Eur. Phys. J. C (2024) 84:388
https://doi.org/10.1140/epjc/s10052-024-12755-0

THE EUROPEAN PHYSICAL JOURNAL C

Letter - Theoretical Physics

# Exact phonon quasibound states around an optical black hole

**David Senjaya**[a]

Department of Physics, Mahidol University, 272 Phraram 6 Street, Ratchathewi, Bangkok 10400, Thailand

Received: 19 February 2024 / Accepted: 1 April 2024


**Abstract** In this work, we present an exact solution of phonon quasibound states in a vortex photon flow that creates an analogous black hole background. We worked out and successfully solve the governing differential equation of the systen that is similar with the relativistic Klein–Gordon equation. The exact radial solutions are discovered in terms of the confluent Heun functions that behaves as ingoing waves near the black hole's horizon and decaying far away from the black hole's horizon. The polynomial condition of the confluent Heun function leads to the discovery of the the complex valued quantized energy levels expression of the quasibound state of which depends on the phonon mass $\Omega_0$, the photon vortex horizon's spin $\Omega_H$ and the azimuthal and the main quantum number $(m, n)$. In the last section, by using the Damour–Ruffini method, the Hawking radiation of the analog black hole's horizon is investigated to derive the phononic radiation distribution function from where, the Hawking temberature is obtained.

## 1 Introduction

Progress in investigations of critical phenomenas in general relativity and quantum field theory in curved space-time has been lacking of experimental feedback. Strong efforts have been strived in search of non-relativistic systems that experimentally testable in the laboratory. In 1981, Unruh [1] investigated the propagation of sound waves in an inhomogeneous flowing fluid and found out that the governing equation was analogous to a massless scalar field in curved space-time. The surface of the fluid flow's velocity has an inward pointing normal component that is equal to the local speed of sound preventing any sound wave propagating through this surface in outward direction. Thus, the surface acts analogously as the event horizon of a black hole. This analogy implies that the analogous black hole should also emits Hawking radiation in the form of phonons. This analogous black hole system makes experimental verification of such phenomenon possible to be done in laboratory.

Many different kinds of analogue models has been proposed after Unruh's prediction such as Bose–Einstein condensates, electromagnetic wave guides, graphene, optical black hole, sonic black hole and ion rings [2–6]. In optical domain, Leonhardt and Piwnicki [7] proposed the idea of an optical black hole for the first time in the year of 2000. The idea is that propagation of light in a moving medium resembles many features of a motion in a curved space-time background. Analogue gravity models have been emerging as a new interest as they suit as test-beds for several aspects relativistic quantum field theory in a curved space-time.

In this letter, we will be focus on photon-fluid system, a non-linear optical system that is represented by hydrodynamic equations of an interacting Bosonic gas [8]. Together with exciton-polariton system and Bose–Einstein condensation of photons in an optical microcavity [9], the photon-fluid system also belongs to the family of the so-called quantum fluids of light [10].

In [11] a photon-fluid with both local and non-local interactions have been investigated, showing that the phonons acquire a finite mass and propagate according to the massive Klein–Gordon equation in a (2+1)-dimensional curved space-time. In this system, with the presence of suitable vortex flows, both quasibound states and stationary states of phonons can be created [12]. These represent the acoustic counterpart of quasibound states and scalar clouds originating from massive fields around Kerr black holes. Scalar clouds in photon fluids have been analytically studied in [ [13] and [14], while the derivation of exact solutions of phonon quasibound states is the subject of the present manuscript.

In general, any inhomogeneous neutrally stable nonlinear system provides an effective curved space-time on which its linear elementary excitations can propagate [15]. For the

[a] e-mail: davidsenjaya@protonmail.com (corresponding author)

Published online: 12 April 2024

longer wavelengths, phonons propagating in an inhomogeneous photon flow has a similar evolution with scalar fields in curved space-time that is governed by a Klein–Gordon like equation, obtained via linearization around the background state. By comparing the governing equation with the Klein–Gordon equationin 2+1 dimension, we can extract the analogous space-time metric as follows [11],

$$ds^2 = -f(r)d(ct)^2 + \frac{dr^2}{g(r)} - 2\Omega_H r_H^2 d\theta d(ct) + r^2 d\theta^2, \tag{1}$$

$$f(r) = 1 - \frac{r_H}{r} - \frac{\Omega_H^2 r_H^4}{r^2}, \tag{2}$$

$$g(r) = 1 - \frac{r_H}{r}, \tag{3}$$

where $r_H$ is the radius of the analog black-hole's horizon that is geometrically interpreted as the circular ring at which the inward radial velocity of the fluid flow is equal to the speed of sound. The $\Omega_H$ is the angular velocity at the horizon. The term $d\theta d(ct)$ represents frame dragging and this term vanishes as $\Omega_H \to 0$.

In [15], it is mentioned that the metric (1) is comparable with equatorial slice of the rotating Kerr black hole geometry in Boyer–Lindquist coordinates where there exists an analog ergoregion that allows phonons to have negative energies leading to energy extraction. The radius of the ergosphere is given by the vanishing of the temporal component, i.e.,

$$r_E = \frac{r_H}{2}\left(1 + \sqrt{1 + 4r_H^2\Omega_H^2}\right). \tag{4}$$

The frame dragging effect itself was firstly investigated in detal by [16,17] in 1918. They consider Coriolis effects up to the first order and calculating such effects near the center of a rotating mass shell and in the far field of a rotating spherical body. In the context of a real black hole possessing accretion disk, the radial precession causes stresses and dissipation and when the torque is strong enough compared to the internal viscous forces, the inner regions of the disk are forced to align with the spin of the central black hole [18]. In these recent years, the frame dragging effect has been successfully measured directly [19–21].

In 2015, the gravitational wave signal of a binary black hole merger was directly detected for the first time [22] that makes black hole spectroscopy a new emerging interest. In this research, we are interested to investigate the Klein–Gordon equation of phonon quasibound states in the vortex flow of photons that resembles a single horizoned black hole [12]. The phonon fields in the analog black hole background is governed by the covariant Klein–Gordon equation. Quantum mechanically, the phonon states have discrete complex spectrum as they tunnel through the potential barrier and captured by the horizon. Thus, the quasibound states are decaying localized solutions lying in the black hole potential wells.

However, due to the complexity of the equations involved, especially the radial equation, analytical methods were used less often and only for certain problems. The vast majority of these studies made use of numerical techniques such as the asymptotical analysis, WKB, and continued fraction to investigate the specific task at hand. Fortunately, very recently, [23–27] successfully finds novel exact scalar quasibound state solutions respectively around an analog systems to the Schwarzschild black hole, charged and chargeless Lense-Thirring black hole and Reissner Nordstrom black hole where the radial equations of the scalar field are successfully solved in terms of the confluent Heun functions. The importance of this particular special function in black hole physics was mentioned in [28].

In this present work, we are going to show in detail analytical derivation of exact solutions of relativistic phonon quasibound states around an analog rotating optical black hole. We successfully solve the radial equation exactly in terms of confluent Heun functions and having the exact solutions in the hand, the complex quantized energy levels expression is obtained from the confluent Heun's polynomial condition. And finally, by applying the Damour–Ruffini method, the Hawking radiation of the apparent black hole's horizon is investigated and the Hawking temperature is obtained.

## 2 The Klein–Gordon equation

The Klein–Gordon equation describes the relativistic quantum mechanics of a scalar field in a curved space-time. The relativistic equation comes from the canonical quantization of the second class constrained of the relativistic classical mechanics [29],

$$\left\{-\hbar^2\left[\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\right] + k^2c^2\right\}\psi = 0, \tag{5}$$

where $E_0 = kc^2$ is the scalar's rest energy per unit mass, so, $k = 1$ represents massive particles and $k = 0$ represents massless particles. For the phonon-photon flow system, the governing equation reads as follows [12],

$$\left\{-\left[\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\right] + \frac{\Omega_0^2}{\hbar^2}\right\}\psi = 0, \tag{6}$$

where $\frac{\Omega_0}{\hbar}$ is the phonon rest energy per unit mass defined as $\frac{\Omega_0}{\hbar} = v_s^2$ where $v_s$ is the speed of sound.

Substituting the metric in (1) and using the ansatz of separation of variables,

$$\psi(t, r, \theta) = e^{-i\frac{E}{\hbar c}ct}e^{im\theta}\frac{R(r)}{\sqrt{r}}, \tag{7}$$

where $m$ is the azimuthal quantum number. After substitution of the ansatz, the radial equation is successfully isolated as follows,

$$\Delta\partial_x\left(\Delta\partial_x\right)R\left(x\right)+\left[\Delta\left(\frac{\Delta}{4x^2}-\frac{1}{2x^3}-\frac{m^2}{x^2}-\Omega_0^2\right)\right.$$
$$\left.+\left(\Omega-\frac{m\Omega_H}{x^2}\right)^2\right]R\left(x\right)=0, \quad (8)$$

where,

$$\Delta=1-\frac{r_H}{r}=1-\frac{1}{x}, \quad (9)$$
$$x=\frac{r}{r_H}, \quad (10)$$
$$\Omega=\frac{Er_H}{\hbar c}. \quad (11)$$

Now, operating the derivatives in the first term and multiplying the whole equation by $\Delta^{-2}$, we obtain this following equation,

$$\partial_x^2R+\left(\frac{\partial_x\Delta}{\Delta}\right)\partial_xR+\left[\Delta^{-1}\left(\frac{\Delta}{4x^2}-\frac{1}{2x^3}-\frac{m^2}{x^2}-\Omega_0^2\right)\right.$$
$$\left.+\Delta^{-2}\left(\Omega-\frac{m\Omega_H}{x^2}\right)^2\right]R=0. \quad (12)$$

And by substituting $\Delta\left(x\right)$ explicitly as(9), we obtain,

$$\frac{\partial_x\Delta}{\Delta}=\frac{1}{x-1}-\frac{1}{x}, \quad (13)$$
$$\partial_x^2R+p\left(x\right)\partial_xR+q\left(x\right)R=0, \quad (14)$$

where,

$$p\left(x\right)=\frac{1}{x-1}-\frac{1}{x}, \quad (15)$$
$$q\left(x\right)=\left(\frac{x}{x-1}\right)\left(\frac{x-1}{4x^3}-\frac{1}{2x^3}-\frac{m^2}{x^2}-\Omega_0^2\right)$$
$$+\left(\frac{x}{x-1}\right)^2\left(\Omega-\frac{m\Omega_H}{x^2}\right)^2. \quad (16)$$

The coefficient of $R$, i.e. $q\left(x\right)$, is a polynomial function that contains of mixture between $x$ and $x-1$. It is possible to express $q\left(x\right)$ in terms of $\frac{1}{x}$ and $\frac{1}{x-1}$ using these following fractional decompositions,

$$\frac{1}{x\left(x-1\right)}=\frac{1}{x-1}-\frac{1}{x}, \quad (17)$$
$$\frac{1}{x^2\left(x-1\right)}=\frac{1}{x-1}-\frac{1}{x}-\frac{1}{x^2}, \quad (18)$$
$$\frac{1}{x^2(x-1)^2}=-\frac{2}{x-1}+\frac{1}{(x-1)^2}+\frac{2}{x}+\frac{1}{x^2}, \quad (19)$$
$$\frac{x}{x-1}=\frac{1}{x-1}+1, \quad (20)$$

and by making use of the decomposition formulas above, we decompose $q\left(x\right)$ as follows,

$$q\left(x\right)=\left(\Omega^2-\Omega_0^2\right)+\frac{1}{x}\left(\frac{1}{2}+m^2\left(1+2\Omega_H^2\right)\right)$$
$$+\frac{1}{x-1}\left(-\frac{1}{2}-m^2\left(1+2\Omega_H^2\right)-\Omega_0^2+2\Omega^2\right)$$
$$+\frac{1}{x^2}\left(\frac{3}{4}+m^2\Omega_H^2\right)+\frac{1}{(x-1)^2}(\Omega-m\Omega_H)^2. \quad (21)$$

We are interested to find solutions in the region outside the horizon, i.e. $r_H\leq r<\infty$ or equivalently $1\leq x<\infty$, that for the sake of notation convenience, we define a new radial variable $z=-\left(x-1\right)\rightarrow\partial_x=-\partial_z$ that shifts the domain of interest to $0\leq z<-\infty$. In the new variable, the radial equation becomes as follows,

$$\partial_z^2R+p\left(z\right)\partial_zR+q\left(z\right)R=0, \quad (22)$$
$$p\left(z\right)=\frac{1}{z}-\frac{1}{z-1}, \quad (23)$$
$$-\frac{1}{2}\partial_zp\left(z\right)=\frac{1}{2}\frac{1}{z^2}-\frac{1}{2}\frac{1}{(z-1)^2}, \quad (24)$$
$$-\frac{1}{4}p^2\left(z\right)=-\frac{1}{4}\frac{1}{z^2}-\frac{1}{4}\frac{1}{(z-1)^2}+\frac{1}{2}\frac{1}{x-1}-\frac{1}{2}\frac{1}{x}, \quad (25)$$
$$q\left(z\right)=\left(\Omega^2-\Omega_0^2\right)-\frac{1}{z-1}\left(\frac{1}{2}+m^2\left(1+2\Omega_H^2\right)\right)$$
$$-\frac{1}{z}\left(-\frac{1}{2}-m^2\left(1+2\Omega_H^2\right)-\Omega_0^2+2\Omega^2\right)$$
$$+\frac{1}{(z-1)^2}\left(\frac{3}{4}+m^2\Omega_H^2\right)+\frac{1}{z^2}(\Omega-m\Omega_H)^2. \quad (26)$$

Following Appendix A, the normal form of the radial equation is obtained as follows,

$$\partial_z^2Y\left(z\right)+K\left(z\right)Y\left(z\right)=0, \quad (27)$$
$$Y\left(z\right)=z^{\frac{1}{2}}\left(z-1\right)^{-\frac{1}{2}}R(r) \quad (28)$$

where,

$$K\left(z\right)=-\frac{1}{2}\partial_zp\left(z\right)-\frac{1}{4}p^2\left(z\right)+q\left(z\right)=\left(\Omega^2-\Omega_0^2\right)$$
$$-\frac{1}{z-1}\left(m^2\left(1+2\Omega_H^2\right)\right)-\frac{1}{z}\left(-m^2\left(1+2\Omega_H^2\right)\right.$$
$$\left.-\Omega_0^2+2\Omega^2\right)$$
$$+\frac{1}{(z-1)^2}\left(m^2\Omega_H^2\right)+\frac{1}{z^2}\left(\frac{1}{4}+(\Omega-m\Omega_H)^2\right). \quad (29)$$

### 2.1 The radial solution

Comparing (27) together with (29) with the normal form of the confluent Heun's differential equation in Appendix B, we find the exact radial solution of the phonon quasibound states in the vortex flow of photons as follows,

$$R = e^{\frac{1}{2}\alpha z} z(z-1)^{\frac{1}{2}\gamma}\left[Az^{\frac{\beta}{2}}\,\mathrm{HeunC}(z) + Bz^{-\frac{\beta}{2}}\,\mathrm{HeunC}'(z)\right], \tag{30}$$

$$z = -\frac{r_H - r}{r_H}, \tag{31}$$

where the confluent Heun function's parameters are algebraically solved as follows,

$$\alpha = 2\sqrt{\Omega_0^2 - \Omega^2}, \tag{32}$$
$$\beta = 2i\,(\Omega - m\Omega_H)\,, \tag{33}$$
$$\gamma = i\sqrt{4m^2\Omega_H^2 - 1} = i\,|\gamma|\,, \tag{34}$$
$$\delta = \Omega_0^2 - 2\Omega^2, \tag{35}$$
$$\eta = \frac{1}{2} - m^2\left(1 + 2\Omega_H^2\right) - \Omega_0^2 + 2\Omega^2. \tag{36}$$

Finally, can write the exact solution of the Klein–Gordon equation (7) as follows,

$$\psi(t,r,\theta) = e^{-i\frac{E}{\hbar c}ct} e^{im\theta} e^{\frac{1}{2}\alpha z} z(z-1)^{\frac{1}{2}(\gamma-1)} \times \times\left[Az^{\frac{\beta}{2}}\,\mathrm{HeunC}(z) + Bz^{-\frac{\beta}{2}}\,\mathrm{HeunC}'(z)\right], \tag{37}$$

where $z = -\frac{r_H - r}{r_H}$.

## 2.2 Energy levels

Polynomial functions have a very important role in quantum mechanics. In the matter of the scalar's radial solution, the polynomial function is essentially responsible for the behavior of scalar field between zero and infinity. Moreover, the degree of the respective polynomial determines how many zeroes the radial wave function has [30]. Now, let us apply the polynomial condition of the confluent Heun function (see Appendix B) to obtain the energy levels expression as follows,

$$\frac{\delta}{\alpha} + \frac{\beta + \gamma}{2} = -(n_r + 1) \qquad n_r + 1 = n = 1, 2, 3, \ldots \tag{38}$$

$$\frac{\Omega_0^2 - 2\Omega^2}{2\sqrt{\Omega_0^2 - \Omega^2}} + i\left(\Omega - m\Omega_H + \frac{|\gamma|}{2}\right) = -n. \tag{39}$$

For the case $\frac{\Omega^2}{\Omega_0^2} \ll 1$, the equation can be solved algebraically as follows,

$$\Omega_{m,n} = \frac{2i}{3}\Omega_0 + \frac{1}{3}\sqrt{-4\Omega_0^2 - 12i\Omega_H m + 6i\,|\gamma| + 12n + 6}. \tag{40}$$

We can compare the exact quasibound states' frequencies by solving the exact formula (39) with the one presented in [12] that are numerically calculated by Numerov integration method. Following [12], we set $\Omega_H = 1.0224$, $n_r = 0$ and $m = 1$ and varying $\Omega_0$ following Table 1 in the aforementioned reference. The comparison between the two are presented in Table 1. Comparing both of the real and imaginary parts of the exact and numerical $\Omega$, we can conclude that the real part of $\Omega$ are in agreement with the numerically estimated values, while for the imaginary part, which is related to the quasibound states' lifetime, our analytic formula provides exact results improving the previously published numerical estimations.

## 2.3 Wave function in extreme limits

In this subsection, investigate the behaviour of the quasibound states solution near the horizon $r \to r_H$ and far away from the horizon $r \to \infty$ will be investigated. In this $r \to r_H$ limit, the confluent Heun functions' argument $r - r_H$ is approaching 0, thus, $\mathrm{HeunC}(0) = \mathrm{HeunC}'(0) \approx 1$. Also $e^{-\frac{1}{2}\alpha\left(\frac{r-r_H}{r_H}\right)} \approx 1$. Thus, we can rewrite the wave function (37) as follows,

$$\psi_{\to r_H} = e^{-i\frac{E}{\hbar c}ct} e^{im\theta}\left[A\left(\frac{r - r_H}{r_H}\right)^{\frac{1}{2}\beta} + B\left(\frac{r - r_H}{r_H}\right)^{-\frac{1}{2}\beta}\right]. \tag{41}$$

Now, for the sake of notation aesthetic, we redefine $\frac{r - r_H}{r_H} = \zeta r - \zeta_0$ and expressing $\beta$ explicitly as in (33). This leads to this following equation,

$$\psi_{\to r_H} = e^{i\frac{E}{\hbar c}ct} Y_\ell^{m_\ell}(\theta,\phi) \left[A(\zeta r - \zeta_0)^{\frac{i|\beta|}{2}} + B(\zeta r - \zeta_0)^{-\frac{i|\beta|}{2}}\right], \tag{42}$$

and using the complex relation $z^i = e^{1\ln(z)}$ together with $\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$, we get,

$$\psi_{\to r_H} = e^{-i\frac{E}{\hbar c}ct} e^{im\theta}\left[C\cos\left(\zeta r - \zeta_0\right)\right], \tag{43}$$

which represent a purely ingoing wave.

Thus, approaching $r_H$ the solution represents a purely ingoing wave while approaching $\infty$, the exponential part of the wave function, i.e. $e^{-\frac{1}{2}\alpha\left(\frac{r-r_H}{r_H}\right)}$, suppresses the confluent Heun polynomial and quenches the whole wave function, i.e. $\psi_{\to\infty} \approx 0$.

# 3 Hawking radiation

With the exact radial solution in hand, in this section, the Damour–Ruffini [31] method will be applied to calculate the Hawking temperature of black hole's horizon. We start by expressing the exact radial solution in the near the horizon limit, i.e. $r \to r_H$. In this limit, $\mathrm{HeunC}(0) = \mathrm{HeunC}'(0) \approx 1$ and $e^{-\frac{1}{2}\alpha(z)} \approx 1$. This condition allows us to rewrite the wave function (37) as follows,

$$\psi_{\to r_H} = e^{-i\frac{E}{\hbar c}ct} e^{im\theta} z\left[Az^{\frac{\beta}{2}} + Bz^{-\frac{\beta}{2}}\right], \tag{44}$$

**Table 1** Comparison of the exact and numerical values of Ω

| $O_0$ | Re (O) ref | Re (O) exact | Im (O) ref | Im (O) exact |
|---|---|---|---|---|
| 0.35202527 | 0.35034321 | 0.345998585 | $0.078\times10^{-5}$ | 0.00176420428 |
| 0.43203101 | 0.42909303 | 0.421994807 | $0.080\times10^{-5}$ | 0.00385925494 |
| 0.52803790 | 0.52298580 | 0.511973344 | $0.230\times10^{-5}$ | 0.00777173635 |
| 0.62582270 | 0.61784984 | 0.602441353 | $0.500\times10^{-5}$ | 0.01340131797 |
| 0.73338598 | 0.72119458 | 0.712366787 | $1.080\times10^{-5}$ | 0.02256533483 |
| 0.83917135 | 0.82167984 | 0.813036093 | $2.040\times10^{-5}$ | 0.03312585948 |
| 0.93517824 | 0.91176974 | 0.904584445 | $3.090\times10^{-5}$ | 0.04442945757 |
| 1.0498531 | 1.0178151 | 1.014315012 | $3.640\times10^{-5}$ | 0.05996517330 |
| 1.1583054 | 1.1163115 | 1.118583814 | $-0.462\times10^{-5}$ | 0.07658340292 |

where the factor $(-1)^{\frac{1}{2}(\gamma-1)}$ has been absorbed into the normalization constant $A$ and $B$.

The wave function (44) can be viewed as consisting of two parts, that are the ingoing and outgoing waves,

$$\psi=\begin{cases}\psi_{+in}=Az^{\frac{1}{2}\beta} & ingoing\\ \psi_{+out}=Bz^{-\frac{1}{2}\beta} & outgoing\end{cases}, \tag{45}$$

When there is an ingoing wave hitting the horizon $r_H$, a particle-antiparticle pair are induced. The particle is reflected and will enhance outgoing wave while the antiparticle counterpart becomes the transmitted wave crossing the horizon. Analytical continuation of the wave function $\psi(z)$ can be calculated by using this following trick,

$$z^{\lambda}\to\left[\left(\frac{r}{r_H}-1\right)+i\epsilon\right]^{\lambda}=\begin{cases}z^{\lambda} & , r>r_H\\ |z|^{\lambda}e^{i\lambda\pi} & , r<r_H\end{cases}. \tag{46}$$

This allows us to obtain the expression for $\psi_{-out}$ as follows,

$$\psi_{-out}=\psi_{+out}, \tag{47}$$

$$z\to ze^{i\pi}, \tag{48}$$

$$z\to -z=ze^{i\pi}, \tag{49}$$

that leads to,

$$\begin{aligned}\psi_{-out}&=B\left(ze^{i\pi}\right)^{-\frac{1}{2}\beta},\\ &=\psi_{+out}e^{-\frac{1}{2}i\pi\beta}\end{aligned} \tag{50}$$

$$\left|\frac{\psi_{-out}}{\psi_{+in}}\right|^2=\left|\frac{\psi_{+out}}{\psi_{+in}}\right|^2e^{-i2\pi\beta}=\left|\frac{\psi_{+out}}{\psi_{+in}}\right|^2e^{4\pi[\Omega-m\Omega_H]}. \tag{51}$$

The normalization of the total probability leads to,

$$\left|\frac{\psi_{-out}}{\psi_{+in}}\right|^2+\left|\frac{\psi_{+out}}{\psi_{+in}}\right|^2=1, \tag{52}$$

i.e. the total probablity of the particle wave going out to infinity and the antiparticle wave going inside the black hole must be equal to 1. Near the $r_H$, thus, we have,

$$\psi_{out}=\begin{cases}\psi_{+out} & , r>r_H\\ \psi_{-out} & , r<r_H\end{cases}, \tag{53}$$

or can be rewritten as follows,

$$\psi_{out}=Bz^{-\frac{1}{2}\beta}\left[\theta(-z)+\theta(z)\,e^{2\pi[\Omega-m\Omega_H]}\right]. \tag{54}$$

The Hawking temperature $T_H$ of the corresponding horizon is to be extracted from the thermal spectrum known as radiation distribution function by calculating the modulus square of the ratio between the normalization constant between the outgoing and incoming waves as follows,

$$\left\langle\frac{\psi_{out}}{\psi_{in}}\middle|\frac{\psi_{out}}{\psi_{in}}\right\rangle=1=\left|\frac{B}{A}\right|^2\left|1-e^{4\pi[\Omega-m\Omega_H]}\right|, \tag{55}$$

$$\left|\frac{B}{A}\right|^2=\frac{1}{e^{4\pi[\Omega-m\Omega_H]}-1}. \tag{56}$$

Now, let us consider this following algebraic modification,

$$4\pi[\Omega-m\Omega_H]=4\pi\left[\frac{Er_H}{\hbar c}-m\Omega_H\right]=\frac{\hbar(\omega-\omega_H)}{\left[\frac{c\hbar}{4\pi r_H}\right]}, \tag{57}$$

$$\omega_H=\frac{cm\Omega_H}{r_H}. \tag{58}$$

Finally, by comparing with the bosonic distribution function $e^{\frac{\hbar\omega}{k_BT}}$ we obtain the analog black hole's horizon temperature as follows,

$$T_H=\frac{c\hbar}{4\pi k_B r_H}. \tag{59}$$

The expression (59) is equal to the Hawking temperature of chargeless Lense-Thirring black hole $T_H=\frac{r_H c\hbar}{4\pi k_B r_s^2}$ in the recent published result [26].

## 4 Conclusions

In this work, the exact analytical phonon quasibound state's quantized energy levels and their wave functions in a vortex photon flow analog black hole are derived in detail. The exact angular solution is found in terms of pure harmonics functions while the radial exact solutions are discovered in terms of confluent Heun functions (30). The obtained exact radial solution is valid for all region of interest, i.e. $r_s \leq r < \infty$, a significant improvement of assymptotical method that can only solves for either region very close to the horizon of very far away from the horizon which was presented in [12]. Investigation of the obtained radial solution around the horizon shows a purely ingoing wave while far away from the horizon, the wave function is decaying exponentially.

By making use of the confluent Heun function's polynomial condition, the quantized energy levels' expression is succesfully derived and presented in (39). Comparison of the values of $\Omega$ between the previously published results [12] and the the exact results by solving Eq. (39) is presented in Table 1 where the real parts of the $\Omega$ are convergent while in this letter, we present for the first time, the exact values of the imaginary part of the $\Omega$. And in low frequncy limit, we obtain a simplified algebraic expression of the energy levels (40). In the last section, after indentifying the ingoing and outgoing component of the wave funtion, the [31] method is applied to find the Hawking temperature of the black hole's horizon (59).

**Data Availability Statement** This manuscript has no associated data or the data will not be deposited. [Authors' comment: There is no associated data used in this manuscript.]

**Code Availability Statement** The manuscript has no associated code/software. [Author's comment: The manuscript has no associated code as it's not a computational or numerical work.]


Funded by SCOAP$^3$.

## A Normal form

An ordinary differential equation is said to be in the normal form if is solved explicitly for the highest derivative [32]. One may start with a general form of the linear second order ordinary differential equation as follows,

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0. \tag{60}$$

In order to bring the linear second order ordinary differential equation above to its normal form, we express the $y(x)$ in this particular form, which is specially designed to remove the first order derivative terms [33],

$$y = Y(x)e^{-\frac{1}{2}\int p(x)dx}, \tag{61}$$

$$\frac{dy}{dx} = \frac{dY}{dx}e^{-\frac{1}{2}\int p(x)dx} - \frac{1}{2}Ype^{-\int p(x)dx}, \tag{62}$$

$$\frac{d^2y}{dx^2} = \frac{d^2Y}{dx^2}e^{-\frac{1}{2}\int p(x)dx} - \frac{1}{2}\frac{dY}{dx}pe^{-\frac{1}{2}\int p(x)dx} - \frac{1}{2}Y\frac{dp}{dx}e^{-\frac{1}{2}\int p(x)dx} + \frac{1}{4}Yp^2e^{-\frac{1}{2}\int p(x)dx}. \tag{63}$$

Substituting the expressions to (60), a lot of things cancel out and it resulted in this following equation where the first order derivative term has been removed,

$$\frac{d^2Y}{dx^2} + \left(-\frac{1}{2}\frac{dp}{dx} - \frac{1}{4}p^2 + q\right)Y = 0, \tag{64}$$

$$Y = ye^{\frac{1}{2}\int p(x)dx}. \tag{65}$$

It is important to mention that if $Q(x) = -\frac{1}{2}\frac{dp}{dx} - \frac{1}{4}p^2 + q < 0$ and $Y(x)$ is a nontrivial solution of (65), then $Y(x)$ does not oscillate at all and has at most one zero. And if $Q(x) = -\frac{1}{2}\frac{dp}{dx} - \frac{1}{4}p^2 + q > 0$ and $\int_1^\infty Q(x)dx = \infty$, then $Y(x)$ has infinitely many zeros on the positive $x$-axis [33].

## B Normal form of confluent Heun equation

The confluent Heun equation is a second-order linear differential equation which has three regular singularities given by in this following form [34],

$$\frac{d^2y}{dx^2} + \left(\alpha + \frac{\beta+1}{x} + \frac{\gamma+1}{x-1}\right)\frac{dy}{dx} + \left(\frac{\mu}{x} + \frac{\nu}{x-1}\right)y = 0, \tag{66}$$

$$\nu = \frac{1}{2}(\alpha + \beta + \gamma + \alpha\beta + \beta\gamma) + \delta + \eta. \tag{67}$$

The solutions of the confluent Heun equation are given in the terms of the five parameters confluent Heun functions,

$$y = A\,\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, x) + Bx^{-\beta}\,\text{HeunC}(\alpha, -\beta, \gamma, \delta, \eta, x), \tag{68}$$

$$y = A\,\text{HeunC}(x) + Bx^{-\beta}\,\text{HeunC}'(x) = \text{HeunC}(x)\,. \tag{69}$$

The confluent Heun function is reduced to be a polynomial function with degree $n_r$ if this following condition is fulfilled,

$$\frac{\delta}{\alpha}+\frac{\beta+\gamma}{2}+1=-n_r, \quad n_r\in\mathbb{Z}. \tag{70}$$

Now, let us express confluent Heun's differential equation in its the normal form. First, we indentify the $p$ and $q$ functions (see Appendix A) as follows,

$$p=\alpha+\frac{\beta+1}{x}+\frac{\gamma+1}{x-1}, q=\frac{\mu}{x}+\frac{\nu}{x-1}, \tag{71}$$

$$y(x)=\mathrm{HeunC}=Y(x)e^{-\frac{1}{2}\alpha x}x^{-\frac{1}{2}(\beta+1)}(x-1)^{-\frac{1}{2}(\gamma+1)}, \tag{72}$$

and this leads to,

$$-\frac{1}{2}\frac{dp}{dx}=\frac{1}{x^2}\left(\frac{\beta+1}{2}\right)+\frac{1}{(x-1)^2}\left(\frac{\gamma+1}{2}\right), \tag{73}$$

$$-\frac{1}{4}p^2=-\frac{\alpha^2}{4}-\frac{1}{x^2}\left(\frac{\beta^2+1+2\beta}{4}\right) -\frac{1}{(x-1)^2}\left(\frac{\gamma^2+1+2\gamma}{4}\right)-\frac{2}{x}\left(\frac{\alpha\beta+\alpha}{4}\right) -\frac{2}{x-1}\left(\frac{\alpha\gamma+\alpha}{4}\right)-\frac{2}{x\,(x-1)}\left(\frac{\beta\gamma+1+\beta+\gamma}{4}\right), \tag{74}$$

$$W(x)=-\frac{1}{2}\frac{dp}{dx}-\frac{1}{4}p^2+q=-\frac{\alpha^2}{4}+\frac{\frac{1}{2}-\eta}{x}+\frac{\frac{1}{4}-\frac{\beta^2}{4}}{x^2} +\frac{-\frac{1}{2}+\delta+\eta}{x-1}+\frac{\frac{1}{4}-\frac{\gamma^2}{4}}{(x-1)^2}. \tag{75}$$

Combining everything, we transform the confluent Heun equation's to be in a normal form as follows,

$$\frac{d^2Y(x)}{dx^2}+W(x)Y(x)=0, \tag{76}$$

$$Y(x)=e^{\frac{1}{2}\alpha x}x^{\frac{1}{2}(\beta+1)}(x-1)^{\frac{1}{2}(\gamma+1)}\,\mathrm{HeunC(x)}\,. \tag{77}$$

## References


1. W.G. Unruh, Experimental black-hole evaporation? Phys. Rev. Lett. **46**, 1351–1353 (1981). https://doi.org/10.1103/PhysRevLett.46.1351
2. A. Mathie, *Black Holes and Analogy* (Springer, Cham, 2023), pp.249–276
3. M. Visser, *Survey of Analogue Spacetimes* (Springer, Berlin, 2013), pp.31–50
4. W.G. Unruh, Dumb holes: analogues for black holes. Philos. Trans. Roy. Soc. A Math. Phys. Eng. Sci. **366**(1877), 2905–2913 (2008). https://doi.org/10.1098/rsta.2008.0062
5. B.S. Kandemir, Hairy BTZ black hole and its analogue model in graphene. Ann. Phys. **413**, 168064 (2020). https://doi.org/10.1016/j.aop.2019.168064
6. B. Horstmann, B. Reznik, S. Fagnocchi, J.I. Cirac, Hawking radiation from an acoustic black hole on an ion ring. Phys. Rev. Lett. **104**, 250403 (2010). https://doi.org/10.1103/PhysRevLett.104.250403
7. U. Leonhardt, P. Piwnicki, Relativistic effects of light in moving media with extremely low group velocity. Phys. Rev. Lett. **84**, 822–825 (2000). https://doi.org/10.1103/PhysRevLett.84.822
8. R. Chiao, J. Boyce, Bogoliubov dispersion relation and the possibility of superfluidity for weakly-interacting photons in a 2D photon fluid. Phys. Rev. A Phys. Rev. A **60**, 4114–4121 (1999). https://doi.org/10.1103/PhysRevA.60.4114
9. J. Klaers, J. Schmitt, F. Vewinger, M. Weitz, Bose–Einstein condensation of photons in an optical microcavity. Nature **468**(7323), 545–548 (2010). https://doi.org/10.1038/nature09567
10. I. Carusotto, C. Ciuti, Quantum fluids of light. Rev. Mod. Phys. **85**, 299–366 (2013). https://doi.org/10.1103/RevModPhys.85.299
11. F. Marino, Massive phonons and gravitational dynamics in a photon-fluid model. Phys. Rev. A **100**, 063825 (2019). https://doi.org/10.1103/PhysRevA.100.063825
12. M. Ciszak, F. Marino, Acoustic black-hole bombs and scalar clouds in a photon-fluid model. Phys. Rev. D (2021). https://doi.org/10.1103/physrevd.103.045004
13. S. Hod, Stationary scalar clouds supported by rapidly-rotating acoustic black holes in a photon-fluid model. Phys. Rev. D **103**, 084003 (2021). https://doi.org/10.1103/PhysRevD.103.084003
14. S. Hod, No-short scalar hair theorem for spinning acoustic black holes in a photon-fluid model. Phys. Rev. D **104**, 104041 (2021). https://doi.org/10.1103/PhysRevD.104.104041
15. M.C. Braidotti, F. Marino, E.M. Wright, D. Faccio, The Penrose process in nonlinear optics. AVS Quantum Sci. **4**(1), 010501 (2022). https://doi.org/10.1116/5.0073218. https://pubs.aip.org/avs/aqs/article-pdf/doi/10.1116/5.0073218/16493402/010501_1_online.pdf
16. J. Lense, H. Thirring, Über den Einfluß der Eigenrotation der Zentralkörper auf die Bewegung der Planeten und Monde nach der Einsteinschen Gravitationstheorie. Physikalische Zeitschrift **19**, 156 (1918). ADS Bibcode: 1918PhyZ...19..156L. Accessed 2023-01-09
17. P. Herbert, On the history of the so-called Lense-Thirring effect. Gen. Relativ. Gravit. **3**, 9 (2007)
18. P.J. Armitage, P. Natarajan, Lense-Thirring precession of accretion disks around compact objects. Astrophys. J. **525**(2), 909–914 (1999). https://doi.org/10.1086/307955
19. P. Bambhaniya, J.V. Trivedi, D. Dey, P.S. Joshi, A.B. Joshi, Lense–Thirring effect and precession of timelike geodesics in slowly rotating black hole and naked singularity spacetimes. Phys. Dark Univ. **40**, 101215 (2023). https://doi.org/10.1016/j.dark.2023.101215
20. L. Iorio, The Lense Thirring effect on the Galilean moons of Jupiter. Universe (2023). https://doi.org/10.3390/universe9070304
21. C. Chakraborty, P. Majumdar, Strong gravity Lense–Thirring precession in Kerr and Kerr-Taub-nut spacetimes. Class. Quantum Gravity **31**(7), 075006 (2014). https://doi.org/10.1088/0264-9381/31/7/075006
22. BPea Abbott, Observation of gravitational waves from a binary black hole merger. Phys. Rev. Lett. **116**, 061102 (2016). https://doi.org/10.1103/PhysRevLett.116.061102
23. H.S. Vieira, K.D. Kokkotas, Quasibound states of Schwarzschild acoustic black holes. Phys. Rev. D **104**, 024035 (2021). https://doi.org/10.1103/PhysRevD.104.024035
24. H.S. Vieira, K. Destounis, K.D. Kokkotas, Analog Schwarzschild black holes of Bose–Einstein condensates in a cavity: quasinormal modes and quasibound states. Phys. Rev. D **107**, 104038 (2023). https://doi.org/10.1103/PhysRevD.107.104038
25. D. Senjaya, Exact massive and massless scalar quasibound states around a charged Lense–Thirring black hole. Phys. Lett. B **849**, 138414 (2024). https://doi.org/10.1016/j.physletb.2023.138414
26. D. Senjaya, Exact analytical quasibound states of a scalar particle around a slowly rotating black hole. J. High Energy Astrophys. **40**, 49–54 (2023). https://doi.org/10.1016/j.jheap.2023.10.002

27. D. Senjaya, Exact analytical quasibound states of a scalar particle around a Reissner–Nordström black hole. Phys. Lett. B **848**, 138373 (2024). https://doi.org/10.1016/j.physletb.2023.138373
28. H.S. Vieira, V.B. Bezerra, Confluent Heun functions and the physics of black holes: resonant frequencies, hawking radiation and scattering of scalar waves. Ann. Phys. **373**, 28–42 (2016). https://doi.org/10.1016/j.aop.2016.06.016
29. D. Senjaya, A.S. Rivera, Canonical quantization of neutral and charged static black hole as a gravitational atom. J. Phys. Conf. Ser. **1719**, 012019 (2021)
30. L.I. Deych, Advanced Undergraduate Quantum Mechanics: Methods and Applications. Springer, Berlin (2018). https://books.google.co.th/books?id=8i3QxgEACAAJ
31. T. Damour, R. Ruffini, Black-hole evaporation in the Klein–Sauter–Heisenberg–Euler formalism. Phys. Rev. D **14**, 332 (1976)
32. F.W.J. Olver, R.F.B.D.W. Lozier, C.W. Clark, *NIST Handbook of Mathematical Functions*, 1st edn. (Cambridge University Press, Cambridge, 2010)
33. G.F. Simmons, *Differential Equations with Application and Historical Notes*, 2nd edn. (Tata Mcgraw Hill, New Delhi, 1991)
34. A. Ronveaux, *Heun's Differential Equations*, 1st edn. (Clarendon Press, Oxford, 1995)