General relativistic neutrino transport with spectral methods

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Abstract. Supernovae numerical simulations are highly demanding in computational power, yet they are also rapidly increasing in complexity. There is now a very high demand in new methods for neutrino treatment, due to the fact that neutrino is at the same time a crucial ingredient and the most expensive one in computational resources. A possibility to explore is the treatment by spectral methods, because they are known to require less resources. It is the purpose of a new code, named `Lorene's Ghost` (for Lorene's gravitational handling of spectral transport), developed to treat the problem of neutrino transport in supernovae. The code can handle full phase-space dependent problems (6D + time), retains the full energy dependence of the neutrinos, and can also handle general relativity in the conformally flat approximation. Several points of the formalism needed to obtain a formulation of the general relativistic Boltzmann equation adapted to our case are discussed, as well as some technical aspects of spectral methods.

1. Introduction

The explosion mechanism of supernovae remains a puzzle. Despite considerable effort and increasing complexity of the numerical models, no convincing consensus has emerged yet. For recent reviews, see e.g. [1–3] and references therein.

Neutrinos are one of the key ingredients in a supernova numerical simulation because they carry out more than 99% of the explosion energy, and because, in the neutrino driven explosion mechanism, they revive the shock by their deposition of energy in the gain layer. Because they decouple from the fluid before reaching this gain layer, a fluid description of the neutrinos would be wrong. One needs a description able to handle all the regimes, from fully opaque (where the neutrinos are coupled to the fluid, in the interior of the proto-neutron star) to fully transparent (after $\sim$ 1000km, where the neutrinos are completely decoupled). This mean solving the radiative transfert equation, or one simplified version of it. On the one hand, solving the radiative transfert equation, or Boltzmann equation, is computationnaly challenging, mostly because of its very high dimensionnality (seven phase space dimensions). On the other hand, it is the only way to have a quantitatively accurate treatment of the semi-transparent regime behind the shock, where the neutrinos can deposit energy to revive it.

One solution to overcome this issue may be the use of spectral methods. Spectral methods are known to have exponential convergence if the represented function is C^∞ (see section 3), meaning that they require less points, and so less computational power to represent the same

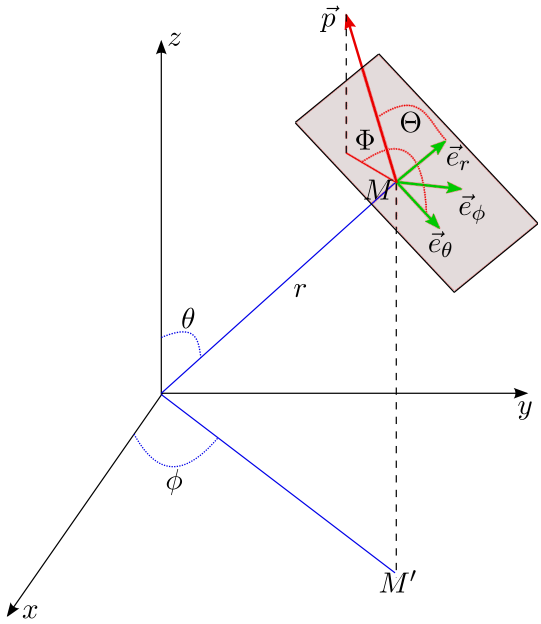

Figure 1. Representation of the 6D double spherical coordinate basis.

function. This idea lead to the writing of the `Lorene's Ghost` code [4]. `Lorene's Ghost` is an extension of the `Lorene` library [5], able to treat 7-dimensional scalar functions.

Greek indices run from 0 to 4, latin indices run from 1 to 4. I adopt the Einstein summation convention and a metric with a $(-,+,+,+)$ signature, and geometrical units in which $c = G = \hbar = 1$.

2. The Boltzmann equation

The general relativistic Boltzmann equation [4; 6; 7] can be written as

$$\epsilon \frac{\partial f}{\partial t} + p^i \frac{\partial f}{\partial x^i} - \Gamma^i{}_{\mu\nu} p^\mu p^\nu \frac{\partial f}{\partial p^i} = \mathcal{C}[f] \ , \tag{1}$$

where f is the distribution function, x^i and t are the coordinate space and time, p^μ is the 4-momentum of the neutrinos, $\Gamma^i{}_{\mu\nu}$ are the Christoffel symbols and $\mathcal{C}[f]$ is the collision operator.

2.1. The distribution function

The distribution function is a scalar [6; 8], defined in principle on the phase space (x^μ, p^μ). Because of the mass shell condition $p^\mu p_\mu = 0$ (that holds with neutrinos, which we suppose have a negligibly small mass), the phase space reduces to seven independent dimensions, (t, x^i, p^i), as in eq.(1).

`Lorene's Ghost` uses the double spherical coordinate basis. Space is represented by a radius and two angles, as usual in a spherical coordinate basis, $x^i = (r, \theta, \phi)$, and momentum space is also represented by a radius and two angles, $p^i = (\epsilon, \Theta, \Phi)$. ϵ is the energy of the neutrinos. On figure 1, a representation of the double spherical coordinate basis is drawn, where we see clearly that the momentum space basis depends on the real space basis.

2.2. Mixing frames

In supernovae numerical simulations, most implementations of the Boltzmann equation or a simplified version of it [9–12] use the mixed frame formalism : while the spacetime variables

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belong to the coordinate basis (or, in non-general relativistic codes, the Eulerian basis, since both are the same if the metric is flat), the momentum space variables belong to the Lagrangian basis. This simplifies a lot the computation of the collision operator, the right hand side term of eq.(1). On the other hand, the connection coefficients are no longer the ones of the coordinate basis (the Christoffel symbols), as in eq.(1), but the connection coefficients of the Lagrangian frame, which are much harder to compute. A special treatment of these have been implemented in `Lorene's Ghost`, which avoids any time derivative (except for $\partial v/\partial t$), known to be hard to handle with spectral methods, and involves the computation of Ricci rotation coefficients (see [4]).

Physically, these connection coefficients of the Lagrangian basis include important terms for the neutrino treatment such as the Doppler shift, the gravitational redshift or the aberration due to the acceleration of the fluid.

3. Spectral methods

Spectral methods (see, *e.g.* [13; 14] and references therein) decompose a function using a global basis function (here, we use Fourier or Chebyshev bases). As a result, convergence is very rapid (exponential, unless the function is not C^∞ or close to it) and one can use fewer points than with finite differences to get to the same resolution. In the case of the neutrino treatment, this shall lead to a substantial decrease in the needed computational time.

3.1. `Lorene's Ghost`

`Lorene's Ghost` advances in time with an explicit Adams–Bashforth scheme of 3rd order. Another finite differencing in time has been tested, namely a Runge–Kutta 2nd order scheme. All six other dimensions are represented with spectral methods.

Two angles (ϕ and Φ) go from 0 to 2π and are fully periodic. These are well suited for a representation on a Fourier basis. For the θ angle, an analytic extension can be performed to recover a periodic dimension. This enables us to also use a Fourier representation. The remaining arguments (r, ϵ and Θ) are decomposed onto Chebyshev bases.

3.2. Gibbs phenomenon and filtering

When one tries to represent a discontinuous function with spectral methods, spurious oscillations appear that can potentially spoil the entire representation of the function. On figure 2 is an illustration of this phenomenon. The red analytical function $f(r)$ is discontinuous, and its representation using 33 points on a Chebyshev basis has several drawbacks : it is oscillating (in the whole domain), the exponential convergence is lost, and the discontinuity is considerably smoothed. Filtering helps getting rid of these problems. While the smoothing remains, filtering can get rid of the two other mentioned drawbacks.

Note also that a non linear operator often populates coefficients of higher order. It means that even when dealing with a smooth function well represented with a finite set of coefficients, non linearities can create oscillations, and so filtering is very often needed in spectral methods.

On figure 3, the same analytic function as on figure 2 is represented, but this time with 257 points in r and with an exponential filter applied to the result, meaning that the i-th coefficient is multiplied by

$$\exp\left(\alpha\left[\frac{i}{n_r}\right]^{2d}\right) , \tag{2}$$

where n_r is the total number of coefficients in r, d is the order taken to be 1 here, and α is usually the machine precision (10^{-16} here).

On figure 3, one can see that no oscillation remains, and that the numerical representation is different from the analytical one only in the vicinity of the discontinuity.

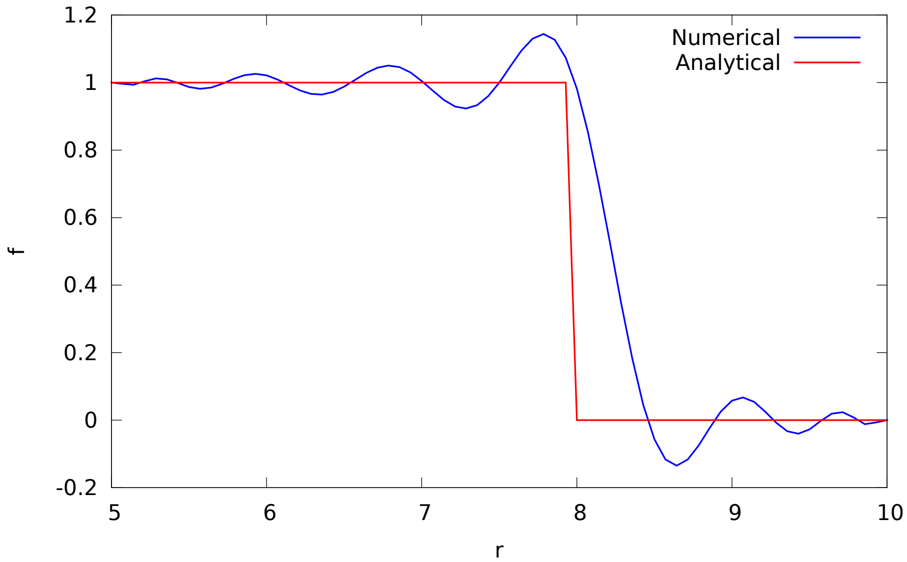

Figure 2. Representation of a discontinuity (red) by Chebyshev spectral methods (blue, 33 points). Gibbs phenomenon (oscillations) is clearly visible.

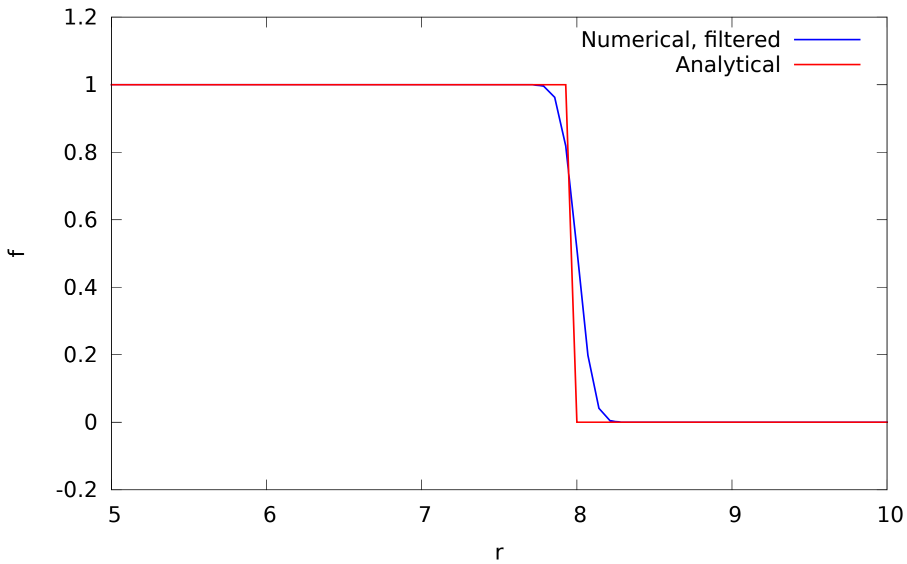

Figure 3. Representation of a discontinuity (red) by Chebyshev spectral methods (blue, 257 points) with a first order exponential filter.

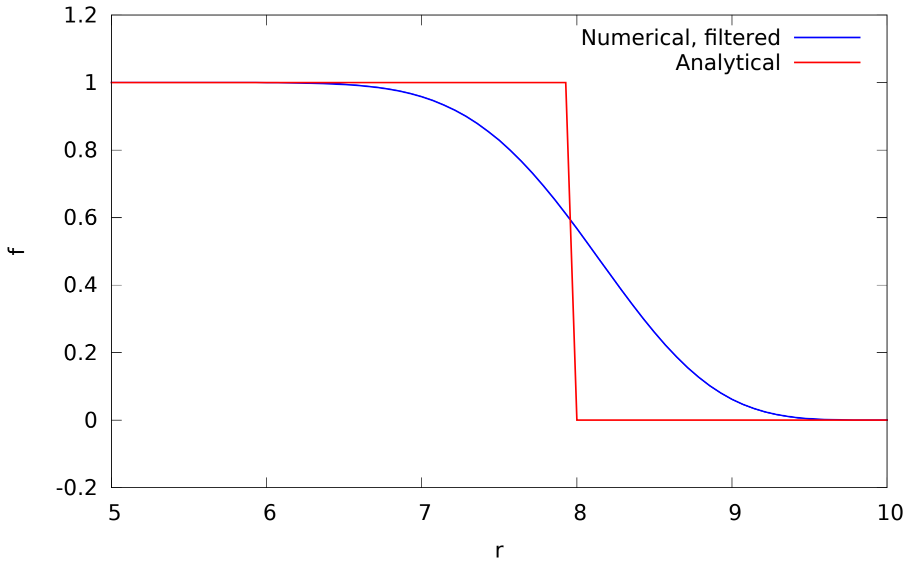

Figure 4. Representation of a discontinuity (red) by Chebyshev spectral methods (blue, 33 points) with a first order exponential filter.

On figure 4 is the same as on figure 3 but this time with 33 points in r. Note the much stronger smoothing that results in a smaller slope. This is because the first order exponential filter changes a lot the function. Other filters or higher order exponential filters would not change the function that much and can be sufficient in many cases, this test of a sharp discontinuity being an extreme case.

4. Conclusion

I showed that the treatment of the neutrino problem in supernovae is possible with spectral methods. They are viable methods with the capability of a strong speedup compared to finite difference methods. The code `Lorene's Ghost` is able to treat test cases, including 6D + time full angle and energy dependent configurations.

I addressed the main drawback of spectral methods, namely the appearance of Gibbs phenomenon due to either discontinuities, strong gradients in the represented function, or aliasing due to non linear terms in the equations. I showed that filtering is a good option to overcome this problem and is capable of getting rid of all oscillations.

`Lorene's Ghost` passed many tests, including the recovery of spectral convergence in all six dimensions, a particle number conservation with a collision term, a gravitational redshift test with a Schwarzschild metric and a so-called searchlight beam test (see [4]). All these tests, including the 7-dimensional ones, are able to run on a single core with a moderate spectral resolution.

A technical issue remains to be adressed, namely the fact that `Lorene's Ghost` cannot go down to $r = 0$ and has to stop at a finite R_{min}. What is missing is the generalization (if possible) of the regularity conditions imposed in `Lorene` near the $r = 0$ point. If they do not exist, an alternative would be to couple `Lorene's Ghost` to an approximate solver that would be used only in the opaque regime. A diffusion equation, for instance, is sufficient in this regime. Another alternative is to treat the r dimension with finite differences.

Next steps include the coupling of `Lorene's Ghost` to an hydrodynamics solver to simulate supernovae, namely the `CoCoNuT` code [15]. Studies such as speed up tests and comparisons of approximated neutrino treatments with our Boltzmann solver will then be possible.

This work can also be extended to simplified treatments that would decrease even more the needed computational resources. It would be easy with `Lorene's Ghost` to reduce, for instance, to a Schwarzschild metric in the whole simulation, to reduce to one spatial dimension and use the so-called ray-by-ray approximation [9] to deal with multi-dimensional problems, or to reduce the velocity dependent terms in the connection coefficients (see section 2.2) to first order in the fluid velocity v.

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