Lie Algebras of Univariate and Bivariate Hermite Polynomials and New Generating Function

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Lie Algebras of Univariate and Bivariate Hermite Polynomials and New Generating Function

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Abstract This paper presents the connections between univariate and bivariate Hermite polynomials and associated differential equations with specific representations of $\mathfrak{sl}(2,R)$ algebra whose Cartan sub-algebras coincide with the differential operators involved in these differential equations . Applying the Baker-Campbell-Hausdorff formula to these algebras, results in new relations and generating functions in one-variable and Bivariate Hermite polynomials. A general form of $\mathfrak{sl}(2,R)$ representation for other special polynomials such as Laguerre and Legendre polynomials is introduced. A new generating function for Hermite polynomials is presented.

MSC codes: 33C45 ; 16S30 ; 05A15

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1. Introduction

Hermite polynomials are among the most applicable special functions. These polynomials arise in diverse fields of probability, physics, numerical analysis, and signal processing. As an example, in quantum mechanics, eigenfunction solutions of quantum harmonic oscillators are described in terms of Hermite polynomials. Bivariate Hermite polynomials are useful in algebraic geometry and two-dimensional quantum harmonic oscillators [1-3,8]. Concerning the field of applications, Hermite polynomials in one variable are divided into probabilist and physicist versions. In the present paper, we focus on one and two-dimensional probabilist Hermite polynomials which are denoted as $H_n^e(x)$ and $H_{m,n}(x,y,\Lambda)$ respectively. In sections 2,3,4 we prove that the differential operators which are involved in the differential equations of Hermits polynomials, constitute representations for $\mathfrak{sl}(2,R)$ algebra. In sections 5,6 by introducing isomorphic Lie algebras whose Cartan sub-algebras are these differential operators and applying the Baker-Campbell-Hausdorff formula, new relations for one variable and bivariate Hermite polynomials has been presented. It is known that all known generating functions for Hermite polynomials contain a factorial term in the denominators. In section 7 we introduce a new generating function for one-variable Hermite polynomials without a factorial denominator.

2. Lie algebra of Hermite polynomials of one variable

Probabilistic Hermite polynomials, presented as:

$$H_n^e(x) = e^{\frac{-D^2}{2}} x^n \tag{2.1}$$

$H_n^e(x)$ are probabilist Hermite polynomials or equivalently the solutions to Hermite differential equations and $D = \frac{d}{dx}$:

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$$\mathbb{D}_H\, H_n^e(x) = (xD - D^2)\, H_n^e(x) = n\, H_n^e(x) \tag{2.2}$$

Where $\mathbb{D}_H$ is denoted as $\mathbb{D}_H = (xD - D^2)$ as Hermite differential operator. This is an eigenvalue problem with positive integer eigenvalues n.

The equation (2.1) is the transformation of basis $(1, x, x^2, x^3, \dots)$ under the action of operator $e^{\frac{-D^2}{2}}$ which is compatible with Rodrigues' formula and results in probabilistic Hermite polynomials $H_n^e(x)$. The monomials x^n expand a polynomial vector space $\mathbb{V}$. The operator $e^{\frac{-D^2}{2}}$ changes the basis x^n into the basis $H_n^e(x)$. Let $\mathfrak{gl}\,(\mathbb{V})$ denote the linear transformation that maps vector space $\mathbb{V}$ onto itself. We present isomorphic Lie algebras of $\mathfrak{sl}(2, R)$ defined by $\mathfrak{sl}(2, R)$ modules on vector space $\mathbb{V}$ which is a linear map ϕ defined by $\phi : \mathfrak{sl}(2, R) \to \mathfrak{gl}\,(\mathbb{V})$ that preserves the commutator relations of $\mathfrak{sl}(2, R)$ [4,5].

$$\phi[a, b] = [\phi(a), \phi(b)] \qquad a, b \in \mathfrak{sl}(2, R) \tag{2.3}$$

This representation is $\mathfrak{sl}(2, R)$ module over vector space $\mathbb{V}$.

First, we review the structure of the irreducible vector space representation of $\mathfrak{sl}(2, R)$.The generators of this algebra in matrix representation are as follows:

$$H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.4}$$

The commutation relations for this representation of $\mathfrak{sl}(2, R)$ are:

$$[X, Y\,] = 2H \;\;,\;\; [H, X\,] = -X \;\;,\;\; [H, Y\,] = Y \tag{2.5}$$

Let's define a representation of $\mathfrak{sl}(2, R)$ as its module over $\mathbb{V}$ that preserves commutation relations by differential operators as its generators [4]:

$$\boldsymbol{h} = xD - \frac{n}{2} \;\;, \qquad \boldsymbol{e} = D = \partial_x \;\;, \qquad \boldsymbol{f} = x^2 D - nx \tag{2.6}$$

With the same commutation relations

$$[\boldsymbol{e}, \boldsymbol{f}] = 2\boldsymbol{h} \quad , \quad [\boldsymbol{h}, \boldsymbol{e}] = -\boldsymbol{e} \quad , [\boldsymbol{h}, \boldsymbol{f}] = \boldsymbol{f} \tag{2.7}$$

The Cartan sub-algebra $H = \boldsymbol{h}$ produces a decomposition of representation space:

$$\mathbb{V} = \oplus\, \mathbb{V}_j \tag{2.8}$$

$\mathbb{V}_j$ are the eigenspace (eigenfunction) of generator $\boldsymbol{h}$ as Cartan sub-algebra of $\mathfrak{sl}(2, R)$, and provides the solutions to the related differential equation.

$$\boldsymbol{h}\mathbb{V}_j = f(j)\mathbb{V}_j \tag{2.9}$$

As an example, monomials x^n are eigenfunctions or eigenspaces of generator $\boldsymbol{h}$**,** realized as eigenspace $\mathbb{V}_n$. The eigenvalues $f(j)$ in most cases equals an integer j or $j(j+1)$ as we observe in Hermite, Laguerre, and Legendre differential equations.

We search for a Lie algebra $\mathfrak{L}_H$ isomorphic to $\mathfrak{sl}(2, R)$ algebra that its generators to be defined based on Hermite differential operators. Here we apply the transformation operator $e^{\frac{-D^2}{2}}$ as described in equation (2.1) for Hermite polynomials to derive similarity transformations (conjugation) of $\mathfrak{sl}(2,R)$ bases as follows:

$$X_1 = e^{\frac{-D^2}{2}} \boldsymbol{h}\, e^{\frac{D^2}{2}}\ ,\ X_2 = e^{\frac{-D^2}{2}} \boldsymbol{e}\, e^{\frac{D^2}{2}}\ ,\ X_3 = e^{\frac{-D^2}{2}} \boldsymbol{f}\, e^{\frac{D^2}{2}} \tag{2.10}$$

Concerning to a theorem in Lie algebra theory, these generators constitute an isomorphic Lie algebra to $\mathfrak{sl}(2,R)$ with similar commutation relations. We call this algebra as " Hermite operator Lie algebra". Due to the equation (2.1) that implies the change of basis x^n to $H_n^e(x)$, the operator xD with eigenfunctions, x^n corresponds to the operator $\mathbb{D}_H$ with eigenfunctions $H_n^e(x)$ and common eigenvalues n through a similarity transformation described by

$$\mathbb{D}_H = e^{\frac{-D^2}{2}} (xD)\, e^{\frac{D^2}{2}} \tag{2.11}$$

Therefore, we have:

$$X_1 = e^{\frac{-D^2}{2}} \boldsymbol{h}\, e^{\frac{D^2}{2}} = e^{\frac{-D^2}{2}} \left(xD - \frac{n}{2}\right) e^{\frac{D^2}{2}} = \mathbb{D}_H - \frac{n}{2} \tag{2.12}$$

Generator X_2 simply is calculated as $\frac{d}{dx} = D$.

Proposition 2.1 For X_3 we have:

$$X_3 = (x - D)(\mathbb{D}_H - n)$$

Proof: by equation (2.11) we have the identity:

$$\mathbb{D}_H = e^{\frac{-D^2}{2}} (xD) e^{\frac{D^2}{2}} = \left(e^{\frac{-D^2}{2}} x\, e^{\frac{D^2}{2}}\right)\left(e^{\frac{-D^2}{2}} D\, e^{\frac{D^2}{2}}\right) = e^{\frac{-D^2}{2}} x e^{\frac{D^2}{2}} D \tag{2.13}$$

Thus :

$$\mathbb{D}_H D^{-1} = e^{\frac{-D^2}{2}} x e^{\frac{D^2}{2}} \tag{2.14}$$

By equation (2.2) we have:

$$(xD - D^2)D^{-1} = (x - D) = e^{\frac{-D^2}{2}} x e^{\frac{D^2}{2}} \tag{2.15}$$

Now for X_3 from (2.10) we have:

$$X_3 = e^{\frac{-D^2}{2}} \boldsymbol{f}\, e^{\frac{D^2}{2}} = e^{\frac{-D^2}{2}} (x^2 D - nx)\, e^{\frac{D^2}{2}} = e^{\frac{-D^2}{2}} (x^2 D) e^{\frac{D^2}{2}} - n e^{\frac{-D^2}{2}} x e^{\frac{D^2}{2}} \tag{2.16}$$

By equations (2.13), (2.14), and (2.15) we get

$$X_3 = \left[e^{\frac{-D^2}{2}} x e^{\frac{D^2}{2}}\right]\left[e^{\frac{-D^2}{2}} (xD) e^{\frac{D^2}{2}}\right] - n(x - D) =$$

$$= (x - D)\mathbb{D}_H - n(x - D) = (x - D)(\mathbb{D}_H - n) \tag{2.17}$$

□

and for generators of this Lie algebra, we have:

$$X_1 = xD - D^2 - \frac{n}{2} = \mathbb{D}_H - \frac{n}{2}\quad ,\ X_2 = D\quad ,\quad X_3 = (x - D)(\mathbb{D}_H - n) \tag{2.18}$$

Where $\mathbb{D}_H$ denotes the Hermite differential operator i.e., $xD - D^2$. The commutation relations coincide with the $\mathfrak{sl}(2,R)$ algebra as follows:

$$[X_1\,, X_2] = -X_2 \quad , [X_1\,, X_3] = X_3 \quad , [X_2, X_3] = 2X_1 \tag{2.19}$$

Proposition 2.2 Hermite polynomials satisfy the equation:

$$e^{\frac{-D^2}{2}}(e^{-1}+x)^n = e^{(\mathbb{D}_H - n - \frac{D}{1-e})}\, H_n^e(x) \tag{2.20}$$

*proof***:** Due to a theorem for the BCH formula [6,7], if $[X\,, Y] = sY$ for $s \in \mathbb{R}$, we have:

$$e^X e^Y = e^{X + \frac{s}{1-e^{-s}}Y} \tag{2.21}$$

Respect to equation (2.19) for $[X_1\,, X_2]$ **,** the BCH formula for X_1 and X_2 generators gives:

$$e^{[xD - D^2]}\, e^D = e^{\left[xD - D^2 - \frac{D}{1-e}\right]} \tag{2.22}$$

The term $-\frac{n}{2}$ in X_1 was omitted because it has no role in commutation relation $[X_1\,, X_2]$.

Multiplying both sides by $H_n^e(x)$

$$e^{[xD - D^2]}\, e^D H_n^e(x) = e^{\left[xD - D^2 - \frac{D}{1-e}\right]}\, H_n^e(x) \tag{2.23}$$

For $e^D H_n^e(x)$, by the identity $DH_n^e(x) = nH_{n-1}^e(x)$ we obtain:

$$e^D H_n^e(x) = \left(1 + D + \frac{D^2}{2!} + \ldots\right) H_n^e(x) = H_n^e(x) + nH_{n-1}^e(x) + \frac{n(n-1)}{2!} H_{n-2}^e(x) + \ldots + 1$$

Thus $$e^D H_n^e(x) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}^e(x) \tag{2.24}$$

Substituting in equation (2.23) and replacing Hermite differential operator $xD - D^2$ with $\mathbb{D}_H$ gives

$$e^{\mathbb{D}_H} \sum_{k=0}^{n} \binom{n}{k} H_{n-k}^e(x) = e^{\mathbb{D}_H - \frac{D}{1-e}} H_n^e(x) \tag{2.25}$$

$$\sum_{k=0}^{n} \binom{n}{k} e^{\mathbb{D}_H} H_{n-k}^e(x) = e^{\mathbb{D}_H - \frac{D}{1-e}} H_n^e(x) \tag{2.26}$$

$$\sum_{k=0}^{n} \binom{n}{k} e^{n-k}\, H_{n-k}^e(x) = e^{\mathbb{D}_H - \frac{D}{1-e}} H_n^e(x)$$

$$\sum_{k=0}^{n} \binom{n}{k} e^{-k}\, e^{\frac{-D^2}{2}} x^{n-k} = e^{(\mathbb{D}_H - n - \frac{\partial_x}{1-e})} H_n^e(x)$$

$$e^{\frac{-D^2}{2}} \sum_{k=0}^{n} \binom{n}{k} (e^{-1})^k\, x^{n-k} = e^{(\mathbb{D}_H - n - \frac{D}{1-e})} H_n^e(x) \tag{2.27}$$

$$e^{\frac{-D^2}{2}} (e^{-1}+x)^n = e^{(\mathbb{D}_H - n - \frac{D}{1-e})}\, H_n^e(x)$$

Or: $$e^{-(\mathbb{D}_H - n - \frac{D}{1-e})} e^{\frac{-D^2}{2}} (e^{-1}+x)^n = H_n^e(x) \tag{2.28}$$

□

It is notable to compare this equation with

$$e^{\frac{-D^2}{2}} x^n = H_n^e(x) \tag{2.29}$$

3. Bivariate Hermite Polynomials

An ordinary definition for bivariate Hermite polynomials is as follows [4-6].

$$H_{n,m}(s,t,\Lambda) = (-1)^{n+m} e^{\varphi(s,t)/2} \frac{\partial^{m+n}}{\partial s^m \partial t^n}[e^{-\varphi(s,t)/2}] \tag{3.1}$$

With definition $\varphi(s,t) = as^2 + 2bst + ct^2$ as a positive definite quadratic form and $\Lambda = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. The equivalent equation for $H_{n,m}(s,t,\Lambda)$ is [4-6]:

$$H_{n,m}(s,t,\Lambda) = \sum_{k=0}^{\min(n,m)} (-1)^k k! \binom{m}{k}\binom{n}{k} a^{(n-k)/2} b^k c^{(m-k)/2} H_{n-k}^e\left(\frac{as+bt}{\sqrt{a}}\right) H_{m-k}^e\left(\frac{bs+ct}{\sqrt{c}}\right) \tag{3.2}$$

With $a, c > 0$, and $ac - b^2 > 0$. By changing variables $x = \frac{as+bt}{\sqrt{a}}$ and $y = \frac{bs+ct}{\sqrt{c}}$

$$\widehat{H}_{n,m}(x,y,\Lambda) = \sum_{k=0}^{\min(n,m)} (-1)^k k! \binom{m}{k}\binom{n}{k} a^{(n-k)/2} b^k c^{(m-k)/2} H_{n-k}^e(x) H_{m-k}^e(y) \tag{3.3}$$

These polynomials satisfy the partial differential equation [4-6]:

$$\left[\left(x\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2}\right) + \left(y\frac{\partial}{\partial y} - \frac{\partial^2}{\partial y^2}\right) - 2\frac{b}{\sqrt{ac}}\frac{\partial^2}{\partial x \partial y}\right] \widehat{H}_{m,n}(x,y,\Lambda) = (m+n)\widehat{H}_{m,n}(x,y,\Lambda) \tag{3.4}$$

Let denote $\mathfrak{D}$ as the differential operator in differential equation (3.4)

$$\mathfrak{D} = \left(x\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2}\right) + \left(y\frac{\partial}{\partial y} - \frac{\partial^2}{\partial y^2}\right) - 2\frac{b}{\sqrt{ac}}\frac{\partial^2}{\partial x \partial y} \tag{3.5}$$

If we denote $\partial_x = \frac{\partial}{\partial x}$ and $\partial_y = \frac{\partial}{\partial y}$, with the identities:

$$e^{\frac{-\partial_x^2}{2}} x^{m-k} = H_{m-k}^e(x) \quad , \quad e^{\frac{-\partial_y^2}{2}} y^{n-k} = H_{n-k}^e(y) \tag{3,6}$$

Equation (3.3) changes to

$$\widehat{H}_{n,m}(x,y,\Lambda) = e^{\frac{-\partial_x^2}{2}} e^{\frac{-\partial_y^2}{2}} \sum_{k=0}^{\min(n,m)} (-1)^k k! \binom{n}{k}\binom{m}{k} a^{(n-k)/2} b^k c^{(m-k)/2} x^{n-k} y^{m-k} \tag{3.7}$$

We denote the new polynomials as $u_{m,n}(x,y)$:

$$u_{n,m}(x,y) = \sum_{k=0}^{\min(n,m)} (-1)^k k! \binom{n}{k}\binom{m}{k} a^{(n-k)/2} b^k c^{(m-k)/2} x^{n-k} y^{m-k} \tag{3.8}$$

These polynomials constitute a set of linearly independent basis, the transformation from these bases to $H_{n,m}(x,y)$ is as follows:

$$\widehat{H}_{n,m}(x,y,\Lambda) = e^{\frac{-\partial_x^2}{2}} e^{\frac{-\partial_y^2}{2}} u_{n,m}(x,y) = e^{-\frac{\partial_x^2+\partial_y^2}{2}} u_{n,m}(x,y) \tag{3.9}$$

Therefore, the corresponding differential operator with $u_{n,m}(x,y)$ as its eigenfunctions could be derived by similarity transformation:

$$\mathfrak{D}' = e^{\frac{\partial_x^2+\partial_y^2}{2}}\,\mathfrak{D}\,e^{-\frac{\partial_x^2+\partial_y^2}{2}} \tag{3.10}$$

$\mathfrak{D}$ is denoted as the differential operator given in the eigenvalue equation (3.4). Thus, we have

$$\mathfrak{D}' = e^{\frac{\partial_x^2+\partial_y^2}{2}}\left[(x\partial_x - \partial_x^2) + (y\partial_y - \partial_y^2) - 2\frac{b}{\sqrt{ac}}\partial_x\partial_y\right]e^{-\frac{\partial_x^2+\partial_y^2}{2}} \tag{3.11}$$

Proposition 3.1 The reduced form of $\mathfrak{D}'$ is:

$$\mathfrak{D}' = x\partial_x + y\partial_y - 2\frac{b}{\sqrt{ac}}\partial_x\partial_y \tag{3.12}$$

Proof : Respect to equations (3.5) and (3.6) and the commutativity of ∂_x^2 and ∂_y^2 we get:

$$\mathfrak{D}' = \left[e^{\frac{\partial_x^2}{2}}(x\partial_x - \partial_x^2)e^{-\frac{\partial_x^2}{2}} + e^{\frac{\partial_y^2}{2}}(y\partial_y - \partial_y^2)e^{-\frac{\partial_y^2}{2}} - 2\frac{b}{\sqrt{ac}}(e^{\frac{\partial_x^2}{2}}\partial_x e^{-\frac{\partial_x^2}{2}})(e^{\frac{\partial_y^2}{2}}\partial_y e^{-\frac{\partial_y^2}{2}})\right] \tag{3.13}$$

With respect to equations (2.2) , (2.11) to (2.15) we have:

$$\mathbb{D}_H(x) = e^{\frac{-\partial_x^2}{2}}x\partial_x\,e^{\frac{\partial_x^2}{2}}$$

$$e^{\frac{\partial_x^2}{2}}\,\mathbb{D}_H\,e^{\frac{-\partial_x^2}{2}} = e^{\frac{\partial_x^2}{2}}(x\partial_x - \partial_x^2)e^{\frac{-\partial_x^2}{2}} = x\partial_x \tag{3.14}$$

Repeating for $\mathbb{D}_H(y)$ we get

$$e^{\frac{\partial_y^2}{2}}\,\mathbb{D}_H\,e^{\frac{-\partial_y^2}{2}} = y\partial_y \tag{3.15}$$

By the identities : $e^{\frac{\partial_x^2}{2}}\,\partial_x\,e^{-\frac{\partial_x^2}{2}} = \partial_x$, $e^{\frac{\partial_y^2}{2}}\,\partial_y\,e^{-\frac{\partial_y^2}{2}} = \partial_y$

Then equation (3.13) reduces to:

$$\mathfrak{D}' = x\partial_x + y\partial_y - 2\frac{b}{\sqrt{ac}}\partial_x\partial_y \qquad \square$$

Therefore, the differential operator $\mathfrak{D}'$ satisfies the differential equation:

$$\mathfrak{D}'u_{n,m}(x,y) = (m+n)u_{n,m}(x,y) \tag{3.16}$$

Its eigenvalues are the same as the differential equation (3.4), because $\mathfrak{D}$ and $\mathfrak{D}'$ are related by the similarity relation (3.10).

4. Bivariate Hermite Polynomials as $\mathfrak{sl}(2,R)$ Modules

In this section we introduce an associated Lie algebra of bivariate Hermite differential operator. First, we search for the compatible $\mathfrak{sl}(2,R)$ algebra in terms of differential operators of two variables. with respect to equations (2.6) and (2.7) the Cartan sub-algebra of $\mathfrak{sl}(2,R)$ can be taken as:

$$\boldsymbol{h} = \frac{1}{2}(x\partial_x + y\partial_y + 1) + \alpha\partial_x\partial_y \tag{4.1}$$

The additional term $\alpha\partial_x\partial_y$ has been chosen to satisfy the required commutation relations. The other generators are proposed as

$$\boldsymbol{e} = \alpha\, \partial_x \partial_y \quad , \quad \boldsymbol{f} = \frac{1}{2\alpha} xy + \frac{1}{2}\left(x\partial_x + y\partial_y\right) + \frac{\alpha}{4}\, \partial_x \partial_y \tag{4.2}$$

These generators satisfy the commutation relations of $\mathfrak{sl}(2,R)$ as described in (2.7):

$$[\boldsymbol{h}\,, \boldsymbol{e}] = -\boldsymbol{e} \quad , \quad [\boldsymbol{h}\,, \boldsymbol{f}] = \boldsymbol{f} \quad , \quad [\boldsymbol{e}\,, \boldsymbol{f}] = 2\boldsymbol{h} \tag{4.3}$$

By substituting $\alpha = \frac{-b}{\sqrt{ac}}$ and respect to equations (3.12) and (3.16) and (4.1), the Cartan sub-algebra $\boldsymbol{h}$ satisfies the differential equation:

$$\boldsymbol{h}\, u_{n,m}(x,y) = \frac{1}{2}\left[\left(x\partial_x + y\partial_y + 1\right) - \frac{2b}{\sqrt{ac}} \partial_x \partial_y\right] u_{n,m}(x,y)$$

$$= \frac{1}{2}\left[(m+n)u_{n,m}(x,y) + u_{n,m}(x,y)\right] = \frac{1}{2}(m+n+1)u_{n,m}(x,y) \tag{4.4}$$

Thus $u_{n,m}(x,y)$ are eigenfunctions or weight vectors of $\boldsymbol{h}$ as Cartan sub-algebra of $\mathfrak{sl}(2,R)$.
According to the equation

$$e^{-\frac{\partial_x^2 + \partial_y^2}{2}} u_{n,m}(x,y) = H_{n,m}(x,y) \tag{4.5}$$

With respect to equations (2.11),(2.12) and (2.15), the similarity transformation of generators $\boldsymbol{h}$, $\boldsymbol{e}$, and $\boldsymbol{f}$ by operator $e^{-\frac{\partial_x^2 + \partial_y^2}{2}}$ yields:

$$\begin{aligned} \boldsymbol{h}' &= \frac{1}{2}(\mathbb{D}_H(x) + \mathbb{D}_H(y) + 1) - \frac{b}{\sqrt{ac}} \partial_x \partial_y \\ \boldsymbol{e}' &= -\frac{b}{\sqrt{ac}} \partial_x \partial_y \\ \boldsymbol{f}' &= -\frac{\sqrt{ac}}{2b}(x - \partial_x)(y - \partial_y) + \frac{1}{2}[\mathbb{D}_H(x) + \mathbb{D}_H(y)] - \frac{b}{4\sqrt{ac}} \partial_x \partial_y \end{aligned} \tag{4.6}$$

The bivariate Hermite polynomials $H_{n,m}(x,y)$ are eigenfunctions of $\boldsymbol{h}'$ with eigenvalues $\frac{1}{2}(m + n + 1)$.

The lowering operator in this algebra is given by:

$$A^- = \boldsymbol{e}' = -\frac{b}{\sqrt{ac}}\, \partial_x \partial_y \tag{4.7}$$

$\boldsymbol{h}'$ represents the Cartan subalgebra of related Lie algebra. One of the commutator relations:

$$[\boldsymbol{h}'\,, A^-] = -A^- = \frac{b}{\sqrt{ac}} \partial_x \partial_y \tag{4.8}$$

$A^- = \boldsymbol{e}'$ is applicable in BCH formula. This operator acts on $H_{n,m}(x,y)$ as lowering operator.

5. BCH Formula and $\mathfrak{sl}(2,\boldsymbol{R})$ Generators of Bivariate Hermite Polynomial

Proposition 5.1 If in equation (3.3), b be replaced by $2b(1-e)$, the resultant bivariate Hermite polynomial $\widehat{H}_{n,m}(x,y,\Lambda')$ satisfies the equation :

$$e^{\mathfrak{D}} \widehat{H}_{n,m}(x,y,\Lambda') = a^{n/2} c^{m/2}\, e^{m+n} H_n^e(x) H_m^e(y) \tag{5.1}$$

Proof. Due to a theorem for the BCH formula [6,7], if $[X\,, Y] = sY$ and $s \in \mathbb{R}$, then we have:

$$e^X e^Y = e^{X + \frac{s}{1 - e^{-s}} Y}$$

we choose X and Y in such a way that the BCH formula is simplified to equations that gives rise to new relations of bivariate Hermite polynomials. Let assume X and Y in a modified form of bases introduced in equations (4.6)

$$X = \mathfrak{D} = \mathbb{D}_H(x) + \mathbb{D}_H(y) - \frac{2b}{\sqrt{ac}}\partial_x\partial_y \quad , \quad Y = \frac{-2(1-e)b}{\sqrt{ac}}\partial_x\partial_y \tag{5.2}$$

Where we used equation (3.14) and identities:

$$\mathbb{D}_H(x) = e^{-\frac{\partial_x^2}{2}}(x\partial_x)e^{\frac{\partial_x^2}{2}} = x\partial_x - \partial_x^2 \ \text{ and } \ \mathbb{D}_H(y) = e^{-\frac{\partial_y^2}{2}}(y\partial_y)e^{\frac{\partial_y^2}{2}} = y\partial_y - \partial_y^2$$

With respect to the commutation relation

$$[X\,,Y] = -Y = \frac{2(1-e)b}{\sqrt{ac}}\partial_x\partial_y \tag{5.3}$$

Then, we have: $$\exp X \exp Y = \exp\left(X - \frac{Y}{1-e}\right) = \exp\left(X + \frac{2b}{\sqrt{ac}}\partial_x\partial_y\right) \tag{5.4}$$

$$\exp[\,\mathbb{D}_H(x) + \mathbb{D}_H(y) - \frac{2b}{\sqrt{ac}}\partial_x\partial_y]\exp\,(\frac{-2(1-e)b}{\sqrt{ac}}\partial_x\partial_y) = \exp\,(\mathbb{D}_H(x) + \mathbb{D}_H(y))$$

$$\exp X \exp\,(\frac{-2b(1-e)}{\sqrt{ac}}\partial_x\partial_y) = \exp\,(\mathbb{D}_H(x) + \mathbb{D}_H(y)) \tag{5.5}$$

Multiplying both sides of (5.5) by $H_n^e(x)H_m^e(y)$, yields:

$$e^{[\mathbb{D}_H(x)+\mathbb{D}_H(y)-\frac{2b}{\sqrt{ac}}\partial_x\partial_y]}\, e^{\left(\frac{-2b(1-e)}{\sqrt{ac}}\partial_x\partial_y\right)} H_n^e(x)H_m^e(y) = e^{(\mathbb{D}_H(x)+\mathbb{D}_H(y))}H_n^e(x)H_m^e(y) \tag{5.6}$$

$$= e^{m+n}H_n^e(x)H_m^e(y) \tag{5.7}$$

$$e^{\mathfrak{D}-(m+n)}\sum_{k=0}^{\min\,(m,n)} k!\binom{n}{k}\binom{m}{k}(-1)^k\left(\frac{2b(1-e)}{\sqrt{ac}}\right)^k H_{n-k}^e(x)H_{m-k}^e(y) = H_n^e(x)H_m^e(y) \tag{5.8}$$

Where $\mathfrak{D}$ is defined in equation (3.5). Comparing this with equation (3.3) shows the change $b \to b' = 2b(1-e)$ yields a new bivariate Hermite polynomial $\widehat{H}_{n,m}(x,y,\Lambda')$:

$$\widehat{H}_{n,m}(x,y,\Lambda) = \sum_{k=0}^{\min\,(n,m)} (-1)^k\, k!\binom{m}{k}\binom{n}{k} a^{(n-k)/2}b^k c^{(m-k)/2}H_{n-k}^e(x)H_{m-k}^e(y)$$

$$\widehat{H}_{n,m}(x,y,\Lambda') = a^{n/2}c^{m/2} \sum_{k=0}^{\min(n,m)} (-1)^k\, k!\binom{m}{k}\binom{n}{k}(\frac{b'}{\sqrt{ac}})^k\, H_{n-k}^e(x)H_{m-k}^e(y)$$

(5.9)

Where $b' = 2b(1-e)$. Substitution of equation (5.8) into (5.9) gives:

$$a^{-n/2}c^{-m/2}\, e^{\mathfrak{D}-(m+n)}\widehat{H}_{n,m}(x,y,\Lambda') = H_n^e(x)H_m^e(y) \tag{5.10}$$

Or: $$e^{\mathfrak{D}}\widehat{H}_{n,m}(x,y,\Lambda') = a^{n/2}c^{m/2}\, e^{m+n}H_n^e(x)H_m^e(y)$$

6. The general form of differential operator representation of $\mathfrak{sl}(2,\boldsymbol{R})$ and BCH formula

Rodrigues' formula for some orthogonal polynomials such as Hermite, Laguerre and Legendre polynomials are defined by the action of specific differential operators on the n-th integer power of some function $B(x)$, with $n \in \mathbb{N}$:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \tag{6.1}$$

$$L_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}(\frac{d}{dx}-1)^n x^n \tag{6.2}$$

The $B(x)$ in these equations is $B(x) = x^2 - 1$ and $B(x) = x$ respectively.

Let $B(x)$ and its integers exponents form a set $\{1\,, B(x),\ B^2(x)\,, \dots\,, B^n(x)\}$ of independent basis in polynomial space. One can interpret these formulas as transformations from polynomial space basis $\{1\,, B(x),\ B^2(x)\,, \dots\,, B^n(x)\}$ to new basis i.e., Legendre, and Laguerre polynomials. The definition of $B(x)$ in Rodrigues' formula limited to a polynomial with degree at most 2. However, in this section we use $B(x)$ with no limitation and as any kind of smooth functions with real variable.

Proposition 6.1 The Lie algebra generators defined by differential operators that represents an isomorphic algebra to sl (2, R), are represented as

$$\boldsymbol{h} = \frac{B}{B\prime}D - \frac{n}{2}\ , \qquad \boldsymbol{e} = \frac{D}{B\prime}\ , \qquad \boldsymbol{f} = \frac{B^2}{B\prime}D - nB \tag{6.3}$$

Where $B' = B'(x)$ is the derivative of $B = B(x)$.

Proof. It is straightforward to prove that these bases satisfy the commutation relations of $\mathfrak{sl}(2, R)$ in (2.7).

The polynomials $B^n(x)$ are eigenfunctions of the operator $\boldsymbol{h}$ as Cartan sub-algebra with integer eigenvalues. As the author proved [9], similarity transformation of $\boldsymbol{h}$ with operators defined in equations (6.1) and (6.2) yields the Legendre and Laguerre differential operators and equations respectively. Solutions of these differential equations are the corresponding eigenfunctions.

We apply the specific case of the BCH formula that has been introduced in equation (2.21) [6,7]:

$$e^{X_1}e^{X_2} = e^{X_1 + \frac{sX_2}{1-e^{-s}}} \tag{6.4}$$

When the generators X_1 and X_2 satisfy the commutation relation

$$[X_1\,, X_2\,] = sX_2 \tag{6.5}$$

With $s \in \mathbb{R}$. Due to the commutation relation of $\mathfrak{sl}(2, R)$, we obtain:

$$[\boldsymbol{h}\,, \boldsymbol{e}] = -\boldsymbol{e} \tag{6.6}$$

With $s = -1$, and $\boldsymbol{e}' = (1-e)\boldsymbol{e}$ The commutation relation becomes:

$$[\boldsymbol{h}\,, \boldsymbol{e}'] = (1-e)[\boldsymbol{h}\,, \boldsymbol{e}] = -\boldsymbol{e}' = -(1-e)\boldsymbol{e} \tag{6.7}$$

BCH formula reads as:

$$\exp \boldsymbol{h} \exp \boldsymbol{e}' = \exp\left(\boldsymbol{h} - \frac{\boldsymbol{e}'}{1-e}\right) = \exp(\boldsymbol{h} - \boldsymbol{e}) \tag{6.8}$$

If $\boldsymbol{h}$ is written as: $$\boldsymbol{h} = \frac{B-1}{B\prime} D - \frac{n}{2} + \frac{D}{B'} = \frac{B-1}{B'} D - \frac{n}{2} + \boldsymbol{e} \tag{6.9}$$

Then we have:

$$\exp\left(\frac{B}{B\prime} D - \frac{n}{2}\right) \exp\left((1-e)\frac{D}{B\prime}\right) = \exp\left(\frac{B-1}{B'} D - \frac{n}{2}\right)$$

$$\exp\left(\frac{B}{B\prime} D\right) \exp\left((1-e)\frac{D}{B\prime}\right) = \exp\left(\frac{B-1}{B'} D\right) \tag{6.10}$$

the inverse of both sides yields:

$$\exp\left((e-1)\frac{D}{B\prime}\right) \exp\left(\frac{-B}{B\prime} D\right) = \exp\left(\frac{1-B}{B'} D\right) \tag{6.11}$$

Example :

For the algebra of Laguerre differential equation and related differential operators $\mathfrak{D}_L'$, the equivalent generators to $\boldsymbol{h}$ and $\boldsymbol{e}'$ are [9]:

$$Y_1 = \boldsymbol{h} = \mathfrak{D}_L' - \frac{n}{2} \quad , \quad Y_2 = \boldsymbol{e}' = (1-e)(\mathfrak{D}_L' - xD) \tag{6.12}$$

Where $\mathfrak{D}_L'$ is called the Laguerre differential operator whose eigenfunctions are Laguerre polynomials as is defined by:

$$\mathfrak{D}_L' = -(xD^2 - xD + D) \tag{6.13}$$

commutation relation reads as:

$$[Y_1\,, Y_2] = -Y_2 \tag{6.14}$$

Thus, for BCH formula we have:

$$\exp\left(\mathfrak{D}_L' - \frac{n}{2}\right) \exp(\mathfrak{D}_L' - xD) = \exp\left(\mathfrak{D}_L' - \frac{n}{2} - \mathfrak{D}_L' + xD\right)$$

$$\exp(\mathfrak{D}_L') \exp[(1-e)(\mathfrak{D}_L' - xD)] = \exp(xD) \tag{6.16}$$

This is an exponential operational equation for $\mathfrak{D}_L'$.

7. A new generating function for Hermite polynomials

A search on all types of generating functions for Hermite polynomials, reveals that all known generating functions contain a factorial term in the denominators. In this section we present a new generating function without factorial term in denominators of the related sum.

Proposition 7.1 A generating function for Hermite polynomials is:

$$g(x,t) = \sum_{n=0}^{\infty} t^n H_n^e(x) = \frac{1}{\sqrt{2}t} e^{\frac{(1-xt)^2}{2t^2}} \Gamma\left(\frac{1}{2}\ , \frac{(1-xt)^2}{2t^2}\right) \tag{7.1}$$

Proof : Acting operator $O = e^{\frac{-\partial_x^2}{2}}$ on the series $\sum_{n=0}^{\infty} t^n x^n = \frac{1}{1-xt}$ due to equation (2.1) yields a generating function for Hermite polynomials:

$$g(x,t) = e^{\frac{-\partial_x^2}{2}} \sum_{n=0}^{\infty} t^n x^n = \sum_{n=0}^{\infty} t^n H_n^e(x) = e^{\frac{-\partial_x^2}{2}} \left(\frac{1}{1-xt}\right) \tag{7.2}$$

Expanding the right side obtains:

$$g(x,t) = \sum_{n=0}^{\infty} (-1)^j \frac{t^{2j}(1-xt)^{-2j-1}(2j)!}{2^j j!} \tag{7.3}$$

With respect to the identity

$$\frac{(2j)!}{2^j j!} = \frac{\sqrt{\pi}\,(-2)^j}{\Gamma(j-\frac{1}{2})} \tag{7.4}$$

and denoting $y = t^{-1}(1-xt)$ we have:

$$g(x,t) = \frac{\sqrt{\pi}}{1-xt} \sum_{n=0}^{\infty} \left(\frac{y^2}{2}\right)^{-j} \Gamma(j-\tfrac{1}{2})^{-1} \tag{7.5}$$

By the identity for n-th derivative of z^s while $n \in \mathbb{N}$ and $s \in \mathbb{R}$

$$\frac{d^n}{dz^n} z^s = \frac{\Gamma(s+1)}{\Gamma(s+1-n)} z^{s-n} \tag{7.6}$$

For $s = -\frac{1}{2}$ we have:

$$\frac{d^n}{dz^n} z^{-\frac{1}{2}} = \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-n)} z^{-\frac{1}{2}-n} \tag{7.7}$$

$$\sum_{n=1}^{\infty} \frac{d^n}{dz^n} z^{-\frac{1}{2}} = \Gamma(\tfrac{1}{2}) z^{-\frac{1}{2}} \sum_{n=1}^{\infty} \frac{z^{-n}}{\Gamma(\frac{1}{2}-n)} \tag{7.8}$$

The strict condition for convergence of this formal series is $z \neq 0$. By rewriting equation (7.8) with $j = n$ on the right side, we obtain:

$$z^{-\frac{1}{2}} + \sum_{n=1}^{\infty} \frac{d^n}{dz^n} z^{-\frac{1}{2}} = \Gamma\left(\frac{1}{2}\right) z^{-\frac{1}{2}} \sum_{j=0}^{\infty} \frac{z^{-j}}{\Gamma\left(\frac{1}{2}-j\right)} \tag{7.9}$$

Comparing right side of this equation with (7.5) and changing variable $\frac{y^2}{2} = z$ and by the identity $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, equation (7.5) becomes:

$$\sqrt{2} t g(x,t) = z^{-\frac{1}{2}} + \sum_{n=1}^{\infty} \frac{d^n}{dz^n} z^{-\frac{1}{2}} \tag{7.10}$$

The right side of equation (7.10) can be calculated as follows:

$$z^{-\frac{1}{2}} + \sum_{n=1}^{\infty} \frac{d^n}{dz^n} z^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{d^n}{dz^n} z^{-\frac{1}{2}} = \frac{1}{1-\frac{d}{dz}}\left(z^{-\frac{1}{2}}\right) \tag{7.11}$$

Then by the integral identity

$$\frac{1}{1-\frac{d}{dz}}\left(z^{-\frac{1}{2}}\right) = -e^z \int e^{-z}\ z^{-\frac{1}{2}} dz = e^z\Gamma(\frac{1}{2}, z) \quad (7.12)$$

And Calculation of the integral, the equation (7.10) becomes:

$$g(x,t) = \frac{1}{\sqrt{2}t} e^z\Gamma(\frac{1}{2}, z) \quad (7.13)$$

Or:

$$g(x,t) = \sum_{n=0}^{\infty} t^n H_n^e(x) = \frac{1}{\sqrt{2}t} e^{\frac{(1-xt)^2}{2t^2}}\Gamma(\frac{1}{2}, \frac{(1-xt)^2}{2t^2}) \quad (7.14)$$

□

This is a new generating function for Hermite polynomials. As a test for validation of equation (7.14), By knowing the limit:

$$\lim_{t\to 0} g(x,t) = \lim_{t\to 0} \sum_{n=0}^{\infty} t^n H_n^e(x) = 1 \quad (7.15)$$

We find the limit of the right side of equation (7.14) when $t \to 0$:

$$\lim_{t\to 0} g(x,t) = \lim_{t\to 0} \sum_{n=0}^{\infty} t^n H_n^e(x) = \lim_{t\to 0} \frac{1}{\sqrt{2}t} e^{\frac{(1-xt)^2}{2t^2}}\Gamma\left(\frac{1}{2}, \frac{(1-xt)^2}{2t^2}\right) = 1 \quad (7.16)$$

For validation of equation (7.16), we calculated the $\lim_{t\to 0} g(x,t)$ by online calculator keisan. Calculations of equation (7.13) for $z = 1200$ and $z = 2000$ while t is small, result in 0.994 and 0.997 for $g(x,t)$ respectively. This verifies the limit in equation (7.16) as expected.

Conclusion

By introducing a method for representation of $\mathfrak{sl}(2,R)$ algebra with differential operators that are involved in Hermite's univariate and bivariate differential equations, a set of new relations for Hermite polynomials are derived. The general form of this method could be applied for other special polynomials like Laguerre and Legendre polynomials. Based on this method, a new generating function for Hermite polynomial without the factorial terms in the denominators is introduced.

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