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Article

Revise the Phase-Space Analysis of the Dynamical Spacetime Unified Dark Energy Cosmology

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Abstract: We analyze the phase-space of an alternate scalar field cosmology that aims to combine the concepts of dark energy and the dark sector. The investigation focuses on stationary points within this phase-space, considering different functional forms of the two potential functions. Our findings indicate that a de Sitter universe is achievable solely when at the asymptotic limit the potential function is constant. For constant potential function, the de Sitter universe is recovered in the finite regime; however, for the exponential potential, the de Sitter universe exists at the infinity regime. The cosmological viability of the present theory is discussed.

Keywords: cosmology; asymptotic dynamics; unified dark energy

1. Introduction

The analysis of the observational data shows that our universe is dominated by dark energy and dark matter. Collectively, these components constitute approximately 96% of the total energy density of the universe, with dark matter making up approximately 28% and dark energy around 68%. Dark matter is a hypothetical form of matter that does not interact with radiation, and it can explain the gravitational phenomena in compact objects [1,2]. On the other hand, the impact of dark energy is observable at large scales. Dark energy has been proposed to explain the rapid acceleration of the universe [3–5], for that dark energy has negative pressure that induces repulsive gravitational forces [6].

The origin of the universe's acceleration is unknown. Over the last few decades, various models have been proposed in the literature. These models can be grouped into two broad categories, those belonging to modified theories of gravity [7–13] and those classified as “dark energy” models [14–19]. The first category offers a geometric explanation for the universe's acceleration. It introduces new geometric invariants into the Einstein–Hilbert Action, resulting in new terms in the field equations that account for the acceleration. On the contrary, “dark energy” models are defined within General Relativity, where an exotic matter source is introduced related to the cosmic acceleration. As far as dark matter is concerned in cosmological studies, it is usually introduced as a pressureless fluid source in the field equations.

There exists a distinct category of models that offer a unified approach to describing the dark sector of the universe [20]. Chaplygin gas and its modifications are a simple mechanism for the unification of dark matter and dark energy [21–25]. Another category of approaches comprises the bulk viscosity models [26,27] and the scalar field theories [28–31]. Within the unified models, both dark matter and dark energy are attributed to particular dynamical components of the effective fluid. In the unified models, there exists an interaction between dark matter and dark energy. This interaction can explain various problems related to cosmological observations [32–35].

In [36], a new mechanism was found to introduce dark matter components in cosmological dynamics. This mechanism is based on two scalar field models, a quintessence scalar



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field and a second scalar field coupled in the Action Integral to the energy momentum tensor of the quintessence. This approach was used in [37] to propose unified dark energy models. From the analysis of the phase space and it was found that the model can describe various eras of the cosmological history. It was found that for the exponential potential, the phase-space has asymptotics which can describe the matter epoch, the de Sitter universe and another scaling solution. Similar results were found for another functional of the potential in [38], while the theory has been used to explain the cosmological observations.

In this study, we revise the phase-space analysis presented in [37]. We demonstrate that for the exponential potential put forth in the same paper, the only asymptotic solutions correspond to the matter-dominated era. Notably, this model lacks any asymptotic solution that describes acceleration. The emergence of a de Sitter universe is solely achieved when the scalar field potential assumes the role of a cosmological constant at the asymptotic limit. For these latter two scenarios, we present analytic solutions for the field equations. Furthermore, acceleration is observed when the second potential function is not constant.

Indeed, the phase-space analysis is a powerful method for the study of proposed gravitational theories [39–41] and it can be applied for the construction of constraints [42–44]. The overview of the paper is as follows.

In Section 2 we outline the cosmological model under consideration and provide the field equations for the homogeneous and isotropic spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) geometry. Section 3 delves into the investigation of closed-form analytic solutions. We both validate previous findings and establish a novel general solution within the theory. The core analysis is presented in Section 4, where we meticulously examine the phase-space properties of the unified dark energy model. We explore three distinct cosmological scenarios, revealing that the de Sitter expansion only emerges as an attractor when all potential functions within the theory remain constant. In Section 5, we study the asymptotics when one of the variables reaches infinity, where we find that there exist two de Sitter solutions; the first is always a source and the second is always an attractor. Finally, our conclusions are presented in Section 6.

2. Field Equations

We follow the approach outlined in [37] and consider the gravitational Action Integral

$$S = \int d^4x \sqrt{-g} \left(R + \chi_{\mu;\nu} T_{(\chi)}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \phi_{;\mu} \phi_{;\nu} - V(\phi) \right). \tag{1}$$

Here, ϕ represents a scalar field that inherits the symmetries of the background space, along with the potential function $V(\phi)$, while χ_μ denotes a dynamical space-time vector; $T_{(\chi)}^{\mu\nu}$ is the energy-momentum tensor [45] defined as

$$T_{(\chi)}^{\mu\nu} = -\frac{1}{2} g^{\mu\kappa} g^{\nu\lambda} \phi_{;\kappa} \phi_{;\lambda} + U(\phi) g^{\mu\nu} \tag{2}$$

where $U(\phi)$ is now a second potential function.

Indeed, when the covariant derivative is defined by the Levi–Civita connection, then variation of (1) with respect to the vector field χ_μ provides the equation $T_{(\chi)}^{\mu\nu}{}_{;\nu} = 0$. The introduction of the term $\chi_{\mu;\nu}$ in the Action Integral is related to the “two measures theory” which has been proposed to solve the cosmological constant problem [46–48]. A similar vector field was introduced in [49]. For a classical analogue, we refer the reader in the discussion presented in [50]. Indeed, the term $\chi_{\mu;\nu} T_{(\chi)}^{\mu\nu}$ in the Action integral indicates the introduction of the constraint equation $T_{(\chi)}^{\mu\nu}{}_{;\nu} = 0$ for the theory.

For the spatially flat FLRW geometry

$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2),$$

with scale factor $a(t)$, Hubble function $H = \frac{\dot{a}}{a}$, and for a homogeneous time-like vector field $\chi^\mu = \chi_0 \delta_t^\mu$, the gravitational field equations read [37]

$$3H^2 = \rho_{eff} \tag{3}$$

$$-2H^2 - 3H^2 = p_{eff} \tag{4}$$

where the effective energy density ρ_{eff} and pressure components p_{eff} are defined as

$$\rho_{eff} = \dot{\phi}^2 \left(\dot{\chi}_0 \left(1 - \frac{3}{2}H \right) - \frac{1}{2} \right) + V(\phi) + \dot{\phi} \dot{\chi}_0 (U_{,\phi} + \ddot{\phi}), \tag{5}$$

$$p_{eff} = \frac{1}{2} \dot{\phi}^2 (\dot{\chi}_0 - 1) - V(\phi) - \chi_0 \dot{\phi} U_{,\phi}. \tag{6}$$

Furthermore, the equations of motion for the two scalar fields, namely ϕ and χ_0 , are expressed as

$$\ddot{\phi} + \frac{3}{2}H\dot{\phi} + U_{,\phi} = 0, \tag{7}$$

$$\ddot{\phi}(\dot{\chi}_0 - 1) + \dot{\phi}(\ddot{\chi} + 3H(\dot{\chi}_0 - 1)) - U_{,\phi}(\dot{\chi}_0 + 3H\chi_0) + V_{,\phi} = 0. \tag{8}$$

This gravitational model was initially proposed as a unified dark energy model in [37]. Subsequently, in [38], the model was employed to elucidate cosmological observations. The findings indicated a slight deviation of the model from Λ -cosmology.

3. Analytic Solutions

In this Section we proceed with the derivation of analytical solutions for the field Equations (3)–(8). We explore two cosmological scenarios based on the two potential functions: (A) $U(\phi) = U_0$, $V(\phi) = V_0$ and (B) $U(\phi) = U_0$ and $V(\phi) = V_0 e^{-\lambda\phi}$.

3.1. Case A: $U(\phi) = U_0$ and $V(\phi) = V_0$

This particular case was previously examined in [37]. We now reconstruct the analytical solution. In fact, by considering (7), we deduce that

$$\phi(t) = \phi_0 + \int \phi_1 a^{-\frac{3}{2}} dt. \tag{9}$$

By replacing in (8) it follows that

$$\chi_0(t) = \chi_0^0 + \int \left(1 + \chi_0^1 a^{-\frac{3}{2}} \right) dt, \tag{10}$$

Therefore, the analytic solution for the Hubble function is given by Equation (3), that is,

$$3H^2 = V_0 + \frac{\phi_1^2}{2} a^{-3} + \phi_1^2 \chi_0^1 a^{-\frac{9}{2}}. \tag{11}$$

We note that the scalar field ϕ introduces the dark matter component a^{-3} , and furthermore, from the second scalar field χ_0 , the component $a^{-\frac{9}{2}}$ emerges. This latter component corresponds to an ideal gas with an equation of state parameter of $\frac{1}{2}$.

3.2. Case B: $U(\phi) = U_0$ and $V(\phi) = V_0 e^{-\lambda\phi}$

Assume now that the scalar field potential $V(\phi) = V_0 e^{-\lambda\phi}$ and $U(\phi) = U_0$. Thus, from (8), the solution (9) follows. Moreover, for the second scalar field we derive

$$\dot{\chi}_0(t) = \frac{2}{\phi_1} \exp(-f_1(t)) \int \exp(f_1(t)) \left(\frac{\lambda \phi_1^2}{a^{\frac{3}{2}}} + 6\lambda a^{\frac{3}{2}} H^2 + 3\phi_1 H \right) dt, \tag{12}$$

where

$$f_1(t) = \ln(a)^{\frac{3}{2}} + \phi_1 \int \frac{2\lambda}{a^{\frac{3}{2}}} dt. \tag{13}$$

Therefore, the Hubble function is calculated:

$$3H^2 = -\frac{\phi_1}{2}a^{-3} + V_0 \exp\left(-\lambda\left(\phi_0 + \int \phi_1 a^{-\frac{3}{2}} dt\right)\right) + \left(a^{-3} \frac{2}{\phi_1} \exp(-f_1(t)) \int \exp(f_1(t)) \left(\frac{\lambda\phi_1^2}{a^{\frac{3}{2}}} + 6\lambda a^{\frac{3}{2}} H^2 + 3\phi_1 H\right) dt\right). \tag{14}$$

We remark that when $\lambda = 0$, the solution of Case A is recovered.

4. Asymptotic Solutions

We proceed with the analysis of the equilibrium points for the field Equations (3)–(8). We select three distinct cases for the analysis: (I) $U(\phi) = U_0, V(\phi) = 0$, (II) $U(\phi) = U_0, V(\phi) = V_0 e^{-\lambda\phi}$ and (III) $U(\phi) = U_0 e^{-\kappa\phi}, V(\phi) = V_0 e^{-\lambda\phi}$.

4.1. Case I: $U(\phi) = U_0$ and $V(\phi) = 0$

For the first case with $U(\phi) = U_0$ and $V(\phi) = 0$, we apply the H -normalization approach and we introduce the new variables

$$x = \frac{\dot{\phi}}{\sqrt{6}H}, \delta = \dot{\chi}_0 - 1, d\tau = Hdt. \tag{15}$$

The field Equations (3)–(8) are expressed as the following first-order dynamical system

$$\frac{dx}{d\tau} = \frac{3}{2}\delta x^3 \tag{16}$$

$$\frac{d\delta}{d\tau} = -\frac{3}{2}\delta \tag{17}$$

with the constraint equation

$$1 - x^2(1 + 2\delta) = 0. \tag{18}$$

In the new variables, the effective equation of the state parameter for the cosmological fluid $w_{eff} = -1 - \frac{2}{3} \frac{\dot{H}}{H^2}$ reads

$$w_{eff}(x, \delta) = x^2\delta. \tag{19}$$

We observe that the dynamical system is symmetric in the change of variables $x \rightarrow -x$. Hence, without loss of generality, we choose to work in the range $x \geq 0$. Therefore, there exists a unique stationary point $A = (x(A), \delta(A))$ for the latter dynamical system at the finite regime with coordinates $A_1^\pm = (\pm 1, 0)$. These two points describe matter-dominated eras, that is, $w_{eff}(A_1^\pm) = 0$.

From (18) we replace in (16) and we find that

$$\frac{dx}{d\tau} = \frac{3}{4}x(1 - x^2). \tag{20}$$

Thus, it is easy to observe that points A_1^2 are always attractors.

4.2. Case II: $U(\phi) = U_0$ and $V(\phi) = V_0 e^{-\lambda\phi}$

For the second case, that is, for the exponential potential $V(\phi) = V_0 e^{-\lambda\phi}, \lambda \neq 0$, we introduce the new variables

$$x = \frac{\dot{\phi}}{\sqrt{6}H}, \delta = \dot{\chi}_0 - 1, y^2 = \frac{V(\phi)}{3H^2}, d\tau = Hdt. \tag{21}$$

Therefore, the field equations read

$$1 - x^2(1 + 2\delta) - y^2 = 0, \tag{22}$$

$$\frac{dx}{d\tau} = \frac{3}{2}(\delta x^3 - xy^2), \tag{23}$$

$$\frac{d\delta}{d\tau} = \frac{\sqrt{6}}{2}\lambda\frac{y^2}{x} - \frac{3}{2}\delta, \tag{24}$$

$$\frac{dy}{d\tau} = -\frac{y}{2}(\sqrt{6}\lambda x + 3(y^2 - 3x^2\delta - 1)). \tag{25}$$

Furthermore, the effective equation of the state parameter is derived

$$w_{eff}(x, y, \delta) = -y^2 + x^2\delta.$$

The stationary points $B = (x(B), \delta(B), y(B))$ of the dynamical system (22)–(25) are

$$B_1^\pm = (\pm 1, 0, 0), B_2 = \left(\frac{\sqrt{6}}{2} \frac{1}{\lambda}, \frac{2\lambda^2 - 3}{9}, \frac{\sqrt{2\lambda^2 - 3}}{\sqrt{6}\lambda}\right).$$

Points B_1^\pm exhibit the same physical characteristics as points A_1^\pm , representing matter-dominated eras. Additionally, for point B_2 , we calculate $w_{eff}(B_2) = 0$, implying that if B_2 exists, i.e., when $\lambda^2 > \frac{3}{2}$, it corresponds to a matter-dominated epoch.

The derivation of point B_2 was previously presented in [37], identified as point D therein. Upon substituting δ from (22) into (25), we obtain expression (33) from [37], indicating its vanishing nature at point D rather than a non-zero constant. As a result, the assertion that point B_2 signifies acceleration is incorrect.

We repeat part of the calculations outlined in [37]. From (22), it follows that

$$x^2(1 + 2\delta) = 1 - y^2, \tag{26}$$

and for $x \neq 0$,

$$\delta = \frac{1}{2}\left(-1 + \frac{1 - y^2}{x^2}\right). \tag{27}$$

It is clear that x can not take the value zero, otherwise the following system is not defined. Indeed, the dynamical system (22)–(25) can be reduced to the following two-dimensional system (with $x \neq 0$)

$$\frac{dx}{d\tau} = -\frac{3}{4}x(x^2 + 3y^2 - 1), \tag{28}$$

$$\frac{dy}{d\tau} = -\frac{1}{4}y(2\sqrt{6}x + 3(x^2 - 3y^2 - 3)), \tag{29}$$

The stationary points within the defined range of parameters x and y consist of B_1^\pm and B_2 . Consequently, we deduce that the point C discussed in [37], characterized by $x = 0$ and $y = 1$, is nonexistent at the finite regime.

The eigenvalues of the two-dimensional system (28), (29) are

$$e_1(B_1^\pm) = -\frac{3}{2}, e_2(B_1^\pm) = \frac{1}{2}(3 \mp \sqrt{6}\lambda),$$

and

$$e_1(B_2) = \frac{3}{4}\left(-1 + \frac{\sqrt{5\lambda^2 - 6}}{\lambda}\right), e_2(B_2) = \frac{3}{4}\left(-1 - \frac{\sqrt{5\lambda^2 - 6}}{\lambda}\right),$$

From the above we conclude that point B_1^+ is an attractor for $\lambda > \sqrt{\frac{3}{2}}$, while point B_1^- is an attractor for $\lambda < -\sqrt{\frac{3}{2}}$. Finally, as far as the point B_2 is concerned, we infer that when it exists, is always a saddle point.

In Figures 1–3 we present phase-portraits in the two- and three-dimensional spaces for the dynamical system (22)–(25) for different values of parameter λ . From the plots, it is clear that the trajectories go in the area where $\delta \rightarrow \infty$; this means that it seems that stationary points exist at the infinity regime. In the following analysis, specifically in Section 5, we study the existence of stationary points for very large values of δ .

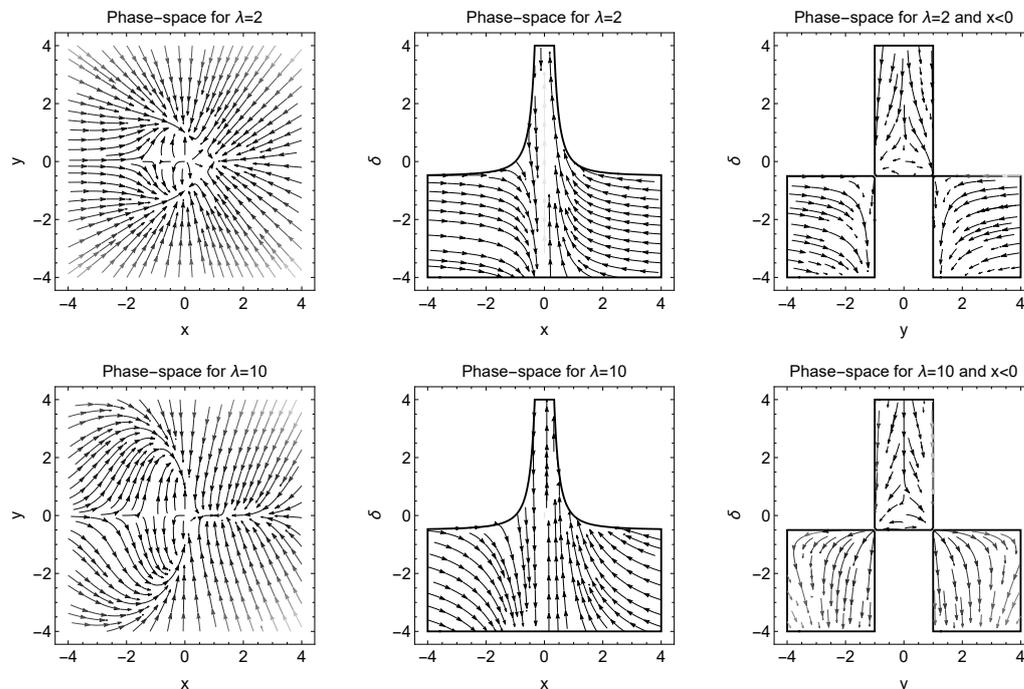


Figure 1. Phase-space portraits of the dynamical system (22), (25) on the two-dimensional planes (x, y) , (x, δ) and (y, δ) for $\lambda = 2$ and $\lambda = 10$.

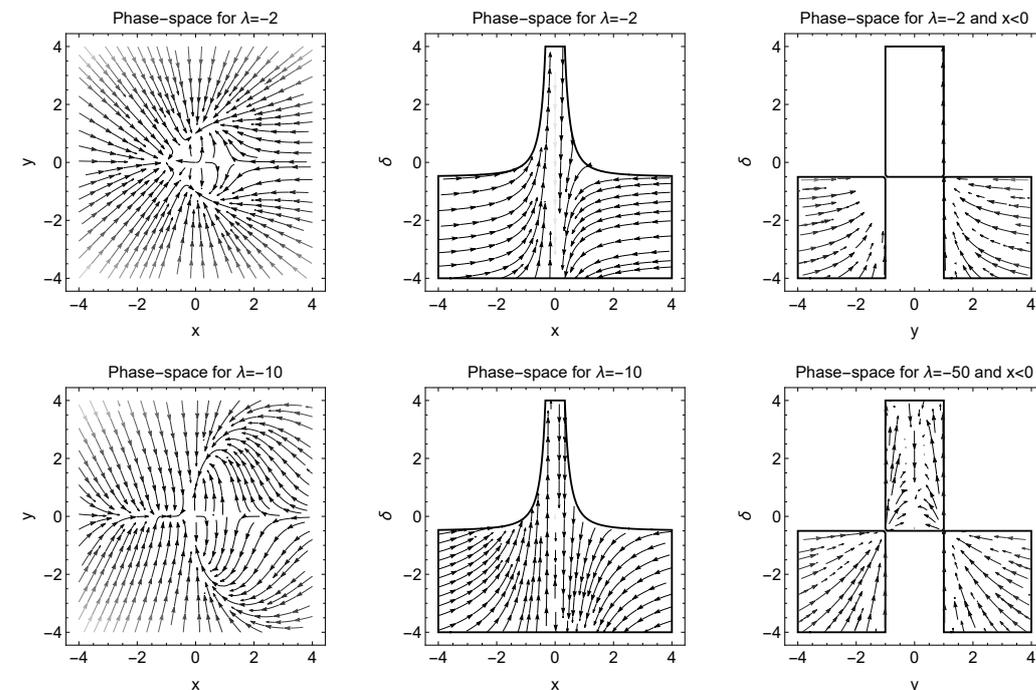


Figure 2. Phase-space portraits of the dynamical system (22), (25) on the two-dimensional planes (x, y) , (x, δ) and (y, δ) for $\lambda = -2$ and $\lambda = -10$.

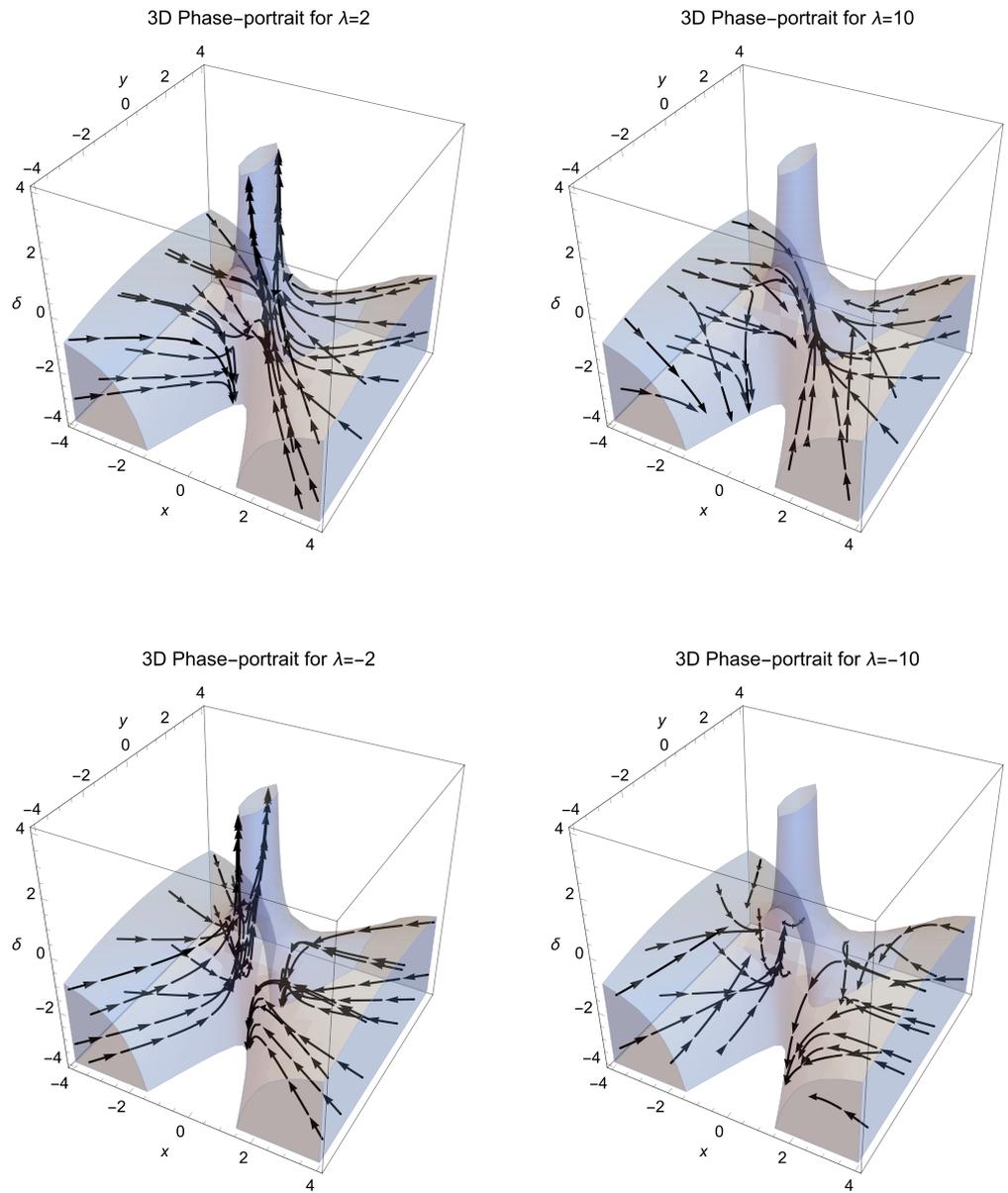


Figure 3. 3D phase-space portraits of the dynamical system (22), (25) for $\lambda = 2$, $\lambda = 10$, $\lambda = -2$ and $\lambda = -10$.

Subcase $\lambda = 0$

Consider the specific scenario where $\lambda = 0$, and the scalar field potential $V(\phi)$ assumes the role of the cosmological constant, i.e., $V(\phi) = V_0$.

Indeed, the dynamical system (23)–(25) becomes

$$\frac{dx}{d\tau} = \frac{3}{2}(\delta x^3 - xy^2), \tag{30}$$

$$\frac{d\delta}{d\tau} = -\frac{3}{2}\delta, \tag{31}$$

$$\frac{dy}{d\tau} = -\frac{y}{2}(3(y^2 - 3x^2\delta - 1)). \tag{32}$$

The stationary points are $B_1^\pm = (\pm 1, 0, 0)$ and $B_3 = (0, 0, 1)$. Point B_3 characterizes the de Sitter solution with $w_{eff}(B_3) = -1$, and it consistently acts as an attractor due to the negative eigenvalues of the linearized system.

In Figure 4 we present the three-dimensional phase-space for the latter dynamical system with $\lambda = 0$. We observe that B_3 is the future attractor of the dynamical system.

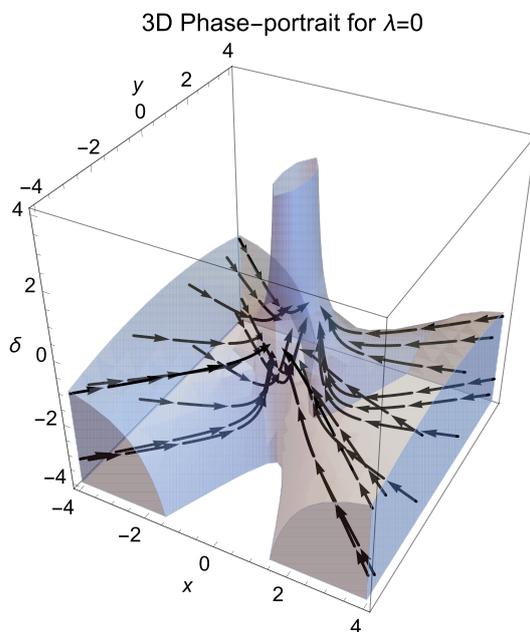


Figure 4. 3D phase-space portraits of the dynamical system (22), (25) for $\lambda = 0$, where B_3 is the unique attractor.

4.3. Case III: $U(\phi) = U_0e^{-\kappa\phi}$ and $V(\phi) = V_0e^{-\lambda\phi}$

Let us assume now the case where $U(\phi) = U_0e^{-\kappa\phi}$ and $V(\phi) = V_0e^{-\lambda\phi}$ and $\kappa \neq \lambda$. For this cosmological model, we introduce the variables

$$x = \frac{\dot{\phi}}{\sqrt{6}H}, \delta = \dot{\chi}_0 - 1, y^2 = \frac{V(\phi)}{3H^2}, \tag{33}$$

$$z^2 = \frac{U(\phi)}{H^2}, M = \chi_0 H, d\tau = H dt. \tag{34}$$

Thus, the field equations are expressed by the following algebraic-differential system

$$1 - x^2(1 + 2\delta) - y^2 = 0, \tag{35}$$

$$\frac{dx}{d\tau} = \frac{1}{6} \left(9x(x^2\delta - y^2) + \sqrt{6}\kappa z^2(1 + 3x^2M) \right), \tag{36}$$

$$\frac{d\delta}{d\tau} = -\frac{1}{6x} \left(\sqrt{6}\kappa z^2(1 + 3M + 2\delta) - 3\sqrt{6}\lambda y^2 + 9x\delta \right), \tag{37}$$

$$\frac{dy}{d\tau} = \frac{1}{2}y \left(3(1 - y^2 + x^2\delta) + \sqrt{6}x(\kappa z M - \lambda) \right), \tag{38}$$

$$\frac{dM}{d\tau} = 1 + \delta - \frac{1}{2}M \left(3(1 - y^2 + x^2\delta) + \sqrt{6}\kappa z^2 x M \right), \tag{39}$$

$$\frac{dz}{d\tau} = \frac{1}{2}z \left(3(1 - y^2 + x^2\delta) + \sqrt{6}\kappa x(Mz^2 - 1) \right), \tag{40}$$

and

$$w_{eff} = -y^2 + x^2\delta + \frac{\sqrt{6}}{3}\kappa Mxz^2. \tag{41}$$

The stationary points $C = (x(C), \delta(C), y(C), M(C), z(C))$ of the dynamical system (36)–(40) which satisfy the constraint Equation (35) are

$$C_1^\pm = \left(\pm 1, 0, 0, \frac{2}{3}, 0 \right), C_2 = \left(\frac{\sqrt{6}}{2} \frac{1}{\lambda}, \frac{2\lambda^2 - 3}{9}, \frac{\sqrt{2\lambda^2 - 3}}{\sqrt{6}\lambda}, \frac{4}{27}(\lambda^2 + 3), 0 \right)$$

$$C_3 = \left(x_3, \frac{1}{2} \left(\frac{1}{x_3} - 1 \right), 0, \left(\frac{1}{2} - \frac{1}{6x_3^2} - \frac{\sqrt{6}}{x_3} \kappa \right), \sqrt{\frac{x^2 \left(\sqrt{17 + 10x_3^2 + 9x_3^4} - 5 - x_3^2 \right)}{4(1 + x^2)}} \right)$$

where x_3 is a real solution of the algebraic equation

$$4x_3\kappa = \sqrt{\frac{3}{2}} \left(1 - 3x_3^2 + \sqrt{17 + 10x_3^2 + 9x_3^4} \right). \tag{42}$$

Points C_1^\pm and C_2 are the extensions of points B_1^\pm in B_2 for the nonconstant potential $U(\phi)$ in the five-dimensional space. Point C_3 is new and describes a universe where potential $U(\phi)$ contributes in the cosmological fluid. C_3 is real and physical, accepted for $\sqrt{17 + 10x_3^2 + 9x_3^4} - 5 - x_3^2 \geq 0$, that is, $x^2 \geq 1$. The effective equation of state parameter $w_{eff}(C_3) = \frac{1}{4} \left(\sqrt{17 + 10x_3^2 + 9x_3^4} - 3(1 + x_3^2) \right)$. We observe that $w_{eff}(C_3) \geq -\frac{1}{3}$, which means that point C_3 can not describe acceleration. In Figure 5, the $\kappa = \kappa(x_3)$ is given as it is presented in Equation (42).

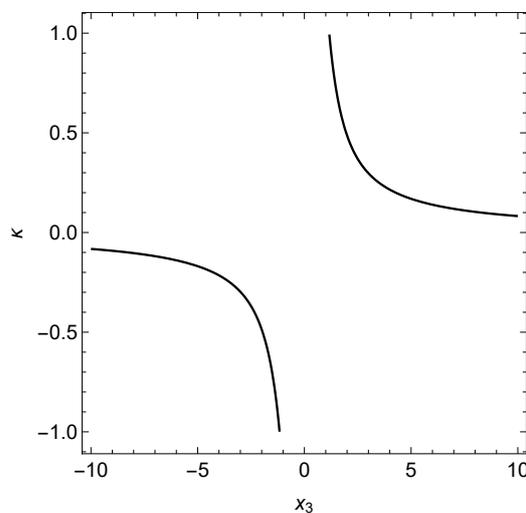


Figure 5. We plot the $\kappa = \kappa(x_3)$ as it is given by the algebraic equation $4x_3\kappa = \sqrt{\frac{3}{2}} \left(1 - 3x_3^2 + \sqrt{17 + 10x_3^2 + 9x_3^4} \right)$.

We observe that in this case where $U(\phi)$ contributes in the cosmological fluid there is not any stationary point which can describes inflation. We omit the presentation of the stability properties and we focus in the special case where $\kappa = \lambda$.

The results of this Section are summarized in Table 1.

Table 1. Stationary points and physical properties at the finite regime.

Model	$U(\phi)$	$V(\phi)$	Point	w_{eff}	Acceleration?
I	U_0	0	A_1^\pm	0	No
II	U_0	$V_0e^{-\lambda\phi}$	B_1^\pm	0	No
			B_2	0	No
$\text{II}_{(\lambda=0)}$	U_0	V_0	B_1^\pm	0	No
			B_3	-1	Yes
III	$U_0e^{-\kappa\phi}$	$V_0e^{-\lambda\phi}$	C_1^\pm	0	No
			C_2	0	No
			C_3	$w_{eff}(C_3) \geq -\frac{1}{3}$	No

5. Analysis at Infinity

Until now we have seen that the de Sitter universe exists only for the asymptotic solutions with $V(\phi)$ a constant, that is, $\lambda = 0$. We have investigated the existence of asymptotic solutions at the finite regime; however, the dynamical variables are constrained by the algebraic equation $1 - x^2(1 + 2\delta) - y^2 = 0$, which means that they can take values at infinity. In order to study the existence of asymptotic solutions at infinity we should introduce Poincare variables.

In case II, we focus on $U(\phi) = U_0$ and $V(\phi) = V_0e^{-\lambda\phi}$. We introduce the new variable $z = x^2\delta$, and the two-dimensional system in the plane $\{x, z\}$ reads

$$\frac{dz}{d\tau} = -\frac{3}{2}(1 - x^2 - 3z), \tag{43}$$

$$\frac{dx}{d\tau} = \frac{1}{2}(\sqrt{6}\lambda x(1 - x^2 - 2z) - 3z(3 - 2x^2 - 6z)). \tag{44}$$

Indeed, there exist the stationary points $B^* = (x(B^*), z(B^*))$ with coordinates $B_1^* = (\frac{\sqrt{6}}{2}\frac{1}{\lambda}, \frac{2\lambda^2-3}{9})$, $B_2^* = (0, \frac{1}{2})$ and $B_3^* = (0, 0)$. Point B_1^* is B_1 , while points B_2^* and B_3^* describe de Sitter solutions with $w_{eff}(B_2^*) = -1$ and $w_{eff}(B_3^*) = -1$.

As far as the stability properties are concerned, the resulting eigenvalues for point B_2^* are $\{\frac{9}{2}, \frac{3}{4}\}$ and for point B_3^* we derive $\{-\frac{9}{2}, -\frac{3}{2}\}$. We infer that B_2^* is always a source and B_3^* is always an attractor.

Because of the constraint equation $y^2 = 1 - x^2 - 2z$, at the new stationary points we calculate $y(B_2^*) = 0$ and $y(B_3^*) = 1$. Thus, point B_3^* is the de Sitter solution described by point C in [37].

In Figure 6, we present the two-dimensional phase-space portrait for the dynamical system (43), (44), from where it is clear that the de Sitter solution described by B_3^* is a future attractor for the dynamical system.

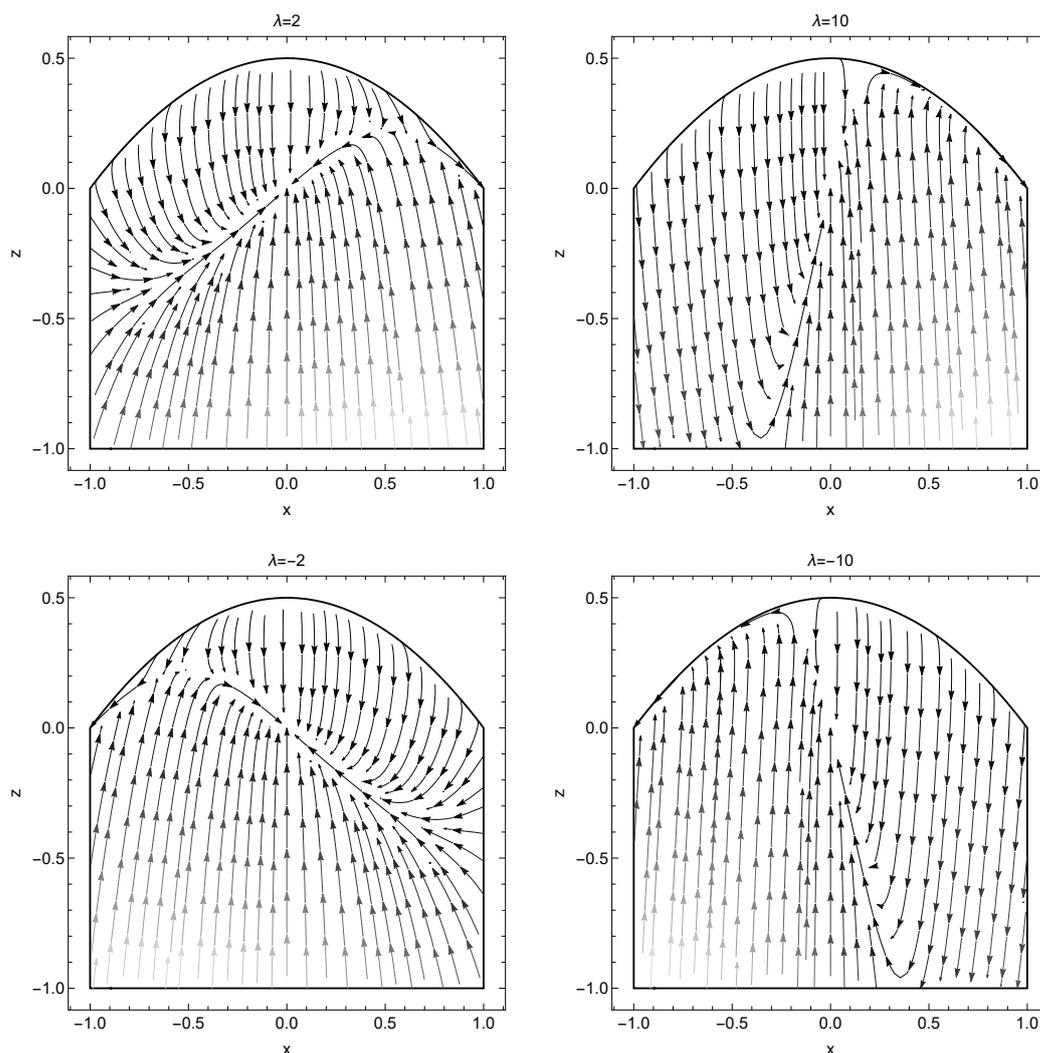


Figure 6. Phase-space portraits of the dynamical system (43), (44) on the two-dimensional planes (x, z) for $\lambda = 2, \lambda = 10, \lambda = -2$ and $\lambda = -10$.

6. Conclusions

In this study, we conducted a thorough phase-space analysis of the cosmological model, considering different forms of the potential functions. Our findings indicate that the cosmological model does not exhibit the de Sitter universe as an attractor, except in the case where the potential function $V(\phi)$ represents the cosmological constant. Specifically, we investigated the following cosmological scenarios for the two potential functions: (I) $U(\phi) = U_0, V(\phi) = 0$, (II) $U(\phi) = U_0, V(\phi) = V_0e^{-\lambda\phi}$ and (III) $U(\phi) = U_0e^{-\kappa\phi}, V(\phi) = V_0e^{-\lambda\phi}$.

Referring to Table 1, for the analysis at the finite regime, we observe that for Model I, two stationary points emerge, characterizing the era of matter dominance. In contrast, Model II, which involves an exponential potential, reveals three stationary points that pertain to matter-dominated epochs. This finding stands in contrast to the outcomes previously reported in [37,38]. However, in the scenario where the exponent λ becomes zero, leading to the cosmological constant limit, a de Sitter solution emerges. A natural question which arises is if there are other forms for the scalar field potential $V(\phi)$ and $U(\phi) = U_0$ in which the de Sitter universe exists as an asymptotic solution. For a general potential function $V(\phi)$ parameter $\lambda = -\frac{V_{,\phi}}{V}$ is not a constant but a dynamical parameter which satisfies the equation

$$\frac{d\lambda}{d\tau} = -\sqrt{6x}\lambda^2(\Gamma(\lambda) - 1), \quad \Gamma(\lambda) = \frac{V(\phi(\lambda))V_{,\phi\phi}(\phi(\lambda))}{(V_{,\phi}(\phi(\lambda)))^2}. \quad (45)$$

Thus, the stationary points depend on the value of λ which solves the algebraic equation $\lambda^2(\Gamma(\lambda) - 1) = 0$. Consequently, a de Sitter point exist when $\lambda = 0$ is a root of the algebraic equation $\lambda^2(\Gamma(\lambda) - 1) = 0$. For the potential $V(\phi) = V_0 e^{-\sigma_0 \phi} + \Lambda$, it follows that $\Gamma(\lambda) = \frac{\sigma_0}{\lambda}$, which means that the algebraic equation reads $\lambda(\sigma_0 - \lambda) = 0$, with solutions $\lambda = \sigma_0$ and $\lambda = 0$. As a result, on the surface $\lambda = 0$, the de Sitter point exist. On the other hand, for the power-law potential $V(\phi) = V_0 \phi^{-\alpha}$, it follows $\Gamma(\lambda) = \frac{\alpha+1}{\alpha}$, and the algebraic equation reads $\lambda^2 = 0$, that is, the de Sitter solution exist. Of course the stability properties of the de Sitter solution depend on the function form of $V(\phi)$. It is important to note that the asymptotic solution with $\lambda = 0$ leads to a scalar field potential $V(\phi)$ which remains constant.

For the third cosmological scenario, Model III, the absence of solutions capable of describing acceleration is notable. An exception is found only in the special case where $\kappa = \lambda$. In this circumstance, a collection of stationary points corresponding to scaling solutions manifests. Although accelerated solutions are present, they do not provide a de Sitter universe at the finite regime.

However, because the dynamical variables of the field equations are not constrained, they can reach infinity. Specifically for Model II, we investigate the existence of stationary points when $\delta \rightarrow \infty$. We found that, at infinity, there exist two stationary points which correspond to two de Sitter solutions. The one solution is always a source, while the other is always an attractor.

The current investigation underscores the potency of phase-space analysis as a robust tool for assessing the viability of gravitational models and constraining proposed theories. Importantly, this analysis can also serve as a classification and selection rule for the initial value problem. For instance, Model II, for $|\lambda| > \sqrt{\frac{3}{2}}$, has two attractors, one of the points B_1^\pm and the stationary point B_3^* . On the other hand, for $|\lambda| < \sqrt{\frac{3}{2}}$, B_3^* is the unique attractor, and the de Sitter universe is the unique future solution.

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