin, José M. Mourão, and João P. Nunes. Complex Time Evolution in Geometric Quantization and Generalized Coherent State Transforms. Preprint, 2012. [arXiv: 1203.4767]
William D. Kirwin, José M. Mourão, and João P. Nunes. Coherent State Transforms and the Mackey-Stone-Von Neumann Theorem. Preprint, 2012. [arXiv: 1211.2145]
João N. Esteves, José M. Mourão, and João P. Nunes. Quantization in Singular Real Polarizations: Kähler Regularization, Maslov Correction and Pairings. Preprint, 2015. [arXiv: 1501.00112]

[50] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of Differential Geometry*, volume 1. Interscience Publishers, first edition, 1963.
Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of Differential Geometry*, volume 2. Interscience Publishers, first edition, 1969.

[51] Tim Koslowski and Hanno Sahlmann. Loop Quantum Gravity Vacuum with Nondegenerate Geometry. *SIGMA*, **8**(026), 2012. [arXiv: 1109.4688]
Alexander Stottmeister and Thomas Thiemann. Structural Aspects of Loop Quantum Gravity and Loop Quantum Cosmology from an Algebraic Perspective. Preprint, 2013. [arXiv: 1312.3657]
Miguel Campiglia and Madhavan Varadarajan. The Koslowski-Sahlmann Representation: Quantum Configuration Space. *Class. Quant. Grav.*, **31**:175009, 2014. [arXiv: 1406.0579]

[52] Karel Kuchař. Parametrized Scalar Field on $\mathbb{R} \times S^1$: Dynamical Pictures, Spacetime Diffeomorphisms, and Conformal Isometries. *Phys. Rev. D*, **39**:1579–1593, 1989.
Karel Kuchař. Dirac Constraint Quantization of a Parametrized Field Theory by Anomaly-free Operator Representations of Spacetime Diffeomorphisms. *Phys. Rev. D*, **39**:2263–2280, 1989.

[53] Suzanne Lanéry and Thomas Thiemann. Projective Limits of State Spaces I. Classical Formalism. Preprint, 2014. [arXiv: 1411.3589]
Suzanne Lanéry and Thomas Thiemann. Projective Limits of State Spaces II. Quantum Formalism. Preprint, 2014. [arXiv: 1411.3590]
Suzanne Lanéry and Thomas Thiemann. Projective Limits of State Spaces III. Toy-Models. Preprint, 2014. [arXiv: 1411.3591]
Suzanne Lanéry and Thomas Thiemann. Projective Loop Quantum Gravity I. State Space. Preprint. [arXiv: 1411.3592]

[54] John M. Lee. *Introduction to Smooth Manifolds*. Number 218 in Graduate Texts in Mathematics. Springer, second edition, 2012.

[55] Jerzy Lewandowski, Andrzej Okołów, Hanno Sahlmann, and Thomas Thiemann. Uniqueness of Diffeomorphism Invariant States on Holonomy-Flux Algebras. *Commun. Math. Phys.*, **267**:703–733, 2006. [arXiv: gr-qc/0504147]
Christian Fleischhack. Representations of the Weyl Algebra in Quantum Geometry. *Commun. Math. Phys.*, **285**:67–140, 2009. [arXiv: math-ph/0407006]

[56] Etera R. Livine and Daniel R. Terno. Reconstructing Quantum Geometry from Quantum Information: Area Renormalisation, Coarse-Graining and Entanglement on Spin Networks. Preprint, 2006. [arXiv: gr-qc/0603008]

Seramika Ariwahjoedi, Jusak Sali Kosasih, Carlo Rovelli, and Freddy P. Zen. How Many Quanta are There in a Quantum Spacetime? Preprint, 2014. [arXiv: 1404.1750]

[57] F. K. Manasse and Charles W. Misner. Fermi Normal Coordinates and Some Basic Concepts in Differential Geometry. *J. Math. Phys.*, **4**:735, 1963.

[58] Paweł Duch, Wojciech Kamiński, Jerzy Lewandowski, and Jedrzej Świeżewski. Observables for General Relativity Related to Geometry. *JHEP*, **05**:077, 2014. [arXiv: 1403.8062]

[59] Elisa Manrique, Robert Oeckl, Axel Weber, and José A. Zapata. Loop Quantization as a Continuum Limit. *Class. Quant. Grav.*, **23**:3393–3404, 2006. [arXiv: hep-th/0511222]
Benjamin Bahr, Rodolfo Gambini, and Jorge Pullin. Discretisations, Constraints and Diffeomorphisms in Quantum Gravity. *SIGMA*, **8**(002), 2012. [arXiv: 1111.1879]
Bianca Dittrich. From the Discrete to the Continuous - Towards a Cylindrically Consistent Dynamics. *New J. Phys.*, **14**:123004, 2012. [arXiv: 1205.6127]
Bianca Dittrich and Sebastian Steinhaus. Time Evolution as Refining, Coarse Graining and Entangling. *New J. Phys.*, **16**:123041, 2014. [arXiv: 1311.7565]
Bianca Dittrich. The Continuum Limit of Loop Quantum Gravity - a Framework for Solving the Theory. Preprint, 2014. [arXiv: 1409.1450]

[60] Donald Marolf and Jose M. Mourao. On the Support of the Ashtekar-Lewandowski Measure. *Commun. Math. Phys.*, **170**:583–606, 1995. [arXiv: hep-th/9403112]
Abhay Ashtekar and Jerzy Lewandowski. Projective Techniques and Functional Integration for Gauge Theories. *J. Math. Phys.*, **36**:2170–2191, 1995. [arXiv: gr-qc/9411046]
Abhay Ashtekar and Jerzy Lewandowski. Differential Geometry on the Space of Connections via Graphs and Projective Limits. *J. Geom. Phys.*, **17**:191–230, 1995. [arXiv: hep-th/9412073]

[61] Gustav F. Mehler. Ueber die Entwicklung einer Function von beliebig vielen Variabeln nach Laplaceschen Functionen höherer Ordnung. *Journal für Reine und Angewandte Mathematik*, **66**:161–176, 1866.
David Slepian. On the Symmetrized Kronecker Power of a Matrix and Extensions of Mehler's Formula for Hermite Polynomials. *SIAM Journal on Mathematical Analysis*, **3**:606–616, 1972.

[62] P. Mitra and R. Rajaraman. Gauge-Invariant Reformulation of an Anomalous Gauge Theory. *Physics Letters B*, **225**:267–271, 1989.
Marc Henneaux and Claudio Teitelboim. *Quantization of Gauge Systems*. Princeton University Press, 1994.

[63] Robert Oeckl. A Positive Formalism for Quantum Theory in the General Boundary Formulation. *Found. Phys.*, **43**:1206–1232, 2013. [arXiv: 1212.5571]

[64] Robert Oeckl. Towards State Locality in Quantum Field Theory: Free Fermions. Preprint, 2013. [arXiv: 1307.5031]

[65] Andrzej Okołów. Hilbert Space Built over Connections with a Non-compact Structure Group. *Class. Quant. Grav.*, **22**:1329–1360, 2005. [arXiv: gr-qc/0406028]

[66] Andrzej Okołów. Quantization of Diffeomorphism Invariant Theories of Connections with a Non-compact Structure Group – an example. *Comm. Math. Phys.*, **289**:335–382, 2009. [arXiv: gr-qc/0605138]

[67] Andrzej Okołów. ADM-like Hamiltonian Formulation of Gravity in the Teleparallel Geome-

try. *Gen. Rel. Grav.*, **45**:2569–2610, 2013. [arXiv: 1111.5498]
Andrzej Okołów. ADM-like Hamiltonian Formulation of Gravity in the Teleparallel Geometry: Derivation of Constraint Algebra. Preprint, 2013. [arXiv: 1309.4685]

[68] Andrzej Okołów. Construction of Spaces of Kinematic Quantum States for Field Theories via Projective Techniques. *Class. Quant. Grav.*, **30**:195003, 2013. [arXiv: 1304.6330]

[69] Andrzej Okołów. Kinematic Quantum States for the Teleparallel Equivalent of General Relativity. *Gen. Rel. Grav.*, **46**:1653, 2014. [arXiv: 1304.6492]

[70] Richard Arnowitt, Stanley Deser, and Charles W. Misner. The Dynamics of General Relativity. In Louis Witten, editor, *Gravitation: an Introduction to Current Research*, chapter 7, pages 227–265. Wiley, 1962. [arXiv: gr-qc/0405109]

[71] Daniele Oriti, Roberto Pereira, and Lorenzo Sindoni. Coherent States for Quantum Gravity: Towards Collective Variables. *Class. Quant. Grav.*, **29**:135002, 2012. [arXiv: 1202.0526]

[72] Richard S. Palais. *Foundations of Global Non-linear Analysis*. Benjamin, 1968.
Walter Rudin. *Functional Analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, second edition, 1991.

[73] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics – Functional Analysis*, volume 1. Academic Press, revised and enlarged edition, 1980.

[74] Paul Renteln and Lee Smolin. A Lattice Approach to Spinorial Quantum Gravity. *Class. Quant. Grav.*, **6**:275, 1989.
Paul Renteln. Some Results of $SU(2)$ Spinorial Lattice Gravity. *Class. Quant. Grav.*, **7**:493, 1990.
Rodolfo Gambini and Jorge Pullin. A Rigorous Solution of the Quantum Einstein Equations. *Phys. Rev. D*, **54**:5935–5938, 1996. [arXiv: gr-qc/9511042]
Hugo Fort, Rodolfo Gambini, and Jorge Pullin. Lattice Knot Theory and Quantum Gravity in the Loop Representation. *Phys. Rev. D*, **56**:2127–2143, 1997. [arXiv: gr-qc/9608033]
Alejandro Corichi and Jose A. Zapata. On Diffeomorphism Invariance for Lattice Theories. *Nucl. Phys. B*, **493**:475–490, 1997. [arXiv: gr-qc/9611034]
Renate Loll. On the Diffeomorphism Commutators of Lattice Quantum Gravity. *Class. Quant. Grav.*, **15**:799–809, 1998. [arXiv: gr-qc/9708025]

[75] Carlo Rovelli. *Quantum Gravity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, second edition, 2007.

[76] Carlo Rovelli. Zakopane Lectures on Loop Gravity. Preprint, 2011. [arXiv: 1102.3660]

[77] Carlo Rovelli, Daniele Colosi, Luisa Doplicher, Winston Fairbairn, Leonardo Modesto, and Karim Noui. Background Independence in a Nutshell. *Class. Quant. Grav.*, **22**:2971–2990, 2005. [arXiv: gr-qc/0408079]
Abhay Ashtekar and Jerzy Lewandowski. Background Independent Quantum Gravity: a Status Report. *Classical and Quantum Gravity*, **21**:53, August 2004. [arXiv: gr-qc/0404018]

[78] Carlo Rovelli and Lee Smolin. Discreteness of Area and Volume in Quantum Gravity. *Nucl. Phys. B*, **442**:593–622, 1995. Erratum: Nucl. Phys. B 456 (1995) 753. [arXiv: gr-qc/9411005]
Renate Loll. The Volume Operator in Discretized Quantum Gravity. *Phys. Rev. Lett.*, **75**:3048–3051, 1995. [arXiv: gr-qc/9506014]

[79] Carlo Rovelli and Lee Smolin. Spin Networks and Quantum Gravity. *Phys. Rev. D*, **52**:5743–5759, 1995. [arXiv: gr-qc/9505006]
John C. Baez. Spin Network States in Gauge Theory. *Adv. Math.*, **117**:253–272, 1996. [arXiv: gr-qc/9411007]
John C. Baez. Spin Networks in Nonperturbative Quantum Gravity. In Louis Kauffman, editor, *The Interface of Knots and Physics*, pages 167–203. A.M.S., Providence, 1996. [arXiv: gr-qc/9504036]
Thomas Thiemann. The Inverse Loop Transform. *J. Math. Phys.*, **39**:1236–1248, 1998. [arXiv: hep-th/9601105]

[80] Hanno Sahlmann. When Do Measures on the Space of Connections Support the Triad Operators of Loop Quantum Gravity? *J. Math. Phys.*, **52**:012503, 2011. [arXiv: gr-qc/0207112]
Madhavan Varadarajan. The Graviton Vacuum as a Distributional State in Kinematic Loop Quantum Gravity. *Class. Quant. Grav.*, **22**:1207–1238, 2005. [arXiv: gr-qc/0410120]

[81] Arthur Sard. The Measure of the Critical Values of Differentiable Maps. *Bull. Amer. Math. Soc.*, **48**:883–890, 1942.

[82] Robert Schatten. *Norm Ideals of Completely Continuous Operators*. Number 27 in Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, 1960.

[83] Lee Smolin. Linking Topological Quantum Field Theory and Nonperturbative Quantum Gravity. *J. Math. Phys.*, **36**:6417–6455, November 1995. [arXiv: gr-qc/9505028]
Marcelo Barreira, Mauro Carfora, and Carlo Rovelli. Physics with Nonperturbative Quantum Gravity: Radiation from a Quantum Black Hole. *Gen. Rel. Grav.*, **28**:1293–1299, 1996. [arXiv: gr-qc/9603064]
Kirill V. Krasnov. On Quantum Statistical Mechanics of a Schwarzschild Black Hole. *Gen. Rel. Grav.*, **30**:53–68, 1998. [arXiv: gr-qc/9605047]

[84] Jean-Marie Souriau. *Structure des Systèmes Dynamiques*. Dunod, 1970.

[85] Alexander Stottmeister and Thomas Thiemann. Coherent States, Quantum Gravity and the Born-Oppenheimer Approximation, III: Applications to Loop Quantum Gravity. Preprint, 2015. [arXiv: 1504.02171]

[86] Zirô Takeda. Inductive Limit and Infinite Direct Product of Operator Algebras. *Tohoku Math. J. (2)*, **7**(1-2):67–86, 1955.

[87] Masamichi Takesaki. *Theory of Operator Algebras I*. Number 124 in Encyclopaedia of Mathematical Sciences. Springer, 2nd printing of the 1st edition, 2002.

[88] Thomas Thiemann. Quantum Spin Dynamics (QSD). *Class. Quant. Grav.*, **15**:839–873, 1998. [arXiv: gr-qc/9606089]

[89] Thomas Thiemann. Quantum Spin Dynamics (QSD): VII. Symplectic Structures and Continuum Lattice Formulations of Gauge Field Theories. *Class. Quant. Grav.*, **18**:3293–3338, 2001. [arXiv: hep-th/0005232]

[90] Thomas Thiemann. Complexifier Coherent States for Quantum General Relativity. *Class. Quant. Grav.*, **23**:2063–2118, 2006. [arXiv: gr-qc/0206037]

[91] Thomas Thiemann. The Phoenix Project: Master Constraint Programme for Loop Quantum Gravity. *Class. Quant. Grav.*, **23**:2211–2248, 2006. [arXiv: gr-qc/0305080]

Thomas Thiemann. Quantum Spin Dynamics VIII. The Master Constraint. *Class. Quant. Grav.*, **23**:2249–2266, 2006. [arXiv: gr-qc/0510011]

Muxin Han and Yongge Ma. Master Constraint Operator in Loop Quantum Gravity. *Phys. Lett. B*, **635**:225–231, 2006. [arXiv: gr-qc/0510014]

[92] Thomas Thiemann. *Modern Canonical Quantum General Relativity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2007.

[93] Thomas Thiemann and Oliver Winkler. Gauge Field Theory Coherent States (GCS): IV. Infinite Tensor Product and Thermodynamical Limit. *Class. Quant. Grav.*, **18**:4997–5054, 2001. [arXiv: hep-th/0005235]

[94] Thomas Thiemann and Antonia Zipfel. Linking Covariant and Canonical LQG II: Spin Foam Projector. Preprint, 2013. [arXiv: 1307.5885]

[95] Casey Tomlin and Madhavan Varadarajan. Towards an Anomaly-Free Quantum Dynamics for a Weak Coupling Limit of Euclidean Gravity. *Phys. Rev. D*, **87**(4):044039, 2013. [arXiv: 1210.6869]

[96] Erik van den Ban and Marius Crainic. Analysis on Manifolds. Lecture notes, 2010.

[97] Madhavan Varadarajan. Photons from Quantized Electric Flux Representations. *Phys. Rev. D*, **64**:104003, 2001. [arXiv: gr-qc/0104051]

Madhavan Varadarajan. Gravitons from a Loop Representation of Linearised Gravity. *Phys. Rev. D*, **66**:024017, 2002. [arXiv: gr-qc/0204067]

Abhay Ashtekar, Stephen Fairhurst, and Joshua L. Willis. Quantum Gravity, Shadow States, and Quantum Mechanics. *Class. Quant. Grav.*, **20**:1031–1062, 2003. [arXiv: gr-qc/0207106]

[98] John von Neumann. Die Eindeutigkeit der Schrödingerschen Operatoren. *Math. Ann.*, **104**:570–578, 1931.

John von Neumann. Über Einen Satz Von Herrn M. H. Stone. *Ann. Math.*, **33**(3):567–573, 1932.

Marshall H. Stone. On One-Parameter Unitary Groups in Hilbert Space. *Ann. Math.*, **33**(3):643–648, 1932.

[99] John von Neumann. On Infinite Direct Products. *Compos. Math.*, **6**:1–77, 1939.

[100] Frank W. Warner. *Foundations of Differentiable Manifolds and Lie Groups*. Scott Foresman, first edition, 1971.

[101] William C. Waterhouse. An Empty Inverse Limit. *Proc. Amer. Math. Soc.*, **36**:618, 1972.

[102] Hermann Weyl. Quantenmechanik und Gruppentheorie. *Zeitschrift für Physik*, **46**(1-2):1–46, 1927.

Eugene Wigner. On the Quantum Correction For Thermodynamic Equilibrium. *Phys. Rev.*, **40**:749–759, Jun 1932.

José E. Moyal. Quantum Mechanics as a Statistical Theory. *Mathematical Proceedings of the Cambridge Philosophical Society*, **45**:99–124, 1 1949.

[103] John A. Wheeler. *Geometrodynamics*. Academic Press, 1963.

[104] Bryce S. DeWitt. Quantum Theory of Gravity. I. The Canonical Theory. *Phys. Rev.*, **160**:1113–1148, 1967.

[105] Nick M. J. Woodhouse. *Geometric Quantization*. Oxford Mathematical Monographs. Oxford University Press, second edition, 1992.