Introductory Lectures on Quantum Field Theory and non-Abelian Gauge Theories

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An Introduction to:

Quantum Fields, Particles and Geometry

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Chapter 0

Notations and Conventions

In this notes, as usual in the literature of particle physics (the leadership of opposition being the relativity community), the metric signature is chosen to be $\eta := \mathrm{diag}\,(+1,-1,...,-1)$. Moreover, natural units, whereby $\hbar \equiv 1$ and $c \equiv 1$, are adopted.

The interval $0 < ... < d < D$ is the set for which the D–dimensional spacetime indices M, N etc. belongs . Let $x^M := \left(x^0, x^1, ..., x^d\right)$ be an inertial coordinate system in the $D = 1+d$ dimensional Minkowski spacetime, $\mathbf{M}^D := \mathbb{R}^{1,d}$.

A convenient notation for the spatial coordinates associated to the spacetime event x^M is $\mathbf{x} := \left(x^1, ..., x^d\right)$. The euclidean *norm* $\|\mathbf{x}\|$ of $\mathbf{x}$ is:

$$\|\mathbf{x}\| := \sqrt{\sum_{k=1}^{d} \left(x^k\right)^2}. \tag{1}$$

Given two events in $\mathbf{M}^D$, say x and y, the lorentz product is written by the juxtaposition of symbols:

$$xy := \eta_{MN} x^M y^N = x^0 y^0 - \mathbf{x} \cdot \mathbf{y}. \tag{2}$$

Recall that the d'Alembert operator reads:

$$\Box_x = \eta_{MN} \partial_x^M \partial_x^N = \left(\frac{\partial}{\partial x^0}\right)^2 - \nabla_x^2. \tag{3}$$

For simplicity's sake, we write $\partial_x^M := \partial / \partial x^M$.

In the absence of ambiguity, the coordinate label may be omitted, say, $\partial_x^M = \partial^M$. Moreover, when only one spacetime event is concerned, the x's timelike coordinate is written as $t := x^0$.

One writes:

$$f(r) \sim \mathscr{O}\left[g(r)\right], \tag{4}$$

when there are $M, L > 0$ for which, when $|r| > L$, it holds that $|f(r)| \leq M\,|g(r)|$.

Chapter 1

Quantum Methods

1.1 Feynman's Pathintegral

The evolution of the quantum state of a particle in the real line is described by the Hamiltonian operator $\mathbf{H}(\mathbf{q},\mathbf{p})$, called hamiltonian from now on. Here, $\mathbf{q}$ (resp. $\mathbf{p}$) is the position (resp. momentum) operator. I am working in Schrödinger–picture.

Take two points in the line, say q and q'. Let $t < t'$ be a time interval. The quantity of interest is:

$$K\left(q',t';q,t\right) := \left\langle q'\right| e^{-i\hbar^{-1}(t'-t)\mathbf{H}(\mathbf{q},\mathbf{p})} \left|q\right\rangle . \tag{1.1}$$

Physically, $K(q',t';q,t)$ is the probability amplitude that an electron, say, detected in a monolithic active pixel sensor (MAPS) at position q when my clock marked t, will be detected at position q' when my clock reads t'. Mathematically, it is the fundamental solution to the Schrödinger equation in the space–representation (space-rep.). Hence the name *kernel* for the beast.

I will partition my MAPS between the points q and q' in N steps, $q_0, q_1 \ldots, q_{N-1}$. Likewise, ignoring what Bishop Berkeley (and Zeno for that matter) warned, discretize time as well. Let $t_0 = t$, $t_N = t'$ and, by induction, define $t_{k+1} = t_k + \varepsilon$ for all $k = 1,\ldots,N-1$. Here $\varepsilon := N^{-1}(t'-t)$.

Use completeness of the basis for state-rep. to derive:

$$\int dq \left|q\right\rangle \left\langle q\right| = 1 \implies K\left(q',t';q,t\right) = \prod_{k=0}^{N-1} \int dq_k \, a_k, \tag{1.2}$$

where:

$$a_k := \left\langle q_{k+1}\right| e^{-i\hbar^{-1}\varepsilon\mathbf{H}(\mathbf{q},\mathbf{p})} \left|q_k\right\rangle = \left\langle q_{k+1}\right| e^{-i\hbar^{-1}\varepsilon[\mathbf{K}(\mathbf{p})+\mathbf{V}(\mathbf{q})]} \left|q_k\right\rangle . \tag{1.3}$$

I assumed in the last equality the separability of the hamiltonian into a kinetic term $\mathbf{K}(\mathbf{p})$ and a potential $\mathbf{V}(\mathbf{q})$. From power series expansion[1] I deduce:

$$e^{\varepsilon(\mathbf{A}+\mathbf{B})} = e^{\varepsilon\mathbf{A}}e^{\varepsilon\mathbf{B}} + \mathscr{O}\left(\varepsilon^2\right) \implies a_k = \left\langle q_{k+1}\right| e^{i\hbar^{-1}\varepsilon\mathbf{K}(\mathbf{p})} e^{i\hbar^{-1}\varepsilon\mathbf{V}(\mathbf{q})} \left|q_k\right\rangle + \mathscr{O}\left(\varepsilon^2\hbar^{-2}\right), \text{ and} \tag{1.4}$$

$$e^{-i\hbar^{-1}\varepsilon\mathbf{V}(\mathbf{q})} \left|q_k\right\rangle = \underbrace{e^{-i\hbar^{-1}\varepsilon V(q_k)}}_{\text{c-number}} \left|q_k\right\rangle \implies a_k = e^{-i\hbar^{-1}\varepsilon V(q_k)} \left\langle q_{k+1}\right| e^{-i\hbar^{-1}\varepsilon\mathbf{K}(\mathbf{p})} \left|q_k\right\rangle + \mathscr{O}\left(\varepsilon^2\hbar^{-2}\right). \tag{1.5}$$

[1] In publication, I may spell the witchery Baker–Campbell–Hausdorff to display my erudition.

Use completeness of the basis for state-rep. to see:

$$\int dp_{k+1} \left|p_{k+1}\right\rangle \left\langle p_{k+1}\right| = 1 \implies \left\langle q_{k+1}\right| e^{i\hbar^{-1}\varepsilon \mathbf{K}(\mathbf{p})} \left|q_k\right\rangle = \int dp_{k+1} \left\langle q_{k+1}\right| e^{i\hbar^{-1}\varepsilon \mathbf{K}(\mathbf{p})} \left|p_{k+1}\right\rangle \left\langle p_{k+1}\right| q_k\rangle , \tag{1.6}$$

Again, from power series expansion applied to an eigenstate equation,

$$e^{i\hbar^{-1}\varepsilon \mathbf{K}(\mathbf{p})} \left|p_{k+1}\right\rangle = \underbrace{e^{i\hbar^{-1}\varepsilon K(p_{k+1})}}_{\text{c-number}} \left|p_{k+1}\right\rangle . \tag{1.7}$$

Moreover, I know $\langle q|\, p\rangle = (2\pi\hbar)^{-1/2} e^{i\hbar^{-1}pq}$, so:

$$\left\langle q_{k+1}\right| e^{-i\hbar^{-1}\varepsilon \mathbf{K}(\mathbf{p})} \left|q_k\right\rangle = \int dp_{k+1}\, e^{-i\hbar^{-1}\varepsilon K(p_{k+1})} \left\langle q_{k+1}\right| p_{k+1}\rangle \times \left\langle p_{k+1}\right| q_k\rangle \tag{1.8}$$

$$= \int \frac{dp_{k+1}}{2\pi\hbar}\, e^{i\hbar^{-1}\varepsilon\left[\frac{p_{k+1}(q_{k+1}-q_k)}{\varepsilon} - K(p_{k+1})\right]}. \tag{1.9}$$

Thence:

$$a_k = \int \frac{dp_{k+1}}{2\pi\hbar}\, e^{i\hbar^{-1}\varepsilon\left[\frac{p_{k+1}(q_{k+1}-q_k)}{\varepsilon} - H(q_{k+1},p_{k+1})\right]} + \mathscr{O}\left(\varepsilon^2\hbar^{-2}\right) \tag{1.10}$$

I can finally write the kernel in discretized form:

$$K\left(q',t';q,t\right) = \prod_{k=0}^{N-1} \left\{ \left(\int dq_k \int \frac{dp_{k+1}}{2\pi\hbar} \right) e^{i\hbar^{-1}\varepsilon\left[\frac{p_{k+1}(q_{k+1}-q_k)}{\varepsilon} - H(q_{k+1},p_{k+1})\right]} \right\} + \mathscr{O}\left(\varepsilon^2\hbar^{-2}\right). \tag{1.11}$$

As the exponential maps a sum into the product of the exponentials,

$$\prod_{k=0}^{N-1} \left(\int dq_k \int \frac{dp_{k+1}}{2\pi\hbar} \right) e^{i\hbar^{-1}\varepsilon\left[\frac{p_{k+1}(q_{k+1}-q_k)}{\varepsilon} - H(q_{k+1},p_{k+1})\right]} \tag{1.12}$$

$$= \int \left(\prod_{k=0}^{N-1} dq_k \right) \times \int \left(\prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar} \right) e^{i\hbar^{-1}\varepsilon \sum_{k=0}^{N-1}\left[\frac{p_{k+1}(q_{k+1}-q_k)}{\varepsilon} - H(q_{k+1},p_{k+1})\right]}. \tag{1.13}$$

This is how I like to arrive at Feynman pathintegral in phasespace[2],

$$\int_{q=q(t)}^{q'=q(t')} [q(t)] \int \left[\frac{dp(t)}{2\pi\hbar}\right] := \lim_{N\longrightarrow\infty} \int \left(\prod_{k=0}^{N-1} dq_k \right) \times \int \left(\prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar} \right) \tag{1.14}$$

By the teachings of Legendre and Riemann, the action can be written as:

$$I := \int_{t_0}^{t_N} dt\, \left[p(t)\,\dot{q}(t) - H(q,p)\right] = \lim_{N\longrightarrow\infty} \sum_{k=0}^{N-1} \frac{t_N - t_0}{N} \left[\frac{p_{k+1}\left(q_{k+1}-q_k\right)}{\varepsilon} - H\left(q_{k+1},p_{k+1}\right) \right], \tag{1.15}$$

[2]There are no mistypes in the sentence you read, I hope.

and thereby our kernel becomes:

$$K\left(q',t';q,t\right)=\lim_{N\longrightarrow\infty}\int\left(\prod_{k=0}^{N-1}dq_k\right)\times\int\left(\prod_{k=1}^{N}\frac{dp_k}{2\pi\hbar}\right)e^{i\hbar^{-1}\sum_{k=0}^{N-1}\varepsilon\left[\frac{p_{k+1}(q_{k+1}-q_k)}{\varepsilon}-H(q_{k+1},p_{k+1})\right]}. \tag{1.16}$$

Lastly, from aesthetic needs, I type:

$$K\left(q',t';q,t\right)=\int_{q=q(t)}^{q'=q(t')}\left[q\left(t\right)\right]\int\left[\frac{dp(t)}{2\pi\hbar}\right]e^{i\hbar^{-1}I[q(t),p(t)]}. \tag{1.17}$$

In nonrelativistic quantum mechanics, the hamiltonian for a mass $M>0$ particle interacting with the potential $V\left(q\right)$ is

$$H\left(q,p\right)=\frac{p^2}{2M}+V(q). \tag{1.18}$$

Completing the circle (however in complex–plane and spelling Jordan),

$$\int dx\,e^{-\frac{1}{2}|A|x^2+|B|x}=\sqrt{\frac{2\pi}{A}}e^{-\frac{B^2}{2|A|}},\ \forall\left(A,B\in\mathbb{R},A\neq0\right).$$

In faith *élan vital* survives analytic continuation, Eqs.(1.10)–(1.17) yields:

$$a_k=e^{-i\hbar^{-1}\varepsilon V(q_{k+1})}\int\frac{dp_{k+1}}{2\pi\hbar}e^{i\hbar^{-1}\varepsilon\left[-\frac{(p_{k+1})^2}{2M}+\frac{(q_{k+1}-q_k)}{\varepsilon}p_{k+1}\right]}=\sqrt{\frac{M}{2\pi i\hbar\varepsilon}}e^{i\hbar^{-1}\varepsilon\left[\frac{M}{2}\left(\frac{q_{k+1}-q_k}{\varepsilon}\right)^2-V\left(q_{q_{k+1}}\right)\right]}, \tag{1.19}$$

$$K\left(q_N,t_N;q_0,t_0\right)=\lim_{N\longrightarrow\infty}\left(\frac{M}{2\pi i\hbar\varepsilon}\right)^{N/2}\int\left(\prod_{k=0}^{N-1}dq_k\right)e^{i\hbar^{-1}\sum_{k=0}^{N-1}\varepsilon\left[\frac{M}{2}\left(\frac{q_{k+1}-q_k}{\varepsilon}\right)^2-V\left(q_{q_{k+1}}\right)\right]}, \tag{1.20}$$

$$I=\int_{t_0}^{t_N}dt\left[\frac{M}{2}\dot{q}\left(t\right)^2-V\left(q\left(t\right)\right)\right]:=\int_{t_0}^{t_N}dt\,L\left(q\left(t\right),\dot{q}\left(t\right)\right), \tag{1.21}$$

$$\int_{q_0=q(t_0)}^{q_N=q(t_N)}\left[q\left(t\right)\right]:=\lim_{N\longrightarrow\infty}\left(\frac{M}{2\pi i\hbar\varepsilon}\right)^{N/2}\int\left(\prod_{k=0}^{N-1}dq_k\right), \tag{1.22}$$

$$K\left(q_N,t_N;q_0,t_0\right)=\int_{q_0=q(t_0)}^{q_N=q(t_N)}\left[q\left(t\right)\right]\exp\left\{\frac{i}{\hbar}\int_{t_0}^{t_N}dt\left[\frac{M}{2}\dot{q}\left(t\right)^2-V\left(q\left(t\right)\right)\right]\right\}. \tag{1.23}$$

So reads Feynman pathintegral in configurationspace[3]:

$$K\left(q_N,t_N;q_0,t_0\right)=\int_{q_0=q(t_0)}^{q_N=q(t_N)}\left[q\left(t\right)\right]e^{i\hbar^{-1}I}. \tag{1.24}$$

[3]There are no mistypes in the sentence you read, I hope.

1.2 Wick's Combinatorics

On what follows, let $A := \left(A_{ij}\right)$ be a real, symmetric and nonsingular $(N \times N)$–matrix. I will denote by $\Delta_{ij} := \left(A^{-1}\right)_{ij}$ the inverse. The permutation group of k–points will be denoted by S_k.

The inner product of vectors $x = (x_1, ..., x_N)$ and $y = (y_1, ..., y_N)$ in N–space reads:

$$(x,y) := \sum_{k=1}^{N} x_k y_k \implies (x,Ay) = \sum_{i,k=1}^{N} x_i A_{ij} y_j = X^T (AY). \tag{1.25}$$

In the last equality, X (resp. Y) is the line matrix representing the vector x (resp. y). The transpose of X is the column matrix X^T.

The "mean" value of a quantity $F(x_1, ..., x_N)$ in the gaussian distribution $C_N^{-1} e^{-\frac{1}{2}(x,Ax)}$ is:

$$\langle F(x) \rangle := \int \frac{d^N x}{C_N} e^{-\frac{1}{2}(x,Ax)} F(x). \tag{1.26}$$

I fix the normalization constant C_N by requiring $\langle 1 \rangle \equiv 1$. This gives to me that:

$$C_N = \frac{\sqrt{|\det A|}}{(2\pi)^{N/2}} \implies \langle F(x) \rangle = \frac{(2\pi)^{N/2}}{\sqrt{|\det A|}} \int d^N x \, e^{-\frac{1}{2}(x,Ax)} F(x). \tag{1.27}$$

Happily, this rather cumbersome expression will not be needed below. (I showed it for completeness, which rather important in quantum mechanics.)

I prove Wick theorem with help from a simple identity. First note:

$$\int d^N x \frac{\partial}{\partial x_j} \left(e^{-\frac{1}{2}(x,Ax)} F(x) \right) = 0. \tag{1.28}$$

Second, apply chain rule:

$$\left\langle \frac{\partial F}{\partial x_j} \right\rangle = \int \frac{d^N x}{C_N} e^{-\frac{1}{2}(x,Ax)} \frac{\partial F}{\partial x_j} = \sum_{k=1}^{N} \int \frac{d^N x}{C_N} e^{-\frac{1}{2}(x,Ax)} F(x) A_{jk} x_k \tag{1.29}$$

Finally, multiply both sides by Δ_{ij} and sum over i:

$$\sum_{j=1}^{N} \Delta_{ij} \left\langle \frac{\partial F}{\partial x_j} \right\rangle = \sum_{j,k=1}^{N} \int \frac{d^N x}{C_N} e^{-\frac{1}{2}(x,Ax)} F(x) \Delta_{ij} A_{jk} x_k = \int \frac{d^N x}{C_N} e^{-\frac{1}{2}(x,Ax)} F(x) x_i = \langle x_i F(x) \rangle \tag{1.30}$$

The result is our trick to prove Wick's theorem:

$$\langle x_i F(x) \rangle = \sum_{j=1}^{N} \Delta_{ij} \left\langle \frac{\partial F}{\partial x_j} \right\rangle. \tag{1.31}$$

As a preliminary, take $F(x) = 1$,

$$G_1(x) = x_i = \sum_j \Delta_{ij} \left\langle \frac{\partial}{\partial x_j} 1 \right\rangle = 0. \tag{1.32}$$

Now a simple notation will prove itself useful, even to *state* Wick theorem in full generality. For a given ordered list of N elements, say $(x_1,...,x_N)$, I will use a hat to denote the ordered list of $N-1$ elements with the point x_j removed,

$$\left(x_1,...,\widehat{x_j},...,x_N\right) := \left(x_1,...,x_{j-1},x_{j+1},...,x_N\right). \tag{1.33}$$

Replace $x_i \mapsto x_{i_{2k+1}}$ and let $F(x) = x_{i_1}...x_{i_{2k}}$ in Eq.(1.31). By $\langle x_i\rangle = 0$ and recursion,

$$\left\langle x_{i_1}...x_{i_{2k+1}}\right\rangle = G_{2k+1}\left(i_1,...,i_{2k+1}\right) = \sum_{j_1=1}^{N} \Delta_{(i_{2k+1})j_1} G_{2k-1}\left(i_1,...,\widehat{i_{j_1}},...,i_{2k}\right) \tag{1.34}$$

$$= \sum_{j_1,j_2=1}^{N} \Delta_{(i_{2k+1})j_1}\Delta_{(i_{2k})j_2} G_{2k-3}\left(i_1,...,\widehat{i_{j_1}},...,\widehat{i_{j_2}},...,i_{2k-1}\right) = ... = 0. \tag{1.35}$$

Likewise, substitute $x_i \mapsto x_{i_{2k}}$ and use $F(x) = x_{i_1}...x_{i_{2k-1}}$ in Eq.(1.31).

$$\langle x_{i_1}...x_{i_{2k}}\rangle = G_{2k}\left(i_1,...,i_{2k}\right) = \sum_{j_1=1}^{N} \Delta_{(i_{2k})j_1} G_{2k-2}\left(i_1,...,,\widehat{i_{j_1}},...,i_{2k-1}\right) \tag{1.36}$$

$$= \sum_{j_1,j_2=1}^{N} \Delta_{(i_{2k})j_1}\Delta_{(i_{2k-1})j_2} G_{2k-4}\left(i_1,...,,\widehat{i_{j_1}},...,\widehat{i_{j_2}},...,i_{2k-2}\right) = ... = \sum_{\varphi\in S_k}\prod_{\sigma=1}^{k} \Delta_{\varphi(\sigma)\sigma}. \tag{1.37}$$

1.3 Propagators: *"A Study in Epsilon"*

> "One of the most remarkable discoveries in elementary particle physics has been that of the complex plane, the theory of functions of complex variables plays the role not of a mathematical tool, but of a fundamental description of nature inseparable from physics."
> – J. Schwinger [Schwinger(1998)]

1.3.1 Remarks on the *D*–dimensional Green Function

In this section, we comment on some issues regarding *a* green function $G^D(x)$ of the Klein-Gordon operator $\Box_x + m^2$ with "mass" $m > 0$:

$$\left(\Box_x + m^2\right) G^{(D)}(x) = -\delta^D(x). \tag{1.38}$$

Here, $\delta^D(x)$ is the D–dimensional Dirac distribution.

1.3.1.1 Feynman's $i\varepsilon^+$

The D–dimensional representation (rep., from now on) of $\delta^D(x)$ is [Friedlander(1998)]:

$$\delta^D(x) = \int \frac{d^Dk}{(2\pi)^D} e^{-ikx}. \tag{1.39}$$

Multiplying both sides of Eqs.(1.38, 1.39) by e^{ikx} and integrating over the momenternegy–space, one finds the *formal* fourier transformed rep. of Δ^D:

$$G^D(x) = \int \frac{d^D k}{(2\pi)^D} \frac{e^{-ik\cdot x}}{k^2 - m^2}. \tag{1.40}$$

The integral of Eq.(1.39) is ill-posed. In order to regularize the latter, we follow Feynman's prescription, well motivated in [Zee(2010), Schwartz(2014), Sec.14.5]. On what follows, this method will be referred to as the $(i\varepsilon^+)$–regularization.

Let ε^+ be a positive infinitesimal [Bell(2008)] and define[4]:

$$K^D(x) = \int \frac{d^D k}{(2\pi)^D} \frac{e^{-ik\cdot x}}{k^2 - m^2 + i\varepsilon^+}. \tag{1.41}$$

Applying the Klein-Gordon operator to Eq.(1.41),

$$\left(\Box_x + m^2\right) K^D(x) = \int \frac{d^D k}{(2\pi)^D} \frac{\left(\Box_x + m^2\right) e^{-ik\cdot x}}{k^2 - m^2 + i\varepsilon^+} \tag{1.42}$$

$$= -\int \frac{d^D k}{(2\pi)^D} \frac{k^2 - m^2}{k^2 - m^2 + i\varepsilon^+} e^{-ik\cdot x} \tag{1.43}$$

$$= -\int \frac{d^D k}{(2\pi)^D} \frac{1}{1 + \frac{i\varepsilon^+}{k^2 - m^2}} e^{-ik\cdot x} \tag{1.44}$$

$$= -\int \frac{d^D k}{(2\pi)^D} \left(1 + \frac{i\varepsilon^+}{k^2 - m^2}\right)^{-1} e^{-ik\cdot x}. \tag{1.45}$$

Using the fact that ε^+ is an infinitesimal, under the assumption $K^D(x)$ is a green function to $\Box_x + m^2$, one derives that:

$$\left(\Box_x + m^2\right) K^D(x) = -\int \frac{d^D k}{(2\pi)^D} \left(1 - \frac{i\varepsilon^+}{k^2 - m^2}\right) e^{-ik\cdot x} = -\delta^{(D)}(x) + i\varepsilon^+ K^D(x), \tag{1.46}$$

and hence:

$$\left(\Box_x + m^2 - i\varepsilon^+\right) K^D(x) = -\delta^{(D)}(x). \tag{1.47}$$

Accordingly, Feynman's regularization in energy-momentum space should be accompanied by a transformation in configuration–space, resp.:

$$p^2 \mapsto p^2 + i\varepsilon^+, \; x^2 \mapsto x^2 - i\varepsilon^+. \tag{1.48}$$

1.3.1.2 Cauchy's Residues

Finally, Eq.(23) of [Zee(2010), Sec.I.3] is generalized to D–dimensional Minkowski spacetime.

Consider the following meromorphic function:

$$f(z) = \frac{e^{-itz}}{z^2 - \left(\mathbf{k}^2 + m^2 - i\varepsilon^+\right)}. \tag{1.49}$$

[4] $K^D(x)$ will be referred to as the $(i\varepsilon^+)$–regulated kernel.

There are two second–order poles are located at:

$$z_{\pm} = \pm (E_k - i\delta), \; E_k := \sqrt{\mathbf{k}^2 + m^2}, \tag{1.50}$$

where the infinitesimal $\delta := \varepsilon^+/2E_k \sim \mathscr{O}(\varepsilon^+)$.

Using the theory of residues [Stein and Shakarchi(2010)], from Eq.(1.41), one derives:

$$K^D(x) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{2\pi} \int dz \left(\theta\left(x^0\right) + \theta\left(-x^0\right)\right) f(z) \tag{1.51}$$

$$= \int \frac{d^d k}{(2\pi)^d} \frac{1}{2\pi} \left\{ -\theta\left(x^0\right) \left[2\pi i \operatorname*{Res}_{z \to z_-} f(z)\right] + \theta\left(-x^0\right) \left[2\pi i \operatorname*{Res}_{z \to z_+} f(z)\right] \right\}. \tag{1.52}$$

Therefore, the final form of the $(i\varepsilon^+)$–regulated kernel is:

$$K^D(x) = \frac{1}{2i(2\pi)^d} \int \frac{dk^d}{E_k} \exp\left[-i\left(E_k |t| - \mathbf{k}\cdot\mathbf{x}\right)\right]. \tag{1.53}$$

1.3.2 Horror vacui

"Nature abhors a vacuum."
– Aristotle, *Phys.* IV, 208a–223b.

This section shows the exponential decay of the $(i\varepsilon^+)$–regulated kernel $K^D(x)$ (cf. Eq.(1.41)), for spacelike separated events $-x^2 > 0$. We study the cases of $D = 4$ ([Zee(2010), Exercises I.3.1], resp.) and $D = 2$ ([Zee(2010), Exercises I.3.2], resp.).

1.3.2.1 "Nature" ($D = 4$)

Applying Eq.(1.53) to the dimensionality of the physical spacetime (as experimentally accepted at the time of writing), the Feynman propagator reads:

$$D_F(t, \mathbf{x}) := K^{(1+3)}(x) = \frac{1}{2i(2\pi)^3} \int \frac{d\mathbf{p}^3}{E_p} e^{-i\left(E_p|t| - \mathbf{p}\cdot\mathbf{x}\right)}. \tag{1.54}$$

We replaced $\mathbf{k} \mapsto \mathbf{p}$ in Eq.(1.54), as this is the familiar momentum–vector in $d = 3$ dimensions.

Likewise, to use the $d = 3$ commonplace notation, let $\vec{x} := \mathbf{x}$ (resp., $\vec{p} := \mathbf{p}$) be a non-zero position–vector (resp., momentum–vector). Define the *polar angle* $\phi(\mathbf{x}, \mathbf{p})$ between $\vec{x}$ and $\vec{p}$ by the implicit function:

$$\cos\phi(\vec{x}, \vec{p}) := \frac{\vec{x} \cdot \vec{p}}{|\vec{x}|\,|\vec{p}|}. \tag{1.55}$$

Moreover,

$$r(\vec{x}) := \sqrt{\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2}, \tag{1.56}$$

is the *radial* coordinate.

With the help of Eqs.(1.55, 1.56), the momentum–space can be parametrized with spherical coordinates, such that the integral of Eq.(1.54) gives:

$$D_F(t,\vec{x}) = \frac{1}{2i(2\pi)^3}\left(\int_0^\infty dp\,k^2\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^\pi \sin\phi\right)\frac{1}{E_p}e^{-i(E_k|t|-pr\cos\phi)} \tag{1.57}$$

$$= \frac{1}{2i(2\pi)^2}\int_0^\infty dp\,\frac{p^2e^{-iE_p|t|}}{E_p}\int_0^\pi \sin\phi\, e^{ipr\cos\phi} = -\frac{1}{2(2\pi)^2 r}\int_{-\infty}^\infty \frac{dp}{E_p}k\exp\left(-iE_p|t|\right)e^{ipr}. \tag{1.58}$$

Using the simple identity:

$$e^{ipr} = -\frac{i}{k}\frac{\partial}{\partial r}e^{ipr}, \tag{1.59}$$

Eq.(1.58) assumes the following compelling form:

$$D_F(t,\vec{x}) = \frac{1}{2(2\pi)^2 r}\frac{\partial}{\partial r}\int_{-\infty}^\infty \frac{dp}{E_p}\exp\left[-i\left(E_p|t|-pr\right)\right] \tag{1.60}$$

$$= \frac{1}{2(2\pi)^2 r}\frac{\partial}{\partial r}\int_{-\infty}^\infty \frac{dp}{E_p}\exp\left[i\left(pr-E_p|t|\right)\right]. \tag{1.61}$$

Now, introduce the energy-momentum parametrization:

$$p(\xi) := m\sinh\xi \implies E(\xi) = m\cosh\xi,\ (-\infty<\xi<\infty). \tag{1.62}$$

From Eqs.(1.62, 1.60)),

$$D_F(t,\vec{x}) = \frac{1}{2(2\pi)^2 r}\frac{\partial}{\partial r}\int_{-\infty}^\infty d\xi\,\exp\left[im\left(r\sinh\xi-|t|\cosh\xi\right)\right]. \tag{1.63}$$

If x is spacelike, then a parameter $0<\beta<\infty$ exists [Naber(2003)], for which:

$$r = \sqrt{-x^2}\cosh(\beta),\, t = \sqrt{-x^2}\sinh(\beta). \tag{1.64}$$

Accordingly, by Eqs.(1.61, 1.64) and the hyperbolic identity:

$$\sinh(\xi-\beta) = \sinh\beta\cosh\xi - \cosh\beta\sinh\xi, \tag{1.65}$$

one deduces:

$$D_F(t,\vec{x}) = \frac{1}{2(2\pi)^2 r}\frac{\partial}{\partial r}\int_{-\infty}^\infty d\xi\,\exp\left[im\sqrt{-x^2}\sinh(\xi-\beta)\right] \tag{1.66}$$

$$= \frac{1}{2(2\pi)^2 r}\frac{\partial}{\partial r}\int_{-\infty}^\infty d\xi\,\exp im\sqrt{-x^2}\sinh\xi. \tag{1.67}$$

Remark 1.1. In line 1.67 we used the constancy of β, which depends only on the given event x.

Remark 1.2. The integral in last equation of Eq.(1.67) is ill-defined, for:

$$\int_{-\infty}^{\infty} d\xi \underbrace{\exp im\sqrt{-x^2}\sinh\xi}_{\cos\left(\sqrt{-x^2}\sinh\xi\right)+i\sinh\sqrt{-x^2}\sinh\xi} = 2\int_0^{\infty} d\xi\ \cos\underbrace{m\sqrt{-x^2}\sinh\xi}_{:=\phi(\xi)}, \tag{1.68}$$

where $\cos\phi(\xi)$ oscillates indefinitely, when $0 \le \phi(\xi) < \infty$ as $0 < \xi < \infty$. In some form or another, equation

$$D_F(t,\vec{x}) = \frac{1}{2(2\pi)^2 r}\frac{\partial}{\partial r}\left(2\int_0^{\infty} d\xi\ \cos m\sqrt{-x^2}\sinh\xi\right), \tag{1.69}$$

is found in many standard textbooks, e.g., [Zee(2010), Greiner and Reinhardt(2008)].

Remark 1.3. Nevertheless, in studying the fourier transform of Coulomb potential, [Schwartz(2014), Sec. 3.5.4, Eq.(3.64)] regulates an integral similar to Eq.(1.69) replacing $|\mathbf{k}| \mapsto |\mathbf{k}| + i\delta^+$ (with $\delta^+ > 0$ an infinitesimal).

This is interesting to show how fourier transform Coulomb potential from momentum to space rep. by deforming the contour in complex k–plane, dispensing with introducing a “fictitious” photon’s mass m_γ.

Here Eq.(1.67) is regulated by Feynman’s prescription in configuration rep., Eq.(1.48):

$$D_F(t,\vec{x}) = \frac{1}{2(2\pi)^2 r}\frac{\partial}{\partial r}\int_{-\infty}^{\infty} d\xi\ \exp\left(im\sqrt{-x^2+i\varepsilon^+}\sinh\xi\right) \tag{1.70}$$

$$= \frac{1}{2(2\pi)^2 r}\frac{\partial}{\partial r}\int_{-\infty}^{\infty} d\xi\ \exp\left(im\sqrt{-x^2}\sinh\xi\right)\underbrace{e^{-m\delta^+\sinh\xi}}_{\text{damping factor}}, \tag{1.71}$$

where $\delta^+ = 2^{-1}\left(-x^2\right)^{1/2}\varepsilon^+ \sim \mathscr{O}(\varepsilon^+)$.

Finally, we work on the asymptotics of $D_F(t,\vec{x})$, starting with the analytic continuation:

$$\xi(\tau) := \tau + \frac{i\pi}{2},\ -\infty < \tau < \infty. \tag{1.72}$$

Consequently, Eq.(1.70) becomes:

$$D_F(t,\vec{x}) = \frac{1}{2(2\pi)^2 r}\frac{\partial}{\partial r}\int_{-\infty+i\pi/2}^{\infty+i\pi/2} d\tau\ \exp\left[im\sqrt{-x^2+i\varepsilon^+}\sinh\left(\tau+\frac{i\pi}{2}\right)\right] \tag{1.73}$$

$$= \frac{1}{2(2\pi)^2 r}\frac{\partial}{\partial r}\int_{\{\Im\mathfrak{m}(z)=\pi/2\}} d\tau\ \exp\left(-m\sqrt{-x^2+i\varepsilon^+}\cosh\tau\right) \tag{1.74}$$

$$= \frac{1}{2(2\pi)^2 r}\frac{\partial}{\partial r}\int_{\{\Im\mathfrak{m}(z)=\pi/2\}} d\tau\ \exp\left[-m\sqrt{r^2-t^2}\left(1+\frac{\tau^2}{2}+\mathscr{O}\left(\tau^4\right)\right)\right] \tag{1.75}$$

$$\underset{r\to\infty}{\sim} \frac{1}{2(2\pi)^2 r}\frac{\partial}{\partial r}\left[e^{-m\sqrt{r^2-t^2}}\int_{-\infty}^{\infty} d\tau\ \exp\left(-\frac{m\sqrt{-x^2}}{2}\tau^2\right)\right] \tag{1.76}$$

$$= \frac{1}{2(2\pi)^2 r}\frac{\partial}{\partial r}\left(\sqrt{\frac{2\pi}{m\sqrt{r^2-t^2}}}e^{-m\sqrt{r^2-t^2}}\right), \tag{1.77}$$

whereupon:

$$D_F(t,\mathbf{x}) \underset{r\to\infty}{\sim} -\frac{\left(2m\sqrt{r^2-t^2}+1\right)}{8\sqrt{2}\pi^{3/2}\left(r^2-t^2\right)^2\sqrt{m\sqrt{r^2-t^2}}}e^{-m\sqrt{r^2-t^2}}, \tag{1.78}$$

which means:

$$D_F(t,\mathbf{x}) \underset{-x^2\to\infty}{\sim} \mathscr{O}\left(e^{-m\sqrt{-x^2}}\right). \tag{1.79}$$

1.3.2.2 Lorentzian's Flatland ($D=2$)

Now we study Eq.(1.53) in a 2–dimensional toy model,

$$\Delta(t,y) := K^{(1+3)}(t,y) = \frac{1}{4\pi i}\int_{-\infty}^{\infty}\frac{dp}{E_p}\exp\left[-i\left(E_p|t|-py\right)\right], \tag{1.80}$$

where $y := x^1$ is *the* spatial coordinate.

Applying the momentum parametrization of Eq.(1.62), our toy model propagator, Eq.(1.80), becomes:

$$\Delta(t,y) = \frac{1}{4\pi i}\int_{-\infty}^{\infty}d\xi\,\exp\left[-im\left(|t|\cosh\xi - y\sinh\xi\right)\right]. \tag{1.81}$$

It is useful to separate $\Delta(t,y)$ in two contributions, those from $y>0$ and $y<0$:

$$\Delta(t,y) = \theta(y)\Delta(t,y)+\theta(-y)\Delta(t,-y) =: \Delta_+(t,y)+\Delta_-(t,y). \tag{1.82}$$

For the purposes of asymptotic analysis, suffices to show both $\Delta_+(t,y)$ and $\Delta_-(t,y)$ decays exponentially in for spacelike events (namely, $y^2-t^2>0$). Since the calculations are similar, we work on $\Delta_+(t,y)$:

$$\Delta_+(t,y) = \frac{1}{4\pi i}\int_{-\infty}^{\infty}d\xi\,\exp\left[-im\left(|t|\cosh\xi - |y|\sinh\xi\right)\right]. \tag{1.83}$$

As in $D=4$, for spacelike separations, there is a parameter $0<\beta<\infty$ for which:

$$y = \sqrt{-x^2}\cosh\beta,\ \ |t| = \sqrt{-x^2}\sinh\beta. \tag{1.84}$$

With this parametrization, one derives from Eqs.(1.83, 1.84) that:

$$\Delta_+(t,y) = \frac{1}{4\pi i}\int_{-\infty}^{\infty}d\xi\,\exp\left[im\sqrt{-x^2}\left(\cosh\beta\sinh\xi-\sinh\beta\cosh\xi\right)\right] \tag{1.85}$$

$$= \frac{1}{4\pi i}\int_{-\infty}^{\infty}d\xi\,\exp im\sqrt{-x^2}\sinh\xi. \tag{1.86}$$

As before[5], Eq.(1.86) is regulated by Feynman's $(i\varepsilon^+)$–prescription in configuration space.

Lastly, performing analytic continuation:

$$\xi(\tau) := \tau+\frac{i\pi}{2}\ \left(-\infty<\mathfrak{Re}\,\tau<\infty\right), \tag{1.87}$$

[5] Recall Eq.(1.71).

and applying steepest descent method, we compute:

$$\Delta_+(t,y) = \frac{1}{4\pi i}\int_{-\infty+i\pi/2}^{\infty+i\pi/2} d\tau \exp\left(-m\sqrt{y^2-t^2}\cosh\tau\right) \tag{1.88}$$

$$= \frac{1}{4\pi i}\int_{\{\mathfrak{Im}(z)=\pi/2\}} d\tau \exp\left[-m\sqrt{y^2-t^2}\left(1+\frac{\tau^2}{2}+\mathscr{O}\left(\tau^4\right)\right)\right] \tag{1.89}$$

$$\underset{y\to\infty}{\sim} \frac{1}{4\pi i}\left(\frac{2\pi}{m\sqrt{y^2-t^2}}\right)^{1/2} e^{-m\sqrt{y^2-t^2}}, \tag{1.90}$$

from which we conclude:

$$\Delta_+(t,y) \underset{\sqrt{y^2-t^2}\to\infty}{\sim} \mathscr{O}\left(e^{-m\sqrt{-x^2}}\right). \tag{1.91}$$

As the same line of reasoning applies equally well to $\Delta_-(t,y)$, toy model propagator, we achived the result:

$$\Delta(t,y) \underset{\sqrt{-x^2}\to\infty}{\sim} \mathscr{O}\left(e^{-m\sqrt{-x^2}}\right). \tag{1.92}$$

1.3.2.3 "Mesotron?"

> *"Since its discovery, the name of the new particle has undergone several evolutionary stages. It was sometimes called 'heavy electron,' sometimes 'light proton,' and then somebody suggested the mesotron, derived from the Greek word mesos, which means 'in between.' But Werner Heisenberg's father, who was a professor of classical languages, objected that the letters 'tr' have no place in that name. Indeed, while the name electron was derived from the Greek word electra, the Greek word mesos has no 'tr' in it. Thus, the name of Yukawa's particle was settled as meson."*
> – G. Gamow [Gamow(1961)]

This section studies the tree–level, effective Yukawa potential in $D = 1+d \geq 2$ with mass $m^2 > 0$. Mathematically, what we are seeking is the fundamental solution to the Helmholtz operator [Sommerfeld(1961)],

$$\left(\nabla^2+m^2\right)\phi^{(d)}(\mathbf{x}) = -\delta^{(d)}(\mathbf{x}). \tag{1.93}$$

A rigorous, modern mathematical treatment of the subject is given by [Evans(2010), §6.5]. Here, we use a trick due to Schwinger.

Generalizing Eqs.(6, 8) of [Zee(2010), Sec.I.4],

$$\phi^{(d)}(\mathbf{x}) = -\int \frac{d^dk}{(2\pi)^d}\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2+m^2}. \tag{1.94}$$

From [Schwartz(2014), App.B], we learn how to write fractions:

$$\frac{p}{q} = \int_0^\infty d\alpha\, pe^{-\alpha q}. \tag{1.95}$$

Here, α is known as Schwinger's parameter.

By Eqs.(1.94, 1.95),

$$\phi^{(d)}(\mathbf{x}) = -\int \frac{d^d k}{(2\pi)^d} \int_0^\infty d\alpha \, \exp\left[-\alpha\left(\mathbf{k}^2+m^2\right)+i\mathbf{k}\cdot\mathbf{x}\right] \tag{1.96}$$

$$= -\int_0^\infty d\alpha \, e^{-\alpha m^2} \int \frac{d^d k}{(2\pi)^d} \exp\sum_{I=1}^{d}\left[-\alpha\left(k^I\right)^2+ik^I x^I\right] \tag{1.97}$$

$$= -\int_0^\infty d\alpha \, e^{-\alpha m^2} \int \frac{d^d k}{(2\pi)^d} \exp\left(-\alpha\mathbf{k}^2+i\mathbf{k}\cdot\mathbf{x}\right) \tag{1.98}$$

$$= -\frac{1}{(2\pi)^d}\int_0^\infty d\alpha \, e^{-\alpha m^2} \prod_{I=1}^{d}\int_{-\infty}^{\infty} dk^I \exp\left[-\alpha\left(k^I\right)^2+ik^I x^I\right] \tag{1.99}$$

$$= -\frac{1}{(2\pi)^d}\int_0^\infty d\alpha \, e^{-\alpha m^2} \prod_{I=1}^{d}\sqrt{\frac{\pi}{\alpha}}e^{\frac{\left(ix^I\right)^2}{4\alpha}} \tag{1.100}$$

$$= -\frac{1}{2^d\pi^{d/2}}\int_0^\infty d\alpha \, \frac{\exp\left\{-\left[\left(\frac{|\mathbf{x}|}{2}\right)^2\frac{1}{\alpha}+m^2\alpha\right]\right\}}{\alpha^{d/2}}. \tag{1.101}$$

After an exercise in *anamnesis*, or consulting [Jeffrey and Zwillinger(2007), §8.407 WA 92(8)], one recalls:

$$\int_0^\infty d\alpha \, \frac{\exp\left[-\left(\frac{A^2}{\alpha}+B^2\alpha\right)\right]}{\alpha^{d/2}} = 2\left(\frac{B}{A}\right)^{\frac{d-2}{2}} K_{\frac{d-2}{2}}\left(2AB\right), \tag{1.102}$$

for all $A,B\in\mathbb{R}$.

Remark. Here $K_\nu(z)$ is the modified Bessel function. In terms of the Hankel function of the second–kind $H_\nu(z)$, the former is given by:

$$K_\nu(z) = -\frac{i\pi}{2}e^{-i\pi\nu/2}H^{(2)}_{-\nu}(-iz), \tag{1.103}$$

where:

$$H^{(2)}_\nu(x) = -\frac{2\exp(i\pi\nu/2)}{i\pi}\int_0^\infty dt\,\exp\left(-ix\cosh t\right)\cosh(\nu t). \tag{1.104}$$

Lastly, in Eq.(1.101) we have $A=|\mathbf{x}|/2$ and $B=m$.
Whereby, the right–hand side of Eq.(1.102) provides:

$$2^{d/2}K_{\frac{d-2}{2}}\left(m|\mathbf{x}|\right). \tag{1.105}$$

The Yukawa potential in d space dimensions is, therefore,

$$\phi^{(d)}(\mathbf{x}) = -\frac{1}{2^{d/2}\pi^{d/2}}\left(\frac{m}{|\mathbf{x}|}\right)^{\frac{d-2}{2}} K_{\frac{d-2}{2}}\left(m|\mathbf{x}|\right). \tag{1.106}$$

Chapter 2

A Song to Faraday and Cavendish

2.1 Gravity at Zürich

Let k^{μ} be an onshell[1] momentum fourvector. The projector operator in momentum–space is defined by equation (eq.):

$$\mathscr{G}^{\mu\nu}(k) := \eta^{\mu\nu} - \frac{k^{\mu}k^{\nu}}{m^2}. \tag{2.1}$$

The projector operator is a symmetric *two-degree tensor* (2-tensor), whose name is justified by it's action upon k^{μ},

$$k^{\mu}\mathscr{G}_{\mu\nu}(k) = k^{\mu}\eta_{\mu\nu} - \frac{(k)^2 k^{\nu}}{m^2} = 0. \tag{2.2}$$

Later on, it'll be useful to know the projector operator's trace[2]:

$$\eta^{\mu\nu}\mathscr{G}_{\mu\nu} = \delta^{\mu}{}_{\mu} - \frac{k^{\mu}k_{\mu}}{m^2} = 3. \tag{2.3}$$

Note that eq. (2.3) holds for every onshell momentum.

Remark 2.1. Let $\varepsilon^{(a)}_{\mu\nu}(k)$ be the polarization symmetric 2–tensors.

In Zürich's Massive Gravity Theory [Fierz and Pauli(1939)], those are labelled by $1 \le a \le 5$ and satisfy:

$$k^{\mu}\varepsilon^{(a)}_{\mu\nu}(k) = 0, \tag{2.4}$$

$$\eta^{\mu\nu}\varepsilon^{(a)}_{\mu\nu}(k) = 0. \tag{2.5}$$

Eqs. (2.4, 2.5) are familiar to students of gravitational waves: they are recognized as the *transverse–traceless* (**TT**) gauge [Maggiore(2008), §2.2]. In passing, let me say that the gravitational analogue of Lorenz gauge in electrodynamics is de Donder gauge [Stewart(1993), §1.13]. Linearizing Einstein equations, de Donder gauge leads to **TT** gauge. I consider the textbook [Schutz(1985), §9.1] a highly readable introduction to the subject.

[1] Meaning: $k^2 := k^{\mu}k_{\mu} = m^2$.

[2] I've no intention of offending my reader's knowledge, but I clarify what it's meant by a *trace* here. Let $\mathscr{T} := T_{\mu\nu}$ be *any* tensor with two Lorentz indices. With help from Minkowski's $\eta_{\mu\nu}$, I define the trace of $\mathscr{T}$ the Lorentz scalar $\mathrm{tr}\,\mathscr{T} := \eta^{\mu\nu}T_{\mu\nu}$.

Anyway, on the remainder of this notes, Eqs.(2.4) will be referred to as the **TT** conditions.

Remark 2.2. Let $\mathscr{P}_{\mu\nu\lambda\sigma}(k)$ be the momentum-space tensorfield of degree four defined by[3]:

$$\mathscr{P}_{\mu\nu\lambda\sigma}(k) := \sum_a \varepsilon^{(a)}_{\mu\nu}(k)\,\varepsilon^{(a)}_{\lambda\sigma}(k)\,. \tag{2.6}$$

I'll allow myself the luxury of writing simply $\mathscr{P} := \left(\mathscr{P}_{\mu\nu\lambda\sigma}\right)$.

$\mathscr{P}$ enjoys the following symmetries:

$$\mathscr{P}_{\mu\nu\lambda\sigma} = \mathscr{P}_{\nu\mu\lambda\sigma},\ \mathscr{P}_{\mu\nu\lambda\sigma} = \mathscr{P}_{\mu\nu\sigma\lambda},\ \mathscr{P}_{\mu\nu\lambda\sigma} = \mathscr{P}_{\lambda\sigma\mu\nu}. \tag{2.7}$$

These follows directly from eq. (2.6) and polarization tensor's $\varepsilon^{(a)}_{\mu\nu}$ symmetry.

The **TT** conditions (2.4), satisfied by $\varepsilon^{(a)}_{\mu\nu}$, implies similar properties satisfied by $\mathscr{P}$, as computed below[4],

$$k^\mu\,\mathscr{P}_{\mu\nu\lambda\sigma}(k) = \sum_a \underbrace{k^\mu\varepsilon^{(a)}_{\mu\nu}(k)}_{\text{Transverse}\implies 0}\ \ \varepsilon^{(a)}_{\lambda\sigma}(k) = 0, \tag{2.8}$$

$$\eta^{\mu\nu}\,\mathscr{P}_{\mu\nu\lambda\sigma} = \sum_a \underbrace{\eta^{\mu\nu}\varepsilon^{(a)}_{\mu\nu}}_{\text{Traceless}\implies 0}\ \ \varepsilon^{(a)}_{\lambda\sigma} = 0. \tag{2.9}$$

Now, I will endeavor to use eqs. (2.8) and (2.9), with the normalization condition:

$$\mathscr{P}_{1212} = 1, \tag{2.10}$$

to find a general expression for $\mathscr{P}$ as linear combination of k^μ and $\mathscr{G}_{\mu\nu}$.

$\mathscr{P}$ should be written as a linear combination of tensorial products of k^μ and $\mathscr{G}_{\mu\nu}$. So, by symmetries (2.7) and combinatorics, $\mathscr{P}$ is generated by the following elements:

$$\mathscr{G}_{\mu\nu}\mathscr{G}_{\lambda\sigma},\ \mathscr{G}_{\mu\lambda}\mathscr{G}_{\nu\sigma} + \mathscr{G}_{\mu\sigma}\mathscr{G}_{\nu\lambda}, \tag{2.11}$$

$$k_\mu k_\nu \mathscr{G}_{\lambda\sigma} + k_\lambda k_\sigma \mathscr{G}_{\mu\nu}, \tag{2.12}$$

$$k_\mu k_\lambda \mathscr{G}_{\nu\sigma} + k_\mu k_\sigma \mathscr{G}_{\nu\lambda} + k_\nu k_\lambda \mathscr{G}_{\mu\sigma} + k_\nu k_\sigma \mathscr{G}_{\mu\lambda}, \tag{2.13}$$

$$k_\mu k_\nu k_\lambda k_\sigma. \tag{2.14}$$

In other words, there are numbers $a_1, \ldots, a_5$ such that:

$$\begin{aligned}
\mathscr{P}_{\mu\nu\lambda\sigma} = {} & a_1\mathscr{G}_{\mu\nu}\mathscr{G}_{\lambda\sigma} + a_2\left(\mathscr{G}_{\mu\lambda}\mathscr{G}_{\nu\sigma} + \mathscr{G}_{\mu\sigma}\mathscr{G}_{\nu\lambda}\right) + a_3\left(k_\mu k_\nu \mathscr{G}_{\lambda\sigma} + k_\lambda k_\sigma \mathscr{G}_{\mu\nu}\right) && (2.15)\\
& + a_4\left(k_\mu k_\lambda \mathscr{G}_{\nu\sigma} + k_\mu k_\sigma \mathscr{G}_{\nu\lambda} + k_\nu k_\lambda \mathscr{G}_{\mu\sigma} + k_\nu k_\sigma \mathscr{G}_{\mu\lambda}\right) && (2.16)\\
& + a_5 k_\mu k_\nu k_\lambda k_\sigma. && (2.17)
\end{aligned}$$

Now, my problem is to find an explicit expression for $\mathscr{P}$.
I proceed in three acts.

[3]In ambiguity's absence I omit the argument k in $\mathscr{P}_{\mu\nu\lambda\sigma}(k)$.

[4]Notwithstanding my commitment to pedantry, the contraction with k^μ in left-hand side of eq. (2.8) vanishes *only if* $\mathscr{P}$ is evaluated at k^μ. So I'll not omit the argument as I promised in fn.(3).

2.1.1 Act I: *Linear Independence*

Using eqs. (2.4, 2.8) in (2.17),

$$k^{\mu}\mathscr{P}_{\mu\nu\lambda\sigma} = a_3 m^2 k_{\nu}\mathscr{G}_{\lambda\sigma} + a_4 m^2\left(k_{\nu}\mathscr{G}_{\lambda\sigma} + k_{\lambda}\mathscr{G}_{\nu\sigma}\right) + a_5 m^2 k_{\nu}k_{\lambda}k_{\sigma} = 0, \tag{2.18}$$

which by linear independence,

$$a_3 = a_4 = a_5 = 0. \tag{2.19}$$

So, I've reduced $\mathscr{P}$ simply to:

$$\mathscr{P}_{\mu\nu\lambda\sigma} = a_1\mathscr{G}_{\mu\nu}\mathscr{G}_{\lambda\sigma} + a_2\left(\mathscr{G}_{\mu\lambda}\mathscr{G}_{\nu\sigma} + \mathscr{G}_{\mu\sigma}\mathscr{G}_{\nu\lambda}\right). \tag{2.20}$$

2.1.2 Act II: *When $\eta^{\mu\nu}$ Walked With $\mathscr{P}$*

Recall, from eq.(2.9), that $\varepsilon^{(a)}_{\mu\nu}$ is traceless. Moreover, in eq. (2.3), I've shown $\eta^{\mu\nu}\mathscr{G}_{\mu\nu} = 3$. Then from Act I, eq.(2.20),

$$\eta^{\mu\nu}\mathscr{P}_{\mu\nu\lambda\sigma} = 3a_1\mathscr{G}_{\lambda\sigma} + 2a_2\mathscr{G}^{\mu}_{\ \lambda}\mathscr{G}_{\mu\sigma}. \tag{2.21}$$

From a straightforward calculation using def. (2.1),

$$\mathscr{G}^{\mu}_{\ \lambda}\mathscr{G}_{\mu\sigma} = \left(\delta^{\mu}_{\ \lambda} - \frac{k^{\mu}k_{\lambda}}{m^2}\right)\left(\eta_{\mu\sigma} - \frac{k_{\mu}k_{\sigma}}{m^2}\right) \tag{2.22}$$

$$= \eta_{\lambda\sigma} - \frac{k_{\lambda}k_{\sigma}}{m^2} - \underbrace{\frac{k_{\sigma}k_{\lambda}}{m^2} + \frac{(k)^2 k_{\sigma}k_{\lambda}}{m^2}}_{\text{vanishes}} \tag{2.23}$$

$$= \mathscr{G}_{\lambda\sigma}. \tag{2.24}$$

As in every crummy novel, I now present the climax.

From eqs.(2.21, 2.24),

$$(3a_1 + 2a_2)\mathscr{G}_{\lambda\sigma} = 0 \implies a_1 = -\frac{2}{3}a_2, \tag{2.25}$$

which gives:

$$\mathscr{P}_{\mu\nu\lambda\sigma} = \left(-\frac{2}{3}\mathscr{G}_{\mu\nu}\mathscr{G}_{\lambda\sigma} + \mathscr{G}_{\mu\lambda}\mathscr{G}_{\nu\sigma} + \mathscr{G}_{\mu\sigma}\mathscr{G}_{\nu\lambda}\right)a_2. \tag{2.26}$$

2.1.3 Act III: *Normalize to Unify*

Let's work in rest frame, such that:

$$k^{\mu} = (m, \mathbf{0}) \underbrace{\implies}_{\text{Eq.(2.1)}} \mathscr{G}_{ij} = \delta_{ij}. \tag{2.27}$$

The normalization of def. (2.10) then gives:

$$\mathscr{P}_{1212} = \left(-\frac{2}{3}\delta_{12}\delta_{12} + \delta_{11}\delta_{22} + \delta_{12}\delta_{12}\right)a_2 = 1 \implies a_2 = 1. \tag{2.28}$$

Finally:

$$\mathscr{P}_{\mu\nu\lambda\sigma} = -\frac{2}{3}\mathscr{G}_{\mu\nu}\mathscr{G}_{\lambda\sigma} + \mathscr{G}_{\mu\lambda}\mathscr{G}_{\nu\sigma} + \mathscr{G}_{\mu\sigma}\mathscr{G}_{\nu\lambda}. \tag{2.29}$$

2.2 Gravity from Cambridge to Göttingen: *"Hypotheses non fungo "*

"I have not as yet been able to discover the reason for these properties of gravity from phenomena, and I do not feign hypotheses. For whatever is not deduced from the phenomena must be called a hypothesis; and hypotheses, whether metaphysical or physical, or based on occult qualities, or mechanical, have no place in experimental philosophy."
– Isaac Newton [Newton(1726)]

One recognizes the fundamental solution to the $d = 3+n$ Poisson equation in fourier representation (rep.) as given (up to a proportionality factor) by:

$$\Delta^{(d)} V(x) \propto -\delta^{(d)}(x) . \tag{2.30}$$

Eq. (2.30) may be considered[5] as the "static" massless limit of [Mol(2020), Eq.(5)], which I've studied in a former problem assignment.

Proportionality constants (ctes.) and physical units will be discussed soon. But before doing so, we make some comments and define the geometry upon which our considerations are based.

2.2.1 Göttingen: Riemann

Let $\left(X^{(d)}, g_{mn}\right)$ be the $d = 3+1$ dimensional space with a flat *riemannian* metric, modeling a spacelike hypersurface of Kaluza-Klein theory with a compactified *large extra-dimension*[6], and let $x^m : X \longrightarrow \mathbb{R}^3 \times [0, 2\pi]$ be a global[7] chart on M:

$$x^m := (\underbrace{x^1, x^2, x^3}_{\vec{x}}, \underbrace{x^4}_{w^0}). \tag{2.31}$$

Here, $\vec{x} := \left(x^1, x^2, x^3\right)$ co–ordinates the non-compact dimensions, while $-a_0/2 \leq w^0 \leq a_0/2$ parametrizes the compact dimension with length a_0. Topologically, it's clear that X is homeomorphic to $\mathbf{E}^3 \times S^1$, where $\mathbf{E}^3$ is the affine geometry of 3–dimensional euclidean space [Arnold(2013), sec. (2.9)].

The line-element describing the riemannian space (X, g_{mn}) is given by[8]:

$$ds^2 = g_{mn} dx^m dx^n = (d\vec{x})^2 + \frac{a_0^2}{4\pi^2} \left(dw^0\right)^2 , \tag{2.32}$$

where, for simplicity, I defined:

$$(d\vec{x})^2 := \sum_{i=1}^{3} \left(dx^i\right)^2 . \tag{2.33}$$

From eq. (2.32), we read the matrix rep. of the metric tensor g_{mn} and it's inverse, resp.:

$$(g_{mn}) = \mathrm{diag}\left(+1, +1, +1, \frac{a_0^2}{4\pi^2}\right), \ (g^{mn}) = \mathrm{diag}\left(+1, +1, +1, \frac{4\pi^2}{a_0^2}\right). \tag{2.34}$$

[5] Of course, with appropriated regularizations in the integral form of fourier rep.
[6] For details, please cf. [Zee(2010), Ex. I.6.1].
[7] In differential geometric language, this means: covering the entire manifold M.
[8] A nice way to understand eq. (2.32) is to consider an analogy with cylindrical coordinates.

Recall that our main object of study is eq. (2.30). To understand it's properties, I need to compute the d'Alembert operator for the metric g_{mn}:

$$\Delta^{(d)} V\left(\vec{x}, w^0\right) = |g|^{-1/2} \partial_m \left(|g|^{1/2} g^{mn} \partial_n V\right) \tag{2.35}$$

$$= \partial_m \partial^m V\left(\vec{x}, w^0\right). \tag{2.36}$$

Remark 2.3. From eq. (2.36), one may study the behavior of the propagation of fields in compactified Kaluza-Klein models using the method of separation of variables,

$$V\left(\vec{x}, w^0\right) \equiv V(\vec{x}) f\left(w^0\right) \implies \frac{\Delta^{(3)} V(\vec{x})}{V(\vec{x})} + \frac{4\pi^2}{a_0^2} \frac{f''\left(w^0\right)}{f\left(w^0\right)} = 0. \tag{2.37}$$

The stability of extra-dimensions from the analysis of eq. (2.37) can be found in [Penrose(2003), Penrose(2017), sec. 1.11].

My main concern now, however, regards the simpler problem of [Zee(2010), ex. I.6.1].

2.2.2 "Pois(s)on" in Planck's Units

Let $\rho : X^{(d)} \longrightarrow \mathbb{R}_{\geq 0}$ be a mass density. Clearly,

$$[\rho] = \frac{M}{L^d} = M^{1-d}. \tag{2.38}$$

The Poisson equation for a potential V in a d–dimensional space, with mass density ρ, is:

$$\Delta^{(d)} V(x) = k^{(d)} G^{(d)} \rho(x), \tag{2.39}$$

where $G^{(d)}$ is the gravitational cte. in d space dimensions.

Nevertheless, note that our potential should have units of energy,

$$[V(x)] = M, \tag{2.40}$$

while the units of the d'Alembert operator are:

$$\left[\Delta^{(d)}\right] = \left[\left(\frac{\partial}{\partial x}\right)^2\right] = \frac{1}{L^2} = M^2. \tag{2.41}$$

Therefore, the units of the gravitational cte. $G^{(d)}$ in a universe with d spatial dimensions follows from eqs. (2.38–2.41):

$$[G^{(d)}] = M^{2-d}. \tag{2.42}$$

$$[G_N] = \frac{1}{M} = L, \; [G^{(4)}] = \frac{1}{M^2}, \tag{2.43}$$

where from now on, $G_N := G^{(3)}$ is Newton's cte. of universal gravitation. In passing, note that in Planck units, $G_N = 1\,\ell_{\mathbf{P}}$.

2.2.3 Render to de Rham what's Stoke's

Let $\Sigma \hookrightarrow X^{(d)}$ be an immersion with an outward–pointing unit–normal $n^m(x)$. The latter condition means:

$$g_{mn}N^mN^n = 1,\ N^mY^ng_{mn} = 0\ \ (\forall Y(p) \in T_p\Sigma,\, p \in \Sigma). \tag{2.44}$$

In physics, one calls Σ a $(d-1)$–brane in the d–dimensional spatial "bulk" $X^{(d)}$.

From a geometric viewpoint, the submanifold Σ inherits a riemannian metric γ_{ab} from $\left(X^{(d)}, g_{mn}\right)$, known as the first fundamental form of Σ. In components, let σ^a be coordinates naturally adapted to Σ, where $1 \leq a, b \leq d-1$. Then,

$$\gamma_{ab} = g_{mn}\frac{\partial x^m}{\partial \sigma^a}\frac{\partial x^n}{\partial \sigma^b}, \tag{2.45}$$

where $\sigma^a \in \Sigma \mapsto x := x^m(\sigma^a) \in X^{(d)}$ is a parametrization of the brane into the bulk.

An important rôle is played by the first fundamental form's determinant (understood as the Jacobian transformation):

$$|\gamma| := \det(\gamma_{ab}). \tag{2.46}$$

Let $Z^m(x)$ be a vectorfield in the bulk $X^{(d)}$. The divergence theorem for higher dimensions (called between mathematicians Stoke's theorem, tough the most generic form of the latter was achived only by de Rham) yields:

$$\int_\Sigma \partial_m Z^m(x)\, d^dx = \oint_{\partial\Sigma} Z^m(x(\sigma^a))\, N_m(x(\sigma^a))\, \sqrt{|\gamma|}d^{(d-1)}\sigma. \tag{2.47}$$

2.2.4 Cambridge Again: I *do* feign hypotheses

Define the static gravitational field–strength in the universe $\left(X^{(d)},\, g_{mn}\right)$ by the d–vectorfield:

$$F^m(x) := -\frac{\partial}{\partial x_m}V(x). \tag{2.48}$$

Moreover, let the "radial" function $r : X^{(d)} \longrightarrow \mathbb{R}_{\geq 0}$ on the *brane* Σ be given by:

$$r := \sqrt{|\vec{x}|^2}, \tag{2.49}$$

and define the unit vectorfield $Z^m(x)$ in $X^{(d)}$,

$$Z^m(x) := \frac{(\vec{x}, 0)}{r(x)}. \tag{2.50}$$

Therefore, with definitions of eqs. (, ,), I hypothesize the following ansatz for the gravitational field-strength:

$$F^m(x) = f(r)\, Z^m(x).$$

Lastly, we apply eqs. (2.39, 2.47)

$$\partial_m \mathbf{g}^m_{(d)}(x) = -k^{(d)}G^{(d)}\rho(x). \tag{2.51}$$

$$\mathbf{g}_{(d)}(x) := \mathbf{g}_{(d)}\left(\vec{x}, w^0\right) = \mathbf{g}^m_{(d)}(|\vec{x}|)\, N_m(x) \tag{2.52}$$

$$\Sigma(r) := \{|\vec{x}| \leq r\} \times S^1\left(\frac{a_0}{2\pi}\right) \tag{2.53}$$

$$\partial\Sigma = \{|\vec{x}| = r\} \times \left\{-\frac{a_0}{2} \leq w^0 \leq \frac{a_0}{2}\right\} \tag{2.54}$$

$$\mathscr{M}^{(4)}_{\text{eff}} := \frac{1}{\sqrt{G_N a_0}}. \tag{2.55}$$

2.3 An Act of Micro-aggression

"The introduction of numbers is an act of violence."
– Hermann Weyl [Weyl(2009)]

According to [Arkani-Hamed(1999)] and [Han(1999)], if there exists one "large" and compact extra–dimension in our spacetime, with a length scale of:

$$a_0 \sim 1\,\text{mm} \approx 6.25 \times 10^{31}\, \mathfrak{l}_{\mathbf{P}}, \tag{2.56}$$

then[9], either by eq. (2.55) (to be motivated in a following version of the present notes), or [Zee(2010), Eq.(3), Sec.(I.6)], the energy–scale at which new physical phenomena would be manifested is of the order:

$$\begin{aligned} M_{TG} := \mathscr{M}^{(4)}_{\text{eff}} = \frac{1}{\sqrt{(1\,\mathfrak{l}_{\mathbf{P}}) \times (6.25 \times 10^{35}\,\mathfrak{l}_{\mathbf{P}})}} &= \left(6.25 \times 10^{35}\right)^{-1/2} \mathfrak{m}_{\mathbf{P}} \qquad (2.57) \\ &= \left(6.25 \times 10^{35}\right)^{-1/2} \mathfrak{m}_{\mathbf{P}} \qquad (2.58) \\ &\sim 10^3\,\text{GeV}. \qquad (2.59) \end{aligned}$$

Unfortunately, as shown in the following "Livingstone" plot [Barletta(2014)], the ideas advocated by those authors are by now shown to be wrong.

[9] Recall that: $1\,\mathfrak{m}_{\mathbf{P}} \approx 1.2 \times 10^{19}\,\text{GeV}$.

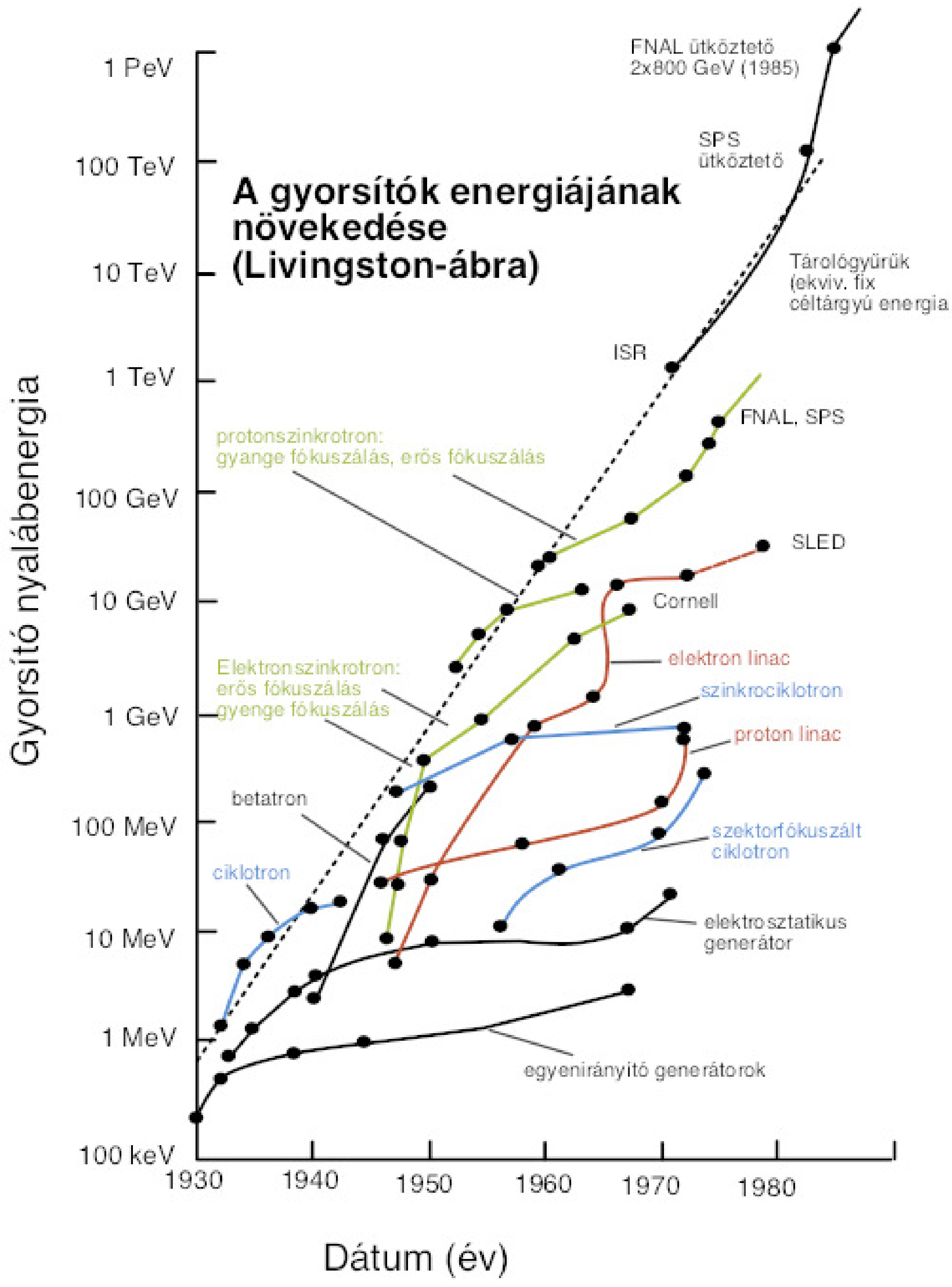


Figure 2.1: *"The so-called Livingston plot illustrates how history of discovery on the energy frontier has been enabled by the history of invention (red arrows) in accelerator science and technology." [Barletta(2014)]*

Chapter 3

Feynman's Motives, Schwinger's *Sources*

"Is the purpose of theoretical physics to be no more than a cataloging of all the things that can happen when particles interact with each other and separate? Or is it to be an understanding at a deeper level in which there are things that are not directly observable (as the underlying quantized fields are) *but in terms of which we shall have a more fundamental understanding?"*
– J. Schwinger, in [Schwinger and Englert(2013)]

"Contrary to what occurs in ordinary topology, one finds oneself confronting a disconcerting abundance of different cohomological theories. One has the distinct impression (but in a sense that remains vague) that each of these theories 'amount to the same thing,' that they 'give the same results'. In order to express this intuition, of the kinship of these different cohomological theories, I formulated the notion of motive *associated to an algebraic variety. By this term, I want to suggest that it is the 'common motive' (or 'common reason') behind this multitude of cohomological invariants attached to an algebraic variety, or indeed, behind all cohomological invariants that are a priori possible."*
– A. Grothendieck, in [Grothendieck(1985)]

The main object in path–integral's quantization of field theories is the partition function $\mathscr{Z}[J]$ of fields coupled to a source J. I open §3.1 offering a philosophical comment on Schwinger's source theory and Bohr's *"Kopenhagener geist"* of quantum theory. You're excused for ignoring those speculations.

In these notes, my aim is to study an interpretation for the perturbative series of $\mathscr{Z}[J]$, stated at in Eq.(3.7), with graph theory's concepts. For simplicity, I use a simple model: "quartic *hermetian* ϕ^4–theory."

A *hermetian* scalar field ϕ is:

1. A section in line–bundle $\mathbf{M}^D \times \mathbb{R} \overset{\pi}{\hookrightarrow} \mathbf{M}^D$,

$$\pi \circ \phi = \mathrm{id}_{\mathbf{M}^D} \text{ (the identity map in } \mathbf{M}^D). \tag{3.1}$$

Alternatively, in algebraic geometry's vainglorious jargon, I simply whisper: short exact sequence:

$$\pi \circ \phi : \mathbf{M}^D \overset{\phi}{\hookrightarrow} \mathbf{M}^D \times \mathbb{R} \overset{\pi}{\hookrightarrow} \mathbf{M}^D, \tag{3.2}$$

splits [Hatcher(2002)].

2. "Hermiticity" is the quality that:
$$\phi(x) = \phi^{\dagger}(x) . \tag{3.3}$$
It's tempting to declare ϕ a "real" field. This isn't a capital sin. Nevertheless, quantum field theory heighten classical fields from mere sections in associated vector–bundles to (distribution–valued) operators. *Hermiticity*, therefore, is the pedant's rightful choice.

3. Physically, hermetian, distribution–valued unitary operators provides a mathematical realization of *neutral* fields.

Lagrange's functional density, or Lagrangian now on[1], for a *hermetian* scalar field with quartic interaction, is given by:
$$\mathscr{L} = \mathscr{L}_0(\phi, \partial\phi) + \mathscr{L}_{\text{int}}(\phi) , \tag{3.4}$$
where the free (interacting, resp.) contributions are:
$$\mathscr{L}_0(\phi, \partial\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2(x) , \ \mathscr{L}_{\text{int}} = -\frac{\lambda}{4!}\phi^4(x) . \tag{3.5}$$

The partition function of ϕ^4 sourced by current J is:
$$\mathscr{Z}[J] = (\mathscr{Z}[0])^{-1} \int [d\phi] \exp\left[i \int d^D x \left(\mathscr{L}_0(\phi, \partial\phi) + J(x)\phi(x) + \frac{i\varepsilon}{2}\phi^2(x) \right) \right] , \tag{3.6}$$
where $[d\phi]$ is the *pseudo*–measure of functional "integral," whatever it is, and $\varepsilon > 0$ an infinitesimal needed to regularize Eq.(3.6).

To simplify our following equations (and with an eye at higher speculations), I omit the dimensionality of spacetime's measure, $dx := d^D x$. Moreover, I refrain from repeating the normalizing factor $(\mathscr{Z}[0])^{-1}$ of Eq.(3.6), by simply replacing the equality sign by $\sim$ below. Sincerely, this' a trick: I want to hide my shame of being unable to define it's precise mathematical meaning.

The basic equation of perturbative quantum field theory, motivated from heuristic arguments in §3.2, is:
$$\mathscr{Z}[J] \sim \exp\left[i \int dx \, \mathscr{L}_{\text{int}} \left(-i \frac{\delta}{\delta J(x)} \right) \right] \mathscr{Z}_0[J] . \tag{3.7}$$

Here, $\mathscr{Z}_0[J]$ is the partition function of the *non-self-interacting sourced hermetian scalar field*.

What a grandiose way of simply referring to Eq.(3.6) with $\lambda = 0$! I studied this case in §3.1. For anxious readers, I prescribe Eq.(3.14) instead of `SSRI`[2].

Lastly, $D(x)$ is the fundamental solution to Feynman's $i\varepsilon$–regulated Klein-Gordon operator:
$$\left(\partial^2 + m^2 - i\varepsilon \right) D(x) = -\delta(x) . \tag{3.8}$$

$D(x)$ is known as the *Feynman propagator*, as fairness isn't a matter for science historians, justifying my ignorance about [Stueckelberg(1938)].

[1] Maybe "Lagrangean," if you wish to publish in Phys. Rev. Lett. [Mermin(1988)].

[2] "Selective serotonin reuptake inhibitor." [DSM-5(2013)]

3.1 The Non–selfish Field

> *"The meme for blind faith secures its own perpetuation by the simple unconscious expedient of discouraging rational inquiry."* — Richard Dawkins, on *"The Selfish Gene"* [Dawkins(1989)]

I find very difficult to name a real scalar field ϕ, when *sourced* by a current J, a "free" theory.

The reason is psychological. A reminiscence from my classical intuition. Quantum mechanically, the observer, whose sources J are *deployed* as a *kapital* for performing a *measurement* (in Bohr's sense [Bohr(1955)]), will reach the conclusion, from absence of particle's creation (or annihilation), that such a theory is non-self-interacting.

Let $\lambda = 0$ in the Lagrangian of Eqs.(3.4, 3.5).

From Eq.(3.6), the partition function becomes:

$$\mathscr{Z}_0[J] := \lim_{\lambda\to 0}\mathscr{Z}[J] \sim \int [d\phi]\exp\left[i\int dx\left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2(x) + \frac{i\varepsilon}{2}\phi^2(x) + J(x)\phi(x)\right)\right]. \tag{3.9}$$

Under "reasonable" assumptions, for spacelike separations:

$$\lim_{-(x-y)^2\to\infty}\phi(x-y) = 0. \tag{3.10}$$

A Whig historiography of science [Butterfield(1965)] is continuous overthrown of reasonable assumptions. I don't claim affiliation to this school, except that assumption (3.10) requires some revision in topological field theories.

In any case, with (3.10), the total derivative:

$$\partial_m(\phi\partial^m\phi), \tag{3.11}$$

may be ignored in the Lagrangian contributing to $\mathscr{Z}_0[J]$.

Thence, I can write Eq.(3.9) as:

$$\mathscr{Z}_0[J] \sim \int [d\phi]\exp\left[i\int dx\left(-\frac{1}{2}\phi(x)D^{-1}\phi(x) + J(x)\phi(x)\right)\right], \tag{3.12}$$

where D^{-1} is the ($i\varepsilon$'s regularized) inverse to Klein-Gordon operator, Eq.(3.8).

Playing by analogy, I know that for finite–dimensional integrals:

$$\int \frac{d^n x}{(2\pi)^{n/2}}\exp\left(-\frac{1}{2}\sum_{a,b=1}^{n} x_a A_{ab} x_b + \sum_{c=1}^{n} J_c x_c\right) \sim \exp\left(-\frac{1}{2}\sum_{a,b,=1}^{n} J_a A^{-1}_{ab} J_b\right). \tag{3.13}$$

Whereby I take a dubious step, nevertheless, one which Nature seems to agree[3].

By Eq.(3.12), I declare:

$$\mathscr{Z}_0[J] \sim \exp\left[-\frac{i}{2}\left(\int dx_1 dx_2 J(x_1) D(x_1 - x_2) J(x_2)\right)\right]. \tag{3.14}$$

[3] A *motive* which requires further investigation [Marcolli(2010)].

3.2 A Small λ's Perturbation

"In the thirties, under the demoralizing influence of quantum-theoretic perturbation theory, the mathematics required of a theoretical physicist was reduced to a rudimentary knowledge of the Latin and Greek alphabets."

– Res Jöst, as quoted in [Streater and Wightman(1964)]

I return to Eqs.(3.4, 3.6) with humility.

Assuming $|\lambda| \ll 1$ and with the power of Euler's series,

$$\mathscr{Z}[J] \sim \int [d\phi] \sum_{n=0}^{\infty} \frac{1}{n!} [i\lambda\phi(x)]^n \exp\left\{ i \int dx \left[\mathscr{L}_0(\phi, \partial\phi) + J(x)\phi(x) \right] \right\}. \tag{3.15}$$

Functional differentiation, something that should better look alike ordinary ones to deserve the name, yields:

$$\frac{\delta}{\delta J(y)} \exp\left\{ i \int dx \left[\mathscr{L}_0(\phi, \partial\phi) + J(x)\phi(x) \right] \right\} = i\lambda\phi(y) \exp\left\{ i \int dx \left[\mathscr{L}_0(\phi, \partial\phi) + J(x)\phi(x) \right] \right\}. \tag{3.16}$$

On the shoulders of Eqs.(3.15, 3.16), I obtain:

$$\mathscr{Z}[J] \sim \int [d\phi] \exp\left[-\int dy\, i\mathscr{L}_{\text{int}} \left(\frac{1}{i} \frac{\delta}{\delta J(x')} \right) \right] \exp\left\{ i \int dy \left[\mathscr{L}_0(\phi, \partial\phi) + J(x)\phi(x) \right] \right\}. \tag{3.17}$$

My lesson from Sec.(3.1) was Eq.(3.14), which gives me a form to compute $\mathscr{Z}[J]$ with more humble tools:

$$\mathscr{Z}[J] \sim \exp\left[-\int dx\, i\mathscr{L}_{\text{int}} \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right) \right] \mathscr{Z}_0[J]. \tag{3.18}$$

Particularizing this result to Eq.(3.5) *motivates* Eq.(3.7), *as I desired.*

3.3 Dyson's Holiday

It was a holiday exercise in combinatorics. At least for an Englishman undergraduate, to find an interpretation of Eq.(3.18) in a language from graph theory [Dyson(1949)].

From the expansion:

$$\mathscr{Z}[J] \sim \sum_{V=0}^{\infty} \frac{1}{V!} \left(-\frac{\lambda}{4!} \int dy \frac{\delta^4}{\delta J(y)^4} \right)^V \sum_{P=0}^{\infty} \frac{1}{P!} \left(\int J(x_1) D(x_1 - x_2) J(x_2)\, dx_1 dx_2 \right)^P, \tag{3.19}$$

he noted that for any term, with a fixed $E := 4V - 2P \geq 0$ number of *unpaired edges* (which otherwise vanishes by functional differentiation), there are:

$$\frac{2P!}{(4V-2P)!},$$

ways for the $4V$ functional derivatives $(\delta/\delta J)$'s act upon $2P$ sources J's.

Any source in P is recognized as an *edge paired* within some functional derivative in V. Those edges are called *propagators*, and functional derivatives *vertices.* I have, in any expansion of Eq.(3.19) for E fixed, 2! choices of propagators and 4! to vertices. Moreover, exchanging vertices (propagators, resp.) between themselves, gives $V!$ ($P!$, resp.) ways to recombine that term in a graph, thereby cancelling any Taylor's coefficients in series (3.19).

Hence, let $\Gamma = (V, P)$ be a graph with V vertices and P paired edges, with symmetry factor $S(\Gamma)$ arising from the possible rearrangements of *unpaired* edges. If between $[\![...]\!]$ one treats all the objects *as if* those were commutative, Eq.(3.19) becomes:

$$\mathscr{Z}[J] \sim \sum_{\Gamma(V,P)\,\in\,\text{graphs}} \frac{(-i\lambda)^V}{S(\Gamma)} \left[\!\!\left[\left(\int dy \frac{\delta^4}{\delta J(y)^4} \right)^V \left[\int dx_1 dx_2 \, J(x_1) D(x_1 - x_2) J(x_2) \right]^P \right]\!\!\right] . \tag{3.20}$$

3.4 Example

In this example we consider four sources, localized around the spacetime events x_1, ..., x_4.

$$J(x) = \prod_{a=1}^{4} \delta(x - x_a) . \tag{3.21}$$

Now my endeavor is to demonstrate how to attribute a diagrammatic interpretation for the perturbation series associated to ϕ^4–theory, Eq.(3.19).

To this end, let $\Gamma := \Gamma(2,4)$ be a graph with vertex number $V = 2$ (interactions) and $E = 4$ unpaired edges (sources), corresponding to the amplitude $\mathscr{M}(\Gamma)$ at order $\mathscr{O}(\lambda^2)$.

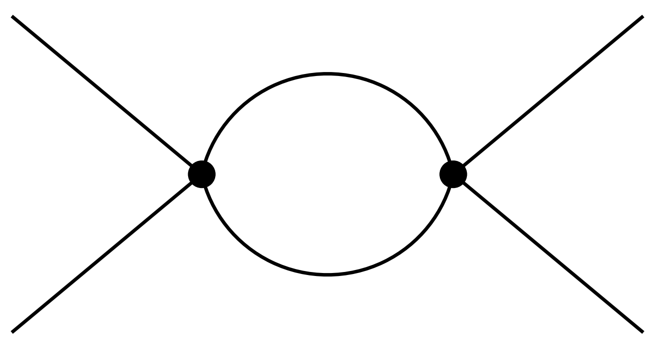


Figure 3.1: Graph Γ with $V = 2$ vertices and $E = 4$ external edges, representing the amplitude $\mathscr{M}(\Gamma)$.

From Eq.(3.19), $\mathscr{M}(\Gamma)$ is given by:

$$\mathscr{M}(\Gamma) = \frac{(i\lambda)^2}{S_\Gamma} \left[\!\!\left[\int dy_1 dy_2 \frac{\delta^4}{\delta J(y_1)^4} \frac{\delta^4}{\delta J(y_2)^4} \left[\prod_{a=1}^{6} \int dx_a \, J(x_a) D(x_{a+1} - x_a) J(x_{a+1}) \right] \right]\!\!\right] \tag{3.22}$$

$$= \frac{(i\lambda)^2}{S_\Gamma} \int dy_1 dy_2 \, \underbrace{D(x_1 - y_1) D(x_2 - y_1) D(x_3 - y_2) D(x_3 - y_2)}_{\text{unpaired edges}} \underbrace{D^2(y_2 - y_1)}_{\text{internal edges}}, \tag{3.23}$$

where $S_\Gamma = 2$ is the symmetry–factor.

Recalling the expression for Feynman propagator in momentum–space,

$$D(x) = \int \frac{d^D k}{(2\pi)^D} \frac{e^{ix\cdot k}}{k^2 - m^2 + i\varepsilon}, \tag{3.24}$$

Eq.(3.23) provides:

$$\mathscr{M}(\Gamma) = \frac{(i\lambda)^2}{S_G} \int dy_1 dy_2 \prod_{a=1}^{6} \left(\int \frac{d^D k_a}{(2\pi)^D} \frac{1}{k_a^2 - m^2 + i\varepsilon} \right) e^{i\sum_{b=1}^{4} x_b\cdot k_b} e^{iy_1\cdot(k_5 - k_2 - k_4 - k_6)} e^{iy_2\cdot(k_5 - k_2 - k_4 - k_6)}. \tag{3.25}$$

Performing the integration in Eq.(3.25) over y_1 and y_2, one obtains a product involving two Dirac's distributions,

$$(2\pi)^{2D} \delta^D (k_5 - k_2 - k_4 - k_6)\, \delta^D (k_5 - k_2 - k_4 - k_6). \tag{3.26}$$

Over the $6D$–dimensional momentum-space, the effect of Eq.(3.26) within measure $\prod_{a=1}^{6} d^D k_a$ is to ensure both total-momenta conservation,

$$\sum_{a=1}^{4} k_a = 0, \tag{3.27}$$

and the parametrization one of the momentum variables, say k_6, as a function of the others:

$$k_6 = k_5 - (k_1 + k_2) = k_5 - (k_3 + k_4). \tag{3.28}$$

Lastly, applying the constraints of Eqs.(3.27, 3.28) to integral representation of $\mathscr{M}(\Gamma)$ in Eq.(3.25) gives:

$$\mathscr{M}(\Gamma) = \frac{1}{2} (2\pi)^D \delta^D \left(\sum_{a=1}^{4} k_a \right) \left(\prod_{b=1}^{4} \int \frac{d^D k_b}{(2\pi)^D} \frac{e^{ix_b\cdot k_b}}{k_b^2 - m^2 + i\varepsilon} \right) \tag{3.29}$$

$$\times \int \frac{d^D k}{(2\pi)^D} \frac{(-i\lambda)^2}{(k^2 - m^2 + i\varepsilon)\left[(k - k_1 - k_2)^2 - m^2 + i\varepsilon\right]}. \tag{3.30}$$

In momentum-space representation, which is obtained after applying the Fourier transformation over the variables k_1, ..., k_4 in Eq.(3.30), the amplitude associated to the graph Γ assumes our final form:

$$\mathscr{M}(\Gamma) = \frac{(-i\lambda)^2}{2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - m^2 + i\varepsilon)\left[(k - k_1 - k_2)^2 - m^2 + i\varepsilon\right]}. \tag{3.31}$$

3.5 Vafa on *"Amateurish Philosophy"* and Chew's Black Box.

"Most of the greatest evils that man has inflicted upon man have come through people feeling quite certain about something which, in fact, was false."

– Bertrand Russell, in Unpopular Essays (1950). [Russell(1958)]

3.5.1 The Bootstrap consolation

In *"Puzzles to Unravel the Universe,"* the *influential* physicist Cunrum Vafa, from Harvard, claims every theoretical physicist to be an *"amateur philosopher."*

I quoted Schwinger's at the beginning of these notes. Also, I hinted my on at Sec.(3.1).

Now, let us follow Geoffrey Chew and Heisenberg's.

> *"The simple framework of S-matrix theory and the restricted set of questions that it presumes to answer constitutes* a major advantage over quantum field theory. *The latter is* burdened with a superstructure*, inherited from classical electromagnetic theory, that seems* designed to answer a host of experimentally unanswerable questions.''
> – G. F. Chew, in [Chew(1966)]. **[Added emphasis.]**

> *"The present paper seeks to establish a basis for theoretical quantum mechanics founded exclusively upon relationships between quantities which are in principle observable."*
> – W. Heisenberg (1925), translation from [Van Der Waerden(2007)].

Abstracting from Feynman, Stueckelberg, Wick and Dyson's rules, based on graph theory, which follows from Eq.(3.20) and was exemplified at Sec.(3.4), in a style *à lá* Veltman's pragmatic approach, we close our discussion of the present Chapter with the following computation of the 4–point function.

3.5.2 Scattering Amplitude's $p_1 p_2 \longrightarrow p_3 p_4$ in $\lambda\phi^4$–Model

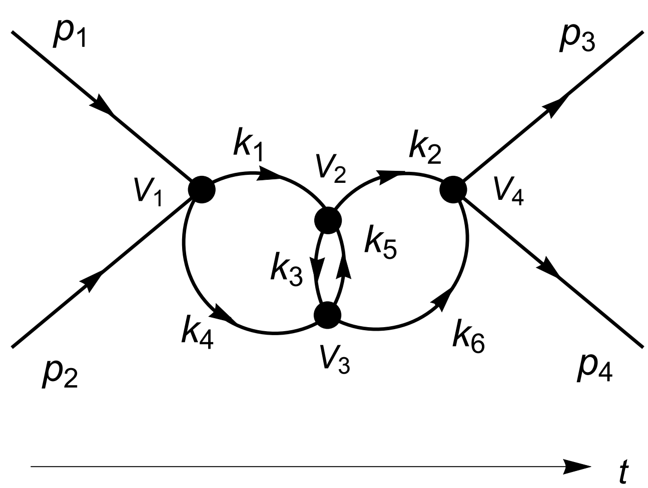


Figure 3.2: Graph $\Gamma := \Gamma(4,2)$ with $V = 4$ vertices, $E = 4$ unpaired–edges (sources), and the amplitude $\mathscr{M}_\Gamma$ contributing to two "neutral mesons's" scattering at order $\mathscr{O}\left(\lambda^4\right)$.

$$\mathscr{M}_\Gamma\left(p_1,p_2,p_3,p_4\right) = (-i\lambda)^4\left(\prod_{a=1}^{6}\int\frac{d^Dk_a}{(2\pi)^D}\,\frac{i}{k_a^2-m^2+i\varepsilon}\right) \tag{3.32}$$

$$\times\underbrace{(2\pi)^D\,\delta^{(D)}\left(p_1+p_2-k_1-k_4\right)}_{\text{Vertex \#1}}\times\underbrace{(2\pi)^D\,\delta^{(D)}\left(k_1+k_5-k_2-k_3\right)}_{\text{Vertex \#2}} \tag{3.33}$$

$$\times\underbrace{(2\pi)^D\,\delta^{(D)}\left(k_3+k_4-k_5-k_6\right)}_{\text{Vertex \#3}}\times\underbrace{(2\pi)^D\,\delta^{(D)}\left(k_2+k_6-p_3-p_4\right)}_{\text{Vertex \#4}}. \tag{3.34}$$

Chapter 4

Emmy Nöther: The *"Queen's Gambit"* of Quantum Fields

4.1 "Master" Equation: An Outline

As is often the case in the practice of theoretical physics, the following strategy is adopted in order to prepare ourselves in dealing with more realistic theories. One begins by considering a simplified model, but which retains some of the mathematical features of the model of our original interest, but which is neither trivial nor so much involved as the former, in such a way that one may hope to obtain a better understanding of both qualitative and mathematical aspects, so as to be better prepared to return one original problem.

In this section, the author outlines a general method, using the formalism of functional integration, to obtain a set of identities relating propagators of gauge field theories, whose importance will be discussed in the section that follows).

To start our investigations, let us consider the simple Lagrangian for a non–hermetian scalar field:

$$\mathscr{L}_0 := \eta^{MN} (\partial_M \varphi) \partial_N \varphi^\dagger - m^2 \varphi^\dagger \varphi. \tag{4.1}$$

The Lagrangian of Eq.(4.1) is invariant under the global phase–transformations:

$$\varphi(x) \mapsto e^{-i\alpha} \varphi(x), \;\; \varphi^\dagger(x) \mapsto e^{i\alpha} \varphi^\dagger(x), \tag{4.2}$$

which in infinitesimal form, reads:

$$\delta_\alpha \varphi = -i\alpha\varphi, \;\; \delta_\alpha \varphi^\dagger = i\alpha\varphi^\dagger. \tag{4.3}$$

Since the parameter α is constant over spacetime, Eqs.(4.2, 4.3) are referred in literature (cf. [Zee(2010), Peskin(2018)]) as "global" *gauge transformations*. This author will adopt this terminology in what follows, as it suffices to motivate the general methods, in our simplified model, before dealing with QED.

Remark 4.1. In the next coming notes to our advanced course in the quantum theory of fields, exactly the same methodology described in the present section, for our "toy–theory," will be generalized in order to be applied in the derivation of BRST cohomology, when dealing with quantization of gauge theories with non-Abelian groups. (Nevertheless, those techniques are still restricted to *compact* Lie groups, given the existence of an invariant Haar measure needed to normalize the pathintegral pseudo-measure. Thereby, even when trying to overcome the *loophole* of the non-renormalizability of General Relativity, when the latter is reformulated as a gauge theory in the Lorentz frame–bundle [Schweizer(1980), Hehl(1980), Blagojevic(2001), Thiemann(2008), Freedman and Van Proeyen(2012)], new complications appears.)

Notation 4.1. For the sake of notational simplicity, the set of fields with their resp. hermetian conjugates and derivatives, will be denoted simply as:

$$\Psi := \left\{ \varphi(x), \partial\varphi(x); \varphi(x), \varphi^{\dagger}(x), \partial\varphi^{\dagger}(x) \right\}. \tag{4.4}$$

Remark 4.2. This student accepts apologies due to his excess of pedantry. Nevertheless, it's again an opportunity to emphasize, once in a while, fundamental differences between canonical and pathintegral quantization. The former method defines fields as operators, taking values in Schwartz space of distributions[1] [Friedlander(1998)]. As for the functional integral formalism, one conceives fields in the classical sense (connections in principle bundles whereby a gauge theory is defined upon, or as sections in the associated vector or spin bundles).

Notice that in pathintegral quantization, however, fields not necessarily are what Dirac's [Dirac(1981b)] would class "*c*–numbers," namely, either classical or commuting objects, as one requires the use of Grassmann algebras for defining Schwinger's sources to obtain a generating functional for fields obeying Fermi–Dirac statistics.

The present notes follows Schwinger's principle of the quantum action and of sources. Then, let $\mathscr{J}(x)$ and $\mathscr{J}^*(x)$ be classical external currents to be coupled, resp., to the *classical* fields $\varphi(x)$ and $\varphi^*(x)$. For the sake of notational simplicity, one includes both currents $\mathscr{J}(x)$ and $\mathscr{J}^*(x)$ as $\mathscr{S} := (\mathscr{J}(x), \mathscr{J}^*(x))$.

Upon such considerations, the action to our simplified model is given by:

$$S[\mathscr{S}] = S_0[\mathscr{S}] + \int d^D x \left[\mathscr{J}^*(x)\varphi(x) + \mathscr{J}(x)\varphi^*(x)\right], \text{ where: } S_0[\Psi] := \int d^D x \mathscr{L}_0[\Psi]. \tag{4.5}$$

In the pathintegral formalism, quantization is performed by introducing the appropriate partition function $\mathscr{Z}[\mathscr{S}]$ with sources,

$$\mathscr{Z}[\mathscr{S}] := \int [d\varphi][d\varphi^*] \exp(iS\{\mathscr{J}\}) \tag{4.6}$$

$$= \int [d\varphi][d\varphi^*]\, e^{iS_0} \exp\left\{ i \int d^D x \left[\mathscr{J}^*(x)\varphi(x) + \mathscr{J}(x)\varphi^*(x)\right] \right\}. \tag{4.7}$$

The usual partition function, namely, the vacuum–to–vacuum transition amplitude, will be denoted as:

$$\mathscr{Z}_0 := \lim_{\|\mathscr{S}\| \longrightarrow 0} \mathscr{Z}[\mathscr{S}].$$

$$\frac{\delta\mathscr{Z}}{\delta\mathscr{J}^*(x)} = \int [d\varphi][d\varphi^*]\, e^{iS_0} \exp\left\{ i \int d^D x \left[\mathscr{J}^*(x)\varphi(x) + \mathscr{J}(x)\varphi^*(x)\right] \right\} \varphi(x), \tag{4.8}$$

$$\frac{\delta\mathscr{Z}}{\delta\mathscr{J}(x)} = \int [d\varphi][d\varphi^*]\, e^{iS_0} \exp\left\{ i \int d^D x \left[\mathscr{J}^*(x)\varphi(x) + \mathscr{J}(x)\varphi^*(x)\right] \right\} \varphi^*(x). \tag{4.9}$$

[1] Recall, for Minkowski space $M := \mathbb{R}^{(1,3)}$, that the space of distributions (or generalized functions), according to L. Schwartz's formulation, is the module (over the ring of germs of smooth functions in M) containing all continuous, linear functionals, defined upon the space of *test functions,* denoted by $\mathscr{C}_0^{\infty}(M)$, of all smooth, real–valued functions with compact support. (To refresh the memory, the support of a function f is the set $\operatorname{supp}(f) := f(M) \setminus f^{-1}(0)$, namely, that subset of f's domain with non–zero values.)

whereby the hermetian field operators are replaced simply complex conjugate of c–valued fields.

$$\delta_\alpha \mathscr{Z}\{\mathscr{J}\} = 0,$$

$$\begin{aligned}
&\delta_\alpha \int [d\varphi][d\varphi^*]\, e^{iS_0} \exp\left\{ i\int d^Dx \left[\mathscr{J}^*(x)\,\varphi(x) + \mathscr{J}(x)\,\varphi^*(x)\right]\right\} \\
&= \int [d\varphi][d\varphi^*]\, e^{iS_0} \times \left(\delta_\alpha \exp\left\{ i\int d^Dx \left[\mathscr{J}^*(x)\,\varphi(x) + \mathscr{J}(x)\,\varphi^*(x)\right]\right\}\right) \\
&= \alpha \int [d\varphi][d\varphi^*]\, e^{iS_0 + i\int d^Dx\left[\mathscr{J}^*(x)\varphi(x)+\mathscr{J}(x)\varphi^*(x)\right]} \int d^Dx \left[\mathscr{J}^*(x)\,\varphi(x) - \mathscr{J}(x)\,\varphi^*(x)\right],
\end{aligned}$$

inverting the order of integration[2] yields:

$$\begin{aligned}
\int d^Dx \Bigg\{ &\mathscr{J}^*(x) \left(\int [d\varphi][d\varphi^*]\, e^{iS_0 + i\int d^Dy\left[\mathscr{J}^*(y)\varphi(y)+\mathscr{J}(y)\varphi^*(y)\right]}\varphi(x)\right) - \\
&\mathscr{J}(x) \left(\int [d\varphi][d\varphi^*]\, e^{iS_0 + i\int d^Dy\left[\mathscr{J}^*(y)\varphi(y)+\mathscr{J}(y)\varphi^*(y)\right]}\varphi^*(x)\right) \Bigg\} = 0.
\end{aligned}$$

$$\int d^Dx \left(\mathscr{J}(x) \frac{\delta \mathscr{Z}\{\mathscr{J}\}}{\delta \mathscr{J}(x)} - \mathscr{J}^*(x) \frac{\delta \mathscr{Z}\{\mathscr{J}\}}{\delta \mathscr{J}^*(x)} \right) = 0. \tag{4.10}$$

The simplicity of Eq.(4.10) may be misleading. Nevertheless, there lies the reason behind it’s importance. Thereby, following [Zinn-Justin(1999)], Eq.(4.10) will be referred to as a *Master* equation.

Our first *Master* equation was derived for a “toy–model,” the free non-hermetian scalar field. Hence, no one should be surprised about the *apparent* lack of information provided by Eq.(4.10). However, as will be shown, in QED on the following, and in the notes that will be submitted in due time, what we have obtained so far is our first non-perturbative result in quantum field theory.

The importance of such a result will become clear when introducing BRST symmetries and, finally, the “renormalization program.”

Exercise. Let the external source in Eq.(4.10) be given by $\mathscr{J}(x) = \delta^{(D)}(x)$. **(I):** Derive: $\langle\Omega|\mathscr{T}\{\varphi(x)\}|\Omega\rangle - \langle\Omega|\mathscr{T}\{\varphi^\dagger(x)\}|\Omega\rangle = 0$. **(II):** Then, check the vanishing of vacuum expectation values of single operators in the free non-hermetian scalar field. Moreover, apply the functional derivative $\delta/\delta\mathscr{J}(y)$ to our Master equation, obtaining similar results for higher–order Green’s functions.

Exercise. Now the reader is invited to consider a more interesting theory: Yukawa complex scalar field φ with mass $\mu > 0$, coupled to real scalar field ϕ with mass $m > 0$, for which the Lagrangian reads:

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{\text{int}}, \tag{4.11}$$

$$\mathscr{L}_0 := \eta^{MN}(\partial_M\varphi)\,\partial_N\varphi^\dagger + \eta^{MN}(\partial_M\phi)\,\partial_N\phi - \mu^2\varphi^\dagger\varphi - m^2\phi^2,\ \mathscr{L}_{\text{int}} := -\lambda\left(\varphi^\dagger\varphi\right)\phi. \tag{4.12}$$

[2]This step should be taken with great care. Rigorously, one should verify that no mathematician (particularly the analysts) are in the room.

Note that both "free" and interacting terms in Eq.(4.12) are globally invariant under a phase-change, see Eq.(4.3). Thereby the Master equation holds good. Apply the steps **(I, II)** delineated in our former exercise to obtain non-trivial relations between the Green's function. (In particular, the 2–point and 3–point vertex amplitudes.)

4.2 The "Master" of Quantum Electrodynamics

4.2.1 Mathematical Formulation

Let A_M be the electromagnetic potential and $\{x^M\}$ Lorentz coordinates in Minkowski spacetime $\mathbb{R}^{1,D-1}$. In this section, we assume $D = 4$.

From a geometric perspective, namely, that of p–bundles, $\mathbf{A} := A_\mu dx^\mu$ is the Maxwell connection 1–form[3]. The covariant derivative associated to $\mathbf{A}$ is then defined as $D_\mu := \partial_\mu + iqA_\mu$, where $q < 0$ is the *elementary unit* of charge, and whereupon the curvature arising from parallel is $\mathbf{F} = (1/2)\,D_{[\mu}A_{\nu]}dx^\mu \wedge dx^\nu$.

Nevertheless, in particle physics, one is usually habituated with the following terminology: $A_\mu(x)$ is the (bosonic) field whose excitations are identified with photons, while the components $F_{\mu\nu}$ of the associated curvature is the (Faraday) field–strength.

Moreover, in the associated spin–bundle, letting $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}$ be generators of the complexified Clifford algebra $(\mathfrak{Cl}_{1,3}(\mathbb{R}))^{\mathbb{C}}$ and using $e^M = dx^M$ as (holonomic) frame-fields, the Dirac–field is represented by the 4–spinor section ψ. The spin-connection is so defined by $\nabla^{\text{spin}} := \gamma^\mu D_\mu$.

The dynamics of QED is governed by the Lagrangian:

$$\mathscr{L}_{\text{QED}} = -\frac{1}{2}\text{Tr}\left(\mathbf{F}\wedge\star\mathbf{F}\right) + i\bar{\psi}\left(\nabla^{\text{spin}} - m\right)\psi, \tag{4.13}$$

or in components, which is more familiar for physicists, one reads:

$$\mathscr{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^\mu D_\mu\psi - m^2\bar{\psi}\psi. \tag{4.14}$$

4.2.2 QED Propagators and Vertices

In this subsection, a brief outline of the scheme of canonical quantization to QED, whose main purpose is simply to *motivate* our definitions of $S(p)$ and Γ^μ. Moreover, a method by means of which those may be computed.[4]

4.2.2.1 Generalities on the Canonical Quantization (for Abelian theories)

The linearity of field equations, a characteristic quality of Abelian gauge theories, such as that of the classical equations of motions following as the Euler-Lagrange of $\mathscr{L}_{\text{QED}}$, written explicit in components in Eq.(4.14),

[3]Since electrodynamics is an Abelian gauge theory, the Maxwell connection is Lie algebra–valued in $\mathfrak{u}(1)$, trivially generated by the nilpotent identity. Hence we omit structure constants and hermetian generators, for now.

[4]Clearly, the matter–field propagator and vertices are easily defined, and sometimes also to calculate, from functional integration. The aim however is to keep the interplay between both formulations of QFT, canonical and functional, exhibiting to the student their mutual complementarity in solving problems particularly difficult in one, nonetheless conceptually or quantitatively clear in the other..

gives ones an expectation that canonical quantization is a feasible method to obtain that set of quantities, such as the Green's functions, needed to compute the physical predictions of the theory and observable, as the *S*–matrix.

Usually, there exists a well-known, succinctly outlined here. One starts with introducing creation and annihilation operators. These satisfies a (Jordan) algebra, determined by the equal–time commutation (or anti–commutation, depending on fields' statistics) relations, given by the nature of the fields themselves. Thence, one is assured, as regards the principle of micro–causality [Wightman and Schweber(1955)], as required by Poincaré invariance, is satisfied.

Now, assume our imaginary theory builder is able to normal–order generators of operators which, when taken together, constitute a *complete set of commuting observables* [Dirac(1981b)]. One of those (hermetian) operators, of fundamental importance when adopting either *Heisenberg representation* or, as required later on, *interaction representation* [Weinberg(1995), Ch.3, §5], is the generator of the evolution operator, the Hamiltonian.

If the Hamiltonian, besides those other operators[5], can be expressed as polynomials of creation and annihilation operators, *particularly* what is to be recognized as the "interaction" term, then one is assured that Wichmann–Eyvind–Crichton [Wichmann and Crichton(1963)] principle of cluster decomposition holds good. For a simplified review, consult [Weinberg(1995), Ch.4].

Finally, observable quantities, such as the *S*–matrix, are obtained applying Dyson's series to the LSZ reduction equation in order to compute time–ordered, vacuum–to–vacuum (known as *vevs.*) expectation values, with the help of Wick's theorem [Itzykson and Zuber(2012), Ch.5, §3.3].

4.2.2.2 Propagators and Vertices in Canonical Quantization

Definition 4.1. Let $(\cdot,\cdot)$ be the inner product of the projective Hilbert space $\mathbf{P}(\mathscr{H})$ of the physical states in the canonically quantized QED. Let Ω_0 denote the ground–state of the canonical QED, through which the naught under–script is employed to empathize our blessed ignorance as far as the Faddeev–Kulish states exists (and their related infrared theorems [Weinberg(1965)]). Close our definitions by denoting with $\mathscr{T}\{\star\star\star\}$ the time–ordering of the arbitrary field–operators $\star$'s.

Recall, from an elementary course in QED [Mandl and Shaw(2010)], that the fermionic propagator $iS(p)$ and the 3–vertex function $\Gamma^\mu(p_1,p_2,k)$ are defined, resp., in energy–momentum representation, as the fourier transforms of *vevs*,

$$iS(p_2-p_1) := \int d^Dx_1 d^Dx_2\, e^{ip\cdot(x_2-x_1)}\left(\Omega_0,\, \mathscr{T}\{\bar{\psi}(x_2)\,\psi(x_1)\}\,\Omega_0\right), \tag{4.15}$$

$$(2\pi)^D\,\delta^{(D)}(p_1+p_2-k)\,\Gamma^\mu(p_1,p_2,k) := \widetilde{\Gamma}^\mu(p_1,p_2,k), \tag{4.16}$$

where:

$$\widetilde{\Gamma}^\mu(p_1,p_2,k) = \int d^Dx_1 d^Dx_2 d^Dy\, e^{i(p_1\cdot x_1+p_2\cdot x_2-k\cdot y)}\left(\Omega_0,\, \mathscr{T}\{A^\mu(y)\,\bar{\psi}(x_2)\,\psi(x_1)\}\,\Omega_0\right). \tag{4.17}$$

[5] Also, this student feels the need for completeness and thereby mention the importance of operators related to symmetries. A prominent rôle is staged by the spacetime–translation operators, and the ones representing the action of Lorentz boots on physical states Ψ (belonging to one's theory Hilbert space, $\Psi\in\mathbf{H}$). Employing a well-known notation, the corresponding generators of spacetime–translations and Lorentz boosts are, resp., $\mathbf{K}^i$ and $\mathbf{J}^i$. These, together with commutation relations acting on $\mathbf{H}$, constitute an algebra isomorphic to the Lie algebra of the (proper–orthochronos Lorentz group) $\mathscr{L}_+^\uparrow$.
For the interested reader, cf. [Weinberg(1995), Ch.2, Vo.I] for a detailed discussion.

Remark 4.3. Note that a proportionality factor $1/(2\pi)^D$, within the measure of energy-momentum space, may “oscillate” non–periodically, according to the tastes of textbook writers. Thankfully, by Poincaré recurrence theorem, mapping one author psychological state to a Hamiltonian system, one proves the measure of equations, sensible to variations of personalities, vanishes, for sufficiently large–times. As there are, at time of writing, no feasible experimental test which could falsify our goal of a future study of superstring theory, such large–times are of no concern.

Applying Wick’s theorem and performing the integrations from the fourier transformations, one arrives at the explicit results:

Conclusion 4.1. The *unrenormalized*[6] Dirac–propagator for QED is given by:

$$S(p) = \frac{\eta_{\mu\nu}\gamma^{\mu}p^{\nu} + m}{p^2 - m^2 + i\varepsilon^{+}}, \tag{4.18}$$

where $\varepsilon^{+} > 0$ is the Feynman’s *protégée* infinitesimal.
(Sometimes, the imaginary unity is present as a factor in both sides, as such is the particular form of the Dirac–propagator suitable to compute amplitudes perturbatively, by means of Feynman–Dyson graphs).

Conclusion 4.2. The *unrenormalized* (as a precise definition is explained in ¶6) 3–vertex connected function $\Gamma^{\mu}(p_1, p_2, k)$ for QED is, indexed as dictated by the matrix–representation of the generators $\{\gamma^{\mu}\}$ of $\mathfrak{Cl}\left(\mathbb{R}^{(1,3)}\right)^{\mathbb{C}}$ (the space–time algebra of Minkowski space),

$$\Gamma^{\mu}(p_1, p_2, k)_{\alpha\beta} = -iq\,(\gamma^{\mu})_{\alpha\beta}\,. \tag{4.19}$$

Notation 4.2. When one is interested in the limiting expression of Eq.(4.19), through which the energy-momentum of ingoing and outgoing states of the Dirac–field are such as $p_1 = p_2 =: p$, it proves itself useful to introduce the notation:

$$V^{\mu}(p;k)_{\alpha\beta} \coloneqq \lim_{p_1 \to p} \Gamma^{\mu}(p_1, -p_1, k)_{\alpha\beta} = -iq\,(\gamma^{\mu})_{\alpha\beta}\,. \tag{4.20}$$

4.2.3 “Effective” Action

It was learned in our discussion of Faddeev-Popov method (recall our former notes [Mol(2021)]) that, to correctly define the pathintegral measure in field space, in the presence of gauge symmetries, one should: (i) replace the phasespace of possible field configurations by it’s factor under the action of gauge symmetry group; (ii) introduce into the pathintegral measure contributions from *geisterfeld*, namely, unobservable fields violating spin-statistics theorem; and finally, (iii) modify the original Lagrangian with a gauge fixing term.

To proceed, an *a posteriori* fact (cf. [Mol(2021)] for details) will be assumed on what follows, the reason being a significant simplification of our calculations. For an Abelian gauge theory, if one chooses the family of covariant gauges (labelled by the real parameter ξ), a procedure which is simply realized by adding to Eq.(4.14) the *gauge-fixing* term:

$$\mathscr{L}_{\mathrm{GF}} \coloneqq -\frac{1}{2\xi}\left(\partial_{\mu}A^{\mu}\right)^2, \tag{4.21}$$

[6] As at this stage in our course, no discussion was delivered concerning the rôle of counter-terms Z’s, neither the renormalization program, one could simply ignore this qualification in these notes. Nevertheless, this student is also taking a class on “people skills” at *Coursera,* and one of my home-works is to exercise humility. Therefore, I use this opportunity to humbly declare that neither of the following results takes account of the need to renormalize the “bare” parameters, examples of which that figures in Eq.(4.18) are, resp., the “bare” mass $m =: m_0$ and charge $q =: q_0$.

any coupling between propagators of *geisterfeld* to bosonic field A_μ is inexistent.

In effect, the final effect, arising from performing the functional integral of *geisterfeld*, ends with a source–independent phase–factor multiplying $\mathscr{Z}$. Henceforth, one should allow oneself to ignore the Faddeev-Popov *geisterfeld*, in so far as an Abelian theory, with a carefully chosen *gauge–fixing*, is concerned.

Let the action for the ordinary QED be denoted by:

$$S_0\left[A_\mu,\psi,\bar{\psi}\right] := \int d^D x\,\mathscr{L}_{\mathrm{QED}}, \tag{4.22}$$

while the spacetime integral over the coupling between the c–number source $\mathscr{J}^\mu$ to A_μ, and the Grassmann-valued sources σ and $\bar{\sigma}$ to the Dirac field, be given by:

$$\mathfrak{J} := \mathfrak{J}\left[\mathscr{J}^\mu,\sigma,\bar{\sigma}\right] = \int d^4x\,\left[\mathscr{J}^\mu(x)A_\mu(x)+\bar{\sigma}(x)\psi(x)+\bar{\psi}(x)\sigma(x)\right]. \tag{4.23}$$

Hence, the "effective" action for our theory is:

$$S_{\mathrm{eff}} = S_0 + \int d^4x\left[-\frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 + \mathscr{J}^\mu(x)A_\mu(x)+\bar{\sigma}(x)\psi(x)+\bar{\psi}(x)\sigma(x)\right]. \tag{4.24}$$

Remark 4.4. Admittedly, the adjective "effective" for the action from Eq.(4.24) is, at this moment of our studies, an abuse of language. At least from the viewpoint of Renormalization (Semi–group) Theory [Collins and Collins(1985)], the true *effective action* capable of rendering QED a fully predictive theory (of course, in the formalism of Lagrangian field theories quantized by functional integration), should include *counter–terms*, as will be discussed soon, in order to renormalize the "bare" parameters appearing in Eq.(4.14).

Remark 4.5. It may be worth mentioning that, besides the framework of Renormalization Group (RG), there are alternative approaches capable of rendering QED a finite and predictive theory, by taking care simultaneously of the rôle played by micro-causality and the mathematical structure of relativistic quantum field theories, in which those fields should be treated as a distribution–valued operators, constituting a module nevertheless not an algebra. It's know as the Epstein-Glaser approach or causal perturbation theory. For original papers, see [Epstein and Glaser(1973)]. A pedagogical introduction is presented detailed by [Scharf(2013)].

Honestly, this author have still been unable to decide if the mathematical complications, which are intrinsic to the Epstein-Glaser approach, provides sufficient reason for any possible replacement of a highly successful and physically motivated research program, such as RG theory.

Moreover, those mathematical complications certainly requires no high level of sophistication and abstraction, such as the ones which culminated in the works of mathematical gauge theories [Atiyah(1988)] and topological QFTs [Witten et al.(1982), Witten(1988)]. The difficulties are mostly technicalities arising from old-fashioned analysis and very long computations.

4.2.4 "Master" Equation

Lastly (but not least!), the generating functional for the Green's functions of QED (taking into account all the reservations which have been hopefully clarified above), can be written as:

$$\mathscr{Z} := \mathscr{Z}\left[\mathscr{J}^\mu,\sigma,\bar{\sigma}\right] = \int \left[dA_\mu\right]\left[d\psi\right]\left[d\bar{\psi}\right]\exp\left(iS_{\mathrm{eff}}\right). \tag{4.25}$$

Remark 4.6. At this point, one should take notice of the fact that, in the limit whereby the external currents vanishes, namely, $\|\mathscr{J}\|, \|\sigma\|, \|\bar{\sigma}\| \to 0$, one recovers from the generating functional (cf. Eq.(4.25)) the well-known *partition function* of one's theory,

$$\mathscr{Z}_0 := \lim_{\|\mathscr{J}^\mu\|, \|\sigma\|, \|\bar{\sigma}\| \to 0} \mathscr{Z}[\mathscr{J}^\mu, \sigma, \bar{\sigma}] = \int [dA_\mu][d\psi][d\bar{\psi}]\, e^{i\int d^4x \left\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^\mu D_\mu\psi - m^2\bar{\psi}\psi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right\}}. \tag{4.26}$$

From our geometric construction of QED in §4.2.1, we should expect that gauge symmetry is an intrinsic feature of the theory, which Eq.(4.25) is supposed to describe after quantization[7].

Using however the more mundane notation of coordinates components, an infinitesimal gauge transformation, generated by a spacetime function $\Lambda : x \in \mathbb{R}^{1,3} \mapsto \Lambda(x) \in \mathbb{R}$ (which, at this step, should be *at least* continuously differentiable), reads:

$$\delta_\Lambda A_\mu(x) = \partial_\mu \Lambda(x), \;\; \delta_\Lambda \psi(x) = -iq\Lambda(x)\psi(x), \;\; \delta_\Lambda \bar{\psi}(x) = iq\Lambda(x)\overline{\psi}(x). \tag{4.27}$$

The basic *postulate* of our work with functional integrals may be so summarized. The symmetries of Eq.(4.27) are preserved after all care have been taking in properly defining our pathintegral (pseudo–)measure, so that:

$$\delta_\Lambda \mathscr{Z} = \int [dA_\mu][d\psi][d\bar{\psi}] \exp(iS_{\text{eff}}) \tag{4.28}$$

$$\times\, \delta_\Lambda i \int d^4x \left[-\frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \mathscr{J}^\mu(x) A_\mu(x) + \bar{\sigma}(x)\psi(x) + \bar{\psi}(x)\sigma(x)\right]. \tag{4.29}$$

Now, rearrange the order of integration in Eq.(4.29)[8], and thereafter, apply the transformations in Eq.(4.27). One thus obtain:

$$\delta_\Lambda \mathscr{Z} = \int d^4x\, i \int [dA_\mu][d\psi][d\bar{\psi}] \exp(iS_{\text{eff}}) \tag{4.30}$$

$$\times \left\{\frac{1}{\xi}\Box_x [\partial_\mu A^\mu(x)] - (\partial_\mu \mathscr{J}^\mu)(x) + iq[\bar{\sigma}(x)\psi(x) - \bar{\psi}(x)\sigma(x)]\right\} \Lambda(x), \tag{4.31}$$

where $\Box_x := \partial_\mu \partial^\mu$.

On the other hand, similarly to Eqs.(4.8, 4.9), the functional derivatives of $\mathscr{Z}[\mathscr{J}^\mu, \sigma, \bar{\sigma}]$ according to each external current are:

$$\frac{\delta \mathscr{Z}}{\delta \mathscr{J}^\nu(y)} = i \int [dA_\mu][d\psi][d\bar{\psi}] \exp(iS_{\text{eff}}) A_\nu(y), \tag{4.32}$$

$$\frac{\delta \mathscr{Z}}{\delta \sigma(y)} = -i \int [dA_\mu][d\psi][d\bar{\psi}] \exp(iS_{\text{eff}})\, \bar{\psi}(y), \tag{4.33}$$

$$\frac{\delta \mathscr{Z}}{\delta \bar{\sigma}(y)} = i \int [dA_\mu][d\psi][d\bar{\psi}] \exp(iS_{\text{eff}})\, \psi(y). \tag{4.34}$$

[7] Anomalies, however, may appear, a subject that should be taken seriously own it's right. Unfortunately, it's beyond the scope of our present purposes. But allow myself to say that, in QCD, chiral anomalies is a very serious technical issue.

[8] Once again, by being very careful with the presence of any mathematician around your study.

Note that the sign in Eq.(4.33) is due to the anti–commutativity of Grassmann–valued operators. In particular, by the process of functionally differentiating Eq.(4.23), the contribution to our “effective” action displayed at Eq.(4.24) due to the external sources, with respect to $\sigma(x)$ is:

$$\frac{\delta \mathfrak{J}}{\delta \sigma(y)} = i\int d^4x \frac{\delta}{\delta\sigma(y)}\left[\bar{\psi}(x)\sigma(x)\right] = i\int d^4x \left[-\bar{\psi}(x)\frac{\delta\sigma(x)}{\delta\sigma(y)}\right] = -i\bar{\psi}(y). \tag{4.35}$$

Therefore, applying Eqs.(4.32, 4.33, 4.34) to the variation of $\delta_\Lambda \mathscr{Z}$ obtained in Eq.(4.31), one arrives at the following result:

$$\delta_\Lambda \mathscr{Z} = \int d^4x \left\{-\frac{1}{\xi}\Box_x\left[\partial_\mu \frac{\delta\mathscr{Z}}{\delta\mathscr{J}_\mu(x)}\right] - \left(\partial_\mu \mathscr{J}^\mu\right)(x)\mathscr{Z} + iq\left[\bar{\sigma}(x)\frac{\delta\mathscr{Z}}{\delta\bar{\sigma}(x)} - \frac{\delta\mathscr{Z}}{\delta\sigma(x)}\sigma(x)\right]\right\}\Lambda(x) = 0. \tag{4.36}$$

As Eq.(4.36) holds true for whatever test function $\Lambda \in \mathscr{C}_0^\infty$ one chooses, the term within curly brackets within the integrand vanishes for all sources. Thence follows our “Master” equation:

$$\frac{1}{\xi}\Box_x\left\{\frac{\partial}{\partial x^\mu}\left[\frac{\delta\mathscr{Z}}{\delta\mathscr{J}_\mu(x)}\right]\right\} + \left(\partial_\mu\mathscr{J}^\mu\right)(x)\mathscr{Z} + iq\left[\sigma(x)\frac{\delta\mathscr{Z}}{\delta\sigma(x)} + \bar{\sigma}(x)\frac{\delta\mathscr{Z}}{\delta\bar{\sigma}(x)}\right] = 0. \tag{4.37}$$

4.2.5 Dictionary to Functional Quantization

In §4.2.2.2, a derivation was given of the Dirac–propagator and the 3–vertex function, using the canonical quantization familiar from an elementary course in QED.

On what follows, we simply recall, as was discussed in more details in our former lecture, the form assumed by those quantities when the pathintegral formalism in chosen as one’s method of quantization.

In the case of the Dirac–propagator, the energy–momentum space representation is defined as usual:

$$(2\pi)^{(D)}\delta^{(D)}(p_2-p_1)\, iS(p_2-p_1) := \int d^Dx_1 d^Dx_2\, e^{ip\cdot(x_2-x_1)}\tilde{S}(x_2-x_1), \tag{4.38}$$

in such a way one is allowed to ignore possibly inconvenient distributions along, while the imperative to the mapping between the canonical and functional methods is:

$$\tilde{S}(x_2-x_1) = \frac{1}{i}\lim_{\|\sigma\|,\|\bar{\sigma}\|\to 0}\left[\frac{\delta^2\mathscr{Z}}{\delta\bar{\sigma}(x_2)\,\delta\sigma(x_1)}\right]. \tag{4.39}$$

With regards to the (unrenormalized) interaction, between the Maxwell and Dirac fields,

$$\mathscr{L}_{\text{int}} = -iq\gamma^\mu A_\mu \bar{\psi}\psi$$

tank-fully, Eq.(4.16) allows oneself to disregard any concerns about those factors of $(2\pi)^D$, besides the energy–momentum conservation factors arising as $\delta's$ distributions.

In spacetime representation,

$$\hat{\Gamma}^\mu(x_1,x_2;y) = \frac{1}{i^3}\lim_{\|\mathscr{J}\|,\|\sigma\|,\|\bar{\sigma}\|\to 0}\left[\frac{\delta^3\mathscr{Z}}{\delta\bar{\sigma}(x_2)\,\delta\sigma(x_1)\,\delta A_\mu(y)}\right], \tag{4.40}$$

which, upon fourier transforming to energy–momentum coordinates, one deduces (up to the proportionality factors mentioned above):

$$\tilde{\Gamma}(p_1,p_2;k) := \frac{1}{i^3} \lim_{\|\mathscr{J}\|,\|\sigma\|,\|\bar{\sigma}\|\to 0} \int d^D x_1 d^D x_2 d^D y \, e^{i(p_2-p_1)\cdot(x_2-x_1)+ik\cdot y} \frac{\delta^3 \mathscr{Z}}{\delta\bar{\sigma}(x_2)\,\delta\sigma(x_1)\,\delta A_\mu(y)}. \tag{4.41}$$

Remark 4.7. In functional quantization, Eqs.(4.38, 4.40, 4.41) are the analogous to those derived in §4.2.2.2, Eqs.(4.15, 4.16, 4.17), using the canonical formulation in the projective Hilbert state–space.

4.2.6 Example: Ward-Takahashi Identities

As a simple application of Eq.(4.37), this section presents a derivation of a relation in the context of QED, the Ward–Takahashi identities [Takahashi(1957)].

First, one should note that, for Grassmann–valued functions, such as the external sources $\sigma(x)$ and $\bar{\sigma}(x)$, their fundamental property of anti–commuting among themselves still holds true [DeWitt(1992)]:

$$\frac{\delta}{\delta\sigma(y)}\frac{\delta}{\delta\bar{\sigma}(z)} = -\frac{\delta}{\delta\bar{\sigma}(z)}\frac{\delta}{\delta\sigma(y)}. \tag{4.42}$$

With the above remarks in mind, one applies Eq.(4.42) to Eq.(4.10) and take the usual limit, whereby external currents $\|\mathscr{J}\|, \|\sigma\|, \|\bar{\sigma}\| \to 0$, to obtain:

$$\lim_{\|\mathscr{J}\|,\|\sigma\|,\|\bar{\sigma}\|\to 0} \frac{1}{\xi}\Box_x \left[\frac{\partial}{\partial x^\mu}\left(\frac{\delta^3\mathscr{Z}}{\delta\bar{\sigma}(z)\,\delta\sigma(y)\,\delta\mathscr{J}_\mu(x)}\right)\right] \tag{4.43}$$

$$= \lim_{\|\sigma\|,\|\bar{\sigma}\|\to 0} iq\left\{\delta^{(D)}(x-z)\frac{\delta^2\mathscr{Z}}{\delta\sigma(y)\,\delta\bar{\sigma}(x)} + \frac{\delta^2}{\delta\bar{\sigma}(z)\,\delta\sigma(x)}\left[\delta^{(D)}(x-z)\right]\right\}. \tag{4.44}$$

Lastly, employing Eqs.(4.15, 4.16) for the Dirac–propagator and 3–vertex function, resp., at our form of the "Master" formula, displayed at Eq.(4.44), one should be able to deduce:

$$S(p_2) - S(p_1) = -\frac{i}{\xi}(p_2-p_1)^\nu S(p_2)\,\eta_{\mu\nu}V^\mu(p_2,p_1)\,S(p_1), \tag{4.45}$$

which is exactly the generalization obtained by Y. Takahashi [Takahashi(1957)] of the well-known Ward identity [Ward(1950)].

Chapter 5

Regularization Schemes: *"An Entrance to Renormalization Theory"*

5.1 Introduction

In our former lectures, we closed the first part of a study on "advanced" quantum field theory, whereby non–Abelian gauge theories have been quantized by means functional integration. Now, we are capable of computing the Green's functions from the generating functionals, which are needed in order to calculate physically observable quantities, the **S**–matrix playing a prominent rôle

In this Chapter, our aim is to review some parts from our last lectures, nevertheless at this time, with a particular attention to the theory of quantum electrodynamics (QED). Moreover, our emphasis will be devoted to those concepts that are needed to a proper appreciation, or *raison d'être* let's say, for the theory of renormalization "group."

For the sake of pedagogy, we choose to work on a concrete problem, both physically and historically important, the calculation of the self–energy of the electron, at order $\sim \mathscr{O}\left(\mathbf{e}^2\right)$, in spinor quantum electrodynamics (QED).

5.2 *Anamnesis*

Let A_μ and ψ the Maxwell and Dirac fields, respectively, and denote by $F_{\mu\nu} := \partial_{[\mu}A_{\nu]}$ the Faraday field–strength tensor.

The complete generating functional to QED, from which all Green's functions can be derived, is:

$$\mathscr{Z}\left[\mathscr{J}^{\mu}(x), \sigma(x), \bar{\sigma}(x)\right] = \int \left[dA_\mu\right]\left[d\psi\right]\left[d\bar{\psi}\right] e^{i\left(S_0 + S_{\mathrm{GF}} + S_{\mathrm{int}}\right)} \tag{5.1}$$

$$\times \exp\left\{ i \int d^4x J^{\mu}(x) A_\mu(x) + \bar{\sigma}(x)\psi(x) + \bar{\psi}(x)\sigma(x) \right\}, \tag{5.2}$$

where:

$$S_0 = \int d^4x \left[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(x)\left(i\gamma^{\mu}\partial_\mu - m\right)\psi(x) \right], \tag{5.3}$$

$$S_{\rm GF} = -\frac{1}{2\xi}\int d^4x\,\left(\partial_\mu A^\mu\right)^2,\ \text{and} \tag{5.4}$$

$$S_{\rm int} = -i\mathbf{e}\int d^4x\,\bar{\psi}(x)\,\gamma^\mu A_\mu(x)\,\psi(x)\,, \tag{5.5}$$

are, respectively, the free, gauge–fixing and interacting contributions to the total action[1].

Perturbation theory consists in the power series expansion of the (imaginary) exponential within Eq.(5.1) in the "coupling constant," $\mathbf{e}$ acting as a proportionality constant to the interaction term, Eq.(5.5).

The *free* generating functional, by which is one means, the limit whereby our theory coupling constant vanishes, is defined by $\mathbf{e}\to 0$.

As we have shown that for QED,

$$\mathscr{Z}_0\left[\mathscr{J}^\mu(x),\sigma(x),\bar{\sigma}(x)\right] = \exp\left\{i\int d^4x d^4y\,\left[\frac{1}{2}J^\mu(y)\,\Delta_{\mu\nu}(y-x)\,J^\nu(x) + \bar{\alpha}^\alpha(y)\,S_{\alpha\beta}(y-x)\,\alpha^\beta(x)\right]\right\}, \tag{5.6}$$

Here, $i\Delta^{(F)}_{\mu\nu}(x) =: S^{(F)}(x)$ and $iS(x) =: S_F(x)$ are, respectively, Feynman propagators for the Maxwell's and Dirac's *electron* fields.

In spacetime representation, one reads:

$$iS_F(y-x) = \int\left[dA_\mu\right]\left[d\psi\right]\left[d\bar{\psi}\right]e^{i(S_0+S_{\rm GF}+S_{\rm int})}\,\psi(y)\,\bar{\psi}(x)\,, \tag{5.7}$$

$$i\Delta^{(F)}_{\mu\nu}(y-x) = \int\left[dA_\mu\right]\left[d\psi\right]\left[d\bar{\psi}\right]e^{i(S_0+S_{\rm GF}+S_{\rm int})}A_\mu(y)\,A_\nu(x)\,. \tag{5.8}$$

Thence, a representation in the form of a perturbation series for Eq.(5.1), from which observables, like the S–matrix, may be expressed as superpositions of connected Green's functions (recall Dyson's series in canonical formalism):

$$\mathscr{Z}\left[\mathscr{J}^\mu(x),\sigma(x),\bar{\sigma}(x)\right] = \exp\left\{-ie\int d^4x\,\left[\frac{\delta}{\delta J^\mu(x)}\frac{\delta}{\delta\sigma(x)}\gamma^\mu\frac{\delta}{\delta\bar{\sigma}(x)}\right]\right\}Z_0\left[J^\mu,\sigma,\bar{\sigma}\right]. \tag{5.9}$$

Lastly, we close our review by defining the generating functional for all the *connected* Green's functions:

$$\mathscr{W}\left[J^\mu,\sigma,\bar{\sigma}\right] := -i\log\mathscr{Z}\left[\mathscr{Z}^\mu(x),\sigma(x),\bar{\sigma}(x)\right], \tag{5.10}$$

as can be shown by induction in [Abers and Lee(1973)].

5.2.1 Self–energy

Let $\mathbf{S}(p)$ be the *complete* propagator of Dirac's field ψ, defined in the energy–momentum representation by:

$$\mathbf{S}(p) := \lim_{\|\mathscr{J}\|,\|\sigma\|,\|\bar{\sigma}\|\to 0}\int d^4x_1 d^4x_2\,e^{-ip\cdot(x_2-x_1)}\frac{\delta^2}{\delta\sigma(x_2)\,\delta\bar{\sigma}(x_1)}\mathscr{W}\left[\mathscr{J}^\mu(x),\sigma(x),\bar{\sigma}(x)\right]. \tag{5.11}$$

[1] Choosing gauge–fixing condition as in Eq.(5.4), Maxwell–field's propagator belongs to the class of covariant gauges, labelled by ξ.

Remark 5.1. From a physical viewpoint, $\mathbf{S}(p)$ is *analogous* to our familiar “free” propagator of the Dirac’s field, simply denoted by $S(p)$. Here, however, our interest relies in the fermion–propagator of the “complete” theory of quantum electrodynamics, whereby *all the interactions are taken into account*. In this sense, $S(p)$ is the contribution of order $\sim \mathscr{O}\left(e^{0}\right)$ to $\mathbf{S}(p)$, once the elementary unit of charge is chosen as the coupling constant of QED.

Remark 5.2. Historically, the fine–structure constant $\alpha \propto e^{2}/4\pi$ played a key rôle in the development of QED. Sometimes, α was taken as the genuine coupling constant of the latter theory. For an illustration, consider M. Born remark in 1935 that [Born(1935)]: *“the explanation of this number [the fine structure–constant α] must be the central problem of natural philosophy.”* Even more drastically, W. Pauli told to his psychoanalyst [Pauli et al.(2013)Pauli, Schlapp, Enz, and Meyenn]: *“When I die, my first question to the devil will be: What is the meaning of the fine structure constant.”*

As will be clarified later, when we study the theory of Renormalization “Semi–group,” (RG, from now on) the fine–structure constant α along with other parameters figuring in the Lagrangian formulation of one’s theory, flows under the action of the “semi–group” of renormalization according to the energy–scale through which the process of interest takes place.

To close this brief appetizer our study of RG, the flow under which the parameters of one’s theory scales as a function of energy, is dictated by what’s known as the Callan–Symanzik equations[Collins and Collins(1985)].

In any case, my hope is to have been able to provide a convincing argument as to why the elementary unit of charge e should be taken as the order–by–order of magnitude in applying the methods of perturbative QFT, as summarized in Eq.(5.9), instead of α, better left today for the crackpots’s online archives.

Applying Eqs.(5.6, 5.9, 5.10) to Eq.(5.11), we obtain the following series expansion:

$$\mathbf{S}(p)=i(\gamma\cdot p+m)^{-1}+i(\gamma\cdot p+m)^{-1}\times\left[i\Sigma^{(2)}(p)\right]\times i(\gamma\cdot p+m)^{-1}+\mathscr{O}\left(e^{4}\right). \tag{5.12}$$

Therefore, $i\Sigma^{(2)}(p)$ is the one–loop correction, of order $\sim \mathscr{O}\left(\mathbf{e}^{2}\right)$, to the self–energy of the electron.

On the following:

1. The propagator of the Maxwell’s field will be regulated by the introduction of a fictitious mass of the photon, $m_{\gamma} \longrightarrow 0^{+}$.
2. The Feynman’s gauge, where $\xi = 1$, is chosen in Eqs.(5.11, 5.12).

$$i\Sigma^{(2)}(p)=\lim_{m_{\gamma}\downarrow 0}\int_{\mathbb{R}^{(1,3)}}\frac{d^{4}k}{(2\pi)^{4}}\underbrace{\left(-ie\gamma^{\mu}\right)}_{1^{\text{st}}\text{ int. vertex}}\times\underbrace{i\left(\gamma\cdot(p+k)-m+i\varepsilon_{1}^{+}\right)^{-1}}_{\text{fermion-propagator's one-loop}}\times\underbrace{\left(-ie\gamma^{\nu}\right)}_{2^{\text{nd}}\text{ int. vertex}} \tag{5.13}$$

$$\times\underbrace{\left(-\frac{i\eta_{\mu\nu}}{k^{2}+m_{\gamma}^{2}}\right)}_{\text{photon-propagator within fermionic loop}}, \tag{5.14}$$

and therefore:

$$i\Sigma^{(2)}(p) = -\lim_{m_\gamma \downarrow 0} \frac{e^2}{8\pi^4} \int_{\mathbb{R}^{(1,3)}} d^4k \frac{2m - \gamma \cdot (p+q)}{\left[(p+k)^2 - m^2\right]\left(k^2 - m_\gamma^2\right)} \tag{5.15}$$

Employing Eqs.(A.24) from App.(A.2), shifting the integration variable by $k^\mu \mapsto k^\mu - \tau p^\mu$ and defining:

$$Q_\tau^2(m, m_\gamma) := \tau m^2 + (1-\tau) m_\gamma^2 - \tau(1-\tau) p^2, \tag{5.16}$$

one finally arrives at the $\sim \mathscr{O}\left(\mathbf{e}^2\right)$ correction to the *complete* Dirac propagator:

$$i\Sigma^{(2)}(p) = -\frac{\mathbf{e}^2}{8\pi^4} \int_0^1 d\tau \left[2m - (1-\tau)\gamma \cdot p\right] \int d^4k \frac{1}{\left[k^2 + Q_\tau^2(m, m_\gamma)\right]^2}. \tag{5.17}$$

Note that Eq.(5.17), which physically one would expect the order $\sim \mathscr{O}\left(\mathbf{e}^2\right)$ correction to the self–energy of the electron, diverges.

5.2.2 *"The Archetype"*

Suppose we are given a QFT, written in Lagrangian form, such that the interaction–term are polynomial functions of the fields and sources. For example, see Eqs.(5.3, 5.4, 5.5) for QED, the archetypical relativistic quantum field theory of the fundamental interactions.

A general problem, both for the student or the professional phenomenologist, is to employ (or deploy, whatever) the Green's functions, obtained from the generating functional, Eq.(5.1), to arrive at physically observable results. The fundamental example being, of course, the **S**–matrix.

In such a task, a common exercise is to compute a "N–loop" integral, a quantum mechanical amplitude contributing to the state of the physical system at hand, at order $\sim \mathscr{O}(N)$ in perturbation theory, which usually takes the form:

$$i\mathscr{M}^{(N)}(p) = \kappa\,(ig)^N \times \int \left(\prod_{i=1}^N \frac{d^4k_i}{(2\pi)^4}\right) \times \prod_{j=1}^N \left(\frac{P\left(k_j, p\right)}{k_j^2 - m^2 + i\varepsilon^+}\right) \times \delta^{(4)}\left(p - \sum_{k=1}^N k_k\right). \tag{5.18}$$

In Eq.(5.18), the following list of parameters and functions, follows directly from an analysis of the generating functional, Eq.(5.10):

- κ is a proportionality factor, depending on the particularities of the theory.
- $g \ll 1$ is a sufficiently small coupling constant, for which perturbation theory, as surmised in Eq.(5.9), may be applied with sufficient confidence[2],.
- $P\left(k_j, p\right)$ is a polynomial, also built upon the details of the Lagrangian, as in the action of Eq.(5.5), being *linear* in the energy–momentum variables k_j, p and, when fermions are present, in the matrices $\gamma \cdot k_j$ and $\gamma \cdot p$.

[2] Mathematically, the meaning of this statement is that, besides failing to be Borel summable, Dyson's series is summable in the sense of Abel [Shawyer(1969)], in such a way that physically meaningful results are still computable in the framework of perturbation theory [Dyson(1958)].

From the general form of Eq.(5.18), no one should be surprised that "loop" integrals diverges.

Indeed, applying a procedure known as Wick's rotation, briefly introduced in App.(A.3), the integrations over the N–dimensional, energy–momentum space, inside Eq.(5.18), and which inherits the Lorentz signature from spacetime representation, are converted into integrations over spaces with Euclidean signature.

Then, assuming the integrand is a function only of the square of the energy–momentum 4–vector, we can separate the integrals into solid angles, as explained in App.(A), and obtain as a proportionality constant to Eq.(5.18) terms of the form[3]:

$$\int_0^\infty dk_E \int_0^\infty d\ell_E \, k_E^3 \ell_E^3 \frac{1}{\left[(k_E+\ell_E)^2+Q_1^2\right]\left(\ell_E^2+Q_2^2\right)}, \tag{5.19}$$

which are clearly logarithmically divergent.

All this being said, the first step which one should assign to oneself to appreciate the renormalization "group" theory, is to learn how to *regularize* integrals such as Eqs.(5.18, 5.19).

Such is the purpose of the present notes.

5.3 Switzerland: *"The Ghosts of Zürich"*

5.3.1 Notation

On what follows, define the following family of functions:

$$Q_E^{(\tau)}\left(m_1,m_2,p_E^2\right) = \tau m_1^2 + (1-\tau)\, m_2^2 + \tau\,(1-\tau)\, p_E^2, \tag{5.20}$$

whose parameters are :

1. The mass of the electron, $m_1 = m$.
2. The ficticious mass of the photon, $m_2 = m_\gamma$, introduced only to regularize the photon–propagator.
3. The Euclideanized momenta squared p_E^2.

Now, noticing that:

$$\int_{\mathbf{E^4}} d^4k_E\, f\left(k_E^2\right) = \left(\int d\Omega^{(3)}\right) \times \int_0^\infty dk_E\, f\left(k_E\right) = 2\pi^2 \int_0^\infty dk_E\, f\left(k_E\right), \tag{5.21}$$

and defining:

$$\bar{Q}_\tau^2\left(m,m_\gamma\right) := Q_\tau^2\left(m,m_\gamma,p_E^2\right), \tag{5.22}$$

Eq.(5.17) gives:

$$i\Sigma^{(2)}(p) = -\frac{ie^2}{8\pi^2}\int_0^1 d\tau \left[2m+(1-\tau)\,\gamma_E\cdot p_E\right]\int_0^\infty dk_E^2 \left\{\frac{1}{\left[k_E^2+\bar{Q}_\tau^2\left(m,m_\gamma\right)\right]^2}-1\right\}. \tag{5.23}$$

[3]The rôle played by the constants Q_1 and Q_2 in Eq.(5.19) will be clear in the development of our next sections.

5.3.2 Description of the Method

Now, let B_μ and χ be, respectively, the *geistfelder* corresponding to Maxwell's and Dirac's fields A_μ and ψ. Moreover, let $G_{\mu\nu} := \partial_{[\mu} B_{\nu]}$ be the pseudo–Faraday tensor.

The regularization scheme known as Pauli–Villars is the following. Defining the pseudo–covariant derivative by $\mathbb{D}_\mu := \partial_\mu - ie\,(A+B)_\mu$, let the Lagrangian:

$$\mathscr{L}_{\text{Zürich}} := \frac{1}{4} G_{\mu\nu} G^{\mu\nu} - \frac{1}{2}\Lambda^2\,(B\cdot B) - \bar{\chi}\,(i\gamma\cdot\mathbb{D} - \Lambda)\,\chi + e\bar{\psi}\,(\gamma\cdot B)\,\psi, \tag{5.24}$$

be added to the generating functional $Z[J^\mu, \sigma, \bar{\sigma}]$.

Clearly, both B_μ and χ violates spin–statistics, and so the reason thereby those are called *geistfelder.*

Nevertheless, the contribution coming from these fields,

$$i\Sigma^{(2)}_{\text{Zürich}}\,(p) = \frac{ie^2}{8\pi^2}\int_0^1 d\tau\,[2m + (1-\tau)\,\gamma_E\cdot p_E]\int_0^\infty dk_E^2 \left\{ \frac{1}{\left[k_E^2 + \bar{Q}_\tau^2\,(m,\Lambda)\right]^2} - 1 \right\}, \tag{5.25}$$

a result obtained by following the very same procedures leading to $i\Sigma^{(2)}\,(p)$, implies a finite contribution to the self–energy of the electron:

$$i\Sigma^{(2)}_{\text{Regulated}}\,(p) = -\frac{ie^2}{8\pi^2}\int_0^1 d\tau\,[2m + (1-\tau)\,\gamma_E\cdot p_E]\log\left[\frac{\bar{Q}_\tau^2\,(m,\Lambda)}{\bar{Q}_\tau^2\,(m,m_\gamma)}\right], \tag{5.26}$$

for any value of $\Lambda \geq m, m_\gamma$.

Finally, returning to Minkowski signature by reverting Wick's rotations, as defined by Eqs.(A.35, A.36), we obtain our final regulated value to electron's self–energy:

$$i\Sigma^{(2)}_{\text{Regulated}}\,(p) = \frac{ie^2}{8\pi^2}\int_0^1 d\tau\,[(1-\tau)\,(\gamma\cdot p) - 2m]\log\left[\frac{Q_\tau^2\,(m,\Lambda)}{Q_\tau^2\,(m,m_\gamma)}\right]. \tag{5.27}$$

5.3.3 Between Vices and Virtues

Technically, the regularization method devised by Pauli–Villars [Pauli and Villars(1949)] and Stückelberg–Rivier [Stueckelberg de Breidenbach and Green(1951), Rivier(1949)] *geistfelder,* suffices to regularize all "fundamental" divergences in QED [Rivier(1949), Jauch and Rohrlich(2012)].

By "fundamental," one should understand those divergences originating from those loop integrals (whose measure is defined upon the energy–momentum space) within the connected proper vertex functions of QED, which are precisely the ones from which all other amplitudes are built upon, by the addition or removal of external lines[4].

For the sake of completeness, those divergences are:

1. The self–energy of the electron, also known as *mass renormalization*, whose order $\mathscr{O}\left(\mathbf{e}^2\right)$ is given by $i\Sigma^{(2)}$, the result of which is displayed at Eq.(5.23).

[4]Or in the terminology of *graph theory*, which one hopes to have learned in meantime, one calls *"unpaired edges."*

2. The energy due to the vacuum polarization, generically denoted by $i\Pi^{(2)}_{\mu\nu}$ and computed, at order $\mathscr{O}\left(e^2\right)$, as in Eq.(5.23):
$$i\Pi^{(2)}_{\mu\nu}(p) := \int d^4x_1 d^4x_2\, e^{-ip\cdot(x_2-x_1)} \left\{\frac{\delta^2\mathscr{W}\left[\mathscr{J}^{\mu}(x),\sigma(x),\bar{\sigma}(x)\right]}{\delta A_{\mu}(x_2)\,\delta A_{\nu}(x_1)}\right\}_{\|\mathscr{J}\|=\|\lambda\|=\|\bar{\lambda}\|=0}$$
$$= \int \frac{d^4k}{(2\pi)^4} \operatorname{tr}\left\{\gamma^{\mu}\times i\left(\gamma\cdot k-m\right)^{-1}\times\gamma_{\nu}\times i\left[\gamma\cdot(k+p)+m\right]^{-1}\right\}. \qquad (5.28)$$
The integral in Eq.(5.28) is, as expected from dimensional analysis, divergent, however is also amenable of being regulated by the Zürich's *geistfelder*.

3. Loop contributions to the vertex–function $\Gamma^{\mu}(p_1,p_2)$, whereby an incoming fermion, say[5] $e^-_{(1)}$, carrying an energy–momentum p^{μ}_1, creates a pair of fermion $e^-_{(2)}$ with energy–momentum $p^{\mu}_1-k^{\mu}$, and a photon $\gamma_{(1)}$ with energy–momentum k^{μ}. Finally, the photon photon γ is absorbed by the fermion $e^-_{(2)}$ and emits a third fermion $e^-_{(3)}$ with energy–momentum
$$\Gamma^{\mu}(p_1,p_2;k) = \int d^4x_1 d^4x_2 d^4y\, e^{-ip\cdot(x_2-x_1)} \left\{\frac{\delta^2\mathscr{W}\left[\mathscr{J}^{\mu}(x),\sigma(x),\bar{\sigma}(x)\right]}{\delta A_{\mu}(x_2)\,\delta A_{\nu}(x_1)}\right\}_{\|\mathscr{J}\|=\|\lambda\|=\|\bar{\lambda}\|=0} \qquad (5.29)$$

Remark 5.3. Physically, the vacuum polarization is an effect due to the self–energy of the photon, the latter of which creates and subsequently annihilates a virtual pair of electron and positrons, respectively. Such a behavior is to be expected from QED's interaction Lagrangian, $\mathscr{L}_{\text{int}} = -i\mathbf{e}\bar{\psi}(x)\left(\gamma\cdot A(x)\right)\psi(x)$.

Nevertheless, the applicability of Pauli–Villars regularization scheme is restricted to gauge theories whose symmetry group is Abelian. Moreover, the introduction of *geistfelder* violates spin–statistics theorem, a fundamental result of relativistic quantum field theories.

Therefore, even as loop integrals, such as Eq.(5.25), arising from the contribution of *geistfelder*, are Lorentz invariant in *appearance*, their very presence in the Lagrangian formulation of the theory is in itself a violation of the principles of special relativity and quantum theory.

Forasmuch as such considerations having being said, one finds necessary to search for a more satisfactory regularization method, which is the subject matter of the section that follows.

5.4 Netherlands: The "Gangsters" of Utrecht

5.4.1 Motivation

To introduce the scheme of regulating our "loop"' integrals known as *dimensional regularization*, consider the following circumstance.

Suppose we are interested, for reasons which may be better left to students of psychoanalysis, in studying, at least perturbatively, a Yang–Mills theory interacting with matter fields, represented by Dirac's 4–spinors, with coupling constant g, similarly to the actions presented in Eqs.(5.3, 5.5).

[5]Where the under-script (n) is employed only for the sake of labelling the particles.

After carefully following the prescriptions delineated in §5.2, from which the generating functional $\mathscr{Z}\left[\mathscr{J}^{\mu}(x),\lambda(x)\right.$ for all Green's functions having been defined, a natural line of inquire is to study the complete fermion–propagator:

$$\mathfrak{G}(p) := \int d^4x_1 d^4x_2\, e^{-ip\cdot(x_2-x_1)} \left\{ \frac{\delta^2 \mathscr{W}\left[\mathscr{J}^{\mu}(x),\lambda(x),\bar{\lambda}(x)\right]}{\delta\sigma(x_2)\,\delta\bar{\sigma}(x_1)} \right\}_{\|\mathscr{J}\|=\|\lambda\|=\|\bar{\lambda}\|=0} \tag{5.30}$$

$$= i\mathscr{S}(p)^{-1} + i\mathscr{S}(p)^{-1} \times \left\{ i\Sigma^{(2)}(p) \right\} \times i\mathscr{S}(p)^{-1} + \mathscr{O}\left(g^4\right), \tag{5.31}$$

where by $i\mathscr{S}(p)$ we denote the "free" fermion–propagator.

As we have learned in our former studies, independently of the particularities of our theory, the amplitude of order $\mathscr{O}\left(g^2\right)$ to $\mathfrak{G}(p)$, denoted in Eq.(5.30) by $i\Sigma^{(2)}(p)$, is of the general form:

$$i\mathscr{M}^{(2)}(\alpha,\mathscr{Q}) := \kappa \times g^2 \lim_{\varepsilon_1,\varepsilon_2\to 0^+} \int_{\mathbb{R}^{(1,3)}} \frac{d^4k}{\left[(p+k)^2 - m^2 + i\varepsilon_1^+\right]\left(k^2 - m^2 + i\varepsilon_2^+\right)}, \tag{5.32}$$

where κ is a proportionality constant, depending upon the details of our theory.

The fundamental observation coming from Eq.(5.32) is that the origin of the logarithmic divergence in $i\mathscr{M}^{(2)}(\alpha,\mathscr{Q})$ relies on the number of spacetime dimensions.

Claim. Analytically continuing the spacetime dimensions to $n = 4-\varepsilon$, where $0<\varepsilon\ll 1$, the divergence of Eq.(5.32) can be:

- Regulated, if one keeps ε finite.
- Or, while one is taking the limit $\varepsilon \longrightarrow 0^+$, the divergence will be "absorbed" into what will be later on called a *counter–term.*
  Therefore, one is left with only physically meaningful contributions.

5.4.2 Mathematical Formalism

The method of dimensional regularization, as far as our present interests in QED are concerned, can be summarized in the analytical continuation of the following integral:

Definition 5.1. Let $\alpha \in \mathbb{N}$, Q^{μ} an energy–momentum 4–vector and $\mathscr{F} := (0,4) \in \mathbb{R}$ a family of indices. Define the family of integrals $\{\mathscr{I}_{\varepsilon}(\alpha,Q^{\mu})\}_{\varepsilon\in\mathfrak{F}}$ defined upon an "analytically continued" measure (a precise meaning of which will be given in the following theorem)[6]:

$$\mathscr{I}_{(n,\varepsilon)}\left(\alpha,Q^2\right) := \lim_{\varepsilon_0\to 0^+,} \int_{\mathbb{R}^{(1,n-1)}} \frac{d^{(n-\varepsilon)}k}{(2\pi)^{(n-\varepsilon)}} \frac{1}{\left(k^2 - Q^2 + i\varepsilon_0\right)^{\alpha}}. \tag{5.33}$$

[6] In Eq.(5.33), as one might have expected, $\varepsilon_0 > 0$ is the infinitesimal following Feynman's prescription, to appropriately choose the contour of integration in order to obtain the correct propagator.
For the sake of brevity, from now on, Feynman's $i\varepsilon_0$–regularization scheme is tacitly understood.

Theorem 5.1. *Letting α a positive integer and Q^μ an energy–momentum 4–vector, for any index $\varepsilon \in \mathfrak{F}$,* Eq.(5.33) *yields:*

$$\mathscr{I}_{(n,\varepsilon)}\left(\alpha, Q^2\right) = (-1)^\alpha \frac{i\pi^{n/2}}{(2\pi)^n} \frac{1}{\left(Q^2\right)^{\alpha - n/2}} \frac{\Gamma(\alpha - n/2)}{\Gamma(\alpha)}. \tag{5.34}$$

Indeed, recalling the unit solid angle of Eq.(A.4) from App.(A), and then Eq.(A.35, A.36) from App.(A.3), we arrive at the following list of computations:

$$\begin{aligned}
\mathscr{I}_{(n,\varepsilon)}\left(\alpha, Q^2\right) &= \int_{\mathbf{E}^n} \frac{(-id^n k_E)}{(2\pi)^n} \frac{1}{\left(-k_E^2 - Q^2\right)^\alpha} && (5.35)\\
&= \frac{(-1)^{\alpha+1} i}{(2\pi)^n} \int_{\mathbf{E}^n} d^n k_E \, \frac{k_E^{n-1}}{\left(k_E^2 + Q^2\right)^\alpha} && (5.36)\\
&= \frac{(-1)^{\alpha+1} i}{(2\pi)^n} \times \underbrace{\left(\int d\Omega^{(n)}\right)}_{\text{Area of n-sphere}} \times \int_0^\infty dk_E \, \frac{1}{\left(k_E^2 + Q^2\right)^\alpha} && (5.37)\\
&= \frac{(-1)^{\alpha} i}{(2\pi)^n} \times \frac{2\pi^{n/2}}{\Gamma(n/2)} \times \int_0^\infty d\left(\frac{k_E}{Q}\right) \times \left(\frac{k_E}{Q}\right)^{n-1} \frac{Q^n}{Q^{2\alpha}} \frac{1}{\left(1 + k_E^2/Q^2\right)^\alpha} && (5.38)\\
&= \frac{(-1)^{\alpha} i}{(2\pi)^n} \times \frac{2\pi^{n/2}}{\Gamma(n/2)} \times \frac{2}{\left(Q^2\right)^{\alpha - n/2}} \int_0^\infty d\tau \, \tau^{n-1} \left(1 + \tau^2\right)^{-\alpha}. && (5.39)
\end{aligned}$$

Finally, use Eq.(A.12, A.14) from App.(A.1) to obtain the desired result.

An exceptionally comprehensive table of integrals generalizing Eq.(5.33) for many arguments and functions in the numerator, useful into the application of dimensional regularization (either in relativistic quantum field theories as in critical phenomena of condensed–matter [Zinn-Justin(2007)]) is available at [Kleinert(2001)].

5.4.3 Application: The Self–energy of the Electron

Definition 5.2. Let [**X**] be any physical quantity, such as a classical field or a quantum distribution–valued operator, even physical constants. We will denote by [**X**] the unit of mass of that quantity.

Claim. From the action functional of QED given by Eqs.(5.3, 5.5), it follows by simple dimensional analysis that:

$$\left[A_\mu\right] = \frac{n-2}{2} = 1 - \frac{\varepsilon}{2}, \tag{5.40}$$

$$[\psi] = [\bar{\psi}] = \frac{n-1}{2} = \frac{3}{2} - \frac{\varepsilon}{2}, \tag{5.41}$$

$$[\mathbf{e}] = \frac{\varepsilon}{2}. \tag{5.42}$$

It is convinient to keep the coupling constant dimensionless.

This being the case, let μ be a "dummy" parameter[7] with scale of mass and apply the following replacements:

$$\mathbf{e} \mapsto \mathbf{e}\mu^{\varepsilon/2}, \tag{5.43}$$

[7] By "dummy" we mean, without physical significance. As a matter of fact, such a parameter will disappear at the final result.

$$D_\mu \psi(x) \mapsto D_\mu \psi(x) = \left(\partial_\mu + i\mathbf{e}\mu^{\varepsilon/2} A_\mu\right) \psi(x) . \tag{5.44}$$

It can be shown that Clifford algebras, for which the Dirac γ's matrices are a representation, can be generalized to n–dimensional spaces [Lounesto et al.(2001)Lounesto, Cassels, Society, and Hitchin]. Thence,

$$\gamma^\mu \gamma_\mu = n,\ \gamma^\mu (\gamma \cdot k) \gamma_\mu = (2-n)(\gamma \cdot k) . \tag{5.45}$$

Now, using Eq.(A.24) from App.(A.2),

$$i\Sigma^{(2)}(p) = -\mu^\varepsilon \mathbf{e}^2 \times \int \frac{d^n k}{(2\pi)^n} \frac{(2-n)\gamma\cdot(k+p) + nm}{\left[(k+p)^2 - m^2\right]\left(k^2 - m_\gamma^2\right)} \tag{5.46}$$

$$= -\mu^\varepsilon \mathbf{e}^2 \int_0^\infty d\tau \int \frac{d^n k}{(2\pi)^n} \frac{(2-n)\gamma\cdot(k+p) + nm}{\left[\left((k+p)^2 - m^2\right)\tau + \left(k^2 - m_\gamma^2\right)(1-\tau)\right]^2} . \tag{5.47}$$

Defining, as usual,

$$Q^2 := \tau m^2 - \tau(1-\tau) p^2,$$

and performing the transformation $k^\mu \mapsto k^\mu + p^\mu$ in the integration variable,

$$i\Sigma^{(2)}(p) = -\mu^\varepsilon \mathbf{e}^2 \int_0^\infty d\tau \left[(2-n)\gamma\cdot(k+p) + nm\right] \times \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 - Q^2)^\alpha} \tag{5.48}$$

$$= -\mu^\varepsilon \mathbf{e}^2 \int_0^\infty d\tau \left[(2-n)\gamma\cdot(k+p) + nm\right] \mathscr{I}_{(n,\varepsilon)}\left(\alpha, Q^2\right) \tag{5.49}$$

$$= -\mu^\varepsilon \mathbf{e}^2 \int_0^\infty d\tau \left[(2-n)\gamma\cdot(k+p) + nm\right] \frac{(-1)^\alpha i\pi^{n/2}\Gamma(\alpha - n/2)}{(2\pi)^n \Gamma(\alpha)(Q^2)^{\alpha - n/2}}, \tag{5.50}$$

Lastly[8], applying the approximation provided by Eq.(A.22), which we obtained in App.(A.1), we have:

$$i\Sigma^{(2)}(p) \approx \frac{ie^2}{8\pi^2} \int_0^\infty d\tau \left[(1-\tau)(\gamma\cdot p) - 2m\right] \left(1 - \frac{\varepsilon}{2}\right) \underbrace{\left(1 + \frac{\varepsilon}{2}\log(4\pi)\right)\left(\frac{2}{\varepsilon} - \gamma_{\text{Euler}}\right)} \left(1 - \frac{\varepsilon}{2}\log\frac{\mathscr{Q}^2}{\mu^2}\right) \tag{5.51}$$

$$\approx \frac{ie^2}{8\pi^2} \int_0^\infty d\tau \left[(1-\tau)(\gamma\cdot p) - 2m\right] \left(1 - \frac{\varepsilon}{2}\right) \left(\frac{2}{\varepsilon} - \gamma_{\text{Euler}} + \log(4\pi)\right) \left(1 - \frac{\varepsilon}{2}\log\frac{\mathscr{Q}^2}{\mu^2}\right) \tag{5.52}$$

$$= \frac{ie^2}{8\pi^2} \int_0^\infty d\tau \left[(1-\tau)(\gamma\cdot p) - 2m\right] \left\{\frac{2}{\varepsilon} - \log\left[\frac{\tau m^2 - \tau(1-\tau)p^2}{4\pi\mu^2}\right] - (\gamma_{\text{Euler}} + 1)\right\} . \tag{5.53}$$

Therefore, Eq.(5.52) provides a finite contribution to the self–energy of the electron for all *finite* and non–negative real $\varepsilon > 0$.

Whence, $i\Sigma^{(2)}(p)$ diverges only in the limit whereby $\varepsilon \longrightarrow 0$.

Upon such considerations, we may finally conclude that:

[8]We also note that:
$$\frac{1}{2^n \pi^{n/2}} = \frac{1}{(4\pi)^{2-\varepsilon/2}} \approx \frac{1}{16\pi^2}\left[1 + \frac{\varepsilon}{2}\log(4\pi)\right].$$

1. The physically meaningful contributions in Eq.(5.52) to the total amplitude $\mathbf{S}(p)$, due to the self–energy of the electron, at order $\sim \mathscr{O}\left(\mathbf{e}^2\right)$,

2. From the “spurious,” that’s to say, unphysical divergences, which are destined to be absorbed into a *counter-term*, a terminology soon to be clarified as we introduce the theory of renormalization.

Chapter 6

A Matter "Self-Regularization"

Figure 6.1: Our modern Hercules could well be our the ambitious students of high–energy physics. As soon as the character master the art infinities under control to keep those now seemingly meaningless (the ones arising from non–ordered canonical operators, or a functional "integral" kept unnormalized), is given as soon as possible his second labour: – Killing under the help of one's modern saber, the counter–terms, Hydra's infinities: "loop" divergences in perturbative quantum field theories.

6.1 Introduction: *Anamnesis*

In this section, our aim is to review our studies with particular attention to quantum electrodynamics (QED), preparing ourselves to renormalization theory.

Let A_μ and ψ the Maxwell and Dirac fields, respectively.

Adopting minimal coupling's prescription, according to which interaction's gauge invariance is preserved simply upon a replacement:

$$\partial_\mu \mapsto D_\mu := \partial_\mu - ieA_\mu. \tag{6.1}$$

and denoting Faraday's field–strength tensor by $F_{\mu\nu} := \partial_{[\mu}A_{\nu]}$, QED's Lagrangian reads:

$$\mathscr{L}_{\text{QED}} := \mathscr{L}_{\text{QED}}\left[A_\mu(x), F_{\mu\nu}(x); \psi(x)\right] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma\cdot D - m)\psi. \tag{6.2}$$

$$\mathscr{Z}\left[\mathscr{J}^\mu(x), \sigma(x), \bar{\sigma}(x)\right] = \int \left[dA_\mu\right][d\psi][\bar{\psi}]\, e^{i\int d^4x \left\{\mathscr{L}_{\text{QED}}\left[A_\mu(x),\, \psi(x),\, \bar{\psi}(x) F_{\mu\nu}(x)\right] - \frac{1}{2\xi}(\partial\cdot A)^2\right\}} \tag{6.3}$$

$$\times \exp\left[i\int d^4y\, \mathscr{J}^\mu(y) A_\mu(y) + \bar{\sigma}(y)\psi(y) + \bar{\psi}(y)\sigma(y)\right] \tag{6.4}$$

$$\mathscr{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - \mathscr{L}_{\text{int}},\ \mathscr{L}_{\text{int}} := -ie\bar{\psi}(\gamma\cdot A)\psi \tag{6.5}$$

$$\mathscr{Z}[\mathscr{S}] = \exp\left\{i\int d^4y\, \mathscr{L}_{\text{int}}\left[\frac{1}{i}\frac{\delta}{\delta\mathscr{S}(y)}\right]\right\} \underbrace{\int [d\Psi] \exp\left[i\int d^4x\, \mathscr{L}_0(\Psi, \partial\Psi) + \langle\mathscr{S}(x), \Psi(x)\rangle\right]}_{\mathscr{Z}_0[\mathscr{S}]} \tag{6.6}$$

$$\mathscr{Z}\left[\mathscr{J}^\mu(x), \sigma(x), \bar{\sigma}(x)\right] = \exp\left\{i\int d^4y\, \mathscr{L}_{\text{int}}\left[-ie\gamma^\mu \frac{\delta^3}{\delta\mathscr{J}_\mu(y)\,\delta\bar{\sigma}(y)\,\delta\sigma(y)}\right]\right\} \mathscr{Z}_0\left[\mathscr{J}^\mu(x), \sigma(x), \bar{\sigma}(x)\right]$$

$$\Phi := \{\mathbf{A}, \psi, \bar{\psi}, \mathbf{F}\},\ \mathscr{S} := \{J^\mu(x), \sigma(x), \bar{\sigma}(x)\} \tag{6.7}$$

$$\log\{W\left[\mathscr{J}^\mu(x), \sigma(x), \bar{\sigma}(x)\right]\} := i\mathscr{Z}[\Phi; \mathscr{S}] \tag{6.8}$$

$$\Gamma[\Phi, \mathscr{S}] := W[\Phi; \mathscr{S}] - \int d^4x\, \left(\mathscr{J}^\mu(x) A_\mu(x) + \bar{\sigma}(x)\psi(x) + \bar{\psi}(x)\sigma(x)\right) \tag{6.9}$$

6.2 Electron (Dirac–field) Self–Energy

$$(2\pi)^4\delta^{(4)}(p_2 - p_1) \times iS_F^{(0)}(p_2 - p_1) := \int d^4x_1 d^4x_2\, e^{i(p_2-p_1)\cdot(x_2-x_1)} \lim_{\|\mathscr{J}^\mu\|\to 0}\ \lim_{\|\sigma\|\to 0}\ \lim_{\|\bar{\sigma}\|\to 0} \left\{\frac{\delta^2\mathscr{Z}_0}{\delta\bar{\sigma}(x_2)\,\delta\bar{\sigma}(x_1)}\right\} \tag{6.10}$$

$$(2\pi)^4\,\delta^{(4)}\left(p_2-p_1\right)\times\underbrace{iS_F^{(0)}\left(p_2\right)}_{\text{incoming fermion-propagator}}\times\underbrace{\Sigma^{(1)}\left(p_2-p_1\right)}_{\text{1-loop self-energy correction}}\times\underbrace{i\left(\gamma\cdot p_1-m+i\delta_1^+\right)^{-1}}_{\text{outgoing fermion-propagator}} \tag{6.11}$$

$$=\int d^4x_1\,e^{\sum_{n=1}^4 i(p_n-p_{n-1})\cdot(x_n-x_{n-1})}\lim_{\|\mathscr{J}^\mu\|\to0}\lim_{\|\sigma\|\to0}\lim_{\|\bar{\sigma}\|\to0}\left\{\frac{\delta^6W\left[\mathscr{J}^\mu(x),\sigma(x),\bar{\sigma}(x)\right]}{\delta\bar{\sigma}(x_6)\,\delta\bar{\sigma}(x_5)\,\delta A_\nu(x_4)\,\delta A^\nu(x_3)\,\delta\bar{\sigma}(x_2)\,\delta\bar{\sigma}(x_1)}\right\} \tag{6.12}$$

$$=\int d^4x_1d^4x_2\,e^{i(p_2-p_1)\cdot(x_2-x_1)}\lim_{\|\mathscr{J}^\mu\|\to0}\lim_{\|\sigma\|\to0}\lim_{\|\bar{\sigma}\|\to0}\left\{\frac{\delta^2W}{\delta\bar{\sigma}(x_2)\,\delta\sigma(x_1)}\right\} \tag{6.13}$$

$$iS_{\mathrm{F}}^{(1)}(p)=\int_{\mathbb{R}^{(1,3)}}\frac{d^4k}{(2\pi)^4}\underbrace{\left(-ie\gamma^\mu\right)}_{1^{\text{st}}\text{ int. vertex}}\times\underbrace{i\left(\gamma\cdot(p+k)-m+i\varepsilon_1^+\right)^{-1}}_{\text{fermion-propagator's one-loop}}\times\underbrace{\left(-ie\gamma^\nu\right)}_{2^{\text{nd}}\text{ int. vertex}}$$
$$\times\underbrace{\frac{i}{k^2+i\varepsilon_2^+}\left(\eta_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}\right)}_{\text{photon-propagator's within fermionic's loop}}$$

$$\xi=1,\ \frac{i}{k^2+i\varepsilon_2^+}\left(\eta_{\mu\nu}-(1-\xi)\frac{k_\mu k_\mu}{k^2}\right)\mapsto\lim_{m_\gamma\to0}\frac{i}{k^2-m_\gamma^2}$$

$$iS_{\mathrm{F}}^{(1)}(p)=-e^2\lim_{m_\gamma\downarrow0}\int_{\mathbb{R}^{(1,3)}}\frac{d^4k}{(2\pi)^4}\frac{(p+q)_\alpha\left(\gamma^\mu\gamma^\alpha\gamma_\mu\right)+m\left(\gamma^\mu\gamma_\mu\right)}{\left[(p+k)^2-m^2+i\varepsilon^+\right]\left(k^2-m_\gamma^2\right)}$$

At this point, there are two reasons to be careful about spatial dimensions and metric's signature. Nonetheless, first some words about notation.

In this notes, $\mathbb{R}^{(1,3)}$ meant either Minkowski spacetime or energy–momentum space. Ambiguity's abscense is due to the fact that which space is at play is clear from context. Moreover, those are related by a fourier transform.

Remark. Nonetheless, at this point, where one is wise in performing anaylitic continuation in k^0, known as Wick's rotation,

$$k^0\mapsto\tau=k_E^0:=-ik^0,\ \Longrightarrow\ dk^0=idk_E^0,\ d^nk:=dk^0d^{(n-1)}\mathbf{k}=id^nk_E$$

$$\left(\eta_{\mu\nu}\gamma^\mu k^\nu-m\right)\psi(k)=0$$
$$\left(-\gamma^0k^0+\delta_{ij}\gamma^ik^j-m\right)\psi(k)$$

aiming at analytic continuation to imaginary time, one should consider metric's signature from Clifford's algebra $\mathscr{C}\ell\left(\mathbb{R}^{(1,3)}\right)$, with usual generators γ^μ's, to $\mathscr{C}\ell\left(\mathbf{E}^4\right)$ with generators γ^m ()defining relations,

$$\gamma^\mu\gamma_\mu=n$$

$$\gamma^{\mu}\gamma^{\alpha}\gamma_{\mu} = (2-n)\,\gamma^{\alpha}$$
$$n=4 \implies \gamma^{\mu}\gamma_{\mu} = 4,\ \gamma^{\mu}\gamma^{\alpha}\gamma_{\mu} = -2\gamma^{\alpha}$$

$$\Sigma(p) \coloneqq e^2 \times \lim_{m_\gamma \downarrow 0} \int_{\mathbb{R}^{(1,3)}} d^4k \, \frac{\gamma\cdot(p+q) - 2m}{\left[(p+k)^2 - m^2 + i\varepsilon^+\right]\left(k^2 - m_\gamma^2\right)} \tag{6.14}$$

$$\frac{1}{AB} = \int_0^1 d\tau \, \frac{1}{(A\tau + B(1-\tau))^2}$$

$$iS_{\mathrm{F}}^{(1)}(p) = \frac{e^2}{8\pi^4} \lim_{m_\gamma\downarrow 0} \int_{\mathbb{R}^{(1,3)}} d^4k$$

$$\begin{aligned}\Sigma(p) &= e^2 \int_{\mathbb{R}^{(1,3)}} \frac{d^4k}{(2\pi)^4} \left(\gamma\cdot(p+q) - 2m\right) \int_0^1 d\tau \, \frac{1}{\left\{\left[(p+k)^2 - m^2 + i\varepsilon^+\right]\tau + \left(k^2 - m_\gamma^2\right)(1-\tau)\right\}^2} \\ &= e^2 \int_{\mathbb{R}^{(1,3)}} \frac{d^4k}{(2\pi)^4} \left(\gamma\cdot(p+q) - 2m\right) \int_0^1 d\tau \, \frac{1}{\left\{\left[(p+k)^2 - m^2 + i\varepsilon^+\right]\tau + \left(k^2 - m_\gamma^2\right)(1-\tau)\right\}^2}\end{aligned}$$

6.3 "A Tale of Two Cities"

On the course of our studies, many divergences appeared. So far, to a all of them, the student could, without difficult, assign either an interpretation, or if one is allowed small moment of honest[1], an "excuse." Such as, infinities arising from canonical quantization in

6.3.1 Switzerland: Zürich's Ghosts

Let B_μ and χ be, respectively, the *geistfelder* corresponding to Maxwell's A_μ and Dirac's ψ fields, and $G_{\mu\nu} \coloneqq \partial_{[\mu}B_{\nu]}$ our pseudo-Faraday tensor.

Following an *analogy* to minimal coupling's prescription, as above, will be proven to be a useful choice.

One one hand, the original gauge invariant interaction between photon and electron's fields A_μ and ψ is preserved. On the other, both the coupling between *geistfelder* B_μ and χ among themselves and their interactions with observable fields A_μ and ψ, are realized simultaneously.

Thence, our replacement to our newly defined derivative operator:

$$\partial_\mu \mapsto \mathbb{D}_\mu \coloneqq \partial_\mu - ie\,(A+B)_\mu\,. \tag{6.15}$$

$$\Delta\mathscr{L}_{\text{Zürich}} \coloneqq \frac{1}{4}G_{\mu\nu}G^{\mu\nu} - \frac{1}{2}\Lambda^2\,(B\cdot B) - \bar{\chi}\,(i\gamma\cdot\mathbb{D} - \Lambda)\,\chi + e\bar{\psi}\,(\gamma\cdot B)\,\psi. \tag{6.16}$$

$$\mathscr{L}_{\text{Zürich}} = \mathscr{L}_{\text{QED}} + \Delta\mathscr{L}_{\text{Zürich}} \tag{6.17}$$

[1] Which, by Heisenberg's uncertainty principle, means that a small moment of honest implies a high precision of the position being taken upon.

$$\mathscr{L}_{\text{geist}} \coloneqq -\bar{\chi}\left(i\gamma^{\mu}\partial_{\mu}-\Lambda\right)\chi \tag{6.18}$$

$$\left(i\gamma^{\mu}\partial_{\mu}-\Lambda\right)\chi = -\frac{\delta\mathscr{L}_{\text{geist}}}{\delta\bar{\chi}(x)} = -\frac{\partial}{\partial x^{\mu}}\left(\frac{\delta\mathscr{L}_{\text{geist}}}{\delta\partial_{\mu}\bar{\chi}(x)}\right) = 0 \tag{6.19}$$

$$\left(i\gamma^{\mu}\partial_{\mu}-\Lambda\right)\chi(x) = -i\delta^{(4)}(x) = -i\int\frac{d^4k}{(2\pi)^4}e^{ik\cdot x}$$

$$\int d^4x\, e^{ik\cdot x}\left(i\gamma^{\mu}\partial_{\mu}+\Lambda\right)\chi(x) = \int d^4x\, e^{ik\cdot x}\left(i\gamma^{\mu}\partial_{\mu}-\Lambda\right)\chi(x)$$

6.4 Netherlands: Utrecht's Gamble

$$\mathscr{I}^{(n)}(a,Q) \coloneqq \int\frac{d^nk}{(2\pi)^n}\frac{1}{\left(k^2-Q+i\varepsilon^{+}\right)^{\alpha}} \tag{6.20}$$

$$\mathscr{I}^{(n)}(a,Q) \coloneqq i\int\frac{d^nk_E}{(2\pi)^n}\frac{1}{\left(k^2-Q+i\varepsilon^{+}\right)^{\alpha}} \tag{6.21}$$

Chapter 7

Classification of Divergences: *"Planck's Principle for the Progress of Science."*

"People have developed a technique for handling the infinities in certain theories. For these theories the infinities can all be collected into certain parameters representing physical constants, which are then renormalized to their experimental values, and so the infinities get discarded. The resulting equations are well-defined and can be used to calculate results that can be compared with experiment. The agreement is often very good, and many physicists are well-satisfied with this situation."
– P. A. M. Dirac (1981), in "Does renormalization make sense?" [Dirac(1981a)]

"The shell game that we play [...] *is called 'renormalization.' But no matter how clever the word, it is what I would call a dippy process! Having to resort to such hocus–pocus has prevented us from proving that the theory of quantum electrodynamics is mathematically self–consistent. It's surprising that the theory still hasn't been proved self-consistent one way or the other by now; I suspect that renormalization is not mathematically legitimate."*
– R. P. Feynman (1985), in "QED: The Strange Theory of Light and Matter." [Feynman(1985)]

"A new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die and a new generation grows up that is familiar with it. "
– M. Planck (1950), in Scientific autobiography. [Planck(1950)]

7.1 Planck's Units

Definition 7.1. Define a system of units, which will be referred to on what follows as the *natural or Planck units*, to be such that: (1) The universal speed of propagation, either of massless particles or the wave–front of fields obeying massless equations of motions[1], to be $c \equiv 1$. (2) Planck's constant, historically[2] called *the quantum of action*, is defined so that $\hbar \equiv 1$.

[1]Recall that in the Hamilton–Jacobi theory, curves everywhere perpendicular to surfaces of constant phases are *analogous* to the worldline followed by physical particles in classical limit [Landau(2013)]. That theory was developed upon considerations on the derivation of geometric optics from the undulatory theory, passing through the foundations of classical dynamical systems, and closing with it's complexification, known as the Schrödinger equation in non–relativistic quantum mechanics [Hamilton(1833), Hamilton and Beaufort(1834)].

[2]In retrospect, after Feynman's insight [Feynman and Brown(2005)] of reformulating quantum theory as a sum over "paths" (field configurations, to be precise), Planck's original terminology seems quite adequate.

Remark 7.1. Denoting by T and ℓ, respectively, the units of time and length in Planck's system, it follows from $c \equiv 1$ that $T = \ell$. Moreover, letting M be the unit of Planck's mass (and therefore by $E = Mc^2$ also of energy), after setting the quantum of action to be such that $\hbar \equiv 1$, one should have $\ell^{-1} = M$. Consequently, in these notes, we express all the physical quantities in dimensions of Planck's mass M.

Definition 7.2. Let $\mathbf{X}$ be any physical quantity, e.g., a classical field on a spacetime manifold, a distribution–valued operator acting on a quantum Hilbert state–space, or a cross–section $\mathbf{X} = (d\sigma/d\Omega)$ from a scattering experiment. We denote by $[\![\mathbf{X}]\!]$ that physical quantity when expressed in terms of the *units* of the scale of Planck's mass M, while:

$$\dim(\mathbf{X}) := \log_M [\![\mathbf{X}]\!], \tag{7.1}$$

will be referred to as the *dimensionality* of $\mathbf{X}$ in Planck's system.

Let $\left\{\phi^{(\alpha)}_{\mu_\alpha} \,|\, \alpha \in \mathscr{A}\right\}$ be a collection of fields satisfying Bose–Einstein statistics, where $\mathscr{A}$ is the family of indices classifying our fields. Here, μ_α labels the whole set of indices which each one of the fields $\phi^{(\alpha)}_{\mu_\alpha}(x)$ may be attributed to according to it's mathematical nature.

Now, let $\left\{\psi^{(\beta)} \,|\, \beta \in \mathscr{B}\right\}$ denote the collection of (matter) fields satisfying Fermi–Dirac statistics, where $\mathscr{B}$ is the family of indices organizing these fields.

A prototypical Lagrangian density for a theory containing the collection of fields $\left\{\phi^{(\alpha)}_{\mu_\alpha}\right\}$ and $\left\{\psi^{(\beta)}\right\}$ interacting with one another is given by:

$$\mathscr{L}\left(\phi^{(\alpha)}_{\mu_\alpha}, \psi^{(\beta)}\right) \equiv \mathscr{L}_0 - \sum_{i=1}^{n} g_i V_i\left(\phi^{(\alpha)}_{\mu_\alpha}, \psi^{(\beta)}\right), \tag{7.2}$$

where g_i denotes the coupling constant of the potential $V_i\left(\phi^{(\alpha)}_{\mu_\alpha}, \psi^{(\beta)}\right)$, responsible for the interactions between the fields $\phi^{(\alpha)}_{\mu_\alpha}$ and $\psi^{(\beta)}$.

7.1.1 Example: Self–Interacting Real Scalar Field [Zee(2010), Ex. III.2.1]

Let $\phi(x)$ be a massive scalar field, living in a d–dimensional spacetime, and $\{\lambda_i\}_{i\geq 4}$ a family of coupling constants, governed by the action:

$$S = \int d^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2(x) + \sum_{\ell\geq 4}\lambda_\ell \phi^\ell(x)\right]. \tag{7.3}$$

As the action should have the same units as $\hbar$, then:

$$\left[\!\left[m^2\phi^2\right]\!\right] = M^2 \times [\![\phi]\!]^2 = M^d, \tag{7.4}$$

which implies that:

$$[\![\phi]\!] = M^{\frac{d-2}{2}}, \text{ or: } \dim(\phi) = \frac{d-2}{2}. \tag{7.5}$$

As to the dimensionality of coupling constants,

$$\left[\!\left[\lambda_\ell \phi^\ell\right]\!\right] = [\![\lambda_\ell]\!] \times [\![\phi]\!]^\ell = [\![\lambda_\ell]\!] \times M^{\frac{n(d-2)}{2}} = M^d, \tag{7.6}$$

and therefore,

$$\dim(\lambda_\ell) = d + \frac{n(2-d)}{2}. \tag{7.7}$$

7.1.2 Example: "Fermi" Theory [Zee(2010), Ex. III.3.1]

Consider a simplified model of Fermi theory for the weak interactions in d–dimensional Minkowski spacetime, whose action functional is given by:

$$S_{\mathrm{F}} = \int d^d x \mathscr{L}_{\mathrm{F}},\ \mathscr{L}_{\mathrm{F}} = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi + G_{\mathrm{F}}^{(d)}\left(\bar{\psi}\psi\right)^2. \tag{7.8}$$

Let ν be the positive integer such that either $d = 2\nu$ or else $d = 2\nu + 1$.
Assume $\psi(x)$ is a 2^{ν}–component spinorfield on $\mathbb{R}^{(1,d-1)}$.
The mass term in Eq.(7.8) requires that:

$$[\![m\bar{\psi}\psi]\!] = M \times [\![\psi]\!]^2 = M^d, \tag{7.9}$$

thence:

$$\dim(\psi) = \frac{d-1}{2}. \tag{7.10}$$

Similarly, the units of the coupling constant $G_F^{(d)}$ can be so determined such that:

$$\left[\!\!\left[G_{\mathrm{F}}^{(d)}\left(\bar{\psi}\psi\right)^2\right]\!\!\right] = \left[\!\!\left[G_{\mathrm{F}}^{(d)}\right]\!\!\right] \times \left(M^{\frac{(d-1)}{2}}\right)^4 = \left[\!\!\left[G_{\mathrm{F}}^{(d)}\right]\!\!\right] \times M^{2(d-1)} = M^d, \tag{7.11}$$

and therefore:

$$\dim\left[G_{\mathrm{F}}^{(d)}\right] = 2 - d. \tag{7.12}$$

Remark 7.2. For the erudite reader, Eq.(7.12) means that, for a $d = 4$ dimensional spacetime, Fermi theory is "non–renormalizable," which historically was one of the reasons whereby the electroweak theory was discovered [Pais(1988)]. Interestingly, as a "toy" model, G_F is dimensionless when $d = 2$.

7.1.3 Example: Classical Electrodynamics

The Maxwell's potential $A_{\mu}(x)$, a four–dimensional spacetime vectorfield whose "covariant curl," known as the Faraday tensor,

$$F_{\mu\nu} := 2\partial_{[\mu}A_{\nu]} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}, \tag{7.13}$$

defines the electromagnetic field–strength. The action for electrodynamics coupled to an external electric current $\mathscr{J}^{\mu}(x)$ is give by:

$$S_{\mathrm{EM}} = \int d^4 x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathscr{J}^{\mu}(x)A_{\mu}(x)\right], \tag{7.14}$$

where $\mathscr{J}^{\mu}(x)$ is an external electric–current. Here, one denotes by **e** the elementary unity of charge, which we take as being dimensionless.

Applying Lord Rayleigh's dimensional analysis [Bridgman(1978)] to Eq.(7.14), one finds that:

$$\dim\left(F_{\mu\nu}\right) = 2,\ \dim\left(A_{\mu}\right) = 1,\ \dim\left(\mathscr{J}^{\mu}\right) = 3. \tag{7.15}$$

7.1.4 Example: Spinor Electrodynamics in $\mathbb{R}^{(1,d-1)}$ spacetime

Let the Maxwell's potential be defined, up to a gauge transformation, by a vectorfield $A_\mu(x)$, while Dirac's matter field be represented by a 2^ν–component spinorfield $\psi(x)$, where either $d = 2\nu$ or $d = 2\nu + 1$.

The Lagrangian density for "spinor electrodynamics," is:

$$\mathscr{L}^{(d)}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left(i\nabla^{\$} - m \right) \psi, \tag{7.16}$$

where $D_\mu := \partial_\mu - i\mathbf{e}A_\mu$ is the covariant derivative and $\nabla^{\$} := \gamma^\mu D_\mu$ inherited by the associated spin–bundle upon which $\psi(x)$ lives as a section.

7.2 Perturbation Theory Revisited

Recall from our previous problem sheets (or as explained at length in [Faddeev(2018), Berazin(2012), Ramond(1997), Abers and Lee(1973)]) that a theory whose Lagrangian density is given by the prototype of Eq.(7.2), the generating functional for the Green's functions are:

$$\mathscr{Z}\left[J^{(\alpha)}_{\mu_\alpha}(x), \sigma^{(\beta)}(x), \bar{\sigma}^{(\beta)}(x)\right] = \int \mathscr{D}\left[\phi^{(\alpha)}_{\mu_\alpha}(x), \psi^{(\beta)}(x)\right] e^{i\int d^4x \mathscr{L}\left(\phi^{(\alpha)}_{\mu_\alpha}, \psi^{(\beta)}\right) + S_{\text{sources}}}, \tag{7.17}$$

where the sources are integrated after being coupled to their respective fields:

$$S_{\text{sources}} = \sum_{\alpha,\beta,\gamma} i \int d^4x \left[J^{(\alpha)\mu_\alpha}(x)\, \phi^{(\alpha)}_{\mu_\alpha}(x) + \bar{\sigma}^{(\beta)}(x)\, \psi^{(\beta)}(x) + \bar{\psi}^{(\gamma)}(x)\, \sigma^{(\gamma)}(x) \right]. \tag{7.18}$$

Moreover, remember that the Green's functions for the generating functional in the "free" theory, namely, the one obtained by letting $g_i = 0$ for all $i = 1, ..., j = N$, is:

$$\mathscr{Z}_0\left[J^{(\alpha)}_{\mu_\alpha}(x), \sigma^{(\beta)}(x), \bar{\sigma}^{(\beta)}(x)\right] = \int \mathscr{D}\left[\phi^{(\alpha)}_{\mu_\alpha}(x), \psi^{(\beta)}(x)\right] e^{i\int d^4x \mathscr{L}_0\left(\phi^{(\alpha)}_{\mu_\alpha}, \psi^{(\beta)}\right) + S_{\text{sources}}}. \tag{7.19}$$

In the canonical formalism, following [Weinberg(1995), Ch.2-5], Ω_0 denotes the vacuum state of the "free" theory, while $\mathscr{T}\{\star\star\star\}$ is the time–ordering operator. The Feynman propagators for ϕ–boson and ψ–fermion are defined, respectively, using the framework of the canonical formalism, by:

$$iD^{(\alpha)}_{\mu_{\alpha 1}\mu_{a_2}}(x_2 - x_1) := \left(\Omega_0,\ \mathscr{T}\{\phi^{(\alpha)}_{\mu_{\alpha_2}}(x_2)\, \phi^{(\alpha)\dagger}_{\mu_{\alpha_1}}(x_1)\}\Omega_0 \right), \tag{7.20}$$

and:

$$iS^{(\beta)}(x_2 - x_1) := \left(\Omega_0,\ \mathscr{T}\{\psi^{(\beta_2)}(x_2) \left(\psi^{\beta_1}\right)^{\dagger}(x_1)\}\Omega_0 \right). \tag{7.21}$$

Equivalently, since Eqs.(7.20, 7.21), being evaluated as vacuum expectation values of the "free" vacuum Ω_0, are equivalent to the fundamental solutions:

$$\frac{\delta \mathscr{L}_0}{\delta \phi^{(\alpha)}_{\mu_\alpha}(x)} - \partial_\mu \left[\frac{\delta \mathscr{L}_0}{\delta \left(\partial_\mu \phi^{(\alpha)}_{\mu_\alpha} \right)(x)} \right] = -i\delta_{\mu_\alpha} \delta^{(4)}(x), \tag{7.22}$$

and:

$$\frac{\delta \mathscr{L}_0}{\delta \psi^\beta(x)} - \partial_\mu \left[\frac{\delta \mathscr{L}_0}{\delta \left(\partial_\mu \psi^\beta \right)(x)} \right] = -\delta^{(4)}(x). \tag{7.23}$$

Remark 7.3. The difference between the right–hand sides of Eqs.(7.22, 7.23), besides the obvious Kronecker’s delta δ_{μ_α} in the first, which depends on the tensorial nature of the bosonic field, resides in the imaginary unit i multiplying the Dirac’s $\delta^{(4)}(x)$ distribution. The reason for this is simply that the fields obeying the Bose–Einstein statistics[3], or at very least, their potentials (or gauge connection) satisfy 2^{nd} order differential equations.

Nevertheless, as Dirac noticed in his revolutionary paper [Dirac(1930)], 2^{nd} order operators can be linearized upon the introduction of *complexifed* Clifford algebras[4], say $\mathscr{C}\ell\left(\mathbb{R}^{(1,3)}\right)^{\mathbb{C}}$. Therefore, the imaginary unit is already included in the algebraic structure of the field equations, explaining it’s absence in Eq.(7.23).

Following [Faddeev(2018)] and [Berazin(2012)], Eqs.(7.17, 7.19) implies the fundamental equation in perturbative field theory:

$$\mathscr{Z}\left[J^{(\alpha)}_{\mu_\alpha}(x),\sigma^{(\beta)}(x),\bar{\sigma}^{(\beta)}(x)\right] \tag{7.24}$$

$$=\exp\left[i\int d^4x'\,\mathscr{L}_{\mathrm{I}}\left(\frac{1}{i}\frac{\delta}{\delta J^{(\alpha)}_{\mu_\alpha}(x'),},\frac{1}{i}\frac{\delta}{\delta\sigma^{(\beta)}(x')},\frac{1}{i}\frac{\delta}{\delta\bar{\sigma}^{(\beta)}(x')}\right)\right]\mathscr{Z}_0\left[J^{(\alpha)}_{\mu_\alpha}(x),\sigma^{(\beta)}(x),\bar{\sigma}^{(\beta)}(x)\right]. \tag{7.25}$$

Moreover, $\mathscr{Z}_0\left[J^{(\alpha)}_{\mu_\alpha}(x),\sigma^{(\beta)}(x),\bar{\sigma}^{(\beta)}(x)\right]$ can be explicitly calculated using Eqs.(7.19, 7.23, 7.22), yielding:

$$\mathscr{Z}_0\left[J^{(\alpha)}_{\mu_\alpha}(x),\sigma^{(\beta)}(x),\bar{\sigma}^{(\beta)}(x)\right] \tag{7.26}$$

$$=\sum_{\alpha,\beta}\int d^4x_1d^4x_2\left[-\frac{i}{2}J^{\mu_{\alpha_2}}(x_2)\,D^{(\alpha)}_{\mu_{\alpha 2}\,\mu_{a_2}}J^{\mu_{\alpha_1}}(x_1)+i\left(\bar{\sigma}^{(\beta)}\right)^a(x_2)\,S_{ab}(x_2-x_1)\left(\sigma^{(\beta)}\right)^b(x_1)\right]. \tag{7.27}$$

Lastly, with the “free” Green’s functions Eqs.(7.22, 7.23), and recognizing that:

$$W\left[J^{(\alpha)}_{\mu_\alpha}(x),\sigma^{(\beta)}(x),\bar{\sigma}^{(\beta)}(x)\right]=-i\log\left(\mathscr{Z}\left[J^{(\alpha)}_{\mu_\alpha}(x),\sigma^{(\beta)}(x),\bar{\sigma}^{(\beta)}(x)\right]\right), \tag{7.28}$$

is the generating functional for connected Green’s functions (see [Abers and Lee(1973)] for an inductive proof), through Eqs.(7.24, 7.27, 7.28), we completed the framework of perturbative quantum field theory.

Upon the replacement of Eq.(7.24) into Eq.(7.28), one might hope to obtain connected Green’s functions from which physically observable quantities, such as the **S**–matrix.

Unfortunately, those functions contains divergent integrals over the energy–momentum space.

In order to understand the nature of these divergences, and therefore to introduce renormalization theory, one should first start by classifying the divergences associated to each amplitude in the perturbative series produced by Eq.(7.24).

[3]At very least, this is the case to the ones which are physically relevant to our present purposes, namely, the theories of Standard Model.

[4]For a pedagogical discussion of this beautiful subject, I recommend the recent book [Vaz and da Rocha(2016)]. A more comprehensive however heavier on the part of the reader, see [Lounesto et al.(2001)Lounesto, Cassels, Society, and Hitchin] or [Hestenes and Lasenby(2015)].

7.3 Degrees of Divergences

In this subsection, we shall introduce the notions of *superficial degree of divergences,* and criterions to qualify a theory as being *renormalizable, super–renormalizable* and *non–renormalizable.*

Let Γ be an amplitude contributing to the perturbative expansion generated by Eq.(7.24, 7.28).

Since W generates connected Green’s functions, Γ is a factor between off–shell propagators and a proper connected vertex, also known as *one particle irreducible diagram*, which we denote by **1PI**.

The off–shell propagators attached to Γ are called *external lines*, while the propagators connecting interactions, called from now on *vertices*, within a **1PI** diagram are called *internal lines*.

- $E_B :=$ # *external lines* corresponding to propagators of fields satisfying Bose–Einstein statistics, e.g., a photon.
- $E_F :=$ # *external lines* corresponding to propagators of fields obeying Dirac-Fermi statistics, e.g., an electron.
- $N_i :=$ # interactions of type $V_i\left(\phi^{(\alpha)}_{\mu_\alpha}, \psi^{(\beta)}\right)$ within the proper connected vertex of Γ.
- $I_B :=$ # internal lines corresponding to propagators of fields satisfying Bose–Einstein statistics, e.g., a virtual photon.
- $I_F :=$ # internal lines corresponding to propagators of fields obeying Dirac-Fermi statistics, e.g., a virtual electron.

Fixing our attention to an interaction of i^{th} type, inserted within the connected vertex of Γ, we also have:

- $d_i :=$ # derivatives in $V_i\left(\phi^{(\alpha)}_{\mu_\alpha}, \psi^{(\beta)}\right)$, each one adding a factor of momentum into the numerator.
- $N_{i(F)} :=$ # Fermi–Dirac fields in the i^{th} interaction vertex.
- $N_{i(B)} :=$ # Bose–Einstein fields in the i^{th} interaction vertex.

Before we can define the notion of superficial degree of divergence, take notice of the following phenomena.

Fields obeying Fermi–Dirac statistics, say $\Psi(x)$, are realized as sections of the spin–bundle associated to the principal bundle where our theory lives.

In the absence of interactions, $\Psi(x)$ is governed by a “free” Lagrangian density which is of 1^{st} order in the covariant derivative,

$$\mathscr{L}_{\text{Fermi-Dirac}} = \bar{\Psi}(x)\left(i\partial^{\$} - m\right)\Psi(x)\,, \tag{7.29}$$

where $\partial^{\$} := \gamma^\mu \partial_\mu$ is the derivative inherited by the spin–bundle associated to our trivial connection.

Now, by Fourier transforming the fundamental solution to Euler–Lagrange Eq.(7.23), one obtains the following propagator for $\Psi(x)$ in d–dimensional spacetime representation[5]:

$$\hat{S}_{\text{Fermi-Dirac}}(x) = \int \frac{d^d p}{(2\pi)^d}\, e^{ip\cdot x} \frac{\gamma\cdot p + m}{p^2 - m^2 + i\varepsilon^+}. \tag{7.30}$$

[5] A hat above a propagator means a Fourier transformation from energy–momentum to spacetime representation.

Therefore, the asymptotics of propagators for Fermi–Dirac fields are, in energy–momentum representation:

$$S_{\text{Fermi-Dirac}}(k) \sim \mathscr{O}\left[\left(\sqrt{k^\mu k_\mu}\right)^{n-1}\right]. \tag{7.31}$$

On the other hand, consider Boson–Einstein fields. These are either sections on a line–bundle, e.g. a scalar field $\phi(x)$, or a connection in the associated vector bundle, say, an Abelian gauge–field $A_\mu(x)$. Typically, Lagrangian densities for these fields are, respectively:

$$\mathscr{L}_{\text{Scalar}} = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2, \tag{7.32}$$

and (with covariant gauge–fixing term only for the sake of convenience)[6],

$$\mathscr{L}_{\text{Abelian-field}} = -\frac{1}{2}\partial_{[\mu}A_{\nu]} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2. \tag{7.33}$$

As above, we Fourier transform the Euler–Lagrange Eq.(7.22) to obtain the spacetime representation of the propagators of the scalar $\phi(x)$ and gauge $A_\mu(x)$ fields

These are, respectively,[7]

$$i\bar{\Delta}_{\text{Scalar}}(x) = \int \frac{d^d p}{(2\pi)^d}\, e^{ip\cdot x}\frac{i}{p^2 - m^2 + i\varepsilon^+}, \tag{7.34}$$

and:

$$i\bar{\Delta}_{\text{Abelian-field}}(x) = \int \frac{d^d p}{(2\pi)^d}\, e^{ip\cdot x}\frac{i}{p^2 - i\varepsilon^+}\left(\eta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right). \tag{7.35}$$

Consequently, Eqs.(7.34, 7.35) indicates the following asymptotic decay for fields obeying Bose–Einstein statistics:

$$\Delta_{\text{Bose-Einstein}}(k) \sim \mathscr{O}\left[\left(\sqrt{k^\mu k_\mu}\right)^{d-2}\right]. \tag{7.36}$$

Let Γ be an amplitude corresponding to a connected Green's function. The *superficial degree of divergence* $D(\Gamma)$ of the latter is obtained by summing the units of powers of energy–momentum in the numerator, and then subtracting the units of powers of energy–momentum in the denumerator, of each propagator within the **1PI** of Γ.

Under the *hypothesis* that the asymptotics of the propagators, as given by Eqs.(7.31, 7.36), can be reliably employed as the units of powers of energy–momentum of each propagator in the ultraviolet, then:

- Each internal propagator of a field satisfying Bose–Einstein statistics weights $(d-2)I_B$ units of mass in $D(\Gamma)$. (Cf. Eq.(7.36).)
- Each internal propagator of a field satisfying Fermi–Dirac statistics weights $(d-1)I_F$ units of mass in $D(\Gamma)$. (Cf. Eq.(7.31).)
- If the interaction $V_i\left(\phi^{(\alpha)}_{\mu_\alpha}, \psi^{(\beta)}\right)$ contains d_i derivatives, then the i^{th} vertex weights $d_i N_i$ units of mass in $D(\Gamma)$.

[6] Recall the anti–symmetrization condition: $\partial_{[\mu}A_{\nu]} := 2^{-1}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)$.

[7] For Bose-Einstein fields, we indicate Fourier transform to spacetime representation by an over-bar.

- Each vertex contains a energy–momentum measure $d^d k$, each of which weights $-d$ units of mass. With a total of $\sum_{i=1}^{n} N_i$ interactions within the **1PI**, one should subtract $-d\sum_{i=1}^{n} N_i$ units.

- A final Dirac’s $\delta^{(4)}$ distribution, due to total energy–momentum conservation, weights d units.

Therefore, the *superficial degree of divergence* is given by:

$$D(\Gamma) := (d-2)\, I_B + (d-1)\, I_F + \sum_{i=1}^{n} d_i N_i - d\sum_{i=1}^{n} N_i + d. \tag{7.37}$$

Nevertheless, a more useful expression to $D(\Gamma)$ can be found by a simple topological consideration.

Since each interaction vertex connects two internal propagators, the totality of external propagators corresponding to fields obeying Bose–Einstein and Fermi–Dirac statistics are, respectively,

$$E_B = \sum_{i=1}^{n} N_{i(E)} N_i - 2I_B, \tag{7.38}$$

and:

$$E_F = \sum_{i=1}^{n} N_{i(F)} N_i - 2I_F. \tag{7.39}$$

Solving Eqs.(7.38, 7.39) for the internal propagators I_B and I_F and replacing into Eq.(7.37), one finds:

$$D(\Gamma) = d - \frac{1}{2}(d-2)\, E_B - \frac{1}{2}(d-1)\, E_F + \sum_{i=1}^{n} \delta_i N_i, \tag{7.40}$$

where we have defined:

$$\delta_i := \frac{1}{2}(d-2)\, N_{i(B)} + \frac{1}{2}(d-1)\, N_{i(F)} + d_i - d, \tag{7.41}$$

known as the *index of the* i^{th} *interaction vertex.*

Relativistic quantum field theories can be classified according to the possible ultraviolet divergences of their connected Green’s functions. Moreover, the latter are arranged according to the family $\{\delta_i\}_{i=1}^{n}$ of indices of their interaction vertices:

- If $\delta_i < 0$ for *some* i^{th} *interaction vertex*, then by Eq.(7.40), even fixing the values of the number of external Bose–Einstein and Fermi–Dirac propagators, respectively, E_B and E_F, there will always exists an amplitude Γ, with a sufficiently large N_i, such that $D(\Gamma) < 0$, yielding a divergent amplitude. Therefore, we call theories with $\delta_i < 0$ for some $i = 1, \ldots, N$ *non–renormalizable.*

- If $\delta_i > 0$ for *all interaction vertices*, Eq.(7.40) implies that, by increasing the number of interactions in the amplitudes of connected Green’s functions linearly increases the superficial degree of divergences in such a way that, for fixed external boson and fermions propagators, E_B and E_F, respectively, there exists only a finite number of diverging amplitudes, all of which can be removed by renormalizing “bare” physical constants. Theories with this property are called *super–renormalizable.*

- Let a theory be such that $\delta_i \geq 0$ for *all vertices* with the exception of some interactions $\delta_j = 0$, for instance. Eq.(7.40) shows that: (1) For those vertices, e.g., the j^{th} interaction, with positive index, $\delta_j > 0$, the superficial degree of divergence will increase as one considers amplitudes with more and more insertions of V_j, and therefore, there are just a finite number of diverging amplitudes;

7.3.1 Example: Yukawa

Let a Yukawa theory in $d=4$ dimensional Minkowski spacetime be governed by the Lagrangian density:

$$\mathscr{L}_Y = \frac{1}{2}(\partial\varphi)^2 - \frac{\mu^2}{2}\varphi^2 + \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - \lambda\varphi^4 + G_Y\varphi\bar{\psi}\psi. \tag{7.42}$$

There are two interaction vertices, such that:

$$g_1 = \lambda,\ V_1 = \varphi^4,\ N_{1(F)} = 0,\ N_{1(B)} = 4,\ d_1 = 0, \tag{7.43}$$

$$g_2 = G_Y,\ V_2 = \varphi\bar{\psi}\psi,\ N_{2(F)} = 2,\ N_{2(B)} = 1,\ d_2 = 0. \tag{7.44}$$

Hence, by Eq.(7.41), the interaction indices are:

$$\delta_1 = N_{1(B)} + \frac{3}{2}N_{1(F)} - 4 = 0, \tag{7.45}$$

$$\delta_2 = N_{2(B)} + \frac{3}{2}N_{2(F)} - 4 = 0. \tag{7.46}$$

Therefore, for the Yukawa theory, whose interaction term is dictated by the Lagrangian of Eq.(7.42), the *superficial degree of divergence* of a connected amplitude Γ is, accordingly to Eq.(7.40),

$$D(\Gamma) = 4 - E_B - \frac{3}{2}E_F. \tag{7.47}$$

7.3.2 Example: Fermi

Let $\psi(x)$ be a 2^ν–component spinorfield living in the d–dimensional Minkowski spacetime $\mathbb{R}^{(1,d-1)}$. The integer ν is defined such that either $d=2\nu$ or else $d=2\nu+1$.

A simplified d–dimensional version of Fermi theory for the weak interactions evolves according to the action functional:

$$S_F = \int d^dx\,\mathscr{L}_F(\psi,\partial\psi), \tag{7.48}$$

with Lagrangian density given by:

$$\mathscr{L}_F = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi + G_F^{(d)}\left(\bar{\psi}\psi\right)^2. \tag{7.49}$$

There is only one interaction term in Eq.(7.49) such that, in the notation of Eq.(7.2),

$$g_1 = G_F^{(d)},\ V_1 = -\left(\bar{\psi}\psi\right)^2, \tag{7.50}$$

which is characterized by:

$$N_{1(B)} = 0,\ N_{1(F)} = 4,\ d_1 = 0. \tag{7.51}$$

Then, the vertex's index determined by Eq.(7.50) in d–dimensional Minkowski spacetime is:

$$\delta^{(d)} := \delta_1 = d-2. \tag{7.52}$$

Consequently, Fermi theory for weak interactions, taken as a realistic model in $d=4$, is non–renormalizable since $\delta^{(4)}=2$. Nonetheless, considered as a "toy" model in a $d=2$ spacetime, the theory becomes renormalizable, as was expected from Eq.(7.12).

7.3.3 Example: QED

Recall that the Lagrangian density for quantum electrodynamics (QED) in Minkowski spacetime with $d=4$ dimensions is:

$$\mathscr{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\nabla^{\$} - m\right)\psi, \tag{7.53}$$

where $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ is the Faraday field–strength tensor, $A_\mu(x)$ the electromagnetic four–vector potential (up to a gauge transformation), and $\psi(x)$ the 4–component spinor of Dirac's matter field.

Moreover, $\nabla^{\$} := \gamma^\mu D_\mu$ is the spin connection, acting in the associated spin–bundle on which $\psi(x)$ lives as a section, inherited from the covariant derivative $D_\mu = \partial_\mu - i\mathbf{e}A_\mu$ for which $A_\mu(x)$ is the connection on the principal–bundle.

Remark 7.4. Employing the geometric language to formulate a field theory is both elegant[8] as, at the same time, ensures that "minimal coupling" principle, derived from the true number of *physical degrees of freedom* of the Bose–Einstein's field involved in the theory, is satisfied at the beginning. However, in order to analyze the theory from a field theoretical viewpoint, e.g., the possibility of divergences in the connected Green's functions, indices of the interaction vertices, or even the mathematical consistency of the quantum theory (e.g., unitarity), one should eventually descend from geometry to particles and fields.

Expanding the covariant derivative of the Lagrangian density in Eq.(7.54), one obtains:

$$\mathscr{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi + \mathbf{e}A_\mu\bar{\psi}\gamma^\mu\psi. \tag{7.54}$$

It follows that there exists only one interaction vertex:

$$g_1 = \mathbf{e},\ V_1 = A_\mu\bar{\psi}\gamma^\mu\psi, \tag{7.55}$$

such that:

$$N_{1(B)} = 1,\ N_{1(F)} = 2,\ d_1 = 0. \tag{7.56}$$

Thence, the index of QED's vertex is:

$$\delta := \delta_1 = N_{1(B)} + \frac{3}{2}N_{1(F)} - 4 = 0, \tag{7.57}$$

and consequently, the theory is renormalizable and the *superficial degree of divergence* of a connected Green's function Γ is given by:

$$D(\Gamma) = 4 - E_B - \frac{3}{2}E_F. \tag{7.58}$$

7.4 Example: Gravitation

Let $\left(M, g_{\mu\nu}\right)$ be a $d=4$ dimensional spacetime without any matter fields, and therefore governed entirely by gravitation, believed to be described by General Relativity (GR).

Let $\left(g^{-1}\right)_{\mu\nu} := g^{\mu\nu}$ be the inverse metric tensor and $R_{\mu\nu}$ is the Ricci tensor derived from Levi–Civita connection uniquely determined[9] by $g_{\mu\nu}$. Here, the student is following the conventions collected at [Straumann(2013), App.D], with some adaptation from [Wald(2010)].

[8]Or to be precise and less philosophical, economical: the field equations are simpler to write down.

[9]Recall that, given any metric tensor $g_{\mu\nu}$ on a manifold M, the Levi–Civita connection ∇_μ associated to $g_{\mu\nu}$ is the *unique* metric–compatible and torsionless covariant derivative, respectively, $\nabla_\lambda g_{\mu\nu} = 0$ and $X^\mu\nabla_\mu Y^\nu - Y^\mu\nabla_\mu X^\nu = [X,Y]^\nu$. See [Wald(2010), Ch.3].

Without loss of generality, the cosmological constant $\Lambda \equiv 0$.

The vacuum Einstein equations follows as Euler–Lagrange equations of Einstein–Hilbert action:

$$S_{EH}\left[g_{\mu\nu}\right] = \frac{1}{16\pi G_{\mathrm{N}}} \int d^4x \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}, \tag{7.59}$$

up to boundary terms, such as [York Jr(1972), Gibbons et al.(1978)Gibbons, Hawking, and Perry], which *might* be relevant to a deeper understanding of black hole thermodynamics. Here, G_{N} is Newton's constant of universal gravitation, for which $\dim\left(G_{\mathrm{N}}\right) = -2$.

We are interested in analyzing the theory of gravitation from the point of view of a quantum field theory. As presented in last paragraph, GR is a *geometric* theory of spacetime. Hence, notions as renormalization of amplitudes and connected Green's functions are ill–posed.

An obvious reason is the following:

- Suppose one wishes to "quantize" the metric tensor $g_{\mu\nu}$ as a field theory. Then it will be necessary to take into account quantum corrections to $g_{\mu\nu}$ as a power series of $\hbar$. Thereby, the causal structure of hypersurface in spacetime should be subjected to "uncertainty" relations. This point was raised in the first paper on quantum gravity by M. Bronstein (1936), republished at [Bronstein(2012)].

- Nevertheless, one of the foundations of relativistic quantum field theories is the principle of micro-causality, which employs the Minkowski metric $\eta_{\mu\nu}$ and the structure of the isometries of Minkowski spacetime, particularly the proper orthochronos Lorentz group $\mathscr{L}_+^\uparrow$, in order to classify the possible degrees of freedom and polarizations of annihilations and creation operators from which the fields are build as polynomials [Wichmann and Crichton(1963), Weinberg(1995)].

In any case, we are interested in considering the possibility of a quantum field theory for gravity, even if to understand why our framework of renormalization is not completely satisfactory to understand the ultraviolet regime of such a theory.

Following [Deser(1970)], the action in Eq.(7.59) can be rewritten as:

$$S_{EH}\left[g_{\mu\nu}\right] = \frac{1}{16\pi G_{\mathrm{N}}} \int d^4x \left[\sqrt{-g} g^{\mu\nu} \left(\Gamma^{\alpha}_{\ \beta\mu}\Gamma^{\beta}_{\ \alpha\nu} - \Gamma^{\alpha}_{\ \mu\nu}\Gamma^{\beta}_{\ \alpha\beta}\right) - \nabla_\mu \mathfrak{K}^\mu\right], \tag{7.60}$$

where $\mathfrak{K}^\mu$ is the Kraichnan [Kraichnan(1955)] vector:

$$\mathfrak{K}^\mu = \sqrt{-g}\left(g^{\mu\nu}\Gamma^{\alpha}_{\ \alpha\nu} - g^{\alpha\beta}\Gamma^{\mu}_{\ \alpha\beta}\right). \tag{7.61}$$

Suppose there exists a world–volume $\Omega \subset M$ such that the Ricci scalar is sufficiently small, whereby we may safely work in a "freely falling coordinate system," or Gaussian normal coordinates $\{x^\mu\} : \Omega \longrightarrow \mathbb{R}^4$, in which there is an event $e \in \Omega$ where $\Gamma^{\mu}_{\ \nu\lambda}(e) = 0$.

Therefore, inside the world–volume Ω, one may ignore the boundary term obtained by Gauss theorem in Eq.(7.60), so that the gravitational action within that world–volume is equivalent to:

$$S_{EH}\left[g_{\mu\nu}\right] \equiv S_G\left[g_{\mu\nu}\right] = \frac{1}{16\pi G_{\mathrm{N}}} \int d^4x \sqrt{-g} g^{\mu\nu} \left(\Gamma^{\alpha}_{\ \beta\mu}\Gamma^{\beta}_{\ \alpha\nu} - \Gamma^{\alpha}_{\ \mu\nu}\Gamma^{\beta}_{\ \alpha\beta}\right). \tag{7.62}$$

On the other hand, it was shown by [Fierz and Pauli(1939)] that the only consistent action for a free, massless particle with spin $s=2$ (which in Wigner's classification of the representations of Lorenz group [Weinberg(1995), Ch.2] is embedded in a symmetric, two–covariant tensor field $h_{\mu\nu}$) is given by:

$$S_{\mathrm{PF}} = \frac{1}{2}\int d^4x \left[-\left(\partial_\rho h_{\mu\nu}\right)\partial^\rho h^{\mu\nu} + 2\left(\partial_\rho h_{\mu\nu}\right)\partial^\nu h^{\mu\rho} - 2\left(\partial_\nu h^{\mu\nu}\right)\partial_\nu h + \left(\partial_\mu h\right)\partial^\mu h\right] \tag{7.63}$$

Applying dimensional analysis, the units of mass dimensions of the *graviton* $h_{\mu\nu}$ is:

$$\left[\!\left[\left(\partial_\rho h_{\mu\nu}\right)\partial^\lambda h^{\sigma\delta}\right]\!\right] = [\![\partial]\!]^2 \times [\![h]\!]^2 = M^2 \times [\![h]\!]^2 = M^4, \tag{7.64}$$

and so:

$$\dim\left(h_{\mu\nu}\right) = 1. \tag{7.65}$$

Therefore, in order to expand the action of Eq.(7.62) in terms of $h_{\mu\nu}$, one needs to introduce a new parameter, say,

$$\kappa := \sqrt{32\pi G_{\mathrm{N}}}, \tag{7.66}$$

such that $\dim(k) = -1$.

Then, under the assumption of low–curvatures within the world–volume Ω, we should study the following background expansion:

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}. \tag{7.67}$$

The structure of Eq.(7.62) when expanded in orders of κ, which should be an analytic function given the nature of the inverse metric and square–root of metric's determinant, is:

$$S_G \sim \int d^4x \left[(\partial h)^2 + \kappa\,(h\partial h\partial h) + \kappa^2\,(h\partial h\partial h\partial h) + \mathscr{O}\left(\kappa^3\right)\right], \tag{7.68}$$

where by $(h\partial h\partial h\cdots\partial h)$ is understood the finite polynomial in $h_{\mu\nu}$, obtained by all possible contractions from the tensorial indices of ∂_μ and $h_{\mu\nu}$. The exact expressions for each of these polynomials can be found in [Veltman(1976)] and [Scharf(2014)]. Thence, by Eqs.(7.40, 7.41), for an amplitude Γ corresponding to a graviton's connected Green's function,

$$D(\Gamma) = 4 - E_B. \tag{7.69}$$

Since there are infinitely many interactions, there are amplitudes with arbitrarily negative superficial degrees of divergences, and for each new order in κ, one obtains new types divergences, by attaching new external propagators. In other words, the geometric version of our theory of gravity, GR and Einstein–Hilbert action of Eq.(7.59), or the field–theoretical viewpoint developed in Eq.(7.63), shown by [Deser(1970)] to recursively imply in the GR action, is not only non–renormalizable, but produces a highly divergent perturbation series.

7.5 Self–Energy

"The *perpetuum mobile* missed by crackpots."
– Folklore

In this section, we give an introductory exposition of some concepts involved in "mass renormalization," a phenomena due to the self–energy of a field satisfying Fermi–Dirac statistics when interacting with other fields, generally those satisfying Boson–Einstein statistics.

On what follows, in order to simplify our notations, let the effective action be denoted by:

$$S = S_0 + S_{\mathrm{I}} + S_{\mathrm{Sources}}. \tag{7.70}$$

Here, S_0 is the action for the "free" fields of the theory of interest:

$$S_0 := \int d^4x \mathscr{L}_0, \tag{7.71}$$

while S_{I} is the interaction term, whose general form is:

$$S_{\mathrm{I}} := \int d^4x \mathscr{L}_{\mathrm{I}}. \tag{7.72}$$

The term S_{Sources} is the integral over the sources coupled to the fields, as required by the definition of the generating functionals, and will be introduced according to the particularities of the theory below.

7.6 Kindergarten

Let's consider a model consisting of a scalar field $\phi\left(x\right)$ with mass μ and a Dirac's 4–component matter spinor-field $\psi\left(x\right)$.

The free Lagrangian density for both fields is:

$$\mathscr{L}_0 = \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{1}{2}\mu^2\phi^2 + \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi, \tag{7.73}$$

while the interactions are determined by:

$$\mathscr{L}_{\mathrm{I}} = \frac{\lambda}{4!}\phi^4 - \mathbf{g}\,\phi\,\bar{\psi}\psi. \tag{7.74}$$

Let $J\left(x\right)$ be an external "classical–valued" source, and let $\sigma\left(x\right)$ and $\bar{\sigma}\left(x\right)$ be "Grassmann–valued" currents. Then the source term in the effective action reads:

$$S_{\mathrm{Sources}} = \int d^4x \left[J\left(x\right)\phi\left(x\right) + \bar{\sigma}\left(x\right)\psi\left(x\right) + \bar{\psi}\left(x\right)\sigma\left(x\right)\right]. \tag{7.75}$$

7.7 Generating Functionals

The generating functional for the Green's functions of the theory defined by Eqs.(7.73, 7.74) is defined to be[10]:

$$\mathscr{Z}\left[J\left(x\right), \sigma\left(x\right), \bar{\sigma}\left(x\right)\right] = \int \mathscr{D}\left[\phi\left(x\right), \psi\left(x\right), \bar{\psi}\left(x\right)\right] \exp\left(iS\right), \tag{7.76}$$

while the generating functional for the "free" Green's functions is:

[10]Here and on what follows, the normalization constant of a pathintegral is taken to be absorbed into integration measure.

$$\mathscr{Z}_0\left[J(x),\sigma(x),\bar{\sigma}(x)\right]=\int\mathscr{D}\left[A_\mu(x),\psi(x),\bar{\psi}(x)\right]\exp i\left(S_0+S_{\text{Sources}}\right). \tag{7.77}$$

According to the prescriptions outlined at the references [Ryder(1996), Chs.6,7.] and [Bailin and Love(1993), Chs4,8.], and in our former problem sheets *"Problem Sheet IV: Feynman's Motives, Schwinger's Sources,"* the functional generator of "free" propagators is given by:

$$\mathscr{Z}_0\left[J(x),\sigma(x),\bar{\sigma}(x)\right] \tag{7.78}$$

$$=\int d^4x_1d^4x_2\left[-\frac{i}{2}J(x_2)D(x_2-x_1)J(x_1)+i\bar{\sigma}^\alpha(x_2)S_{\alpha\beta}(x_2-x_1)\sigma^\beta(x_1)\right]. \tag{7.79}$$

In §7.2, the propagators for fields satisfying Bose–Einstein and Fermi–Dirac statistics, respectively, $D(x)$ and $S(x)$, have already been defined, both from the formalism of canonical quantization in Eqs.(7.20, 7.21), and also as the fundamental solutions to the Euler–Lagrange Eqs.(7.22, 7.23).

We found the explicit spacetime representation of the scalar field propagator $J(x)$ and Dirac 4–spinor propagator $S_{\alpha\beta}(x)$ in Eqs.(7.34) and (7.30), respectively.

7.8 Perturbation Theory

The key formula of perturbative quantum field theory, applied our "toy" theory (described by the Lagrangian density defined at Eqs.(7.73, 7.74)) reads:

$$\mathscr{Z}\left[J(x),\sigma(x),\bar{\sigma}(x)\right]=\exp\left[i\int d^4x\,\mathscr{L}_{\text{I}}\left(\frac{1}{i}\frac{\delta}{\delta J(x)},\frac{1}{i}\frac{\delta}{\delta\sigma(x)},\frac{1}{i}\frac{\delta}{\delta\bar{\sigma}(x)}\right)\right]\mathscr{Z}_0\left[J(x),\sigma(x),\bar{\sigma}(x)\right]. \tag{7.80}$$

Remark 7.5. In *"Problem Sheet IV,"* we gave motivation of Eq.(7.80) with a particular emphasis to theories involving Boson–Einstein fields and "classical–valued" external sources. The subject is also explained at numerous references on quantum gauge theories, nevertheless a particularly clear and short exposition can be found at [Ramond(1997), Ch.8] and [Ryder(1996), §6.8], where Fermi–Dirac fields and "Grassmann–valued" currents are employed. For a more precise argument, see [Berazin(2012)].

Recalling the generating functional for connected Green's functions:

$$W\left[J(x),\sigma(x),\bar{\sigma}(x)\right]=-i\log\left\{\mathscr{Z}\left[J(x),\sigma(x),\bar{\sigma}(x)\right]\right\}, \tag{7.81}$$

the propagator for the Fermi–Dirac field $\mathbf{S}(p)$ in energy–momentum representation reads:

$$(2\pi)^4\delta^{(4)}(p_2-p_1)\prod_{n=1}^{2}\left(\frac{1}{\gamma\cdot p_n-m}\right)\times\mathbf{S}(p_2-p_1) \tag{7.82}$$

$$:=\int d^4x_1d^4x_2\,e^{i(p_1\cdot x_1-p_2\cdot x_2)}\left[\!\left[\frac{\delta^2W}{\delta\sigma(y)\,\bar{\sigma}(y)}\right]\!\right]_{\|J\|=\|\sigma\|=\|\bar{\sigma}\|=0}. \tag{7.83}$$

Applying the perturbative expansion defined by Eq.(7.80) to Eq.(7.82) at $\sim\mathscr{O}\left(\mathbf{g}^2\right)$ yields:

$$\mathbf{S}_F(p) = S_F^{(0)}(p) + \sum_{n=1}^{\infty} S_F^{(0)}(p) \times \prod_{m=1}^{n} \left[i\Sigma^{(2)}(p) S_F^{(0)}(p) \right] + \mathscr{O}\left(\mathbf{g}^4\right)$$

$$\mathbf{S}_F(p) = \frac{i}{\gamma \cdot p - m} + \frac{1}{\gamma \cdot p - m} \left[i\Sigma^{(2)}(p) \right] \frac{i}{\gamma \cdot p - m} \tag{7.84}$$

$$+ \frac{i}{\gamma \cdot p - m} \left[i\Sigma^{(2)}(p) \right] \frac{i}{\gamma \cdot p - m} \times \frac{i}{\gamma \cdot p - m} \left[i\Sigma^{(2)}(p) \right] \frac{i}{\gamma \cdot p - m} \tag{7.85}$$

$$+ \sum_{n>2}^{\infty} \frac{i}{\gamma \cdot p - m} \times \prod_{m=1}^{n} \left[i\Sigma^{(2)}(p) \frac{i}{\gamma \cdot p - m} \right] + \mathscr{O}\left(\mathbf{g}^4\right) \tag{7.86}$$

$$= \mathbf{S}_F^{(2)}(p) + \mathscr{O}\left(\mathbf{g}^4\right), \tag{7.87}$$

where:

$$\mathbf{S}_F^{(2)}(p) = \frac{1}{\gamma \cdot p - \left(m + \Sigma^{(2)}(p)\right)}, \tag{7.88}$$

and:

$$i\Sigma^{(2)}(p) = -\mathbf{g}^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma \cdot (p+k) + m}{\left[(k+p)^2 - m^2\right] (k^2 - \mu^2)}. \tag{7.89}$$

7.9 Self–Energy Insertion $\Sigma^{(2)}(p)$

Now, our objective is to compute $\Sigma^{(2)}(p)$, the self–energy insertion at $\sim \mathscr{O}\left(\mathbf{g}^2\right)$.

From linear algebra and Lorentz invariance, $\Sigma^{(2)}(p)$ should be a linear combination of $\gamma \cdot p$ and $\mathbb{I}$ with coefficients functions of the squared energy–momentum p^2,

$$\Sigma^{(2)}(p) = A\left(p^2\right) \gamma \cdot p + B\left(p^2\right). \tag{7.90}$$

To compute $\Sigma^{(2)}(p)$ from Eq.(7.89), one should apply the standard ritual:

- Feynman's parameters, in such a way that the denumerator is a function depending only in the square of the energy–momentum variable upon which one is integrating. In the case of Eq.(7.89), a function only of k^2. Hence, the integration can be separated into a solid–angle and an ordinary integral depending only in the magnitude of k, namely, $\sqrt{|k^2|}$.

- Use Wick's rotation to analytically continue k^0 into $\mathbb{C}$, obtaining an integral over an Euclidean space.

- Introduce a *cutoff* Λ in the integral over the magnitude of energy–momentum, and finally, perform the integration.

- Apply the inverse of Wick's rotation to return the variables to Lorentz signature.

Each one of these steps are performed at App.(**??**), and the final result is:

$$\Sigma^{(2)}(p) \underset{\Lambda\to\infty}{\sim} \frac{\mathbf{g}^2}{16\pi^2}\int_0^1 d\tau\,((1-\tau)\,\gamma\cdot p+m)\left[1+\log\left(\frac{\Lambda^2}{\mathscr{Q}^2(m,\mu;p^2,\tau)}\right)+\mathscr{O}\left(\frac{1}{\Lambda^2}\right)\right], \tag{7.91}$$

where we have defined a new function depending on the masses and the incoming (or outgoing) energy–momentum, given by:

$$\mathscr{Q}^2(m,\mu;p,\tau) := \tau(\tau-1)\,p^2+\left(m^2-\mu^2\right)\tau+\mu^2. \tag{7.92}$$

Hence, the coefficients are given by:

$$A\left(p^2\right) \underset{\Lambda\to\infty}{\sim} \frac{\mathbf{g}^2}{16\pi^2}\int_0^1 d\tau\,(1-\tau)\left[1+\log\left(\frac{\Lambda^2}{\mathscr{Q}^2(m,\mu;p^2,\tau)}\right)\right], \tag{7.93}$$

$$B\left(p^2\right) \underset{\Lambda\to\infty}{\sim} m\times\frac{\mathbf{g}^2}{16\pi^2}\int_0^1 d\tau\,(1-\tau)\left[1+\log\left(\frac{\Lambda^2}{\mathscr{Q}^2(m,\mu;p^2,\tau)}\right)\right]. \tag{7.94}$$

Finally, we can prove that $B\left(p^2\right)\to 0$ as $m\to 0$

$$\lim_{m\to 0}\left|B\left(p^2\right)\right| \underset{\Lambda\to\infty}{\leq} \lim_{m\to 0}\left(m\times\int_0^1 d\tau\,(1-\tau)\left|\log\left[\mathscr{Q}^2\left(m,\mu;p^2,\tau\right)\right]\right|\right) \tag{7.95}$$

$$= \lim_{m\to 0}\left(m\times\int_0^1 d\tau\,(1-\tau)\left|\log\left[\tau(\tau-1)\,p^2+\left(m^2-\mu^2\right)\tau+\mu^2\right]\right|\right), \tag{7.96}$$

and noticing the upper–bound,

$$\lim_{m\to 0}\int_0^1 d\tau\,(1-\tau)\left|\log\left[\tau(\tau-1)\,p^2+\left(m^2-\mu^2\right)\tau+\mu^2\right]\right| \tag{7.97}$$

$$= \int_0^1 d\tau\,(1-\tau)\left|\log\left[\tau(\tau-1)\,p^2+(1-\tau)\,\mu^2\right]\right| < \infty \tag{7.98}$$

follows our desired result:

$$\lim_{m\to 0} B\left(p^2\right) = 0. \tag{7.99}$$

7.10 Quantum Electrodynamics

7.10.1 *Anamnesis*

Recall that the action for spinor electrodynamics, namely, the theory of Maxwell's potential $A_\mu(x)$ minimally coupled to Dirac's matter fields $\psi(x)$, with the latter being a 4–spinor representation of $SL(2,\mathbb{C})$ acted upon by the covariant derivative $D_\mu = \partial_\mu - i\mathbf{e}A_\mu$, is given by:

$$S_{\text{QED}}\left[A_\mu(x),\psi(x),\bar\psi(x)\right] := \int d^4x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar\psi(x)\left(i\nabla^{\$}-m\right)\psi(x)\right]. \tag{7.100}$$

Here, $\mathbf{e}$ is the elementary unit of charge, and $\nabla^{\$} := \gamma^\mu D_\mu$ is the connection inherited by the associated spin–bundle, where $\psi(x)$ belongs to the space of sections.

Replacing the definition of covariant derivative D_μ into the QED action, one sees that Eq.(7.100) can be written as a contribution of the "free" action:

$$S_0\left[A_\mu(x),\psi(x),\bar{\psi}(x)\right]=\int d^4x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\psi}(x)\left(i\gamma^\mu\partial_\mu-m\right)\psi(x)\right], \tag{7.101}$$

together with the interaction:

$$S_{\mathrm{I}}\left[A_\mu(x),\psi(x),\bar{\psi}(x)\right]=\mathbf{e}\int d^4xA_\mu(x)\bar{\psi}(x)\gamma^\mu\psi(x). \tag{7.102}$$

Now, let $\mathscr{J}^\mu(x)$ be a "classical–valued" external source, and denote by by $\sigma(x)$ and $\bar{\sigma}(x)$ "Grassmann–valued" currents.

Recall that the coupling between the external sources S_{Sources} to Maxwell's gauge potential $A_\mu(x)$ and Dirac's matter field $\psi(x)$ is defined by:

$$S_{\mathrm{Sources}}=\int d^4x\left[\mathscr{J}^\mu(x)A_\mu(x)+\bar{\sigma}(x)\psi(x)+\bar{\psi}(x)\sigma(x)\right]. \tag{7.103}$$

We finally complete our brief review of elementary QED.

Remembering the definitions of perturbative quantum field theory discussed in §7.2 and §7.8, the generating functional for all Green's functions of QED is defined by:

$$\mathscr{Z}\left[\mathscr{J}^\mu(x),\sigma(x),\bar{\sigma}(x)\right]=\int\mathscr{D}\left[A_\mu(x),\psi(x),\bar{\psi}(x)\right]e^{i\left(S_0+S_{\mathrm{I}}+S_{\mathrm{Sources}}\right)}. \tag{7.104}$$

The generating functional for the "free" Green's function is given by:

$$\mathscr{Z}_0\left[\mathscr{J}^\mu(x),\sigma(x),\bar{\sigma}(x)\right] \tag{7.105}$$
$$=\int\mathscr{D}\left[A_\mu(x),\psi(x),\bar{\psi}(x)\right]e^{i\left(S_0+S_{\mathrm{Sources}}\right)} \tag{7.106}$$
$$=\int d^4x_1d^4x_2\left[-\frac{i}{2}\mathscr{J}^\mu(x_2)D_{\mu\nu}\mathscr{J}^\nu(x_1)+i\left(\bar{\sigma}^{(\beta)}\right)^a(x_2)S_{ab}(x_2-x_1)\left(\sigma^{(\beta)}\right)^b(x_1)\right], \tag{7.107}$$

where the Feynman's and Dirac's propagators (in spacetime representation) have already been computed, respectively, in Eqs.(7.35) and (7.30).

Lastly, the key formula for perturbation theory (cf. Eqs.(7.24, 7.27)) yields:

$$\mathscr{Z}\left[\mathscr{J}^\mu(x),\sigma(x),\bar{\sigma}(x)\right] \tag{7.108}$$
$$=\exp\left(-\mathbf{e}\int d^4x\frac{\delta}{\delta\mathscr{J}^\mu(x)}\frac{\delta}{\delta\bar{\sigma}(x)}\gamma^\mu\frac{\delta}{\delta\sigma(x)}\right)\mathscr{Z}_0\left[J(x),\sigma(x),\bar{\sigma}(x)\right] \tag{7.109}$$
$$=\exp\left(-\mathbf{e}\int d^4x\frac{\delta}{\delta\mathscr{J}^\mu(x)}\frac{\delta}{\delta\bar{\sigma}(x)}\gamma^\mu\frac{\delta}{\delta\sigma(x)}\right) \tag{7.110}$$
$$\times\int d^4x_1d^4x_2\left[-\frac{i}{2}\mathscr{J}^\mu(x_2)D_{\mu\nu}(x_2-x_1)\mathscr{J}^\nu(x_1)+i\left(\bar{\sigma}^{(\beta)}\right)^a(x_2)S_{ab}(x_2-x_1)\left(\sigma^{(\beta)}\right)^b(x_1)\right]. \tag{7.111}$$

7.11 *Dressed* Amplitudes

The complete[11] Dirac's propagator $\mathbf{G}(p)$ of QED is defined by:

$$(2\pi)^4\,\delta^{(4)}(p_2-p_1)\prod_{n=1}^{2}\left(\frac{1}{\gamma\cdot p_n-m}\right)\times\mathbf{G}(p_2-p_1) \tag{7.112}$$

$$:=\int d^4x_1 d^4x_2\, e^{-i(p_1\cdot x_1-p_2\cdot x_2)}\left[\!\!\left[\frac{\delta^2 W\left[\mathscr{J}^{\mu}(x),\sigma(x),\bar{\sigma}(x)\right]}{\delta\sigma(x_2)\,\bar{\sigma}(x_1)}\right]\!\!\right]_{\|\mathscr{J}^{\mu}\|=\|\sigma\|=\|\bar{\sigma}\|=0}, \tag{7.113}$$

while the complete *vertex function* $\Gamma^{\mu}(p_1,p_2;k)$ of QED is:

$$(2\pi)^4\,\delta^{(4)}(p_2+k-p_1)\left\{\prod_{n=1}^{2}\left(\frac{1}{\gamma\cdot p_n-m}\right)\frac{-i}{k^2+i\varepsilon^+}\left[\eta_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}\right]\right\}\times\Gamma^{\mu}(p_1,p_2;k) \tag{7.114}$$

$$:=\int d^4x_1 d^4x_2 d^4y\, e^{i(p_2\cdot x_2+k\cdot y-p_1\cdot x_1)}\left[\!\!\left[\frac{\delta^3\mathscr{W}\left[\mathscr{J}^{\mu}(x),\sigma(x),\bar{\sigma}(x)\right]}{\delta\mathscr{J}_{\mu}(y)\,\delta\sigma(x_2)\,\bar{\sigma}(x_1)}\right]\!\!\right]_{\|\mathscr{J}^{\mu}\|=\|\sigma\|=\|\bar{\sigma}\|=0}. \tag{7.115}$$

For *onshell* photon–electron vertices, we simply write $\Gamma^{\mu}(p_1,p_2):=\Gamma^{\mu}(p_1,p_2;p_2-p_1)$.

Using the perturbative expansion at Eq.(7.108) into the definition given by Eq.(7.112) of the complete Dirac's propagator $\mathbf{G}(p)$, one obtains a geometric series of the form:

$$i\mathbf{G}(p)=i(\gamma\cdot p-m)^{-1}+i(\gamma\cdot p-m)^{-1}\times\sum_{n=1}^{\infty}\prod_{m=1}^{n}\left[i\Sigma(p)\right]\times i(\gamma\cdot p-m)^{-1} \tag{7.116}$$

$$=i(\gamma\cdot p-m-\Sigma(p))^{-1}, \tag{7.117}$$

where we readily recognize $iS_F(p)=i(\gamma\cdot p-m)^{-1}$ as the "free" fermion propagator, and $\Sigma(p)$ is the summation of all the proper connected amplitudes (**1PI**) contributing to the Green's function $\mathbf{G}(p)$ for the complete, interacting, quantum electrodynamics.

From now one, $\Sigma(p)$ will be referred to as the *self–energy insertion*.

7.11.1 Radiative Correction $\sim\mathscr{O}\left(\mathbf{e}^2\right)$ [Zee(2010), Ex. III.3.3(ii)]

In *"Problem Sheet IX: A Problem in Self–Regularization,"* where we discussed the method of dimensional regularization, the radiative correction[12] to Dirac's propagator was calculated as an illustration of that scheme.

Recall that we obtained[13]:

$$\Sigma^{(2)}(p)=\frac{\mathbf{e}^2}{8\pi^2}\int_0^1 d\tau\left[(1-\tau)(\gamma\cdot p)-2m\right]\left\{\frac{2}{\varepsilon}-\log\left[\frac{\tau m^2-\tau(1-\tau)p^2}{4\pi\mu^2}\right]-(\gamma_{\mathrm{Euler}}+1)\right\}. \tag{7.118}$$

[11] By which we mean, the non–perturbative definition which takes into account the contributions derived from all order in the perturbation expansion.

[12] Namely, at order $\sim\mathscr{O}(\alpha)$, where $\alpha=\mathbf{e}^2/4\pi$ was known in the old literature as the *fine–structure constant.* Today, after an improved understanding of quantum field theory with renormalization semi–group, the exact calculation of α is now left to crackpots.

[13] In Eq.(7.118), μ is an unphysical mass scale introduced only to preserve the dimensionality of the elementary unit of charge $\mathbf{e}$ as we analytically continued spacetime's dimensions. Also, we recall that Euler's constant γ_{Euler} was discussed in App.(B), Eq.(69) of [Mol(1st March 2021)].

As expected from linear algebra and Lorentz invariance, $\Sigma^{(2)}(p)$ is as a linear combination of $\gamma \cdot p$ and $\mathbb{I}$ whose coefficients are functions of squared energy–momentum p^2,

$$\Sigma^{(2)}(p) = a\left(p^2\right)\gamma \cdot p + b\left(p^2\right)\mathbb{I}. \tag{7.119}$$

Therefore, reading directly from Eq.(7.118),

$$a\left(p^2\right) = \frac{\mathbf{e}^2}{8\pi^2}\int_0^1 d\tau\,(1-\tau)\left\{\frac{2}{\varepsilon} - \log\left[\frac{\tau m^2 - \tau(1-\tau)p^2}{4\pi\mu^2}\right] - (\gamma_{\text{Euler}}+1)\right\}, \tag{7.120}$$

$$b\left(p^2\right) = m \times \frac{\mathbf{e}^2}{4\pi^2}\int_0^1 d\tau\left\{\frac{2}{\varepsilon} - \log\left[\frac{\tau m^2 - \tau(1-\tau)p^2}{4\pi\mu^2}\right] - (\gamma_{\text{Euler}}+1)\right\}. \tag{7.121}$$

Now, take note that:

$$\lim_{m\to 0}\int_0^1 d\tau \log\left[\frac{\tau m^2 - \tau(1-\tau)p^2}{4\pi\mu^2}\right] = \lim_{m\to 0}\int_0^1 d\tau \log\left[\frac{\tau(1-\tau)p^2}{4\pi\mu^2}\right] < \infty, \tag{7.122}$$

and thereby, Eqs.(7.121, 7.122) implies the limit is well–defined:

$$\lim_{m\to 0} b\left(p^2\right) = 0,$$

as we wanted to show.

7.12 Non–perturbative Analysis

7.12.1 Types of Divergences

On the following, it will prove itself useful to introduce the notion of "different types of infinities."

Let $\Lambda_i > 0$ denote a *cutoff*, which in each expression that follows, is tacitly assumed to be taken as a limit to infinity, $\Lambda_i \longrightarrow \infty$.

Then we may classify different types "infinity" by asymptotics:

$$A_i \sim \mathscr{O}\left(f\left(\Lambda_i\right)\right), \tag{7.123}$$

and we say that the coefficient A_i, of some polynomial to be defined below, "diverges as f."

As an example, if

$$A_i \sim \mathscr{O}\left(\log \Lambda_i\right), \tag{7.124}$$

then A_i diverges logarithmically.

7.12.2 Generalized Ward–Takahashi Identity

In our former *"Problem Sheet VIII,"* we deduced a generalization of Ward-Takahashi identities using gauge invariance symmetry within the generating functional for the Green's functions:

$$\lim_{\|\mathscr{J}^\mu\|,\|\sigma\|,\|\bar{\sigma}\|\to 0}\frac{1}{\xi}\Box_x\left[\frac{\partial}{\partial x^\mu}\left(\frac{\delta^3\mathscr{Z}}{\delta\bar{\sigma}(z)\,\delta\sigma(y)\,\delta\mathscr{J}_\mu(x)}\right)\right] \tag{7.125}$$

$$= \lim_{\|\sigma\|,\|\bar{\sigma}\|\to 0} iq\left\{\delta^{(D)}(x-z)\frac{\delta^2\mathscr{Z}}{\delta\sigma(y)\,\delta\bar{\sigma}(x)} + \frac{\delta^2}{\delta\bar{\sigma}(z)\,\delta\sigma(x)}\left[\delta^{(D)}(x-z)\right]\right\}. \tag{7.126}$$

Fourier transforming Eqs.(7.126) from spacetime to energy–momentum representation, and applying the definitions from Eqs.(7.112, 7.114), one obtains the familiar version of Ward–Takahashi identity:

$$(p_2 - p_1)_\mu \mathbf{G}(p_1)\Gamma^\mu(p_1,p_2)\mathbf{G}(p_2) = i\mathbf{G}(p_2) - i\mathbf{G}(p_1). \tag{7.127}$$

Now, multiplying both sides of Eq.(7.127) by $[\mathbf{G}(p_1)]^{-1}$ from the left and by $[\mathbf{G}(p_2)]^{-1}$ from the right, we derive:

$$i[\mathbf{G}(p_1)]^{-1} - i[\mathbf{G}(p_1)]^{-1} = (p_2 - p_1)_\mu \Gamma^\mu(p_1,p_2). \tag{7.128}$$

Lastly, taking the partial derivative of Eq.(7.128) with respect to p_2, and then letting $p_1 = p_2 =: p$, one arrives at the differential form of Ward–Takahashi identity:

$$\frac{\partial}{\partial p_\mu}[\mathbf{G}(p)]^{-1} = i\Gamma^\mu(p,p). \tag{7.129}$$

Remark 7.6. Curiously, Eq.(7.129) posses the same form of the original identity found by J. C. Ward in [Ward(1950)]. Nevertheless, our derivation holds for all orders in perturbation theory, while Ward's original formula was a tree–level identity.

7.12.3 Divergence's Degrees

First, let $\Gamma^\mu_{(\ell)} := \Gamma^\mu_{(\ell)}(p,p)$ be a connected Green's function (in energy–momentum representation) corresponding to photon–electron vertex, as defined by Eqs.(7.114, 7.115), up to order $\sim \mathscr{O}\left(\mathbf{e}^{2\ell}\right)$ in perturbation theory (see Eq.(7.108)).

Since $\Gamma^\mu_{(\ell)}$ contains $E_B = 1$ photon's propagator and $E_F = 2$ Dirac's propagators, the superficial degree of divergence of $\Gamma^\mu_{(\ell)}$ given by Eq.(7.58) is:

$$D\left(\Gamma^\mu_{(\ell)}\right) = 4 - E_B - \frac{3}{2}E_F = 0. \tag{7.130}$$

Hence, the divergence contained in $\Gamma^\mu_{(\ell)}$ is logarithmic and independent of the incoming (or outgoing, whatever) electron or positron energy–momentum.

Thence, using Lorentz invariance and linear algebra, $\Gamma^\mu_{(\ell)}$ can be decomposed into two terms:

$$\Gamma^\mu_{(\ell)}(p,p) = \mathscr{O}(\log\Lambda)\gamma^\mu + \Gamma^\mu_{(f)}(p,p), \tag{7.131}$$

where $\mathscr{O}(\log\Lambda_1)$ represents a logarithmic divergence, as explained at §7.12.1, and $\Gamma^\mu_{(f)}(p,p)$ is the remaining finite contribution.

Now, we proceed similarly to the self–energy insertion up to order $\sim \mathscr{O}\left(\mathbf{e}^{2\ell}\right)$, denoted by $\Sigma_{(\ell)}(p)$.

This amplitude is the formal sum of all connected Green's functions with $E_B = 0$ and $E_F = 2$ containing powers of the elementary unit of charge up to $\mathbf{e}^{2\ell}$. Hence, Eq.(7.58) yields:

$$D(\Sigma) = 4 - E_B - \frac{3}{2}E_F = 1, \tag{7.132}$$

which implies that $\Sigma_{(\ell)}(p)$ is a linear polynomial in $\gamma\cdot p - m$:

$$\Sigma_{(\ell)}(p) = A_0 + A_1(\gamma\cdot p - m) + \Sigma_{(f)}(p). \tag{7.133}$$

Here, $\Sigma_{(f)}(p)$ is the remaining finite part of $\Sigma_{(\ell)}(p)$.

Finally, imposing the *onshell* constraints:

$$\Sigma(p)\Big|_{\gamma\cdot p=m}=0, \tag{7.134}$$

$$\frac{\partial}{\partial(\gamma\cdot p)}\Sigma(p)\Big|_{\gamma\cdot p=m}=0, \tag{7.135}$$

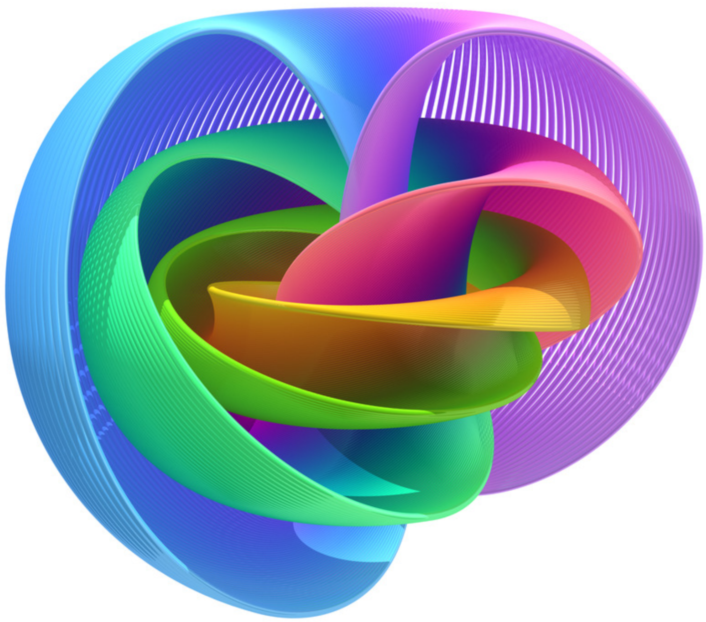
we can eliminate A_0, and moreover, replacing Eqs.(7.131, 7.133) into the generalized Ward–Takahashi identity, see Eq.(7.129), we deduce that:

$$A_1\sim\mathscr{O}(\log\Lambda). \tag{7.136}$$

In other words, we have just shown that the self–energy shift is logarithmically divergent.

Chapter 8

Under the Spell of the Gauge Principle



In this Lecture, we discuss “gauge fields” belonging to the family Yang–Mills theories, namely, those with a compact and semi–simple symmetry group.

This classes of theories, originally conceived in the physics literature mainly due to the pioneer work of H. Weyl [Weyl(1923)], still in the earlier days of classical “unified field theories,” appeared in a more familiar form in [Yang and Mills(1954)], and the unpublished manuscript [Heisenberg and Pauli(1993)] (1958).

8.1 Notations

On what follows, let $M_4 := \mathbb{R}^{(1,3)}$ be the Minkowski space, $\{x^\mu\}$ Lorentz (i.e., inertial) coordinates and $\hat{\theta}^\mu := dx^\mu$ our choice of (holonomic) orthonormal frames, to be called *vielbein*.

In many cases, we shall be interested in linear transformations involving a given *vielbein.* In such cases, it will prove useful to introduce the row–matrix $\hat{\theta} := \left(\hat{\theta}^{\mu}\right)_{0\leq\mu\leq 3}$ and the column–matrix $\left(\hat{\theta}\right)^{\mathrm{T}}$ defined by the transpose of the former.

8.2 Lie Group and Associated Algebras

Letting G denote a compact and semi–simple Lie group, we denote by script letters $\mathfrak{g}$ the associated (complexified) Lie algebra and $T^a \in \mathfrak{g}$ (for all $1 \leq a \leq \dim G$) a basis for $\mathfrak{g}$ consisting in Hermite generators with normalization:

$$\mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}, \tag{8.1}$$

and structure constants:

$$\left[T^a, T^b\right] = c^{abc}T^c. \tag{8.2}$$

For a concise nonetheless precise exposition of the structures of group theory appurtenant to particle physics, see [O'Raifeartaigh(1986)].

In these notes, we adopt Einstein strong summation convention for the Lie algebra indices.

8.3 Minimal Coupling

On what follows, denote by $\rho : G \longrightarrow V$ the fundamental representation of the Lie group G.

Let $\mathbf{A} := A_{\mu}^{(a)}\hat{\theta}^{\mu} \otimes T^a \in \Omega^1\left(M_4, \mathfrak{g}\right)$ be a Lie–algebra valued one–form, and $\nabla_{\mu} \in \mathrm{End}\,\Gamma\left(M_4\right)$ the derivation acting on the module $\Gamma\left(M_4\right)$, whose action is given by:

$$\nabla_{\mu}\Phi := \partial_{\mu}\Phi + igA_{\mu}^{a}T^{a}\Phi,\ \forall\Phi \in \Gamma\left(M_4, V\right). \tag{8.3}$$

We shall refer to $\mathbf{A}$ as a *gauge potential* and ∇_{μ} the covariant derivative associated to the latter. The physical meaning of those objects should become more familiar after studying the examples that follows in §(8.3.1, 8.3.2).

Let $U \in \mathscr{C}^{\infty}\left(M_4, G\right)$ be a smooth mapping from Minkowski space into an element of the Lie group G.

Then $\mathbf{U} := \rho \circ U \in \mathscr{C}^{\infty}\left(M_4, V\right)$ is a smooth mapping from Minkowski space into the linear space whose fundamental representation ρ acts upon.

A change of local trivialization, also known as the *gauge transformation* induced by $\mathbf{U}$ is performed under the simultaneous application of the mappings:

$$\begin{cases} \hat{\theta} \mapsto \hat{\theta}' = \left(\hat{\theta}\right)^{\mathrm{T}}\mathbf{U},\ \text{and:} \\ \sigma \mapsto \sigma' := \mathbf{U}\sigma, \end{cases} \tag{8.4}$$

where $\sigma \in \Gamma\left(M_4, V\right)$ is a cross–section over the trivial vector–bundle[1] $\mathscr{B} := \left(M_4 \times V, M_4, \pi, G\right)$.

[1] As usual, $\pi : (x, v) \in M_4 \times V \mapsto v \in V$ is the projector into the first factor.

8.3.1 Example: N–Tuple of Fermi–Dirac Fields

Let Ψ be an N–Tuple of Fermi–Dirac fields all members of which possess the same mass $m > 0$.

Each each component of Ψ, say, $x \in M_4 \mapsto \Psi_\ell(x)$ where $1 \le \ell \le N$, is a cross–section[2] belonging to the spinor representation of[3] $\mathrm{SL}(2;\mathbb{C}) \oplus \overline{\mathrm{SL}(2;\mathbb{C})}$, which we denote by ρ.

From the spin–statistics theorem [Fierz(1939), Pauli(1940)], fields arising from spinor representations satisfies Fermi–Dirac statistics, and thence our name attributed to Ψ.

Let the Lagrange's functional density, which underlies the dynamics of those Fermi–Dirac fields Ψ, be given by:

$$\mathscr{L}_{\mathrm{FD}}(\Psi, \nabla\Psi) = \bar{\Psi}(i\gamma \cdot \nabla - m)\Psi, \tag{8.5}$$

so that the action functional becomes:

$$S_{\mathrm{FD}}[\Psi] = \int_\Omega d^4x \mathscr{L}_{\mathrm{FD}}(\Psi, \nabla\Psi) \;\; (\text{where}\, \Omega \subset M_4). \tag{8.6}$$

where the world–volume $\Omega \subset M_4$ is sufficiently "large," and the asymptotics of Ψ are such that the contributions coming from boundary terms[4] can be safely neglected.

Following the *minimal coupling principle*, the action functional is required to be invariant under a local change *vielbein*, namely, a *gauge transformation*, as defined by Eq.(8.4).

A sufficient condition to satisfy the *principle of minimal coupling* is to assume that, under a gauge transformation, the Lagrangian is itself an invariant:

$$\mathscr{L}_{\mathrm{FD}}(\Psi, \nabla\Psi) = \mathscr{L}_{\mathrm{FD}}\left(\Psi', \nabla'\Psi'\right), \tag{8.7}$$

where the transformation law for the *vielbein* (see Eq.(8.4)) induces, respectively, the following transformations for the spin cross–section and the corresponding covariant derivate,

$$\begin{cases} \Psi \mapsto \Psi' = (\rho \circ \mathbf{U})\Psi, \\ \nabla\Psi \mapsto \nabla'\Psi' = (\rho \circ \mathbf{U})\nabla\Psi. \end{cases} \tag{8.8}$$

Applying the transformation laws from Eq.(8.4) into the definition of the Lagrangian for Ψ, given by Eq.(8.48), and expanding both sides of Eq.(8.7) in their respective *vielbein* and the basis T^a for $\mathfrak{g}$ one finds, after canceling the binomials proportional to the masses, that:

$$i\bar{\Psi}\gamma^\mu \partial_\mu \Psi - g\bar{\Psi}\gamma^\mu A^{(a)}_\mu T^a \Psi = i\bar{\Psi}U^\dagger \gamma^\mu \left(\partial_\mu U\right)\Psi + i\bar{\Psi}U^\dagger \gamma^\mu U\left(\partial_\mu \Psi\right) - g\bar{\Psi}U^\dagger \gamma^\mu A'^{(a)}_\mu T^a U\Psi. \tag{8.9}$$

To simplify the notation, we have chosen to write $U = \rho \circ \mathbf{U}$.

After an algebraic manipulation of Eq.(8.9), one finds the mapping $\mathbf{A} = A^{(a)}_\mu \hat{\theta}^\mu T^a \mapsto \mathbf{A}' = A'^{(a)}_\mu \hat{\theta}'^\mu T^a$ from the connection form written in the original choice of *vielbein* to the one obtained after a gauge transformation:

$$A'^{(a)}_\mu T^a = ig^{-1}\left(\partial_\mu U\right)U^\dagger + UA^{(a)}_\mu U^\dagger. \tag{8.10}$$

[2]Formally, Ψ is a section living in the spin–bundle associated to $\mathscr{B}$, where $\gamma \cdot \nabla$ happens to be the induced spin–covariant derivative.

[3]A mathematically rigorous and readable account of spinor representations, clarifying the geometric meaning of spinors as natural objects underlying the structure of Minkowski spacetime, is available in [Naber(2003), Ch.3].

[4]This assumption should be revised in a quantum theory based upon Euclidean path-integrals.

8.3.2 Example: $\mathrm{SU}(N)$ Theory for Bose–Einstein Fields

Now, suppose $\Phi \in \Gamma\left(M_4, V^{\mathbb{C}}\right)$ represents a N–tuple of massive (with $m > 0$ for all members of the N–tuple), non–Hermite physical fields in the $\mathrm{SU}(N)$ representation, whose dynamics is governed by the Lagrange's functional density:

$$\mathscr{L}_{\mathrm{BE}}(\Phi, \nabla\Phi) = \left(\nabla_\mu \Phi\right)^\dagger (\nabla^\mu \Phi) - m^2 \Phi^\dagger \Phi. \tag{8.11}$$

By the principle of *minimal coupling*, one wishes to impose an invariance of the action functional, governing the dynamics of the Φ–field, under the transformations of Eq.(8.4).

In order to satisfy the *minimal coupling* principle, a sufficient condition is to impose the following law for the transformation of the covariant derivative:

$$\left(\nabla_\mu \Phi\right)' = \mathbf{U}\left(\nabla_\mu \Phi\right), \tag{8.12}$$

such that the Lagrange's functional density given by Eq.(8.11) is an "invariant" under *gauge transformations*, or more precisely:

$$\mathscr{L}_{\mathrm{BE}}(\Phi, \nabla\Phi) = \mathscr{L}_{\mathrm{BE}}\left(\Phi', \nabla'\Phi\right). \tag{8.13}$$

Remark 8.1. The condition stated by Eq.(8.12) is only sufficient for the invariance of the action functional:

$$S_\Omega[\Phi] := \int_\Omega d^4x \, \mathscr{L}_{\mathrm{BE}}(\Phi, \nabla\Phi), \ (\text{where}\, \Omega \subset M_4), \tag{8.14}$$

nonetheless not necessary. One could still conceive an additional boundary term arising from a total derivative, which for the time being, will be ignored under the assumption that the asymptotic behavior of Φ is such that the contributions on $\partial\Omega$ may be safely neglected.

A change of *vielbein* induces a transformation of the connection form, or in other words, a *gauge transformation* generates the mappings:

$$\begin{cases} \hat{\theta} \mapsto \hat{\theta}' = \left(\hat{\theta}\right)^{\mathrm{T}} \mathbf{U}, \text{ and:} \\ \mathbf{A} = A_\mu^{(a)} \hat{\theta}^\mu \otimes T^a \mapsto \mathbf{A}' = A_\mu^{'(a)} \hat{\theta}'^\mu \otimes T^a. \end{cases} \tag{8.15}$$

As in Eq.(8.3), the new covariant derivative ∇' associated to the connection form $\mathbf{A}'$ acts upon cross–sections as:

$$\left(\nabla_\mu \Phi\right)' := \partial_\mu \Phi' + igA_\mu^{'a} T^a \Phi', \ \forall \Phi \in \Gamma(M_4, V). \tag{8.16}$$

Now, replacing the mappings from Eq.(8.4) into Eqs.(8.12, 8.16) yields:

$$\left(\nabla_\mu \Phi\right)' = \partial_\mu (\mathbf{U}\Phi) + igA_\mu^{'a} T^a \mathbf{U}\Phi = \left(\partial_\mu \mathbf{U}\right)\Phi + \mathbf{U}\left(\partial_\mu \Phi\right) + igA_\mu^{'a} T^a \mathbf{U}\Phi \tag{8.17}$$

$$= \mathbf{U}\left(\nabla_\mu \Phi\right) = \mathbf{U}\partial_\mu \Phi + igA_\mu^a \mathbf{U} T^a \Phi, \tag{8.18}$$

which after simplification, gives:

$$\left(\partial_\mu \mathbf{U} + igA_\mu^{'a} T^a \mathbf{U} - igA_\mu^a \mathbf{U} T^a\right)\Phi = 0, \tag{8.19}$$

for all cross–sections $\Phi \in \Gamma(M_4, V)$.

Hence:

$$\partial_\mu \mathbf{U} + igA'^a_\mu T^a \mathbf{U} - igA^a_\mu \mathbf{U} T^a = 0. \tag{8.20}$$

Moreover, note that T^a being Hermite's generators for the Lie algebra $\mathfrak{su}(N)$, it follows that a similarity transformation $\mathbf{U}T^a\mathbf{U}^\dagger$ belongs to another set of Hermite's matrices spanning the adjoint representation.

Since $\mathbf{U} \in \mathrm{SU}(N)$ takes values in the fundamental representation, and recalling our adoption of the Einstein *strong* summation convention for indices labelling elements of the Lie algebra, the middle term of Eq.(8.20) can be expressed as:

$$A^a_\mu \mathbf{U} T^a = \sum_{1 \le a \le N} A'^a_\mu T^a \mathbf{U} = \sum_{1 \le a \le N} A'^a_\mu \mathbf{U} T^a = A'^a_\mu \mathbf{U} T^a, \tag{8.21}$$

keeping, of course, Einstein's convention.

Then, by virtue of Eqs.(8.21, 8.20),

$$\partial_\mu \mathbf{U} + \left(igA'^a_\mu T^a - igA^a_\mu T^a \right) \mathbf{U} = 0. \tag{8.22}$$

Lastly, multiplying Eq.(8.22) on the right by $\mathbf{U}^\dagger$, one arrives at the transformation law for the connection coefficients:

$$A'^a_\mu T^a = A^a_\mu T^a + ig^{-1} \left(\partial_\mu \mathbf{U} \right) \mathbf{U}^{-1}. \tag{8.23}$$

8.4 Cartan, Killing and Bianchi

In this subsection, for simplicity sake, let us consider only a trivial vector–bundle $\mathscr{B}$, with Minkowski space M_4 as a base and whose fibers are precisely isomorphic to the linear space V on which the adjoint representation ρ of G acts upon.

Definition 8.1. Let $\mathbf{A} := A^{(a)}_\mu \hat{\theta}^\mu \otimes T^a \in \Omega^1(M_4, \mathfrak{g})$ be a one–form taking values in the (complexifed) Lie algebra of a compact and semi–simple Lie group G, denoted by $\mathfrak{g}^{\mathbb{C}}$ on what follows. A necessary and sufficient condition for $\mathbf{A}$ to be *connection form* is that, under a change of orthonormal frames induced by the local "Lorentz" transformations:

$$\begin{cases} \mathbf{U} := \left(U^\mu_\nu \right) \in \mathscr{C}^\infty (M_4, \mathrm{SO}(1,3)), \\ \hat{\theta}'^\mu_x = U^\mu_\nu (x) \hat{\theta}^\nu_x, \ \forall x \in M_4, \end{cases} \tag{8.24}$$

the transformation law for the components of the connection forms follows the prescription provided by Eq.(8.23), namely:

$$\begin{cases} \hat{\theta} \mapsto \hat{\theta}' = (\hat{\theta})^{\mathrm{T}} \mathbf{U}, \\ \mathbf{A} \mapsto \mathbf{A}' = \left(A^a_\mu T^a + ig^{-1} \left(\partial_\mu \mathbf{U} \right) \mathbf{U}^{-1} \right) \hat{\theta}'^\mu \otimes T^a. \end{cases} \tag{8.25}$$

Definition 8.2. Let $\mathbf{A} \in \Omega^1(M_4, \mathfrak{g})$ be a connection form, as in Def.(8.2). The *covariant derivative* ∇ associated to $\mathbf{A}$, acting on the space of cross–sections[5] $\Gamma(\mathscr{B})$, which is a module over the ring of smooth functions $\mathscr{C}^\infty(M_4)$, is defined by:

$$\nabla_\mu \sigma := \partial_\mu \sigma + igA^{(a)}_\mu T^a \sigma, \ \forall \sigma \in \Gamma(\mathscr{B}). \tag{8.26}$$

Sometimes, we call $\mathbf{A}$ simply a connection, the nature of a Lie–algebra valued exterior differential form being understood.

[5]Recall that, given a generic topological bundle $(E, \mathscr{B}, \pi)$, where E is the total space being projected onto the base $\mathscr{B}$ by the surjection π, a cross–section σ is any mapping $\sigma : \mathscr{B} \longrightarrow E$ such that: $\pi \circ \sigma = \mathrm{id}$.

Definition 8.3. Followings Defs.(8.1, 8.2), let $\mathbf{A} := A_\mu^{(a)} \hat{\theta}^\mu \otimes T^a \in \Omega^1(M_4, \mathfrak{g})$ be a connection form with covariant derivative ∇. The curvature two–form (or simply *curvature form*):

$$\mathbf{F} := \frac{1}{4} F_{\mu\nu}^{(a)} \left(\hat{\theta}^\mu \wedge \hat{\theta}^\nu\right) \otimes T^{(a)} \in \Omega^2(M_4, \mathfrak{g}), \tag{8.27}$$

induced by the connection $\mathbf{A}$, is defined accordingly to our chosen holonomic *vielbein* $\hat{\theta}^\mu$ by:

$$\mathscr{F}_{\mu\nu} := F_{\mu\nu}^{(a)} \otimes T^a = \nabla_\mu(\mathbf{A})_\nu - \nabla_\nu(\mathbf{A})_\mu = \partial_\mu(\mathbf{A})_\nu - \partial_\nu(\mathbf{A})_\mu + \left[(\mathbf{A})_\mu, (\mathbf{A})_\nu\right]. \tag{8.28}$$

Using the structure constants of the Lie algebra $\mathfrak{g}$, Eq.(8.27) can be rewritten in the following more useful form:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)} - g c^{abc} A_\mu^{(b)} A_\nu^{(c)}. \tag{8.29}$$

Remark 8.2. For the relativity connoisseur, the subscript μ in ∇_μ indeed means the derivative along the coordinate vector field $\partial/\partial x^\mu$,

$$\nabla_\mu \sigma = \nabla_{\frac{\partial}{\partial x^\mu}} \sigma, \tag{8.30}$$

for any given cross–section σ.

Proposition 8.1. (Cartan–Killing 1^{st} Structure Equation.) *Let* $\mathbf{A} \in \Omega^1(M_4, \mathfrak{g})$ *be a connection form with covariant derivative* ∇, *as stated in* Defs.(8.3, 8.3), *and* $\mathbf{F} := \mathscr{F}_{\mu\nu} \hat{\theta}^\mu \wedge \hat{\theta}^\nu \in \Omega^2(M_4, \mathfrak{g})$ *the corresponding curvature form.*

Then:

$$\left[\nabla_\mu, \nabla_\nu\right] = ig\mathscr{F}_{\mu\nu}. \tag{8.31}$$

In fact, for any cross–section $\sigma \in \Gamma(\mathscr{B})$,

$$\begin{aligned}
\left[\nabla_\mu, \nabla_\nu\right]\sigma &= \partial_\mu(\nabla_\nu\sigma) + ig\mathscr{A}_\mu(\nabla_\nu\sigma) - \partial_\mu(\nabla_\nu\sigma) - ig\mathscr{A}_\nu\left(\nabla_\mu\sigma\right) && (8.32)\\
&= \partial_\mu(\partial_\nu\sigma + ig\mathscr{A}_\nu\sigma) + ig\mathscr{A}_\mu(\partial_\nu\sigma + ig\mathscr{A}_\nu\sigma) - \partial_\nu\left(\partial_\mu\sigma + ig\mathscr{A}_\mu\sigma\right) - ig\mathscr{A}_\nu\left(\partial_\mu\sigma + ig\mathscr{A}_\mu\sigma\right) && (8.33)\\
&= ig\left(\partial_\mu\mathscr{A}_\nu - \partial_\nu\mathscr{A}_\mu\right)\sigma + (ig)^2\left[\mathscr{A}_\mu, \mathscr{A}_\nu\right]\sigma = ig\left(2\partial_{[\mu}\mathscr{A}_{\nu]} + igA_\mu^{(a)}A_\nu^{(b)}\left[T^a, T^b\right]\right) && (8.34)\\
&= ig\left(2\partial_{[\mu}\mathscr{A}_{\nu]} - igc_{abc}A_\mu^{(a)}A_\nu^{(b)}T^c\right). && (8.35)
\end{aligned}$$

Now, replacing the components of the curvature form, see Eq.(8.29), into Eq.(8.35) yields:

$$\left[\nabla_\mu, \nabla_\nu\right]\sigma = ig\mathscr{F}_{\mu\nu}\sigma, \ \forall\sigma \in \Gamma(\mathscr{B}),$$

proving the Cartan–Killing first structure Eq.(8.31).

Corollary 8.1. (Curvature Form Tranformation Law.) *Following the notations of* Prop.(8.1), *let* $\mathbf{A}$ *be a curvature form with associated covariant derivative* ∇ *and curvature form* $\mathbf{F}$. *Then under a gauge transformation, as in* Eq.(8.4), *the curvature form changes by a similarity transformation:*

$$\begin{cases} \hat{\theta} \mapsto \hat{\theta}' = \left(\hat{\theta}\right)^T \mathbf{U}, \\ \mathbf{F} \mapsto \mathbf{F}' = \mathbf{U}\mathbf{F}\mathbf{U}^\dagger. \end{cases} \tag{8.36}$$

Indeed, let $\sigma \in \Gamma(\mathscr{B})$ be a cross–section. Applying the Cartan–Killing first structure form of Eq.(8.31), recall Prop.(8.1),

$$\mathscr{F}_{\mu\nu}\sigma = -ig^{-1}\left[\nabla_{\mu},\nabla_{\nu}\right]\sigma. \tag{8.37}$$

On the other hand, by a *gauge transformation*, or a change of *vielbein*, the covariant derivative ∇ transforms as described by Eq.(8.12), namely:

$$\begin{cases} \hat{\theta} \mapsto \hat{\theta}' = \left(\hat{\theta}\right)^{\mathrm{T}}\mathbf{U}, \\ \nabla_{\mu}\sigma \mapsto \left(\nabla_{\mu}\sigma\right)' = \mathbf{U}\left(\nabla_{\mu}\sigma\right). \end{cases} \tag{8.38}$$

Thence, Eqs.(8.37, 8.38) implies the following transformation law $\mathscr{F}_{\mu\nu} \mapsto \mathscr{F}'_{\mu\nu}$ for the curvature form:

$$ig\mathscr{F}_{\mu\nu}\sigma = \left[\nabla_{\mu},\nabla_{\nu}\right]\sigma \mapsto ig\mathscr{F}'_{\mu\nu}\sigma' = \left(\left[\nabla_{\mu},\nabla_{\nu}\right]\sigma\right)' = \mathbf{U}\left(\left[\nabla_{\mu},\nabla_{\nu}\right]\sigma\right) = \mathbf{U}\left(ig\mathscr{F}_{\mu\nu}\sigma\right), \tag{8.39}$$

and by Eq.(8.4), we arrive at:

$$\mathscr{F}'_{\mu\nu}\sigma' = \mathscr{F}'_{\mu\nu}\left(\mathbf{U}\sigma\right) = \mathbf{U}\mathscr{F}_{\mu\nu}\sigma,\ \forall\sigma \in \Gamma\left(\mathscr{B}\right), \tag{8.40}$$

implying that:

$$\mathscr{F}'_{\mu\nu}\mathbf{U} = \mathbf{U}\mathscr{F}_{\mu\nu}. \tag{8.41}$$

Lastly, multiplying both sides of Eq.(8.41) by $\mathbf{U}^{\dagger}$ on the right, we deduce the change of curvature form under our *gauge transformation*,

$$\mathscr{F}'_{\mu\nu} = \mathbf{U}\mathscr{F}_{\mu\nu}\mathbf{U}^{\dagger}, \tag{8.42}$$

therefore proving our assertion.

Corollary 8.2. (Bianchi's Identity). *Following the notations of* Prop.(8.1)*, let* $\mathbf{A}$ *be a curvature form with associated covariant derivative* ∇ *and curvature form* $\mathbf{F}$*. Then the following algebraic identity holds good*[6]*:*

$$\nabla_{(\lambda}\mathscr{F}_{\mu\nu)} = 0. \tag{8.43}$$

For, denoting by $\varepsilon^{\mu\nu\lambda\rho}$ the Levi–Civita's totally anti–symmetric symbol, let the vector field $X = X^{\mu}\left(\partial/\partial x^{\mu}\right)$ be given by:

$$X^{\mu} := \frac{1}{3!}ig\varepsilon^{\mu\nu\lambda\rho}\left(\nabla_{\lambda}\mathscr{F}_{\mu\nu} + \nabla_{\nu}\mathscr{F}_{\lambda\mu} + \nabla_{\mu}\mathscr{F}_{\nu\lambda}\right). \tag{8.44}$$

Employing the Cartan–Killing first structure form, recalling Eq.(8.31) from Prop.(8.1), one can rewrite Eq.(8.44) as an operator equation:

$$X^{\mu} = \frac{1}{3!}\varepsilon^{\mu\nu\lambda\rho}\left(\nabla_{\lambda}\left[\nabla_{\mu},\nabla_{\nu}\right] + \nabla_{\nu}\left[\nabla_{\lambda},\nabla_{\mu}\right] + \nabla_{\mu}\left[\nabla_{\nu},\nabla_{\lambda}\right]\right) = \varepsilon^{\mu\nu\lambda\rho}\left[\nabla_{\mu},\left[\nabla_{\nu},\nabla_{\lambda}\right]\right], \tag{8.45}$$

and lastly, applying Jacobi's identity to Eq.(8.45), we deduce:

$$X^{\mu} = 0 \implies \nabla_{\lambda}\mathscr{F}_{\mu\nu} + \nabla_{\nu}\mathscr{F}_{\lambda\mu} + \nabla_{\mu}\mathscr{F}_{\nu\lambda} = 0, \tag{8.46}$$

as required.

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[6]Here and from now on, the symmetrizing notation for indices are adopted, so that:

$$\nabla_{(\lambda}\mathscr{F}_{\mu\nu)} = \frac{1}{3!}\left(\nabla_{\lambda}\mathscr{F}_{\mu\nu} + \nabla_{\nu}\mathscr{F}_{\lambda\mu} + \nabla_{\mu}\mathscr{F}_{\nu\lambda}\right).$$

8.5 Classical Yang–Mills Theories

A Non–Abelian field theory, living in the Minkowski spacetime M_4, obeying the action from an N–dimensional symmetry Lie group G (with complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$), and interacting with matter fields satisfying either Bose–Einstein or Fermi–Dirac statistics, is constituted by:

1. A *gauge potential*, mathematically realized by a connection form $\mathbf{A} := A_{\mu}^{(a)} \hat{\theta}^{\mu} \otimes T^{a} \in \Omega^{1}(M_4, \mathfrak{g})$, as in Def.(8.1), with associated covariant derivative ∇ and curvature form $\mathbf{F}$, the latter being detailed at Def.(8.3).
   The dynamics of the *gauge potential* $\mathbf{A}$ is governed by the following density of Lagrange's functional:
   $$\mathscr{L}_{\text{Gauge}}(\mathbf{A}, \nabla \mathbf{A}) = -\frac{1}{2} \operatorname{Tr}\left[\mathscr{F}_{\mu\nu} \mathscr{F}^{\mu\nu}\right] = -\frac{1}{4} F_{\mu\nu}^{(a)} F_{(a)}^{\mu\nu}. \tag{8.47}$$

2. An N–Tuple of Fermi–Dirac fields Ψ (recall §(8.3.1)), with Lagrange's functional density:
   $$\mathscr{L}_{\text{FD}}\left(\Psi, \bar{\Psi}; \nabla \Psi, \nabla \bar{\Psi}\right) = \bar{\Psi}\left(i\gamma^{\mu} \nabla_{\mu} - m\right) \Psi. \tag{8.48}$$

3. An N–Tuple of Bose–Einstein fields Φ (recall §(8.3.2)), whose dynamics is given by the Lagrange's functional density:
   $$\mathscr{L}_{\text{BE}}\left(\Phi, \bar{\Phi}; \nabla \Phi, \nabla \bar{\Phi}\right) = \left(\nabla_{\mu} \Phi\right)^{\dagger} \left(\nabla^{\mu} \Phi\right) - m^{2} \Phi^{\dagger} \Phi. \tag{8.49}$$

If the Lie group G is compact, then the non–Abelian theory, whose complete Lagrangian is defined by:

$$\mathscr{L}_{\text{YM}} := \mathscr{L}_{\text{Gauge}} + \mathscr{L}_{\text{FD}} + \mathscr{L}_{\text{BE}} = -\frac{1}{4} F_{\mu\nu}^{(a)} F_{(a)}^{\mu\nu} + \bar{\Psi}\left(i\gamma^{\mu} \nabla_{\mu} - m\right) \Psi + \left(\nabla_{\mu} \Phi\right)^{\dagger} \left(\nabla^{\mu} \Phi\right) - m^{2} \Phi^{\dagger} \Phi, \tag{8.50}$$

will be called *a Yang–Mills theory* with *gauge group G*, after C.-N. Yang and R. L. Mills [Yang and Mills(1954)].

The fundamental importance of Yang–Mills theories relies in the following deep result due to S. Lie and W. Killing:

stating that: we state and defer the prove to [O'Raifeartaigh(1986), §(3.5)].

8.6 Classical Theory

In this subsection, we are interested in computing the classical field equations and (covariant) conserved currents applying the Euler–Lagrange method.

At this stage, the reader is assumed to be familiar with relativistic wave equations for fields with arbitrary spin minimally coupled to background fields. If otherwise, as nothing is better than to learn form the masters themselves, see [Dyson and Derbes(2011)][7].

Therefore, the most interesting field equations for us are the ones governing the *gauge field–strength*.

[7] Other well–known presentations are [Bjorken and Drell(1964), Greiner(2012)].

8.6.1 Euler–Lagrange Field Equation

Using "old–school" covariant formulation of variational calculus[8], our equations of motion reads:

$$\nabla_\mu \left[\frac{\delta \mathscr{L}_{\text{YM}}}{\delta \left(\nabla_\mu A^\nu_{(a)} \right)} \right] = \frac{\delta \mathscr{L}_{\text{YM}}}{\delta A^\nu_{(a)}}, \tag{8.51}$$

where $A^{(a)}_\mu$ and $\nabla_\mu A^{(a)}_\nu$ are considered independent field variables by the functional derivatives.

From Eq.(8.47), the Lagrange's functional density for the *gauge–field*, one derives:

$$\frac{\delta \mathscr{L}_{\text{Gauge}}}{\delta \left(\nabla_\mu A^{(a)}_\nu \right)} = -\frac{1}{2} \left(\nabla^\alpha A^\beta_{(a)} - \nabla^\beta A^\alpha_{(a)} \right) \frac{\delta \left(\nabla^\alpha A^\beta_{(a)} - \nabla^\beta A^\alpha_{(a)} \right)}{\delta \left(\nabla_\mu A^{(a)}_\nu \right)} = -F^{\mu\nu}_{(a)}. \tag{8.52}$$

On the other hand, the Lagrangian for matter fields in Eqs.(8.48, 8.49) are independent from $\nabla_\mu A^{(a)}_\nu$.

Thence, now we should aim our attention in computing the variations of $\mathscr{L}_{\text{FD}}$ and $\mathscr{L}_{\text{BE}}$ only accordingly to the *gauge potential* $A^{(a)}_\mu$.

As a first step, expand the Lagrange's functional density for the N–Tuple of Fermi–Dirac Ψ spinor fields in terms of the covariant derivative by the replacement $\nabla_\mu \mapsto \partial_\mu + igA^{(a)}_\mu T^a$ in Eq.(8.48), yielding:

$$\mathscr{L}_{\text{FD}} = i\bar{\Psi}\gamma^\mu \partial_\mu \Psi - m^2 \bar{\Psi}\Psi - gA^{(a)}_\mu \bar{\Psi}\gamma^\mu T^a \Psi. \tag{8.53}$$

Similarly, for Bose–Einstein fields Φ belonging to the fundamental representation $\mathrm{SU}(N)$, one obtains:

$$\mathscr{L}_{\text{BE}} = \left(\partial_\mu \Phi \right)^\dagger \left(\partial^\mu \Phi \right) - m^2 \Phi^\dagger \Phi + ig \left(\partial_\mu \Phi \right)^\dagger A^\mu_{(a)} T^a \Phi - igA^{(a)}_\mu \Phi^\dagger T^a \left(\partial^\mu \Phi \right) + g^2 A^{(a)}_\mu A^{(b)\mu} \Phi^\dagger T^a T^b \Phi. \tag{8.54}$$

Lastly, we are able to easily compute the following functional derivatives:

$$\frac{\delta \mathscr{L}_{\text{FD}}}{\delta A^\nu_{(a)}} = -g\bar{\Psi}\gamma_\nu T^a \Psi, \tag{8.55}$$

and:

$$\frac{\delta \mathscr{L}_{\text{BE}}}{\delta A^\nu_{(a)}} = \underbrace{ig \left(\partial^\nu \Phi \right)^\dagger T^a \Phi + g^2 A_{\nu(b)} \Phi^\dagger T^b T^a \Phi}_{\textbf{(\#1)}} - \underbrace{ig\Phi^\dagger T^a \left(\partial^\nu \Phi \right) + g^2 A^\nu_{(b)} \Phi^\dagger T^a T^b \Phi}_{\textbf{(\#2)}}. \tag{8.56}$$

Interestingly, before one might be tempted to finally obtain our desired field equations, assembling Eq.(8.51) from the results obtained from Eqs.(8.52, 8.55, 8.56), note that the later variation can rewritten in covariant form.

Indeed, term (#**1**) from Eq.(8.56) is given, in covariant form, by:

$$ig \left(\nabla^\nu \Phi \right)^\dagger T^a \Phi = ig \left(\left(\partial^\nu \Phi \right)^\dagger - igA^\nu_{(b)} \Phi^\dagger T^b \right) T^a \Phi \tag{8.57}$$

$$= ig \left(\partial^\nu \Phi \right)^\dagger T^a \Phi + g^2 A_{\nu(b)} \Phi^\dagger T^b T^a \Phi, \tag{8.58}$$

[8] A comprehensive discussion of those methods can be found in [Anderson and Anderson(1967)] and [Lovelock and Rund(2012)]. Nonetheless, a more mathematically sophisticated reformulation of classical field theories, using the techniques from "jet–bundles," is exposed at length at [Binz and Sniatycki(2011), Olver(2012)].

while the analogous result also holds for term (#**2**),

$$ig\Phi^{\dagger}T^{a}\left(\nabla^{\nu}\Phi\right)=ig\Phi^{\dagger}T^{a}\left(\partial^{\nu}\Phi+igA^{\nu}_{(b)}T^{b}\Phi\right) \tag{8.59}$$

$$=ig\Phi^{\dagger}T^{a}\left(\partial^{\nu}\Phi\right)-g^{2}A^{\nu}_{(b)}\Phi^{\dagger}T^{a}T^{b}\Phi. \tag{8.60}$$

Henceforth, upon the substitution of Eqs.(8.58, 8.60) within the functional derivative obtained at Eq.(8.56), one arrives at:

$$\frac{\delta\mathscr{L}_{\mathrm{BE}}}{\delta A^{\nu}_{(a)}}=ig\left(\nabla^{\nu}\Phi\right)^{\dagger}T^{a}\Phi-ig\Phi^{\dagger}T^{a}\left(\nabla^{\nu}\Phi\right). \tag{8.61}$$

Consequently, upon the replacement of Eqs.(8.52, 8.55, 8.56) into the Euler–Lagrange form, as in Eq.(8.51), one deduces the equations of motion for the *gauge field–strength*,

$$\nabla^{\mu}F^{(a)}_{\mu\nu}=g\bar{\Psi}\gamma_{\nu}T^{a}\Psi+ig\left(\nabla_{\nu}\Phi\right)^{\dagger}T^{a}\Phi-ig\Phi^{\dagger}T^{a}\left(\nabla_{\nu}\Phi\right). \tag{8.62}$$

Remark 8.3. The equation of motion for the *gauge field* obtained above, namely, Eq.(8.62), is the generalization of the inhomogeneous Maxwell's equations for gauge theories whose symmetry group G is non–Abelian and semi-simple. Indeed, if one chooses the symmetry group to be $G=\mathrm{U}(1)$, and only takes into account the contributions coming Fermi–Dirac fields are taken into account, our developments above reduces to the Maxwell–Lorentz electrodynamics, which upon quantization, yields QED.

We close by taking notice that, with help from Bianchi's identity (recall Eq.(8.43)) proved in Cor.(8.2), and Jacobi's identity satisfied by Lie algebra–valued differential operators, one easily derives that:

$$\left[\nabla_{\mu},\nabla_{\nu}\right]\mathscr{F}_{\rho\sigma}=\left[\mathscr{F}_{\mu\nu},\mathscr{F}_{\rho\sigma}\right]. \tag{8.63}$$

Indeed, let $\mathscr{J}:=J^{(a)}_{\mu}\hat{\theta}^{\mu}\otimes T^{a}\in\Omega^{1}\left(M_{4},\mathfrak{g}\right)$ be the *covariant current density*:

$$J^{(a)}_{\mu}=g\bar{\Psi}\gamma_{\nu}T^{a}\Psi+ig\left(\nabla_{\nu}\Phi\right)^{\dagger}T^{a}\Phi-ig\Phi^{\dagger}T^{a}\left(\nabla_{\nu}\Phi\right), \tag{8.64}$$

which upon contraction with the Hermite generators T^{a} of our Lie algebra $\mathfrak{g}$, denote: $\mathscr{J}_{\mu}=J^{(a)}_{\mu}T^{a}$.

To obtain our conservation law for the covariant current density, first multiply both sides of Eq.(8.63) by $\eta^{\mu\rho}\eta^{\nu\sigma}$ to obtain:

$$\left[\nabla_{\mu},\nabla_{\nu}\right]\mathscr{F}^{\mu\nu}=\left[\mathscr{F}^{\mu\nu},\mathscr{F}_{\mu\nu}\right]=0, \tag{8.65}$$

and then note :

$$\left[\nabla_{\mu},\nabla_{\nu}\right]\mathscr{F}^{\mu\nu}=\nabla_{\mu}\nabla_{\nu}\mathscr{F}^{\mu\nu}-\nabla_{\nu}\nabla_{\mu}\mathscr{F}^{\mu\nu}=2\nabla_{\mu}\nabla_{\nu}\mathscr{F}^{\mu\nu}=0. \tag{8.66}$$

Finally, applying Eq.(8.66) into the field equation (recall Eq.(8.62)), one derives:

$$\nabla_{\mu}\mathscr{J}^{\mu}=0, \tag{8.67}$$

as desired.

Remark 8.4. The reason as to why the current $\mathscr{J}^{\mu}=J^{\mu}_{(a)}T^{a}$ defined in Eq.(8.64) is called *covariant* is simply it's written in a covariant fashion relatively to the connection $\mathbf{A}$. Nevertheless, in order to integrate Eq.(8.67) and obtain a *conserved charge*, from the divergence theorem, one needs to take into account the contributions from the non–Abelian gauge field itself. Indeed, since:

$$\nabla^{\mu}\mathscr{F}_{\mu\nu}=\partial^{\mu}\mathscr{F}_{\mu\nu}+ig\left[\mathscr{A}^{\mu},\mathscr{F}_{\mu\nu}\right]=\partial^{\mu}\mathscr{F}_{\mu\nu}-gc^{abc}A^{\mu(a)}F^{(b)}_{\mu\nu}T^{c}, \tag{8.68}$$

the spacetime conserved current density is:

$$\partial^{\mu}F^{(a)}_{\mu\nu}=gc^{abc}A^{\rho(b)}F^{(c)}_{\rho\nu}+g\bar{\Psi}\gamma_{\nu}T^{a}\Psi+ig\left(\nabla_{\nu}\Phi\right)^{\dagger}T^{a}\Phi-ig\Phi^{\dagger}T^{a}\left(\nabla_{\nu}\Phi\right).$$

8.6.2 Nöther's Canonical Current Density

The action functional of the Yang–Mills theory discussed above (cf. Eq.(8.50)), with each contribution explicitly expressing their functional dependence, is given by:

$$S_{\text{Matter}} = \int_{\Omega} d^4x \left[\mathscr{L}_{\text{Gauge}} (\mathbf{A}, \nabla \mathbf{A}) + \mathscr{L}_{\text{FD}} \left(\Psi, \bar{\Psi}; \nabla \Psi \right) + \mathscr{L}_{\text{BE}} \left(\Phi, \bar{\Phi}; \nabla \Phi, \nabla \Phi^{\dagger} \right) \right]. \tag{8.69}$$

Assuming that the Euler–Lagrange equations are satisfied (namely, the fields are *onshell*), a variation of Eq.(8.69), parametrized by some Lie algebra–valued spacetime function $x \in M_4 \mapsto \beta_{(a)}(x) T^a$, can be written as:

$$\delta S = \int_{\Omega} d^4x \nabla_{\mu} j^{\mu} = \int_{\partial \Omega} d\Sigma \, \mathbf{n}_{\mu} j^{\mu}, \tag{8.70}$$

where $d\Sigma$ is the area element of the hypersurface $\partial \Omega$, $\mathbf{n}_{\mu}$ the outward–pointing unit normal, and the *canonical current density* is given by:

$$j^{\mu} = \frac{\delta \mathscr{L}_{\text{Gauge}}}{\delta \left(\nabla_{\mu} A_{\nu}^{(b)} \right)} \frac{\delta A_{\nu}^{(b)}}{\delta \beta_a} + \frac{\delta \mathscr{L}_{\text{BE}}}{\delta \left(\nabla_{\mu} \Phi \right)} \frac{\delta \Phi}{\delta \beta_a} + \left(\frac{\delta \Phi}{\delta \beta_a} \right)^{\dagger} \frac{\mathscr{L}_{\text{BE}}}{\delta \left(\nabla_{\mu} \Phi^{\dagger} \right)} + \frac{\delta \mathscr{L}_{\text{FD}}}{\delta \left(\nabla_{\mu} \Psi \right)} \overline{\frac{\delta \Psi}{\delta \beta_a}}. \tag{8.71}$$

Remark 8.5. Note that, in Eq.(8.71) for the total covariant current density, there is no term involving directly the Clifford[9] conjugate associated to the N–tuple of fields Ψ, differently from the N–tuple of fields Φ. The reason lies simply in the fact that, for spinor fields, the Lagrangian (and obviously, the field equations themselves) contains only first order differential operators.

Let $x \in M_4 \mapsto \beta_{(a)}(x) \in \mathbb{R}$ be the family of spacetime functions parametrizing the "local" *gauge transformation*:

$$x \in M_4 \mapsto \mathbf{U}(x) = \exp\left(-ig\beta_a(x) T^a \right) \in G. \tag{8.72}$$

Thence, the corresponding infinitesimal transformations for the N–tuple of fields Φ and Ψ (and their conjugates) are, respectively:

$$\begin{cases} \delta \Phi = -ig\beta_a(x) T^a \Phi, \\ \delta \Phi^{\dagger} = ig\beta_a(x) \Phi^{\dagger} T^a, \\ \delta \Psi = -ig\beta_a(x) T^a \Psi, \\ \delta \bar{\Psi} = ig\beta_a(x) T^a \bar{\Psi}. \end{cases} \tag{8.73}$$

On the other hand, recalling Eq.(8.23) for the transformation law of the connection forms, one deduces that:

$$\delta A_{\mu}^{a} = gc^{abc} A^{\rho(b)} F_{\rho\mu}^{(c)}. \tag{8.74}$$

Replacing the variations, from Eqs.(8.73, 8.74), into the Nöther's formula obtained from the variational calculus at Eq.(8.71), yields:

$$j_{\mu}^{(a)} = gc^{abc} A^{\rho(b)} F_{\rho\nu}^{(c)} + g\bar{\Psi}\gamma_{\mu} T^a \Psi + ig \left(\nabla_{\mu} \Phi \right)^{\dagger} T^a \Phi - ig\Phi^{\dagger} T^a \left(\nabla_{\mu} \Phi \right), \tag{8.75}$$

the canonical current density.

[9] We call *Clifford conjugate*, since each component of the N–tuple of fields Ψ belongs to a representation of the (complexified) Clifford algebra $\mathscr{C}\ell(1,3)$, being therefore the analog (indeed, the *generalization*) of the Hermite conjugate.

8.7 Quantum Yang–Mills Theories

Let $\mathscr{M}$ be the configuration space of the set of fields $\left\{A_\mu^{(a)}, \Phi, \Psi\right\}$ for the *classical* Yang–Mills theory, with structural group G, discussed at §(8.5).

Recall that, from Hamilton's stationary action principle [Hamilton(1833)], the classical dynamics of those fields is governed by the Lagrange's functional densities of Eqs.(8.47, 8.48, 8.49). The classical field equations, following the Euler–Lagrange method, are:

$$\begin{cases} \nabla_\mu \nabla^\mu \Phi = m^2 \Phi, \\ i\gamma^\mu \nabla_\mu \Psi = m\Psi, \\ \nabla^\mu \mathscr{F}_{\mu\nu} = \mathscr{J}_\nu, \end{cases} \tag{8.76}$$

where the *gauge field–strength*, generalizing Faraday electromagnetic tensor, is the *curvature form* $\mathscr{F} := F_{\mu\nu}^{(a)} \left(\hat{\theta}^\mu \wedge \hat{\theta}^\nu\right) \otimes T^a$ (defined by Eq.(8.27) in §(8.3)), associated to the *connection* $\mathscr{A} = A_\mu^{(a)} \hat{\theta}^\mu \otimes T^a$.

From the perspective of the *quantum* action principle, further developed by Feynman [Feynman and Brown(2005)] after Dirac [Dirac(2005)], the existence of a symmetry group G, acting over $\mathscr{M}$ according to the transformation laws of Eqs.(8.4, 8.15), requires the introduction of a *gauge–fixing* procedure, so as one might define a well–posed pathintegral over the quotient space $\mathscr{M}/G$[10].

Remark 8.6. Classical field theories with phasespace[11] $T^*(\mathscr{M})$, exhibiting symmetries under the action of the structure group G (called by physicists the *gauge group*) accordingly to the infinitesimal transformations of Eqs.(8.4, 8.15), from the Hamiltonian viewpoint of constrained dynamics [Dirac(2001)], should be quantized rigorously by the methods of BRST geometric quantization [Henneaux and Teitelboim(2020)]. The generalization of those methods for more sophisticated theories, whose symmetry group is replaced only by the infinitesimal transformations laws, generated by Hermite operators whose graded Lie algebra is not closed, requires the more formal methods of Batalin–Vilkovisky, also known as the BV–BRST geometric quantization [Barnich et al.(2000)Barnich, Brandt, and Henneaux].

In order to do so, following [Berazin(2012), Faddeev(2018)],

1. Let the Lagrangian density $\mathscr{L}_{\mathrm{YM}}$ of Yang–Mills theory (see Eq.(8.50)) be supplemented by a gauge–fixing term, for given arbitrary real $\xi > 0$:

$$\mathscr{L}_{\mathrm{GF}} := -\frac{1}{\xi} \mathrm{Tr}\left[\left(\partial_\mu \mathscr{A}^\mu\right)^2\right] = -\frac{1}{2\xi}\left(\partial_\mu A^\mu_{(a)}\right)^2. \tag{8.77}$$

2. In addition, let the Faddeev–Popov [Faddeev and Popov(1967)] and de Witt [DeWitt(1967)] *geisperfeld* $\mathbf{c}^{(a)}(x)$ and $\bar{\mathbf{c}}^{(a)}(x)$ play the role of the gauge–fixing functional determinant, the effect of which reduces to the introduction of an *auxiliary* Lagrangian density, *only within the pathintegral*, for those fields:

$$\mathscr{L}_{\mathrm{geisper}} = \bar{\mathbf{c}}^{(a)} \partial^\mu \left(\delta_{ab}\partial_\mu - g c^{abc} A_\mu^{(c)}\right) \mathbf{c}^{(b)} = \bar{\mathbf{c}}^{(a)} \nabla_\mu \mathbf{c}^{(a)}. \tag{8.78}$$

[10] Geometrically, the action of G over each "point" of M generates the "orbit" of physically indistinguishable configurations.

[11] Letting $\mathscr{M}$ be the configuration space of a Hamiltonian system, the co–tangent bundle $T^*(\mathscr{M})$ defines the theory *phasespace*, the area over which the symplectic structure is built upon. See [Arnold(2013), Arnold et al.(2007)Arnold, Kozlov, and Neishtadt].

Therefore, letting the sources associated to the set of *quantum* fields[12] $\left\{A_\mu^{(a)}, \Phi, \Phi^\dagger, \Psi, \bar{\Psi}\right\}$ and, including the *geisperfeld* $\left\{\mathbf{c}^{(a)}, \bar{\mathbf{c}}^{(a)}\right\}$, be denoted, respectively, by the collection:

$$J := \left\{(J_\mathbf{A})^a_\mu, J_\Phi, J_{\Phi^\dagger}, J_\Psi, J_{\bar{\Psi}}, (J_\mathbf{c})^a, (J_{\bar{\mathbf{c}}})^a\right\}, \tag{8.79}$$

the generating functional $Z[J]$ is given by:

$$Z[J] = \int_{\mathscr{M}/G} \mathscr{D}\left[A_\mu^{(a)}, \Phi, \Phi^\dagger, \Psi, \bar{\Psi}, \mathbf{c}^{(a)}, \bar{\mathbf{c}}^{(a)}\right] \exp i\left\{\int d^4x \mathscr{L}_\mathrm{E} + \mathscr{L}_\mathrm{BE} + \mathscr{L}_\mathrm{FD}\right. \tag{8.80}$$

$$\left. + A_\mu^{(a)} (J_\mathbf{A})^\mu_{(a)} + J^\dagger_\Phi \Phi + \Phi^\dagger J_\Phi + J_\Psi \Psi + \bar{\Psi} J_\Psi + (J_\mathbf{c})^*_a \mathbf{c}^{(a)} + \bar{\mathbf{c}}^{(a)} (J_{\bar{\mathbf{c}}})_a\right\}, \tag{8.81}$$

where the *effective* Lagrangian density for the gauge potential follows from the contributions of Eqs.(8.50, 8.77, 8.78):

$$\mathscr{L}_E = -\frac{1}{4} F^{(a)}_{\mu\nu} F^{\mu\nu}_{(a)} - \frac{1}{2\xi}\left(\partial \cdot A^{(a)}_\mu\right)^2 + \partial^\mu \bar{\mathbf{c}}^{(a)} \nabla_\mu \mathbf{c}_{(a)}. \tag{8.82}$$

8.8 BRST Symmetries

$$\begin{cases} \delta_\theta A_\mu^{(a)} = \theta g^{-1} \nabla_\mu \zeta^{(a)}, \\ \delta_\theta \zeta^{(a)} = \theta g^{-1} c^{abc} \zeta^{(b)} \zeta^{(c)}, \\ \delta_\theta \bar{\zeta}^{(a)} = -\theta g^{-1} \frac{1}{\xi} (\partial \cdot A)^{(a)}. \end{cases} \tag{8.83}$$

$$\delta_\varepsilon A_\mu^{(a)} := \delta_\theta A_\mu^{(a)} = \frac{\theta}{g}\left(D_\mu c\right)^{(a)} = \frac{1}{g} \partial_\mu \varepsilon^{(a)} - g \mathfrak{f}^{abc} A_\mu^{(b)} \varepsilon^{(c)}, \tag{8.84}$$

where:

$$\varepsilon^{(a)}(x) := \theta c^{(a)}(x) \in \mathbb{R},\ \forall x \in \mathbb{R}^{(1,3)}, \tag{8.85}$$

is a cross–section on the bundle over which the gauge theory is defined, with values in a commutative field, since the product of Grassmann–numbers θ and $c^{(a)}(x)$ yields a commutative field.

One recognizes in Eq.(8.84) a *gauge transformation* parametrized by $\varepsilon^{(a)}(x)$.

Thence, variations described by Eq.(8.83) are such that the Lagrange's functional density for Yang–Mills field, namely Eq.(**??**), is invariant under the infinitesimal transformation:

$$\delta_\theta(\mathscr{L}_\mathrm{YM}) = \delta_\varepsilon(\mathscr{L}_\mathrm{YM}) = 0. \tag{8.86}$$

Moreover, taking notice of the following facts (where Jacobi's identity for the structure constants of $\mathfrak{g}$ and the transformations of Eq.(8.83) are systematically applied):

$$\delta_\theta\left(\frac{1}{2}\mathfrak{f}^{abc} c^{(b)} c^{(c)}\right) = \mathfrak{f}^{abc}\left(\delta_\theta c^{(b)}\right) c^{(c)} = \frac{\theta}{2} \mathfrak{f}^{abc} \mathfrak{f}^{bpq} c^{(p)} c^{(q)} c^{(c)} = \frac{\theta}{3!} \mathfrak{f}^{(abc} \mathfrak{f}^{bpq)} c^{(p)} c^{(q)} c^{(c)} = 0, \tag{8.87}$$

[12]Now, outside the pathintegral, those are distribution–valued operators acting on the Hilbert state–space of the theory, while within the pathintegral, those are still considered as *classical*–valued fields.

$$\delta_\theta\left(D_\mu c^{(a)}\right)=D_\mu\left(\delta_\theta c^{(a)}\right)-g\mathfrak{f}^{abc}\left(\delta_\theta A_\mu^{(b)}\right)c^{(c)}=\frac{1}{2}\theta D_\mu\left(\mathfrak{f}^{abc}c^{(b)}c^{(c)}\right)-\theta\mathfrak{f}^{abc}\left(D_\mu c^{(b)}\right)c^{(c)}=0, \quad (8.88)$$

$$\delta_\theta\left(\partial_\mu A_{(a)}^\mu\right)=\frac{\theta}{g}\partial_\mu D^\mu c_{(a)}=0. \quad (8.89)$$

Now, using Eqs.(8.87, 8.88, 8.89), one is able to take apply the transformations, whose infinitesimal form are given by Eq.(8.83), to the effective Lagrangian (recall Eq.(**??**)):

$$\delta_\theta\mathscr{L}_{\mathrm{E}}^{(1)}=-\frac{\theta}{g\xi}\left(\partial_\mu A^{\mu(a)}\right)\partial_\nu D^\nu c^{(a)}-\frac{\theta}{g\xi}\partial_\mu\left(\partial_\nu A^{\nu(a)}\right)D_\mu c^{(a)}=-\partial_\mu\left[\frac{\theta}{g\xi}\left(\partial_\nu A^{\nu(a)}\right)D_\mu c^{(a)}\right]. \quad (8.90)$$

On the other hand, as our final goal is to arrive at a generalization of the Ward–Takahashi identities [Ward(1950), Takahashi(1957)] for non–Abelian gauge theories. Within the framework of the pathintegral formalism, our final result should be valid non–perturbatively.

Therefore, it will prove itself more convenient to modify the *gauge–fixing term* of the Lagrangian from Eq.(**??**) to:

$$\mathscr{L}_{\mathrm{GF}}:=\frac{\xi}{2}\mathbf{F}^{(a)}\mathbf{F}^{(a)}+\left(\partial_\mu\mathbf{F}^{(a)}\right)A^{\mu(a)}, \quad (8.91)$$

where $\mathbf{F}^{(a)}$ is nothing except an auxiliary field.

Indeed, applying the Euler–Lagrange equation to our new *gauge–fixing $\mathscr{L}_{\mathrm{GF}}$ term*, one sees that:

$$\xi\mathbf{F}^{(a)}=\frac{\delta\mathscr{L}_{\mathrm{GF}}}{\delta\mathbf{F}^{(a)}}=\partial_\mu\left[\frac{\delta\mathscr{L}_{\mathrm{GF}}}{\delta\left(\partial^\mu\mathbf{F}^{(a)}\right)}\right]=\partial_\mu A^{\mu(a)}, \quad (8.92)$$

which upon replacement into Eq.(**??**), yields our former effective Lagrangian $\mathscr{L}_{\mathrm{E}}^{(1)}$.

Thence, one would be wise in defining the final form of the effective Lagrange's functional density as:

$$\mathscr{L}_{\mathrm{E}}=-\frac{1}{4}F_{\mu\nu}^{(a)}F_{(a)}^{\mu\nu}+\frac{\xi}{2}\mathbf{F}^{(a)}\mathbf{F}^{(a)}+\left(\partial_\mu\mathbf{F}^{(a)}\right)A^{\mu(a)}+\partial^\mu\bar{c}^{(a)}D_\mu c^{(a)}, \quad (8.93)$$

which is invariant under the set of transformations whose infinitesimal form are given by the variations:

$$\delta_\theta A_\mu^{(a)}=\frac{\theta}{g}\left(D_\mu c\right)^{(a)},\ \delta_\theta c^{(a)}=\frac{\theta}{2}\mathfrak{f}^{abc}c^{(b)}c^{(c)},\ \delta_\theta\bar{c}^{(a)}=-\frac{\theta}{g}\mathbf{F}^{(a)},\ \delta\mathbf{F}^{(a)}=0. \quad (8.94)$$

Those transformations are known as BRST, after [Becchi et al.(1976)Becchi, Rouet, and Stora] and [Tyutin(1975)].

8.9 Slavnov–Taylor: *The "Master" Formula*

In the following a functional derivation of the Slavnov–Taylor identities [Slavnov(1972), Taylor(1971)] will be presented based on the BRST symmetries of *gauge theories* whose symmetry group is non–Abelian.

As it should be emphasized, those identities are a generalization of the Ward–Takahashi, studied in our former problem sheet, from Quantum Electrodynamics, which is an Abelian *gauge theory*, to theories of the Yang–Mills family. The latter are understood, at least in the language used in the present notes, as *gauge theories* such that the symmetry group of their principal–bundle is compact and non–Abelian.

Given the necessity of introducing the Faddeev–Popov *geisperfeld* [Faddeev(2018)], one is not able to derive the Slavnov–Taylor identities directly from a variation of the Legendre transform of the generating functional associated to connected Green's functions.

Instead, one should apply the invariance over BRST transformations within the functional integral defining the generating

$$\mathscr{S} := \mathscr{I}\left[\mathscr{J}^{(a)}_{\mu}, \mathbf{J}^{(a)}, \zeta^{(a)}, \bar{\zeta}^{(a)}, \mathfrak{J}^{(a)}_{\mu}, \mathfrak{J}^{(a)}\right] \tag{8.95}$$

$$= \mathscr{J}^{(a)}_{\mu} A^{\mu}_{(a)} + \mathbf{J}^{(a)}\mathbf{F}^{(a)} + i\left(\bar{\zeta}^{(a)} c^{(a)} - \bar{c}^{(a)}\zeta^{(a)}\right) + \mathfrak{J}^{\mu(a)}\left(D_{\mu}c\right)^{(a)} + \frac{g}{2}\mathfrak{f}^{abc}\mathfrak{J}^{(a)}c^{(b)}c^{(c)}. \tag{8.96}$$

$$\mathscr{Z}[\mathscr{S}] := \exp i\mathscr{W}[\mathscr{S}]$$
$$= \int_{\mathscr{M}/\mathrm{SU}(N)} \mathscr{D}\left[A^{(a)}_{\mu}(x), \mathbf{F}^{(a)}, c^{(a)}(x), \bar{c}^{(a)}(x)\right] \exp i\int d^4x \left(\mathscr{L}_{\mathrm{E}} + \mathscr{I}\right).$$

$$\int d^4x \left[\frac{\delta\Gamma}{\delta A^{(a)}_{\mu}(x)}\frac{\delta\Gamma}{\delta K^{\mu}_{(a)}(x)} + \frac{\delta\Gamma}{\delta c^{(a)}(x)}\frac{\delta\Gamma}{\delta K_{(a)}(x)} - \mathbf{F}^{(a)}(x)\frac{\delta\Gamma}{\delta\bar{c}^{(a)}(x)}\right] = 0. \tag{8.97}$$

$$\int d^4p d^4q\, \delta^{(4)}(p+q)\left(\frac{\delta\Gamma}{\delta\hat{A}^{(a)}_{\mu}(p)}\frac{\delta\Gamma}{\delta\hat{\mathfrak{J}}^{\mu(a)}(q)} + \frac{\delta\Gamma}{\delta\hat{c}^{(a)}(p)}\frac{\delta\Gamma}{\delta\hat{\mathfrak{J}}_{(a)}(q)}\right) = \int d^4p\,\hat{\mathbf{F}}^{(a)}(p)\frac{\delta\Gamma}{\delta\hat{\bar{c}}^{(a)}(p)}. \tag{8.98}$$

The Slavnov–Taylor identity is obtained, in functional form, by applying the differential operator:

$$\frac{\delta^2}{\delta\mathbf{F}^{(b)}(p)\,\delta c^{(c)}(q)}, \tag{8.99}$$

to Eq.(8.98), yielding:

$$\frac{\delta^2\Gamma}{\delta\hat{\mathbf{F}}^{b}(p)\,\delta\hat{A}^{(a)}_{\mu}(q)}\frac{\delta^2\Gamma}{\delta\hat{c}^{(c)}(q)\,\delta\mathfrak{J}^{\mu(a)}(p)} = \frac{\delta^2\Gamma}{\delta\hat{c}^{(c)}(q)\,\delta\hat{\bar{c}}^{(b)}(p)}. \tag{8.100}$$

8.10 Interactions

Let us recall the *effective* density of Lagrange functional for a Yang–Mills theory, defined on the principal–bundle:

$$\mathbf{P} := \left(\mathbb{R}^{(1,3)} \times \mathrm{SU}(N), \mathbb{R}^{(1,3)}, \pi_1, \mathrm{SU}(N)\right), \tag{8.101}$$

where the mapping onto the base space π_1 is simply the projection into the first factor, as defined by Eq.(**??**).

The module of cross–sections over the principal–bundle **P** will be denoted, as usual, by $\Gamma(\mathbf{P})$.

Let the Yang–Mills connection over **P** be given by the covariant derivative: $\nabla : \Gamma(\mathscr{B}) \times \Gamma(\mathscr{B}) \longrightarrow \mathrm{End}\Gamma(\mathscr{B})$, defining a connection over our principal–bundle.

$$\mathscr{L}_{\mathrm{E}} = \frac{1}{2g^2}\mathrm{Tr}\left(\mathscr{F} \wedge \star\mathscr{F}\right) - \frac{1}{2\xi}\left(\partial\cdot\mathbf{A}\right)^2 + \partial^{\mu}\bar{c}^{(a)}D_{\mu}c^{(a)}. \tag{8.102}$$

In components, the effective Lagrange's functional density reads:

$$\mathscr{L}_{\mathrm{E}} = -\frac{1}{4} F^{(a)}_{\mu\nu} F^{\mu\nu}_{(a)} - \frac{1}{2\xi}\left(\partial^{\mu} A^{(a)}_{\mu}\right)^{2} + \partial^{\mu}\bar{c}^{(a)}\left(\partial_{\mu} + igA^{(a)}_{\mu}\mathfrak{t}_{(a)}\right)c^{(a)},$$

where:

$$\mathscr{F} =: \frac{1}{2} F^{(a)}_{\mu\nu}\left(dx^{\mu}\wedge dx^{\nu}\right)\otimes T^{a} \in \Omega\left(\mathbb{R}^{(1,3)}, \mathfrak{g}\right), \tag{8.103}$$

defines the components of the exterior 2–form of curvature, such that:

$$F^{(a)}_{\mu\nu}(x) = 2\frac{\partial}{\partial x^{[\mu}} A^{(a)}_{\nu]} - g\mathfrak{f}^{abc} A^{(b)}_{\mu}(x) A^{(c)}_{\nu}(x). \tag{8.104}$$

$$\mathscr{L}_{\mathrm{I}} \coloneqq \mathscr{L}_{\mathrm{E}} - \mathscr{L}_{0} \tag{8.105}$$

$$= g\mathfrak{f}^{abc}\left(\partial_{\mu} A^{(a)}_{\nu}\right) A^{\mu(b)} A^{\nu(c)} - \frac{g^{2}}{4!}\mathfrak{f}^{abp}\mathfrak{f}^{cdp} A^{(a)}_{\mu} A^{(b)}_{\nu} A^{\mu(c)} A^{\nu(d)} - g\mathfrak{f}^{abc} A^{(a)}_{\mu}\left(\partial^{\mu}\bar{c}^{(c)}\right)c^{(b)}. \tag{8.106}$$

$$S_{\mathrm{I}} \coloneqq \int d^{4}x\,\mathscr{L}_{\mathrm{I}}.$$

$$\mathscr{L}^{(3)}_{I}(x_1,x_2,x_3) \coloneqq g\mathfrak{f}^{abc}\left(\partial_{\mu} A^{(a)}_{\nu}\right)(x_1) A^{\mu(b)}(x_3) A^{\nu(c)}(x_3) - g\mathfrak{f}^{abc} A^{\mu(a)}(x_1)\frac{\partial\bar{c}^{(c)}(x_2)}{\partial x^{\mu}_{2}} c^{(b)}(x_3), \tag{8.107}$$

$$\mathscr{L}^{(4)}_{I}(x_1,x_2,x_3,x_4) \coloneqq -\frac{g^{2}}{4!}\mathfrak{f}^{abp}\mathfrak{f}^{cdp} A^{(a)}_{\mu}(x_1) A^{(b)}_{\nu}(x_2) A^{\mu(c)}(x_3) A^{\nu(d)}(x_4).$$

$$S_{\mathrm{I}} \coloneqq \int d^{4}x_1 d^{4}x_2 d^{4}x_3\,\delta^{(3)}(x_1+x_2+x_3)\,\mathscr{L}^{(3)}_{I}(x_1,x_2,x_3), \tag{8.108}$$

$$+\int d^{4}x_1 d^{4}x_2 d^{4}x_3 d^{4}x_4\,\delta^{(3)}(x_1+x_2+x_3+x_4)\,\mathscr{L}^{(4)}_{I}(x_1,x_2,x_3,x_4). \tag{8.109}$$

$$\hat{S}_{\mathrm{I}} \coloneqq (2\pi)^{4}\delta^{(3)}(p_1+p_2+p_3)\int d^{4}x_1 d^{4}x_2 d^{4}x_3\, e^{i(p_1\cdot x_1+p_2\cdot x_2+p_3\cdot x_3)}\mathscr{L}^{(3)}_{I}(x_1,x_2,x_3) \tag{8.110}$$

$$+(2\pi)^{4}\delta^{(3)}(p_1+p_2+p_3+p_4)\int d^{4}x_1 d^{4}x_2 d^{4}x_3 d^{4}x_4\, e^{i(p_1\cdot x_1+p_2\cdot x_2+p_3\cdot x_3+p_4\cdot x_4)}\mathscr{L}^{(4)}_{I}(x_1,x_2,x_3,x_4). \tag{8.111}$$

$$\hat{S}_{\mathrm{I}} = -i(p_1)_{\mu}\hat{A}^{(a)}_{\nu}(p_1)\hat{A}^{\mu(b)}(p_2)\hat{A}^{\nu(c)}(p_3) - i(p_3)^{\mu}\hat{A}^{(a)}_{\mu}(p_1)\hat{\bar{c}}^{(c)}(p_3)\hat{c}^{(b)}(p_2) \tag{8.112}$$

$$-\frac{g^{2}}{4!}\mathfrak{f}^{abp}\mathfrak{f}^{cdp}\hat{A}^{(a)}_{\mu}(p_1)\hat{A}^{(b)}_{\nu}(p_2)\hat{A}^{\mu(c)}(p_3)\hat{A}^{\nu(d)}(p_4). \tag{8.113}$$

$$\Gamma^{(abc)}_{\mu\nu\lambda}(p_1,p_2,p_3) = \frac{1}{i^{3}}\left[\!\!\left[\frac{\delta^{3}\hat{S}_{\mathrm{I}}}{\delta\hat{A}^{\mu}_{(a)}(p_1)\,\delta\hat{A}^{\nu}_{(b)}(p_2)\,\delta\hat{A}^{\lambda}_{(c)}(p_3)}\right]\!\!\right]_{\mathbf{A}=0} \tag{8.114}$$

$$= \mathfrak{f}^{abc}\left[\eta^{\mu\nu}(p_1-p_2)^{\lambda} + \eta^{\nu\lambda}(p_2-p_3)^{\mu} + \eta^{\lambda\mu}(p_3-p_1)^{\nu}\right]. \tag{8.115}$$

$$\Gamma^{(abcd)}_{\mu\nu\lambda\rho}\left(p_1,p_2,p_3\right)=\frac{1}{i^4}\left[\!\!\left[\frac{\delta^4\hat{S}_{\mathrm{I}}}{\delta\hat{A}^{\mu}_{(a)}\left(p_1\right)\delta\hat{A}^{\nu}_{(b)}\left(p_2\right)\delta\hat{A}^{\lambda}_{(c)}\left(p_3\right)\delta\hat{A}^{\rho}_{(d)}\left(p_4\right)}\right]\!\!\right]_{\mathbf{A}=0} \tag{8.116}$$

$$=-g^2\left[\mathfrak{f}^{abp}\mathfrak{f}^{cdp}\left(\eta^{\mu\lambda}\eta^{\nu\rho}-\eta^{\mu\rho}\eta^{\nu\lambda}\right)+\mathfrak{f}^{acp}\mathfrak{f}^{dpb}\left(\eta^{\mu\rho}\eta^{\nu\rho}-\eta^{\mu\nu}\eta^{\lambda\rho}\right)\right. \tag{8.117}$$

$$\left.+\mathfrak{f}^{2abp}\mathfrak{f}^{bcp}\left(\eta^{\mu\nu}\eta^{\lambda\rho}-\eta^{\mu\lambda}\eta^{\nu\rho}\right)\right]. \tag{8.118}$$

$$\Gamma^{(abc)}_{\mu}\left(p_1,p_2,p_3\right)=-\frac{1}{i^3}\left[\!\!\left[\frac{\delta^3\hat{S}_{\mathrm{I}}}{\delta\hat{A}^{\mu}_{(a)}\left(p_1\right)\delta\bar{\hat{c}}_{(c)}\left(p_3\right)\delta\hat{c}_{(b)}\left(p_2\right)}\right]\!\!\right]_{\mathbf{A}=\sigma=\bar{\sigma}=0}. \tag{8.119}$$

Remark. It's worth to mention that the presence of a minus sign in Eq.(8.119), which taken in the sights of Eq.(8.114), deservers an explanation. Since the *geisperfeld* $\hat{c}_{(b)}\left(p_2\right)$ is Grassmann–valued, the law of multiplication between the latter and their conjugate $\bar{\hat{c}}_{(c)}\left(p_3\right)$, requires the Lie algebra indices b and c be exchanged from the left–hand side of Eq.(8.119) to the right–hand side. The reason being, of course, the composition law for generic Grassmann–valued fields, say $\zeta^{(a)}\left(x\right)$, such that:

$$x\in\mathbb{R}^{(1,3)}\mapsto\bar{\zeta}^{(a)}\left(x\right)\zeta^{(b)}\left(x\right)\in\mathbb{R},$$

defines a real–valued field, employed at the functional derivative to obtain the above 3–vertex Green's function.

In any case, since formalities are better left to lawyers, Eq.(8.119) yields:

$$\Gamma^{(abc)}_{\mu}\left(p_1,p_2,p_3\right)=g\mathfrak{f}^{abc}\left(p_3\right)^{\mu}.$$

Appendix A

$\int d\Omega^{(n)}$: The Angle Who Wished to Be Solid

Recall that, in these notes, we are denoting by $\mathbf{E}^N := \left(\mathbb{R}^N, \langle\, , \,\rangle\right)$ the N–dimensional Euclidean space, whose canonical inner product is defined by:

$$\mathbf{x}, \mathbf{y} \in \mathbb{R}^N \mapsto \langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^T \mathbf{y} = \sum_{i=1}^{N} x_i y_i, \tag{A.1}$$

where: $\mathbf{x} = (x_1, ..., x_N)^T$ and $\mathbf{y} = (y_1, ..., y_N)^T$ are, respectively, the matrix representations of $\mathbf{x}$ and $\mathbf{y}$.

Letting the norm of an element $\mathbf{x} \in \mathbf{E}^N$ as $\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{y} \rangle}$, the space $\left(\mathbf{E}^N, \|\cdot\|\right)$ acquires the structure of a Banach space.

Now, let the area of an n–dimensional hypersphere $\mathbb{S}^n$, immersed into $\mathbf{E}^{n+1}$ by:

$$\mathbb{S}^n := \left\{ \|\mathbf{x}\| = 1 \,\middle|\, \mathbf{x} = (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} \right\} \hookrightarrow \mathbf{E}^{n+1}, \tag{A.2}$$

be called the n–dimensional *unit solid–angle*, defined by:

$$\int d\Omega^{(n)} := \int_{x_1^2 + ... + x_{n+1}^2 = 1} dx_1 ... dx_{n+1}. \tag{A.3}$$

Remark A.1. The above nomenclature is outdated at an archeological scale. Nonetheless, it's still alive in the common language of the ordinary physicists [Whittaker(1988)]. Under the assumption that Planck's principle[1] holds true, I afford to pay my due respect to tradition.

Applying gaussian integration in n–dimensions, we shall prove that:

$$\int d\Omega^{(n)} = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \tag{A.4}$$

where $\Gamma(\zeta)$ is the *gamma–function*, cf. Eq.(A.11) of Def.(A.1) in App.(A.1), and the discussions therein.

Indeed, recall from standard gaussian integrals that:

[1] "A new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die and a new generation grows up that is familiar with it."
– Max Planck, Scientific autobiography, 1950, p. 33, 97.

$$\int_{\mathbf{E}^n} d^n\xi \, e^{-\left(\xi_1^2+\ldots+\xi_n^2\right)} = \left(\int_{-\infty}^{\infty} d\xi \, e^{-\xi^2}\right)^n = \pi^{n/2}. \tag{A.5}$$

On the other hand, using polar–coordinates in $\mathbf{E}^n$, one obtains:

$$\begin{aligned}
\int_{\mathbf{E}^n} d^n\xi \, e^{-\left(\xi_1^2+\ldots+\xi_n^2\right)} &= \left(\int d\Omega^{(n)}\right) \times \int_0^{\infty} d\rho \, \rho^{n-1} \exp\left(-\rho^2\right) && \text{(A.6)}\\
&= \frac{1}{2}\int_{-\infty}^{\infty} d\rho \, \rho^{n-1} \exp\left(-\rho^2\right) \times \left(\int d\Omega^{(n)}\right) && \text{(A.7)}\\
&= \left(\frac{1}{2}\int_0^{\infty} dy \, y^{n/2-1} e^{-y}\right) \times \left(\int d\Omega^{(n)}\right) && \text{(A.8)}\\
&= \frac{1}{2}\Gamma\left(\frac{n}{2}\right) \times \left(\int d\Omega^{(n)}\right), && \text{(A.9)}
\end{aligned}$$

and thence Eq.(A.4) follows upon equating the left–hand sides of Eqs.(A.5, A.9).

A.1 *A Tour in* $B\left(\zeta_1, \zeta_2\right)$ *and* $\Gamma\left(z\right)$

This Appendix is a brief tour in the wonderland of holomorphic functions.

Our rides will cover some definitions and well–known theorems about the beta $B\left(\zeta_1, \zeta_2\right)$ and gamma $\Gamma\left(\zeta_j\right)$ functions.

As we shall see, these functions are related to each other in a non–trivial way. Even more surprisingly, they are holomorphic for all $\mathfrak{Re}\left(\zeta_j\right) > 0$ ($j = 1, 2$), belonging, therefore, to the "miraculous" half–plane of the complex numbers[2]:

$$\mathbb{H} := \left\{\zeta \in \mathbb{C} \,\middle|\, \mathfrak{Re}\left(\zeta\right) > 0\right\}. \tag{A.10}$$

Definition A.1. Let $s > 0$ be a positive real number. Then the *gamma–function* is defined by the integral:

$$\Gamma\left(s\right) := \int_0^{\infty} d\tau \, e^{-\tau} \tau^{s-1}. \tag{A.11}$$

Lemma A.1. *The mapping $s \in \mathbb{R}_{>0} \mapsto \Gamma\left(s\right)$, defined by the function of* Eq.(A.11)*, can be analytically continued to $\mathbb{H}$. Moreover, Γ is still functionally defined by Eq.(*A.11)*, now generalized to a mapping $\zeta \in \mathbb{H} \mapsto \Gamma\left(\zeta\right) \in \mathbb{C}$.*

For an elementary proof of the latter statement, we refer the reader to [Stein and Shakarchi(2010), §6.1].

Definition A.2. The *beta–function* is the mapping $\alpha, \beta \in \mathbb{H} \mapsto B\left(\alpha, \beta\right)$ functionally defined by:

$$B\left(\alpha, \beta\right) := 2\int_0^{\infty} d\tau \, \tau^{2\alpha-1} \left(1+\tau^2\right)^{-\left(\alpha+\beta\right)}. \tag{A.12}$$

Lemma A.2. *The* beta–function *is holomorphic in $\mathbb{H}$.*

[2]Miraculous, to the financially inclined reader, as far as the Clay's Institute is willing to pay a million dollar for a proof of the Riemann hypothesis.

As a matter of principle, I decline any `cryptocurrency` method of payment.

The simplest proof that the mapping $(\alpha, \beta) \in \mathbb{H} \mapsto B(\alpha, \beta)$ is holomorphic can still be found in the classic [Narayan(1966)].

Theorem A.1. *The analytically continued holomorphic $\zeta \in \mathbb{H} \mapsto \Gamma(\zeta)$ mapping can be further generalized to a meromorphic function on the entire complex plane, with singularities the simple poles at the non–positive integers $s \in \{0, -1, -2, ...\}$. Moreover, the residues are simply given by:*

$$\underset{\zeta \to s}{Res}\Gamma(\zeta) = \frac{(-1)^n}{n!}, \; n := -\zeta, s \in \mathbb{Z}_{\leq 0}. \tag{A.13}$$

Once again, a straightforward argument to a relatively non–trivial statement as the latter result, is available in [Narayan(1966)]. For a heuristic argument, we refer the reader to [Stein and Shakarchi(2010), p.163].

Theorem A.2. *The beta and gamma–functions satisfies the following identity:*

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}, \tag{A.14}$$

for all α and β belonging to the miraculous *half–plane $\mathbb{H}$.*

In fact, from definition, namely, Eqs.(A.11, A.12), it follows immediately that:

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty \int_0^\infty d\tau d\zeta \, e^{-(\tau+\zeta)} \tau^{\alpha-1} \zeta^{\beta-1}, \tag{A.15}$$

whereby the desired result follows simply by a change of variables: $\zeta(u, v) := uv, \; \tau(u, v) := u(1 - v)$.

Definition A.3. Let γ_{Euler} be *formally* given by the limit:

$$\gamma_{\text{Euler}} := \lim_{N \to \infty} \left[\sum_{n=1}^{N} \frac{1}{n} - \log N \right], \tag{A.16}$$

known as the *Euler's constant*[3].

Claim. The limit of Eq.(A.16) is well–defined.

Indeed, take notice of the following identity:

$$\sum_{n=1}^{N} \frac{1}{n} - \log N = \sum_{n=1}^{N} \frac{1}{n} - \int_1^N ds \frac{1}{s} \tag{A.17}$$

$$= \sum_{n=1}^{N-1} \int_n^{n+1} ds \left(\frac{1}{n} - \frac{1}{s} \right) + \frac{1}{N}. \tag{A.18}$$

Nonetheless, for all $n \leq s \leq n+1$, an elementary application of the mean–value theorem to the function $s \in \mathbb{R}_{>0} \mapsto f(s) := 1/s$, implies that:

$$a_n := \left| \frac{1}{n} - \frac{1}{s} \right| \leq \frac{1}{n^2}. \tag{A.19}$$

[3]Many textbooks call such a "constant" (we use commas for pedantry, since the convergence of the limit defined by Eq.(A.16, see however our next Claim) remains to be proven) Euler–Mascheroni. Interestingly, Lorenzo Mascheroni made a mistake in his calculations, published in his treatise *"Adnotationes ad calculum integrale Euleri."* It was left to Gaß's student, Friedrich Nicolai, to obtain the correct result [Gourdon and Sebah(2004)].

forasmuch as convergence is concerned, $a_n \leq 1/n^2$ for all $n \in \mathbb{N}$ implies:

$$\exists \left(\lim_{N\to\infty} \sum_{n=1}^{N} a_n \right), \tag{A.20}$$

whence establishing the convergence of Eq.(A.16) and, finally, the existence of Euler's constant.

Theorem A.3. (**1st Hadamard Factorization Theorem.**) Let $\zeta \in \mathbb{H}$ and $n \in \mathbb{N}$ any positive integer. Thence:

$$\frac{1}{\zeta\,\Gamma(\zeta)} = \exp\left(\zeta\,\gamma_{\text{Euler}}\right)\left[\prod_{n=1}^{\infty}\left(1+\frac{\zeta}{n}\right)e^{-\zeta/n}\right]. \tag{A.21}$$

For a quite transparent proof of such a result as ponderous as an equation could be, in the present case, Eq.(A.21), the reader is referred to [Stein and Shakarchi(2010), §6.1, p.166].

Finally, from Eq.(A.21), one can easily derive the asymptotic behavior for the gamma–function required by the method of dimensional regularization:

Corollary A.1. *Let $\varepsilon > 0$ be a positive real number. Then:*

$$\Gamma(\varepsilon) \sim \frac{1}{\varepsilon} - \gamma_{Euler} + \mathscr{O}(\varepsilon)\,,\ \varepsilon \ll 1. \tag{A.22}$$

A.2 Feynman's Parameters

Let A_1, ..., A_N be any sequence of numbers or non–singular functions.

The Feynman's parameters states that the denominator:

$$\frac{1}{A_1 \times .... \times A_N}, \tag{A.23}$$

can be written as a product of the N–dimensional integral:

$$\prod_{i=1}^{N}\frac{1}{A_i} = \left(\prod_{j=1}^{N}\int_0^1 d\tau_j\right)\frac{\delta\left(1-\sum_{k=1}^{N}\tau_k\right)}{\left(\sum_{\ell=1}^{N}\tau_\ell A_\ell\right)^2}. \tag{A.24}$$

To prove the above identity, we apply induction in $N \geq 2$.

For $N = 2$, the result is simply an elementary integral:

$$\frac{1}{A_1A_2} = \int_0^1 d\tau\,\frac{1}{\left(A_1\tau + A_2(1-\tau)\right)^2} = \int_0^1 d\tau_1\int_0^1 d\tau_2\,\frac{\delta(1-\tau_1-\tau_2)}{\left(A_1\tau_1 + A_2\tau_2\right)^2}. \tag{A.25}$$

Now, assume that Eq.(A.24) holds good for all $2 \leq N-1 < N$ and apply our hypothesis to the following product:

$$\prod_{i=1}^{N}\frac{1}{A_i} = \left(\prod_{i=1}^{N-1}\frac{1}{A_i}\right)\frac{1}{A_N} = \left[\left(\prod_{j=1}^{N-1}\int_0^1 d\tau_j\right)\frac{\delta\left(1-\sum_{k=1}^{N-1}\tau_k\right)}{\left(\sum_{k=1}^{N}\tau_k A_k\right)^2}\right]\times\frac{1}{A_N} \tag{A.26}$$

$$= \left(\prod_{j=1}^{N-1}\int_0^1 d\tau_j\right)\delta\left(1-\sum_{k=1}^{N-1}\tau_k\right)\underbrace{\frac{1}{\left(\sum_{k=1}^{N}\tau_k A_k\right)^2\times A_N}} \tag{A.27}$$

Consequently, upon the replacement of Eq.(A.25) into the factor of the last line of Eq.(A.27), marked by curly brackets, it follows that:

$$\prod_{i=1}^{N}\frac{1}{A_i} = \left(\prod_{j=1}^{N-1}\int_0^1 d\tau_j\right)\delta\left(1-\sum_{k=1}^{N-1}\tau_k\right)\int_0^1 d\tau_N\int_0^1 d\tau_{N+1}\,\frac{\delta\left(1-\tau_N-\tau_{N+1}\right)}{\left(A_N\tau_N+\left(\sum_{k=1}^{N}\tau_k A_k\right)^2\tau_{N+1}\right)^2} \tag{A.28}$$

$$= \int_0^1 d\tau_{N+1}\,\delta\left(1-\tau_N-\tau_{N+1}\right)\left(\prod_{j=1}^{N}\int_0^1 d\tau_j\right)\delta\left(1-\sum_{k=1}^{N-1}\tau_k\right)\times\frac{1}{\left(A_N\tau_N+\left(\sum_{k=1}^{N}\tau_k A_k\right)^2\tau_{N+1}\right)^2}. \tag{A.29}$$

Finally, integrating over $\int_0^1 d\tau_{N+1}\delta\left(1-\tau_N-\tau_{N+1}\right)$ and replacing $\tau_{N+1}=1-\tau_N$, Eq.(A.29) yields:

$$\prod_{i=1}^{N}\frac{1}{A_i} = \left(\prod_{j=1}^{N}\int_0^1 d\tau_j\right)\delta\left(1-\sum_{k=1}^{N-1}\tau_k\right)\times\frac{1}{\left(A_N\tau_N+\left(\sum_{k=1}^{N}\tau_k A_k\right)^2\left(1-\tau_N\right)\right)^2} \tag{A.30}$$

$$= \left(\prod_{j=1}^{N}\int_0^1 d\tau_j\right)\frac{\delta\left(1-\sum_{k=1}^{N}\tau_k\right)}{\left(\sum_{\ell=1}^{N}\tau_\ell A_\ell\right)^2}, \tag{A.31}$$

completing our proof.

A.3 Wick's Gymnastics

Let us consider a perturbative quantum field theory, with coupling constant $g \ll 1$. The Green's functions of order $\sim \mathscr{O}\left(g^N\right)$ are N–dimensional "loop" integrals in energy–momentum space.

The loop integrals associated to the Green's functions of higher order in perturbation theory are easier to perform in Euclidean spaces than in the original energy–momentum space, carrying with itself the Lorentz signature, a consequence of the fact that their conjugate variables are in the space–time representation.

The reason is simple: once space and time are treated in equal footing, as variables in the 4–dimensional Euclidean space $\mathbf{E}^4$, an integral of a function depending *exclusively upon the squared energy–momentum* 4–*vector* can be separated into an integral over a solid angle, as discussed in App.(A), and a simple 1–dimensional integral over the magnitude of the "Wick–rotated" momenta.

Conjecture A.1. *Let $\{x^\mu\}$ be an inertial coordinate system in Minkowski space.*

Assume all classical fields configurations (regarded as sections on the associated vector or spin bundles) *and field operators* (defined as distribution–valued operators acting upon the rays of Hilbert state space), *are defined so that:*

1. The Dyson–Schwinger equations, and in particular, the classical equations of motion without contact–terms, and:

2. The quantum corrections to the Green's functions,

allows *analytical continuation for the timelike coordinate*:

$$x_E^0 := -ix^0,\ \mathbf{x}_E := \mathbf{x}, \tag{A.32}$$

$$k_E^0 := ik^0,\ \mathbf{k}_E := \mathbf{k}, \tag{A.33}$$

$$\gamma_E^0 := i\gamma^4. \tag{A.34}$$

Definition A.4. The simultaneous application of the mappings given by Eqs.(A.32, A.33, A.34) are called a *Wick's rotation.*

Claim A.1. Under the assumptions of Conjecture (A.1), the effect upon the employment of a Wick's rotation on the measure of integration, squared energy–momentum 4–vector and, finally, either the contraction between 4–vectors and the generators $\{\gamma^\mu\}_{0\leq\mu\leq 3}$ are, respectively:

$$d^4k = -id^4k_E, \tag{A.35}$$

$$k^2 = -k_E^2, \tag{A.36}$$

$$\begin{aligned} k^\mu q_\mu &= (k_E)^\mu\,(q_E)_\mu\,, \\ \gamma\cdot k &= \gamma_E\cdot k_E. \end{aligned} \tag{A.37}$$

Bibliography

[Abers and Lee(1973)] Ernest S Abers and Benjamin W Lee. 16. generating functionals for green's functions and proper vertices. *Physics Reports*, 9(1):91–99, 1973.

[Anderson and Anderson(1967)] J.L. Anderson and J.L. Anderson. *Principles of Relativity Physics*. Academic Press, 1967. ISBN 9780120584505. URL `https://books.google.com.br/books?id=mAVRAAAAMAAJ`.

[Arkani-Hamed(1999)] et.al. Arkani-Hamed, N. Phenomenology, astrophysics, and cosmology of theories with submillimeter dimensions and tev scale quantum gravity. *Physical Review D*, 59(8):086004, 1999.

[Arnold(2013)] V.I. Arnold. *Mathematical Methods of Classical Mechanics*. Graduate Texts in Mathematics. Springer New York, 2013. ISBN 9781475716931. URL `https://books.google.com.br/books?id=UOQlBQAAQBAJ`.

[Arnold et al.(2007)Arnold, Kozlov, and Neishtadt] Vladimir I Arnold, Valery V Kozlov, and Anatoly I Neishtadt. *Mathematical aspects of classical and celestial mechanics*, volume 3. Springer Science & Business Media, 2007.

[Atiyah(1988)] M. Atiyah. *Michael Atiyah Collected Works: Volume 5: Gauge Theories*. Collected Works. Clarendon Press, 1988. ISBN 9780198532798. URL `https://books.google.com.br/books?id=8Js3-h0JJBUC`.

[Bailin and Love(1993)] D. Bailin and A. Love. *Introduction to Gauge Field Theory Revised Edition*. Graduate Student Series in Physics. Taylor & Francis, 1993. ISBN 9780750302814. URL `https://books.google.com.br/books?id=A9MU9pvcEGQC`.

[Barletta(2014)] et.al. Barletta. Planning the future of us particle physics (snowmass 2013): Chapter 6: Accelerator capabilities. *arXiv preprint arXiv:1401.6114*, 2014.

[Barnich et al.(2000)Barnich, Brandt, and Henneaux] Glenn Barnich, Friedemann Brandt, and Marc Henneaux. Local brst cohomology in gauge theories. *Physics Reports*, 338(5):439–569, 2000.

[Becchi et al.(1976)Becchi, Rouet, and Stora] Carlo Becchi, Alain Rouet, and Raymond Stora. Renormalization of gauge theories. *Annals of Physics*, 98(2):287–321, 1976.

[Bell(2008)] J.L. Bell. *A Primer of Infinitesimal Analysis*. Cambridge University Press, 2008. ISBN 9780521887182. URL `https://books.google.com.br/books?id=cIVq97UAvOMC`.

[Berazin(2012)] F.A. Berazin. *The Method of Second Quantization*. Pure and applied physics. Elsevier Science, 2012. ISBN 9780323159401. URL `https://books.google.com.br/books?id=mF6e6XlqnXYC`.

[Binz and Sniatycki(2011)] E. Binz and J. Sniatycki. *Geometry of Classical Fields*. Dover Publications, 2011. ISBN 9780486150444. URL `https://books.google.com.br/books?id=h77cHq1mRh4C`.

[Bjorken and Drell(1964)] J.D. Bjorken and S.D. Drell. *Relativistic Quantum Mechanics*. International series in pure and applied physics. McGraw-Hill, 1964. ISBN 9780070054936. URL `https://books.google.com.br/books?id=pAdRAAAAMAAJ`.

[Blagojevic(2001)] M. Blagojevic. *Gravitation and Gauge Symmetries*. Series in High Energy Physics, Cosmology and Gravitation. CRC Press, 2001. ISBN 9781420034264. URL `https://books.google.com.br/books?id=Ls-fBwAAQBAJ`.

[Bohr(1955)] N. Bohr. *Atomic Physics and Human Knowledge*. Dover Books on Physics. Dover Publications, 1955. ISBN 9780486479286. URL `https://books.google.com.br/books?id=tPEuDQAAQBAJ`.

[Born(1935)] Max Born. The mysterious number 137. In *Proceedings of the Indian Academy of Sciences-Section A*, volume 2, pages 533–561. Springer, 1935.

[Bridgman(1978)] P.W. Bridgman. *Dimensional Analysis*. AMS Press, 1978. ISBN 9780404147747. URL `https://books.google.com.br/books?id=ahhcKQAACAAJ`.

[Bronstein(2012)] Matvei Bronstein. Republication of: Quantum theory of weak gravitational fields. *General Relativity and Gravitation*, 44(1):267–283, 2012.

[Butterfield(1965)] H. Butterfield. *The Whig Interpretation of History*. Norton Library, N318. Norton, 1965. ISBN 9780393003185. URL `https://books.google.com.br/books?id=Ia6IpQKrx58C`.

[Chew(1966)] G.F. Chew. *The Analytic S Matrix: A Basis for Nuclear Democracy*. W. A. Benjamin, 1966. URL `https://books.google.com.br/books?id=CEF2AAAAIAAJ`.

[Collins and Collins(1985)] J.C. Collins and J.C. Collins. *Renormalization: An Introduction to Renormalization, the Renormalization Group and the Operator-Product Expansion*. AN INTRODUCTION TO RENORMALIZATION, THE RENORMALIZATION GROUP AND THE OPERATOR-PRODUCT EXPANSION. Cambridge University Press, 1985. ISBN 9780521311779. URL `https://books.google.com.br/books?id=60wX3xfdGRAC`.

[Dawkins(1989)] R. Dawkins. *The Selfish Gene*. Oxford paperbacks. Oxford University Press, 1989. ISBN 9780192860927. URL `https://books.google.com.br/books?id=WkHO9HI7koEC`.

[Deser(1970)] Stanley Deser. Self-interaction and gauge invariance. *General Relativity and gravitation*, 1(1): 9–18, 1970.

[DeWitt(1992)] B. DeWitt. *Supermanifolds*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1992. ISBN 9780521423779. URL `https://books.google.com.br/books?id=0NinYnTQa-gC`.

[DeWitt(1967)] Bryce S DeWitt. Quantum theory of gravity. ii. the manifestly covariant theory. *Physical Review*, 162(5):1195, 1967.

[Dirac(1981a)] PAM Dirac. Does renormalization make sense? In *AIP Conference Proceedings*, volume 74, pages 129–130. American Institute of Physics, 1981a.

[Dirac(1981b)] P.A.M. Dirac. *The Principles of Quantum Mechanics*. Comparative Pathobiology - Studies in the Postmodern Theory of Education. Clarendon Press, 1981b. ISBN 9780198520115. URL `https://books.google.com.br/books?id=XehUpGiM6FIC`.

[Dirac(1930)] Paul Adrien Maurice Dirac. A theory of electrons and protons. *Proceedings of the Royal Society of London. Series A, Containing papers of a mathematical and physical character*, 126(801):360–365, 1930.

[Dirac(2001)] Paul Adrien Maurice Dirac. *Lectures on quantum mechanics*, volume 2. Courier Corporation, 2001.

[Dirac(2005)] Paul AM Dirac. The lagrangian in quantum mechanics. In *Feynman's Thesis?A New Approach To Quantum Theory*, pages 111–119. World Scientific, 2005.

[DSM-5(2013)] American Psychiatric Association DSM-5. *Diagnostic and Statistical Manual of Mental Disorders (DSM-5®)*. American Psychiatric Publishing, 2013. ISBN 9780890425572. URL `https://books.google.com.br/books?id=-JivBAAAQBAJ`.

[Dyson and Derbes(2011)] F.J. Dyson and D. Derbes. *Advanced Quantum Mechanics*. Fisica y astronomia. World Scientific Publishing Company, 2011. ISBN 9789814383424. URL `https://books.google.com.br/books?id=Wme6CgAAQBAJ`.

[Dyson(1949)] Freeman J Dyson. The radiation theories of tomonaga, schwinger, and feynman. *Physical Review*, 75(3):486, 1949.

[Dyson(1958)] Freeman J Dyson. Integral representations of causal commutators. *Physical Review*, 110(6): 1460, 1958.

[Epstein and Glaser(1973)] Henri Epstein and Vladimir Glaser. The role of locality in perturbation theory. In *Annales de l'IHP Physique théorique*, volume 19, pages 211–295, 1973.

[Evans(2010)] L.C. Evans. *Partial Differential Equations*. Graduate studies in mathematics. American Mathematical Society, 2010. ISBN 9780821849743. URL `https://books.google.com.br/books?id=Xnu0o_EJrCQC`.

[Faddeev(2018)] L.D. Faddeev. *Gauge Fields: An Introduction To Quantum Theory, Second Edition*. CRC Press, 2018. ISBN 9780429962370. URL `https://books.google.com.br/books?id=03FQDwAAQBAJ`.

[Faddeev and Popov(1967)] Ludvig D Faddeev and Victor N Popov. Feynman diagrams for the yang-mills field. *Physics Letters B*, 25(1):29–30, 1967.

[Feynman(1985)] R.P. Feynman. *QED: The Strange Theory of Light and Matter*. Alix G. Mautner memorial lectures. Princeton University Press, 1985. ISBN 9780691125756. URL `https://books.google.com.br/books?id=Uv-uxB0sRKEC`.

[Feynman and Brown(2005)] R.P. Feynman and L.M. Brown. *Feynman's Thesis: A New Approach to Quantum Theory*. World Scientific, 2005. ISBN 9789812563668. URL `https://books.google.com.br/books?id=5kowI7YgFbEC`.

[Fierz and Pauli(1939)] M Fierz and W Pauli. Proc. roy. soc.(london). *Math. Phys.*, 173:211, 1939.

[Fierz(1939)] Markus Fierz. Über die relativistische Theorie kräftefreier Teilchen mit beliebigem Spin. *Helvetica Physica Acta*, 12:3–37, January 1939.

[Freedman and Van Proeyen(2012)] D.Z. Freedman and A. Van Proeyen. *Supergravity*. Cambridge University Press, 2012. ISBN 9780521194013. URL `https://books.google.com.br/books?id=7ANrOHJLTjMC`.

[Friedlander(1998)] G Friedlander. *Introduction to the Theory of Distributions*. Cambridge University Press, 1998.

[Gamow(1961)] G. Gamow. *The Great Physicists from Galileo to Einstein*. Dover Publications, 1961. ISBN 9780486136813. URL `https://books.google.com.br/books?id=XZq8AQAAQBAJ`.

[Gibbons et al.(1978)Gibbons, Hawking, and Perry] Gary W Gibbons, Stephen W Hawking, and Malcolm J Perry. Path integrals and the indefiniteness of the gravitational action. *Nuclear Physics B*, 138(1): 141–150, 1978.

[Gourdon and Sebah(2004)] Xavier Gourdon and Pascal Sebah. The euler constant: γ. *Young*, 1:2n, 2004.

[Greiner(2012)] W. Greiner. *Relativistic Quantum Mechanics: Wave Equations*. Springer Berlin Heidelberg, 2012. ISBN 9783642880827. URL `https://books.google.com.br/books?id=a6_rCAAAQBAJ`.

[Greiner and Reinhardt(2008)] W. Greiner and J. Reinhardt. *Quantum Electrodynamics*. Physics and Astronomy. Springer Berlin Heidelberg, 2008. ISBN 9783540875611. URL `https://books.google.com.br/books?id=5Kd3dBL8a64C`.

[Grothendieck(1985)] A. Grothendieck. *Récoltes et semailles: réflexions et témoignage sur un passé de mathématicien*. Number v. 1-3 in Récoltes et semailles: réflexions et témoignage sur un passé de mathématicien. Université des sciences et techniques du Languedoc et Centre national de la recherche scientifique, 1985. URL `https://books.google.com.br/books?id=3zcbMQAACAAJ`.

[Hamilton and Beaufort(1834)] William Rowan Hamilton and Francis Beaufort. Xv. on a general method in dynamics; by which the study of the motions of all free systems of attracting or repelling points is reduced to the search and differentiation of one central relation, or characteristic function. *Philosophical Transactions of the Royal Society of London*, 124:247–308, 1834. doi: 10.1098/rstl.1834.0017. URL `https://royalsocietypublishing.org/doi/abs/10.1098/rstl.1834.0017`.

[Hamilton(1833)] W.R. Hamilton. *On a General Method of Expressing the Paths of Light, & of the Planets, by the Coefficients of a Characteristic Function*. P.D. Hardy, 1833. URL `https://books.google.gl/books?id=--44AAAAMAAJ`.

[Han(1999)] et.al. Han, T. Kaluza-klein states from large extra dimensions. *Physical Review D*, 59(10): 105006, 1999.

[Hatcher(2002)] A. Hatcher. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002. ISBN 9780521795401. URL `http://pi.math.cornell.edu/~hatcher/AT/ATpage.html`.

[Hehl(1980)] Friedrich W Hehl. Four lectures on poincaré gauge field theory. In *Cosmology and gravitation*, pages 5–61. Springer, 1980.

[Heisenberg and Pauli(1993)] W Heisenberg and W Pauli. On the isospin group in the theory of the elementary particles. In *Original Scientific Papers/Wissenschaftliche Originalarbeiten*, pages 337–351. Springer, 1993.

[Henneaux and Teitelboim(2020)] Marc Henneaux and Claudio Teitelboim. *Quantization of gauge systems*. Princeton university press, 2020.

[Hestenes and Lasenby(2015)] D. Hestenes and A. Lasenby. *Space-Time Algebra*. Springer International Publishing, 2015. ISBN 9783319184135. URL `https://books.google.com.br/books?id=63i6CAAAQBAJ`.

[Itzykson and Zuber(2012)] Claude Itzykson and Jean-Bernard Zuber. *Quantum field theory*. Dover Edition, 2012.

[Jauch and Rohrlich(2012)] J.M. Jauch and F. Rohrlich. *The Theory of Photons and Electrons: The Relativistic Quantum Field Theory of Charged Particles with Spin One-half*. Theoretical and Mathematical Physics. Springer Berlin Heidelberg, 2012. ISBN 9783642809514. URL `https://books.google.com.br/books?id=Epn6CAAAQBAJ`.

[Jeffrey and Zwillinger(2007)] A. Jeffrey and D. Zwillinger. *Table of Integrals, Series, and Products*. Elsevier Science, 2007. ISBN 9780080471112. URL `https://books.google.com.br/books?id=aBgFYxKHUjsC`.

[Kleinert(2001)] et.al. Kleinert, H. *Critical Properties of [Greek Letter Phi]4-theories*. World Scientific, 2001. ISBN 9789810246594. URL `https://books.google.com.br/books?id=rBfWk4vVi-cC`.

[Kraichnan(1955)] Robert H Kraichnan. Special-relativistic derivation of generally covariant gravitation theory. *Physical Review*, 98(4):1118, 1955.

[Landau(2013)] L.D. Landau. *The Classical Theory of Fields*. COURSE OF THEORETICAL PHYSICS. Elsevier Science, 2013. ISBN 9781483293288. URL `https://books.google.com.br/books?id=HudbAwAAQBAJ`.

[Lounesto et al.(2001)Lounesto, Cassels, Society, and Hitchin] P. Lounesto, J.W.S. Cassels, London Mathematical Society, and N.J. Hitchin. *Clifford Algebras and Spinors*. Lecture note series / London mathematical society. Cambridge University Press, 2001. ISBN 9780521005517. URL `https://books.google.com.br/books?id=DTecU6UpkSgC`.

[Lovelock and Rund(2012)] D. Lovelock and H. Rund. *Tensors, Differential Forms, and Variational Principles*. Dover Books on Mathematics. Dover Publications, 2012. ISBN 9780486131986. URL `https://books.google.com.br/books?id=uUi8AQAAQBAJ`.

[Maggiore(2008)] M. Maggiore. *Gravitational Waves: Volume 1: Theory and Experiments*. Gravitational Waves. OUP Oxford, 2008. ISBN 9780198570745. URL `https://books.google.com.br/books?id=mk-1DAAAQBAJ`.

[Mandl and Shaw(2010)] Franz Mandl and Graham Shaw. *Quantum field theory*. John Wiley & Sons, 2010.

[Marcolli(2010)] M. Marcolli. *Feynman Motives*. World Scientific, 2010. ISBN 9789814271219. URL `https://books.google.com.br/books?id=HJ5clJKW9L0C`.

[Mermin(1988)] N David Mermin. What's wrong with this lagrangean? *PhT*, 41(4):9, 1988. URL `https://physicstoday.scitation.org/doi/pdf/10.1063/1.2811374`.

[Mol(1st March 2021)] I. Mol. *Problem Sheet X: A Problem of Self-Regularization*. Quantum Field Theory II. Boschi's Academic Press, 1st March 2021.

[Mol(2020)] I Mol. *Problem Sheet II: A Study in Epsilon*. Quantum Field Theory II. Boschi's Academic Press, 2020.

[Mol(2021)] I Mol. *Teoria Quântica de Campos II (Semestre Pandêmico): Lista VI*. Quantum Field Theory II. Boschi's Academic Press, 2021.

[Naber(2003)] G.L. Naber. *The Geometry of Minkowski Spacetime: An Introduction to the Mathematics of the Special Theory of Relativity*. Applied mathematics sciences. Dover Publications, 2003. ISBN 9780486432359. URL `https://books.google.com.br/books?id=pNfRHzwdVZ0C`.

[Narayan(1966)] S. Narayan. *Theory of Functions of a Complex Variable*. S. Chand, 1966. URL `https://books.google.com.br/books?id=TYhGAAAAYAAJ`.

[Newton(1726)] I.. Newton. *The Principia: Mathematical Principles of Natural Philosophy*. The Principia: Mathematical Principles of Natural Philosophy. University of California Press, 1726. ISBN 9780520088160. URL `https://books.google.com.br/books?id=kmglDQAAQBAJ`.

[Olver(2012)] P.J. Olver. *Applications of Lie Groups to Differential Equations*. Graduate Texts in Mathematics. Springer New York, 2012. ISBN 9781468402742. URL `https://books.google.com.br/books?id=qC_oBwAAQBAJ`.

[O'Raifeartaigh(1986)] L. O'Raifeartaigh. *Group Structure of Gauge Theories*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1986. ISBN 9780521347853. URL `https://books.google.com.br/books?id=BbNdTT0GUlEC`.

[Pais(1988)] A. Pais. *Inward Bound: Of Matter and Forces in the Physical World*. Oxford paperbacks. Clarendon Press, 1988. ISBN 9780198519973. URL `https://books.google.com.br/books?id=mREnwpAqz-YC`.

[Pauli et al.(2013)Pauli, Schlapp, Enz, and Meyenn] W. Pauli, R. Schlapp, C.P. Enz, and K. Meyenn. *Writings on Physics and Philosophy*. Springer Berlin Heidelberg, 2013. ISBN 9783662029947. URL `https://books.google.com.br/books?id=TXrpCAAAQBAJ`.

[Pauli(1940)] Wolfgang Pauli. The connection between spin and statistics. *Physical Review*, 58(8):716, 1940.

[Pauli and Villars(1949)] Wolfgang Pauli and Felix Villars. On the invariant regularization in relativistic quantum theory. *Reviews of Modern Physics*, 21(3):434, 1949.

[Penrose(2003)] Roger Penrose. On the instability of extra space dimensions. *ftpc*, pages 185–201, 2003.

[Penrose(2017)] Roger Penrose. *Fashion, faith, and fantasy in the new physics of the universe*. Princeton University Press, 2017.

[Peskin(2018)] M.E. Peskin. *An Introduction To Quantum Field Theory*. CRC Press, 2018. ISBN 9780429983184. URL https://books.google.com.br/books?id=9EpnDwAAQBAJ.

[Planck(1950)] M. Planck. *Scientific Autobiography: And Other Papers*. Philosophical Library/Open Road, 1950. ISBN 9781497675889. URL https://books.google.com.br/books?id=F3iMBAAAQBAJ.

[Ramond(1997)] P. Ramond. *Field Theory*. Frontiers in Physics. Avalon Publishing, 1997. ISBN 9780813346564. URL https://books.google.com.br/books?id=Ctr9K61fY4kC.

[Rivier(1949)] D Rivier. A method for the elimination of infinities in the theory of quantized fields. application to the magnetic moment of the neutron. *Helvetica Physica Acta (Switzerland)*, 22, 1949.

[Russell(1958)] B. Russell. *Unpopular Essays*. Allen and Unwin, 1958. URL https://books.google.com.br/books?id=5sY8AAAAYAAJ.

[Ryder(1996)] L.H. Ryder. *Quantum Field Theory*. Cambridge University Press, 1996. ISBN 9780521478144. URL https://books.google.com.br/books?id=nnuW_kVJ500C.

[Scharf(2013)] G. Scharf. *Finite Quantum Electrodynamics*. Theoretical and Mathematical Physics. Springer Berlin Heidelberg, 2013. ISBN 9783662011874. URL https://books.google.com.br/books?id=bkftCAAAQBAJ.

[Scharf(2014)] G. Scharf. *Finite Quantum Electrodynamics: The Causal Approach, Third Edition*. Dover Books on Physics. Dover Publications, 2014. ISBN 9780486492735. URL https://books.google.com.br/books?id=zxnqAgAAQBAJ.

[Schutz(1985)] B.F. Schutz. *A First Course in General Relativity*. Series in physics. Cambridge University Press, 1985. ISBN 9780521277037. URL https://books.google.com.br/books?id=qhDFuWbLlgQC.

[Schwartz(2014)] M.D. Schwartz. *Quantum Field Theory and the Standard Model*. Quantum Field Theory and the Standard Model. Cambridge University Press, 2014. ISBN 9781107034730. URL https://books.google.com.br/books?id=HbdEAgAAQBAJ.

[Schweizer(1980)] M.A. Schweizer. *Gauge Theory and Gravitation*. University of Zurich, 1980. URL https://books.google.com.br/books?id=qlUSAQAAIAAJ.

[Schwinger and Englert(2013)] J. Schwinger and B.G. Englert. *Quantum Mechanics: Symbolism of Atomic Measurements*. Springer Berlin Heidelberg, 2013. ISBN 9783662045893. URL https://books.google.com.br/books?id=fDX6CAAAQBAJ.

[Schwinger(1998)] J.S. Schwinger. *Particles, Sources, and Fields*. Number v. 3 in Addison-Wesley series in physics. Advanced Book Program, Perseus Books, 1998. ISBN 9780738200552. URL `https://books.google.com.br/books?id=1tHvAAAAMAAJ`.

[Shawyer(1969)] B. L. R. Shawyer. On the relation between the abel and borel-type methods of summability. *Proceedings of the American Mathematical Society*, 22(1):15–19, 1969. ISSN 00029939, 10886826. URL `http://www.jstor.org/stable/2036914`.

[Slavnov(1972)] Andrei Alekseevich Slavnov. Ward identities in gauge theories. *Theoretical and Mathematical Physics*, 10(2):99–104, 1972.

[Sommerfeld(1961)] A. Sommerfeld. *Partial Differential Equations in Physics*. Pure and Applied Mathematics - Academic Press. Academic Press, 1961. ISBN 9788187169499. URL `https://books.google.com.br/books?id=xHW3_2TcbOAC`.

[Stein and Shakarchi(2010)] E.M. Stein and R. Shakarchi. *Complex Analysis*. Princeton lectures in analysis. Princeton University Press, 2010. ISBN 9781400831159. URL `https://books.google.com.br/books?id=0ECHh9tjPUAC`.

[Stewart(1993)] J.M. Stewart. *Advanced General Relativity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1993. ISBN 9780521449465. URL `https://books.google.com.br/books?id=LZrjJ2P1J1sC`.

[Straumann(2013)] N. Straumann. *General Relativity: With Applications to Astrophysics*. Theoretical and Mathematical Physics. Springer Berlin Heidelberg, 2013. ISBN 9783662118276. URL `https://books.google.com.br/books?id=5iH4CAAAQBAJ`.

[Streater and Wightman(1964)] R.F. Streater and A.S. Wightman. *PCT, Spin and Statistics, and All That*. Princeton Landmarks in Mathematics and Physics. Princeton University Press, 1964. ISBN 9781400884230. URL `https://books.google.com.br/books?id=FyCTDAAAQBAJ`.

[Stueckelberg(1938)] ECG Stueckelberg. Forces of interaction in electrodynamics and in the field theory of nuclear forces.(parts ii and iii). *Helv. Phys. Acta*, 11:299–328, 1938.

[Stueckelberg de Breidenbach and Green(1951)] Ernst Carl Gerlach Stueckelberg de Breidenbach and T A Green. Elimination des constantes arbitraires dans la theorie relativiste des quanta. *Helv. Phys. Acta*, 24: 153–174, 1951. doi: 10.5169/seals-112211. URL `https://cds.cern.ch/record/406756`.

[Takahashi(1957)] Yasushi Takahashi. On the generalized ward identity. *Il Nuovo Cimento (1955-1965)*, 6(2): 371–375, 1957.

[Taylor(1971)] JC Taylor. Ward identities and charge renormalization of the yang-mills field. *Nuclear Physics B*, 33(2):436–444, 1971.

[Thiemann(2008)] T. Thiemann. *Modern Canonical Quantum General Relativity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2008. ISBN 9781139467599. URL `https://books.google.com.br/books?id=p-sXBRd1iOIC`.

[Tyutin(1975)] Igor V Tyutin. Gauge invariance in field theory and statistical physics in operator formalism. *Preprint of P.N. Lebedev PhysicalInstitute, No. 39*, 1975.

[Van Der Waerden(2007)] Bartel Leendert Van Der Waerden. *Sources of quantum mechanics*. Courier Corporation, 2007.

[Vaz and da Rocha(2016)] J. Vaz and R. da Rocha. *An Introduction to Clifford Algebras and Spinors:*. Oxford University Press, 2016. ISBN 9780198782926. URL `https://books.google.com.br/books?id=VNMTDAAAQBAJ`.

[Veltman(1976)] Martin JG Veltman. Quantum theory of gravitation. *Méthodes en théorie des champs*, pages 266–328, 1976.

[Wald(2010)] R.M. Wald. *General Relativity*. University of Chicago Press, 2010. ISBN 9780226870373. URL `https://books.google.com.br/books?id=9S-hzg6-moYC`.

[Ward(1950)] John Clive Ward. An identity in quantum electrodynamics. *Physical Review*, 78(2):182, 1950.

[Weinberg(1995)] S. Weinberg. *The Quantum Theory of Fields*. Number v. 1 in Quantum Theory of Fields, Vol. 2: Modern Applications. Cambridge University Press, 1995. ISBN 9780521550017. URL `https://books.google.com.br/books?id=doeDB3_WLvwC`.

[Weinberg(1965)] Steven Weinberg. Infrared photons and gravitons. *Physical Review*, 140(2B):B516, 1965.

[Weyl(2009)] H. et.al. Weyl. *Philosophy of Mathematics and Natural Science*. Princeton University Press, 2009. ISBN 9780691141206. URL `https://books.google.com.br/books?id=565kXGJPkiYC`.

[Weyl(1923)] Hermann Weyl. Gravitation und elektrizität. In *Das Relativitätsprinzip*, pages 147–159. Springer, 1923.

[Whittaker(1988)] E.T. Whittaker. *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*. Cambridge Mathematical Library. Cambridge University Press, 1988. ISBN 9780521358835. URL `https://books.google.com.br/books?id=epH1hCB7N2MC`.

[Wichmann and Crichton(1963)] Eyvind H Wichmann and James H Crichton. Cluster decomposition properties of the s matrix. *Physical Review*, 132(6):2788, 1963.

[Wightman and Schweber(1955)] Arthur Strong Wightman and Silvan S Schweber. Configuration space methods in relativistic quantum field theory. i. *Physical Review*, 98(3):812, 1955.

[Witten(1988)] Edward Witten. Topological quantum field theory. *Communications in Mathematical Physics*, 117(3):353–386, 1988.

[Witten et al.(1982)] Edward Witten et al. Supersymmetry and morse theory. *Journal of Differential Geometry*, 17(4):661–692, 1982.

[Yang and Mills(1954)] Chen-Ning Yang and Robert L Mills. Conservation of isotopic spin and isotopic gauge invariance. *Physical review*, 96(1):191, 1954.

[York Jr(1972)] James W York Jr. Role of conformal three-geometry in the dynamics of gravitation. *Physical review letters*, 28(16):1082, 1972.

[Zee(2010)] A. Zee. *Quantum Field Theory in a Nutshell: Second Edition*. In a Nutshell. Princeton University Press, 2010. ISBN 9781400835324. URL `https://books.google.com.br/books?id=n8Mmbjtco78C`.

[Zinn-Justin(2007)] J. Zinn-Justin. *Phase Transitions and Renormalization Group*. Oxford Graduate Texts. OUP Oxford, 2007. ISBN 9780199227198. URL `https://books.google.com.br/books?id=mUMVDAAAQBAJ`.

[Zinn-Justin(1999)] Jean Zinn-Justin. Renormalization of gauge theories and master equation. *Modern Physics Letters A*, 14(19):1227–1235, 1999.