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Quantum aspects of Pohlmeyer-reduced $AdS_5 \times S^5$ superstring

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Declaration of originality

The materials presented in the thesis "Quantum aspects of Pohlmeyer-reduced $AdS_5 \times S^5$ superstring" are entirely the result of my research under the supervision of Arkady Tseytlin. All of the publications used in this thesis are fully referenced.

The list of my publications discussed in this thesis is as follows:

[HIT] B. Hoare, Y, Iwashita and A. A. Tseytlin, "Pohlmeyer-reduced form of string theory in $AdS_5 \times S^5$: Semiclassical expansion," J. Phys. A **42** (2009) 375204 [arXiv:0906.3800].

[IWA] Y. Iwashita, "One-loop corrections to $AdS_5 \times S^5$ superstring partition function via Pohlmeyer reduction," J. Phys. A **43**, 345403 (2010) [arXiv:1005.4386].

[IRT] Y. Iwashita, R. Roiban and A. A. Tseytlin, "Two-loop corrections to partition function of Pohlmeyer-reduced theory for $AdS_5 \times S^5$ superstring," Phys. Rev. D **84**, 126017 (2011) [arXiv:1109.5361].

Abstract

The $AdS_5 \times S^5$ superstring action is constructed by the Green-Schwarz formalism. For quantization it is necessary to eliminate unphysical degrees of freedom from the action by solving the Virasoro constraints and fixing the fermionic kappa-symmetry, which can be achieved by the Pohlmeyer reduction preserving the two-dimensional Lorentz invariance and the integrability. The resulting system is a gauged Wess-Zumino-Witten (gWZW) model deformed with a certain integrable potential and two-dimensional fermions. This thesis explores the quantum relation between the $AdS_5 \times S^5$ superstring theory and the deformed gWZW model by evaluating the reduced theory quantum partition functions for respective classical string configurations.

To understand the quantum relation between the original string theory and the reduced theory, the one-loop computation in the reduced theory is first studied for homogeneous and inhomogeneous string configurations localized in subspaces. For these classical backgrounds we demonstrate that the reduced theory partition function is exactly the same as the string theory one, then they are equivalent at one-loop level.

Next we investigate the two-loop relation between the original string theory the reduced theory. The two-loop computation in the reduced theory is performed by considering the long folded string localized in AdS_3. We show that the nontrivial finite terms of the two-loop partition functions of the two theories match, exhibiting the same patterns of the bosonic contributions and fermionic contributions. This is a strong indication that the $AdS_5 \times S^5$ GS string and its reduced form are closely related at the quantum level.

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Chapter 1

Introduction

Overview and motivation

The original motivation to introduce the one-dimensional object to elementary particle physics was to construct a phenomenological model of strong interactions. This idea was forgotten after the discovery of quantum chromodynamics (QCD) and due to shortcomings of the string theory in four dimensions. Later physicists considered QCD string as a convenient tool for studying strongly coupled regime of gauge interactions. While perturbative QCD is very successful, to address the questions about its strong-coupling dynamics one needs alternative tools. String theory is one of them.

A revival of string theory, as a unified theory of all the fundamental interactions, was driven by the discovery that string theory can be a consistent, i.e., anomaly free, theory [1]. The perturbative string theory in the first revolution during the 1980's is summarized in [2] and references therein.

A remarkable progress during the 1990's, including a discovery of a new solitonic object, D-brane [11], and string duality web accommodated by eleven-dimensional M-theory [3, 4, 5, 6, 7, 8, 9, 10], allowed for nonperturbative exploration not only in string theory but also in gauge theory. The D-brane is a hypersurface where open strings can attach. In the low energy limit open strings are described by gauge theory on the worldvolume of the D-brane. At the same time, a D-brane deforms the spacetime around it; when the number of the branes is large, the stack of D-branes can be described as a supergravity soliton. Given the two different descriptions of the D-brane, one might expect a connection between gauge theory and closed string theory [12, 13].

The conjectured duality, the anti-de Sitter/conformal field theory (AdS/CFT) correspondence, elaborates this expectation and states the equivalence of a particular closed string theory and quantum field theory. Strong/weak coupling nature of the AdS/CFT allows us to investigate strongly coupled field theories in terms of weekly coupled gravity, which stimulated its application to broad range of areas such as the holographic approach to QCD (see review article [14] and references therein) and the connection between condensed matter and gravitational physics (recent reviews are given in [15, 16]).

The most celebrated example in the AdS/CFT correspondence is the equivalence between type IIB superstring theory in $AdS_5 \times S^5$ and the $\mathcal{N} = 4$ $SU(N)$ Super Yang-Mills (SYM) theory. On the gauge theory side, the theory is controlled by the Yang-Mills coupling $g_{\rm YM}$ and the rank of the gauge group N, whereas the parameters on the string theory side are the string coupling g_s and string length ℓ_s. Together with the identical radii of AdS_5 and S^5, R, these parameters are related in the AdS/CFT duality as

$$g_{\rm YM}^2 = 4\pi g_s \,, \qquad R^4 = 4\pi g_s N \ell_s^4 \,. \tag{1.1}$$

These relations explicitly show that the AdS/CFT is the strong/weak type duality. For example, when the gauge theory is strongly coupled $\lambda = g_{\rm YM}^2 N \gg 1$, the string theory is at weak coupling $R/\ell_s \gg 1$, that is, the type IIB supergravity is a good approximation. The dictionary of the AdS/CFT correspondence provides a map between a string state in $AdS_5 \times S^5$ and a local gauge invariant operator in SYM; one can identify the energy of a given string state, E, with the conformal dimension of the dual operator, $\triangle$ [17, 18],

$$E(R/\ell_s, g_s) = \triangle(\lambda, 1/N) \,. \tag{1.2}$$

Hence examining this formula in various regions in the parameter space is a nontrivial check of the AdS/CFT duality. At early stage, this mapping was only confirmed for a small subset of the operators; the chiral primaries in the gauge theory dual to the string states that survive in the supergravity limit. In general, one should quantize strings in $AdS_5 \times S^5$, express the string energy spectrum in terms of R/ℓ_s, g_s and show its equivalence to gauge theory spectrum.

A nontriviality in formulating superstring theory in $AdS_5 \times S^5$ stems from the existence of the Ramond-Ramond (RR) background. Superstring action in a curved background supported by nonvanishing RR fields can be constructed using the Green-Schwarz (GS) formalism leading to the action which is invariant under the spacetime supersymmetry transformation [19, 20] (see [21, 22] for GS action in a general supergravity background). The paper [23] introduced the reinterpretation of the GS action as a two-dimensional sigma model on the coset superspace F/G, where F is the target superspace isometry and G is its subgroup. Using the coset approach to the GS superstring, the $AdS_5 \times S^5$ case was considered in [24, 25]. In this case the coset is

$$F/G = \frac{PSU(2,2|4)}{Sp(2,2) \times Sp(4)} \,. \tag{1.3}$$

The authors there employed the exponential parametrization for the coset elements and solved the Maurer-Cartan equations, then showed that the resulting action possesses the local fermionic κ symmetry by introducing the three-dimensional topological Wess-Zumino (WZ) term. For quantization we need to eliminate unphysical degrees of freedom. Fixing the local fermionic κ symmetry and choosing the light-cone gauge necessarily causes the problem that the resulting action contains terms quadratic and quartic in the fermions and the gauge-fixed action does not exhibit the two-dimensional Lorentz invariance beyond the quadratic level [26, 27].

Although the $AdS_5 \times S^5$ superstring sigma model is in general nonlinear, it can be simplified by taking a specific limit [28, 29], where the background geometry takes the form of the plane wave metric supported by a RR 5-from field. This is one of the three maximally supersymmetric solutions of type IIB supergravity, the other two are flat spacetime and $AdS_5 \times S^5$ spacetime [30, 31]. In the plane wave background the GS string becomes a free massive theory once one chooses the light-cone gauge, and so, it is straightforwardly quantized as in the flat background.

Furthermore, in [32], Berenstein, Maldacena and Nastase (BMN) proposed the duality relating type IIB superstrings in the maximally supersymmetric plane wave background to the four-dimensional $\mathcal{N} = 4$ SYM theory in the so-called BMN limit,

$$N \to \infty, \text{ and then, } J \to \infty \text{ with } \tilde{\lambda} \equiv \frac{\lambda}{J^2} \text{ fixed,} \tag{1.4}$$

where J is an R-charge in the SYM side, or equivalently, an angular momentum in S^5 in the string side. With the agreement in the asymptotic behavior between the spectra of $E - J$ in the string theory side and $\triangle - J$ in the gauge theory side, a concrete AdS/CFT dictionary relating string states and SYM operators in the near-BPS sector was constructed.

Despite the large amount of supporting evidence and the wide range of applications of the AdS/CFT correspondence, no rigorous proof of it was given so far. This is due to the difficulty of the string quantization in a nontrivial background. It is integrability that allows us to deal with this difficulty and provides a better way of understanding both the $AdS_5 \times S^5$ superstring theory and $\mathcal{N} = 4$ SYM theory.[1] On the string theory side, the classical integrability for the $AdS_5 \times S^5$ bosonic string theory was found based on a coset model on $F/G = \frac{SU(2,2)\times SU(4)}{Sp(2,2)\times Sp(4)}$ in [34] and it was shown in [35] the classical integrability also exists in the full $AdS_5 \times S^5$ GS superstring theory. Using the classical integrability, the algebraic curve approach to the classical and semiclassical aspects of finite-gap string solutions was developed, e.g., in [36, 37, 38, 39] (for further quantum generalizations see e.g., [40, 41]).

In this thesis we are going to explore an approach to string theory in $AdS_5 \times S^5$ based on Pohlmeyer reduction. Initiated by Pohlmeyer's original work where it was shown that the equation of motion of the chiral S^2 sigma model is reduced to the sin-Gordon equation [42], the reduction technique was applied to conformal-gauge bosonic string theory, which involves studying the integrability of classical string motion in de Sitter spacetime [43, 44] and finding classical string configurations localized in subspace of $AdS_5 \times S^5$ [45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55]. The integrability property allows for powerful methods to generate soliton solutions

[1] The integrability on the gauge theory side is not discussed in this thesis. See the review article [33] and references therein for the progress in this direction.

in the reduced theory, and string solutions can be constructed from the solutions in the reduced theory. For example, string theory in $R \times S^3$ and string theory in $AdS_3 \times S^1$ are reduced to the complex sin-Gordon model and the complex sinh-Gordon model, respectively. More recently the Pohlmeyer reduction of AdS_3 string theory was used in evaluating the minimal area of an open string surface ending on null Wilson loop, which is related to the strong-coupling limit of $\mathcal{N} = 4$ SYM theory [56, 57, 58, 59, 60]. For this theory it is known that the reduced form is the generalized sinh-Gordon model. Although one can easily switch off the S^1 sector of most of the classical string solutions stretching in $AdS_3 \times S^1$, the relation between the complex sinh-Gordon model and the generalized sinh-Gordon model is understood only at the level of equation of motion. The extension of the above argument on the AdS_3 theory to the $AdS_n \times S^m$ case for general $n,\ m$ is nontrivial. Open string solutions stretching in AdS_4, AdS_5 or $AdS_3 \times S^3$ are discussed in [58, 59, 60, 61, 62, 63].

The Pohlmeyer-type reduction of the full $AdS_5 \times S^5$ superstring sigma model was proposed in [64, 65, 66] based on the Lagrangian formulation of the generalized sin-Gordon models reduced from sigma models in symmetric space [67]. In the construction of the reduced theory, new variables are algebraically related to supercoset current components, the Virasoro conditions are automatically solved, and the local fermionic κ symmetry is fixed. The resulting system is expressed as gauged Wess-Zumino-Witten (gWZW) model associated with the coset,

$$G/H = \frac{Sp(2,2) \times Sp(4))}{[SU(2)]^4}\,, \tag{1.5}$$

and deformed with an integrable potential and two-dimensional fermionic fields. The reduced Lagrangian exhibits the two-dimensional Lorentz invariance, and after integrating out the gauge fields, the reduced theory involves 8 bosonic degrees of freedom and 16 fermionic degrees of freedom, i.e., involves only physical degrees of freedom.

While the Pohlmeyer reduction connects the two theories at the level of equations of motion, the deformed gWZW model and the original superstring theory are classically different in the sense that they are governed by different Hamiltonians and Poisson bracket structures. However, this does not necessarily imply that their quantum theories are different. For bosonic string theory in the $R \times S^3$ subspace of $AdS_5 \times S^5$, the Faddeev-Reshetikhin reduction of the $SU(2)$ principal chiral sigma model is admitted [68], where the Hamiltonian and Poisson brackets for the original sigma model are replaced by new Hamiltonian and Poisson brackets. After quantization of this new system, the original sigma model is recovered in a certain limit. In this thesis we shall investigate the quantum relation between the full $AdS_5 \times S^5$ superstring sigma model and its Pohlmeyer-reduced form in terms of their partition functions.[2]

If the quantum relation between the two theories is uncovered, quantum aspects of the $AdS_5 \times S^5$ GS superstring could be understood by studying the Pohlmeyer-reduced form instead of the original string theory. There are several advantages in discussing the Pohlmeyer-reduced form of the $AdS_5 \times S^5$ superstring sigma model.

The primary point is that one solves the Virasoro constraints of the original theory preserving the Lorentz invariance on the worldsheet and the classical integrability. Hence the Pohlmeyer-reduced theory could be regarded as a starting point for a "first-principles" solution of the $AdS_5 \times S^5$ superstring and used to derive S-matrices for elementary excitation on the worldsheet with the two-dimensional Lorentz invariance [73, 74, 75, 76, 77, 78] (see also [53, 54, 55, 79, 80, 81] for the Pohlmeyer-reduced approach to sigma models on other symmetric spaces). This should be in contrast with lack of the $2d$ Lorentz invariance in the light-cone gauge $AdS_5 \times S^5$ superstring S-matrix determining the full quantum string spectrum once the quantum integrability is assumed [82, 83, 84, 85].[3] Further expectation in this direction could be to find an exact Lorentz-invariant S-matrix as for other two-dimensional Lorentz-invariant theories [86, 87].

Another advantage of the Pohlmeyer reduction is that the reduced model has a simpler quantum

[2]The $AdS_5 \times S^5$ superstring sigma model and the deformed gWZW model have common properties. As noticed in [64], the bosonic interaction potential and fermionic "Yukawa" term in the reduced theory are exactly the same as the original $AdS_5 \times S^5$ superstring Lagrangian expressed in terms of the new variables. Also, the UV finiteness was confirmed up to two-loop order for the $AdS_5 \times S^5$ GS superstring theory in [69, 70, 71] and for the Pohlmeyer-reduced theory in [72].

[3]In [77, 78] the S-matrix in the Pohlmeyer-reduced theory was studied by using the quantum group deformation, where the S-matrix describing the magnon excitations for the original string theory is realized as a limiting case and the relativistic reduced theory S-matrix is obtained in another limit. However, the direct connection between the light-cone gauge $AdS_5 \times S^5$ superstring S-matrix and the relativistic reduced theory S-matrix is still unclear. Also, there are significant differences between the quantum deformed S-matrix and the perturbative S-matrix in the reduced theory [75].

structure than the original theory does. In the bosonic subsector, this is because the reduced theory contains only physical degrees of freedom and one need not consider the interactions between physical and unphysical fluctuations (after gauge degrees of freedom are properly integrated out). For fermions, the Lagrangian in the reduced theory contains only quadratic terms in fermions in the "Yukawa" term, which means that fermionic loop never becomes higher than one even in higher-loop computation of the deformed gWZW model. Therefore, the reduced theory could be a useful alternative for the purpose of investigating the quantum strings in $AdS_5 \times S^5$.

Original contributions

This thesis is devoted to elucidating the quantum relation between $AdS_5 \times S^5$ superstring theory (ST) and its Pohlmeyer-reduced theory (PRT) by comparing their partition functions. The original research was published in [HIT, IWA, IRT].

In [HIT] it was conjectured that the one-loop equivalence of the two theories should be realized at the level of the quantum partition functions,

$$Z^{(1)}_{\rm ST} = Z^{(1)}_{\rm PRT}\,, \tag{1.6}$$

where (1) represents that both the reduced theory partition function and string theory partition function are at one-loop level. While the one-loop equivalence may be naively expected from that of the equations of motion, the explicit demonstration of (1.6) is needed for individual string configurations. The contribution of the paper [HIT] is to show that characteristic frequencies of quadratic fluctuations found in the GS superstring in the conformal gauge agree with those found in the reduced theory for cases of any classical string localized $AdS_2 \times S^2$ and homogeneous strings in $R \times S^3$ or $AdS_3 \times S^1$. All of the results there support the conjecture (1.6) .

In [IWA] the semiclassical equivalence of the original theory and reduced theory was investigated at the level of the fluctuation Lagrangian for more nontrivial classical string configurations lying in $AdS_3 \times S^1$. The examples of the classical solutions include the (S, J) folded string, the (S, J) circular string, the (S, J) spiky string and the generalized folded string with both the orbital momentum and the winding on a big circle of S^5. Especially, it was shown that the Lagrangian for quadratic fluctuations in the reduced theory is exactly the same as the fluctuation Lagrangian found by perturbing the Nambu action in the static gauge for the (S, J) folded string. This statement is stronger than (1.6).

In [IRT] the two-loop relation of the $AdS_n \times S^n$ GS string theory $(n = 3, 5)$ and its reduced form was investigated by evaluating the two-loop effective action of the reduced theory for the folded string solution localized in AdS_3. Then it was proposed that the PRT side of the relation (1.6) should be modified as

$$\begin{aligned} &\Gamma_{\rm PRT} = -\ln Z_{\rm PRT} = \frac{1}{2\pi}\ {\rm f}(k)\ V_2\,, \qquad {\rm f}(k) = {\rm a}_{1_{\rm PRT}} + \frac{2{\rm a}_{2_{\rm PRT}}}{k}\ +\ O(\frac{1}{k^2})\,, \\ &{\rm a}_{1_{\rm PRT}} = {\rm a}_{1_{\rm ST}}\,, \qquad {\rm a}_{2_{\rm PRT}} = {\rm a}_{2_{\rm ST}} + \frac{1}{4}{\rm a}_{1_{\rm ST}}^2\,, \end{aligned} \tag{1.7}$$

where V_2 is the string worldvolume and k is the coupling constant in the reduced theory assumed to be proportional to the effective string tension $\sqrt{\lambda}$. Although the two-loop effective action in the reduced theory contains extra term $\frac{1}{4}{\rm a}_{1_{\rm ST}}^2$, the reproduction of the nontrivial part ${\rm a}_{2_{\rm ST}}$ implies that these two theories are nontrivially related at the quantum level.

This thesis also contains two original results which were not published.

In chapter 3 we discuss the one-loop computation for several string configurations in $AdS_3 \times S^1$. If a classical string possesses both the orbital momentum and the winding on a big circle on of S^5, the reduced theory has $\mu_+ \neq \mu_-$ (see section 2.1 for the definition of $\mu_\pm$). In such a case, we did not get the exact agreement of characteristic frequencies of individual fluctuations in [IWA], while the sum of the frequencies in PRT agrees with the string theory result. One solution to this is given in section 3.5, where we consider the (S, J) circular string solution as an example.

The other result is the one-loop computation for a homogeneous solution in $R \times S^5$ in appendix C. The cases we discussed so far are classical string solutions localized in subspaces of $AdS_3 \times S^3$. Hence, this is the first exploration in the semiclassical structure of the reduced theory for a string truly stretching in $R \times S^5$. As expected, the one-loop equivalence is demonstrated, although the structure of the reduced model is slightly different from the cases mentioned before.

Contents of this thesis

We shall start in the next chapter with a review of the classical Pohlmeyer reduction for the $AdS_3 \times S^1$ bosonic string theory (section 2.1) and the $AdS_n \times S^n$ GS string theory (section 2.2) following the papers [HIT, 64]. For the bosonic string theory in $AdS_3 \times S^1$ one can explicitly write the relation between the embedding coordinates in the original string theory and two fields of the reduced theory. There we will introduce two reduced models called the coth model and the tanh model, which are connected by the nonlocal "T-duality" transformation. We shall show that these two models are embedded in the reduced model of the $AdS_5 \times S^5$ superstring theory and are related by the $H \times H$ gauge transformation rather than the H gauge transformation in the full reduction, giving rise to much simplification of reducing classical string solutions to the corresponding reduced theory solutions.

The main goal of chapter 3 is to demonstrate the one-loop equivalence of the original string theory and the reduced theory in terms of their quantum partition functions.

In section 3.1 we shall discuss the (S, J) folded string. A folded string in pure AdS_3 was first studied as the simplest string state whose classical energy grows logarithmically with the spacetime spin in AdS_3 [88], and soon after, this solution was extended to the (S, J) folded string solution in $AdS_3 \times S^1$ where the S^1 sector is the pointlike string moving along a big circle of S^5, i.e., the BMN state [89]. Its quadratic fluctuations were found from the Nambu action in the static gauge and in the Polyakov action in the conformal gauge [89, 90], and the equivalence of these two approaches was shown at the one-loop level in [91]. Here one may ask which type of the fluctuation Lagrangian arises in the reduced theory. We shall show that the fluctuation Lagrangian derived from the Nambu action is related to that of the coth model.

In section 3.2 we shall study the homogeneous (S, J) circular string solution which has both the angular momentum and the winding on a big circle of S^5, and gives us the first example of $\mu_+ \neq \mu_-$ (see section 2.1 for the definition of $\mu_\pm$). In [38, 92, 93] semiclassical expansions around the circular string were worked out, and in particular, its fermionic fluctuations are evaluated by carrying out the worldsheet computation in [93] and by employing the algebraic curve method in [38] (see also [94]). We shall show that the fluctuation Lagrangian of the reduced theory is equivalent to the Lagrangian found in [93]. As the circular string is homogeneous, we can evaluate characteristic frequencies of its quadratic fluctuations also in the reduced theory. In the case of the (S, J) circular string the total sum of the bosonic and fermionic frequencies in the reduced theory agrees with the string theory result although some of the individual frequencies appear to disagree.

A spiky string solution in AdS_3 was first found in [95] as a generalization of the folded string solution, and extended to a solution stretching in $AdS_3 \times S^1$ in [96]. In section 3.3 we shall discuss the (S, J) spiky string solution and derive its fluctuation Lagrangian of the reduced theory. Our result on the semiclassical expansions for the (S, J) spiky string is expected to agree with the one in the original string theory. As the (S, J) spiky string solution is also the case of $\mu_+ \neq \mu_-$ (see section 2.1 for the definition of $\mu_\pm$), we consider that the discrepancy could happen for the individual fluctuations, but the total sum the fluctuations should be the same as in the string theory. We shall also discuss the fluctuation Lagrangian in the limiting cases of the (S, J) spiky string; spiky string without motion or stretching in S^5, and its folded string limit.

In the large spin limit, spikes of the spiky string approach the conformal boundary of AdS_3, and the solution becomes locally equivalent to the scaling limit of the folded string. The existence of this homogeneous limit is shown in [96] where the expression for its string energy is similar to that for the homogeneous (S, J) folded string. A generalization of the (S, J) folded string solution leads to the explicit construction of another limiting solution of the (S, J) spiky string; the new folded string has both the orbital momentum and the winding in the S^1 sector [97]. This solution again has the scaling limit where the solution becomes homogeneous, which is the case we shall study in section 3.4. As happened in the circular string case, some individual characteristic frequencies in the reduced theory disagree with the result in [97], but the sum of the frequencies is the same as in the string theory.

The disagreement of characteristic frequencies of the individual fluctuations found in the $\mu_+ \neq$

μ_- cases in section 3.2 and section 3.4 can be resolved by modifying the relation between the embedding coordinates in the original string theory and two complex sinh-Gordon fields in the reduced theory. This prescription will be discussed in section 3.5 by considering the (S, J) circular string solution.

Chapter 4 will be devoted to investigating the two-loop relation between $AdS_n \times S^n$ GS string and its reduced theory for $n = 3, 5$. Since the two-loop computation for a general string configuration is complicated, we shall consider the infinite spin limit of the folded string in AdS_3. This case was studied in [69, 70] in the original string theory.

We will give a summary of the two-loop computation for the long spinning string in the original string theory and the reduced theory in section 4.1. While the non-trivial parts of the two two-loop partition functions agree, the reduced theory partition function contains an extra two-loop term proportional to the square of the one-loop coefficient in both the $n = 3$ and $n = 5$ cases.

Section 4.2 will review the action of reduced theory for string theory in $AdS_n \times S^n$ and then explain its perturbative expansion around a general classical configuration. We shall follow an approach based on the Polyakov-Wiegmann identity, by which the unphysical degrees of freedom contained in the gauge fields are isolated. We will present the fluctuation Lagrangian in this approach up to quartic order and show the Feynman diagrams that contribute to the two-loop partition functions in the reduced theory. In general, individual diagram contributions are gauge-dependent, so some graphs may or may not appear depending on the gauge choices. In the original string theory non-1PI diagrams did not contribute in the conformal gauge [69, 70], but did in the light-cone gauge [97]. In the present case we will also have nonvanishing contributions of the non-1PI diagrams in the reduced theory.

In section 4.3 we will consider the $AdS_3 \times S^3$ reduced theory using two approaches. The first approach is to use the Polyakov-Wiegmann identity mentioned above, where the unphysical degrees of freedom are still involved (approach I). The second approach is to impose a gauge on $g \in G$ and integrate out the gauge fields from the deformed gWZW Lagrangian (approach II). The resulting system contains only physical degrees of freedom and is described by sum of the complex sinh-Gordon and the complex sine-Gordon models coupled to two-dimensional fermions. We shall compare the results of the two approaches and suggest a resolution such that they provide the same result.

In section 4.4 we shall discuss the two-loop computation in the reduced $AdS_5 \times S^5$ theory. As approach II in $AdS_3 \times S^3$ case based on integrating out gauge fields first appears to be difficult, we will follow approach I using the Polyakov-Wiegmann identity. We will first show that the one-loop partition function in the reduced theory will match again the corresponding string theory result. Then we will discuss the two-loop computation and present the final expression for the two-loop coefficient (4.11) by a direct analogy with the $AdS_3 \times S^3$ case.

Chapter 5 will contain a summary and remarks on open problems.

The $\mathfrak{psu}(n, n|2n)$ superalgebra for $n = 1, 2$ will be summarized in appendix A, which will be used when we introduce component fields of the fluctuations in the reduced theory. In appendix B we shall relate the parametrization of the supercoset $\frac{PSU(2,2|4)}{Sp(2,2)\times Sp(4)}$ to the embedding coordinates in $AdS_5 \times S^5$. Appendix C will deal with the one-loop computation in the reduced theory for strings in $R \times S^5$. By the analogous computation in chapter 3 we will show the one-loop equivalence for a pulsating string in $R\times S^2$ in appendix C.1, a two-spin circular string in $R\times S^3$ in appendix C.2 and a short two-spin in $R \times S^5$ in appendix C.3. Appendix D will contain remarks on the computation for $AdS_5 \times S^5$ in chapter 4, including the investigation on nonlocal transformations such that physical modes decouple from unphysical modes in appendix D.1, the one-loop computation with an alternative gauge choice in appendix D.2, and the BMN limit in the two-loop computation in appendix D.3.

Chapter 2

Pohlmeyer reduction

This chapter is devoted to a description of the Pohlmeyer reduction of bosonic string theory in $AdS_3 \times S^1$ and GS string theory in $AdS_3 \times S^3$ and in $AdS_5 \times S^5$. Pohlmeyer originally proposed a way of reducing the equations of motion of the sigma model on $R \times S^2$ to the sin-Gordon equation [42]. This work is extended to the case of a sigma model on $R \times S^3$ and its reduced model is the complex sin-Gordon theory. The construction of these reduced model can be applied for the $AdS_3 \times S^1$ bosonic string theory with a slight modification. In section 2.1 we shall review the Pohlmeyer reduction of bosonic string theory in $AdS_3 \times S^1$ being based on the coset model on $SO(2,2)/SO(1,2)$. We will particularly focus on two reduced models called the coth model and the tanh model of the complex sinh-Gordon theory, which are related by the "T-duality" transformation as discussed in detail in [64] for complex sin-Gordon model. In section 2.2 we shall review the Pohlmeyer reduction of GS $AdS_n \times S^n$ string theory for $n = 3, 5$ following the papers [HIT, 64].

2.1 Pohlmeyer reduction of bosonic string theory in $AdS_3 \times S^1$

Our starting point is the worldsheet Lagrangian for a bosonic string propagating in $AdS_3 \times S^1$ spacetime,

$$L = L_{\rm AdS} + L_{\rm S} \,, \tag{2.1}$$

with

$$\begin{aligned} L_{\rm AdS} &= \partial_+ Y^P \partial_- Y_P - \Lambda \left(Y^P Y_P + 1 \right) \,, \\ L_{\rm S} &= \partial_+ X^M \partial_- X_M - \tilde{\Lambda} \left(X^M X_M - 1 \right) \,, \end{aligned} \tag{2.2}$$

where $\partial_\pm = \partial_\tau \pm \partial_\sigma$ and the contraction is defined by using $\eta = {\rm diag}(-1, 1, 1, -1)$ for $P, Q, \cdots = 0, 1, 2, 3$ indices (AdS_3 sector) and $\delta = {\rm diag}(1, 1)$ for $M, N, \cdots = 1, 2$ indices (S^1 sector).

Reflecting the fact that the S^1 sector of a string solution in $AdS_3 \times S^1$ is always homogeneous, the AdS part of the stress tensor satisfies $T^{\rm AdS}_{\pm\pm} = -\mu_\pm^2$ with constant $\mu_\pm$. Because the worldsheet of a closed string is a cylinder, it is not necessarily allowed to set $\mu_+ = \mu_- \equiv \mu$.[1] Instead we introduce the mass scale in the reduced theory by $\mu = \sqrt{\mu_+ \mu_-}$. This prescription will be used in the case of classical string with both the orbital momentum and the winding in the S^1 sector (see sections 3.2, 3.3, 3.4).

In the case of the $AdS_3 \times S^1$ bosonic string theory, we can explicitly write the relation between the embedding coordinates and scalar fields of the reduced theory. Let us first look at the construction of the coth model. Introduce a set of $O(2,2)$ vectors by Y_P, $\partial_+ Y_Q$, $\partial_- Y_R$ and $K_P \equiv \epsilon_{QRSP} Y^Q \partial_+ Y^R \partial_- Y^S$, and define ϕ and $\theta_{\scriptscriptstyle A}$ by

$$\begin{aligned} \partial_+ Y^P \partial_- Y_P &= -\mu^2 \cosh 2\phi_{\scriptscriptstyle A} \,, \\ K_P \, \partial_\pm^2 Y^P &= 4\mu^3 \cosh^2\phi_{\scriptscriptstyle A} \, \partial_\pm \chi_{\scriptscriptstyle A} \,, \end{aligned} \tag{2.3}$$

[1] We have $\mu_+ = \mu-$ in several cases, e.g., when the S^1 sector is the BMN vacuum.

then these ϕ_A and χ_A satisfy the complex sinh-Gordon equations,

$$\begin{aligned} &\partial_+\partial_-\phi_A + \tfrac{\cosh\phi_A}{\sinh^3\phi_A}\partial_+\chi_A\partial_-\chi_A + \tfrac{1}{2}\mu^2\sinh 2\phi_A = 0\,, \\ &\partial_+(\coth^2\phi_A\,\partial_-\chi_A) + \partial_-(\coth^2\phi_A\,\partial_+\chi_A) = 0\,, \end{aligned} \tag{2.4}$$

which follow from the Lagrangian of the coth model,

$$L_{\rm coth} = \partial_+\phi_A\partial_-\phi_A + \coth^2\phi_A\,\partial_+\chi_A\partial_-\chi_A - \frac{1}{2}\mu^2\cosh 2\phi_A\,. \tag{2.5}$$

Another model of the complex sinh-Gordon theory is the tanh model obtained by replacing (2.3) by

$$\begin{aligned} \partial_+Y^P\partial_-Y_P &= -\mu^2\cosh 2\phi_A\,, \\ K_P\,\partial_\pm^2 Y^P &= \mp 4\mu^3\sinh^2\phi_A\,\partial_\pm\theta_A\,. \end{aligned} \tag{2.6}$$

The resulting equations describe the tanh model,

$$\begin{aligned} &\partial_+\partial_-\phi_A - \tfrac{\sinh\phi_A}{\cosh^3\phi_A}\partial_+\theta_A\partial_-\theta_A + \tfrac{1}{2}\mu^2\sinh 2\phi_A = 0\,, \\ &\partial_+(\tanh^2\phi_A\,\partial_-\theta_A) + \partial_-(\tanh^2\phi_A\,\partial_+\theta_A) = 0\,, \end{aligned} \tag{2.7}$$

whose Lagrangian is

$$L_{\rm tanh} = \partial_+\phi_A\partial_-\phi_A + \tanh^2\phi_A\,\partial_+\theta_A\partial_-\theta_A - \frac{1}{2}\mu^2\cosh 2\phi_A\,. \tag{2.8}$$

As pointed out in [64] for the complex sin-Gordon model, these two models are related at the level of equations of motion by the "T-duality" transformation,

$$\partial_\pm\chi_A = \mp\tanh^2\phi_A\,\partial_\pm\theta_A\,. \tag{2.9}$$

Because this transformation is non-local, a classical solution in the reduced theory might take a complicated form in one model even if it is simple in the other model.

As we will see in the next chapter, the perturbation in the complex sinh-Gordon model describes a part of the fluctuations in the reduced model of the $AdS_5 \times S^5$ GS string. Moreover, it can be a useful tool for evaluating a complicated subsector of the physical fluctuations in the reduced $AdS_5 \times S^5$ theory.

2.2 Pohlmeyer reduction of $AdS_n \times S^n$ Green-Schwarz string

The $AdS_n \times S^n$ GS string can be described by the F/G coset sigma model where $F = PSU(1,1|2) \times PSU(1,1|2)$, $G = SU(1,1) \times SU(2)$ for $n = 3$ and the $F = PSU(2,2|4)$, $G = Sp(2,2) \times Sp(4)$ for $n = 5$ [24]. The essential feature of the superalgebra $\mathfrak{psu}(1,1|2)$, $\mathfrak{psu}(2,2|4)$ is that it admits a Z_4 automorphism Ω such that the condition $Z_4(F) = F$ determines the maximal subgroup to be G.[2] Regarding a coset element f as a map from the string world-sheet into the graded group F, the current the left-invariant current $J = f^{-1}df$ belongs to the superalgebra of F, then is decomposed as

$$J = f^{-1}d\,f = \mathcal{A} + Q_1 + P + Q_2\,, \qquad \mathcal{A} \in \mathfrak{f}_0, \quad Q_1 \in \mathfrak{f}_1, \quad P \in \mathfrak{f}_2, \quad Q_2 \in \mathfrak{f}_3\,. \tag{2.10}$$

For the latter purpose let us introduce new notations for the bosonic components, $\mathfrak{g} = \mathfrak{f}_0$ and $\mathfrak{p} = \mathfrak{f}_2$. $\mathcal{A}$ is the algebra of the subgroup G defining the F/G coset, P is the bosonic "coset" component, and Q_1, Q_2 are the fermionic currents.

The GS action in the conformal gauge is written in terms of these components,

$$I_{ST} = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma L_{GS}\,, \qquad L_{GS} = \mathrm{STr}\Big[P_+P_- + \frac{1}{2}(Q_{1+}Q_{2-} - Q_{1-}Q_{2+})\Big]\,, \tag{2.11}$$

where $\partial_\pm = \partial_\tau \pm \partial_\sigma$. The conformal-gauge (Virasoro) constraints are

$$\mathrm{STr}\,(P_\pm P_\pm) = 0\,. \tag{2.12}$$

This system is invariant under a local G gauge transformation, $f \to fg$. The equations of

[2] Important aspects of the $\mathfrak{psu}(1,1|2)$ algebra and the $\mathfrak{psu}(2,2|4)$ algebra are summarized in appendix A.

motion are derived by varying (2.11) by f,

$$\begin{aligned} &\partial_+ P_- + [\mathcal{A}_+, P_-] + [Q_{2+}, Q_{2-}] = 0\,, \\ &\partial_- P_+ + [\mathcal{A}_-, P_+] + [Q_{1-}, Q_{1+}] = 0\,, \\ &[P_+, Q_{1-}] = 0\,, \qquad [P_-, Q_{2+}] = 0\,, \end{aligned} \tag{2.13}$$

which should be supplemented by the conformal gauge constraints (2.12) and by the Maurer-Cartan equation,

$$\partial_- J_+ - \partial_+ J_- + [J_-, J_+] = 0\,. \tag{2.14}$$

Under the Z_4 decomposition the Maurer-Cartan equation (2.14) decomposes as follows

$$\begin{aligned} &\partial_- P_+ - \partial_+ P_- + [\mathcal{A}_-, P_+] + [Q_{1-}, Q_{1+}] + [P_-, \mathcal{A}_+] + [Q_{2-}, Q_{2+}] = 0\,, \\ &\partial_- \mathcal{A}_+ - \partial_+ \mathcal{A}_- + [\mathcal{A}_-, \mathcal{A}_+] + [Q_{1-}, Q_{2+}] + [P_-, P_+] + [Q_{2-}, Q_{1+}] = 0\,, \\ &\partial_- Q_{1+} - \partial_+ Q_{1-} + [\mathcal{A}_-, Q_{1+}] + [Q_{1-}, \mathcal{A}_+] + [P_-, Q_{2+}] + [Q_{2-}, P_+] = 0\,, \\ &\partial_- Q_{2+} - \partial_+ Q_{2-} + [\mathcal{A}_-, Q_{2+}] + [Q_{1-}, P_+] + [P_-, Q_{1+}] + [Q_{2-}, \mathcal{A}_+] = 0\,. \end{aligned} \tag{2.15}$$

Note that the first equation is the sum of the first and second equations in (2.13), then automatically solved by the equations of motion. The Pohlmeyer reduction procedure involves solving the equations of motion and Virasoro constraints by introducing new variables parametrising the physical degrees of freedom. For the new variables, the final three equations in the decomposed Maurer-Cartan equation (2.15) yield nontrivial equations which will be the equations of motion in the reduced theory.

The following is the brief description on the Pohlmeyer reduction (for more details see [64]). Before solving the Virasoro constraints, let us fix the G gauge and find a useful form of $P_\pm$ for the reduction. This can be done by the polar decomposition theorem which states that for any $k \in \mathfrak{p}$ there exists $g_0 \in G$ such that $g_0^{-1} k g_0 \in \mathfrak{a}$ (see appendix A for the definition of $\mathfrak{p}$ and $\mathfrak{a}$). Hence it is allowed to use a G gauge transformation to put P_+ into the form,

$$P_+ = p_1 T_1 + p_2 T_2\,, \tag{2.16}$$

where p_1, p_2 are real functions and $T_1,\ T_2 \in \mathfrak{a}$ are

$$T_1 = \tfrac{i}{2}\,\mathrm{diag}\,(1,\,-1,\,0,\,0)\ ,\quad T_2 = \tfrac{i}{2}\,\mathrm{diag}\,(0,\,0,\,1,\,-1)\ ,\quad \text{for } n=3\,,$$
$$T_1 = \tfrac{i}{2}\,\mathrm{diag}\,(1,\,1,\,-1,\,-1,\,0,\,0,\,0,\,0)\ ,\quad T_2 = \tfrac{i}{2}\,\mathrm{diag}\,(0,\,0,\,0,\,0,\,1,\,1,\,-1,\,-1)\ ,\quad \text{for } n=5\,. \tag{2.17}$$

From the Virasoro constraints (2.12) we have the condition $p_1^2 - p_2^2 = 0$. Then we set $p_1 = p_2 = p_+$, that is,

$$P_+ = p_+ T\,,\quad T = T_1 + T_2 = \frac{i}{2}\,\mathrm{diag}\,(1,\,1,\,-1,\,-1,\,1,\,1,\,-1,\,-1)\ , \tag{2.18}$$

We can apply the polar decomposition theorem to P_-; there exists $g \in G = Sp\,(2,\,2)\times Sp\,(4)$ such that

$$P_- = p_- g^{-1} T g\,, \tag{2.19}$$

where p_- is a real function. It should be noted that T is an element of the maximal abelian subalgebra of $\mathfrak{p}$ and induces the further orthogonal decomposition. The group H whose algebra is $\mathfrak{h}$ is then defined as the subgroup of G which stabilizes T, $\left[h,\,T\right] = 0$, $h \in H$. Due to this property there is an arbitrariness in the choice of g since P_- is invariant under $g \to hg$ for $h \in H$.

Next we shall show that $p_\pm$ are constants by solving the equations of motion in (2.13). If one fixes the κ-symmetry gauge to project the fermionic currents as $Q_1 = Q_1^{\parallel}$ and $gQ_2g^{-1} = (gQ_2g^{-1})^{\parallel}$, the last two equations in (2.13) implies $Q_{1-} = Q_{2+} = 0$ because $\left[T, \mathfrak{f}_{1,3}^{\parallel}\right] = 2T\mathfrak{f}_{1,3}^{\parallel}$. Then the first and second equations in (2.13) are simplified

$$\partial_+ P_- + [\mathcal{A}_+, P_-] = 0\,,\qquad \partial_- P_+ + [\mathcal{A}_-, P_+] = 0\,. \tag{2.20}$$

Noticing that $P_+ \in \mathfrak{h}$, the second equation is decomposed as

$$\partial_- P_+ + \left[(\mathcal{A}_-)_{\mathfrak{h}}\,, P_+\right] = 0\,,\qquad [(\mathcal{A}_-)_{\mathfrak{m}}\,, P_+] = 0\,. \tag{2.21}$$

Due to the block diagonal nature of P_+, $\mathcal{A}_-$ these equation should hold in the AdS_n part and

S^n part separately. In particular, the first equation in (2.21) reads $\partial_-(P_+)^1 + \left[\left(\mathcal{A}^1_-\right)_{\mathfrak{h}}, P^1_+\right] = 0$,[3] that ends up with the conservation laws in the individual subparts, $\partial_- \mathrm{tr}^1\left(P_+ P_+\right) = 0$, $\partial_- \mathrm{tr}^2\left(P_+ P_+\right) = 0$. Since $\mathrm{tr}^{1,2}(T^2) \neq 0$, we can conclude $\partial_- p_+ = 0$. The same trick for the first equation in (2.20) leads to $\partial_+ p_- = 0$. Using the residual conformal diffeomorphism symmetry it is always possible to set $p_\pm = \mu_\pm = \mathrm{const}$. Hence we get

$$P_+ = \mu_+ T\,, \qquad P_- = \mu_- g^{-1} T g\,. \tag{2.22}$$

As seen in the reduction of the $AdS_3 \times S^1$ bosonic string theory, there is an argument on the constants $\mu_\pm$; if the sigma model was defined on $2d$ Minkowski space then we could use a Lorentz transformation to set $\mu_+ = \mu_- = \mu$ as was done in [64]. However, if we are interested in the case of the closed string when the worldsheet is a cylinder, then this is not possible. It is still useful to define the following combination of μ_+ and μ_-,

$$\mu = \sqrt{\mu_+ \mu_-}\,. \tag{2.23}$$

With the solutions for $P_\pm$, (2.22), it is possible to rewrite $\mathcal{A}_\pm$. Under the Z_2 decomposition,

$$\mathcal{A}_\pm = A_\pm + \left(\mathcal{A}_\pm\right)_{\mathfrak{m}}\,, \qquad A_\pm = \left(\mathcal{A}_\pm\right)_{\mathfrak{h}}\,, \tag{2.24}$$

the second equation in (2.21) implies $\left(\mathcal{A}_-\right)_{\mathfrak{m}} = 0$. Plugging (2.22) into the first equation in (2.20), one finds that it is solved by

$$\mathcal{A}_+ = g^{-1}\partial_+ g + g^{-1} + g^{-1} A_+ g\,. \tag{2.25}$$

So the equations of motion (2.13) and the Virasoro constraints (2.12) have been totally solved. All of the bosonic degrees of freedom have been encoded into an group element $g \in G$ and

[3] Here we use the notation ^{1}, ^{2} introduced in [64]. The index ^{1} represents the 4×4 $\mathfrak{su}(2,2)$ part and the index ^{2} does the 4×4 $\mathfrak{su}(4)$ part in the 8×8 supermatrix representation of $\mathfrak{psu}(2,2|4)$ described in appendix A (similarly they respectively represent the 2×2 $\mathfrak{su}(1,1)$ part and the 2×2 $\mathfrak{su}(2)$ part for $\mathfrak{psu}(1,1|2)$). In terms of this notation the supertrace STr is expressed as $\mathrm{STr} = \mathrm{tr}^1 - \mathrm{tr}^2$.

$A_\pm$ taking values in the algebra $\mathfrak{h}$ of H, i.e., $\left[A_\pm, T\right] = 0$. Finally, we make the following redefinitions of the non-vanishing fermionic fields

$$\begin{aligned} \Psi_{_R} &= \tfrac{1}{\sqrt{\mu_+}}(Q_{1+})^{\parallel} \,, \\ \Psi_{_L} &= \tfrac{1}{\sqrt{\mu_-}}(gQ_{2-}g^{-1})^{\parallel} \,. \end{aligned} \tag{2.26}$$

Then we have rewritten the original currents in terms of a new set of fields, $(g\,, A_\pm,\, \Psi_{_R},\, \Psi_{_L})$, describing only the physical degrees of freedom of the system.

Substituting these fields into the second, third and fourth equations in (2.15) we obtain the following set of equations of motion for the reduced theory

$$\begin{aligned} &\partial_-\left(g^{-1}\partial_+ g + g^{-1}A_+ g\right) - \partial_+ A_- + [A_-, g^{-1}\partial_+ g + g^{-1}A_+ g] \\ &\qquad = -\mu^2\,[g^{-1}Tg, T] - \mu\,[g^{-1}\Psi_{_L} g, \Psi_{_R}]\ , \end{aligned} \tag{2.27}$$

$$D_-\Psi_{_R} = \mu\,[T, g^{-1}\Psi_{_L} g]\ , \qquad D_+\Psi_{_L} = \mu\,[T, g\Psi_{_R} g^{-1}]\ , \qquad D_\pm = \partial_\pm + [A_\pm,]\ .$$

The set of these equations exhibits $H \times H$ gauge symmetry,

$$\begin{aligned} &g \to h^{-1}g\bar{h}\,, \qquad A_+ \to h^{-1}A_+ h + h^{-1}\partial_+ h, \qquad A_- \to \bar{h}^{-1}A_-\bar{h} + \bar{h}^{-1}\partial_-\bar{h} \\ &\Psi_{_R} \to \bar{h}^{-1}\Psi_{_R}\bar{h}\,, \qquad \Psi_{_L} \to h^{-1}\Psi_{_L} h\,. \end{aligned} \tag{2.28}$$

In order to write down a Lagrangian for the equations of motion (2.27) we should partially fix the $H \times H$ gauge symmetry as in [HIT],

$$\begin{aligned} \tau\left(A_+\right) &= \left(g^{-1}\partial_+ g + g^{-1}A_+ g - \tfrac{1}{2}\left[\left[T, \Psi_{_R}\right], \Psi_{_R}\right]\right)_{\mathfrak{h}}\ , \\ \tau^{-1}\left(A_-\right) &= \left(-\partial_- g g^{-1} + gA_- g^{-1} - \tfrac{1}{2}\left[\left[T, \Psi_{_L}\right], \Psi_{_L}\right]\right)_{\mathfrak{h}}\ . \end{aligned} \tag{2.29}$$

Here τ is a supertrace-preserving[4] automorphism of the algebra $\mathfrak{h}$. This partial gauge fixing

[4] $\mathrm{STr}\left(\tau\left(u_1\right)\tau\left(u_2\right)\right) = \mathrm{STr}\left(u_1 u_2\right),\ \ u_{1,2} \in \mathfrak{h}$.

reduces the $H \times H$ gauge symmetry to the following asymmetric H gauge symmetry,

$$g \to h^{-1} g \hat{\tau}(h) \ , \qquad A_+ \to h^{-1} A_+ h + h^{-1}\partial_+ h, \qquad A_- \to \hat{\tau}(h)^{-1} A_- \hat{\tau}(h) + \hat{\tau}(h)^{-1} \partial_- \hat{\tau}(h)$$
$$\Psi_R \to \hat{\tau}(h)^{-1} \Psi_R \hat{\tau}(h) \ , \qquad \Psi_L \to h^{-1} \Psi_L h \, , \tag{2.30}$$

where $\hat{\tau}$ is a lift of τ from $\mathfrak{h}$ to H. Hereafter we shall consider a special case where τ, $\hat{\tau}$ are identity. The equations of motion, (2.27), and the gauge field equations, (2.29), follow from the Lagrangian,

$$\begin{aligned} L_{\text{dWZW}} &= L_{\text{gWZW}} + \mu^2 \, \text{STr}(g^{-1} T g T) \\ &\quad + \tfrac{1}{2} \text{STr} \left(\Psi_L \left[T, D_+ \Psi_L \right] + \Psi_R \left[T, D_- \Psi_R \right] \right) + \ \mu \, \text{STr} \left(g^{-1} \Psi_L g \Psi_R \right) \ , \end{aligned} \tag{2.31}$$

where L_{gWZW} is the Lagrangian of the gauged G/H WZW model,

$$\begin{aligned} I_{\text{gWZW}} &= \int \frac{d^2\sigma}{4\pi} \text{STr}(g^{-1}\partial_+ g g^{-1} \partial_- g) - \int \frac{d^3\sigma}{12\pi} \text{STr}(g^{-1} dg g^{-1} dg g^{-1} dg) \\ &+ \ \int \frac{d^2\sigma}{2\pi} \text{STr} \left(A_+ \, \partial_- g g^{-1} - A_- \, g^{-1} \partial_+ g - g^{-1} A_+ g A_- + A_+ A_- \right) \ . \end{aligned} \tag{2.32}$$

This Lagrangian is invariant under the gauge transformations (2.30). It is this system that will be referred to as the Pohlmeyer-reduced theory (PRT) computed with the original string theory (ST). The reduced theory is the G/H gauged WZW model with a gauge invariant integrable potential and fermionic extension. We have $G = Sp(2,2) \times Sp(4)$, $H = [SU(2)]^4$ in the $AdS_5 \times S^5$ case and $G = SU(1,1) \times SU(2)$, $H = [U(1)]^2$ in the $AdS_3 \times S^3$ case.

Consequently the reduced model is described by the action,

$$I_{PRT} = \frac{k}{8\pi} \int d\sigma^2 L_{\text{dWZW}} \, , \tag{2.33}$$

where the coupling constant k is undetermined classically.[5] However, if the the GS string theory and the Pohlmeyer-reduced theory are related at the quantum level, k should be related to the string coupling constant $\sqrt{\lambda}$. In particular, it is observed that the μ-dependent term in the

[5] Generally the constant in front of the action is given by $\frac{k}{8\pi s}$, where s is the index of the representation. For the representation we will use in this thesis, we have $s = 1/2$ for $AdS_5 \times S^5$ and $s = 1$ for $AdS_3 \times S^3$. The detailed discussion is given in [73, 74, 75].

reduced theory can be directly obtained by substituting (2.22) into the GS action (2.11). Then one may conjecture that

$$k = 2\sqrt{\lambda}\,. \tag{2.34}$$

While the one-loop computation is insensitive to the identification of the coupling constants (chapter 3), the two-loop computation depends on the coupling constant (chapter 4).

In order to study fluctuations around a particular classical solution by using the reduced theory one needs to find corresponding g_0, $A_{0\pm}$ by fixing the G gauge and partially fixing the $H \times H$ gauge. Generally it is not easy to find a gauge such that g_0, $A_{0\pm}$ solve the gauge equations (2.29) and take a convenient form for extracting physical part of the perturbation. Moreover, it becomes much harder if the classical string solution is inhomogeneous, which is also the case we will discuss in the next chapter. One can circumvent this gauge fixing task by using an embedding of the complex sinh-Gordon model into the deformed gWZW model.

Hence we shall next show how the complex sinh-Gordon model is realized in the framework of the Pohlmeyer reduction of the $AdS_5 \times S^5$ superstring theory. Because we will consider classical solutions whose S^5 part is localized in S^1 of the S^5, the S^5 parts of g_0 and $A_{0\pm}$ are the vacuum solution, and accordingly, the bosonic fluctuations in the S^5 sector are massive fields with the masses $\pm\mu$ as shown in appendix C.2. Hence we will focus on the AdS sector, then g_0 and $A_{0\pm}$ below always mean matrices for the AdS sector.

A classical string solution in $AdS_3 \times S^1$ can be expressed as a classical solution g_0 which takes value on $SU(1,1)$ in the gWZW model. One natural parameterization of $\mathfrak{su}(1,1)$ as a subalgebra of $\mathfrak{sp}(2,2)$ in [66] is

$$R_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad R_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}. \tag{2.35}$$

Then the classical solution is expressed in terms of the Euler angles, $\phi_{_A}$ and $\chi_{_A}$,

$$\bar{g}_0 = g_2 g_1 g_2 \,, \quad g_1 = \exp(\phi_{_A} R_1) \,, \quad g_2 = \exp(\frac{1}{2}\chi_{_A} R_2) \,, \tag{2.36}$$

and the matrix elements of $\bar{g}_0$ are written as,

$$\bar{g}_0 = \begin{pmatrix} e^{i\chi_A}\cosh\phi_{_A} & 0 & 0 & \sinh\phi_{_A} \\ 0 & e^{-i\chi_A}\cosh\phi_{_A} & \sinh\phi_{_A} & 0 \\ 0 & \sinh\phi_{_A} & e^{i\chi_A}\cosh\phi_{_A} & 0 \\ \sinh\phi_{_A} & 0 & 0 & e^{-i\chi_A}\cosh\phi_{_A} \end{pmatrix} . \tag{2.37}$$

The corresponding gauge fields are obtained by solving the gauge field equations (2.29),

$$\bar{A}_{0\pm} = \frac{i}{2}\,\bar{a}_\pm R_2 \,, \quad \bar{a}_+ = -\coth^2\phi_{_A}\partial_+\chi_{_A} \,, \quad \bar{a}_- = \coth^2\phi_{_A}\partial_-\chi_{_A} \,. \tag{2.38}$$

Plugging these into the gWZW Lagrangian in [64], one finds that the Lagrangian of the coth model, (2.5), is recovered.

On the other hand, the Lagrangian of the tanh model is obtained if we choose the following parameterization of g_0,

$$g_0 = \begin{pmatrix} 0 & e^{i\theta_A}\cosh\phi_{_A} & -e^{i\theta_A}\sinh\phi_{_A} & 0 \\ -e^{-i\theta_A}\cosh\phi_{_A} & 0 & 0 & e^{-i\theta_A}\sinh\phi_{_A} \\ e^{i\theta_A}\sinh\phi_{_A} & 0 & 0 & -e^{i\theta_A}\cosh\phi_{_A} \\ 0 & -e^{-i\theta_A}\sinh\phi_{_A} & e^{-i\theta_A}\cosh\phi_{_A} & 0 \end{pmatrix} . \tag{2.39}$$

The gauge equations (2.29) are solved by

$$A_{0\pm} = \frac{i}{2}\,a_\pm R_2 \,, \quad a_+ = -2\frac{\cosh 2\phi_{_A}\,\partial_+\theta_{_A}}{1+\cosh 2\phi_{_A}} \,, \quad a_- = -\mathrm{sech}^2\phi_{_A}\,\partial_-\theta_{_A} \,. \tag{2.40}$$

Note that $\bar{g}_0$ and g_0 are related by an $H \times H$ gauge transformation rather than an H gauge transformation. One example of the H gauge transformation is

$$\bar{g}_0 \to g_0 = h_L^{-1} \bar{g}_0 h_R \,, \tag{2.41}$$

where

$$h_L = \begin{pmatrix} 0 & -e^{\frac{3}{2}i\theta_A} & 0 & 0 \\ e^{-\frac{3}{2}i\,\theta_A} & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{\frac{3}{2}i\theta_A} \\ 0 & 0 & e^{-\frac{3}{2}i\theta_A} & 0 \end{pmatrix}, \quad h_R = \begin{pmatrix} e^{-\frac{1}{2}i\theta_A} & 0 & 0 & 0 \\ 0 & e^{\frac{1}{2}i\theta_A} & 0 & 0 \\ 0 & 0 & -e^{-\frac{1}{2}i\theta_A} & 0 \\ 0 & 0 & 0 & -e^{\frac{1}{2}i\theta_A} \end{pmatrix}. \tag{2.42}$$

Chapter 3

One-loop computation: strings in $AdS_3 \times S^1$

In chapter 2 we reviewed the Pohlmeyer reduction of the $AdS_3 \times S^1$ bosonic string theory and $AdS_n \times S^n$ GS string theory, and introduced particular ways of embedding the reduced $AdS_3 \times S^1$ model into the Pohlmeyer-reduced GS string theory. Bosonic string theory in $AdS_3 \times S^1$ is classically equivalent to the complex sinh-Gordon theory and the $AdS_n \times S^n$ GS string is reduced to the deformed gWZW model at the classical level. In this chapter we shall extend this equivalence to the one-loop level. For this purpose we will first discuss perturbative expansions around a general classical solution in the reduced models.

Lagrangian for quadratic fluctuations in the coth model of the complex sinh-Gordon theory (2.5) is obtained by considering the perturbation, $\phi_{_A} \to \phi_{_A} + \delta\phi_{_A}$ and $\chi_{_A} \to \chi_{_A} + \delta\chi_{_A}$. Then we have

$$\begin{aligned}\mathcal{L}_{\coth(2)} = \partial_+\delta\phi_{_A}\partial_-\delta\phi_{_A} + \left(\tfrac{3+2\sinh^2\phi_A}{\sinh^4\phi_A}\partial_+\chi_{_A}\partial_-\chi_{_A} - \mu^2\cosh 2\phi_{_A}\right)(\delta\phi_{_A})^2 \\ + \coth^2\phi_{_A}\,\partial_+\delta\chi_{_A}\partial_-\delta\chi_{_A} - \tfrac{2\cosh\phi_A}{\sinh^3\phi_A}(\partial_+\chi_{_A}\partial_-\delta\chi_{_A} + \partial_+\delta\chi_{_A}\partial_-\chi_{_A})\delta\phi_{_A}\,.\end{aligned} \tag{3.1}$$

It is also useful to discuss perturbation in the T-dualized model, the tanh model (2.8). By

$\phi_A \to \phi_A + \delta\phi_A$ and $\theta_A \to \theta_A + \delta\theta_A$ in (2.8), we obtain the Lagrangian for quadratic fluctuations,

$$\begin{aligned}\mathcal{L}_{\tanh(2)} &= \partial_+\delta\phi_A\partial_-\delta\phi_A + \left(\tfrac{3-2\cosh^2\phi_A}{\cosh^4\phi_A}\partial_+\theta_A\partial_-\theta_A - \mu^2\cosh 2\phi_A\right)(\delta\phi_A)^2 \\ &\quad + \tanh^2\phi_A\,\partial_+\delta\theta_A\partial_-\delta\theta_A + \tfrac{2\sinh\phi_A}{\cosh^3\phi_A}(\partial_+\theta_A\partial_-\delta\theta_A + \partial_+\delta\theta_A\partial_-\theta_A)\delta\phi_A\,.\end{aligned} \tag{3.2}$$

Note that the the T-duality transformation given in (2.9) works even at the semiclassical level, that is, (3.2) is T-dual to (3.1), which we will use in section 3.1.

On general ground there is no reason to expect that the complex sinh-Gordon theory is equivalent to the $AdS_3 \times S^1$ bosonic string theory at the quantum level. As far as the one-loop corrections are concerned, on may think this expectation might be true because the first order corrections directly follow from the equations of motion. However, this is not correct for the case of $\mu_+ \neq \mu_-$ if we use the reduction relation (2.3) for the coth model or (2.6) for the tan model, and it is necessary to take into account all the bosonic and fermionic fluctuations in order to obtain a correct set of physical fluctuations (see sections 3.2, 3.3 and 3.4). This issue can be resolved if we modify the reduction (2.3), (2.6) leaving the reduced model (2.5), (2.8) unchanged. We will come back to this point in section 3.5.

Next let us consider perturbation in the deformed gWZW model (2.31). We shall introduce fluctuations around a classical solution, g_0, $A_{0\pm}$, Ψ_{R0}, Ψ_{L0}, as follows

$$\begin{aligned} &g = g_0 e^\eta = g_0(1 + \eta + \tfrac{1}{2}\eta^2 + \mathcal{O}(\eta^3))\,, \\ &A_+ = A_{+0} + \delta A_+\,, \qquad A_- = A_{-0} + \delta A_-\,, \\ &\Psi_R = \Psi_{R0} + \delta\Psi_R\,, \qquad \Psi_L = \Psi_{L0} + \delta\Psi_L\,. \end{aligned} \tag{3.3}$$

Hereafter we will focus on classical solutions with vanishing fermions, i.e., $\Psi_{R0} = \Psi_{L0} = 0$. Under this perturbation the quadratic fluctuations of the Lagrangian for the deformed gWZW model (2.31) are described by

$$\begin{aligned}\mathcal{L}_{\mathrm{dWZW}(2)} = \mathrm{STr}\Big[&\tfrac{1}{2}\mathscr{D}_+\eta D_-\eta - \mathscr{D}_+\delta A_- - g_0^{-1}\delta A_+ g_0 D_-\eta - g_0^{-1}\delta A_+ g_0\delta A_- + \delta A_+\delta A_- \\ &-\tfrac{\mu^2}{2}\left[\eta, g_0^{-1}Tg_0\right][\eta, T] + \tfrac{1}{2}\delta\Psi_R\,[T, D_-\delta\Psi_R] + \tfrac{1}{2}\delta\Psi_L\,[T, D_+\delta\Psi_L] + \mu g_0^{-1}\delta\Psi_L g_0\delta\Psi_R\Big]\,,\end{aligned} \tag{3.4}$$

where the derivative operators $\mathscr{D}$, D are defined by $\mathscr{D}_+ = \partial_+ + \left[g_0^{-1}\partial_+ g_0 + g_0^{-1}A_+ g_0,\ \right]$, $D_\pm = \partial_\pm + [A_\pm,\ ]$. This Lagrangian was derived in [HIT].

Below we will evaluate the fluctuations in the whole $AdS_5 \times S^5$ around classical strings stretching in the $AdS_3 \times S^1$ subspace by using the embedding of the complex sinh-Gordon model into the deformed gWZW model, i.e., (2.39) and (2.37). Although g_0 for the coth model is constructed by a standard gauging, the coth model has several issues when we calculate quadratic fluctuations, and we will mainly use the embedding of the tanh model.

The remaining part of this chapter is organized as follows.

In section 3.1 we shall discuss the (S, J) folded string. Its quadratic fluctuations were found from the Nambu action in the static gauge and in the Polyakov action in the conformal gauge in the original string theory [89, 90]. We shall show that the fluctuation Lagrangian in the coth model of the reduced theory is related to that derived from the Nambu action.

In section 3.2 we shall study the homogeneous (S, J) circular string solution which has both the angular momentum and the winding on a big circle of S^5, and gives us the first example of $\mu_+ \neq \mu_-$. We will show that the fluctuation Lagrangian of the reduced theory agrees with the string theory result [93]. Since the circular string is homogeneous, we can evaluate characteristic frequencies of the quadratic fluctuations also in the reduced theory. In the case of the (S, J) circular string the total sum of the bosonic and fermionic frequencies in the reduced theory agrees with the string theory result although some of the individual frequencies appear to disagree.

In section 3.3 we shall discuss the (S, J) spiky string solution and derive its fluctuation Lagrangian of the reduced theory. Our result on the semiclassical expansions for the (S, J) spiky string is expected to agree with the one in the original string theory. We shall also discuss the fluctuation Lagrangian in the limiting cases of the (S, J) spiky string; spiky string without motion or stretching in S^5, and its folded string limit.

In section 3.4 we will consider a generalization of the (S, J) folded string solution with both the orbital momentum and the winding in the S^1 sector found in [97]. This solution again has the scaling limit where the solution becomes homogeneous. As happened in the circular string case, some individual characteristic frequencies in the reduced theory disagree with the result in [97], but the sum of the frequencies is the same as in the string theory.

We will find the disagreement of characteristic frequencies of the individual fluctuations in the $\mu_+ \neq \mu_-$ cases in section 3.2 and section 3.4. It can be resolved by modifying the reduction relations (2.3) and (2.6). This prescription will be discussed in section 3.5 by considering the (S, J) circular string solution.

3.1 Folded string

In this section we shall evaluate the Lagrangian for quadratic fluctuations around the (S, J) folded string. In the original string theory the semiclassical expansions are carried out in the Nambu action in the static gauge and in the Polyakov action in the conformal gauge in [89], and the equivalence of these two approaches is shown in [91]. Although the Pohlmeyer reduced form of the $AdS_5 \times S^5$ is constructed from the conformal gauge string theory, it is expected that the fluctuation Lagrangian of the reduced theory takes a similar form to the effective Nambu action as both the reduced model and the Nambu action involve only the physical degrees of freedom after choosing a gauge.

We shall derive the Lagrangian for bosonic fluctuations in section 3.1.1 and the Lagrangian for fermionic fluctuations in section 3.1.2. To compare our result with the original string theory we shall carry out the perturbation in the Nambu action in the static gauge, and show how this approach is related to the coth model and tanh model in the reduced theory in section 3.1.3.

Let us first review the (S, J) folded string in $AdS_3 \times S^1$ which is expressed in terms of the embedding coordinates,

$$Y_0 + iY_3 = \cosh\rho\, e^{i\kappa\tau}\,, \quad Y_1 + iY_2 = \sinh\rho\, e^{iw\tau}\,, \quad X_1 + iX_2 = e^{i\nu\tau}\,, \tag{3.5}$$

where κ, w and ν are constants, and ρ is a function of σ, $\rho = \rho(\sigma)$. The equation of motion and the conformal gauge constrains read

$$\rho'' = (\kappa^2 - w^2)\,\sinh\rho\,\cosh\rho\,, \quad \rho'^2 = \kappa^2\cosh^2\!\rho - w^2\sinh^2\!\rho - \nu^2\,. \tag{3.6}$$

Next we shall derive the corresponding classical solution in the reduced theory. The mass scale of the reduced theory μ can be found by observing the AdS part of the stress tensor. In the present case we have $T^{\rm AdS}_{\pm\pm} = -\nu^2$ meaning that this is the case of $\mu_+ = \mu_-$. Then we set

$$\mu = \nu\,. \tag{3.7}$$

We have two ways of the reduction, the embedding of the coth model or the embedding of the tanh model. As mentioned below it is more convenient to employ the tanh approach. The classical solution, $\phi_{_A}$ and $\theta_{_A}$, in the tanh model is given by the relations in (2.6),

$$\phi_{_A} = \log\left(\tfrac{\rho'+\sqrt{\nu^2+\rho'^2}}{\nu}\right)\,, \qquad \theta_{_A} = \tfrac{w\kappa}{\nu}\tau\,. \tag{3.8}$$

Substituting these into the formula of g_0 and $A_\pm$, we have

$$g_0 = \begin{pmatrix} 0 & \upsilon\frac{\sqrt{\nu^2+\rho'^2}}{\nu} & -\upsilon\frac{\rho'}{\nu} & 0 \\ -\upsilon^*\frac{\sqrt{\nu^2+\rho'^2}}{\nu} & 0 & 0 & \upsilon^*\frac{\rho'}{\nu} \\ \upsilon\frac{\rho'}{\nu} & 0 & 0 & -\upsilon\frac{\sqrt{\nu^2+\rho'^2}}{\nu} \\ 0 & -\upsilon^*\frac{\rho'}{\nu} & \upsilon^*\frac{\sqrt{\nu^2+\rho'^2}}{\nu} & 0 \end{pmatrix}\,, \qquad \upsilon = e^{\frac{iw\kappa\tau}{\nu}}\,, \tag{3.9}$$

and the gauge field equations (2.29) are solved by the following $A_{\pm 0}$,

$$A_{\pm 0} = \frac{i}{2}\,a_{\pm 0}R_2\,, \quad a_{+0} = 2w\kappa\left(-\frac{1}{\nu} + \frac{\nu}{2\,(\nu^2+\rho'^2)}\right)\,, \;\; a_{-0} = -\frac{w\kappa\nu}{(\nu^2+\rho'^2)}\,. \tag{3.10}$$

The reason we use the tanh model here is as follows. If we plug the folded string solution (3.5) into the reduction equation (2.3) with $\mu = \nu$, we obtain the classical solution of the coth model

$$\phi_{_A} = \log\left(\frac{\rho' + \sqrt{\nu^2 + \rho'^2}}{\nu}\right) , \qquad \partial_\pm \chi_{_A} = \mp\left(\frac{w\kappa}{\nu} - \frac{w\kappa\nu}{\nu^2 + \rho'^2}\right) . \tag{3.11}$$

Here $\chi_{_A}$ can be expressed in an integral form. This can be also understood by substituting $\phi_{_A}$ into the relation of $\chi_{_A}$ and $\theta_{_A}$ in (2.9). On the right hand side, $\partial_\pm\theta_{_A}$ is constant, but $\tanh\phi_{_A}$ has the $\rho(\sigma)$ dependence, and then, $\chi_{_A}$ ends up with the complicated form.

As far as the calculation of the quadratic fluctuations is concerned, the expression of $\chi_{_A}$ in (3.11) does not seem to cause a serious problem because only its derivative, $\partial_+\chi_{_A}$, appears in to the Lagrangian for the quadratic fluctuations if the $H \times H$ gauge is properly chosen. However, $\bar{g}_0$ for the coth model in (2.37) is not this case; due to the peculiar form of $\bar{g}_0$ some components of $\bar{g}_0^{-1}\partial_+\bar{g}_0$ and $\bar{g}_0^{-1}\bar{A}_{0+}\bar{g}_0$ contain the $e^{i\chi_A}$ factor, and consequently, the perturbed Lagrangian has the $\chi_{_A}$ dependence. Of course this $e^{i\chi_A}$ factor can be removed from the perturbed Lagrangian by redefinition of fluctuation fields, but such redefinition is not trivial. Also, the form of g_0 for the tan model in (2.39) is very convenient to integrate out the gauge fields and decouple the physical fields from the unphysical fields. Therefore, we will basically continue to use the tanh model in the following sections.

Before moving on to the one-loop computation, let us add some remarks on the open string counterpart of the folded string solution in the scaling limit where the (S, J) folded string solution becomes homogeneous. The (S, J) folded string in this limit was studied in the reduced theory [HIT]. In [98] it was shown that the two classical string solutions are connected by the $SO(2,4)$ rotation and analytic continuation on the worldsheet $\tau \to -i\tau$, and then, quantum corrections to the scaling function calculated in the open string picture are the same as those in the folded string picture.[1] As the isometry of AdS_5, $SO(2,4)$, becomes obscure by the Pohlmeyer reduction, any two solutions related by an $SO(2,4)$ transformation are encoded into a single solution in the reduced theory. Hence we obtain a reduced theory solution corresponding

[1]In [98] the equivalence of the closed string and the open string was shown for $\nu = 0$ classically and semiclassically. A $\nu \neq 0$ open string solution can be easily constructed by adding the BMN vacuum in the S^1 sector [70], which is the counterpart of the scaling limit of the (S, J) folded string (see also [97] for a folded string and its corresponding open string surface with both the orbital momentum and the winding in S^1 of S^5).

to the null cusp solution in the original theory by the analytic continuation $\tau \to -i\tau$ in g_0 (3.9) (after taking the scaling limit, $w \to \kappa$ and $\rho \to \ell\sigma$ with $\kappa^2 - \nu^2 = \ell^2$). In general, the equivalence of characteristic frequencies of the quadratic fluctuations in the reduced theory comes from that of the classical solutions g_0. In the reduced theory, therefore, one can trivially check the frequencies for the two classical solutions match.

3.1.1 Bosonic fluctuations in reduced theory

The argument for the S^5 sector is the same as the case of the scaling limit studied in [HIT] (see appendix C.2). Four bosonic fluctuations in the S^5 sector are massive fields with $m_B^2 = \nu^2$.

In the AdS_5 sector we shall discuss the bosonic fluctuations by using the tanh model by the above reason. To express the quadratic fluctuations in terms of components fields, let us introduce bosonic fields by

$$\eta^{\parallel} = \begin{pmatrix} 0 & 0 & a_1 + ia_2 & a_3 + ia_4 \\ 0 & 0 & a_3 - ia_4 & -a_1 + ia_2 \\ a_1 - ia_2 & a_3 + ia_4 & 0 & 0 \\ a_3 - ia_4 & -a_1 - ia_2 & 0 & 0 \end{pmatrix}, \tag{3.12}$$

which correspond to physical fields in the reduced theory.

$$\eta^{\perp} = \begin{pmatrix} ih_1 & h_2 + ih_3 & 0 & 0 \\ -h_2 + ih_3 & -ih_1 & 0 & 0 \\ 0 & 0 & ih_4 & h_5 + ih_6 \\ 0 & 0 & -h_5 + ih_6 & -ih_4 \end{pmatrix}, \tag{3.13}$$

$$
\delta A_+ = \begin{pmatrix} ia_{+1} & (a_{+2}+ia_{+3})\, v^2 & 0 & 0 \\ -(a_{+2}-ia_{+3})\, v^{*2} & -ia_{+1} & 0 & 0 \\ 0 & 0 & ia_{+4} & (a_{+5}+ia_{+6})\, v^2 \\ 0 & 0 & -(a_{+5}-ia_{+6})\, v^{*2} & -ia_{+4} \end{pmatrix},
$$

$$
\delta A_- = \begin{pmatrix} ia_{-1} & a_{-2}+ia_{-3} & 0 & 0 \\ -a_{-2}+ia_{-3} & -ia_{-1} & 0 & 0 \\ 0 & 0 & ia_{-4} & a_{-5}+ia_{-6} \\ 0 & 0 & -a_{-5}+ia_{-6} & -ia_{-4} \end{pmatrix}. \tag{3.14}
$$

These are unphysical fields in the reduced theory and to be gauged away or integrated out from the fluctuation Lagrangian.

An advantage of using g_0 (2.39) is that the system decouples into two subsectors: One contains a_1 and a_2 coupling to the diagonal parts of $\eta^\perp$ and $\delta A_\pm$, while the other contains a_3 and a_4 coupling to the off-diagonal components of $\eta^\perp$ and $\delta A_\pm$. One can fix the H gauge such that the physical fields a_i decouple from the unphysical fields,

$$
h_1 + h_4 = \text{const}\,, \tag{3.15}
$$

which is the same as one of the three gauge conditions in the long string limit case in [HIT], and we should impose two more conditions,

$$
\begin{aligned}
&\sqrt{(-w^2+\nu^2+\rho'^2)(-\kappa^2+\nu^2+\rho'^2)} \\
&\times\Big[w\kappa\nu^3(h_3+h_6) + (\nu^2+\rho'^2)\Big((\nu^2+2\rho'^2)(a_{+2}+a_{+5}) + \nu^2(\partial_- h_2 + \partial_- h_5 - a_{-2} - a_{-5})\Big)\Big] \\
&+\rho'\Big[-w^2\kappa^2\nu^2(h_2+h_5) + (\nu^2+\rho'^2)\Big(w\kappa\nu(\partial_- h_3 + \partial_- h_6 - a_{-3} - a_{-6}) \\
&\qquad\qquad + (\nu^2+\rho'^2)(\partial_- a_{+2} + \partial_- a_{+5})\Big)\Big] = 0\,,
\end{aligned} \tag{3.16}
$$

and

$$\sqrt{\left(-w^2+\nu^2+\rho'^2\right)\left(-\kappa^2+\nu^2+\rho'^2\right)}$$
$$\times\left[w\kappa\nu^3(-h_2+h_5)-\left(\nu^2+\rho'^2\right)\Big(\left(a_{+3}-a_{+6}\right)\left(\nu^2+2\rho'^2\right)+\nu^2\left(-\partial_- h_3+\partial_- h_6+a_{-3}-a_{-6}\right)\Big)\right]$$
$$+\rho'\Big[w^2\kappa^2\nu^2(-h_3+h_6)+\left(\nu^2+\rho'^2\right)\Big(w\kappa\nu\left(-\partial_- h_2+\partial_- h_5+a_{-2}-a_{-5}\right)$$
$$-\left(\nu^2+\rho'^2\right)\left(\partial_- a_{+3}-\partial_- a_{+6}\right)\Big)\Big]=0\,. \tag{3.17}$$

Under this gauge choice the Lagrangian for a sector with a_1 and a_2 is

$$\mathcal{L}_1=2\sum_{i=1,2}\left(\partial_+ a_i\partial_- a_i-\left(\nu^2+2\rho'^2\right)a_i^2\right)\,, \tag{3.18}$$

and the sector with a_3 and a_4 has the Lagrangian,

$$\mathcal{L}_2=2\Big[\partial_- a_3\partial_+ a_3-\left(\nu^2+2\rho'^2+\tfrac{2w^2\kappa^2}{\nu^2+\rho'^2}-\tfrac{3w^2\kappa^2\nu^2}{(\nu^2+\rho'^2)^2}\right)a_3^2+\partial_- a_4\partial_+ a_4$$
$$+\left(\tfrac{2w\kappa\nu\partial_- a_3}{\nu^2+\rho'^2}+\tfrac{2w\kappa\nu\partial_+ a_3}{\nu^2+\rho'^2}\right)a_4-\nu^2\left(-1+\tfrac{2\left(w^2+\kappa^2\right)}{(\nu^2+\rho'^2)}-\tfrac{3w^2\kappa^2}{(\nu^2+\rho'^2)^2}\right)a_4^2\Big]\,. \tag{3.19}$$

This Lagrangian does not diverge at turning points of the folded string, i.e., at $\rho'=0$. This observation is different from the case of $\nu=0$ in [89]. The long string limit case is recovered if we take $w\to\kappa$ and $\rho\to\ell\sigma$, and replace the conformal gauge constraint by $\kappa^2-\nu^2=\ell^2$. In this limit the resulting Lagrangian yields the correct frequencies [90].

If we take $\nu\to 0$ limit in (3.18) and (3.19), the Lagrangians become

$$\mathcal{L}_1=2\sum_{i=1,2}\left(\partial_+ a_i\partial_- a_i-2\rho'^2 a_i^2\right)\,, \tag{3.20}$$

and

$$\mathcal{L}_2=2\left[\partial_- a_3\partial_+ a_3-\frac{2\left(w^2\kappa^2+\rho'^4\right)}{\rho'^2}a_3^2+\partial_- a_4\partial_+ a_4\right]\,. \tag{3.21}$$

The sum of these two Lagrangians is the exactly the same as the Lagrangian (5.6) in [89], which is found by perturbing the Nambu action in the static gauge. We find that a_1 and a_2 correspond to β_i while a_3 corresponds to ϕ_A and a_4 is interpreted as a massless fluctuation denoted as $\tilde{\varphi}$ in

[89].

So far we have carried out the perturbation in the full reduced theory by embedding the tanh model into the gWZW model and shown that the perturbed Lagrangian takes the Nambu-Goto type in the original string theory. One may expect two of the fluctuations are captured by perturbing the tanh model directly. Plugging the folded string solution (3.8) into the fluctuation Lagrangian (3.2) and rescaling $\delta\theta_{_A}$ by

$$\delta\theta_{_A} \to \frac{\delta\theta_{_A}}{\sqrt{1-\frac{\nu^2}{\nu^2+\rho'^2}}}\,, \tag{3.22}$$

then we have the following Lagrangian,

$$\begin{aligned}\mathcal{L}_{\rm tanh} = \partial_-\delta\phi_{_A}\partial_+\delta\phi_{_A} - \left(\nu^2+2\rho'^2+\tfrac{2w^2\kappa^2}{\nu^2+\rho'^2}-\tfrac{3w^2\kappa^2\nu^2}{(\nu^2+\rho'^2)^2}\right)\delta\phi_{_A}^2 + \partial_-\delta\theta_{_A}\partial_+\delta\theta_{_A}\\ + \left(\tfrac{2w\kappa\nu\partial_-\delta\phi_A}{\nu^2+\rho'^2}+\tfrac{2w\kappa\nu\partial_+\delta\phi_A}{\nu^2+\rho'^2}\right)\delta\theta_{_A} - \nu^2\left(-1+\tfrac{2\left(w^2+\kappa^2\right)}{(\nu^2+\rho'^2)}-\tfrac{3w^2\kappa^2}{(\nu^2+\rho'^2)^2}\right)\delta\theta_{_A}^2\,.\end{aligned} \tag{3.23}$$

Noticing that this Lagrangian takes the same form as (3.19), we find that $\delta\phi_{_A}$ and $\delta\theta_{_A}$ correspond to a_3 and a_4, respectively. Hence it turns out that the perturbation in the tanh model describes the most complicated part of the bosonic fluctuation Lagrangian of the deformed gWZW model. This fact is useful when we consider more complicated classical solutions (see sections 3.2, 3.3 and 3.4).

3.1.2 Fermionic fluctuations

For a consistency with the original string theory, the masses of the fermionic fluctuations in the reduced theory should match those found in the original theory. We define component fields of the fermionic fluctuations in the following way,

$$\delta\Psi_{_R} = \begin{pmatrix} 0 & \mathfrak{X}_R \\ \mathfrak{Y}_R & 0 \end{pmatrix}, \quad \delta\Psi_{_L} = \begin{pmatrix} 0 & \mathfrak{X}_L \\ \mathfrak{Y}_L & 0 \end{pmatrix}, \tag{3.24}$$

where

$$\mathfrak{X}_R = \begin{pmatrix} 0 & 0 & \alpha_1 + i\alpha_2 & \alpha_3 + i\alpha_4 \\ 0 & 0 & -\alpha_3 + i\alpha_4 & \alpha_1 - i\alpha_2 \\ \alpha_5 + i\alpha_6 & \alpha_7 - i\alpha_8 & 0 & 0 \\ \alpha_7 + i\alpha_8 & -\alpha_5 + i\alpha_6 & 0 & 0 \end{pmatrix}, \quad (3.25)$$

$$\mathfrak{Y}_R = \begin{pmatrix} 0 & 0 & -\alpha_6 - i\alpha_5 & -\alpha_8 - i\alpha_7 \\ 0 & 0 & \alpha_8 - i\alpha_7 & -\alpha_6 + i\alpha_5 \\ \alpha_2 + i\alpha_1 & \alpha_4 - i\alpha_3 & 0 & 0 \\ \alpha_4 + i\alpha_3 & -\alpha_2 + i\alpha_1 & 0 & 0 \end{pmatrix}, \quad (3.26)$$

and

$$\mathfrak{X}_L = \begin{pmatrix} 0 & 0 & (\beta_1 + i\beta_2)v & (\beta_3 + i\beta_4)v \\ 0 & 0 & (\beta_3 - i\beta_4)v^* & (-\beta_1 + i\beta_2)v^* \\ (\beta_5 + i\beta_6)v & (-\beta_7 + i\beta_8)v & 0 & 0 \\ (\beta_7 + i\beta_8)v^* & (\beta_5 - i\beta_6)v^* & 0 & 0 \end{pmatrix}, \quad (3.27)$$

$$\mathfrak{Y}_L = \begin{pmatrix} 0 & 0 & (-\beta_6 - i\beta_5)v^* & (-\beta_8 - i\beta_7)v \\ 0 & 0 & (-\beta_8 + i\beta_7)v^* & (\beta_6 - i\beta_5)v \\ (\beta_2 + i\beta_1)v^* & (-\beta_4 + i\beta_3)v & 0 & 0 \\ (\beta_4 + i\beta_3)v^* & (\beta_2 - i\beta_1)v & 0 & 0 \end{pmatrix}, \quad (3.28)$$

where all component fields are real Grassmann. The extra factor $v = \exp(iw\kappa\tau/\nu)$ is introduced in $\mathfrak{X}_L$ and $\mathfrak{Y}_L$ such that the exponential factor does not appear in the Lagrangian. The resulting Lagrangian is

$$\begin{aligned} \mathcal{L}_{\mathrm{F}} = 2\Bigg[&\sum_{i=1}^{8} (\alpha_i \partial_- \alpha_i + \beta_i \partial_+ \beta_i) \\ &+ \tfrac{w\kappa\nu}{\nu^2 + \rho'^2} (\alpha_1\alpha_2 + \alpha_3\alpha_4 + \alpha_5\alpha_6 - \alpha_7\alpha_8 - \beta_1\beta_2 - \beta_3\beta_4 - \beta_5\beta_6 + \beta_7\beta_8) \\ &+ 2\sqrt{\nu^2 + \rho'^2}\, (-\alpha_3\beta_1 - \alpha_1\beta_3 + \alpha_7\beta_5 - \alpha_5\beta_7 + \alpha_4\beta_2 + \alpha_2\beta_4 + \alpha_8\beta_6 - \alpha_6\beta_8) \Bigg], \end{aligned} \quad (3.29)$$

After a careful observation we find that this system decouples into four subsectors which exhibit an identical structure containing two of α_i fields and two of β_i fields. Let us focus on the subsector containing α_1, α_2, β_3 and β_4, described by a part of the Lagrangian,

$$\mathcal{L} = \alpha_1 \partial_- \alpha_1 + \alpha_2 \partial_- \alpha_2 + \beta_3 \partial_+ \beta_3 + \beta_4 \partial_+ \beta_4 + \tfrac{w\kappa\nu}{\nu^2+\rho'^2} \left(\alpha_1 \alpha_2 - \beta_3 \beta_4\right) + 2\sqrt{\nu^2+\rho'^2} \left(-\alpha_1 \beta_3 + \alpha_2 \beta_4\right) , \tag{3.30}$$

which is simplified as

$$\mathcal{L} = \bar{\psi} \gamma^a \partial_a \psi + \frac{1}{2} \frac{w\kappa\nu}{\nu^2+\rho'^2} \bar{\psi} \Gamma_1 \psi - \sqrt{\nu^2+\rho'^2} \bar{\psi} \Gamma_2 \psi , \tag{3.31}$$

where

$$\psi = \begin{pmatrix} \beta_3 \\ \beta_4 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} , \quad \gamma^\tau = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} , \quad \gamma^\sigma = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} , \quad \bar{\psi} = \psi^\dagger \gamma^\tau , \tag{3.32}$$

and

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} , \quad \Gamma_2 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} . \tag{3.33}$$

The same prescription applies for the other subsectors, and consequently, our result agrees with the string theory result [89].

3.1.3 Bosonic fluctuations from Nambu action of original string theory

In [89] the authors studied fluctuations around the folded string without the S^1 part ($\nu = 0$ in (3.5)) and found that the mass of one bosonic fluctuation found in the Nambu action contains a $1/\rho'^2$ term. However, our calculation in the reduced theory in section 3.1.1 shows that the Lagrangian has no $1/\rho'^2$ term if the string solution has the S^1 sector.

It might be considered that this difference appears because the fluctuations in the AdS_5 part nontrivially couple to a fluctuation in S^5 part in the $\nu \neq 0$ case. So we shall first carry out the perturbation in the Nambu action with nonvanishing ν. However we will find that the nontrivial coupling is not all of the reason for the difference; the fluctuations in the tanh model and in the Nambu action are related by the T-duality transformation rather than rescaling or rotation of fluctuation fields. Since the partition functions of any two theories connected by the T-duality transformation are the same, this is a nontrivial support for our conjecture on the one-loop equivalence (1.6).

In the reduced theory the T-dual of the tanh model is the coth model. So we shall also show that the perturbation in the coth model recovers the quadratic fluctuations found by perturbing the Nambu action in another gauge. In section 2.1 we showed that the tanh model and coth model are realized as the different ways of $H \times H$ gauge fixing in the full $AdS_5 \times S^5$ reduction. Hence the result supports that the partition function is $H \times H$ gauge independent.

The Nambu action for the bosonic string in $AdS_5 \times S^5$ is given by

$$S_{\rm N} = -\int d\tau d\sigma \sqrt{-{\rm det} h_{ab}}\,, \tag{3.34}$$

where the induced metric on the worldsheet h_{ab} is

$$h_{ab} = g_{\mu\nu}(x)\partial_a x^\mu \partial_b x^\nu\,. \tag{3.35}$$

In the present case the classical forms of $g_{\mu\nu}(x)$ and x^μ are respectively

$$\begin{gathered} g_{\mu\nu}(x)dx^\mu dx^\nu = -\cosh^2\!\rho\, dt^2 + d\rho^2 + \sinh^2\!\rho\left(d\beta_i^2 + d\phi^2\right) + d\psi_s^2 + d\varphi^2 \quad (i=1,2,\ s=1,2,3,4)\,, \\ t = \kappa\tau\,, \quad \rho = \rho(\sigma)\,, \quad \beta_i = 0\,, \quad \phi = w\tau\,, \quad \psi_s = 0\,, \quad \varphi = \nu\tau\,, \end{gathered} \tag{3.36}$$

which are related by the Virasoro constraints and equation of motion (3.6). Imposing the static gauge where the fluctuations of t and ρ are set to zero we have the following perturbation,

$$t = \kappa\tau\,, \quad \rho = \rho(\sigma)\,, \quad \beta_i = \tfrac{1}{\lambda^{1/4}}\tilde\beta_i\,, \quad \phi = w\tau + \tfrac{1}{\lambda^{1/4}}\tilde\phi\,, \quad \psi_s = \tfrac{1}{\lambda^{1/4}}\tilde\psi_s\,, \quad \varphi = \nu\tau + \tfrac{1}{\lambda^{1/4}}\tilde\varphi\,, \tag{3.37}$$

Expanding the Lagrangian (3.34) gives the following action for the quadratic fluctuations,

$$S_{\rm N} = \tfrac{1}{2}\int d\tau d\sigma \Big[\sinh^2\!\rho\, \partial_+\tilde{\beta}_i\partial_-\tilde{\beta}_i - w^2\sinh^2\!\rho\,\tilde{\beta}_i^2 + \partial_+\tilde{\psi}_s\partial_-\tilde{\psi}_s - \nu^2\tilde{\psi}_s^2 \\ + \sinh^2\!\rho \left(1+\tfrac{w^2\sinh^2\rho}{\rho'^2}\right)\partial_+\tilde{\phi}\partial_-\tilde{\phi} + \left(1+\tfrac{\nu^2}{\rho'^2}\right)\partial_+\tilde{\varphi}\partial_-\tilde{\varphi} + \tfrac{w\nu\sinh^2\rho\left(\partial_+\tilde{\varphi}\partial_-\tilde{\phi}+\partial_+\tilde{\phi}\partial_-\tilde{\varphi}\right)}{\rho'^2}\Big]\,. \tag{3.38}$$

This shows that the fluctuations $\tilde{\psi}_s$ have $m^2_{\tilde{\psi}_s} = \nu^2$. They describe the four fluctuations in the S^5 sector in the reduced theory. By rescaling $\tilde{\beta}_i$ by $\tilde{\beta}_i \to \sinh^{-1}\rho\,\tilde{\beta}_i$, we find that $\tilde{\beta}_i$ have $m^2_{\tilde{\beta}_i} = \nu^2 + 2\rho'^2$. Hence the fluctuations β_i correspond to a_1 and a_2 in (3.18).

Let us focus on the other two fields, $\tilde{\phi}$ and $\tilde{\varphi}$, which should be compared with a_3 and a_4 in (3.19). The corresponding part in (3.38) is

$$\mathcal{L}_{\rm N} = F_1\partial_+\tilde{\phi}\partial_-\tilde{\phi} + F_2\partial_+\tilde{\varphi}\partial_-\tilde{\varphi} + F_3\left(\partial_+\tilde{\varphi}\partial_-\tilde{\phi} + \partial_+\tilde{\phi}\partial_-\tilde{\varphi}\right)\,, \tag{3.39}$$

where

$$F_1 = \sinh^2\!\rho\left(1+\frac{w^2\sinh^2\rho}{\rho'^2}\right)\,,\quad F_2 = 1+\frac{\nu^2}{\rho'^2}\,,\quad F_3 = \frac{w\nu\sinh^2\rho}{\rho'^2}\,, \tag{3.40}$$

Even if we rescale $\tilde{\phi}$ and $\tilde{\varphi}$ such that the coefficients of their kinetic terms are one, the resulting Lagrangian does not match the Lagrangian in (3.19). In fact (3.39) and (3.19) are related by the T-duality transformation. Since the Lagrangian (3.39) only depends on the derivatives of the fields ϕ and φ, it is allowed to carry out the T-duality transformation at the level of the Lagrangian in the present case. In order to derive the T-dualized Lagrangian we first denote $\partial_\pm\tilde{\varphi}$ by $\mathcal{A}_\pm$ and introduce a Lagrange multiplier $\tilde{x}$. Then the Lagrangian (3.39) is becomes

$$\mathcal{L}_{\rm N} = F_1\partial_+\tilde{\phi}\partial_-\tilde{\phi} + F_2\mathcal{A}_+\mathcal{A}_- + F_3\left(\partial_+\tilde{\phi}\mathcal{A}_- + \mathcal{A}_+\partial_-\tilde{\phi}\right) + \tilde{x}(\partial_+\mathcal{A}_- - \partial_-\mathcal{A}_+)\,. \tag{3.41}$$

The Lagrange multiplier $\tilde{x}$ will become a new physical field in the T-dual picture. Integrating out $\mathcal{A}_+$ yields

$$\mathcal{L}_{\rm TN} = \frac{1}{F_2}\partial_+\tilde{x}\partial_-\tilde{x} + \left(F_1 - \frac{F_3^2}{F_2}\right)\partial_+\tilde{\phi}\partial_-\tilde{\phi} + \frac{F_3}{F_2}\left(\partial_+\tilde{x}\partial_-\tilde{\phi} - \partial_+\tilde{\phi}\partial_-\tilde{x}\right)\,. \tag{3.42}$$

Finally, rescaling $\tilde{x} \to F_2^{1/2}\tilde{x}$ and $\tilde{\phi} \to \left(F_1 - \frac{F_3^2}{F_2}\right)^{-1/2}\tilde{\phi}$, we have the following Lagrangian,

$$\begin{aligned}\mathcal{L}_{\rm TN} &= \partial_-\tilde{\phi}\partial_+\tilde{\phi} - \left(\nu^2 + 2\rho'^2 + \tfrac{2w^2\kappa^2}{\nu^2+\rho'^2} - \tfrac{3w^2\kappa^2\nu^2}{(\nu^2+\rho'^2)^2}\right)\tilde{\phi}^2 + \partial_-\tilde{x}\partial_+\tilde{x} \\ &+ \left(\tfrac{2w\kappa\nu\partial_-\tilde{\phi}}{\nu^2+\rho'^2} + \tfrac{2w\kappa\nu\partial_+\tilde{\phi}}{\nu^2+\rho'^2}\right)\tilde{x} - \nu^2\left(-1 + \tfrac{2(w^2+\kappa^2)}{(\nu^2+\rho'^2)} - \tfrac{3w^2\kappa^2}{(\nu^2+\rho'^2)^2}\right)\tilde{x}^2 \,.\end{aligned} \tag{3.43}$$

This is exactly the same as the fluctuated Lagrangian (3.19), and also (3.23). Since the characteristic frequencies are invariant under the T-duality transformation, the fluctuated Lagrangian of the long string limit in the reduced theory produces the correct frequencies.

Recalling that the T-dual of the tanh model is the coth model in the reduced theory, one can expect the fluctuation Lagrangian of the coth model exactly agrees with that of the Nambu action. We are now interested in the nontrivial sector described by the complex sinh-Gordon model. Hence, for our present purpose, it is enough to perturb the Lagrangian of the coth model rather than the deformed gWZW model. Plugging the classical solution (3.11) into the fluctuation Lagrangian (3.1) and rescaling χ_A such that its kinetic term has unit coefficient, we obtain the following Lagrangian,

$$\begin{aligned}\mathcal{L}_{\rm coth(2)} &= \partial_+\delta\phi\partial_-\delta\phi - \left(\nu^2 + 2\rho'^2 + \tfrac{2w^2\kappa^2}{\nu^2+\rho'^2} + \tfrac{w^2\kappa^2\nu^2}{(\nu^2+\rho'^2)^2}\right)(\delta\phi)^2 \\ &+\partial_-\delta\chi_A\partial_+\delta\chi_A - \left(\nu^2 + \tfrac{2(w^2-\nu^2)(\kappa^2-\nu^2)}{\rho'^2} - \tfrac{2w^2\kappa^2}{\nu^2+\rho'^2} + \tfrac{w^2\kappa^2\nu^2}{(\nu^2+\rho'^2)^2}\right)(\delta\chi_A^2) \\ &- \tfrac{4w\kappa\nu^3\rho''}{\rho'(\nu^2+\rho'^2)^2}\delta\chi_A\delta\phi + \tfrac{2w\kappa\nu(\partial_-\delta\chi_A - \partial_+\delta\chi_A)}{\nu^2+\rho'^2}\delta\phi \,.\end{aligned} \tag{3.44}$$

In the Nambu action in the original string theory this Lagrangian is obtained if two fluctuations are introduced in the following way,

$$\begin{aligned} t &= \kappa\tau + N_1^1 z_1 + N_2^1 z_2 \,, \qquad \rho = \rho(\sigma) + N_1^2 z_1 + N_2^2 z_2 \,, \\ \phi &= \omega\tau + N_1^3 z_1 + N_2^3 z_2 \,, \qquad \varphi = \nu\tau + N_1^4 z_1 + N_2^4 z_2 \,, \end{aligned} \tag{3.45}$$

where N_1^i and N_2^i are defined as

$$\begin{aligned} N_1 &= \left(\tfrac{w\tanh\rho(\sigma)}{\sqrt{\rho'(\sigma)^2+\nu^2}}, 0, \tfrac{w\coth\rho(\sigma)}{\sqrt{\rho'(\sigma)^2+\nu^2}}, 0\right) \,, \\ N_2 &= \left(\tfrac{\kappa\nu}{\rho'(\sigma)\sqrt{\rho'(\sigma)^2+\nu^2}}, 0, \tfrac{w\nu}{\rho'(\sigma)\sqrt{\rho'(\sigma)^2+\nu^2}}, \tfrac{\sqrt{\rho'(\sigma)^2+\nu^2}}{\rho'(\sigma)}\right) \,. \end{aligned} \tag{3.46}$$

Substituting these into the Nambu action (3.34), one finds that z_1 corresponds to $\delta\phi$ and z_2 does $\delta\chi_A$. Alternatively one can reproduce the same Lagrangian by applying $O(2)$ rotation to the two fields, $\tilde{\phi}$ and $\tilde{\varphi}$, in the Lagrangian (3.38). Hence it turns out that the perturbation in the coth model corresponds to the perturbation of the Nambu action in the specific gauge.

In the original theory or in the coth model, the Lagrangian possesses the term proportional to $1/\rho'^2$, while such a term does not appear in the tanh model. This can be understood by looking at the T-duality transformation in the reduced theory, (2.9). For the present ϕ_A in (3.8), we have $\tanh^2 \phi_A = \rho'^2/(\nu^2 + \rho'^2)$ and the T-duality transformation is singular at turning points of the folded string. Since the partition function is invariant under the T-duality transformation, the partition function of the reduced theory is the same as that of the original string theory for the (S, J) folded string, which supports our conjecture on the quantum partition function (1.6).

Due to this direct relation between the coth model and the Nambu action one might think that it would be better to use the embedding of the coth model, (2.37), (2.38) rather than tanh model, (2.39), (2.40), when comparing the fluctuations in the reduced theory and original string theory. However, as mentioned earlier, (2.37) is a bad $H \times H$ gauge fixing for the perturbation. Hence we will basically continue to use the tanh model.

In this section we have shown that the relation of the perturbations in the coth model, the tanh model and the Nambu action in the original string theory. Their fluctuation Lagrangians are seemingly different and the difference in the reduced theory follows from the choices of the complex sinh-Gordon model, or in the words of the deformed gWZW model, the choices of the $H \times H$ gauge. However, if a classical string solution in a subsector is described by the sin(sinh)-Gordon model, or equivalently, if the subgroup H for the subsector is trivial, this freedom of choice does not exist, and so, the fluctuation Lagrangian of the reduced theory should always match that found by perturbing the Nambu action. This is the case for the bosonic string theory in $R \times S^2$, which is studied in appendix C.1

3.2 Circular string

In this section we shall discuss semiclassical quantization in the reduced theory for the (S, J) circular string solution. The parallel computation in the original string theory was studied in [92, 93, 38, 94]. It is expected that the perturbation in the reduced theory reproduces the result of [93].

We shall derive the Lagrangian for the quadratic fluctuations by using the embedding of the tanh model, and evaluate their characteristic frequencies for the bosonic fluctuations in section 3.2.1 and for the fermionic fluctuations in section 3.2.2. However a set of the characteristic frequencies in the reduced theory is not identical to the result in [93]. Then we shall show that the sum of the frequencies matches the original string theory result perturbatively in section 3.2.3, and find a $2d$ Lorentz transformation such that each of the frequencies in the reduced theory is the same as the corresponding frequency in the original theory in section 3.2.4. It may be considered that this happens because the (S,J) circular string is the case of $\mu_+ \neq \mu_-$ as a reflection of the existence of winding on a big circle of S^5. We will find an analogous discrepancy for another example of $\mu_+ \neq \mu_-$ in section 3.4.

We shall start the discussion with introducing the (S,J) circular string solution in the embedding coordinates,

$$Y_0 + iY_3 = r_0\, e^{i\kappa\tau}\,, \quad Y_1 + iY_2 = r_1\, e^{i\mathrm{w}\tau + ik\sigma}\,, \quad X_1 + iX_2 = e^{i\omega\tau + im\sigma}\,, \tag{3.47}$$

where $r_0 = \cosh\rho_0$ and $r_1 = \sinh\rho_0$ with a constant radius ρ_0. The parameters m and k are integer winding numbers in the AdS_3 subspace and the S^1 subspace, respectively. This solution has three Cartan charges, $(E,S,J) = \sqrt{\lambda}(\mathcal{E},\mathcal{S},\mathcal{J})$,

$$\mathcal{E} = r_0^2\kappa\,, \quad \mathcal{S} = r_1^2\mathrm{w}\,, \quad \mathcal{J} = \omega\,. \tag{3.48}$$

The equations of motion read

$$\mathrm{w}^2 = \kappa^2 + k^2\,, \quad \omega^2 = \nu^2 + m^2\,, \quad \nu^2 = -\Lambda\,, \quad \kappa^2 = \tilde{\Lambda}\,, \tag{3.49}$$

whereas the conformal gauge constraints are written as

$$2\kappa\mathcal{E} - \kappa^2 = 2\sqrt{k^2+\kappa^2} + \mathcal{J}^2 + m^2\,, \qquad k\mathcal{S} + m\mathcal{J} = 0\,. \tag{3.50}$$

They are supplemented by the identity $r_0^2 - r_1^2 = 1$, which can be rewritten as

$$\frac{\mathcal{E}}{\kappa} - \frac{\mathcal{S}}{\sqrt{k^2+\kappa^2}} = 1\,. \tag{3.51}$$

The relations (3.49), (3.50) and (3.51) show that only three of these parameters are independent. When calculating quantum fluctuations it is convenient to use κ, k, r_1.[2] From the latter three expressions,

[2] Since k can be absorbed in $\mathcal{S}$, it is possible to set k to be 1. Here we leave k arbitrary in order to make it

(3.50) and (3.51), we obtain the following relations,

$$\nu^2 = \sqrt{\kappa^4 - 4k^2\kappa^2 r_1^2(1+r_1^2)}\,, \qquad m^2 = \tfrac{1}{2}\left(\kappa^2 - 2k^2 r_1^2 - \nu^2\right)\,, \tag{3.52}$$

whcih will be useful when we compute fluctuations.

Now let us move on to the reduction of the (S, J) circular string solution. The mass scale of the reduced theory μ is determined by the stress tensor in the original string theory. For the (S, J) circular string solution (3.47) we have

$$T_{\pm\pm}^{\text{AdS}} = -\left(\kappa^2 - 2k\left(k \pm \sqrt{k^2+\kappa^2}\right) r_1^2\right)\,, \tag{3.53}$$

which imply that one needs introduce $\mu_\pm^2 = \sqrt{\kappa^2 - 2k\left(k \pm \sqrt{k^2+\kappa^2}\right) r_1^2}$. Since the closed string theory is defined on a cylinder rather than a plane, it is not allowed to set $\mu_+ = \mu_-$ by using a $2d$ Lorentz transformation. Instead we proceed with a single μ defined by $\mu = \sqrt{\mu_+\mu_-} = \sqrt{\kappa}\left(\kappa^2 - 4k^2 r_1^2 - 4k^2 r_1^4\right)^{1/4}$. Using this μ, the relations in (2.6) for the tanh model give $\phi_{_A}$ and $\theta_{_A}$,

$$\phi_{_A} = \tfrac{1}{2}\log\left(\frac{\kappa - 2kr_1\sqrt{1+r_1^2}}{\sqrt{\kappa^2 - 4k^2 r_1^2(1+r_1^2)}}\right)\,, \qquad \theta_{_A} = A\left(\sqrt{k^2+\kappa^2}\,\tau + k\sigma\right)\,, \tag{3.54}$$

where

$$A = \frac{2k^2 r_1^2(1+r_1^2)}{\sqrt{\kappa}\left(\kappa^2 - 4k^2 r_1^2 - 4k^2 r_1^4\right)^{1/4}\left(\kappa - \sqrt{\kappa^2 - 4k^2 r_1^2 - 4k^2 r_1^4}\right)}\,, \tag{3.55}$$

then the corrsponding classical solution in the deformed gWZW model is obtained by substituting these into (2.39),

$$g_0 = \begin{pmatrix} 0 & vB_+ & -vB_- & 0 \\ -v^*B_+ & 0 & 0 & v^*B_- \\ vB_- & 0 & 0 & -vB_+ \\ 0 & -v^*B_- & v^*B_+ & 0 \end{pmatrix}\,, \tag{3.56}$$

where

$$v = e^{iA\left(\sqrt{k^2+\kappa^2}\,\tau + k\sigma\right)}\,, \qquad B_\pm = \tfrac{1}{2}\left(\frac{\sqrt{\kappa - 2kr_1\sqrt{1+r_1^2}}}{\left(\kappa^2 - 4k^2 r_1^2(1+r_1^2)\right)^{1/4}} \pm \frac{\left(\kappa^2 - 4k^2 r_1^2(1+r_1^2)\right)^{1/4}}{\sqrt{\kappa - 2kr_1\sqrt{1+r_1^2}}}\right)\,. \tag{3.57}$$

easier to compare our result with [93].

By solving the gauge field equations (2.29) we have the classical gauge fields, $A_{\pm 0} = \frac{i}{2}\, a_{\pm 0} R_2$ with

$$a_{+0} = -\frac{\sqrt{\kappa}\big(k+\sqrt{k^2+\kappa^2}\big)}{2\big(\kappa^2-4k^2r_1^2(1+r_1^2)\big)^{1/4}}\,, \qquad a_{-0} = \frac{\big(k-\sqrt{k^2+\kappa^2}\big)\big(\kappa^2-4k^2r_1^2(1+r_1^2)\big)^{1/4}}{2\sqrt{\kappa}}\,. \tag{3.58}$$

We find that $g_0^{-1}\partial_+ g_0$ and $g_0^{-1}A_{+0}g_0$ are constant with these expressions. Hence this is a good starting point to discuss quantum fluctuations around the homogeneous string solution.

3.2.1 Bosonic fluctuations in reduced theory

The S^5 sector is rather simple; bosonic fluctuations in the S^5 sector for a string solution in $AdS_3 \times S^1$ are massive fields with masses $\pm\mu$. In the present case we have $\mu^2 = \mu_+\mu_-$, then we obtain characteristic frequencies for the four massive fields,

$$\pm\sqrt{n^2+\mu^2} = \pm\sqrt{n^2+\sqrt{\kappa^4-4k^2\kappa^2 r_1^2(1+r_1^2)}}\,, \tag{3.59}$$

which are exactly the same as the result in [93].

As done in the case of the folded string, we introduce the fluctuation fields by (3.12), (3.13) and (3.14),[3] integrate out the diagonal parts of the gauge field fluctuations, and then, use the H gauge freedom such that physical fluctuations decople from unphysical fluctuations. The physical part of the quadratic fluctuations is described by the Lagrangian containing a_1 and a_2,

$$\mathcal{L}_1 = 2\sum_{i=1,2}\left(\partial_+ a_i \partial_- a_i - \kappa^2 a_i^2\right)\,, \tag{3.60}$$

and the Lagrangian containing a_3 and a_4,

$$\begin{aligned}\mathcal{L}_2 = 2\Big[&\partial_- a_3 \partial_+ a_3 - 2\kappa\left(\kappa - \sqrt{\kappa^2-4k^2r_1^2\left(1+r_1^2\right)}\right)a_3^2 + \partial_- a_4\partial_+ a_4 \\ &+ \frac{2\big(\kappa^2-4k^2r_1^2(1+r_1^2)\big)^{1/4}}{\sqrt{\kappa}}\left(\left(k+\sqrt{k^2+\kappa^2}\right)\partial_- a_3 - \left(k-\sqrt{k^2+\kappa^2}\right)\partial_+ a_3\right)a_4\Big]\,.\end{aligned} \tag{3.61}$$

Now it is clear that a_1 and a_2 correspond to $\tilde{Y}_2$ in [93], whose frequencies are

$$\pm\sqrt{n^2+\kappa^2}\,. \tag{3.62}$$

[3] Note that v in the expression (3.14) should be replaced by v in (3.57).

However the second Lagrangian $\mathcal{L}_2$ does not describe the remaining two fluctuations in AdS_5 in [93]. In fact, by substituting $e^{i(\Omega\tau - n\sigma)}$ into the fluctuation Lagrangian (3.61), one finds that the condition that the determinant of the mass matrix vanishes reads

$$\begin{aligned}(n^2-\Omega^2)^2 - \tfrac{8k\sqrt{(k^2+\kappa^2)\left(\kappa^2-4k^2r_1^2(1+r_1^2)\right)}}{\kappa}n\Omega + \tfrac{2\left(\kappa^3-(\kappa^2+2k^2)\sqrt{\kappa^2-4k^2r_1^2(1+r_1^2)}\right)}{\kappa}n^2 \\ -\tfrac{2\left(\kappa^3+(\kappa^2+2k^2)\sqrt{\kappa^2-4k^2r_1^2(1+r_1^2)}\right)}{\kappa}\Omega^2 = 0\,,\end{aligned} \tag{3.63}$$

which is different from the corresponding equation in the original theory (cf. Eq. (4.15) in [93]),

$$\left(\Omega^2-n^2\right)^2 + 4\Omega^2\kappa^2 r_1^2 - 4\left(1+r_1^2\right)\left(\Omega\sqrt{k^2+\kappa^2}+kn\right)^2 = 0\,, \tag{3.64}$$

and consequently, the characteristic frequencies do not match.[4] Although the equation (3.63) can not be solved for a genral case, it is possible to solve the equation (3.63) approximately for large $\mathcal{J}$ with $u = \mathcal{S}/\mathcal{J}$, k fixed. The four roots $\Omega_{I;n}$ are

$$\begin{aligned}\Omega_{I=1,2;n} &= \tfrac{-2kn\pm\sqrt{n^2(n^2+4k^2u(1+u))}}{2\mathcal{J}} + O\left(\tfrac{1}{\mathcal{J}^3}\right)\,,\\ \Omega_{I=3,4;n} &= \pm 2\mathcal{J} \pm \tfrac{n^2\mp 2kn+2k^2(1+u)}{2\mathcal{J}} + O\left(\tfrac{1}{\mathcal{J}^3}\right)\,.\end{aligned} \tag{3.65}$$

For $n=0$ the equation (3.63) can be solved exactly,

$$\Omega_{I,0} = \{0,0,\Omega_0,-\Omega_0\}\,,\quad \Omega_0 = \sqrt{\frac{2\kappa^3+2(\kappa^2+2k^2)\sqrt{\kappa^2-4k^2r_1^2\left(1+r_1^2\right)}}{\kappa}}\,. \tag{3.66}$$

These frequencies are totally different from those found in [93]. Therefore, as far as the individual characteristic frequencies are concerned, the reduced theory does not reproduce the result of the original string theory even for the zero modes.

For consistency we shall discuss the perturbation in the coth model which should describe the nontrivial sector containing a_3 and a_4, (3.61). Using the reduction (2.3) and the fluctuation Lagrangian of the

[4] Note that the equation (3.64) is obtained from the equation (4.15) in [93] by $\Omega \to -\Omega$. This difference originates from the fact we use the mode expansion $e^{i(\Omega\tau - n\sigma)}$ rather than $e^{i(\Omega\tau + n\sigma)}$.

coth model (3.1) we obtain the following Lagrangian,

$$
\begin{aligned}
\mathcal{L}_{\mathrm{coth}} = {} & \partial_+\delta\phi\partial_-\delta\phi - 2\kappa\left(\kappa + \sqrt{\kappa^2 - 4k^2 r_1^2\left(1+r_1^2\right)}\right)\delta\phi^2 + \frac{\kappa+\sqrt{\kappa^2-4k^2r_1^2\left(1+r_1^2\right)}}{\kappa-\sqrt{\kappa^2-4k^2r_1^2\left(1+r_1^2\right)}}\partial_-\delta\chi_A\partial_+\delta\chi_A \\
& +16k^2r_1^2\left(1+r_1^2\right)\left(\kappa^2-4k^2r_1^2(1+r_1^2)\right)^{1/4}\left(\kappa-2kr_1\sqrt{1+r_1^2}\right) \\
& \times\frac{\left(\kappa-2kr_1\sqrt{1+r_1^2}+\sqrt{\kappa^2-4k^2r_1^2(1+r_1^2)}\right)\left(\left(k+\sqrt{k^2+\kappa^2}\right)\partial_-\delta\chi_A+\left(k-\sqrt{k^2+\kappa^2}\right)\partial_+\delta\chi_A\right)}{\sqrt{\kappa}\left(\kappa-2kr_1\sqrt{1+r_1^2}-\sqrt{\kappa^2-4k^2r_1^2(1+r_1^2)}\right)^3\left(\kappa+\sqrt{\kappa^2-4k^2r_1^2(1+r_1^2)}\right)}\delta\phi\,,
\end{aligned}
\tag{3.67}
$$

which yields the same equation for characteristic frequencies as in (3.63). Hence the bosonic frequencies for this subsector are model-independent.

Since the other six bosonic frequencies match the corresponding six frequencies of the original theory, the discrepancy in these two frequencies seems to be a serious problem. However, in the following subsections, we will show that the total sum of the characteristic frequencies including the fermionic contributions is the same as that of the original string theory.

3.2.2 Fermionic fluctuations in reduced theory

We define component fields of the fermionic fluctuations as in the folded string case (3.24) - (3.28).[5] Then the fermionic part of the quadratic fluctuations takes the form,

$$
\begin{aligned}
\mathcal{L}_f = 2\Big[& \textstyle\sum_{i=1}^8\left(\alpha_i\partial_-\alpha_i+\beta_i\partial_-\beta_i\right)+\frac{\left(k-\sqrt{k^2+\kappa^2}\right)\left(\kappa^2-4k^2r_1^2(1+r_1^2)\right)^{1/4}}{\sqrt{\kappa}}\left(-\alpha_1\alpha_2-\alpha_3\alpha_4-\alpha_5\alpha_6+\alpha_7\alpha_8\right) \\
& +\frac{\left(k+\sqrt{k^2+\kappa^2}\right)\left(\kappa^2-4k^2r_1^2(1+r_1^2)\right)^{1/4}}{\sqrt{\kappa}}\left(-\beta_1\beta_2-\beta_3\beta_4-\beta_5\beta_6+\beta_7\beta_8\right) \\
& +\sqrt{\kappa}\left(\sqrt{\kappa-2kr_1\sqrt{1+r_1^2}}+\sqrt{\kappa+2kr_1\sqrt{1+r_1^2}}\right) \\
& \quad\times\left(-\alpha_3\beta_1-\alpha_1\beta_3+\alpha_7\beta_5-\alpha_5\beta_7+\alpha_4\beta_2+\alpha_2\beta_4+\alpha_8\beta_6-\alpha_6\beta_8\right)\Big]\,.
\end{aligned}
\tag{3.68}
$$

As expected the coefficients of each term in the Lagrangian is totally constant. Then we can evaluate frequencies in a straightforward way,

$$
\pm\sqrt{(n\pm c)^2+a^2}\pm d\,, \tag{3.69}
$$

[5] Here we should use v in (3.57) for v in the expression of $\delta\Psi_L$.

where

$$\begin{aligned} a^2 &= \tfrac{\kappa}{2}\left(\kappa + \sqrt{\kappa^2 - 4k^2 r_1^2(1+r_1^2)}\right) , \\ c &= \frac{k\left(\kappa^2-4k^2r_1^2\left(1+r_1^2\right)\right)^{1/4}}{2\sqrt{\kappa}} , \\ d &= \frac{\sqrt{k^2+\kappa^2}\left(\kappa^2-4k^2r_1^2\left(1+r_1^2\right)\right)^{1/4}}{2\sqrt{\kappa}} . \end{aligned} \tag{3.70}$$

While a agrees with a in [93], the other two do not match.

Despite of the discrepancies in the individual frequencies, one might still expect that the sums of all the frequencies match, which could provide a nontrivial support that the partition functions in the two theories are equivalent in the case of the (S, J) circular string at one-loop level. To show this we shall evaluate the sum of the frequencies perturbatively in large $\mathcal{J}$ in the next subsection.

3.2.3 Sum of frequencies

The procedure of calculating sum of the characteristic frequencies directly follows from the computation of the one-loop energy correction discussed in [93]. If given a set of N fluctuations, we obtain $2N$ roots $\Omega_{I;n}$ $(I = 1, \ldots, 2N)$ by solving the conditions that the determinant of the corresponding $N \times N$ mass matrix vanishes. The zero modes appear in pairs, $\Omega_{I;0} = \pm\Omega_{p;0}$ $(p = 1, \ldots, N)$, and the non-zero modes can be paired by the condition $\Omega_{I;n} = -\Omega_{I;n}$. Then the frequencies should be summed up as

$$\sum_{p=1}^{N} \hat{\Omega}_{p;0} + \sum_{n=1}^{\infty} \sum_{I=1}^{2N} \hat{\Omega}_{I;n} , \tag{3.71}$$

with

$$\hat{\Omega}_{p;0} = \text{sign}(C_p)\Omega_{p;0} , \qquad \hat{\Omega}_{I;n} = \text{sign}(C_I^{(n)})\Omega_{I;n} , \tag{3.72}$$

$$C_p = \frac{1}{2m_{11}(\Omega_{p;0})\Omega_{p;0}\prod_{q\neq p}(\Omega_{p;0}^2 - \Omega_{q;0}^2)} , \quad C_I^{(n)} = \frac{1}{m_{11}(\Omega_{I;n})\prod_{J\neq I}(\Omega_{I;n} - \Omega_{J;n})} , \tag{3.73}$$

where m_{11} is a minor of the mass matrix, i.e., the determinant of the matrix obtained from the mass matrix by removing the first row and first column. Note that the fermionic frequencies contribute to the partition function negatively. For the (S, J) string we find that the structure of $\text{sign}C_I^{(n)}$ becomes simple because half of the frequencies are positive and half of the frequencies are negative.

The zero modes contribution is

$$4\nu + 2\kappa + \Omega_0 - 8\sqrt{c^2 + a^2} , \tag{3.74}$$

and nonzero modes are

$$2\sum_{n=1}^{\infty}\left[4\sqrt{n^2+\nu^2}+2\sqrt{n^2+\kappa^2}+\frac{1}{2}\sum_{I=1}^{4}\text{sign}(C_I^{(n)})\Omega_{I;n}-\ 4\big(\sqrt{(n+c)^2+a^2}+\sqrt{(n-c)^2+a^2}\ \big)\right], \tag{3.75}$$

where Ω_0 is shown in (3.66) and $\Omega_{I;n}$ is expanded in terms of large $\mathcal{J}$ in (3.65). Using the formula for the large $\mathcal{J}$ expansion of κ,

$$\kappa=\mathcal{J}+\frac{k^2u(2+u)}{2\mathcal{J}}-\frac{k^4u\left(4+12u+8u^2+u^3\right)}{8\mathcal{J}^3}+\cdots, \tag{3.76}$$

where $u=\mathcal{S}/\mathcal{J}$, we find the zero mode part,

$$-\frac{k^2u(1+u)}{\mathcal{J}}+O\left(\frac{1}{\mathcal{J}^3}\right). \tag{3.77}$$

On the other hand, the contribution from the non-zero modes is expanded as

$$-\sum_{n=1}^{\infty}\frac{n^2+2k^2u(1+u)-n\sqrt{n^2+4k^2u(1+u)}}{2\mathcal{J}}+O\left(\frac{1}{\mathcal{J}^3}\right). \tag{3.78}$$

These two results are the same as [93].

It is still mysterious that the characteristic frequencies of the individual fluctuations do not match. In [93] the expansions of the Landau-Lifshitz Lagrangian is also discussed as a useful tool for extracting the part of the fluctuation frequencies in the string theory, $\Omega_{I=1,2}$. Although each frequency in the Landau-Lifshitz model is different from the corresponding one in the string theory, the sum of the two frequencies agrees with the string theory result. So one might expect that the sum of the frequencies of a_3 and a_4 would agree with that of the two fluctuations in the original string theory in the large $\mathcal{J}$ expansion. However this does not happen. Actually, for the nonzero modes of a_3 and a_4, we have

$$\frac{1}{2}\sum_{I=1}^{4}\text{sign}(C_I^{(n)})\Omega_{I;n}=2\mathcal{J}+\frac{2k^2(1+u)+n\left(n+\sqrt{n^2+4k^2u(1+u)}\right)}{2\mathcal{J}}+O\left(\frac{1}{\mathcal{J}^3}\right), \tag{3.79}$$

On the other hand, the corresponding sum in the original string theory is

$$2\mathcal{J}+\frac{2k^2\left(1+3u+u^2\right)+n\left(n+\sqrt{n^2+4k^2u(1+u)}\right)}{2\mathcal{J}}+O\left(\frac{1}{\mathcal{J}^3}\right), \tag{3.80}$$

which is different from (3.79) and the discrepancy is

$$\frac{k^2 u(2+u)}{\mathcal{J}}\,. \tag{3.81}$$

This cancels with the discrepancy found in the fermionic sector, and then the total sums of the frequencies are the same.

3.2.4 $2d$ Lorentz boost

Generally one can evaluate the characteristic frequencies and confirm the agreement of the total sums of the frequencies order by order in large $\mathcal{J}$. However it is technically hard to continue the calculation to higher orders in $1/\mathcal{J}$. Here we shall find a $2d$ Lorentz transformation on the worldsheet, which does not change the total sum of the bosonic and fermionic frequencies, but can change some of the frequencies.

As the Lagrangians for the two fluctuations a_1, a_2 in the AdS_5 sector and all of the four fluctuations in the S^5 sector do not contain a first derivative term, their frequencies are obviously invariant under any Lorentz boost on the worldsheet. Hence one can expect a certain $2d$ Lorentz boost allows us to modify the equations for the fermionic frequencies and the other two bosonic frequencies such that they yield the frequencies found in [93]. Let us introduce the $2d$ Lorentz boost by

$$\tau \to p_1\tau + q_1\sigma\,, \quad \sigma \to q_1\tau + p_1\sigma\,, \quad \text{with} \quad p_1^2 - q_1^2 = 1\,. \tag{3.82}$$

In order to set the equations for frequencies of the two bosonic fluctuations a_3 and a_4 to be the same as the corresponding equation in the original theory (3.64), we choose

$$\begin{aligned} p_1 &= \sqrt{\frac{\kappa^2-2k^2r_1^2+\kappa\sqrt{\kappa^2-4k^2r_1^2\left(1+r_1^2\right)}}{2\kappa\sqrt{\kappa^2-4k^2r_1^2\left(1+r_1^2\right)}}}\,,\\ q_1 &= -\sqrt{\frac{\kappa^2-2k^2r_1^2-\kappa\sqrt{\kappa^2-4k^2r_1^2\left(1+r_1^2\right)}}{2\kappa\sqrt{\kappa^2-4k^2r_1^2\left(1+r_1^2\right)}}}\,. \end{aligned} \tag{3.83}$$

By this $2d$ boost the fermionic frequencies are changed into

$$\pm\sqrt{(n\pm c)^2 + a^2} \pm d\,, \tag{3.84}$$

where

$$a^2 = \frac{\left(\kappa^2+\kappa\sqrt{\kappa^2-4k^2r_1^2\left(1+r_1^2\right)}\right)}{2} ,$$
$$c = \frac{\sqrt{\kappa^2+2k^2\left(1+r_1^2\right)-\kappa\sqrt{\kappa^2-4k^2r_1^2\left(1+r_1^2\right)}}}{2\sqrt{2}} , \tag{3.85}$$
$$d = \frac{\sqrt{\kappa^2+2k^2\left(1+r_1^2\right)+\kappa\sqrt{\kappa^2-4k^2r_1^2\left(1+r_1^2\right)}}}{2\sqrt{2}} ,$$

which are exactly the same as [93]. Therefore all of the fluctuations in the reduced theory agree with the result of [93] in this frame. As the sum of the frequencies should be invariant under the $2d$ boost, our result implies that the sum of the frequencies in the reduced theory, (3.74) and (3.75), recovers the result of the original string theory calculation to all orders in $1/\mathcal{J}$. Then the quantum equivalence of the partition functions (1.6) has been demonstrated for the (S, J) circular string at one-loop level.

Let us discuss the meaning of this $2d$ Lorentz boost. Since the form of the classical solution (3.47) is not $2d$ Lorentz invariant, the fluctuation Lagrangians (3.61) and (3.68) are not either, and consequently, each fluctuation frequency may change by the $2d$ Lorentz transformation. If we apply the Lorentz transformation (3.83) to the classical solution (3.47), the solution becomes

$$Y_0 + iY_3 = r_0\, e^{i\kappa p_1\tau + i\kappa q_1\sigma} , \quad Y_1 + iY_2 = r_1\, e^{i(\mathrm{w}p_1+kq_1)\tau + i(kp_1+\mathrm{w}q_1)\sigma} ,$$
$$X_1 + iX_2 = e^{i(\omega p_1+mq_1)\tau+i(mp_1+\omega q_1)\sigma} . \tag{3.86}$$

With this classical solution the stress tensor takes the following form,

$$T^{\mathrm{AdS}}_{\pm\pm} = -\kappa\sqrt{\kappa^2 - 4k^2r_1^2\left(1+r_1^2\right)} , \tag{3.87}$$

which implies $\mu_+ = \mu_- = \sqrt{\kappa}\left(\kappa^2 - 4k^2r_1^2\left(1+r_1^2\right)\right)^{1/4}$. Hence it turns out that the $2d$ Lorentz boost we applied to both the bosonic fluctuations and the fermionic fluctuations is the same as the one setting $\mu_+ = \mu-$. However, it is still mysterious why the mixing of the bosonic and fermionic frequencies occurs by the Pohlmeyer reduction and why it is necessary to set $\mu_+ = \mu_-$ for the agreement of the frequencies of the individual fluctuations.

3.3 Spiky string

We shall discuss semiclassical expansion around the (S, J) spiky string solution in $AdS_3 \times S^1$ by using the embedding of the tanh model into the deformed gWZW model for the bosonic sector in section 3.3.1 and for the fermionic sector in section 3.3.2. Classical aspects of the spiky string solution were studied in the bosonic string theory [95, 96] and in the Pohlmeyer-reduced form [51, 52]. In [HIT] and in the earlier sections of this chapter we have seen that the semiclassical computation in the reduced theory perfectly recovers the one-loop corrections to the string partition function, and moreover, it has a huge advantage as the reduced theory has simple structures of both the bosonic and fermionic fluctuations after properly fixing the H gauge. Hence we expect that our result will agree with the string theory side, and the computation here will be much simpler than the standard worldsheet approach in the conformal gauge string theory.

First we shall review the (S, J) spiky string solution in $AdS_3 \times S^1$ found in the paper [96]. The solution is expressed in terms of the embedding coordinates,[6]

$$Y_0 + iY_3 = r_0(u)\, e^{iw_0\tau + i\varphi_0(u)}\,, \quad Y_1 + iY_2 = r_1(u)\, e^{iw_1\tau + i\varphi_1(u)}\,, \quad X_1 + iX_2 = e^{i\psi(u)}\,, \tag{3.88}$$

with

$$u = \alpha\sigma + \beta\tau\,, \qquad r_0^2 - r_1^2 = 1\,. \tag{3.89}$$

Here w_0 and w_1 are real constants. The S^1 part is explicitly written as

$$\psi = \nu\tau + \frac{D - \beta\nu}{\beta^2 - \alpha^2}u\,, \tag{3.90}$$

while φ_0 and φ_1 are expressed in differential form,

$$\varphi_0' = -\tfrac{1}{\beta^2-\alpha^2}\left(\tfrac{C_0}{r_0^2} + w_0\beta\right)\,, \qquad \varphi_1' = \tfrac{1}{\beta^2-\alpha^2}\left(\tfrac{C_1}{r_1^2} + w_1\beta\right)\,, \tag{3.91}$$

[6] Here we have assumed r_0 and r_1 have the $u(= \alpha\sigma + \beta\tau)$ dependence whereas they are constants in the case of the circular string (3.47).

where ν, D, C_0 and C_1 are real constants. The Virasoro constraints read

$$-r_0'^2(\beta^2-\alpha^2)-\tfrac{C_0^2}{r_0^2(\beta^2-\alpha^2)}-\tfrac{r_0^2\alpha^2w_0^2}{\beta^2-\alpha^2}+r_1'^2(\beta^2-\alpha^2)+\tfrac{C_1^2}{r_1^2(\beta^2-\alpha^2)}+\tfrac{r_1^2\alpha^2w_1^2}{\beta^2-\alpha^2}+\tfrac{D^2}{\beta^2-\alpha^2}+\tfrac{\alpha^2\nu^2}{\beta^2-\alpha^2}=0\,, \tag{3.92}$$

and

$$w_0C_0+w_1C_1+D\nu=0\,. \tag{3.93}$$

The first constraint is nothing but the condition that the Hamiltonian of this system should vanish. Using $r_0^2-r_1^2=1$ we rewrite (3.92) into an equation for r_1,

$$(\beta^2-\alpha^2)^2r_1'^2=(1+r_1^2)\left(\frac{C_0^2}{1+r_1^2}+\alpha^2w_0^2(1+r_1^2)-\frac{C_1^2}{r_1^2}-\alpha^2w_1^2r_1^2-D^2-\alpha^2\nu^2\right). \tag{3.94}$$

Here it is necessary to assume $w_0^2<w_1^2$ such that the string does not attach the boundary.

The spiky string solution with n spikes consists of $2n$ arcs, each of which should possess two turning points ($r_1'=0$) at some finite values of r_1. Let us introduce a new radial variable $v(u)$ by

$$v=\frac{1}{1+2r_1^2}=\frac{1}{\cosh 2\rho}\,, \tag{3.95}$$

where ρ is the radial coordinate in the global coordinate system of AdS_3. Assume that v' vanishes at $v=v_1,v_2,v_3$ with $v_1\le 0\le v_2\le v_3\le 1$, then the equation for r_1 (3.94) is rewritten as

$$v'=\frac{\sqrt{2vP(v)}}{\alpha^2-\beta^2}\,,\qquad P(v)=\frac{w_1^2-w_0^2}{v_1v_2v_3}\left(v-v_1\right)\left(v-v_2\right)\left(v-v_3\right)\,. \tag{3.96}$$

The constant v_1 is not arbitrary but is a function of v_2 and v_3,

$$v_1=-\frac{v_2v_3}{v_2+v_3+v_2v_3\frac{w_1^2+w_0^2-2(\nu^2+D)}{w_0^2-w_1^2}}\,. \tag{3.97}$$

C_1 and C_2 are expressed in terms of v_1, v_2 and v_3

$$C_0^2=\tfrac{w_0^2-w_1^2}{8}\tfrac{(1+v_1)(1+v_2)(1+v_3)}{v_1v_2v_3}\,,\qquad C_1^2=\tfrac{w_0^2-w_1^2}{8}\tfrac{(1-v_1)(1-v_2)(1-v_3)}{v_1v_2v_3}\,. \tag{3.98}$$

Under our assumption $v_1\le 0\le v_2\le v_3\le 1$ and $w_0^2<w_1^2$, we have $C_0^2\ge 0$ and $C_1^2\ge 0$, then our choice of the roots is consistent. Depending on further conditions for v_2 and v_3, the solution has two

possible regimes; the spike is at the minimum r_1 in one regime and at the maximum r_1 in the other regime.

Next let us discuss the reduction of the (S, J) spiky string solution. The mass scale μ in the reduced theory is determined by the AdS_3 part of the stress tensor, $T^{\text{AdS}}_{\pm\pm}$, in the original string theory. For the (S, J) spiky string solution, due to its complicated structure in the AdS sector, it is convenient to calculate $T^{\text{AdS}}_{\pm\pm}$ from the S^1 part of the stress tensor $T^{\text{S}}_{\pm\pm}$ by using the Virasoro constraint, $T^{\text{AdS}}_{\pm\pm}+T^{\text{S}}_{\pm\pm} = 0$. Then we have

$$T^{\text{AdS}}_{\pm\pm} = -T^{\text{S}}_{\pm} = -\left(\tfrac{\alpha\nu\mp D}{\alpha\mp\beta}\right)^2 , \tag{3.99}$$

which show that this is also the case of $T^{\text{AdS}}_{++} \neq T^{\text{AdS}}_{--}$. So we introduce $\mu_\pm$ by $T^{\text{AdS}}_{\pm\pm} = -\mu^2_\pm$, and define the mass scale μ as

$$\mu = \sqrt{\mu_+\mu_-} = \sqrt{\frac{\alpha^2\nu^2 - D^2}{\alpha^2 - \beta^2}} . \tag{3.100}$$

By this definition of μ we have implicitly assumed $\alpha^2\nu^2 > D^2$ and $\alpha^2 > \beta^2$ (or $\alpha^2\nu^2 < D^2$ and $\alpha^2 < \beta^2$) so that μ is real. Below we will work on the reduction in these parameter regions.

Following the standard procedure of the Pohlmeyer reduction for the tanh model, (2.6), we obtain the sinh-Gordon angle $\phi_{_A}$,[7]

$$\phi_{_A} = \tfrac{1}{2}\log\left[\tfrac{M_2+2\alpha\sqrt{M_2M_3}+\alpha^2M_3}{\alpha^2\nu^2-D^2}\right] , \tag{3.101}$$

and $\theta_{_A}$,

$$\partial_\pm\theta_{_A} = \alpha^2\sqrt{\tfrac{\alpha^2-\beta^2}{\alpha^2\nu^2-D^2}}\,\tfrac{(C_0^2+C_1^2)w_0w_1+C_0C_1\left(w_0^2+w_1^2\right)\pm C_0w_1\alpha\left(M_1-w_0^2\right)\pm C_1w_0\alpha\left(M_1-w_1^2\right)+w_0w_1\left(\alpha^2M_3-D^2\right)}{(\alpha\mp\beta)^2M_2} , \tag{3.102}$$

where we introduced M_i $(i = 1, 2, 3)$ as functions of r_1,

$$M_1 = w_0^2 + \left(w_0^2 - w_1^2\right) r_1^2 , \quad M_2 = M_1\alpha^2 - D^2 , \quad M_3 = M_1 - \nu^2 . \tag{3.103}$$

Hereafter let us consider the case of positive M_1.[8] As the radial coordinate r_1 is a function of $u =$

[7]For notational simplicity we will use r_1 rather than v which is used in the original paper [96]. Although expressing the classical solution in terms of r_1' makes it easy to understand the behavior of the fluctuations at spikes, we avoid to employ r_1' for the same reason.

[8]This assumption is related to a condition for the existence of the solution of the equation (3.94) for $D = \nu = 0$. Naively the equation (3.94) has a solution if its right hand side is positive. A sufficient condition for this is exactly the same as $M_1 \geq 0$ because we have $C_0^2 \geq C_1^2$.

If we consider the other case where M_1 is negative, the $D, \nu \to 0$ limit in the fluctuation Lagrangians yields a wrong answer. This problem is solved by the following prescription. Generally the reduction equation for $\phi_{_A}$

$\alpha\tau + \beta\sigma$, θ_A can be expressed in an integral form.[9] However, this is not a serious problem as far as quadratic fluctuations are concerned, since θ_A appears only as $\partial_\pm\theta_A$ in the fluctuation Lagrangian.

If we substitute (3.101) and (3.102) into (2.39) we find that the corresponding classical solution in the gWZW model takes the following form,

$$g_0 = \begin{pmatrix} 0 & vP_1 & -vP_2 & 0 \\ -v^*P_1 & 0 & 0 & v^*P_2 \\ vP_2 & 0 & 0 & -vP_1 \\ 0 & -v^*P_2 & v^*P_1 & 0 \end{pmatrix} . \tag{3.105}$$

v is given by

$$v = e^{i\theta_A} , \tag{3.106}$$

where θ_A solves the differential equations in (3.102). P_1 and P_2 are written as

$$P_1 = \frac{M_2 + \alpha\sqrt{M_2M_3}}{\sqrt{\left(M_2+\alpha^2M_3+2\alpha\sqrt{M_2M_3}\right)(\alpha^2\nu^2-D^2)}} , \quad P_2 = \frac{\alpha^2M_3 + \alpha\sqrt{M_2M_3}}{\sqrt{\left(M_2+\alpha^2M_3+2\alpha\sqrt{M_2M_3}\right)(\alpha^2\nu^2-D^2)}} . \tag{3.107}$$

The classical gauge field equations (2.29) are solved by $A_{\pm 0} = \frac{i}{2}\, a_{\pm 0} R_2$ with

$$a_{\pm 0} = -\frac{\left(M_2 \pm \alpha^2 M_3\right)\partial_\pm\theta_A}{M_2} , \tag{3.108}$$

where $\partial_\pm\theta_A$ are given by (3.102).

At the level of the classical solution (3.105) we can not take the $D, \nu \to 0$ limit in which the string is not stretching or moving in S^5. It is because $P_{1,2}$ in (3.107) diverge in this limit due to the factor of $\sqrt{\alpha^2\nu^2 - D^2}$ in their denominators. We will show that the $D, \nu \to 0$ limit can be taken once we derive the Lagrangian for quadratic fluctuations, which is the same situation as the (S, J) folded string case.

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in (2.6) has four solutions, two of which are real. Employing the other real branch for ϕ_A,

$$\frac{1}{2}\log\left[\frac{M_2 - 2\alpha\sqrt{M_2M_3} + \alpha^2M_3}{\alpha^2\nu^2 - D^2}\right] , \tag{3.104}$$

one can check the $D, \nu \to 0$ limit becomes well-defined for $M_1 < 0$ at the level of the fluctuation Lagrangians. This observation implies that we should choose ϕ_A appropriately depending on the parameters, α and w_0.

Another reason we have assumed $M_1 \geq 0$ is that the solution has a smooth limit to the folded string in AdS_3 under this assumption. In fact the folded string limit corresponds to $M_1 \to \rho'^2$ which is positive-definite.

[9] Making the T-duality transformation does not help in the present case; $\partial_\pm\chi_A$ in the coth model are not constant.

3.3.1 Bosonic fluctuations in reduced theory

Bosonic fluctuations in the S^5 sector for any string solution in $AdS_3 \times S^1$ are massive fields with masses $\pm\mu$. From (3.100) we find that the Lagrangian for the quadratic fluctuations in the S^5 sector, b_i ($i = 1, \ldots, 4$), is

$$\mathcal{L} = 2\sum_{i=1}^{4}\left(\partial_+ b_i \partial_- b_i - \frac{\alpha^2\nu^2 - D^2}{\alpha^2 - \beta^2} b_i^2\right) . \tag{3.109}$$

In this sector all the fluctuations have the constant masses.

The procedure for deriving the physical part of the fluctuated Lagrangian in the AdS_5 sector is the same as that in the (S, J) folded string case and the circular string case; introduce the fluctuation fields by (3.12), (3.13) and (3.14), integrate out the diagonal parts of the gauge field fluctuations, and then, use the H gauge freedom such that physical fluctuations decouple from unphysical fluctuations.[10] Consequently we find the Lagrangian containing a_1 and a_2,

$$\mathcal{L}_1 = 2\sum_{i=1,2}\left(\partial_+ a_i \partial_- a_i - \frac{M_2 + M_3\alpha^2}{\alpha^2 - \beta^2} a_i^2\right) . \tag{3.110}$$

As shown for the previous classical solutions, one shortcut way to the Lagrangian containing a_3 and a_4 is to perturb the tanh model Lagrangian directly. For the spiky string solution this approach has a big advantage because of the complicated expression for the classical solution. Substituting the classical solution (3.102), (3.101) into the perturbed Lagrangian of the tanh model (3.2), we obtain the Lagrangian for a_3 and a_4,

$$\mathcal{L}_2 = 2\Big[\partial_- a_3 \partial_+ a_3 + A_{33} a_3^2 + \partial_- a_4 \partial_+ a_4 + A_{44} a_4^2 + \left(A_+ \partial_+ a_4 + A_- \partial_- a_4\right) a_3 + A_{34} a_3 a_4\Big] , \tag{3.111}$$

where

$$\begin{aligned} A_{33} = &-\tfrac{M_2+M_3\alpha^2}{\alpha^2-\beta^2} + \tfrac{\alpha^4\left(M_2-3M_3\alpha^2\right)}{M_2^4(\alpha^2-\beta^2)}\Big[w_0^2 w_1^2 (C_0^4 + C_1^4) + 2C_0C_1 w_0 w_1 (C_0^2 + C_1^2)\left(w_0^2 + w_1^2\right) \\ &+ C_0^2C_1^2\left(w_0^4 + 4w_0^2w_1^2 + w_1^4\right) + (C_0^2 w_1^2 + C_1^2 w_0^2)\left(-M_1^2\alpha^2 + 2w_0^2\left(M_2 + M_3\alpha^2\right) - w_0^4\alpha^2\right) \\ &+ w_0^2 w_1^2\left(D^2 - M_3\alpha^2\right)^2 + 2C_0C_1w_0w_1\left(-M_1^2\alpha^2 + (M_2 + M_3\alpha^2)\left(w_0^2 + w_1^2\right) - w_0^2w_1^2\alpha^2\right)\Big] , \end{aligned} \tag{3.112}$$

[10] It should be noted that we use v in (3.106) with (3.102) for v in the expression (3.14).

$$A_{44} = -\frac{\left(\alpha^2\nu^2-D^2\right)^2}{M_2^2M_3^2(\alpha^2-\beta^2)}\Big[C_0^2\left(M_1-w_0^2\right)\left(w_0^2-w_1^2\right)-C_1^2\left(M_1-w_1^2\right)\left(w_0^2-w_1^2\right) \\ +(M_1^2-M_1\left(w_0^2+w_1^2\right)+w_0^2w_1^2)(M_3\alpha^2-D^2)\Big], \tag{3.113}$$

and

$$A_{34} = \frac{4\alpha^2\left(w_0^2-w_1^2\right)\left(\alpha^2\nu^2-D^2\right)^{3/2}}{M_2^3M_3(\alpha^2-\beta^2)^{3/2}}\sqrt{(M_3\alpha^2-D^2)\,r_1^4+\left(C_0^2-C_1^2+M_3\alpha^2-D^2\right)r_1^2-C_1^2} \\ \times\Big[w_0w_1\beta(C_0^2+C_1^2+M_3\alpha^2-D^2)+C_0C_1\beta(w_0^2+w_1^2)+\alpha^2(M_1-w_0w_1)(C_1w_0+C_0w_1)\Big], \tag{3.114}$$

$$A_{\pm} = \frac{2\alpha(\alpha\mp\beta)}{M_2(\alpha\pm\beta)}\sqrt{\frac{\alpha^2\nu^2-D^2}{M_2^2(\alpha^2-\beta^2)}}\Big[w_0w_1\alpha(C_0^2+C_1^2)+w_0w_1\alpha\left(M_3-D^2\alpha^2\right) \\ \mp(C_1w_0+C_0w_1)\left(M_1\alpha^2-w_1^2\alpha^2\right)+C_0C_1\alpha\left(w_0^2+w_1^2\right)\Big]. \tag{3.115}$$

Finally we shall consider the case of no stretching in S^5, which is achieved by taking the limit $D,\nu\to 0$, and correspondingly, $M_2\to\alpha^2M_1$, $M_3\to M_1$. In this case the fluctuations in the S^5 (3.109) become massless. In the AdS_5 sector we have the following Lagrangian,

$$\mathcal{L} = 2\left[\sum_{i=1,2}\left(\partial_+a_i\partial_-a_i-\frac{2\alpha^2M_1}{\alpha^2-\beta^2}a_i^2\right)+\partial_+a_3\partial_-a_3-\frac{2f^2\left(w_0^2-w_1^2\right)^2-2\alpha^2\left(M_1^2+w_0^2w_1^2\right)}{M_1(\alpha^2-\beta^2)}a_3^2+\partial_+a_4\partial_-a_4\right], \tag{3.116}$$

where we have rewritten C_0 and C_1 as $C_0=w_1f$ and $C_1=-w_0f$, respectively, such that they solve the second Virasoro constraint (3.93) with $D=\nu=0$. Hence it turns out that the $D,\nu\to 0$ limit is well defined at the level of the fluctuation Lagrangian in the bosonic sector .

Since the folded string solution in pure AdS_3 is realized as a special case of the spiky string solution, the Lagrangian (3.116) should recover the fluctuation Lagrangians for the folded string, (3.20) and (3.21). In fact the folded string solution corresponds to the limit $f\to 0$, $M_1\to\rho'^2$, $w_0\to\kappa$, $w_1\to w$ and $\alpha,\beta\to 0$, and one can find that the Lagrangian (3.116) reduces to the sum of (3.20) and (3.21) in this limit.

3.3.2 Fermionic fluctuations in reduced theory

Given the component fields of the fermionic fluctuations by (3.24) - (3.28), one can write down the fermionic fluctuation Lagrangian in terms of α_i and β_i,[11]

$$\begin{aligned}\mathcal{L}_{\rm F} \;=\; & 2\Big[\textstyle\sum_{i=1}^{8}(\alpha_i\partial_-\alpha_i+\beta_i\partial_-\beta_i)+A_{\alpha\alpha}\,(-\alpha_1\alpha_2-\alpha_3\alpha_4-\alpha_5\alpha_6+\alpha_7\alpha_8)\\ & +A_{\beta\beta}\,(-\beta_1\beta_2-\beta_3\beta_4-\beta_5\beta_6+\beta_7\beta_8)\\ & +A_{\alpha\beta}\,(-\alpha_3\beta_1-\alpha_1\beta_3+\alpha_7\beta_5-\alpha_5\beta_7+\alpha_4\beta_2+\alpha_2\beta_4+\alpha_8\beta_6-\alpha_6\beta_8)\Big]\,.\end{aligned} \tag{3.117}$$

where

$$A_{\alpha\alpha}=-\tfrac{1}{2}A_+\,,\quad A_{\beta\beta}=\tfrac{1}{2}A_-\,,\quad A_{\alpha\beta}=2\frac{M_2+\sqrt{M_2M_3}\alpha}{\sqrt{\big(M_2+2\sqrt{M_2M_3}\alpha+M_3\alpha^2\big)(\alpha^2-\beta^2)}}\,. \tag{3.118}$$

$A_\pm$ are defined in the bosonic sector, (3.115). The Lagrangian (3.117) again describes four decoupled systems, each of which has four fermionic component fields.

Let us now discuss the special case where the string is in pure AdS_3, that is, $D,\nu\to 0$, $M_2\to\alpha^2 M_1$ and $M_3\to M_1$. Recalling that we are considering that case of $M_1>0$, then we find that the coefficient $A_{\alpha\beta}$ becomes ($A_{\alpha\alpha}, A_{\beta\beta}\to 0$ in this limit)

$$\sqrt{\frac{\alpha^2 M_1}{\alpha^2-\beta^2}}\,. \tag{3.119}$$

Hence the $D,\mu\to 0$ limit is well-defined at the level of the fluctuation Lagrangian in the fermionic sector.

For consistency the coefficient (3.119) should recover the case of the folded string in pure AdS_3. Taking the corresponding limit $M_1\to\rho'^2$ and $\alpha,\beta\to 0$ yields the mass term with a coefficient ρ' which agrees with [89]. Hence the fermionic part of the quadratic fluctuations around the folded string without the S^5 sector is recovered.

[11] Again we should use v in (3.106) with (3.102) for $\delta\Psi_{_L}$ instead of the original v for the (S,j) folded string.

3.4 Folded string with orbital momentum and winding in S^1 of S^5

In this section we shall study semiclassical expansion around the generalized folded string solution with both the orbital momentum and the winding in S^1 of S^5 which was constructed in [97]. After reviewing how to achieve the generalized homogeneous folded string from the (S, J) spiky string, we will evaluate the characteristic frequencies for quadratic fluctuations in the bosonic sector in section 3.4.1 and in the fermionic sector in section 3.4.2 in order to compare them with the result in [97].

The generalized folded string solution also has the open string counterpart, a null cusp solution, which is obtained by a combination of analytic continuation on the worldsheet and $SO(2, 4)$ rotation from the generalized folded string solution [97]. As the $SO(2, 4)$ symmetry is obscure in the reduced theory, the generalized folded string and its open string counterpart are connected by the analytic continuation in the reduced theory. As the $SO(2, 4)$ symmetry is obscure in the reduced theory, the generalized folded string and its open string counterpart are connected by the analytic continuation in the reduced theory. Hence the equivalence between the two classical solutions becomes trivial by the Pohlmeyer reduction.

Because of the homogeneous nature of the generalized folded string, its fluctuation Lagrangian has constant coefficients, and thus, quantum corrections to the partition function can be computed. In fact, in [97], the expansion around the null cusp solutions were discussed and, the one-loop and two-loop corrections were determined. Directly from the equivalence of the generalized folded string and the null cusp solution, these quantum corrections are the same as those of the folded string solution.

Below we shall evaluate the characteristic frequencies of the quadratic fluctuations and compare them with the one loop computation in [97]. Reflecting the fact the generalized folded string has both the orbital momentum and the winding in the S^1 sector, this is another case of $\mu_+ \neq \mu_-$ in the reduced theory. Therefore, the conclusion of the one loop computation is very similar to the (S, J) circular string case in section 3.2; two of eight bosonic frequencies which can be derived by the perturbation in the complex sinh-Gordon model do not agree with the string theory result. The discrepancy is covered by the fermionic contributions, and the total sum of the frequencies agrees with the string theory result. In order to show this we shall find a $2d$ Lorentz transformation such that all of the bosonic and fermionic frequencies become the same as the frequencies of the corresponding fluctuations in the original theory in section 3.4.3.

We shall first review the generalized folded string solution with both the orbital momentum and the winding in S^1 of S^5. Introduce the global coordinates in $AdS_3 \times S^1$ by

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\theta^2 + d\varphi^2 \,, \tag{3.120}$$

then we expressed the solution in the conformal gauge,

$$t = \kappa\tau \,, \qquad \rho = \rho(\sigma) \,, \qquad \theta = \kappa\tau + \vartheta(\sigma) \,, \qquad \varphi = \nu\tau + m\sigma \,, \tag{3.121}$$

where

$$\begin{aligned} &\cosh\rho(\sigma) = \sqrt{1+\gamma^2}\cosh(\ell\sigma) \,, \qquad \tan\vartheta(\sigma) = \gamma\coth(\ell\sigma) \,, \qquad \gamma \equiv \tfrac{\nu m}{\kappa\ell} \,, \\ &\kappa^2 = \ell^2 + \nu^2 + m^2 \,. \end{aligned} \tag{3.122}$$

Here we are considering the following limit,

$$\ell \gg 1 \,, \qquad \nu \gg 1 \;\;, \qquad \kappa, \ell, \nu, m \gg 1 \,, \qquad \frac{\nu}{\ell} = \text{fixed} \,, \qquad \frac{m}{\ell} = \text{fixed} \,. \tag{3.123}$$

In this solution only three parameters are independent. We will use κ, ν and m when we calculate quadratic fluctuations. If we set the winding in a large circle of S^5, m, to be zero, the solution reduces to the (S, J) folded string solution discussed in section 3.1.

To compare this solution with the (S, J) spiky string solution in section 3.3, we rewrite the generalized folded string solution in terms of another radial coordinate v introduced in (3.95) and derive the equation for v',

$$v' = -2\ell v \sqrt{(1+v)\,(1-v-2\gamma^2 v)} \,, \tag{3.124}$$

which shows that v' vanishes at $v = -1, \ 0, \ \frac{1}{1+2\gamma^2}$ corresponding to the three roots v_1, v_2 and v_3 with $v_1 \le 0 \le v_2 \le v_3 \le 1$. Then we find that the generalized folded string solution is realized by taking the following limit in the (S, J) spiky string solution,

$$v_1 = -1 \,, \quad v_2 = 0 \,, \quad v_3 = \frac{1}{1+2\gamma^2} \,, \tag{3.125}$$

which is different from the three limiting cases considered in [96].[12]

Let us next construct the corresponding solution in the reduced theory. For this purpose we express the generalized folded string solution (3.121) in terms of the embedding coordinates,

$$Y_0 + iY_3 = \cosh\rho(\sigma)\, e^{i\kappa\tau}\,, \quad Y_1 + iY_2 = \sinh\rho(\sigma)\, e^{i\vartheta(\sigma)}\,, \quad X_5 + iX_6 = e^{i(\nu\tau + m\sigma)}\,, \tag{3.126}$$

where $\rho(\sigma)$ and $\vartheta(\sigma)$ are given in (3.122), and related to the parameters in the S^1 sector. The mass scale of the reduced theory μ can be extracted from $T^{\rm AdS}_{++}$ and $T^{\rm AdS}_{--}$. In the present case we have

$$T^{\rm AdS}_{\pm\pm} = -(\nu \pm m)^2\,, \tag{3.127}$$

which imply $\mu_\pm = \nu \pm m$, and then, μ should be introduced by their product,

$$\mu = \sqrt{\mu_+\mu_-} = \sqrt{\nu^2 - m^2}\,. \tag{3.128}$$

As done in the semiclassical computation for the other solutions we use the tanh model in the reduced theory. Once we obtain the mass scale μ, the solution in the original theory, (3.126), is encoded into a solution of the tanh model in the reduced theory by using (2.6),

$$\phi_{_A} = \tfrac{1}{2}\log\left[\tfrac{2\kappa^2-\nu^2-m^2+2\sqrt{(\kappa^2-m^2)(\kappa^2-\nu^2)}}{\nu^2-m^2}\right]\,, \qquad \theta_{_A} = \tfrac{\kappa^2-m^2}{\sqrt{\nu^2-m^2}}\tau\,. \tag{3.129}$$

Plugging these into (2.39) and (2.40) we obtain the classical solution for the deformed gWZW model,

$$g_0 = \begin{pmatrix} 0 & vV_1 & -vV_2 & 0 \\ -v^*V_1 & 0 & 0 & v^*V_2 \\ vV_2 & 0 & 0 & -vV_1 \\ 0 & -v^*V_2 & v^*V_1 & 0 \end{pmatrix}\,, \tag{3.130}$$

[12] The connection of the n-spike string and its long string limit was studied in [99] in the context of recovering $AdS_3 \times S^1$ string solutions from the asymptotic $SL(2)$ Bethe Ansatz equations. For $n = 2$ the limiting solution reduces to the generalized folded string solution.

where

$$
\begin{aligned}
v &= e^{\frac{i\left(\kappa^2-m^2\right)}{\sqrt{\nu^2-m^2}}\tau}\,, \\
V_1 &= \frac{\kappa^2-m^2+\sqrt{(\kappa^2-m^2)(\kappa^2-\nu^2)}}{\nu^2-m^2}\sqrt{\frac{\nu^2-m^2}{2\kappa^2-\nu^2-m^2+2\sqrt{(\kappa^2-m^2)(\kappa^2-\nu^2)}}}\,, \\
V_2 &= -\frac{\kappa^2-\nu^2+\sqrt{(\kappa^2-m^2)(\kappa^2-\nu^2)}}{\nu^2-m^2}\sqrt{\frac{\nu^2-m^2}{2\kappa^2-\nu^2-m^2+2\sqrt{(\kappa^2-m^2)(\kappa^2-\nu^2)}}}\,.
\end{aligned}
\tag{3.131}
$$

and the corresponding classical gauge fields are $A_{\pm 0} = \frac{i}{2}\, a_{\pm 0} R_2$ with

$$
a_{+0} = \frac{m^2-2\kappa^2+\nu^2}{\sqrt{\nu^2-m^2}}\,, \qquad a_{-0} = -\sqrt{\nu^2-m^2}\,. \tag{3.132}
$$

With this choice, $g_0^{-1}\partial_+ g_0$ and $g_0^{-1}A_+ g_0$ are constants, and then, the physical part of the quadratic fluctuation Lagrangian has constant coefficients after properly choosing the H gauge. Hence we can straightforwardly evaluate characteristic frequencies of the quadratic fluctuations, which should be compared with the fluctuation frequencies in the original string theory.

3.4.1 Bosonic fluctuations in reduced theory

Four physical modes in the S^5 sector yield four bosonic fluctuations with the masses $\pm\mu$. Since we have $\mu^2 = \nu^2 - m^2$, their frequencies are

$$
\pm\sqrt{n^2+\nu^2-m^2}\,, \tag{3.133}
$$

which are consistent with [97].

For the AdS_5 sector one easy way to obtain the constant coefficient Lagrangian for physical fluctuations is again to introduce the component fields of η and $\delta A_\pm$ as in (3.12), (3.13) and (3.14), and then, to use the H gauge symmetry such that the physical fields decouple from the unphysical fields as in the folded string case.[13] Two of the bosonic fluctuations in the AdS_5 sector are described by the Lagrangian,

$$
\mathcal{L}_1 = 2\sum_{i=1,2}\left(\partial_+ a_i \partial_- a_i - (2\kappa^2-\nu^2-m^2)a_i^2\right)\,, \tag{3.134}
$$

Then their frequencies are

$$
\pm\sqrt{n^2+2\kappa^2-\nu^2-m^2}\,, \tag{3.135}
$$

[13] The original v in (3.14) should be replaced by v in (3.131).

which agree with the result in [97]. The problem is found in the sector of the other two bosonic fluctuations, which can also be captured by the perturbation in the complex sinh-Gordon model,

$$\mathcal{L}_2 = 2\left[\partial_- a_3 \partial_+ a_3 - 4\left(\kappa^2 - \nu^2\right) a_3^2 + \partial_- a_4 \partial_+ a_4 - 2\sqrt{\nu^2 - m^2}\left(\partial_+ a_3 + \partial_- a_3\right) a_4\right]. \tag{3.136}$$

The equation for the characteristic frequencies derived from this Lagrangian is

$$\Omega^4 - 2\left(n^2 + 2\kappa^2 - 2m^2\right)\Omega^2 + n^2\left(n^2 + 4\kappa^2 - 4\nu^2\right) = 0\,, \tag{3.137}$$

which is not the same as the corresponding equation in the original string theory [97],

$$\Omega^4 - 2\Omega^2\left(n^2 + 2\kappa^2\right) + 8\nu m n\Omega + n^2\left(n^2 + 4\kappa^2 - 4\nu^2 - 4m^2\right) = 0\,. \tag{3.138}$$

Therefore, characteristic frequencies of the reduced theory and the original theory are different in this subsector. However, this does not imply that our conjecture on the quantum equivalence between the original string theory and the reduced theory, (1.6), breaks down. Later we will show that this discrepancy should cancel with that of fermionic fluctuations by finding a specific Lorentz transformation making all of the frequencies the same as those found in the original string theory.

3.4.2 Fermionic fluctuations in reduced theory

The parameterization for the fermionic fields (3.24) - (3.28) in the fermionic part of the fluctuation Lagrangian (3.4) gives the constant coefficient Lagrangian,[14]

$$\begin{aligned}\mathcal{L}_f = 2\Big[&\textstyle\sum_{i=1}^{8}\left(\alpha_i\partial_-\alpha_i + \beta_i\partial_-\beta_i\right)\\ &+\sqrt{\nu^2 - m^2}\left(\alpha_1\alpha_2 + \alpha_3\alpha_4 + \alpha_5\alpha_6 - \alpha_7\alpha_8 - \beta_1\beta_2 - \beta_3\beta_4 - \beta_5\beta_6 + \beta_7\beta_8\right)\\ &+\frac{2\left(\kappa^2 - m^2 + \sqrt{(\kappa^2-m^2)(\kappa^2-\nu^2)}\right)}{\sqrt{2\kappa^2 - m^2 - \nu^2 + 2\sqrt{(\kappa^2-m^2)(\kappa^2-\nu^2)}}}\left(\alpha_1\beta_4 + \alpha_2\beta_3 - \alpha_3\beta_2 - \alpha_4\beta_1 - \alpha_5\beta_8 + \alpha_6\beta_7 + \alpha_7\beta_6 - \alpha_8\beta_5\right)\Big].\end{aligned} \tag{3.139}$$

The fermionic characteristic frequencies are given by solving the following equation,

$$\Omega^4 - \frac{1}{2}\left(4n^2 + 4\kappa^2 + \nu^2 - 5m^2\right)\Omega^2 + \frac{1}{16}\left(4n^2 + 4\kappa^2 - \nu^2 - 3m^2\right)^2 = 0\,, \tag{3.140}$$

[14] The original v in $\delta\Psi_L$ should be replaced by v in (3.131).

which is not the same as the equation for the fermionic frequencies in the original theory,

$$\begin{aligned}&\Omega^4 - \tfrac{1}{2}\left(4n^2 + 4\kappa^2 + \nu^2 - 3m^2\right)\Omega^2 + 2n\nu m\Omega \\ &\quad + \tfrac{1}{16}\left[9m^4 + \left(4n^2 + 4\kappa^2 - \nu^2\right)^2 - 2m^2\left(20n^2 + 12\kappa^2 - 3\nu^2\right)\right] = 0\,,\end{aligned} \tag{3.141}$$

and consequently, the characteristic frequencies do not match.

3.4.3 $2d$ Lorentz boost

In section 3.2.3 we carried out the large $\mathcal{J}$ expansion and showed that the sum of the frequencies in the reduced theory is the same as that in the original string theory up to the order $1/\mathcal{J}^2$. Since the sum of the frequencies should be invariant under $2d$ Lorentz boost on the worldsheet, the sums of the frequencies of these two theories are the same if all of the individual frequencies in the reduced theory become the same as the corresponding frequencies in the original theory by a single $2d$ Lorentz boost. The (S, J) circular string is the case.

The present situation is very similar. Six of bosonic frequencies, (3.133) and (3.135), are Lorentz invariant, while the others frequencies described by (3.137) and (3.140) are changed by a Lorentz transformation. If we introduce a $2d$ Lorentz transformation in the reduced theory by

$$\tau \to p_2\tau + q_2\sigma\,, \quad \sigma \to q_2\tau + p_2\sigma\,, \quad p_2^2 - q_2^2 = 1\,, \tag{3.142}$$

with

$$p_2 = \frac{\nu}{\sqrt{\nu^2 - m^2}}\,, \qquad q_2 = -\frac{m}{\sqrt{\nu^2 - m^2}}\,, \tag{3.143}$$

then (3.137) and (3.140) become (3.138) and (3.141), respectively. Hence the frequencies match for all of the individual fluctuations. This implies that the total sum of the quantum corrections to the partition function in the reduced theory agree with the string theory result, and so, supports our conjecture in (1.6).

It is worth mentioning that the transformation (3.142), (3.143) is exactly the same as the one setting $\mu_+ = \mu_-$ in (3.128). In fact, applying the $2d$ Lorentz boost to the generalized folded string solution

(3.126), we find that the stress tensor becomes

$$T^{\rm AdS}_{\pm\pm} = -(\nu^2 - m^2)\,, \tag{3.144}$$

that is, $\mu_+ = \mu_- = \sqrt{\nu^2 - m^2}$. This is the same observation as in the circular string case in section 3.2.

3.5 Remarks on $\mu_+ \neq \mu_-$ case

In sections 3.2, 3.3 and 3.4, we discussed semiclassical quantization for the solutions with $\mu_+ \neq \mu_-$ in the reduced theory. In particular, the characteristic frequencies of the individual fluctuations in sections 3.2 and 3.4 do not agree with those found in the string theory computation, while the total sums of the frequencies match. Then we showed that the agreement of the individual frequencies is achieved by applying the $2d$ Lorentz transformation to the fluctuation Lagrangians.

An alternative resolution to this problem is to modify the reduction procedure. Here we will only discuss the embedding of the tanh model, but the same technique works for the coth model because they are related by the T-duality transformation. In stead of the reduction relation (2.6), let us employ the following ansatz,

$$\begin{aligned} \partial_+ Y^P \partial_- Y_P &= -\mu^2 \cosh 2\phi_{\scriptscriptstyle A}\,, \\ K_P\, \partial_\pm^2 Y^P &= \mp 4\mu_\pm \mu^2 \sinh^2\!\phi_{\scriptscriptstyle A}\, \partial_\pm \theta_{\scriptscriptstyle A}\,. \end{aligned} \tag{3.145}$$

Even with this reduction, the reduced model is still the same as (2.8) since $\mu_\pm$ enter the reduced theory equations of motion as the square root of their product, $\mu = \sqrt{\mu_+\mu_-}$, but it modifies the corresponding reduced theory solutions. To confirm that this reduction relation leads the agreement of the individual fluctuation frequencies, we shall again look at the (S, J) circular string (3.47). In the present case, the complex sinh-Gordon fields take the following form

$$\begin{aligned} \phi_{\scriptscriptstyle A} &= \tfrac{1}{2}\log\left(\frac{\kappa - 2kr_1\sqrt{1+r_1^2}}{\sqrt{\kappa^2 - 4k^2 r_1^2 (1+r_1^2)}}\right)\,, \\ \theta_{\scriptscriptstyle A} &= C_- \tau + C_+ \sigma\,, \end{aligned} \tag{3.146}$$

where

$$C_{\pm} = \frac{k^2 r_1^2 \left(1+r_1^2\right)}{\kappa - \sqrt{\kappa^2 - 4k^2 r_1^2 \left(1+r_1^2\right)}} \left(\pm \frac{k-\sqrt{k^2+\kappa^2}}{\sqrt{\kappa^2+2k\left(-k+\sqrt{k^2+\kappa^2}\right)r_1^2}} + \frac{k+\sqrt{k^2+\kappa^2}}{\sqrt{\kappa^2-2k\left(k+\sqrt{k^2+\kappa^2}\right)r_1^2}} \right), \tag{3.147}$$

then the classical solution in the reduced theory (3.56) is slightly modified. This modification changes one of the Lagrangians for the bosonic fluctuations, (3.61), and that for the fermionic fluctuations, (3.68), leaving the other Lagrangians unchanged. With these new Lagrangians, one can show that the characteristic frequencies of the individual fluctuations exactly agree with the string theory result.

One can also check that all of the characteristic frequencies for the generalized folded string in section 3.4 agree with those found in the original string theory if the reduction relation (3.145) is used; one gets (3.138), (3.141) instead of (3.137), (3.140).

Chapter 4

Two-loop computation

Our aim in this chapter is to explore the two-loop relation of the $AdS_n \times S^n$ GS string theory (ST) and its Pohlmeyer-reduced theory (PRT) by evaluating the two-loop partition function of the reduced theory for $n = 3, 5$ [IRT]. Since the computation of the partition function for a nontrivial string configuration is complicated, we shall consider the scaling limit of the folded string localized in the AdS_3 subspace.

4.1 Summary of two-loop computation

Here let us summarize the original string theory result of the partition function in the long spinning string solution background and show our results in the Pohlmeyer-reduced theory.

4.1.1 Quantum partition function in string theory

Initiated by the exploration of the folded string solution in AdS_3 [17], the folded string solution was extended to the (S, J) folded string carrying nonzero momentum along S^1 of S^5. The semiclassical expansion around the (S, J) folded string was discussed and the one-loop corrections to the string energy were evaluated in [89] and [90] for the case of the scaling limit where the string becomes homogeneous, and the one-loop corrections without taking the limit were studied in [91]. The two-loop computation in the original string theory was first discussed in [69] for the folded string in AdS_3, and in [70] by considering the open string counterpart of the folded string. The two-loop corrections in the case of the (S, J) folded string were evaluated by using the conformal gauge in [71], and by using the light-cone gauge in [97, 100].

When the folded string is localized in AdS_3, nontrivial part of two-loop corrections is characterized by the Catalan's constant K [69, 70]. The logarithm of the resulting quantum partition function is given by

$$\Gamma_{\rm ST} = -\ln Z_{\rm ST} = \frac{1}{2\pi}\ f(\lambda)\ V_2\ , \tag{4.1}$$

$$f(\lambda) = a_1 + \frac{a_2}{\sqrt{\lambda}}\ +\ O(\frac{1}{(\sqrt{\lambda})^2})\ , \tag{4.2}$$

$$a_1 = -3\ln 2\ , \qquad\qquad a_2 = a_{2B} + a_{2F} = {\rm K} - 2{\rm K} = -{\rm K}\ . \tag{4.3}$$

Here a_1 is the one-loop and a_2 is the two-loop contributions (K is the Catalan's constant), and V_2 is the two-dimensional worldvolume $V_2 = \int d\tau' d\sigma' = \kappa^2 \bar{V}_2$. In a_2 we indicated separately the part coming from purely bosonic graphs (a_{2B}) and graphs involving fermions (a_{2F}). Contributions proportional to K originate from two-loop "sunset" graphs with three propagators that are expressed in terms of the following momentum integrals

$$I[m_i^2, m_j^2, m_k^2] \equiv \int \frac{d^2q_i d^2q_j d^2q_k}{(2\pi)^4} \frac{\delta^{(2)}(q_i + q_j + q_k)}{(q_i^2 + m_i^2)(q_j^2 + m_j^2)(q_k^2 + m_k^2)}\ , \tag{4.4}$$

$$I[4,2,2] = \frac{1}{(4\pi)^2}{\rm K}\ , \qquad\qquad I[2,1,1] = \frac{2}{(4\pi)^2}{\rm K}\ . \tag{4.5}$$

Recalling that the system possesses one AdS_3 mode with $m^2 = 4$, two AdS_5 modes transverse to AdS_3 with $m^2 = 2$ and five S^5 modes with $m^2 = 0$ [89], one finds that both the bosonic $I[4, 2, 2]$ and the fermionic $I[2, 1, 1]$ contributions involve the transverse AdS_5 modes with $m^2 = 2$. As these AdS_5 modes are absent, the Catalan's constant is not produced for the $AdS_3 \times S^3$ superstring theory [IRT],

$$AdS_3 \times S^3 : \qquad\qquad {\rm a}_1 = -2\ln 2\ , \qquad\qquad {\rm a}_2 = 0\ . \tag{4.6}$$

4.1.2 Quantum partition function in reduced theory

In this section we will summarize our results of the two-loop computation in the reduced theory.

Although our objective is to evaluate the quantum effective action of the folded string solution localized in AdS_3, it is not allowed to study it directly in the reduced theory. As in [IWA, 66], the reduction of string solutions in pure AdS_3 space is not yet known within the framework of the full $AdS_5 \times S^5$ reduction. So we will start with the (S, J) folded string solution lying in $AdS_3 \times S^1$ and take $J \to 0$ limit eventually.

Let us define the embedding coordinates Y_M $(M = -1, 0, \ldots, 4)$ on $R^{4,2}$ for AdS_5 and X_I $(I = 1, 2, \ldots, 5)$ on R^6 for S^5 with the constraints $\eta_{MN} Y^M Y^N = -1$, $\delta_{IJ} X^I X^J = 1$ where the metrics are $\eta = \mathrm{diag}(-1, 1, \ldots, 1, -1)$, $\delta = \mathrm{diag}(1, \ldots, 1)$. The (S, J) folded string solution in the scaling limit is expressed in terms of the embedding coordinates,

$$\begin{gathered} Y_0 + iY_5 = \cosh(\ell\sigma)\, e^{i\kappa\tau}\,, \qquad Y_1 + iY_2 = \sinh(\ell\sigma)\, e^{i\kappa\tau}\,, \qquad Y_3 = Y_4 = 0\,, \\ X_1 = X_2 = X_3 = X_4 = 0\,, \qquad X_5 + iX_6 = e^{i\mu\tau}\,, \end{gathered} \tag{4.7}$$

where κ, ℓ and μ are constants related by the Virasoro constraints,

$$\kappa^2 = \ell^2 + \mu^2\,. \tag{4.8}$$

The $J \to 0$ case is realized by the limit $\mu \to 0$.

While it was shown in chapter 3 that the limit $\mu \to 0$ where the string solution is localized in AdS_3 is well-defined in the fluctuation Lagrangian at the quadratic level in the "decoupling gauge" in which physical fluctuations decouple from unphysical fields, this limit is not well-defined in another gauge which we will take in this chapter. We will keep the S^1 sector during the intermediate steps and take the limit at the level of the two-loop integral. Then we will get the PRT counterpart of the two-loop partition function for the spinning string solution with $J = 0$. In this case the PRT quantum partition function takes a similar form as in the string theory (4.1), (4.2),

$$\Gamma_{\mathrm{PRT}} = -\ln Z_{\mathrm{PRT}} = \frac{1}{2\pi}\, \mathrm{f}(k)\; V_2\ , \tag{4.9}$$

$$\mathrm{f}(k) = \mathrm{a}_1 + \frac{2\mathrm{a}_2}{k}\ +\ O(\frac{1}{k^2})\,. \tag{4.10}$$

The coefficients a_n that we found are

$$\mathrm{a}_1 = -3\ln 2\ , \qquad \mathrm{a}_2 = \bar{\mathrm{a}}_2 + \tilde{\mathrm{a}}_2\ , \qquad \bar{\mathrm{a}}_2 = -\mathrm{K}\ , \qquad \tilde{\mathrm{a}}_2 = -\frac{1}{4}(\mathrm{a}_1)^2 = -\frac{9}{4}(\ln 2)^2\,. \tag{4.11}$$

The value of the one-loop coefficient a_1 matches the string theory one in (4.3), in agreement with (1.6).

The Catalan's constant term in the two-loop corrections a_2 contains exactly the same coefficient as in the string partition function in (4.3) if we assume the identification of the couplings in (2.34),

$$k_5 = k = 2\sqrt{\lambda} \tag{4.12}$$

Moreover, the bosonic and fermionic contributions are reproduced in the same way as in string theory, that is, $a_{2B} + a_{2F} = +\mathrm{K} - 2\mathrm{K} = -\mathrm{K}$.

It should be emphasized here that the equivalence of the mass spectra of the quadratic fluctuations is not sufficient for obtaining the same structure of the Catalan's constant term, because nontrivial cubic vertices in the reduced theory could generate the additional nontrivial contributions, e.g., a finite term proportional to $I[4,4,4]$ in (4.4). However, our computation shows that such nontrivial finite term does not appear and this is a strong indication that the $AdS_5 \times S^5$ ST and PRT are closely related at the quantum level.

The two-loop result in the reduced theory also contains an additional $\tilde{\mathrm{a}}_2 \sim (\ln 2)^2$ term which is absent in the ST two-loop coefficient a_2. To be precise, we did not manage to derive the value of the coefficient of $(\ln 2)^2$ term directly in the $AdS_5 \times S^5$ case: We also obtained an IR divergent result $\tilde{\mathrm{a}}_2 = -\frac{5}{4}(\ln 2)^2 - \ln 2 \ln m_0$, where $m_0 \to 0$ is an IR cutoff, and assumed a close analogy with $AdS_3 \times S^3$ does work for $AdS_5 \times S^5$. We believe that this IR divergence should be an artifact of involving unphysical massless fluctuations in our approach.

Our expectation to the origin of the IR divergent term is based on the study on the reduced $AdS_3 \times S^3$ theory, which allows for an alternative approach to the two-loop computation, in which the unphysical modes are integrated out from the classical PRT Lagrangian [66]. If we use the same approach as used in $AdS_5 \times S^5$ case, an IR divergent coefficient is obtained, $\tilde{\mathrm{a}}_2^{(1)} = -\frac{2}{3}(\ln 2)^2 - \frac{4}{3}\ln 2 \ln m_0$. On the other hand, the other approach led to a consistent finite two-loop result,

$$\mathrm{AdS}_3 \times \mathrm{S}^3: \qquad \mathrm{f}(k_3) = \mathrm{a}_1 + \frac{2\mathrm{a}_2}{k_3} + O(\frac{1}{k_3^2}) , \tag{4.13}$$

$$\mathrm{a}_1 = -2\ln 2 , \qquad \mathrm{a}_2 = -\frac{1}{4}(\mathrm{a}_1)^2 = -(\ln 2)^2 . \tag{4.14}$$

The coupling constants in the $AdS_5 \times S^5$ and $AdS_3 \times S^3$ are related by

$$k = k_5 = 2k_3 \ . \tag{4.15}$$

Once again, the one-loop coefficient here is the same as in (4.6) and the absence of the more complicated contributions like the Catalan's constant is also consistent with the vanishing of the string theory two-loop coefficient in (4.6).

It remains to be understood if the apparent disagreement of the two-loop coefficients a_2 and a_2 in string and reduced theories by precisely the square of the one-loop coefficient is still suggesting some relation between the two universal scaling functions.

The remaining part of this chapter is organized as follows.

In section 4.2 we shall first review the structure of the reduced theory and explain the approach to perturbative calculations based on a field redefinition using the Polyakov-Wiegmann identity and gauge-fixing on the gauge fields. We shall also present the fluctuation Lagrangian and list the basic types of two-loop diagrams which we will compute later.

In section 4.3 we will consider the $AdS_3 \times S^3$ reduced theory using the two approaches and compare the results of the two approaches. The first approach is to use the Polyakov-Wiegmann identity, where the unphysical degrees of freedom are still involved (approach I). The second approach is to impose a gauge on $g \in G$ and integrate out the gauge fields from the deformed gWZW Lagrangian (approach II). Only the physical degrees of freedom are present in the resulting system in this approach. Then a resolution of the IR divergence problem found in approach I will be proposed such that it restores the equivalence between the two approaches. The resulting finite two-loop coefficient is given in (4.14).

In section 4.4 we will present the analogous computation in the reduced $AdS_5 \times S^5$ theory by using the Polyakov-Wiegmann identity. We will first discuss the one-loop approximation where the result for the partition function matches the string theory result, and then, consider the two-loop computation based on approach I. Using a direct analogy with the $AdS_3 \times S^3$ case, the final expression for the two-loop coefficient is given by the same Catalan's constant term as found in the original string theory plus an additional term proportional to the square of the one-loop coefficient (4.11).

4.2 Expansion of reduced theory Lagrangian

In this section we shall explain perturbative expansions around a general classical configuration in the Pohlmeyer-reduced form of string theory in $AdS_n \times S^n$ for $n = 3, 5$. The reduced form is a gauged Wess-Zumino-Witten model associated with G/H and deformed with an integrable potential and two-dimensional fermionic fields where $G = SU(1,1) \times SU(2)$, $H = [U(1)]^2$ for $n = 3$, and $G = Sp(2,2) \times Sp(4)$, $H = [SU(2)]^4$ for $n = 5$.

4.2.1 Gauge fixing and parameterization based on the Polyakov-Wiegmann identity

In chapter 3 the perturbation in the Lagrangian (2.31) was discussed for evaluating the one-loop corrections to the partition function, where the we fixed the H gauge in a specific way so that physical modes decouple from unphysical modes in the fluctuation Lagrangian. At one-loop level, this gauge choice is always possible if a classical string is localized in the $AdS_3 \times S^3$ subspace, and the physical part of the fluctuation Lagrangian in such gauge agrees with the fluctuation Lagrangian found by perturbing the Nambu action in the original string theory. However, the decoupling is not expected at two-loop level; physical fluctuations couple with unphysical fluctuations more complicatedly in the cubic and quartic terms. Here we shall consider alternative strategies.

Another H gauge choice for the fluctuation fields is that we impose fluctuations of one of the two gauge fields vanishes, $\delta A_+ = 0$, which was employed in [73, 74, 75] in the case of expansions around the vacuum solution. Solve the gauge equations derived by varying the fluctuation Lagrangian by δA_-, then one obtains the Lagrangian involving only physical fluctuations. However, for the long spinning string, solutions for the gauge equations are expressed in terms of nonlocal functions of physical fluctuations, and the nonlocality can not be eliminated from the cubic and quartic terms by simple field redefinition, which makes it harder to perform the two-loop computation.

In this chapter we will discuss two alternative strategies. The first strategy is to fix the H gauge for the classical solution in the deformed gWZW model and use the Polyakov-Wiegmann (PW) identity by which gWZW model reduces into two sets of WZW models at the level of the classical Lagrangian (approach I) [64, 72]. The equivalence between the gWZW model and the WZW models can be extended to the quantum level once one appropriately includes all of the relevant functional determinant contributions. By the PW identity, the unphysical degrees of freedom contained in the gauge fields decouple from the physical degrees of freedom, but we should note that the unphysical modes contained in an element of G are still present in the system and couple with the physical modes.

In the other strategy we choose a specific parameterization for a coset element in G and integrate out the gauge fields at the classical level (approach II). Then the system contains the physical degrees of freedom only. For $AdS_3 \times S^3$, the reduced system becomes the sum of the complex sin-Gordon model and the complex sinh-Gordon model coupled with fermionic fields. We will discuss approach II only for the reduced $AdS_3 \times S^3$ theory because very involved fluctuation Lagrangian is found by the native extension of the specific parameterization of a coset element in G/H to the $AdS_5 \times S^5$ case (cf. [104]). Moreover, it is unclear whether the reduced model for the $AdS_5 \times S^5$ GS string allows us to decouple physical fields from unphysical fields beyond the classical level.

Now let us review the PW identity. It is always possible to rewrite the gauge fields as

$$A_+ = U\partial_+ U^{-1}\,, \quad A_- = \tilde{U}\partial_- \tilde{U}^{-1}\,, \tag{4.16}$$

where $U,\ \tilde{U} \in H$. The coupling of the gauge fields and $g \in G$ is eliminated by the following redefinition of g,

$$\tilde{g} = U^{-1} g\, \tilde{U}\,. \tag{4.17}$$

The coupling to the fermionic fields are absorbed into the rotation,

$$\tilde{\Psi}_L = U^{-1}\Psi_L U\,, \qquad \tilde{\Psi}_R = \tilde{U}^{-1}\Psi_R\tilde{U}\,. \tag{4.18}$$

Finally, the deformed gWZW Lagrangian becomes

$$\begin{aligned} L_{\mathrm{gWZW}} &= L_{\mathrm{WZW}}(\tilde{g}) - L_{\mathrm{WZW}}(U^{-1}\tilde{U}) + \mu^2 \mathrm{STr}\left(\tilde{g}^{-1}T\tilde{g}T\right) \\ &+ \mathrm{STr}\left(\tilde{\Psi}_L T\partial_+\tilde{\Psi}_L + \tilde{\Psi}_R T\partial_-\tilde{\Psi}_R\right) + \mu \mathrm{STr}\left(\tilde{g}^{-1}\tilde{\Psi}_L\tilde{g}\tilde{\Psi}_R\right)\,. \end{aligned} \tag{4.19}$$

The advantage in this form is that physical degrees of freedom on g are not contained in the WZW term on the subgroup H, $L_{\mathrm{WZW}}(U^{-1}\tilde{U})$, and accordingly, we find the physical fluctuations by perturbing the other parts, i.e., the deformed WZW model,

$$\begin{aligned} L_{\tilde g,\tilde\Psi} &= L_{\rm WZW}(\tilde g) + \mu^2 {\rm STr}\left(\tilde g^{-1} T \tilde g T\right) \\ &\quad + {\rm STr}\left(\tilde\Psi_L T\partial_+ \tilde\Psi_L + \tilde\Psi_R T \partial_- \tilde\Psi_R\right) + \mu {\rm STr}\left(\tilde g^{-1}\tilde\Psi_L \tilde g \tilde\Psi_R\right) . \end{aligned} \tag{4.20}$$

Not giving any physical contribution, the other WZW term should be considered to show the cancellation of the all of the unphysical modes at one-loop level,

$$L_{U,\tilde U} = -L_{\rm WZW}(U^{-1}\tilde U) . \tag{4.21}$$

Note that the H gauge symmetry (2.30) is not apparent in the derived Lagrangian (4.19). Under the the H gauge transformation, we find that $U, \tilde U$ changes $U \to U' = h^{-1}U$, $\tilde U \to \tilde U' = h^{-1}\tilde U$. By redefining $\tilde g$, $\tilde\Psi$ with the new U', $\tilde U'$, we again obtain the Lagrangian (4.19).

Also, the Lagrangian (4.20) still contains unphysical degrees of freedom. Taking the $AdS_5 \times S^5$ case, we find that the group element $\tilde g$ contains $10+10$ parameters equal to the dimensions of $Sp(2,2)\times Sp(4)$, $6+6$ of which are unphysical corresponding to the subgroup $H = [SU(2)]^4$. Thus one needs discuss fluctuations for both the physical and unphysical fields.

4.2.2 Structure of quantum corrections

We shall consider perturbation in the Lagrangian (4.20) and derive the fluctuation Lagrangian up to quartic order. We will first investigate the contributions of diagrams which do not involve fermionic propagators (we call them the bosonic contribution) in detail. For these diagrams, the system decouples into the AdS_n sector and the S^n sector as in the original string theory. Using the same technique, we will study the other class of diagrams containing fermionic propagators (we call them the fermionic contribution). Hereafter we will omit the tilde on g, Ψ, i.e., $\tilde g \to g$, $\tilde\Psi \to \Psi$.

In order to obtain the expansion of the deformed WZW model (4.20) to quartic order we shall introduce the fluctuations around a classical fields, g_0 and $\Psi_{R0} = \Psi_{L0} = 0$,

$$g = g_0 e^{\eta} = g_0(1 + \eta + \tfrac{1}{2}\eta^2 + \tfrac{1}{3!}\eta^3 + \tfrac{1}{4!}\eta^4 + \mathcal{O}(\eta^5)) , \qquad \eta \in \mathfrak{g} , \tag{4.22}$$

and fermionic fluctuations Ψ_R and Ψ_L. The quadratic, cubic and quartic terms in the fluctuation

Lagrangian are, respectively,

$$\mathcal{L}^{(2)} = \mathrm{STr}\Big[\tfrac{1}{2}\mathcal{D}_+\eta\partial_-\eta - \tfrac{\mu^2}{2}\left[\eta, g_0^{-1}Tg_0\right][\eta,T] + \Psi_R T\partial_-\Psi_R + \Psi_L T\partial_+\Psi_L + \mu g_0^{-1}\Psi_L g_0\Psi_R\Big], \quad (4.23)$$

$$\mathcal{L}^{(3)} = \mathrm{STr}\Big[-\tfrac{1}{6}\left[\eta, \mathcal{D}_+\eta\right]\partial_-\eta - \tfrac{\mu^2}{6}\left[\eta, g_0^{-1}Tg_0\right][\eta,\, [\eta, T]] + \mu\left(g_0^{-1}\Psi_L g_0\eta\Psi_R - \eta g_0^{-1}\Psi_L g_0\Psi_R\right)\Big], \quad (4.24)$$

$$\begin{aligned}\mathcal{L}^{(4)} = \mathrm{STr}\Big[&\tfrac{1}{24}\left[\eta,\, [\eta, \mathcal{D}_+\eta]\right]\partial_-\eta + \tfrac{\mu^2}{24}\left[\eta,\, \left[\eta, g_0^{-1}Tg_0\right]\right][\eta,\, [\eta, T]] \\ &+\mu\left(\tfrac{1}{2}g_0^{-1}\Psi_L g_0\eta^2\Psi_R + \tfrac{1}{2}\eta^2 g_0^{-1}\Psi_L g_0\Psi_R - \eta g_0^{-1}\Psi_L g_0\eta\Psi_R\right)\Big],\end{aligned} \quad (4.25)$$

where the derivative operator $\mathcal{D}_+$ is defined by using the classical field g_0,

$$\mathcal{D}_+ = \partial_+ + \left[g_0^{-1}\partial_+ g_0\,,\ \right]. \quad (4.26)$$

Physical fluctuations couple with unphysical fluctuations in the fluctuation Lagrangian because the fluctuation η contains both the physical part and unphysical part. Under the Z_2 decomposition deduced by T (see appendix A), we have $\eta = \eta^{\parallel} + \eta^{\perp}$ where $\eta^{\parallel} \in \mathfrak{m}$ and $\eta^{\perp} \in \mathfrak{h}$. It is reasonable to assume that the physical fluctuations are always found in $\eta^{\parallel}$ corresponding to the coset G/H [HIT, IWA, 73, 74, 75].

One-loop and two-loop corrections to the partition function for the long folded string can be computed by using the Euclidean signature on the worldsheet by analytic continuation $\tau \to i\tau$ and rescaling the worldsheet coordinate $\sigma \to \ell\sigma$ with $\ell \to \infty$,

$$\hat{\kappa} = \frac{\kappa}{\ell}, \qquad \hat{\mu} = \frac{\mu}{\ell}, \quad (4.27)$$

where κ, ℓ and μ are the parameters in the (S, J) folded string solution (4.7). After rescaling by ℓ the period of spatial coordinate on the worldsheet is $2\pi\ell$, and it becomes infinite in the $\ell \to \infty$ limit, and then, the spatial component in the momentum space becomes continuous. In the $\hat{\mu} \to 0$ limit where string has no rotation in S^1 of S^5, we have $\kappa = \ell$ from the relation (4.8), that is, the rescaling becomes the same as the one employed in [69].

In the remaining part of this section we will explain the structure of the quantum corrections based on the fluctuation Lagrangians (4.23), (4.24) and (4.25), for the bosonic contribution and fermionic contribution separately.

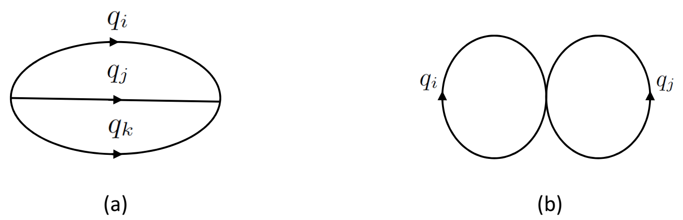

Figure 4.1: Bosonic two-loop diagrams; sunset diagram (a) and double-bubble diagram (b). Bosonic propagators are denoted by solid lines. In (a) the vertex momentum conservation gives the constraint $q_i + q_j + q_k = 0$.

The bosonic contributions are described by the following parts of the Lagrangians (4.23), (4.24) and (4.25),

$$\mathcal{L}_B^{(2)} = \mathrm{STr}\Big[\tfrac{1}{2}\mathcal{D}_+\eta\partial_-\eta - \tfrac{\mu^2}{2}\left[\eta, g_0^{-1}Tg_0\right][\eta, T]\Big]\,, \tag{4.28}$$

$$\mathcal{L}_B^{(3)} = \mathrm{STr}\Big[-\tfrac{1}{6}\left[\eta, \mathcal{D}_+\eta\right]\partial_-\eta - \tfrac{\mu^2}{6}\left[\eta, g_0^{-1}Tg_0\right][\eta,\, [\eta, T]]\Big]\,, \tag{4.29}$$

$$\mathcal{L}_B^{(4)} = \mathrm{STr}\Big[\tfrac{1}{24}\left[\eta,\, [\eta, \mathcal{D}_+\eta]\right]\partial_-\eta + \tfrac{\mu^2}{24}\left[\eta,\, [\eta, g_0^{-1}Tg_0]\right][\eta,\, [\eta, T]]\Big]\,. \tag{4.30}$$

Since all the bosonic matrices $\eta \in \mathfrak{g}$ are written in the block-diagonal form, the AdS_n sector decouples from the S^n sector in the bosonic part. These fluctuation Lagrangians show that the two-loop contributions to the partition function are given by the Feynman diagrams of the topologies shown in Figure 4.1. The sunset diagram in Figure 4.1(a) comes from the cubic terms in $\mathcal{L}_B^{(3)}$, whereas the double-bubble diagram in Figure 4.1(b) comes from the quartic terms in $\mathcal{L}_B^{(4)}$.

To compute these diagrams we shall derive cubic and quartic vertices. Let us denote the fluctuation fields by Φ_I symbolically. The vertices are written as

$$V_{IJK} = \frac{\partial^3\mathcal{L}^{(3)}}{\partial\Phi_I\partial\Phi_J\partial\Phi_K}\,, \quad V_{IJKL} = \frac{\partial^4\mathcal{L}^{(4)}}{\partial\Phi_I\partial\Phi_J\partial\Phi_K\partial\Phi_L}\,, \tag{4.31}$$

then the Lagrangians, (4.23), (4.24) and (4.25), can be expressed as

$$\mathcal{L}_B^{(2)} + \mathcal{L}_B^{(3)} + \mathcal{L}_B^{(4)} = \frac{1}{2}\Phi_I\triangle_{IJ}\Phi_J + \frac{1}{3!}V_{IJK}\Phi_I\Phi_J\Phi_K + \frac{1}{4!}V_{IJKL}\Phi_I\Phi_J\Phi_K\Phi_L\,. \tag{4.32}$$

With the notation (4.31) and (4.32), the one-loop effective action is obtained by computing the quantity $\frac{1}{2}\mathrm{Tr}\ln\triangle$. On the other hand, the two-loop effective action involves the contributions of the cubic terms and the quartic terms. They are computed by

$$\Gamma^{(3)} = -\frac{1}{12}\frac{8\pi}{k}V_2\int\frac{d^2q_id^2q_j}{(2\pi)^2}V_{IJK}V_{I'J'K'}\triangle^{-1}_{II'}\triangle^{-1}_{JJ'}\triangle^{-1}_{KK'}\,, \tag{4.33}$$

$$\Gamma^{(4)} = \frac{1}{8}\frac{8\pi}{k}V_2\int\frac{d^2q_id^2q_j}{(2\pi)^2}V_{IJKL}\triangle^{-1}_{IJ}\triangle^{-1}_{KL}\,, \tag{4.34}$$

where $-\frac{1}{12}$ in $\Gamma^{(3)}$ and $\frac{1}{8}$ in $\Gamma^{(4)}$ are the combinatorial factors, and the factor $\frac{8\pi}{k}$ comes from the overall factor in front of the Lagrangian (2.33).

Following the string theory computation [69, 70], we assume that the power divergent terms can be regularized by an analytic regularization scheme.[1] Thus the two-loop integrals consist of the logarithmic divergent terms and finite terms are remaining. Some of the two-loop integrals are simplified into a product of the one-loop integrals,

$$I[m^2] = \int\frac{d^2q}{(2\pi)^2}\frac{1}{q^2+m^2}\,. \tag{4.35}$$

It is useful to rewrite this UV divergent integral in terms of $I[1]$ by using the relation,

$$I[m^2] = I[1] - \frac{1}{4\pi}\ln m^2\,. \tag{4.36}$$

The second term is divergent if $m = 0$, which is the IR divergence of the loop integral for massless fluctuation fields. Also the following integral is obtained from the sunset diagrams,

$$I[m_i^2, m_j^2, m_k^2] = \int\frac{d^2q_i\,d^2q_j\,d^2q_k}{(2\pi)^2}\frac{\delta^{(2)}(q_i+q_j+q_k)}{(q_i^2+m_i^2)(q_j^2+m_j^2)(q_k^2+m_k^2)}\,. \tag{4.37}$$

Note that $I[m_i^2, m_j^2, m_k^2]$ is UV and IR finite for nonzero m_i, m_j and m_k. In special cases, the finite integral $I[m_i^2, m_j^2, m_k^2]$ gives the Catalan's constant K.

We shall move on to the fermionic contributions. The Lagrangian for the quadratic fermionic fluctuations is

$$\mathcal{L}_F^{(2)} = \mathrm{STr}\Big[\Psi_R T\partial_-\Psi_R + \Psi_L T\partial_+\Psi_L + \mu g_0^{-1}\Psi_L g_0\Psi_R\Big]\,. \tag{4.38}$$

In the fluctuation Lagrangian (4.24) and (4.25), the interaction between bosonic fields and fermionic

[1] Power divergent terms should cancel out provided all contributions are properly accounted for.

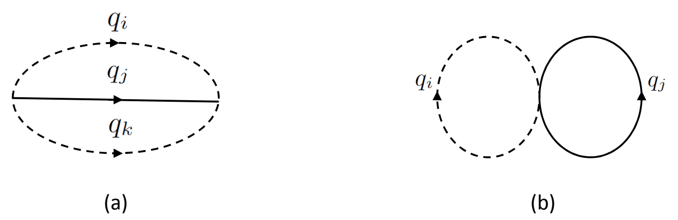

Figure 4.2: Diagrams with bosonic and fermionic propagators. Bosonic propagators are denoted by solid lines and fermionic ones are denoted by dashed lines.

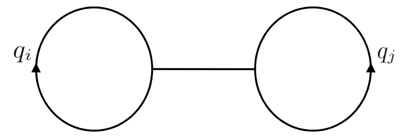

Figure 4.3: Tadpole diagrams. The solid lines can be bosonic or fermionic.

fields is described by the following parts,

$$\mathcal{L}_F^{(3)} = \mathrm{STr}\left[\mu\left(g_0^{-1}\Psi_L g_0\eta\Psi_R - \eta g_0^{-1}\Psi_L g_0\Psi_R\right)\right], \tag{4.39}$$

$$\mathcal{L}_F^{(4)} = \mathrm{STr}\left[\mu\left(\tfrac{1}{2}g_0^{-1}\Psi_L g_0\eta^2\Psi_R + \tfrac{1}{2}\eta^2 g_0^{-1}\Psi_L g_0\Psi_R - \eta g_0^{-1}\Psi_L g_0\eta\Psi_R\right)\right]. \tag{4.40}$$

Unlike the computation in the original string theory, quartic terms in fermions are absent in the approach based on the PW identity in the reduced theory. The contributions of the cubic and quartic interaction terms are given by the diagrams of the two topologies depicted in Figure 4.2: the fermionic sunset diagram in Figure 4.2(a) is the contribution of the cubic interaction (4.39) and the other diagram in Figure 4.2(b) is the fermionic double-bubble arising from the quartic interaction (4.40). These two-loop contributions are expressed in terms of the integrals (4.35) and (4.37).

Generally, individual diagram contributions are gauge-dependent. In the original string theory, non-1PI diagrams are not relevant in the two-loop computation in the conformal gauge [69, 70], but nonvanishing non-1PI contributions are found in the light-cone gauge [97, 100]. In the reduced theory we will obtain the nonvanishing non-1PI contributions. The non-1PI diagram is shown in Figure 4.3. The loops can be bosonic or fermionic, and the intermediate line connecting the two loops is bosonic. The two-loop integrals for the tadpole diagrams are summarized as a product of the integrals (4.35).

In Figure 4.3 the intermediate bosonic line connecting the two loops has zero momentum due to momentum conservation. As several physical components of the propagators (see section 4.3 and section 4.4) vanish by taking the zero-momentum limit, we will set the momentum of the intermediate line to be zero once the integration in the two loops is completed.

4.3 Reduced $AdS_3 \times S^3$ theory

In this section we shall discuss the reduced model of the $AdS_3 \times S^3$ GS string theory and compute the quantum corrections to the PRT partition function.

In the reduced theory, the number of physical degrees of freedom for bosons is the same as the dimension of the coset $(SU(1,1)\times SU(2))/[U(1)]^2$, i.e., $2+2$ degrees of freedom, whereas the fermionic fields contain $4+4$ real Grassmann components. In addition to these physical degrees of freedom, the reduced model involves unphysical degrees of freedom; $1+1$ bosons come from $\eta^{\perp}$ in the algebra $\mathfrak{h}$ of the subgroup $H = [U(1)]^2$ and $2+2$ bosons come from the gauge fields $A_{\pm}$. In section 4.3.1 we will discuss the computation of the quantum corrections in the deformed gWZW by using the PW identity such that the unphysical degrees of freedom in the gauge fields decouple from the physical fields (approach I).

For the reduced $AdS_3 \times S^3$ string theory, the gauge fields are easily integrated out at the classical level. It was shown in [66] that the resulting system is the sum of the complex sinh-Gordon model and the complex sin-Gordon model coupled with the fermionic part by employing a specific parameterization for the the coset element $g \in G = SU(1,1) \times SU(2)$. Thus the other approach to the two-loop computation in the reduced theory, addressed in section 4.3.2, is to consider the fluctuations in the generalized sin-Gordon system (approach II).

Although our final objective is to compute the quantum corrections for the folded string localized in AdS_3, the classical solution we will consider is the (S, J) folded string solution stretching in $AdS_3 \times S^1$ [89, 90, 69], and we will eventually take the limit where the string has no momentum in S^1. The reason for starting with the (S, J) folded string is that it is not well-understood how to embed the classical AdS_3 bosonic theory into the full $AdS_5 \times S^5$ reduction [IWA, 66]. In approach II, we will consider a nontrivial background in the S^3 sector in section 4.3.2 in order to realize the regularity in the perturbation.

4.3.1 Approach I: PW identity and gauge-fixing A

The first strategy is to compute the quantum corrections to the PRT partition function by using the PW identity. The fluctuation Lagrangian is shown in section 4.2.2.

We shall choose a specific parameterization of g in G corresponding to the vector gauging in the paper [64]. The AdS_3 part and S^3 part of g are in the fundamental representations of $SU(1,1)$ and $SU(2)$, respectively. Since the S^3 sector of the (S, J) folded string solution is the vacuum, the S^3 part is the identity matrix. Let us choose the basis in $\mathfrak{su}(1,1)$ by $\bar{R}_1 = \sigma_1$, $\bar{R}_2 = i\sigma_3$ and $\bar{R}_3 = \sigma_2$, where σ_i are the Pauli matrices. One can parameterize the coset element g in G in terms of the Euler angles (ϕ, χ), $\exp\left(\frac{1}{2}\chi\bar{R}_2\right)\exp\left(\phi\bar{R}_1\right)\exp\left(\frac{1}{2}\chi\bar{R}_2\right)$. Then any classical solution with its S^3 part in the vacuum state is written in the form,

$$g = \begin{pmatrix} g_A & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \qquad g_A = \begin{pmatrix} e^{i\chi}\cosh\phi & \sinh\phi \\ \sinh\phi & e^{-i\chi}\cosh\phi \end{pmatrix}. \tag{4.41}$$

With this parameterization the gauge equations derived by varying $A_\pm$ in the deformed gWZW Lagrangian (2.31) solved by

$$A_\pm = \begin{pmatrix} A_{\pm A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad A_{+A} = -\tfrac{i}{2}\partial_+\chi\sigma_3\,, \qquad A_{-A} = \tfrac{i}{2}\cosh^2\!\phi\,\partial_-\chi\sigma_3\,. \tag{4.42}$$

In fact, the parameterization of g in (4.41) is related to the coth model of the complex sinh-Gordon theory in the AdS_3 sector. One can confirm this by substituting (4.41) and (4.42) into the classical deformed gWZW Lagrangian (2.31). The fields in the complex sinh-Gordon theory, ϕ and χ, are written in terms of the embedding coordinates in AdS_3,

$$\begin{aligned} \partial_+ Y^P \partial_- Y_P &= -\mu^2\cosh 2\phi\,, \\ K_P\,\partial_\pm^2 Y^P &= 4\mu^3\cosh^2\!\phi\,\partial_\pm\chi\,, \end{aligned} \tag{4.43}$$

where $K_P \equiv \epsilon_{QRSP}Y^Q\partial_+Y^R\partial_-Y^S$. Substituting the (S, J) folded string solution in (4.7) into (4.43) we obtain the classical values of ϕ and χ. Then the classical solution in the deformed gWZW is written as

$$g_0 = \begin{pmatrix} g_{0A} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \qquad g_{0A} = \begin{pmatrix} \frac{\kappa}{\mu} v_\sigma^* & \frac{\ell}{\mu} \\ \frac{\ell}{\mu} & \frac{\kappa}{\mu} v_\sigma \end{pmatrix}, \tag{4.44}$$

$$v_\tau = e^{\frac{i\kappa^2 \tau}{\mu}}, \qquad v_\sigma = e^{\frac{i\ell^2 \sigma}{\mu}}, \tag{4.45}$$

with the classical gauge fields,

$$A_{0\pm} = \begin{pmatrix} A_{0\pm A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad A_{0\pm A} = \frac{i\kappa^2}{2\mu} \sigma_3 \,. \tag{4.46}$$

To decouple the gauge fields $A_\pm$ from the physical fields, we shall apply the PW identity described in section 4.2.1. First we find U and $\tilde{U}$ defined in (4.16),

$$U = \begin{pmatrix} u & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \qquad \tilde{U} = \begin{pmatrix} \tilde{u} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \qquad u = \tilde{u} = \begin{pmatrix} v_\tau^{*1/2} & 0 \\ 0 & v_\tau^{1/2} \end{pmatrix}, \tag{4.47}$$

then the new classical solution takes the form,

$$\tilde{g}_0 = \begin{pmatrix} \tilde{g}_{0A} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \qquad \tilde{g}_{0A} = \begin{pmatrix} \frac{\kappa}{\mu} v_\sigma^* & \frac{\ell}{\mu} v_\tau \\ \frac{\ell}{\mu} v_\tau^* & \frac{\kappa}{\mu} v_\sigma \end{pmatrix}. \tag{4.48}$$

Here we used v_τ and v_σ defined in (4.45).

In the remaining part, we will study the quantum fluctuations around the classical solution (4.48) with the fluctuation Lagrangians in (4.23), (4.24) and (4.25).

One-loop computation

Let us first focus on the bosonic sector (4.28). We will express the Lagrangian for the bosonic fluctuations in terms of component fields, derive their characteristic frequencies, and compute the functional determinant contribution to the partition function.

Following the parametrization described in appendix A, we introduce the bosonic fluctuation fields,

$$\eta^{\|} = \begin{pmatrix} \eta_A^{\|} & 0 \\ 0 & \eta_S^{\|} \end{pmatrix},$$
$$\eta_A^{\|} = \begin{pmatrix} 0 & w(a_1+ia_2) \\ w^*(a_1-ia_2) & 0 \end{pmatrix}, \qquad \eta_S^{\|} = \begin{pmatrix} 0 & b_1+ib_2 \\ b_1-ib_2 & 0 \end{pmatrix}, \tag{4.49}$$

which correspond to physical fields in $\mathfrak{m}$ coming from the coset part G/H, and

$$\eta^{\perp} = \begin{pmatrix} \eta_A^{\perp} & 0 \\ 0 & \eta_S^{\perp} \end{pmatrix},$$
$$\eta_A^{\perp} = \begin{pmatrix} ic & 0 \\ 0 & -ic \end{pmatrix}, \qquad \eta_S^{\perp} = \begin{pmatrix} id & 0 \\ 0 & -id \end{pmatrix}, \tag{4.50}$$

which are unphysical fields in $\mathfrak{h}$ of the group H. In (4.49), the rescaling factor w for the fluctuations a_1 and a_2 is

$$w = v_\tau v_\sigma = e^{i\frac{\kappa^2\tau+\ell^2\sigma}{\mu}}\,, \tag{4.51}$$

which are introduced such that the resulting fluctuation Lagrangian involves only constant coefficients. Then the derivation of the Lagrangian for quadratic fluctuations is straightforward; plugging (4.49) and (4.50) into the bosonic part of the fluctuation Lagrangian (4.28) yields

$$\mathcal{L}^{(2)} = \mathcal{L}^{(2)}_{AdS_3} + \mathcal{L}^{(2)}_{S^3}\,, \tag{4.52}$$

where the AdS_3 sector is described by

$$\mathcal{L}^{(2)}_{AdS_3} = \sum_{i=1,2} \partial_+ a_i \partial_- a_i + 2\left(\mu\partial_+ a_2 + M_2 \partial_- a_2\right) a_1 - \partial_+ c\, \partial_- c - 4M_1 \partial_- c\, a_1\,, \tag{4.53}$$

with the constants M_1, M_2 defined by

$$M_1 = \frac{\kappa\sqrt{\kappa^2-\mu^2}}{\mu^2}\,, \qquad M_2 = \frac{2\kappa^2-\mu^2}{\mu}\,, \tag{4.54}$$

and the S^3 part is

$$\mathcal{L}^{(2)}_{S^3} = \sum_{i=1,2} \left(\partial_+ b_i \partial_- b_i - \mu^2 b_i^2\right) + \partial_+ d \partial_- d \,. \tag{4.55}$$

One can confirm the agreement with the string theory result by deriving characteristic frequencies of the physical fluctuations from the Lagrangians (4.53), (4.55). The Lagrangian (4.53) describes one unphysical massless fluctuation c and two physical fluctuations with frequencies,

$$\sqrt{n^2 + 2\kappa^2 \pm 2\sqrt{\kappa^4 + n^2\mu^2}}\,, \tag{4.56}$$

which describe the part of the fluctuations found in the original string theory [90], while the other Lagrangian (4.55) contains one unphysical massless field d and two physical fields whose characteristic frequencies are

$$\sqrt{n^2 + \mu^2}\,. \tag{4.57}$$

They are exactly a part of the frequencies in the S^5 sector in the original string theory [90].

From the Lagrangians (4.53) and (4.55) we find that the functional determinant is different from the string theory result by the massless field contributions,

$$\Big([\det(\partial_+\partial_-)]^2 \left[\det\left(\partial_+\partial_- + \mu^2\right)\right]^2 \det\left(\partial_+^2\partial_-^2 + 2\partial_+\partial_-\left(2\kappa^2 - \mu^2\right) + \left(\partial_+^2 + \partial_-^2\right)\mu^2\right) \Big)^{-1/2} . \tag{4.58}$$

One can show that the redundant factor $\det\left(\partial_+^2\partial_-^2\right)$ is eliminated by considering the two more contributions; the first contribution comes from the Jacobian for the field redefinition (4.16), and quantum fluctuations found in WZW term in (4.21) gives the other contribution. As a result, the functional determinant in the bosonic sector is

$$Z^{(1B)} = \Big(\left[\det\left(\partial_+\partial_- + \mu^2\right)\right]^2 \det\left(\partial_+^2\partial_-^2 + 2\partial_+\partial_-\left(2\kappa^2 - \mu^2\right) + \left(\partial_+^2 + \partial_-^2\right)\mu^2\right) \Big)^{-1/2} . \tag{4.59}$$

Next we will discuss the fermionic sector. To derive the Lagrangian for quadratic fluctuations in the fermionic sector we introduce the following component fields,

$$\Psi_R = \begin{pmatrix} 0 & \mathfrak{X}_R \\ \mathfrak{Y}_R & 0 \end{pmatrix}, \quad \Psi_L = \begin{pmatrix} 0 & \mathfrak{X}_L \\ \mathfrak{Y}_L & 0 \end{pmatrix}, \tag{4.60}$$

where

$$\mathfrak{X}_R = \begin{pmatrix} 0 & (\alpha_1 + i\alpha_2)t_{1+} \\ (\alpha_3 + i\alpha_4)t^*_{2+} & 0 \end{pmatrix}, \tag{4.61}$$

$$\mathfrak{Y}_R = \begin{pmatrix} 0 & (-i\alpha_3 - \alpha_4)t_{2+} \\ (i\alpha_1 + \alpha_2)t^*_{1+} & 0 \end{pmatrix}, \tag{4.62}$$

and

$$\mathfrak{X}_L = \begin{pmatrix} 0 & (\beta_1 + i\beta_2)t_{1-} \\ (\beta_3 + i\beta_4)t^*_{2-} & 0 \end{pmatrix}, \tag{4.63}$$

$$\mathfrak{Y}_L = \begin{pmatrix} 0 & (-i\beta_3 - \beta_4)t_{2-} \\ (i\beta_1 + \beta_2)t^*_{1-} & 0 \end{pmatrix}, \tag{4.64}$$

where all of the component fields are real Grassmann. The rescaling factors $t_{1\pm}$, $t_{2\pm}$ are defined by

$$t_{1\pm} = e^{i\frac{\ell^2(\tau\pm\sigma)}{2\mu}}, \qquad t_{2\pm} = e^{i\frac{\left(\kappa^2+\mu^2\right)\tau\pm\ell^2\sigma}{2\mu}}. \tag{4.65}$$

These exponential factors will be reused for the fermionic fluctuations in the $AdS_5 \times S^5$ case in order to realize the constant-coefficient Lagrangian. By the substitution of the matrices (4.60)-(4.64) into the Lagrangian (4.38) we obtain

$$\mathcal{L}_{\mathrm{F}} = \sum_{i=1}^{4} (\alpha_i\partial_-\alpha_i + \beta_i\partial_+\beta_i) + 2\mu\,(\alpha_3\alpha_4 + \beta_3\beta_4) + 2\kappa\,(\alpha_1\beta_2 - \alpha_2\beta_1 - \alpha_3\beta_4 + \alpha_4\beta_3)\,. \tag{4.66}$$

Characteristic frequencies of the fermionic fluctuations are

$$\begin{aligned} 2 \quad &\times \quad \sqrt{n^2+\kappa^2}\,, \\ 1 \quad &\times \quad \sqrt{n^2+\kappa^2}+\mu\,, \\ 1 \quad &\times \quad \sqrt{n^2+\kappa^2}-\mu\,. \end{aligned} \tag{4.67}$$

They are consistent with the result of the original $AdS_3 \times S^3$ string theory in [IRT] because the shifts μ in the frequencies can be removed by rescaling the fermionic fluctuations. From the Lagrangian

(4.66), we find that the functional determinant in the fermionic sector is

$$Z^{(1F)} = \left[\det\left(\partial_+\partial_- + \kappa^2\right)\right]^2 \det\left(\partial_+^2\partial_-^2 + 2\partial_+\partial_-\kappa^2 + \left(\partial_+^2 + \partial_-^2\right)\mu^2 + \left(\kappa^2 - \mu^2\right)^2\right) . \tag{4.68}$$

Here the determinant of the 4-th order operator can be factorized as follows:

$$\begin{aligned} &\det\Big(\left[\partial_+\partial_- + i\mu\left(\partial_+ + \partial_-\right) - \mu^2 + \kappa^2\right]\left[\partial_+\partial_- - i\mu\left(\partial_+ + \partial_-\right) - \mu^2 + \kappa^2\right]\Big) \\ &= \det\left[e^{-i\mu\tau}\left(\partial_+\partial_- + \kappa^2\right)e^{i\mu\tau}\right]\ \det\left[e^{i\mu\tau}\left(\partial_+\partial_- + \kappa^2\right)e^{-i\mu\tau}\right] . \end{aligned} \tag{4.69}$$

and thus the fermionic one-loop contribution is equivalent.

As described in section 4.2.2 the one-loop partition function is computed by using the Euclidean signature on the worldsheet and rescaling the worldsheet coordinate with the limit (4.27). Taking the sum of the bosonic sector (4.59) and fermionic sector (4.68) and performing the momentum integral gives the one-loop correction,

$$\begin{aligned} \Gamma^{(1)} &= \frac{1}{2}\bar{V}_2 \int \frac{d^2q}{(2\pi)^2}\left[\ln(q^2 + 4\kappa^2) + 3\ln q^2 - 4\ln(q^2 + \kappa^2)\right] \\ &= 2\kappa^2\bar{V}_2\big(I[4] - I[1]\big) = \frac{1}{2\pi}(-2\ln 2)V_2 , \end{aligned} \tag{4.70}$$

which is exactly the same as the result in the $AdS_3 \times S^3$ GS string theory [IRT].

Finally let us comment on the $\mu \to 0$ limit when the classical folded string is localized in AdS_3. In the last chapter it was demonstrated that the limit is well-defined in the fluctuation Lagrangian. However, the limit can not be taken at the level of the fluctuation Lagrangian here because the constants M_1, M_2 in (4.54) diverge in this limit. We can understand that this discrepancy is caused by the difference in the H gauge choice. Before the H gauge is fixed in a very specific way such that the unphysical degrees of freedom decouple from the physical ones. On the other hand, the physical and unphysical fields are coupled in our approach based on the PW identity.

Two-loop computation

Here we shall discuss the two-loop computation in the reduced theory using the fluctuation Lagrangian, (4.23), (4.24) and (4.25).

As seen in the one-loop computation, the bosonic fluctuations involve the $1+1$ unphysical fluctuations in addition to the $2+2$ physical fluctuations, and they couple with each other. We will consider all of the fluctuations and compute the diagrams for the $3+3$ bosonic fields, rather than to block-diagonalize the system into the physical part and the unphysical part.

Throughout the two-loop computation, we will use the Euclidean signature on the worldsheet. To compute the two-loop diagrams we shall derive the bosonic propagator. Once we denote the bosonic fluctuation fields as

$$\Phi_I = \{\Phi_{A\,i}, \Phi_{S\,j}\}\,, \tag{4.71}$$

where we reordered the bosonic fields in the AdS_3 sector in $\Phi_{A\,i}$

$$\Phi_{A\,i} = \{a_1, a_2, c\}\,, \qquad \Phi_{S\,i} = \{b_1, b_2, d\}\,, \tag{4.72}$$

the bosonic propagator for the AdS_3 sector is written as

$$\triangle_A^{-1}(q) = \frac{1}{D_2}\begin{pmatrix} -\frac{q^2}{2} & -\frac{\hat\kappa^2 q_- + iq^1\hat\mu^2}{\hat\mu} & -\frac{\hat\kappa q_-\sqrt{\hat\kappa^2-\hat\mu^2}}{\hat\mu} \\ \frac{\hat\kappa^2 q_- + iq^1\hat\mu^2}{\hat\mu} & \frac{q_+\left(q_+^2\hat\mu^2 - 4\hat\kappa^2\left(\hat\kappa^2-\hat\mu^2\right)\right)}{2q_-\hat\mu^2} & \frac{2\hat\kappa\sqrt{\hat\kappa^2-\mu^2}\left(\hat\kappa^2 q_- + iq_1\hat\mu^2\right)}{q_+\hat\mu^2} \\ \frac{\hat\kappa q_-\sqrt{\hat\kappa^2-\hat\mu^2}}{\hat\mu} & \frac{2\hat\kappa\sqrt{\hat\kappa^2-\hat\mu^2}\left(\hat\kappa^2 q_- + iq_1\hat\mu^2\right)}{q_+\hat\mu^2} & \frac{q^4\hat\mu^2 - 4\left(q^1\hat\mu^2 - i\hat\kappa^2 q_-\right)^2}{2q^2\hat\mu^2} \end{pmatrix}, \tag{4.73}$$

$$q_\pm = q_0 \pm iq_1\,, \qquad q^2 = q_0^2 + q_1^2\,, \qquad D_2 = q^4 + 4\hat\kappa^2 q^2 - 4\hat\mu^2 {q_1}^2\,,$$

and the propagator for the S^3 sector is

$$\triangle_S^{-1}(q) = \mathrm{diag}\left(-\frac{1}{2\left(q^2+\hat\mu^2\right)}, -\frac{1}{2\left(q^2+\hat\mu^2\right)}, -\frac{1}{2q^2}\right)\,, \tag{4.74}$$

where we omitted the overall factor $\frac{8\pi}{k}$ which will be restored later.

Using the bosonic propagator and fermionic propagator following from (4.66), the one-loop effective action is evaluated by $\frac{1}{2}\mathrm{Tr}\ln\triangle$, which indeed agrees with our previous computation (4.70). At the level of the propagator one can not take the $\hat\mu \to 0$ ($\mu \to 0$) limit because several components of the propagator diverge in this limit. The $\hat\mu \to 0$ can be smoothly taken after simplifying the integrands of the two-loop integrals, where the physical mass spectrum includes one bosonic mode $m^2 = 4$, four fermionic modes $m^2 = 1$ and three massless modes. The two-loop integrals for the sunset diagrams in

Figure 4.1(a) and Figure 4.2(a) can be expressed as

$$\mathcal{I}_{m_i^2 m_j^2 m_k^2} = \int \frac{d^2q_i\, d^2q_j\, d^2q_j}{(2\pi)^4} \frac{\mathcal{F}(q_i, q_j, q_k)}{q_i^{n_i} q_j^{n_j} q_k^{n_k} (q_i^2 + m_i^2)(q_j^2 + m_j^2)(q_k^3 + m_k^2)}\,, \quad m_i, m_j, m_k = 0, 1 \text{ or } 4\,, \tag{4.75}$$

and for the double-bubble diagrams in Figure 4.1(b) and Figure 4.2(b) the integrals take the form,

$$\mathcal{I}_{m_i^2 m_j^2} = \int \frac{d^2q_i\, d^2q_j}{(2\pi)^4} \frac{\mathcal{F}(q_i, q_j)}{q_i^{n_i} q_j^{n_j} (q_i^2 + m_i^2)(q_j^2 + m_j^2)}\,, \quad m_i, m_j = 0,\ 1 \text{ or } 4\,, \tag{4.76}$$

where $\mathcal{F}(q_i, q_j, q_k)$, $\mathcal{F}(q_i, q_j)$ are certain polynomial functions of q. The Catalan's constant is not obtained in this system due to absence of the physical modes with $m^2 = 2$, and the integrals (4.75) and (4.76) are written in terms of $I[m^2]$ in (4.35). The bosonic sunset diagrams and the bosonic double-bubble diagrams give the following contributions,

$$\begin{aligned} J_{\text{boson sunset}} &= -\frac{1}{12}(6I[4]I[0] + 6I[4]I[4])\,, \\ J_{\text{boson double-bubble}} &= \frac{1}{8}(-4I[4]I[0] - 4I[4]I[4])\,, \end{aligned} \tag{4.77}$$

whereas the contributions of the fermionic sunset and the fermionic double-bubble are

$$\begin{aligned} J_{\text{fermion-boson sunset}} &= \frac{1}{4}(12I[1]I[0] + 4I[4]I[1] - 8I[1]I[1])\,, \\ J_{\text{fermion-boson double-bubble}} &= -\frac{1}{4}(8I[1]I[0] - 8I[4]I[1])\,. \end{aligned} \tag{4.78}$$

In addition to the 1PI diagram contributions above, it is necessary to consider the non-1PI diagram contributions in Figure 4.3,

$$\begin{aligned} J_{\text{boson-boson tadpole}} &= -\frac{1}{8}(-\frac{8}{3}I[4]I[0] - \frac{16}{3}I[4]I[4])\,, \\ J_{\text{boson-fermion tadpole}} &= \frac{1}{8}(-\frac{8}{3}I[1]I[0] - \frac{40}{3}I[4]I[1])\,, \\ J_{\text{fermion-fermion tadpole}} &= -\frac{1}{8}(-8I[1]I[1])\,. \end{aligned} \tag{4.79}$$

Inserting the overall constant in front of the Lagrangian, the two-loop effective action is

$$\Gamma_2^{(3)} = \frac{8\pi}{k_3} V_2 \sum J_n\,, \tag{4.80}$$

where J_n are the contributions of the different types of diagrams given in (4.77), (4.78) and (4.79). For the latter purpose, we will take the sums of the 1PI and non-1PI (tadpole) contributions separately,

$$J_{\rm 1PI} = -\frac{1}{2}\big(I[4] - I[1]\big)\big(I[4] + I[0] - 2I[1]\big)\ , \tag{4.81}$$

$$J_{\rm tadpole} = \frac{1}{6}\big(I[4] - I[1]\big)\big(2I[4] + I[0] - 3I[1]\big)\ . \tag{4.82}$$

These are separately UV finite but IR divergent due to the presence of $I[0]$. Notice also that both expressions are proportional to the coefficient $I[4] - I[1]$ appearing in the one-loop result (4.70).

The total coefficient is then (using (4.36))

$$\begin{aligned}\sum_n J_n &= -\frac{1}{6}\big(I[4] - I[1]\big)\big(I[4] + 2I[0] - 3I[1]\big) \\ &= -\frac{1}{24\pi^2}\left(\ln 2\right)^2 - \frac{1}{12\pi^2}\ln 2 \ln m_0\ . \end{aligned} \tag{4.83}$$

This expression is still IR divergent: we introduced an IR cutoff $m_0 \to 0$ to rewrite $I[0]$.

It is contrary to our expectation that the finite term $(\ln 2)^2$ is obtained in the reduced theory while the two-loop contributions vanish in the original string theory [IRT]. We believe that the appearance of the IR divergent term should be an artifact of our computation. The unphysical fluctuations are included in the present approach, so one may be able to interpret the IR divergence is the contribution of the massless unphysical fields. Also, another possible explanation is that the IR divergence would be an artifact due to the $\mu \to 0$ limit. This limit is well-defined at the level of integrands of the two-loop integrals although it is not at the level of the fluctuation Lagrangian.

To support this statement, in the next subsection we shall repeat the above two-loop computation using a different approach: we fix the gauge on g and integrate out $A_\pm$ so that all unphysical degrees of freedom are explicitly eliminated from the Lagrangian.

4.3.2 Approach II: Integrating out gauge fields

In this section we will discuss the computation for the one-loop and two-loop partition functions in the reduced theory of the $AdS_3 \times S^3$ string theory in the approach where only the physical degrees of freedom are present in the resulting system.

Imposing a gauge on $g \in G$ and integrating out the gauge fields $A_\pm$ gives the classical action for physical degrees of freedom only. The bosonic part of the resulting model derived in [66] is the sum of the complex sinh-Gordon model and the complex sin-Gordon model, and it couples with two-dimensional fermions. Depending on the gauge choice on g, distinct models are obtained. Here we will consider two models. The first model is the "tanh-tan" model, the product of the tanh model of the complex sinh-Gordon theory and the tan model of the complex sin-Gordon theory coupling with two-dimensional fermionic fields (t-t),

$$\begin{aligned}\mathcal{L}_{\mathrm{t-t}} = {} & \partial_+\varphi\partial_-\varphi + \tan^2\varphi\partial_+\theta\partial_-\theta + \partial_+\phi\partial_-\phi + \tanh^2\phi\partial_+\chi\partial_-\chi + \tfrac{\mu^2}{2}\left(\cos 2\varphi - \cosh 2\phi\right) \\ & +\alpha\partial_-\alpha + \beta\partial_-\beta + \gamma\partial_-\gamma + \zeta\partial_-\zeta + \lambda\partial_+\lambda + \xi\partial_+\xi + \rho\partial_+\rho + \sigma\partial_+\sigma \\ & + \tan^2\varphi\left[\partial_+\theta\left(\lambda\xi - \rho\sigma\right) - \partial_-\theta\left(\alpha\beta - \gamma\zeta\right)\right] - \tanh^2\phi\left[\partial_+\chi\left(\lambda\xi - \rho\sigma\right) - \partial_-\chi\left(\alpha\beta - \gamma\zeta\right)\right] \\ & - \left(\alpha\beta - \gamma\zeta\right)\left(\lambda\xi - \rho\sigma\right)\left(\tfrac{1}{\cos^2\varphi} - \tfrac{1}{\cosh^2\phi}\right) - 2\mu\Big(\cosh\phi\cos\varphi\left(\lambda\gamma + \xi\zeta - \rho\alpha - \sigma\beta\right) \\ & + \sinh\phi\sin\varphi\big[\cos(\chi+\theta)\left(-\rho\zeta + \sigma\gamma + \lambda\beta - \xi\alpha\right) - \sin(\chi+\theta)\left(\lambda\alpha + \xi\beta + \rho\gamma + \sigma\zeta\right)\big]\Big)\,.\end{aligned} \tag{4.84}$$

The other model is the "coth-cot" model whose bosonic sector is the combination of the coth model of the complex sinh-Gordon theory and the cot model of the complex sin-Gordon theory (c-c),

$$\begin{aligned}\mathcal{L}_{\mathrm{c-c}} = {} & \partial_+\varphi\partial_-\varphi + \cot^2\varphi\partial_+\theta\partial_-\theta + \partial_+\phi\partial_-\phi + \coth^2\phi\partial_+\chi\partial_-\chi + \tfrac{\mu^2}{2}\left(\cos 2\varphi - \cosh 2\phi\right) \\ & +\alpha\partial_-\alpha + \beta\partial_-\beta + \gamma\partial_-\gamma + \zeta\partial_-\zeta + \lambda\partial_+\lambda + \xi\partial_+\xi + \rho\partial_+\rho + \sigma\partial_+\sigma \\ & - \cot^2\varphi\left[\partial_+\theta\left(\lambda\xi - \rho\sigma\right) - \partial_-\theta\left(\alpha\beta - \gamma\zeta\right)\right] + \coth^2\phi\left[\partial_+\chi\left(\lambda\xi - \rho\sigma\right) - \partial_-\chi\left(\alpha\beta - \gamma\zeta\right)\right] \\ & - \left(\alpha\beta - \gamma\zeta\right)\left(\lambda\xi - \rho\sigma\right)\left(\tfrac{1}{\sin^2\varphi} + \tfrac{1}{\sinh^2\phi}\right) - 2\mu\Big(\sinh\phi\sin\varphi\left(\lambda\gamma + \xi\zeta - \rho\alpha - \sigma\beta\right) \\ & + \cosh\phi\cos\varphi\big[\cos(\chi+\theta)\left(\rho\zeta - \sigma\gamma - \lambda\beta + \xi\alpha\right) - \sin(\chi+\theta)\left(\lambda\alpha + \xi\beta + \rho\gamma + \sigma\zeta\right)\big]\Big)\,.\end{aligned} \tag{4.85}$$

In these Lagrangians, ϕ, θ describe bosonic degrees of freedom in the AdS_3 sector, φ, χ do bosonic degrees of freedom in the S^3 sector, and α, β, γ, ζ, λ, ξ, ρ, σ are real fermionic fields. The mass scale μ is determined by the stress tensor in the original string theory,

$$-T^{AdS}_{\pm\pm} = T^{S}_{\pm\pm} = \mu^2\,. \tag{4.86}$$

Since the Lagrangians (4.84) and (4.85) exhibit that the point $\varphi, \theta = 0$ where the S^3 sector lies in the vacuum is special and small expansions around this point are not well-defined, we will start with a nontrivial background stretching in $AdS_3 \times S^3$, and take the limit eventually so that the classical solution becomes localized in AdS_3. The solution we will consider is a superposition of the the long spinning string in AdS_3 [69, 89, 90] and the circular two-spin string in S^3 [92, 93, 101, 102]

$$\begin{aligned} Y_0 + iY_{-1} &= \cosh(\ell\sigma)\, e^{i\kappa\tau}\,, \qquad Y_1 + iY_2 = \sinh(\ell\sigma)\, e^{i\kappa\tau}\,, \\ X_1 + iX_2 &= \tfrac{1}{\sqrt{2}} e^{i\omega\tau + i\mathrm{k}\sigma}\,, \qquad X_3 + iX_4 = \tfrac{1}{\sqrt{2}} e^{i\omega\tau - i\mathrm{k}\sigma}\,, \\ \kappa^2 &= \ell^2 + \mu^2\,, \qquad \mu^2 = \mathrm{k}^2 + \omega^2\,. \end{aligned} \tag{4.87}$$

The string solution localized in AdS_3 is obtained by taking the limit $\omega, \mu \to 0$.

The explicit relations between the embedding coordinates and the bosonic fields in the reduced theory are given in (2.3) for the t-t model,

$$\begin{aligned} \partial_+ Y^P \partial_- Y_P &= -\mu^2 \cosh 2\phi\,, \\ K_P\, \partial_\pm^2 Y^P &= \mp 4\mu^3 \sinh^2\phi\, \partial_\pm \chi\,. \end{aligned} \tag{4.88}$$

and in (2.6) for the c-c model,

$$\begin{aligned} \partial_+ Y^P \partial_- Y_P &= -\mu^2 \cosh 2\phi\,, \\ K_P\, \partial_\pm^2 Y^P &= 4\mu^3 \cosh^2\phi\, \partial_\pm \chi\,, \end{aligned} \tag{4.89}$$

where $K_P \equiv \epsilon_{QRSP} Y^Q \partial_+ Y^R \partial_- Y^S$. The relations for the S^3 part are given by replacing the hyperbolic functions by the trigonometric functions, i.e., $\cosh\phi \to \cos\varphi$ and $\sinh\phi \to \sin\varphi$, and also, $\chi \to \theta$.

These relations allow us to reduce the classical solution (4.87) into the following solution for the t-t model,

$$\begin{aligned} \phi_0 &= \log\left(\tfrac{\kappa + \sqrt{\kappa^2 - \mu^2}}{\mu}\right)\,, \qquad \chi_0 = \tfrac{\kappa^2}{\mu}\tau\,, \\ \varphi_0 &= \tfrac{1}{2}\arccos\left(\tfrac{2\omega^2}{\mu^2} - 1\right)\,, \qquad \theta_0 = \tfrac{\omega^2}{\mu}\tau\,, \end{aligned} \tag{4.90}$$

and for the c-c model,

$$\begin{aligned} \phi_0 &= \log\left(\tfrac{\kappa + \sqrt{\kappa^2 - \mu^2}}{\mu}\right)\,, \qquad \chi_0 = \tfrac{\mu^2 - \kappa^2}{\mu}\sigma\,, \\ \varphi_0 &= \tfrac{1}{2}\arccos\left(\tfrac{2\omega^2}{\mu^2} - 1\right)\,, \qquad \theta_0 = \tfrac{\omega^2 - \mu^2}{\mu}\sigma\,. \end{aligned} \tag{4.91}$$

Below we will discuss expansions around these classical solutions in the t-t model and c-c model, respectively.

One-loop computation

We shall expand the reduced theory Lagrangians (4.84), (4.85) near the classical solutions (4.90), (4.91),

$$\begin{aligned} \phi &= \phi_0 + \delta\phi\,, \qquad \chi = \chi_0 + \delta\chi\,, \\ \varphi &= \varphi_0 + \delta\phi\,, \qquad \theta = \theta_0 + \delta\theta\,. \end{aligned} \tag{4.92}$$

This leads to the Lagrangian for quadratic fluctuations in the t-t model,

$$\begin{aligned} \mathcal{L}^B_{\rm t-t} &= \partial_-\delta\phi\partial_+\delta\phi + \partial_-\delta\varphi\partial_+\delta\varphi - 4\left(\kappa^2-\mu^2\right)\delta\phi^2 - 4\left(\omega^2-\mu^2\right)\delta\varphi^2 + \frac{\kappa^2-\mu^2}{\kappa^2}\partial_-\delta\chi\partial_+\delta\chi \\ &+ \frac{2\mu\sqrt{\kappa^2-\mu^2}}{\kappa}\delta\phi\left(\partial_-\delta\chi+\partial_+\delta\chi\right) + \frac{\mu^2-\omega^2}{\omega^2}\partial_-\delta\theta\partial_+\delta\theta + \frac{2\mu\sqrt{\mu^2-\omega^2}}{\omega}\delta\varphi\left(\partial_-\delta\theta+\partial_+\delta\theta\right)\,, \end{aligned} \tag{4.93}$$

and in the c-c model,

$$\begin{aligned} \mathcal{L}^B_{\rm c-c} &= \partial_-\delta\phi\partial_+\delta\phi + \partial_-\delta\varphi\partial_+\delta\varphi - 4\kappa^2\delta\phi^2 - 4\omega^2\delta\varphi^2 + \frac{\kappa^2}{\kappa^2-\mu^2}\partial_-\delta\chi\partial_+\delta\chi \\ &+ \frac{2\kappa\mu}{\sqrt{\kappa^2-\mu^2}}\delta\phi\left(\partial_-\delta\chi-\partial_+\delta\chi\right) + \frac{\omega^2}{\mu^2-\omega^2}\partial_-\delta\theta\partial_+\delta\theta + \frac{2\mu\omega}{\sqrt{\mu^2-\omega^2}}\delta\varphi\left(\partial_-\delta\theta-\partial_+\delta\theta\right)\,. \end{aligned} \tag{4.94}$$

One can confirm that these Lagrangians describe the same set of fluctuations by computing the corresponding functional determinant contributions to the partition function,

$$\begin{aligned} Z^{(1B)}_{\rm t-t} = Z^{(1B)}_{\rm c-c} = \Big(&\det\left(\partial_+^2\partial_-^2 + 2(2\kappa^2-\mu^2)\partial_+\partial_- + \mu^2\partial_+^2 + \mu^2\partial_-^2\right) \\ &\times\det\left(\left(\partial_+^2\partial_-^2 + 2(2\omega^2-\mu^2)\partial_+\partial_- + \mu^2\partial_+^2 + \mu^2\partial_-^2\right)\Big)^{-1/2}\,, \end{aligned} \tag{4.95}$$

which read characteristic frequencies of the four bosonic fluctuations,

$$\begin{aligned} &\sqrt{n^2+2\kappa^2\pm 2\sqrt{\kappa^4+n^2\mu^2}}\,, \\ &\sqrt{n^2+2\omega^2\pm 2\sqrt{n^2\mu^2+\omega^4}}\,. \end{aligned} \tag{4.96}$$

These are exactly the same as two of the four bosonic frequencies in AdS_5 and two of the four bosonic frequencies in S^5 [90, 92, 103]. In the $\mu,\omega\to 0$ limit where the background becomes the folded string in AdS_3, the functional determinant (4.95) reduces into

$$Z^{(1B)} = \left[\det\left(\partial_+\partial_-\right)\right]^{-3/2}\left[\det(\partial_+\partial_- + 4\kappa^2)\right]^{-1/2}\,. \tag{4.97}$$

Let us discuss the fermionic sector. It is observed in the Lagrangians (4.84), (4.85) that the coupling terms of χ, θ and the fermionic fields involve $\cos(\chi+\theta)$ and $\sin(\chi+\theta)$, which give nonconstant coefficients to the fluctuation Lagrangian for the fermionic fluctuations. Thus we rotate the fermionic fluctuations in the following way,

$$\begin{aligned} \alpha + i\beta &\to (\alpha+i\beta)e^{\mathcal{B}}\,, \qquad \gamma + i\zeta \to (\gamma+i\zeta)e^{\mathcal{B}^*}\,, \\ \lambda + i\xi &\to (\lambda+i\xi)e^{\mathcal{B}^*}\,, \qquad \rho + i\sigma \to (\rho+i\sigma)e^{\mathcal{B}}\,, \end{aligned} \tag{4.98}$$

where $\mathcal{B} = i\frac{\kappa^2+\omega^2}{2\mu}\tau$ for the t-t model and $\mathcal{B} = i\frac{\kappa^2-\omega^2}{2\mu}\sigma$ for the c-c model. The functional determinants for the t-t model and the c-c model are

$$Z^{(1F)}_{\mathrm{t-t}} = \left[\det\left(\partial_+^2\partial_-^2 + \left(\partial_+^2+\partial_-^2\right)\mu^2 + 2\partial_+\partial_-\left(\kappa^2-\mu^2+\omega^2\right) + \left(\kappa^2-\omega^2\right)^2\right)\right]^2\,, \tag{4.99}$$

$$Z^{(1F)}_{\mathrm{c-c}} = \left[\det\left(\partial_+\partial_- + \kappa^2 - \mu^2 + \omega^2\right)\right]^4\,. \tag{4.100}$$

Although they look different, one can check the equivalence of these expressions by applying a two-dimensional Lorentz transformation, i.e., $SO(1,1)$ transformation to the worldsheet coordinates (τ, σ). In the $\mu, \omega \to 0$ limit in which the classical string is localized in AdS_3, (4.99) and (4.100) coincide,

$$Z^{(1F)} = \left[\det\left(\partial_+\partial_- + \kappa^2\right)\right]^4\,, \tag{4.101}$$

which agrees with the string theory computation [89, 90].

Combining the bosonic contributions (4.97) and the fermionic contributions (4.101), one finds that the one-loop partition function is the same as the string theory computation. Hence approach I in the previous subsection and approach II here provide the same one-loop result.

Two-loop computation

In approach I the one-particle irreducible contributions are given by the diagrams depicted in Figure 4.1 and Figure 4.2. As we integrated out the gauge fields in approach II, the resulting fluctuation Lagrangian involves quartic terms in fermions (see (4.84) and (4.85)). Thus we have to compute an additional diagram depicted in Figure 4.4.

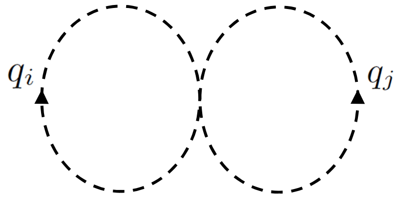

Figure 4.4: Fermionic two-loop diagram with two fermionic loops.

In the two-loop computation we should note the following two points. Recalling that the Lagrangians (4.84) and (4.85) are obtained by integrating out the gauge fields at the level of the classical deformed gWZW Lagrangian, the nontrivial quantum contribution of the counterterm is in general relevant at two-loop level as pointed out in [73, 74, 75]. However, one can check all of the contribution from the counter term is power-divergent, then they do not contribute to the final result. The other point is on the $\mu, \omega \to 0$ limit. While it is expected that the final result is irrespective of the order of the limits $\mu \to 0$ and $\omega \to 0$, individual diagram contributions may have the ambiguity. To show this explicitly we shall introduce a parameter $r = \omega/\mu$.

The contributions of individual diagrams in the t-t model are

$$\begin{aligned}
J_{\text{boson double-bubble}} &= \tfrac{1}{8}\left(-8I[4]I[4]\right) , \\
J_{\text{fermion-boson sunset}} &= \tfrac{1}{4}\left(8I[1]I[0] + 4I[4]I[1] - 2\tfrac{1+2r^2}{r^2}I[1]I[1]\right) , \\
J_{\text{fermion-boson double-bubble}} &= -\tfrac{1}{4}\left(8I[1]I[0] - 4I[4]I[1]\right) , \\
J_{\text{fermion-fermion double-bubble}} &= \tfrac{1}{8}\left(4\tfrac{1-2r^2}{r^2}I[1]I[1]\right) , \\
J_{\text{tadpole}} &= -\tfrac{1}{8}\left(-8I[1]I[1]\right) ,
\end{aligned} \tag{4.102}$$

and in the c-c model we obtain

$$\begin{aligned}
J_{\text{boson double-bubble}} &= \tfrac{1}{8}\left(-8I[4]I[4]\right) , \\
J_{\text{fermion-boson sunset}} &= \tfrac{1}{4}\left(8I[1]I[0] + 4I[4]I[1] - \tfrac{6-4r^2}{1-r^2}I[1]I[1]\right) , \\
J_{\text{fermion-boson double-bubble}} &= -\tfrac{1}{4}\left(8I[1]I[0] - 4I[4]I[1]\right) , \\
J_{\text{fermion-fermion double-bubble}} &= \tfrac{1}{8}\left(-4\tfrac{1-2r^2}{1-r^2}I[1]I[1]\right) , \\
J_{\text{tadpole}} &= -\tfrac{1}{8}\left(-8I[1]I[1]\right) ,
\end{aligned} \tag{4.103}$$

where $I[m^2]$ is defined in (4.35). It is found that the limit $r \to 0$ is not well-defined in the t-t model, whereas several diagram contributions blow up by the limit $r \to 1$ in the c-c model.

As done in approach I, we sum up the 1PI contributions and non-1PI contributions separately, then find that IR-divergent and r-dependent terms cancel out and we get the same result in the two models,

$$J_{\rm 1PI} = -\frac{1}{2} I[4]I[4] + I[4]I[1] - I[1]I[1] \ , \tag{4.104}$$

$$J_{\rm tadpole} = \frac{1}{2} I[1]I[1] \ , \tag{4.105}$$

so that the total is (using (4.36))

$$\Gamma^{(2)} = \frac{8\pi}{k_3} V_2 \sum_n J_n \ , \qquad \sum_n J_n = -\frac{1}{2}\big(I[4] - I[1]\big)^2 = -\frac{1}{8\pi^2}\left(\ln 2\right)^2 \ . \tag{4.106}$$

Combining everything together we find that the effective action for the $AdS_3 \times S^3$ model is (cf. (4.14))

$$\Gamma^{(2)} = \frac{1}{\pi k_3} \, {\rm a}_2 \, V_2 \ , \qquad {\rm a}_2 = -\frac{1}{4}({\rm a}_1)^2 = -(\ln 2)^2 \ . \tag{4.107}$$

Let us now compare these results with those (4.81),(4.82),(4.83) found in approach I in section 4.3.1. It is observed that the 1PI contributions in (4.81) and (4.104) contain the same $I[4]I[4]$ and $I[1]I[1]$ terms, also the fermion-fermion $I[1]I[1]$ tadpole terms are the same. Assuming that the final result should be both UV and IR finite, the expression in approach II (4.106) is more natural. This suggests that it is the tadpole contribution (4.82) in approach I that is to be blamed for the IR problem: it should not actually contain the $I[4]I[4]$ term if the two approaches are to agree. Then if instead of (4.81) one would take

$$\mathcal{J}'_{\rm tadpole} = \frac{1}{6}\big(I[4] - I[1]\big)\big(3I[0] - 3I[1]\big) = \frac{1}{2}\big(I[4] - I[1]\big)\big(I[0] - I[1]\big) \ , \tag{4.108}$$

then the sum of (4.108) with the 1PI contribution (4.81) in the first approach would exactly match the result (4.106) of the second approach.[2]

Another interesting point is that the final two-loop result in (4.106) is proportional to the square of the one-loop coefficient in (4.70), while this terms does not appear in the original string theory

[2] The replacement of (4.82) by (4.108) is formally achieved by replacing $I[4]$ in the second factor in (4.82) by $I[0]$. This may be related to a subtlety in how the two massive AdS_3 modes (which are mixed for $\mu \neq 0$) are treated in the tadpole contributions in the limit when $\mu \to 0$: in that limit one of them has $m^2 = 4$ and the other one becomes massless.

[IRT]. These observations will guide us in interpreting and fixing the two-loop result in the case of the reduced theory for the $AdS_5 \times S^5$ GS string where we will only consider approach I.

4.4 Reduced $AdS_5 \times S^5$ theory

In this section we shall compute the quantum corrections to the partition function of the reduced $AdS_5 \times S^5$ theory in the approach based on the PW identity. The one-loop correction will be discussed in section 4.4.1 where we will first rewrite the fluctuation Lagrangian (4.23) in terms of component fields of the fluctuations, then derive their functional determinant contribution. The form of the resulting fluctuation Lagrangian in the bosonic sector is different from the one found in the original string theory, while both of them give the same functional determinant. In appendix D.1 we will discuss the detail of the relation between the fluctuation Lagrangians. In section 4.4.2 the quantum correction at the next order will be evaluated and the nontrivial Catalan's constant will be derived in agreement with the string theory computation. IR finite two-loop result will be achieved by a close analogy with the $AdS_3 \times S^3$ case.

Using the parameterization of $PSU(2,2|4)$ introduced in appendix B, then we have the reduced theory solution corresponding to the (S, J) folded string (4.7),[3]

$$g_0 = \begin{pmatrix} g_A & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} , \qquad g_A = \begin{pmatrix} 0 & \frac{\kappa}{\mu} v_\tau^* & -\frac{\ell}{\mu} v_\tau^* & 0 \\ -\frac{\kappa}{\mu} v_\tau & 0 & 0 & \frac{\ell}{\mu} v_\tau \\ \frac{\ell}{\mu} v_\tau^* & 0 & 0 & -\frac{\kappa}{\mu} v_\tau^* \\ 0 & -\frac{\ell}{\mu} v_\tau & \frac{\kappa}{\mu} v_\tau & 0 \end{pmatrix} , \qquad v_\tau = e^{i\frac{\kappa^2 \tau}{\mu}} , \tag{4.109}$$

[3]In general, a choice of the reduced theory solution corresponding to a given string theory solution is not unique as one may apply an on-shell $H \times H$ gauge transformation. For example, one may start with a σ-dependent solution,

$$g'_A = \begin{pmatrix} \frac{\kappa}{\mu} v_\sigma^* & 0 & 0 & \frac{\ell}{\mu} \\ 0 & \frac{\kappa}{\mu} v_\sigma & \frac{\ell}{\mu} & 0 \\ 0 & \frac{\ell}{\mu} & \frac{\kappa}{\mu} v_\sigma^* & 0 \\ \frac{\ell}{\mu} & 0 & 0 & \frac{\kappa}{\mu} v_\sigma \end{pmatrix} , \qquad A'_{+A} = A'_{-A} = \frac{i\kappa^2}{2\mu} \mathrm{diag}\,(1, -1, 1, -1) \, ,$$

One may expect that the result for the quantum partition function for the two solutions should be the same. In fact, we have checked that the individual diagram contributions in the two cases are indeed the same.

$$A_{\pm 0} = \begin{pmatrix} A_{\pm A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} , \quad A_{+A} = \frac{i(\ell^2+\kappa^2)}{2\mu}\mathrm{diag}\,(1,-1,1,-1)\ , \quad A_{-A} = \frac{i\mu}{2}\mathrm{diag}\,(1,-1,1,-1)\ , \tag{4.110}$$

and, by computing the stress tensor for the solution (4.7), we find that the mass scale in the reduced theory is μ.

In order to derive the new classical solution $\tilde{g}$ in the system (4.19), let us first find U and $\tilde{U}$. In the case of the (S, J) folded string, the S^5 sector is in the vacuum where the classical gauge fields $A_\pm$ are vanishing. Then U and $\tilde{U}$ take the following form,

$$U = \begin{pmatrix} u & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} , \quad \tilde{U} = \begin{pmatrix} \tilde{u} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} . \tag{4.111}$$

One choice for u and $\tilde{u}$ is such that we have $u = \tilde{u}$ then $U = \tilde{U}$. With this choice the background for the second WZW term in (4.19) is identity. This is achieved by

$$u = \tilde{u} = \begin{pmatrix} w^{*1/2} & 0 & 0 & 0 \\ 0 & w^{1/2} & 0 & 0 \\ 0 & 0 & w^{*1/2} & 0 \\ 0 & 0 & 0 & w^{1/2} \end{pmatrix} , \tag{4.112}$$

then $\tilde{g}$ is given by

$$\tilde{g} = \begin{pmatrix} \tilde{g}_A & 0 \\ 0 & 1 \end{pmatrix} , \quad \tilde{g}_A = \begin{pmatrix} 0 & \frac{\kappa}{\mu}v_\sigma & -\frac{\ell}{\mu}v_\tau^* & 0 \\ -\frac{\kappa}{\mu}v_\sigma^* & 0 & 0 & \frac{\ell}{\mu}v_\tau \\ \frac{\ell}{\mu}v_\tau^* & 0 & 0 & -\frac{\kappa}{\mu}v_\sigma \\ 0 & -\frac{\ell}{\mu}v_\tau & \frac{\kappa}{\mu}v_\sigma^* & 0 \end{pmatrix} , \tag{4.113}$$

where w, v_τ and v_σ are defined in (4.51) and (4.45), respectively. One can confirm that the rotated solution $\tilde{g}$ solves the equation of motion derived from the Lagrangian (4.19), which ensures that this is an appropriate starting point for perturbation.

4.4.1 One-loop computation

Absence of the mixing term between bosonic fluctuations and fermionic fluctuations in the Lagrangian for quadratic fluctuations (4.23) allows us to evaluate the bosonic contribution and fermionic contribution separately at one-loop level. In section 4.4.1 we shall discuss the bosonic quadratic fluctuations described by the Lagrangian (4.28) and show that our result agrees with the string theory computation in terms functional determinant contributions to the partition function. Next we shall show the agreement with the original string theory for the fermionic quadratic fluctuations in (4.38).

The argument below is based on approach I used for the reduced $AdS_3 \times S^3$ theory. As an alternative approach, the one-loop computation in the $\delta A_+ = 0$ gauge is discussed in appendix D.2. Both approaches provide the same functional determinant contribution, so it is confirmed that they are equivalent at least at one loop.

Bosonic fluctuations

Since of the homogeneous nature of the folded string in the scaling limit, one obtains constant-coefficient Lagrangian for the quadratic fluctuations by introducing component fields of the bosonic fluctuations appropriately. One possible choice is

$$\eta^{\|} = \begin{pmatrix} \eta^{\|}_A & 0 \\ 0 & \eta^{\|}_S \end{pmatrix},$$

$$\eta^{\|}_A = \begin{pmatrix} 0 & 0 & a_1 + ia_2 & (a_3 + ia_4)w \\ 0 & 0 & (a_3 - ia_4)w^* & -a_1 + ia_2 \\ a_1 - ia_2 & (a_3 + ia_4)w & 0 & 0 \\ (a_3 - ia_4)w^* & -a_1 - ia_2 & 0 & 0 \end{pmatrix}, \qquad (4.114)$$

$$\eta^{\|}_S = \begin{pmatrix} 0 & 0 & b_1 + ib_2 & b_3 + ib_4 \\ 0 & 0 & -b_3 + ib_4 & b_1 - ib_2 \\ -b_1 + ib_2 & b_3 + ib_4 & 0 & 0 \\ -b_3 + ib_4 & -b_1 - ib_2 & 0 & 0 \end{pmatrix}$$

which correspond to physical fields in $\mathfrak{m}$ coming from the coset part G/H, and

$$\eta^{\perp}=\begin{pmatrix}\eta^{\perp}_{A} & 0\\ 0 & \eta^{\perp}_{S}\end{pmatrix},$$
$$\eta^{\perp}_{A}=\begin{pmatrix} ic_1 & (c_2+ic_3)w & 0 & 0\\ (-c_2+ic_3)w^{*} & -ic_1 & 0 & 0\\ 0 & 0 & ic_4 & (c_5+ic_6)w\\ 0 & 0 & (-c_5+ic_6)w^{*} & -ic_4\end{pmatrix}, \tag{4.115}$$
$$\eta^{\perp}_{S}=\begin{pmatrix} id_1 & d_2+id_3 & 0 & 0\\ -d_2+id_3 & -id_1 & 0 & 0\\ 0 & 0 & id_4 & d_5+id_6\\ 0 & 0 & -d_5+id_6 & -id_4\end{pmatrix},$$

which are unphysical fields living in the algebra $\mathfrak{h}$ of the subgroup H. We rotated several bosonic fluctuations by using w defined in (4.51) in order to have the constant-coefficient fluctuation Lagrangian.

We shall write down Lagrangian for quadratic fluctuations in terms of the component fields and compute their characteristic frequencies. While the resulting Lagrangian looks different from the original string theory result, we will show that characteristic frequencies in our present approach in the reduced theory agree with those found in the string theory computation.

Plugging (4.114) and (4.115) into the Lagrangian for quadratic fluctuations (4.23), we obtain the constant-coefficient Lagrangian containing $4+4$ physical fields and $6+6$ unphysical fields. At one-loop level the AdS_5 sector and the S^5 sector are decoupled. As the S^5 part of the classical solution g_0 is the vacuum, the structure of the fluctuation Lagrangian in the S^5 sector is simple,

$$\mathcal{L}^{(2)}_{S^5}=2\sum_{i=1}^{4}\left(\partial_+ b_i\partial_- b_i-\mu^2 b_i^2\right)+\sum_{j=1}^{6}\partial_+ d_j\partial_- d_j\,. \tag{4.116}$$

This shows that the four physical fields b_i are massive fluctuations with characteristic frequencies,

$$\sqrt{n^2+\mu^2}\,, \tag{4.117}$$

which agree with [90]. On the other hand, the six unphysical fields d_j coming from an element in $\mathfrak{h}$ of the subgroup $H = [SU(2)]^2$ are massless.

Next let us discuss the AdS_5 sector. As explicitly shown in chapter 3 for several string configurations, for a classical solution expressed in the form of (4.113), the system for quadratic fluctuations decouples into two smaller subsectors. One subsector contains a_1, a_2 and the off-diagonal part of $\eta_A^\perp$, c_2, c_3, c_5 and c_6, while the other subsector contains a_3, a_4 and the diagonal part of $\eta_A^\perp$, c_1 and c_4. The total Lagrangian in the AdS_5 sector is written as

$$\mathcal{L}^{(2)}_{AdS_5} = \mathcal{L}^{(2)}_1 + \mathcal{L}^{(2)}_2 \,, \tag{4.118}$$

where the physical fluctuations a_1 and a_2 are governed by the Lagrangian $\mathcal{L}^{(2)}_1$,

$$\begin{aligned}\mathcal{L}^{(2)}_1 = {} & 2\sum_{i=1,2}\left(\partial_+ a_i \partial_- a_i - \left(2\kappa^2-\mu^2\right)a_i^2\right) - 4M_1\left(\mu c_2 + \partial_- c_3 + \mu c_5 + \partial_- c_6\right)a_1 \\ & +4M_1\left(\partial_- c_2 - \mu c_3 - \partial_- c_5 + \mu c_6\right)a_2 - \sum_{j=2,3,5,6}\left(\partial_+ c_j \partial_- c_j + \left(2\kappa^2-\mu^2\right)c_j^2\right) \\ & -2\left(\mu\partial_+ c_3 + M_2 \partial_- c_3\right)c_2 - 2\left(\mu\partial_+ c_6 + M_2\partial_- c_6\right)c_5 \,.\end{aligned} \tag{4.119}$$

and the other subsector described by the Lagrangian $\mathcal{L}^{(2)}_2$ is

$$\mathcal{L}^{(2)}_2 = 2\sum_{i=3,4}\partial_+ a_i \partial_- a_i + 4\left(\mu\partial_+ a_4 + M_2\partial_- a_4\right)a_3 - \sum_{j=1,4}\partial_+ c_j\partial_- c_j + 4M_1(\partial_- c_1 + \partial_- c_4)a_3 \,, \tag{4.120}$$

where M_1 and M_2 are constants which we introduced in the $AdS_3 \times S^3$ case,

$$M_1 = \frac{\kappa\sqrt{\kappa^2-\mu^2}}{\mu^2}\,, \qquad M_2 = \frac{2\kappa^2-\mu^2}{\mu}\,. \tag{4.121}$$

It is worth checking the $\mu \to 0$ ($J \to 0$) limit where the (S, J) folded string has no stretching or rotation in S^5. It is clear that the constants M_1, M_2 in (4.121) are singular by this limit, so the Lagrangians (4.119), (4.120) are also singular. The $\mu \to 0$ limit is not well-defined even after applying the nonlocal transformation discussed in appendix D.1. Thus we will proceed with μ nonzero.

We will compute characteristic frequencies of the fluctuations given in (4.119) and (4.120). Although our fluctuation Lagrangians look different from the fluctuation Lagrangian found by perturbing the Nambu action [97], it will turn out that the characteristic frequencies and the functional determinant contributions of the original string theory are recovered from the Lagrangians (4.119), (4.120). The detailed discussion on the one-loop computation and the comparison to the original string theory at the level of the fluctuation Lagrangian are given in appendix D.1.

In order to compute characteristic frequencies we substitute $e^{i(\Omega\tau+n\sigma)}$ into all the fluctuation fields in the Lagrangians (4.119) and (4.120). For the former Lagrangian we obtain a 6×6 mass matrix. The condition that the determinant of the mass matrix vanishes gives six solutions for Ω. We find that four of the six fluctuation fields are massless, and frequencies of the remaining two massive fluctuations are

$$\sqrt{n^2+2\kappa^2-\mu^2}\,, \tag{4.122}$$

which agree with [90]. For the other Lagrangian (4.120), we have a 4×4 mass matrix, but only two of the four fluctuations in this Lagrangian are massive with frequencies,

$$\sqrt{n^2+2\kappa^2\pm 2\sqrt{\kappa^4+n^2\mu^2}}\,, \tag{4.123}$$

which agree with [90].

Let us compute the functional determinant for both the AdS_5 sector and the S^5 sector. The S^5 sector is described by the Lagrangian (4.116) and its functional determinant is

$$Z_S^{(1B)}=\Big([\det(\partial_+\partial_-)]^6\left[\det\left(\partial_+\partial_-+\mu^2\right)\right]^4\Big)^{-1/2}\,. \tag{4.124}$$

Here $\left(\partial_+\partial_-+\mu^2\right)^4$ comes from the four massive fluctuations and $(\partial_+\partial_-)^6$ is the contribution from the six unphysical fluctuations. On the other hand, the functional determinant for the AdS_5 sector is derived from the Lagrangian (4.118) with (4.119) and (4.120),[4]

$$\begin{aligned}Z_A^{(1B)}=\Big(&\left[\det\left(\partial_+\partial_-+2\kappa^2-\mu^2\right)\right]^2[\det(\partial_+\partial_-)]^2\left[\det\left(\partial_-^2+\mu^2\right)\right]^2\left[\det\left(\partial_+^2+\mu^2\right)\right]^2\\&\times\det\left(\partial_+^2\partial_-^2+\partial_+^2\mu^2+\left(4\kappa^2-2\mu^2\right)\partial_+\partial_-+\mu^2\partial_-^2\right)\Big)^{-1/2}\,.\end{aligned} \tag{4.125}$$

The functional determinant contributions from the massive fluctuations, $\det\left(\partial_+\partial_-+2\kappa^2-\mu^2\right)^2$ and $\det\left(\partial_+^2\partial_-^2+\partial_+^2\mu^2+\left(4\kappa^2-2\mu^2\right)\partial_+\partial_-+\mu^2\partial_-^2\right)$, are the same as the functional determinant found by perturbing the Nambu action. In the remaining part in (4.125), $\det(\partial_+\partial_-)^2$ corresponds to two of the six unphysical massless fluctuations and the other four massless fluctuations do the factor

[4]The same determinant is obtained by using the Lagrangians after the nonlocal transformations, that is, by using (D.6) and (D.11).

$\det\left(\partial_-^2+\mu^2\right)^2\left(\partial_+^2+\mu^2\right)^2$ because

$$\begin{aligned}\det\left(\partial_\pm^2+\mu^2\right) &= \det\left(\partial_\pm+i\mu\right)\det\left(\partial_\pm-i\mu\right)\\ &= \det\left(e^{-i\mu\tau}\partial_\pm e^{i\mu\tau}\right)\det\left(e^{i\mu\tau}\partial_\pm e^{-i\mu\tau}\right) ,\end{aligned}$$

and the exponential factors cancel out in the determinant. One can confirm that the unphysical factor cancels with those of the Jacobian arising from the transformation (4.16) and quantum fluctuations found in the other WZW term in (4.20). Hence the total bosonic contribution to the one-loop partition function is obtained by collecting the determinant of the physical fluctuations,

$$\begin{aligned}Z^{(1B)} = Z_A^{(1B)}Z_S^{(1B)} = \Big(&\left[\det\left(\partial_+\partial_-+2\kappa^2-\mu^2\right)\right]^2\left[\det\left(\partial_+\partial_-+\mu^2\right)\right]^4\\ &\times\det\left(\partial_+^2\partial_-^2+\mu^2\left(\partial_-^2\partial_+^2\right)+\left(4\kappa^2-2\mu^2\right)\partial_+\partial_-\right)\Big)^{-1/2} ,\end{aligned} \tag{4.126}$$

which will be used to compute the total one-loop correction to the partition function.

Fermionic fluctuations

Given the agreement in the bosonic sector, the original string theory result for the fermionic fluctuations should be recovered in the reduced theory in the approach based on the PW identity.

Constant-coefficient Lagrangian is derived also in the fermionic sector if we define component fields of the fermionic fluctuations in the following way,

$$\Psi_R = \begin{pmatrix} 0 & \mathfrak{X}_R \\ \mathfrak{Y}_R & 0 \end{pmatrix} , \quad \Psi_L = \begin{pmatrix} 0 & \mathfrak{X}_L \\ \mathfrak{Y}_L & 0 \end{pmatrix} , \tag{4.127}$$

where

$$\mathfrak{X}_R = \begin{pmatrix} 0 & 0 & (\alpha_1+i\alpha_2)f_+ & (\alpha_3+i\alpha_4)f_+ \\ 0 & 0 & (-\alpha_3+i\alpha_4)f_+^* & (\alpha_1-i\alpha_2)f_+^* \\ (\alpha_5+i\alpha_6)f_+ & (\alpha_7-i\alpha_8)f_+ & 0 & 0 \\ (\alpha_7+i\alpha_8)f_+^* & (-\alpha_5+i\alpha_6)f_+^* & 0 & 0 \end{pmatrix} , \tag{4.128}$$

$$\mathfrak{Y}_R = \begin{pmatrix} 0 & 0 & (-\alpha_6 - i\alpha_5)f_+^* & (-\alpha_8 - i\alpha_7)f_+ \\ 0 & 0 & (\alpha_8 - i\alpha_7)f_+^* & (-\alpha_6 + i\alpha_5)f_+ \\ (\alpha_2 + i\alpha_1)f_+^* & (\alpha_4 - i\alpha_3)f_+ & 0 & 0 \\ (\alpha_4 + i\alpha_3)f_+^* & (-\alpha_2 + i\alpha_1)f_+ & 0 & 0 \end{pmatrix}, \tag{4.129}$$

and

$$\mathfrak{X}_L = \begin{pmatrix} 0 & 0 & (\beta_1 + i\beta_2)f_-^* & (\beta_3 + i\beta_4)f_-^* \\ 0 & 0 & (\beta_3 - i\beta_4)f_- & (-\beta_1 + i\beta_2)f_- \\ (\beta_5 + i\beta_6)f_-^* & (-\beta_7 + i\beta_8)f_-^* & 0 & 0 \\ (\beta_7 + i\beta_8)f_- & (\beta_5 - i\beta_6)f_- & 0 & 0 \end{pmatrix}, \tag{4.130}$$

$$\mathfrak{Y}_L = \begin{pmatrix} 0 & 0 & (-\beta_6 - i\beta_5)f_- & (-\beta_8 - i\beta_7)f_-^* \\ 0 & 0 & (-\beta_8 + i\beta_7)f_- & (\beta_6 - i\beta_5)f_-^* \\ (\beta_2 + i\beta_1)f_-^* & (-\beta_4 + i\beta_3)f_- & 0 & 0 \\ (\beta_4 + i\beta_3)f_-^* & (\beta_2 - i\beta_1)f_- & 0 & 0 \end{pmatrix}, \tag{4.131}$$

where all component fields are real Grassmann and $f_\pm = e^{i\frac{\kappa^2\tau \pm \ell^2\sigma}{2\mu}}$. The extra factors are introduced in both Ψ_R and Ψ_L such that all coefficients in the Lagrangian become constant. The resulting Lagrangian is

$$\mathcal{L}_\mathrm{F} = 2\Bigg[\sum_{i=1}^{8}(\alpha_i\partial_-\alpha_i + \beta_i\partial_+\beta_i) - \mu\,(\alpha_1\alpha_2 + \alpha_3\alpha_4 + \alpha_5\alpha_6 - \alpha_7\alpha_8 - \beta_1\beta_2 - \beta_3\beta_4 - \beta_5\beta_6 + \beta_7\beta_8) \\ +2\kappa\,(\alpha_1\beta_4 + \alpha_2\beta_3 - \alpha_3\beta_2 - \alpha_4\beta_1 - \alpha_5\beta_8 + \alpha_6\beta_7 + \alpha_7\beta_6 - \alpha_8\beta_5)\Bigg], \tag{4.132}$$

which describes fermionic fluctuations with frequencies (up to trivial shift),

$$\sqrt{n^2 + \kappa^2}\,. \tag{4.133}$$

They are the same as the fermionic frequencies in the original string theory [69]. The determinant for the fermionic fluctuations is

$$\det\left(\partial_+^2\partial_-^2 + 2\partial_+\partial_-\kappa^2 + \frac{1}{4}\left(\partial_+^2 + \partial_-^2\right)\mu^2 + \frac{1}{16}\left(4\kappa^2 - \mu^2\right)^2\right). \tag{4.134}$$

Again, by the same argument as in (4.69), this functional determinant is reduced into the from $\big[\det(\partial_+\partial_- + \kappa^2)\big]^2$.

The final expression for the one-loop partition function is thus the same as in string theory [90]. In the $\mu \to 0$ limit we get the familiar result [89] (see (4.11))

$$\begin{aligned}\Gamma^{(1)} &= \frac{1}{2}\bar{V}_2 \int \frac{d^2q}{(2\pi)^2}\big[\ln(q^2+4\kappa^2) + 2\ln(q^2+2\kappa^2) + 5\ln q^2 - 8\ln(q^2+\kappa^2)\big] \\ &= 2\kappa^2\bar{V}_2\big(I[4]+I[2]-2I[1]\big) = \frac{1}{2\pi}\mathrm{a}_1 V_2\ , \qquad \mathrm{a}_1 = -3\ln 2\,. \end{aligned} \tag{4.135}$$

4.4.2 Two-loop computation

In this section we will discuss the two-loop computation using the fluctuation Lagrangians (4.23), (4.24) and (4.25) for the reduced $AdS_5 \times S^5$ theory. First we will study the contributions of diagrams which involve only bosonic propagators. In this case the system is the direct sum of the AdS_5 sector and the S^5 sector. Next we will discuss diagrams containing fermionic propagators. Then tadpole contributions will be evaluated, where we will find that the reduced model gives the nonvanishing tadpole contributions of both bosonic and fermionic loops. Finally the results of the bosonic, fermionic and tadpole contributions will be combined. Throughout this section we will use the Euclidean signature on the worldsheet.

Bosonic 1PI contributions

The bosonic contributions are described by the Lagrangians (4.28), (4.29) and (4.30). These fluctuation Lagrangians show that the two-loop contributions to the two-loop partition function are given by the Feynman diagrams of the topologies shown in Figure 4.1.

To compute the two-loop diagrams we shall derive the bosonic propagator. In the AdS_5 sector, as shown in the discussion on the one-loop computation in section 4.4.1 and appendix D.1, the quadratic terms contain the off-diagonal mixing terms of physical and unphysical fields, which can be diagonalized by the nonlocal transformation. However, the diagonalization is not necessarily useful in the two-loop computation because this transformation produces a complicated nonlocal Lagrangian. Thus we shall consider all of the fluctuations and compute the diagrams for the 10+10 fields (4+4 physical and 6+6 unphysical fluctuations) rather than the block-diagonalization of the system such that the physical part and the unphysical part are decoupled. It is useful to make a set of $O(2)$ transformations of the unphysical fluctuations in the AdS_5 sector in (D.3) and (D.9) in appendix D.1 to obtain four decoupled subsectors. Once we denote the bosonic fluctuation fields as

$$\Phi_I = \{\Phi_{A\,i}, \Phi_{S\,j}\} \tag{4.136}$$

where we reordered the bosonic fields in the AdS_5 sector in $\Phi_{A\,i}$

$$\Phi_{A\,i} = \{a_1, c_2, c_3, a_2, c_5, c_6, a_3, a_4, c_1, c_4\}\,, \qquad \Phi_{S\,i} = \{b_1, \ldots, b_4, d_1, \ldots, d_6\}\,, \tag{4.137}$$

then the bosonic propagator for the AdS_5 sector is written as

$$\triangle_A^{-1}(q) = \begin{pmatrix} M_1(q) & 0 & 0 & 0 \\ 0 & M_1(q) & 0 & 0 \\ 0 & 0 & M_2(q) & 0 \\ 0 & 0 & 0 & \frac{1}{2q^2} \end{pmatrix}, \tag{4.138}$$

where $M_1(q)$ and $M_2(q)$ are 3×3 matrices,

$$M_1(q) = \frac{1}{D_1}\begin{pmatrix} -\frac{4\hat\kappa^4-4\hat\kappa^2\hat\mu^2+\hat\mu^2\left(q_+^2+\hat\mu^2\right)}{4\hat\mu^2\left(q_-^2+\hat\mu^2\right)} & \frac{\hat\kappa\sqrt{\hat\kappa^2-\hat\mu^2}\left(2\hat\kappa^2-\hat\mu^2\right)}{\sqrt{2}\hat\mu^2\left(q_+^2+\hat\mu^2\right)} & -\frac{\hat\kappa\sqrt{\hat\kappa^2-\hat\mu^2}q_+}{\sqrt{2}\hat\mu\left(q_+^2+\hat\mu^2\right)} \\ \frac{\hat\kappa\sqrt{\hat\kappa^2-\hat\mu^2}\left(2-\hat\mu^2\right)}{\sqrt{2}\hat\mu^2\left(q_+^2+\hat\mu^2\right)} & -\frac{4\hat\kappa^2\left(\hat\kappa^2-\hat\mu^2\right)\left(q_-^2+\hat\mu^2\right)+\hat\mu^2\left(q^4-\hat\mu^4\right)}{2\hat\mu^2\left(q^4+2\hat\mu^2q^2-4\hat\mu^2q_0^2+\hat\mu^4\right)} & \frac{\hat\kappa^2q_+\left(q_-^2+\hat\mu^2\right)+iq^1\hat\mu^2\left(q^2-\hat\mu^2\right)}{\hat\mu\left(q^4+2\hat\mu^2q^2-4\hat\mu^2q_0^2+\hat\mu^4\right)} \\ \frac{\hat\kappa\sqrt{\hat\kappa^2-\hat\mu^2}q_+}{\sqrt{2}\hat\mu\left(q_+^2+\hat\mu^2\right)} & -\frac{\hat\kappa^2q_+\left(q_-^2+\hat\mu^2\right)+iq^1\hat\mu^2\left(q^2-\hat\mu^2\right)}{\hat\mu\left(q^4+2\hat\mu^2q^2-4\hat\mu^2q_0^2+\hat\mu^4\right)} & \frac{q^4-\hat\mu^4}{2\left(q^4+2\hat\mu^2q^2-4\hat\mu^2q_0^2+\hat\mu^4\right)} \end{pmatrix},$$

$$M_2(q) = \frac{1}{D_2}\begin{pmatrix} -\frac{q^2}{4} & -\frac{\hat\kappa^2q_-+iq^1\hat\mu^2}{2\hat\mu} & \frac{\hat\kappa q_-\sqrt{\hat\kappa^2-\hat\mu^2}}{\sqrt{2}\hat\mu} \\ \frac{\hat\kappa^2q_-+iq^1\hat\mu^2}{2\hat\mu} & \frac{q_+\left(q_+^2\hat\mu^2-4\hat\kappa^2\left(\hat\kappa^2-\hat\mu^2\right)\right)}{4q_-\hat\mu^2} & -\frac{\sqrt{2}\hat\kappa\sqrt{\hat\kappa^2-\mu^2}\left(\hat\kappa^2q_-+iq_1\hat\mu^2\right)}{q_+\hat\mu^2} \\ -\frac{\hat\kappa q_-\sqrt{\hat\kappa^2-\hat\mu^2}}{\sqrt{2}\hat\mu} & -\frac{\sqrt{2}\hat\kappa\sqrt{\hat\kappa^2-\hat\mu^2}\left(\hat\kappa^2q_-+iq_1\hat\mu^2\right)}{q_+\hat\mu^2} & \frac{q^4\hat\mu^2-4\left(q^1\hat\mu^2-i\hat\kappa^2q_-\right)^2}{2q^2\hat\mu^2} \end{pmatrix},$$

$$q_\pm = q_0 + iq_1\,, \qquad q^2 = q_0^2 + q_1^2\,, \qquad D_1 = q^2 + 2\hat\kappa^2 - \hat\mu^2\,, \qquad D_2 = q^4 + 4\hat\kappa^2 q^2 - 4\hat\mu^2 {q_1}^2\,, \tag{4.139}$$

and the propagator for the S^5 sector is

$$\triangle_S^{-1}(q) = \text{diag}\left(-\frac{1}{4\,(q^2+\hat{\mu}^2)}, -\frac{1}{4\,(q^2+\hat{\mu}^2)}, -\frac{1}{4\,(q^2+\hat{\mu}^2)}, -\frac{1}{4\,(q^2+\hat{\mu}^2)}, -\frac{1}{2q^2}, \dots, -\frac{1}{2q^2}\right)\,, \quad (4.140)$$

where we have omitted the overall factor $\frac{8\pi}{k}$ which will be restored later.

Below we shall evaluate the two-loop contributions diagram by diagram. One particle irreducible contributions of the bosonic fluctuations are given by the bosonic sunset (Figure 4.1(a)) and bosonic double-bubble (Figure 4.1(b)). It should be emphasized that the main difference between the present case and the $AdS_3 \times S^3$ case in section 4.3.1 is the appearance of the Catalan's constant as a consequence of involving the bosonic fluctuations transverse to the AdS_3 subspace in the AdS_5 sector.

Similar to the original string theory computation in the light-cone gauge [97, 100], the bosonic fluctuations also provide the nonvanishing tadpole contributions, which will be discussed later together with tadpoles containing fermionic propagators (Figure 4.3).

Note that the $\hat{\mu} \to 0$ ($\mu \to 0$) limit can be smoothly taken once we simplify the integrands of the two-loop integrals. In this limit we have $\kappa = \ell$ then $\hat{\kappa} \to 1$. We will explicitly show the two-loop PRT partition function only for $\hat{\mu} \to 0$. The two-loop computation for a general $\hat{\mu}$ in the reduced theory is still an open problem.

By plugging the bosonic fluctuation fields (4.114) and (4.115) into $\mathcal{L}^{(3)}$ in (4.24) one finds vertices for the sunset diagrams in Figure 4.1(a). We shall make use of the fact that the AdS_5 part decouples from the S^5 part as the case of the one-loop computation, and first discuss the complicated AdS_5 sector where the Catalan's constant will arise. Corresponding to the three propagators in the sunset diagrams, the two-loop integral for the diagrams can be characterized by three mass parameters, m_i, m_j and m_k,

$$\mathcal{I}_{m_i^2 m_j^2 m_k^2} = \int \frac{d^2q_i\, d^2q_j\, d^2q_j}{(2\pi)^4} \frac{\mathcal{F}(q_i, q_j, q_k)}{q_i^{n_i} q_j^{n_j} q_k^{n_k} (q_i^2+m_i^2)(q_j^2+m_j^2)(q_k^3+m_k^2)}\,, \quad (4.141)$$

where we symbolically denote a polynomial function of q as $\mathcal{F}(q_i, q_j, q_k)$. In the AdS_5 sector the vertices contained in the fluctuation Lagrangian are of the three types. The first type includes the vertices $V_{A\,ijk}$ with $(i,j,k) = (\{7,8,9\},\{1,2,3\},\{1,2,3\})$ or $(i,j,k) = (\{7,8,9\},\{4,5,6\},\{4,5,6\})$. In the $\mu \to 0$ limit we have $D_1 \to q^2+2$ and $D_2 \to (q^2+4)q^2$, so using these vertices we obtain the integral $\mathcal{I}_{422}$ containing the Catalan's constant given by the type of integral $I[4,2,2]$. The vertices

$V_{A\,ijk}$ with $(i,j,k) = (\{7,8,9\},\{7,8,9\},\{7,8,9\})$ lead to the integral $\mathcal{I}_{444}$. For the agreement with the string theory result, the nontrivial integral $I[4,4,4]$ contained in $\mathcal{I}_{444}$ should not appear. The last type is $V_{A\,ijk}$ with $(i,j,k) = (10,\{1,2,3\},\{4,5,6\})$ yielding the integral $\mathcal{I}_{022}$.

The two-loop integrals, $\mathcal{I}_{422}$, $\mathcal{I}_{444}$ and $\mathcal{I}_{022}$, respectively, are simplified as

$$\begin{aligned}\mathcal{I}_{422} &= 2I[4,2,2] - \tfrac{1}{2}I[2]I[0] - \tfrac{3}{2}I[4]I[2]\,,\\ \mathcal{I}_{444} &= -\tfrac{1}{4}I[4]I[0] - \tfrac{1}{4}I[4]I[4]\,,\\ \mathcal{I}_{022} &= -\tfrac{1}{2}I[2]I[2]\,,\end{aligned} \tag{4.142}$$

where we included the combinatorial factor $-1/12$ and used the notation introduced in (4.35).

The computation in the S^5 sector is trivial since the S^5 part of the classical solution in the reduced theory (4.113) is the vacuum. The S^5 part of $\Gamma^{(3)}$ in (4.33) in the $\hat{\mu} \to 0$ limit is given by the following integral,

$$\int \frac{d^2q_i\, d^2q_j}{(2\pi)^4}\left[\frac{q_i^+ q_j^- - q_i^- q_j^+}{q_i^2 q_j^2 q_k^2} + \frac{q_i^+ q_j^- - q_i^- q_j^+}{q_i^2 q_j^2 q_k^2} + (\text{permutation of }(i,j,k))\right]. \tag{4.143}$$

Due to obvious symmetry of the momentum-space integral under interchange $i \leftrightarrow j$, we find that each term vanishes separately.

For the diagrams 4.1(b), the integral is characterized by two mass parameters, m_i and m_j, corresponding to the two propagators with the momenta q_i and q_j,

$$\mathcal{I}_{m_i^2 m_j^2} = \int \frac{d^2q_i\, d^2q_j}{(2\pi)^4}\frac{\mathcal{F}(q_i,q_j)}{q_i^{n_i} q_j^{n_j}(q_i^2+m_i^2)(q_j^2+m_j^2)}\,. \tag{4.144}$$

In the AdS_5 part the nonvanishing elements are $\mathcal{I}_{22}$ and $\mathcal{I}_{44}$. After simplification, they are written as

$$\begin{aligned}\mathcal{I}_{22} &= -\tfrac{1}{2}I[2]I[2]\,,\\ \mathcal{I}_{44} &= -\tfrac{1}{4}I[4]I[0] - \tfrac{1}{4}I[4]I[4]\,,\end{aligned} \tag{4.145}$$

where the combinatorial factor 1/8 is included. Although the quartic terms for the S^5 sector are found in the fluctuation Lagrangian, the corresponding two-loop integrals do not lead to nontrivial contribution.

By summing up (4.142) and (4.145) we find that the bosonic contribution to the quantum partition function is

$$J_{\text{boson}} = 2I[4,2,2] - \frac{1}{2}I[2]I[0] - \frac{1}{2}I[4]I[0] - I[2]I[2] - \frac{3}{2}I[4]I[2] - \frac{1}{2}I[4]I[4]\,. \tag{4.146}$$

The nontrivial finite part of the bosonic contribution is $2I[4,2,2]$,

$$2I[4,2,2] = \frac{1}{8\pi^2}K\,, \tag{4.147}$$

where K is the Catalan's constant. Note that the two-loop contributions from the other WZW term $\mathcal{L}_{\text{WZW}}(U^{-1}\tilde{U})$ lead to power-divergent terms, which should cancel with power-divergent terms from other diagrams.

Fermionic 1PI contributions

The fermionic contributions of the cubic and quartic interaction terms are given by the diagrams of the two topologies depicted in Figure 4.2. As done for the bosonic contributions, the two-loop integrals are written in terms of $\mathcal{I}_{m_i^2 m_j^2 m_k^2}$ for the sunset diagram in Figure 4.2(a) and $\mathcal{I}_{m_i^2 m_j^2}$ for the double-bubble diagram in Figure 4.2(b).

As the fermionic fluctuations have the mass $\hat{\kappa} = 1$ in the $\hat{\mu} \to 0$ limit, the integrals arising from the fermionic sunset are the type of $\mathcal{I}_{m^2 11}$, where m is the mass of the bosonic fluctuation. We find that the nonvanishing contributions are

$$\begin{aligned}
\mathcal{I}_{011} &= 9I[1][0] - \tfrac{9}{2}I[1]I[1]\,, \\
\mathcal{I}_{211} &= -2I[2,1,1] + I[1]I[1] - 2I[2]I[1]\,, \\
\mathcal{I}_{411} &= I[4]I[1] - \tfrac{1}{2}I[1]I[1]\,.
\end{aligned} \tag{4.148}$$

Here we obtained the Catalan's constant, $I[2,1,1]$. It is also important that the integral of the $I[4,1,1]$ type is not derived from $\mathcal{I}_{411}$. This is consistent with the original theory result.

The fermionic double-bubble diagrams give the $\mathcal{I}_{m^2 1}$ type, where the mass of the bosonic field m^2 can be 0, 2 or 4. They are summarized as

$$
\begin{aligned}
\mathcal{I}_{01} &= -9I[1]I[0]\,, \\
\mathcal{I}_{21} &= 6I[2]I[1]\,, \\
\mathcal{I}_{41} &= I[1]I[0] + 2I[4]I[1]\,.
\end{aligned}
\tag{4.149}
$$

Combining (4.148) and (4.149) we obtain the total fermionic contribution,

$$
J_{\text{fermion}} = -2I[2,1,1] + I[1]I[0] - 4I[1]I[1] + 4I[2]I[1] + 3I[4]I[1]\,. \tag{4.150}
$$

The finite part $-2I[2,1,1]$ is rewritten as

$$
-2I[2,1,1] = -\frac{1}{4\pi^2}K\,, \tag{4.151}
$$

where K is the Catalan's constant. More importantly, this value is -2 of the Catalan's constant obtained in the bosonic sector (4.147). This is exactly the same as the observation in the original string theory (4.3).

Combining the bosonic (4.146) and the fermionic (4.150) 1PI contributions together we find

$$
J_{\text{1PI}} = -\frac{1}{8\pi^2}\mathrm{K} - \frac{1}{2}\big(I[4] + I[2] - 2I[1]\big)\big(I[4] + 2I[2] + I[0] - 4I[1]\big)\,. \tag{4.152}
$$

We observe that as in the $AdS_3 \times S^3$ reduced theory case (4.81), the second term in (4.152) is UV finite but IR divergent and is proportional to the same combination $I[4] + I[2] - 2I[1]$ which appears in the one-loop result (4.135).

Tadpole contributions and total result for the two-loop coefficient

The non-1PI diagram relevant in the present case is shown in Figure 4.3. The loops can be bosonic or fermionic, and the intermediate line connecting the two loops is bosonic.

When computing this diagram, we should note that it is not allowed to set the momentum for the intermediate line to be zero initially. $M_2(q)$ of the bosonic propagator in (4.138) and (4.139) vanishes by setting $q = 0$, but this part of the propagator contains the physical degrees of freedom and may contribute to the two-loop corrections. Thus we shall start with nonzero momentum for the intermediate line and take the zero momentum limit after the integration in the two loops integrals is done.

Since the tadpole contributions are necessarily written as a product of the one-loop integral (4.35), the nontrivial finite integral is not obtained from this sector. The non-1PI diagrams with two bosonic loops give

$$J_{\rm boson-boson\ tadpole} = -\frac{1}{8}\left(-\frac{4}{3}I[2]I[0] - \frac{4}{3}I[4]I[0] - 4I[2]I[2] - \frac{20}{3}I[4]I[2] - \frac{8}{3}I[4]I[4]\right) . \quad (4.153)$$

The contributions of non-1PI diagrams with one bosonic loop and one fermionic loop are

$$J_{\rm boson-fermion\ tadpole} = \frac{1}{8}\left(-\frac{8}{3}I[1]I[0] - 16I[2]I[1] - \frac{40}{3}I[4]I[1]\right) . \quad (4.154)$$

Also we obtain the contribution of the non-1PI diagrams with two fermionic loops

$$J_{\rm fermion-fermion\ tadpole} = -\frac{1}{8}\left(-16I[1]I[1]\right) . \quad (4.155)$$

Hence the total tadpole contribution is

$$J_{\rm tadpole} = \frac{1}{6}\big(I[4] + I[2] - 2I[1]\big)\big(2I[4] + 3I[2] + I[0] - 6I[1]\big) . \quad (4.156)$$

Combining together (4.152) and (4.156) we find the following expression for the coefficient in the two-loop effective action,

$$\Gamma^{(2)} = \frac{8\pi}{k}V_2 \sum_n J_n , \quad (4.157)$$

where

$$\sum_n J_n = J_{\rm 1PI} + J_{\rm tadpole} = \bar{J} + \tilde{J} , \qquad \bar{J} = -\frac{1}{8\pi^2}{\rm K} , \quad (4.158)$$

$$\tilde{J} = -\frac{1}{6}\big(I[4] + I[2] - 2I[1]\big)\big(I[4] + 3I[2] + 2I[0] - 6I[1]\big) . \quad (4.159)$$

The resulting two-loop coefficient contains, $\bar{\rm a}_2 = -{\rm K}$, $\tilde{\rm a}_2 = 8\pi^2\tilde{J} = -\frac{5}{4}(\ln 2)^2 - \ln 2\, \ln m_0$ which is IR divergent.

Recalling that the $I[4]I[4]$ term in the tadpole contribution disturbs the IR finite result in approach I for reduced $AdS_3 \times S^3$ theory due to the subtlety in how the tadpole contribution was computed, it is natural to expect that the same subtlety arises in the $AdS_5 \times S^5$ case. Indeed, the results in the reduced $AdS_3 \times S^3$ and reduced $AdS_5 \times S^5$ theories are in direct agreement in what concerns $I[4]I[4]$ contributions coming from the 1PI graphs: this term enters (4.152) with the same coefficient $-\frac{1}{2}$ as in (4.81) or (4.104). If we accept the prescription employed in the $AdS_3 \times S^3$ case where $I[4]$ in the second factor in the tadpole contribution (4.82) is replaced by $I[0]$, here $I[4]$ in the second factor in (4.159) should be replaced by $I[0]$, then the tadpole contribution becomes

$$\begin{aligned} J'_{\text{tadpole}} &= \frac{1}{6}\big(I[4] + I[2] - 2I[1]\big)\big(3I[2] + 3I[0] - 6I[1]\big) \\ &= \frac{1}{2}\big(I[4] + I[2] - 2I[1]\big)\big(I[2] + I[0] - 2I[1]\big) \,, \end{aligned} \tag{4.160}$$

and the sum of (4.152) and (4.160) leads to the finite integral. Then $\tilde{\mathrm{a}}_2$ is finite and again proportional to the square of the one-loop coefficient,

$$\tilde{J} = -\frac{1}{2}\big(I[4] + I[2] - 2I[1]\big)^2 \,, \qquad \text{i.e.} \qquad \tilde{\mathrm{a}}_2 = 8\pi^2 \tilde{J} = -\frac{1}{4}(\mathrm{a}_1)^2 = -\frac{9}{4}(\ln 2)^2 \,. \tag{4.161}$$

Chapter 5

Summary and future directions

In this thesis we discussed quantum aspects of the Pohlmeyer-reduced form of the $AdS_5 \times S^5$ GS superstring theory. The Pohlmeyer reduction is a technique to eliminate unphysical degrees of freedom at the level of equations of motion preserving the Lorentz symmetry and the integrability. In the context of string theory in $AdS_5 \times S^5$, these are great advantages because of the breaking of the $2d$ Lorentz invariance by a straightforward way of gauge fixing in the $AdS_5 \times S^5$ GS action.

After the review of the Pohlmeyer reduction of $AdS_3 \times S^1$ bosonic string theory and the $AdS_5 \times S^5$ GS superstring theory in chapter 2, the one-loop computations for homogeneous and inhomogeneous backgrounds were considered in chapter 3 (and also in appendix C). We demonstrated that the reduced theory partition function is the same as the string theory one for the respective string configurations at one-loop level. The two-loop relation between these two theories was explored in chapter 4. We found a strong indication that the $AdS_5 \times S^5$ GS superstring and its Pohlmeyer-reduced form are closely related at the quantum level: the reduced $AdS_5 \times S^5$ superstring theory correctly produces the same nontrivial constant as the original string theory, i.e., the Catalan's constant, under the specific identification of the coupling constants of the two theories.

Nevertheless, the two-loop partition function in the reduced theory also contains a finite term proportional to the square of the one-loop result. Also, our final result in the $AdS_5 \times S^5$ case is inferred by the comparison between the two approaches in the $AdS_3 \times S^3$ case. A natural suggestion is that we could perform the two-loop computation for $AdS_5 \times S^5$ based on approach II of $AdS_3 \times S^3$, while integrating out A_+, A_- first and gauge-fixing g also leads to more involved fluctuation action than the

$AdS_3 \times S^3$ case.

As a consequence of the absence of the unphysical modes, one virtue in studying the reduced model instead of the original theory is its simpler one-loop structure (see chapter 3). In addition to several string configurations in $AdS_3 \times S^1$ which we discussed in this thesis, there are various kinds of solutions localized in pure AdS_3. It was pointed out in [66, IWA] that the reduced AdS_3 theory is not embedded in the full reduction of the $AdS_5 \times S^5$ GS superstring theory at the level of the classical Lagrangian. However, our investigation in the semiclassical expansions in chapter 3 shows that the one-loop computation for strings in AdS_3 can be studied in the full reduction by starting with their generalized solutions in $AdS_3 \times S^1$ and eventually taking the limit where stretching and motion in S^1 are eliminated. In particular, it is allowed to take such limit in the fluctuation Lagrangian if one fixes the H gauge nicely. This statement is expected to be true for general solutions in $AdS_3 \times S^1$ by the argument analogous to [66]. Let us discuss it briefly. By the replacement $\phi_{_A} \to \phi_{_A} + \ln \frac{\mu}{\sqrt{2}}$, $\theta_{_A} \to \frac{1}{2\mu}\theta_{_A}$ in the fluctuation Lagrangian of the tanh model (3.2), the $\mu \to 0$ limit becomes well-defined in the fluctuation Lagrangian. After the limit and a further replacing $\phi_{_A} \to \phi_{_A} - \frac{1}{4}\ln \partial_+\theta_{_A}\partial_-\theta_{_A}$, the resulting Lagrangian is

$$\mathcal{L}_{\tanh(2)} = \partial_+\delta\phi_{_A}\partial_-\delta\phi_{_A} + \partial_+\delta\theta_{_A}\partial_-\delta\theta_{_A} - 2\cosh 2\phi_{_A}\sqrt{\partial_+\theta_{_A}\partial_-\theta_{_A}}\,(\delta\phi_{_A})^2\,. \tag{5.1}$$

One can confirm that this is the correct fluctuation Lagrangian by plugging the counterpart of the folded string solution in AdS_3,

$$\partial_\pm\theta_{_A} = 2w\kappa\,, \qquad \phi_{_A} = \frac{1}{2}\ln 2\rho'^2 - \frac{1}{4}\ln \partial_+\theta_{_A}\partial_-\theta_{_A}\,, \tag{5.2}$$

and finding that it describes one massless fluctuation and one massive fluctuation with the coefficient of mass term $\frac{2(w^2\kappa^2+\rho'^4)}{\rho'^2}$, which agrees with (3.21) and [89]. This prescription also works for the full reduction in the decoupling gauge; one gets four massless fluctuations and two massive fluctuations with $m^2 = 2\rho'^2$ for bosons and the correct fermionic modes as in [89].

One possible application of this technique is to study semiclassical expansions around the minimal area of an open string ending on n-cusp null Wilson loop on the conformal boundary of AdS_3 [61, 62]. Since it is very hard to reconstruct the corresponding string theory solution from the reduced theory solution which was found in [61, 62], computing the one-loop corrections to the area in the reduced theory leads to much simplification while the one-loop computation on the string theory side involves a problem of the IR regulator [98].

Although the Pohlmeyer-reduced form of the $AdS_5 \times S^5$ GS superstring is integrable, the integrable structure is different from the one found in the original string theory. The integrability property in the classical reduced theory was studied in [105] in terms of Poisson structure and the realization of the quantum integrability was discussed perturbatively [75]. As one might expect that the relation of quantum partition functions of the two theories could directly come from that of the integrable structures, it would be interesting to uncover the nontrivial relation between these two forms of the integrability.

Another open problem is the classical relation between the reduced AdS_n theory and the reduced $AdS_n \times S^1$ theory. While the limit where string has no stretching or momentum in S^1, i.e., $\mu \to 0$, can be smoothly taken in the original string theory, the corresponding limit in the reduced theory is nontrivial especially at the classical level (the semiclassical case was discussed in chapter 3 and appendix C). This situation is very similar to the algebraic curve approach to the finite-gap solution; there is no simple limit connecting the structures of $AdS_3 \times S^1$ strings and AdS_3 strings [106]. Only the $n = 3$ case was discussed in [66] where the authors rescaled the complex sinh-Gordon fields first, then took the limit in order to obtain the reduced AdS_3 theory from the reduced $AdS_3 \times S^1$ theory. Beyond this case the situation is unclear, but it may be possible to find an analog of [107], where the strong/weak duality between $2d$ integrable theories in terms of their S-matrices was found.

A related problem is that the $\mu \to 0$ limit in the deformed gWZW model (2.31), (2.32) is not yet understood. If one naively takes the limit, the potential terms $\mu^2 \mathrm{STr}(g^{-1}TgT)$, $\mu \mathrm{STr}\left(g^{-1}\Psi_L g \Psi_R\right)$disappear from the Lagrangian.

Finally, one straightforward extension of our work is to compute the three-loop corrections to the reduced theory partition function. It would be very interesting if we obtain terms such as $\mathrm{a}_1\mathrm{a}_2$, $(\mathrm{a}_1)^3$ in the three-loop coefficients. However, the three-loop computation should be first completed in the original string theory following the qualitative argument given in [70].

We hope that further study of the Pohlmeyer-reduced version of string theory in $AdS_5 \times S^5$ may lead to important insights into the structure of underlying quantum theory.

Appendix A

Matrix superalgebra

In this appendix we will summarize the superalgebra and its parameterization for the cases of $\mathfrak{psu}(2,2|4)$ and $\mathfrak{psu}(1,1|2)$.[1] We will first review the $\mathfrak{psu}(2,2|4)$ superalgebra, then explain the $\mathfrak{psu}(1,1|2)$ superalgebra by a slight modification of the $\mathfrak{psu}(2,2|4)$ case.

We shall start with the $\mathfrak{su}(2,2|4)$ superalgebra as it is spanned by 8×8 supermatrices $\mathfrak{f}$. In general $\mathfrak{f}$ is expressed in terms of 4×4 matrices,

$$\mathfrak{f} = \begin{pmatrix} \mathfrak{A} & \mathfrak{X} \\ \mathfrak{Y} & \mathfrak{D} \end{pmatrix}, \tag{A.1}$$

where the matrices $\mathfrak{A}$, $\mathfrak{D}$ are Grassmann even and $\mathfrak{X}$, $\mathfrak{Y}$ are Grassmann odd. The superalgebra $\mathfrak{su}(2,2|4)$ requires that the matrix $\mathfrak{f}$ should vanish by taking the supertrace,

$$\mathrm{STr}\mathfrak{f} \equiv \mathrm{tr}\mathfrak{A} - \mathrm{tr}\mathfrak{D} = 0\,, \tag{A.2}$$

and satisfy the reality condition,

$$\mathfrak{f}^\dagger H + H\mathfrak{f} = 0\,, \tag{A.3}$$

[1]In this paper we follow the notation of [64].

with a Hermitian matrix H. It is convenient to choose H to be the following form,

$$H = \begin{pmatrix} \Sigma & 0 \\ 0 & \mathbf{1} \end{pmatrix} , \tag{A.4}$$

where Σ is expressed as

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \tag{A.5}$$

and $\mathbf{1}$ is the 4×4 identity matrix. With this choice of H, the reality condition (A.3) gives the relations,

$$\mathfrak{A}^\dagger = -\Sigma\mathfrak{A}\Sigma\,, \quad \mathfrak{D}^\dagger = -\mathfrak{D}\,, \quad \mathfrak{Y} = -\mathfrak{X}^\dagger\Sigma\,. \tag{A.6}$$

So we see that the bosonic matrix $\mathfrak{A}$ belongs to $\mathfrak{u}(2,2)$ and the other bosonic matrices $\mathfrak{D}$ belong to $\mathfrak{u}(4)$. The only one combination of each $\mathfrak{u}(1)$ generator, $i\mathbf{1}$, satisfies the reality condition (A.3) and supertraceless condition (A.2). Hence the bosonic subalgebra of $\mathfrak{su}(2,2|4)$ is decomposed as

$$\mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1)\,. \tag{A.7}$$

The superalgebra $\mathfrak{psu}(2,2|4)$ is defined as the quotient algebra of $\mathfrak{su}(2,2|4)$ over this $\mathfrak{u}(1)$ factor.

One important property of the $\mathfrak{psu}(2,2|4)$ superalgebra is that it admits a Z_4 automorphism such that the condition $Z_4(\mathfrak{f}) = \mathfrak{f}$ determines the subgroup $G = Sp(2,2) \times Sp(4)$ of $F = PSU(2,2|4)$. Define the automorphism $\mathfrak{f} \to \Omega(\mathfrak{f})$ by

$$\Omega(\mathfrak{f}) = -\begin{pmatrix} K\mathfrak{A}^{\mathrm{t}}K & -K\mathfrak{Y}^{\mathrm{t}}K \\ K\mathfrak{X}^{\mathrm{t}}K & K\mathfrak{D}^{\mathrm{t}}K \end{pmatrix} , \tag{A.8}$$

where the 4×4 matrix K is chosen to be

$$K = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{A.9}$$

which satisfies $K^2 = -\mathbf{1}$, then any matrix in $\mathfrak{psu}(2,2|4)$ can be decomposed as

$$\mathfrak{f} = \mathfrak{f}_0 \oplus \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \mathfrak{f}_3 \,, \tag{A.10}$$

where $\mathfrak{f}_k$ are eigenstates of Ω,

$$\Omega(\mathfrak{f}_k) = i^k \mathfrak{f}_k \,, \tag{A.11}$$

and given by

$$\begin{aligned} \mathfrak{f}_0 &= \frac{1}{4}\left(\mathfrak{f} + \Omega(\mathfrak{f}) + \Omega^2(\mathfrak{f}) + \Omega^3(\mathfrak{f})\right) = \frac{1}{2}\begin{pmatrix} \mathfrak{A} - K\mathfrak{A}^{\mathrm{t}}K & 0 \\ 0 & \mathfrak{D} - K\mathfrak{D}^{\mathrm{t}}K \end{pmatrix}, \\ \mathfrak{f}_1 &= \frac{1}{4}\left(\mathfrak{f} - i\Omega(\mathfrak{f}) - \Omega^2(\mathfrak{f}) + i\Omega^3(\mathfrak{f})\right) = \frac{1}{2}\begin{pmatrix} 0 & \mathfrak{X} - iK\mathfrak{Y}^{\mathrm{t}}K \\ \mathfrak{Y} + iK\mathfrak{X}^{\mathrm{t}}K & 0 \end{pmatrix}, \\ \mathfrak{f}_2 &= \frac{1}{4}\left(\mathfrak{f} - \Omega(\mathfrak{f}) + \Omega^2(\mathfrak{f}) - \Omega^3(\mathfrak{f})\right) = \frac{1}{2}\begin{pmatrix} \mathfrak{A} + K\mathfrak{A}^{\mathrm{t}}K & 0 \\ 0 & \mathfrak{D} + K\mathfrak{D}^{\mathrm{t}}K \end{pmatrix}, \\ \mathfrak{f}_3 &= \frac{1}{4}\left(\mathfrak{f} + i\Omega(\mathfrak{f}) - \Omega^2(\mathfrak{f}) - i\Omega^3(\mathfrak{f})\right) = \frac{1}{2}\begin{pmatrix} 0 & \mathfrak{X} + iK\mathfrak{Y}^{\mathrm{t}}K \\ \mathfrak{Y} - iK\mathfrak{X}^{\mathrm{t}}K & 0 \end{pmatrix}. \end{aligned} \tag{A.12}$$

They satisfy the following commutation relation,

$$[\mathfrak{f}_i, \mathfrak{f}_j] \subset \mathfrak{f}_{i+j \mod 4} \,. \tag{A.13}$$

Let us denote $\mathfrak{g} = \mathfrak{f}_0$ and $\mathfrak{p} = \mathfrak{f}_2$. Then $\mathfrak{g}$ is the algebra of the subgroup G of F defining the coset F/G, and $\mathfrak{p}$ corresponds to the bosonic coset component. The other two components, $\mathfrak{f}_1$, $\mathfrak{f}_3$, are fermionic parts.

Now let us discuss a further Z_2 decomposition, which defines the group H and the coset G/H. Here we shall introduce an element T of the maximal Abelian subalgebra of $\mathfrak{p}$ by

$$T = \frac{i}{2}\,\mathrm{diag}\,(1,\,1,\,-1,\,-1,\,1,\,1,\,-1,\,-1)\;. \tag{A.14}$$

The Z_2 decomposition is then given by

$$\mathfrak{f}_r^{||} = -\,[T,[T,\mathfrak{f}_r]]\;, \qquad\qquad \mathfrak{f}_r^{\perp} = -\{T,\{T,\mathfrak{f}_r\}\}\,. \tag{A.15}$$

It should be noted that this is an orthogonal decomposition, that is

$$\begin{aligned}\mathfrak{f} &= \mathfrak{f}^{||}\oplus\mathfrak{f}^{\perp}\,,\\ \mathrm{STr}(\mathfrak{f}^{||}\mathfrak{f}^{\perp}) &= 0\,.\end{aligned} \tag{A.16}$$

and they form the following commutation relation,

$$\left[\mathfrak{f}^{\perp},\,\mathfrak{f}^{\perp}\right]\subset\mathfrak{f}^{\perp}\,,\qquad \left[\mathfrak{f}^{\perp},\,\mathfrak{f}^{||}\right]\subset\mathfrak{f}^{||}\,,\qquad \left[\mathfrak{f}^{||},\,\mathfrak{f}^{||}\right]\subset\mathfrak{f}^{\perp}\,. \tag{A.17}$$

Identify $\mathfrak{h}=\mathfrak{f}_0^{\perp}$, $\mathfrak{m}=\mathfrak{f}_0^{||}$, $\mathfrak{a}=\mathfrak{f}_2^{\perp}$, $\mathfrak{n}=\mathfrak{f}_2^{||}$. In fact $\mathfrak{a}$ is the maximal Abelian subspace of $\mathfrak{p}$, and the algebra $\mathfrak{h}$ of the subgroup H of G is defined as the stabilizer of T in $\mathfrak{g}$, i.e., $[\mathfrak{h},T]=0$. Together with the commutation relations (A.13) and (A.17), one finds these elements satisfy

$$[\mathfrak{a},\mathfrak{a}]\subset 0\,,\quad [\mathfrak{a},\mathfrak{h}]\subset 0\,,\quad [\mathfrak{h},\mathfrak{h}]\subset\mathfrak{h}\,,\quad [\mathfrak{m},\mathfrak{m}]\subset\mathfrak{h}\,,\quad [\mathfrak{m},\mathfrak{h}]\subset\mathfrak{m}\,,\quad [\mathfrak{m},\mathfrak{a}]\subset\mathfrak{n}\,,\quad [\mathfrak{n},\mathfrak{a}]\subset\mathfrak{m}\,. \tag{A.18}$$

For the specific choice of the matrices H, K and T in (A.4), (A.9) and (A.14), respectively, then one can uniquely express general elements of $\mathfrak{m}$, $\mathfrak{h}$, $\mathfrak{a}$, $\mathfrak{n}$, $\mathfrak{f}_1^{||}$, $\mathfrak{f}_1^{\perp}$, $\mathfrak{f}_3^{||}$ and $\mathfrak{f}_3^{\perp}$ in terms of their matrix components. The following four components are relevant when we determine the fluctuation Lagrangian of the reduced theory. The subspace $\mathfrak{m}$ of $\mathfrak{g}$ corresponds to physical fluctuations in the reduced theory,

$$\mathfrak{m} = \begin{pmatrix} \mathfrak{m}_A & 0 \\ 0 & \mathfrak{m}_S \end{pmatrix},$$
$$\mathfrak{m}_A = \begin{pmatrix} 0 & 0 & a_1 + ia_2 & a_3 + ia_4 \\ 0 & 0 & a_3 - ia_4 & -a_1 + ia_2 \\ a_1 - ia_2 & a_3 + ia_4 & 0 & 0 \\ a_3 - ia_4 & -a_1 - ia_2 & 0 & 0 \end{pmatrix}, \tag{A.19}$$
$$\mathfrak{m}_S = \begin{pmatrix} 0 & 0 & b_1 + ib_2 & b_3 + ib_4 \\ 0 & 0 & -b_3 + ib_4 & b_1 - ib_2 \\ -b_1 + ib_2 & b_3 + ib_4 & 0 & 0 \\ -b_3 + ib_4 & -b_1 - ib_2 & 0 & 0 \end{pmatrix}.$$

Unphysical fluctuations lie in the subspace $\mathfrak{h}$ of $\mathfrak{g}$, which should be gauged away or integrated out from the fluctuation Lagrangian of the reduced theory,

$$\mathfrak{h} = \begin{pmatrix} \mathfrak{h}_A & 0 \\ 0 & \mathfrak{h}_S \end{pmatrix},$$
$$\mathfrak{h}_A = \begin{pmatrix} ic_1 & c_2 + ic_3 & 0 & 0 \\ -h_c + ic_3 & -ic_1 & 0 & 0 \\ 0 & 0 & ic_4 & c_5 + ic_6 \\ 0 & 0 & -c_5 + ic_6 & -ic_4 \end{pmatrix}, \tag{A.20}$$
$$\mathfrak{h}_S = \begin{pmatrix} id_1 & d_2 + id_3 & 0 & 0 \\ -d_2 + id_3 & -id_1 & 0 & 0 \\ 0 & 0 & id_4 & d_5 + id_6 \\ 0 & 0 & -d_5 + id_6 & -id_4 \end{pmatrix}.$$

The κ-symmetry allows us to set fermionic fields to take values in $\mathfrak{f}^{||}$,

$$\mathfrak{f}^{||}_1 = \begin{pmatrix} 0 & \mathfrak{X}_1 \\ \mathfrak{Y}_1 & 0 \end{pmatrix}, \quad \mathfrak{f}^{||}_3 = \begin{pmatrix} 0 & \mathfrak{X}_3 \\ \mathfrak{Y}_3 & 0 \end{pmatrix}, \tag{A.21}$$

where

$$\mathfrak{X}_1 = \begin{pmatrix} 0 & 0 & \alpha_1 + i\alpha_2 & \alpha_3 + i\alpha_4 \\ 0 & 0 & -\alpha_3 + i\alpha_4 & \alpha_1 - i\alpha_2 \\ \alpha_5 + i\alpha_6 & \alpha_7 - i\alpha_8 & 0 & 0 \\ \alpha_7 + i\alpha_8 & -\alpha_5 + i\alpha_6 & 0 & 0 \end{pmatrix}, \tag{A.22}$$

$$\mathfrak{Y}_1 = \begin{pmatrix} 0 & 0 & -\alpha_6 - i\alpha_5 & -\alpha_8 - i\alpha_7 \\ 0 & 0 & \alpha_8 - i\alpha_7 & -\alpha_6 + i\alpha_5 \\ \alpha_2 + i\alpha_1 & \alpha_4 - i\alpha_3 & 0 & 0 \\ \alpha_4 + i\alpha_3 & -\alpha_2 + i\alpha_1 & 0 & 0 \end{pmatrix}, \tag{A.23}$$

and

$$\mathfrak{X}_3 = \begin{pmatrix} 0 & 0 & \beta_1 + i\beta_2 & \beta_3 + i\beta_4 \\ 0 & 0 & \beta_3 - i\beta_4 & -\beta_1 + i\beta_2 \\ \beta_5 + i\beta_6 & -\beta_7 + i\beta_8 & 0 & 0 \\ \beta_7 + i\beta_8 & \beta_5 - i\beta_6 & 0 & 0 \end{pmatrix}, \tag{A.24}$$

$$\mathfrak{Y}_3 = \begin{pmatrix} 0 & 0 & -\beta_6 - i\beta_5 & -\beta_8 - i\beta_7 \\ 0 & 0 & -\beta_8 + i\beta_7 & \beta_6 - i\beta_5 \\ \beta_2 + i\beta_1 & -\beta_4 + i\beta_3 & 0 & 0 \\ \beta_4 + i\beta_3 & \beta_2 - i\beta_1 & 0 & 0 \end{pmatrix}. \tag{A.25}$$

When we derive the fluctuation Lagrangian for fermionic fluctuations, we rescale components of $\Psi_{_R} \in \mathfrak{f}_1^{||}$ and $\Psi_{_L} \in \mathfrak{f}_3^{||}$.

Now we shall discuss the $\mathfrak{psu}(1,1|2)$ superalgebra by modifying the above argument for the $\mathfrak{psu}(2,2|4)$ superalgebra. In the $\mathfrak{psu}(1,1|2)$ case, the matrix $\mathfrak{f}$ in (A.1) is a 4×4 supermatrix spanned by 2×2 matrices, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{X}$ and $\mathfrak{Y}$. Also, H, Σ in (A.4), (A.5) and K in (A.9) should be replaced by

$$H = \begin{pmatrix} \Sigma & 0 \\ 0 & \mathbf{1}_{2\times 2} \end{pmatrix}, \qquad \text{with} \qquad \Sigma = K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.26}$$

whereas T in (A.14) for the further Z_2 decomposition should be chosen as

$$T = \frac{i}{2}\,\mathrm{diag}\,(1,\ -1,\ 1,\ -1)\ . \tag{A.27}$$

Hence general elements of $\mathfrak{m}$, $\mathfrak{h}$, $\mathfrak{a}$, $\mathfrak{n}$, $\mathfrak{f}_1^{||}$, $\mathfrak{f}_1^{\perp}$, $\mathfrak{f}_3^{||}$ and $\mathfrak{f}_3^{\perp}$ in $\mathfrak{psu}(1,1|2)$ are expressed explicitly. The following four components are relevant when we determine the fluctuation Lagrangian of the reduced theory.

$$\begin{gathered}\mathfrak{m} = \begin{pmatrix} \mathfrak{m}_A & 0 \\ 0 & \mathfrak{m}_S \end{pmatrix} , \\ \mathfrak{m}_A = \begin{pmatrix} 0 & a_1 + ia_2 \\ a_1 - ia_2 & 0 \end{pmatrix} , \qquad \mathfrak{m}_S = \begin{pmatrix} 0 & b_1 + ib_2 \\ b_1 - ib_2 & 0 \end{pmatrix} ,\end{gathered} \tag{A.28}$$

which lie in the subspace $\mathfrak{m}$ of $\mathfrak{g}$ corresponding to physical fluctuations in the reduced theory and

$$\begin{gathered}\mathfrak{h} = \begin{pmatrix} \mathfrak{h}_A & 0 \\ 0 & \mathfrak{h}_S \end{pmatrix} , \\ \mathfrak{h}_A = \begin{pmatrix} ic & 0 \\ 0 & -ic \end{pmatrix} , \qquad \mathfrak{h}_S = \begin{pmatrix} id & 0 \\ 0 & -id \end{pmatrix} ,\end{gathered} \tag{A.29}$$

which are unphysical fields taking value on the subspace $\mathfrak{h}$ of $\mathfrak{g}$.

Among the fermionic currents, we use the following two currents,

$$\mathfrak{f}_1^{||} = \begin{pmatrix} 0 & \mathfrak{X}_1 \\ \mathfrak{Y}_1 & 0 \end{pmatrix} , \qquad \mathfrak{f}_3^{||} = \begin{pmatrix} 0 & \mathfrak{X}_3 \\ \mathfrak{Y}_3 & 0 \end{pmatrix} , \tag{A.30}$$

where

$$\mathfrak{X}_1 = \begin{pmatrix} 0 & \alpha_1 + i\alpha_2 \\ \alpha_3 + i\alpha_4 & 0 \end{pmatrix} , \qquad \mathfrak{Y}_1 = \begin{pmatrix} 0 & -i\alpha_3 - \alpha_4 \\ i\alpha_1 + \alpha_2 & 0 \end{pmatrix} , \tag{A.31}$$

and

$$\mathfrak{X}_3 = \begin{pmatrix} 0 & \beta_1 + i\beta_2 \\ \beta_3 + i\beta_4 & 0 \end{pmatrix} , \qquad \mathfrak{Y}_3 = \begin{pmatrix} 0 & -i\beta_3 - \beta_4 \\ i\beta_1 + \beta_2 & 0 \end{pmatrix} , \tag{A.32}$$

When we derive the fluctuation Lagrangian for fermionic fluctuations, we rescale components of $\Psi_{_R} \in \mathfrak{f}_1^{||}$ and $\Psi_{_L} \in \mathfrak{f}_3^{||}$.

Appendix B

Parametrization of $PSU(2,2|4)$ in terms of embedding coordinates

Here we shall discuss the relation between the embedding coordinates in $AdS_5 \times S^5$ and parametrization of the $PSU(2,2|4)$ coset elements (see [85] for details).

Let us define six real coordinates Y^P on $R^{4,2}$ $(P = 0,\ 1,\ \ldots,\ 5)$ and six real coordinates X^M on R^6 $(I = 1,\ 2,\ \ldots,\ 6)$. To define AdS_5 and S^5 embedded in $R^{4,2}$ and R^6 we impose

$$\begin{aligned} \eta_{PQ} Y^P Y^Q &= -1\,, & \eta_{MN} X^M X^N &= 1\,, \\ \eta &= \mathrm{diag}\,(-1,\ -1,\ 1,\ 1,\ 1,\ 1)\ , & \delta &= \mathrm{diag}\,(1,\ 1,\ 1,\ 1,\ 1,\ 1)\ . \end{aligned} \tag{B.1}$$

and define another set of unconstrained coordinates, $t,\ y_i$ on AdS_5 and $\theta_A,\ x_i$ on S^5, $i = 1,\ 2,\ 3,\ 4$:

$$Y^1 + iY^2 = \frac{y_1 + iy_2}{1 - \frac{y^2}{4}}\,, \qquad Y^3 + iY^4 = \frac{y_3 + iy_4}{1 - \frac{y^2}{4}}\,, \tag{B.2}$$

$$Y^0 + iY^{-1} = \frac{1 + \frac{y^2}{4}}{1 - \frac{y^2}{4}} e^{it}\,,$$

$$X^1 + iX^2 = \frac{x_1 + ix_2}{1 + \frac{x^2}{4}}\,, \qquad X^3 + iX^4 = \frac{x_3 + ix_4}{1 + \frac{x^2}{4}}\,, \tag{B.3}$$

$$X^5 + iX^6 = \frac{1 - \frac{x^2}{4}}{1 + \frac{x^2}{4}} e^{i\theta_A}\,.$$

such that $y^2 = y_i y_i$ and $x^2 = x_i x_i$. The corresponding metrics of AdS_5 and S^5 in terms of $t,\ y_i,\ \theta_A,\ x_i$ are

$$\begin{aligned} \eta^{4,2}_{MN} dY^M dY^N &= -\left(\frac{1+\frac{y^2}{4}}{1-\frac{y^2}{4}}\right)^2 dt^2 + \frac{dy_i dy_i}{\left(1-\frac{y^2}{4}\right)^2}\,, \\ \eta^{6,0}_{IJ} dX^I dX^J &= \left(\frac{1-\frac{x^2}{4}}{1+\frac{x^2}{4}}\right)^2 d\theta_A^2 + \frac{dx_i dx_i}{\left(1+\frac{x^2}{4}\right)^2}\,. \end{aligned} \tag{B.4}$$

A suitable choice of bosonic coset element f would be such that $\mathrm{STr}\left(f^{-1}df\right)^2$ coincides with the sum of the two metrics in (B.4). This allows us to relate the embedding coordinates with the bosonic coset element directly:

$$\begin{aligned} f &= \begin{pmatrix} f_A & \mathbf{0}_4 \\ \mathbf{0}_4 & f_S \end{pmatrix} \\ &= \begin{pmatrix} \exp\left(\frac{i}{2} t\gamma_5\right) & \mathbf{0}_4 \\ \mathbf{0}_4 & \exp\left(\frac{i}{2}\theta_A\gamma_5\right) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1-\frac{y^2}{4}}}\left(\mathbf{1}_4 + \frac{1}{2} y_i\gamma_i\right) & \mathbf{0}_4 \\ \mathbf{0}_4 & \frac{1}{\sqrt{1+\frac{x^2}{4}}}\left(\mathbf{1}_4 + \frac{i}{2} x_i\gamma_i\right) \end{pmatrix}\,. \end{aligned} \tag{B.5}$$

Here γ_k are the $\mathfrak{so}\,(5)$ Dirac matrices chosen as

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{B.6}$$

$$\gamma_4 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Appendix C

One-loop computation: strings in $R \times S^5$

Supplementary to chapter 3, we shall perform the one-loop computation in the reduced theory for strings in $R \times S^5$. We have three examples; the pulsating string in $R \times S^2$, the circular two-spin string in $R \times S^3$ and the short two-spin string in $R \times S^5$. In all the cases, the one-loop results in the original string theory are reproduced.

C.1 Pulsating string in $R \times S^2$

The equivalence of equations of motion for the quadratic fluctuations in the original string theory in conformal gauge and in the reduced theory was shown for an arbitrary classical solution localized in the $AdS_2 \times S^2$ subsector in [HIT]. In this sector there are many classical string configurations; the pointlike string moving along a big circle of S^2, the unstable wrapping static string and inhomogeneous solutions such as pulsating strings, folded strings and magnons, e.g., [46, 47, 48, 109, 110, 111, 112]. In this appendix we shall consider the pulsating string in $R \times S^2$ as a nontrivial example in this subsector, and derive the Lagrangian for its quadratic fluctuations which should be directly compared with the fluctuation Lagrangian found by perturbing the Nambu action in the original string theory.

In section 3.1 we showed that the fluctuations from the Nambu action are related to those of the tanh model by the T-duality transformation and the same as those of the coth model. This variety in the reduced theory originates from the freedom in introducing the second field, i.e., θ_A for the tanh model or χ_A for the coth model. Since bosonic string theory in $R \times S^2$ is reduced to the sin-Gordon model which possesses a single field ϕ_S, it is expected that the Lagrangians in the original theory and in the reduced theory exactly match.

The pulsating string solution in S^2 studied in [109, 111] is expressed in terms of the embedding coordinates for $AdS_5 \times S^5$,

$$\begin{aligned} Y_0 + iY_5 = e^{i\kappa\tau}\,, \quad Y_1 = Y_2 = 0\,, \\ X_1 + X_2 = \sin\psi(\tau) e^{im\sigma}\,, \quad X_3 = \cos\psi(\tau)\,, \quad X_4 = X_5 = X_6 = 0\,, \end{aligned} \tag{C.1}$$

where m is the winding number on S^2. The Virasoro constraints give one nontrivial constraint,

$$\dot{\psi}^2 + m^2 \sin^2\psi(\tau) = \kappa^2\,, \tag{C.2}$$

and the equation of motion is also derived by its derivative,

$$\ddot{\psi}^2 + \frac{m^2}{2}\sin 2\psi(\tau) = 0\,, \tag{C.3}$$

where the dot represents the derivative with respect to τ.

The detail of the reduction for the case of the $AdS_2 \times S^2$ subsector is described in [HIT]. In the deformed gWZW model the AdS_5 part of the corresponding element in G takes a trivial form, diag$(i, -i, i, -i)$, while the S^5 part is given by

$$g = \begin{pmatrix} i\cos\phi_S & 0 & 0 & i\sin\phi_S \\ 0 & -i\cos\phi_S & i\sin\phi_S & 0 \\ 0 & i\sin\phi_S & i\cos\phi_S & 0 \\ i\sin\phi_S & 0 & 0 & -i\cos\phi_S \end{pmatrix}, \tag{C.4}$$

where

$$\partial_+ X_a \partial_- X^a = \kappa^2 \cos 2\phi_S\,. \tag{C.5}$$

Here we used the fact the mass scale of the reduced theory is $\mu = \kappa$. One can check that the classical gauge fields $A_{\pm 0}$ totally vanish by solving the gauge field equation (2.29). Substituting (C.1) into (C.5), then we obtain the classical solution in the S^5 sector,

$$g_0 = \begin{pmatrix} \frac{i\dot{\psi}}{\kappa} & 0 & 0 & i\sqrt{1-\frac{\dot{\psi}^2}{\kappa^2}} \\ 0 & -\frac{i\dot{\psi}}{\kappa} & i\sqrt{1-\frac{\dot{\psi}^2}{\kappa^2}} & 0 \\ 0 & i\sqrt{1-\frac{\dot{\psi}^2}{\kappa^2}} & \frac{i\dot{\psi}}{\kappa} & 0 \\ i\sqrt{1-\frac{\dot{\psi}^2}{\kappa^2}} & 0 & 0 & -\frac{i\dot{\psi}}{\kappa} \end{pmatrix}, \quad A_{\pm 0} = 0\,. \tag{C.6}$$

C.1.1 Bosonic fluctuations

Let us first discuss the bosonic fluctuations. As the AdS_5 part of the classical solution takes the trivial form in the present case, the bosonic fluctuations in the AdS_5 sector are massive fields with $m_B^2 = \kappa^2$. Then their Lagrangian is

$$\mathcal{L}_1 = 2\sum_{i=1}^{4}\left(\partial_+ a_i \partial_- a_i - \kappa^2 a_i^2\right)\,. \tag{C.7}$$

Introduce the components fields of the S^5 part of the physical fluctuation $\eta^{\parallel}$ as follows,[1]

$$\eta^{\parallel} = \begin{pmatrix} 0 & 0 & b_1+ib_2 & b_3+ib_4 \\ 0 & 0 & -b_3+ib_4 & b_1-ib_2 \\ -b_1+ib_2 & b_3+ib_4 & 0 & 0 \\ -b_3+ib_4 & -b_1-ib_2 & 0 & 0 \end{pmatrix}. \tag{C.8}$$

As shown in [HIT] one can partially fix the H gauge such that the physical fields decouple from the unphysical fields. Then the resulting Lagrangian for the physical fluctuations is,

$$\mathcal{L}_2 = 2\left[\sum_{i=1}^{3}\left(\partial_+ b_i \partial_- b_i - \left(2\dot{\psi}^2 - \kappa^2\right) b_i^2\right) + \partial_+ b_4 \partial_- b_4 - \kappa^2\left(1 - \frac{2m^2}{\kappa^2 - \dot{\psi}^2}\right) b_4^2\right]\,. \tag{C.9}$$

The sum of these two Lagrangians in the reduced theory, $\mathcal{L}_1 + \mathcal{L}_2$, is exactly the same as the Lagrangian for the quadratic fluctuations found by perturbing the Nambu action in the original string theory [108].

[1] Both the AdS_5 and S^5 subsectors of H are $[SU(2)]^2$. Hence component fields of the unphysical parts, $\eta^{\perp}$ and $\delta A_{\pm}$, are introduced in the same way as the case of $AdS_3 \times S^1$ in (3.13) and (3.14).

C.1.2 Fermionic fluctuations

The agreement of the bosonic fluctuations in the original theory and reduced theory requires that of fermionic fluctuations. So we shall next check this for completeness. We write down the fermionic fluctuation Lagrangian in terms of the component fields of the fermionic fluctuations introduced in (3.24) - (3.28) with $v = 1$,

$$\begin{aligned} \mathcal{L}_{\rm F} \;=\; & 2\Big[\textstyle\sum_{i=1}^{8} (\alpha_i \partial_- \alpha_i + \beta_i \partial_- \beta_i) \\ & + 2\dot{\psi}\,(-\alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_4\beta_3 - \alpha_3\beta_4 + \alpha_6\beta_5 - \alpha_5\beta_6 - \alpha_8\beta_7 + \alpha_7\beta_8) \Big] \,. \end{aligned} \tag{C.10}$$

This Lagrangian can be separated into four parts and each of them is rewritten in terms of a real four component spinor Ψ,

$$\mathcal{L}_{\rm f} = \bar{\Psi}\gamma^a \partial_a \Psi - \dot{\psi}\bar{\Psi}\Gamma_2 \Psi \,. \tag{C.11}$$

Thus we find that the fermionic fluctuations have the mass term with a coefficient $\dot{\psi}$, which again agrees with the string theory result [108].

C.2 Circular two-spin string in $R \times S^3$

One example of a simple string theory solution we shall consider here is the rigid circular two-spin string on S^3 in S^5 discussed in [92, 93, 101, 102]. Using the embedding coordinates in appendix B, i.e. Y_P $(P = -1,\ 0, \dots,\ 4)$ of $\mathbb{R}^{4,2}$ for the AdS_5 part and X_M $(M = 1,\ 2,\ \dots,\ 6)$ of $\mathbb{R}^6$ for the S^5 part, this bosonic string solution is

$$\begin{aligned} & Y_0 + iY_{-1} = e^{i\kappa\tau} \,, \qquad Y_1 = Y_2 = Y_3 = Y_4 = 0 \,, \\ & X_1 + iX_2 = \tfrac{1}{\sqrt{2}} e^{i\omega\tau + im\sigma} \,, \qquad X_3 + iX_4 = \tfrac{1}{\sqrt{2}} e^{i\omega\tau - im\sigma} \,. \qquad X_5 = X_6 = 0 \,, \end{aligned} \tag{C.12}$$

The Virasoro constraints imply that the three parameters, κ, ω and m, are related by

$$\kappa^2 = m^2 + \omega^2 \ . \tag{C.13}$$

Using the parameterizations discussed in appendix B we obtain the corresponding bosonic coset element f,

$$
\begin{gathered}
f = \begin{pmatrix} f_A & \mathbf{0} \\ \mathbf{0} & f_S \end{pmatrix}, \\
f_A = \begin{pmatrix} e^{\frac{i\kappa\tau}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{i\kappa\tau}{2}} & 0 & 0 \\ 0 & 0 & e^{-\frac{i\kappa\tau}{2}} & 0 \\ 0 & 0 & 0 & e^{-\frac{i\kappa\tau}{2}} \end{pmatrix}, \\
f_S = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{i}{2}e^{-i\omega\tau+im\sigma} & -\frac{i}{2}e^{-i\omega\tau-im\sigma} \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{2}e^{i\omega\tau+im\sigma} & \frac{i}{2}e^{i\omega\tau-im\sigma} \\ \frac{i}{2}e^{i\omega\tau-im\sigma} & \frac{i}{2}e^{-i\omega\tau-im\sigma} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{i}{2}e^{i\omega\tau+im\sigma} & \frac{i}{2}e^{-i\omega\tau+im\sigma} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.
\end{gathered}
\tag{C.14}
$$

The corresponding solution of the reduced theory is [2]

$$
g_0 = \begin{pmatrix} g_A & \mathbf{0} \\ \mathbf{0} & g_S \end{pmatrix}, \qquad V \equiv e^{i\frac{\kappa^2-m^2}{\kappa}\tau}, \tag{C.15}
$$

$$
\begin{gathered}
g_A = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad g_S = \begin{pmatrix} 0 & \frac{\omega}{\kappa}V & -i\frac{m}{\kappa}V & 0 \\ -\frac{\omega}{\kappa}V^* & 0 & 0 & i\frac{m}{\kappa}V^* \\ i\frac{m}{\kappa}V & 0 & 0 & -\frac{\omega}{\kappa}V \\ 0 & -i\frac{m}{\kappa}V^* & \frac{\omega}{\kappa}V^* & 0 \end{pmatrix}, \\
\Psi_{R0} = \Psi_{L0} = 0,
\end{gathered}
\tag{C.16}
$$

and the gauge equations are solved by

$$
\begin{gathered}
A_{\pm\, 0} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_{\pm S} \end{pmatrix}, \\
A_{+S} = i\left(\frac{m^2}{\kappa} - \frac{\kappa}{2}\right)\mathrm{diag}\,(1,-1,1,-1)\,, \quad A_{-S} = -i\frac{\kappa}{2}\mathrm{diag}\,(1,-1,1,-1)\,.
\end{gathered}
\tag{C.17}
$$

[2]The μ parameter of the reduced theory here is identified as κ.

Note that the point-like string (BMN vacuum) solution is a particular case of (C.12), that is when $m = 0$ and $\omega = \kappa$. In the reduced theory the corresponding limit of (C.15) is the vacuum solution in a specific gauge.

C.2.1 Bosonic fluctuations

Since the classical fermionic fields vanish, the bosonic AdS_5 sector, the bosonic S^5 sector and the fermionic sector all decouple at the level of the action and we will discuss them separately.

Here the AdS_5 part of g_0 lives in H and is constant.[3] This is a vacuum solution of this sector. The resulting fluctuation Lagrangian in the bosonic AdS_5 sector is

$$\mathcal{L}_A = \mathrm{STr}\left[\frac{1}{2}\partial_+\eta\partial_-\eta - \delta A_-\partial_+\eta + \delta A_+ g_0 \partial_-\eta g_0^{-1} + \delta A_+\delta A_- - g_0^{-1}\delta A_+ g_0 \delta A_- + \kappa^2\left(\eta\eta T^2 - \eta T\eta T\right)\right] \tag{C.18}$$

We partially fix the H gauge symmetry by setting the diagonal components of $\eta^\perp$ to zero.[4] After integrating out $\delta A_\pm$ the Lagrangian describing only the physical fluctuations is

$$\mathcal{L}_A = \mathrm{STr}\left[\frac{1}{2}\partial_+\eta^\parallel\partial_-\eta^\parallel + \kappa^2\left(\eta^\parallel\eta^\parallel T^2 - \eta^\parallel T\eta^\parallel T\right)\right] . \tag{C.19}$$

Let us introduce the component fields of $\eta^\parallel$ as

$$\eta^\parallel = \begin{pmatrix} 0 & 0 & a_1 + ia_2 & a_3 + ia_4 \\ 0 & 0 & a_3 - ia_4 & -a_1 + ia_2 \\ a_1 - ia_2 & a_3 + ia_4 & 0 & 0 \\ a_3 - ia_4 & -a_1 - ia_2 & 0 & 0 \end{pmatrix} . \tag{C.20}$$

Then (C.19) becomes

$$\mathcal{L}_A = 2\sum_{i=1}^{4}\left(\partial_+ a_i \partial_- a_i - \kappa^2 a_i^2\right) , \tag{C.21}$$

which describes four bosonic fluctuations with frequency $\sqrt{n^2 + \kappa^2}$.

[3] In the AdS_5 case we shall assume that the field is just in the top left 4×4 matrix of the original (8×8) field and similarly for the S^5 case the field will be just in the bottom right 4×4 matrix.

[4] This is to completely remove the degeneracy of expanding around this vacuum.

Now let us consider the S^5 sector. We introduce the following parametrization of $\eta^{\parallel}$, $\eta^{\perp}$ and $\delta A_{\pm}$,

$$\eta^{\parallel} = \begin{pmatrix} 0 & 0 & b_1 + ib_2 & b_3 + ib_4 \\ 0 & 0 & -b_3 + ib_4 & b_1 - ib_2 \\ -b_1 + ib_2 & b_3 + ib_4 & 0 & 0 \\ -b_3 + ib_4 & -b_1 - ib_2 & 0 & 0 \end{pmatrix}, \tag{C.22}$$

$$\eta^{\perp} = \begin{pmatrix} ih_1 & h_2 + ih_3 & 0 & 0 \\ -h_2 + ih_3 & -ih_1 & 0 & 0 \\ 0 & 0 & ih_4 & h_5 + ih_6 \\ 0 & 0 & -h_5 + ih_6 & -ih_4 \end{pmatrix}, \tag{C.23}$$

$$\begin{aligned} \delta A_+ &= \begin{pmatrix} ia_{+1} & (a_{+2} + ia_{+3})\, v^2 & 0 & 0 \\ -(a_{+2} - ia_{+3})\, v^{*2} & -ia_{+1} & 0 & 0 \\ 0 & 0 & ia_{+4} & (a_{+5} + ia_{+6})\, v^2 \\ 0 & 0 & -(a_{+5} - ia_{+6})\, v^{*2} & -ia_{+4} \end{pmatrix}, \\ \delta A_- &= \begin{pmatrix} ia_{-1} & a_{-2} + ia_{-3} & 0 & 0 \\ -a_{-2} + ia_{-3} & -ia_{-1} & 0 & 0 \\ 0 & 0 & ia_{-4} & a_{-5} + ia_{-6} \\ 0 & 0 & -a_{-5} + ia_{-6} & -ia_{-4} \end{pmatrix}. \end{aligned} \tag{C.24}$$

When we substitute this into the bosonic part of the quadratic fluctuation Lagrangian, (3.4) the fields decouple into two smaller sectors. These are, firstly, a sector containing b_3, b_4 and the diagonal components of $\eta^{\perp}$, $\delta A_{\pm}$, which has a Lagrangian with constant coefficients, and secondly, a sector containing b_1,b_2 and the off-diagonal components of $\eta^{\perp}$, $\delta A_{\pm}$. The coefficients in this sector have some τ dependence, arising from the $\delta A_+ \delta A_-$ term, (v defined in (C.15) depends on τ).

If the gauge field fluctuations are integrated out first, we end up with a Lagrangian that has τ-dependent coefficients. To avoid this complication, i.e. to construct an action containing only physical fluctuations and having constant coefficients we choose the following partial gauge fixing

$$
\begin{aligned}
&h_1 + h_4 = \text{const}\,, \\
&\kappa(a_{-2} - a_{-5}) - \kappa^2(h_3 - h_6) - \partial_-(a_{+3} - a_{+6}) - \kappa\partial_-(h_2 - h_5) = 0\,, \\
&\kappa(a_{-3} + a_{-6}) + \kappa^2(h_2 + h_5) - \partial_-(a_{+2} + a_{+5}) - \kappa\partial_-(h_3 + h_6) = 0\,.
\end{aligned}
\tag{C.25}
$$

Then we can easily integrate out the diagonal components of $\delta A_\pm$ to get a Lagrangian for b_3 and b_4 in the desired form. The second two gauge constraints are chosen to decouple b_1 and b_2 from the unphysical fluctuations. By using the remaining gauge freedom one should be able to ensure that the unphysical fields give only trivial contributions to the partition function.

The resulting Lagrangian for this sector is then

$$
\mathcal{L}_S = 2\Big[\sum_{i=1}^{4}\partial_- b_i\partial_+ b_i + \sum_{i=1}^{2}(2m^2 - \kappa^2)b_i^2 + 4m^2 b_4^2 + 2\kappa(b_4\partial_+ b_3 + b_4\partial_- b_3)\Big]\,. \tag{C.26}
$$

This Lagrangian describes two decoupled fluctuations, b_1, b_2, with frequencies

$$
\sqrt{n^2 + \kappa^2 - 2m^2}\,, \tag{C.27}
$$

and two coupled fluctuations, b_3, b_4, with frequencies

$$
\sqrt{n^2 + 2\kappa^2 - 2m^2 \pm 2\sqrt{n^2\kappa^2 + (m^2 - \kappa^2)^2}}\,. \tag{C.28}
$$

They agree with the string theory result.

C.2.2 Fermionic fluctuations

The fermionic sector is described by

$$
\begin{aligned}
\mathcal{L}_F = \text{STr}\Big(\, &\tfrac{1}{2}\delta\Psi_R\,[T, \partial_-\delta\Psi_R + [A_{0-}, \delta\Psi_R]] \\
&+\tfrac{1}{2}\delta\Psi_L\,[T, \partial_+\delta\Psi_L + [A_{0+}, \delta\Psi_L]] + \kappa g_0^{-1}\delta\Psi_L g_0 \delta\Psi_R\Big)\,.
\end{aligned}
\tag{C.29}
$$

To make coefficients in this Lagrangian constant we may rotate some of the fermionic fields to cancels the contribution of g_0 and g_0^{-1} in the "Yukawa" interaction term. This can be achieved by parameterizing the matrix components of $\delta\Psi_R$ and $\delta\Psi_L$ as follows

$$\delta\Psi_R = \begin{pmatrix} 0 & \mathfrak{X}_R \\ \mathfrak{Y}_R & 0 \end{pmatrix}, \quad \delta\Psi_L = \begin{pmatrix} 0 & \mathfrak{X}_L \\ \mathfrak{Y}_L & 0 \end{pmatrix}, \tag{C.30}$$

where

$$\mathfrak{X}_R = \begin{pmatrix} 0 & 0 & \alpha_1 + i\alpha_2 & \alpha_3 + i\alpha_4 \\ 0 & 0 & -\alpha_3 + i\alpha_4 & \alpha_1 - i\alpha_2 \\ \alpha_5 + i\alpha_6 & \alpha_7 - i\alpha_8 & 0 & 0 \\ \alpha_7 + i\alpha_8 & -\alpha_5 + i\alpha_6 & 0 & 0 \end{pmatrix}, \tag{C.31}$$

$$\mathfrak{Y}_R = \begin{pmatrix} 0 & 0 & -\alpha_6 - i\alpha_5 & -\alpha_8 - i\alpha_7 \\ 0 & 0 & \alpha_8 - i\alpha_7 & -\alpha_6 + i\alpha_5 \\ \alpha_2 + i\alpha_1 & \alpha_4 - i\alpha_3 & 0 & 0 \\ \alpha_4 + i\alpha_3 & -\alpha_2 + i\alpha_1 & 0 & 0 \end{pmatrix}, \tag{C.32}$$

$$\mathfrak{X}_L = \begin{pmatrix} 0 & 0 & (\beta_1 + i\beta_2)V^* & (\beta_3 + i\beta_4)V \\ 0 & 0 & (\beta_3 - i\beta_4)V^* & (-\beta_1 + i\beta_2)V \\ (\beta_5 + i\beta_6)V^* & (-\beta_7 + i\beta_8)V & 0 & 0 \\ (\beta_7 + i\beta_8)V^* & (\beta_5 - i\beta_6)V & 0 & 0 \end{pmatrix}, \tag{C.33}$$

$$\mathfrak{Y}_L = \begin{pmatrix} 0 & 0 & (-\beta_6 - i\beta_5)V & (-\beta_8 - i\beta_7)V \\ 0 & 0 & (-\beta_8 + i\beta_7)V^* & (\beta_6 - i\beta_5)V^* \\ (\beta_2 + i\beta_1)V & (-\beta_4 + i\beta_3)V & 0 & 0 \\ (\beta_4 + i\beta_3)V^* & (\beta_2 - i\beta_1)V^* & 0 & 0 \end{pmatrix}. \tag{C.34}$$

Here α_k and β_k are 8+8 real anticommuting functions and V is defined in (C.15). The Lagrangian (C.29) then takes the form

$$\begin{aligned}\mathcal{L}_F = 2\Big[& \textstyle\sum_{i=1}^{8} (\alpha_i \partial_- \alpha_i + \beta_i \partial_+ \beta_i) \\ & + \sqrt{\kappa^2 + m^2}\, (-\alpha_1\alpha_2 + \alpha_3\alpha_4 - \alpha_5\alpha_6 - \alpha_7\alpha_9 + \beta_1\beta_2 - \beta_3\beta_4 + \beta_5\beta_6 + \beta_7\beta_8) \\ & + \sqrt{\kappa^2 - m^2}\, (\alpha_1\beta_3 + \alpha_3\beta_1 - \alpha_5\beta_7 - \alpha_7\beta_5 - \beta_2\alpha_4 + \beta_4\alpha_2 + \beta_6\alpha_8 - \beta_8\alpha_6) \Big] ,\end{aligned} \tag{C.35}$$

which describes 8 fermionic fluctuations with 4+4 sets of the frequencies,

$$\sqrt{n^2 - m^2 + \frac{5\kappa^2}{4} \pm \sqrt{\kappa^4 + n^2\kappa^2 - m^2\kappa^2}}\,. \tag{C.36}$$

The characteristic frequencies found above directly from the reduced theory action are exactly the same as found [92, 103] from the $AdS_5 \times S^5$ string theory action expanded near the solution (C.12).[5]

We conclude that expanding the superstring action near the homogeneous 2-spin solution in $R \times S^3$ and expanding the reduced theory action near its counterpart in the reduced theory one finds the same set of characteristic frequencies and thus the same one-loop contribution to the respective partition functions.

C.3 Short two-spin string in $R \times S^5$

Let us consider the two-spin "short-string" solution in S^5 discussed in [101, 113, 114]. In the conformal-gauge string theory, the solution takes the form,

$$\begin{aligned} Y_0 + iY_{-1} = e^{i\kappa\tau}\,, \qquad & Y_1 = Y_2 = Y_3 = Y_4 = 0\,, \\ X_1 + iX_2 = \sin\gamma_0 e^{i(\tau+\sigma)}\,, \qquad X_3 + iX_4 = \sin\gamma_0 e^{i(\tau-\sigma)}\,. \qquad & X_5 + iX_6 = \cos\gamma_0\,, \end{aligned} \tag{C.37}$$

where the Virasoro constraints relate the two parameters as $\kappa^2 = 2\sin\gamma_0^2$. Setting $\gamma_0 = \pi/2$ is allowed if $\kappa^2 \geq 2$, in which the solution reduces to a special case of a two-spin solution in $R \times S^3$ discussed in [101, 113, 92, 102] in the original string theory and in [HIT] (appendix C.2) in the reduced theory. On the other hand, the string with $\kappa^2 < 2$ lies in the S^5 and this is the case we will discuss hereafter.

[5] Starting with the string solution in the form (C.12) used in [92] one finds that the fermions are naturally periodic [102].

Using the parameterizations discussed in appendix B, we obtain the corresponding bosonic coset element f,

$$f = \begin{pmatrix} f_A & \mathbf{0} \\ \mathbf{0} & f_S \end{pmatrix}, \qquad f_A = \begin{pmatrix} e^{\frac{i\kappa\tau}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{i\kappa\tau}{2}} & 0 & 0 \\ 0 & 0 & e^{-\frac{i\kappa\tau}{2}} & 0 \\ 0 & 0 & 0 & e^{-\frac{i\kappa\tau}{2}} \end{pmatrix},$$
$$f_S = \frac{1}{\sqrt{5-3\cos\gamma_0}} \begin{pmatrix} \sqrt{1+\cos\gamma_0} & 0 & ie^{-i(\tau-\sigma)}\sin\frac{\gamma_0}{2} & -ie^{-i(\tau+\sigma)}\sin\frac{\gamma_0}{2} \\ 0 & \sqrt{1+\cos\gamma_0} & ie^{i(\tau+\sigma)}\sin\frac{\gamma_0}{2} & ie^{i(\tau-\sigma)}\sin\frac{\gamma_0}{2} \\ ie^{i(\tau-\sigma)}\sin\frac{\gamma_0}{2} & ie^{-i(\tau+\sigma)}\sin\frac{\gamma_0}{2} & \sqrt{1+\cos\gamma_0} & 0 \\ -ie^{i(\tau+\sigma)}\sin\frac{\gamma_0}{2} & ie^{-i(\tau-\sigma)}\sin\frac{\gamma_0}{2} & 0 & \sqrt{1+\cos\gamma_0} \end{pmatrix}. \tag{C.38}$$

After fixing the $H \times H$ gauge symmetry such that the classical gauge fields solve the gauge equations, we find that the corresponding solution of the reduced theory is

$$g_0 = \begin{pmatrix} g_A & \mathbf{0}_4 \\ \mathbf{0}_4 & g_S \end{pmatrix}, \qquad g_A = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix},$$
$$g_S = \frac{1}{B} \begin{pmatrix} e^{\frac{iB_2\sigma}{2}} & e^{\frac{iB_2\tau}{2}}\cos\gamma_0 & -ie^{\frac{iB_2\sigma}{2}}\cos\gamma_0 & -ie^{\frac{iB_2\tau}{2}} \\ -e^{-\frac{iB_2\tau}{2}}\cos\gamma_0 & e^{-\frac{iB_2\sigma}{2}} & -ie^{-\frac{iB_2\tau}{2}} & ie^{-\frac{iB_2\sigma}{2}}\cos\gamma_0 \\ -e^{\frac{iB_2\sigma}{2}}\cos\gamma_0 & e^{\frac{iB_2\tau}{2}} & ie^{\frac{iB_2\sigma}{2}} & -ie^{\frac{iB_2\tau}{2}}\cos\gamma_0 \\ -e^{-\frac{iB_2\tau}{2}} & -e^{-\frac{iB_2\sigma}{2}}\cos\gamma_0 & -ie^{-\frac{iB_2\tau}{2}}\cos\gamma_0 & -ie^{-\frac{iB_2\sigma}{2}} \end{pmatrix}.$$

where we defined the constant B_n by

$$B_n = \sqrt{n + \cos 2\gamma_0}\,. \tag{C.39}$$

There are several ways of partially fixing the $H \times H$ gauge. The advantage in taking the present choice is that classical gauge fields vanishes,

$$A_{0-} = A_{0+} = 0\,, \tag{C.40}$$

It is obvious that the fermionic fields are zero directly from the vanishing classical fermions in the original string theory,

$$\Psi_{R0} = \Psi_{L0} = 0\,. \tag{C.41}$$

For the reduced-theory solution (C.3), it is not possible to decouple the physical fields from the unphysical fields by imposing a gauge condition on the fluctuation fields. Hence, unlike the previous two cases, we shall use the PW identity to eliminate the degrees of freedom of the gauge fields $A_\pm$. From the vanishing gauge fields (C.40), it turns out that the transformation $A_{0\pm} \to U, \tilde{U}$ in (4.16) is trivial, $U = \tilde{U} = \mathbf{1}_{8\times 8}$, which means that the classical background g does not change,

$$\tilde{g}_0 = g_0\,. \tag{C.42}$$

Finding that this is a good starting point of perturbation in the Lagrangian (4.19) in the sense that $\tilde{g}_0$ solves the equation of motion derived from the classical Lagrangian, then we shall consider the Lagrangian for quadratic fluctuations (4.23).

C.3.1 Bosonic fluctuations

It is a general property of the reduced theory that the bosonic sector decouples into two subsectors, the AdS_5 sector and the S^5 sector at one-loop level.

Let us define component fields of the bosonic fluctuation η in the following way,

$$\eta^{\|} = \begin{pmatrix} \eta^{\|}_A & 0 \\ 0 & \eta^{\|}_S \end{pmatrix}, \qquad \eta^{\|}_A = \begin{pmatrix} 0 & 0 & a_1 + ia_2 & a_3 + ia_4 \\ 0 & 0 & a_3 - ia_4 & -a_1 + ia_2 \\ a_1 - ia_2 & a_3 + ia_4 & 0 & 0 \\ a_3 - ia_4 & -a_1 - ia_2 & 0 & 0 \end{pmatrix},$$
$$\eta^{\|}_S = \begin{pmatrix} 0 & 0 & b_1 + ib_2 & (b_3 + ib_4)e^{iB_3\frac{\tau-\sigma}{2}} \\ 0 & 0 & (-b_3 + ib_4)e^{-iB_3\frac{\tau-\sigma}{2}} & b_1 - ib_2 \\ -b_1 + ib_2 & (b_3 + ib_4)e^{iB_3\frac{\tau-\sigma}{2}} & 0 & 0 \\ (-b_3 + ib_4)e^{-iB_3\frac{\tau-\sigma}{2}} & -b_1 - ib_2 & 0 & 0 \end{pmatrix} \tag{C.43}$$

which correspond to physical fields in $\mathfrak{m}$ coming from the coset part G/H, and

$$\eta^{\perp} = \begin{pmatrix} \eta^{\perp}_{A} & 0 \\ 0 & \eta^{\perp}_{S} \end{pmatrix} , \qquad \eta^{\perp}_{A} = \begin{pmatrix} ic_1 & c_2 + ic_3 & 0 & 0 \\ -c_2 + ic_3 & -ic_1 & 0 & 0 \\ 0 & 0 & ic_4 & c_5 + ic_6 \\ 0 & 0 & -c_5 + ic_6 & -ic_4 \end{pmatrix} ,$$

$$\eta^{\perp}_{S} = \begin{pmatrix} id_1 & (d_2 + id_3) e^{iB_3 \frac{\tau-\sigma}{2}} & 0 & 0 \\ (-d_2 + id_3) e^{-iB_3 \frac{\tau-\sigma}{2}} & -id_1 & 0 & 0 \\ 0 & 0 & id_4 & (d_5 + id_6) e^{iB_3 \frac{\tau-\sigma}{2}} \\ 0 & 0 & (-d_5 + id_6) e^{-iB_3 \frac{\tau-\sigma}{2}} & -id_4 \end{pmatrix} , \tag{C.44}$$

where we introduced the rotations such that the fluctuation Lagrangian become constant. The constant B_3 is defined in (C.39).

By substituting the solution (C.42), i.e., (C.3) and the fluctuation fields (C.43), (C.44) into the Lagrangian (4.23), we obtain the Lagrangian for quadratic fluctuations expressed in terms of the component fields. As the AdS_5 sector of the classical solution is the vacuum, the Lagrangian for the bosonic fluctuations in the AdS_5 sector takes the simple form,

$$\mathcal{L} = 2 \sum_{i=1}^{4} \left(\partial_+ c_i \partial_- c_i - \kappa^2 c_i^2 \right) - \sum_{j=1}^{6} \partial_+ d_j \partial_- d_j \, , \tag{C.45}$$

from which we find characteristic fluctuations of 4 massive fluctuations c_i,

$$\sqrt{n^2 + \kappa^2} \, . \tag{C.46}$$

This agrees with the string theory result in [101, 113, 114]. The unphysical fluctuations d_j are totally massless.

The S^5 sector exhibits the involved structure reflecting the fact that all the components in the classical solution (C.42) (and (C.3)) are nonvanishing. The resulting Lagrangian contains four physical fluctuations b_i and six unphysical fluctuations d_j which complicatedly couple with each other. Two of the physical fluctuations are massive with frequencies,

$$\sqrt{4 + n^2 - \kappa^2 \pm \sqrt{16n^2 - 4n^2\kappa^2 + \kappa^4}}\,, \tag{C.47}$$

and the other two physical fluctuations are massless, in agreement with the string the result [101, 113, 114]. Together with the 6 unphysical fluctuations the massless frequencies receive shift due to the nontrivial rotation in (C.43) and (C.44),

$$\begin{aligned} 4 \quad &\times \quad n\,, \\ 2 \quad &\times \quad n \pm \sqrt{4 - \kappa^2}\,. \end{aligned} \tag{C.48}$$

Note that we used the relation $\kappa^2 = 2\sin\gamma_0^2$ in order to eliminate γ_0.

C.3.2 Fermionic fluctuations

Fermionic sector

As done in the case of the homogeneous solution in S^3, in order to achieve the fluctuated action with constant coefficients, we rescale $\delta\Psi_{_L}$ only,

$$\delta\Psi_{_R} = \begin{pmatrix} 0 & \mathfrak{X}_R \\ \mathfrak{Y}_R & 0 \end{pmatrix}, \quad \delta\Psi_{_L} = \begin{pmatrix} 0 & \mathfrak{X}_L \\ \mathfrak{Y}_L & 0 \end{pmatrix}, \tag{C.49}$$

where

$$\mathfrak{X}_R = \begin{pmatrix} 0 & 0 & (\alpha_1 + i\alpha_2)v_+^* & (\alpha_3 + i\alpha_4)v_+ \\ 0 & 0 & (-\alpha_3 + i\alpha_4)v_+^* & (\alpha_1 - i\alpha_2)v_+ \\ (\alpha_5 + i\alpha_6)v_+^* & (\alpha_7 - i\alpha_8)v_+ & 0 & 0 \\ (\alpha_7 + i\alpha_8)v_+^* & (-\alpha_5 + i\alpha_6)v_+ & 0 & 0 \end{pmatrix}, \tag{C.50}$$

$$\mathfrak{Y}_R = \begin{pmatrix} 0 & 0 & (-\alpha_6 - i\alpha_5)v_+ & (-\alpha_8 - i\alpha_7)v_+ \\ 0 & 0 & (\alpha_8 - i\alpha_7)v_+^* & (-\alpha_6 + i\alpha_5)v_+^* \\ (\alpha_2 + i\alpha_1)v_+ & (\alpha_4 - i\alpha_3)v_+ & 0 & 0 \\ (\alpha_4 + i\alpha_3)v_+^* & (-\alpha_2 + i\alpha_1)v_+^* & 0 & 0 \end{pmatrix}, \tag{C.51}$$

and

$$\mathfrak{X}_L = \begin{pmatrix} 0 & 0 & (\beta_1 + i\beta_2)v_-^* & (\beta_3 + i\beta_4)v_- \\ 0 & 0 & (\beta_3 - i\beta_4)v_-^* & (-\beta_1 + i\beta_2)v_- \\ (\beta_5 + i\beta_6)v_-^* & (-\beta_7 + i\beta_8)v_- & 0 & 0 \\ (\beta_7 + i\beta_8)v_-^* & (\beta_5 - i\beta_6)v_- & 0 & 0 \end{pmatrix}, \quad \text{(C.52)}$$

$$\mathfrak{Y}_L = \begin{pmatrix} 0 & 0 & (-\beta_6 - i\beta_5)v_- & (-\beta_8 - i\beta_7)v_- \\ 0 & 0 & (-\beta_8 + i\beta_7)v_-^* & (\beta_6 - i\beta_5)v_-^* \\ (\beta_2 + i\beta_1)v_- & (-\beta_4 + i\beta_3)v_- & 0 & 0 \\ (\beta_4 + i\beta_3)v_-^* & (\beta_2 - i\beta_1)v_-^* & 0 & 0 \end{pmatrix}, \quad \text{(C.53)}$$

$$v_\pm = \frac{i}{4}\sqrt{3 + 2\cos 2\gamma_0}\,(\tau \pm \sigma)\ , \quad \text{(C.54)}$$

where α_k and β_k are 8+8 real anticommuting functions. Plugging $\delta\Psi_{_R}$ and $\delta\Psi_{_L}$ into the fermionic part of (4.23), we obtain the Lagrangian for quadratic fermionic fluctuations. The Lagrangian again contains constant coefficients only. Characteristic frequencies derived from the resulting Lagrangian are

$$\sqrt{n^2 + 1 + \frac{\kappa^2}{4} \pm \sqrt{(4 - \kappa^2)n^2 + \kappa^2}}\,, \quad \text{(C.55)}$$

where we used the constraint $\kappa^2 = 2\sin\gamma_0^2$. These frequencies again agree with the result in [101, 113, 114].

Appendix D

Details of computation in chapter 4

D.1 Comments on one-loop computation in section 4.4

The quadratic fluctuation Lagrangian in section 4.4.1 looks different from the corresponding one in $AdS_5 \times S^5$ string theory but these two Lagrangians lead to equivalent sets of characteristic frequencies and the one-loop determinants. Here we shall comment on the structure of subsectors of the bosonic fluctuation Lagrangian (4.118). Let us start with $\mathcal{L}_1^{(2)}$ in (4.118) containing a_1 and a_2. Integrating out c_2, c_3, c_5 and c_6 gives[1]

$$\tilde{\mathcal{L}}_1^{(2)} = 2 \sum_{i=1,2} \left[\partial_+ a_i \partial_- a_i - \left(2\kappa^2 - \mu^2\right) a_i^2 + 4M_1^2 a_i \frac{\partial_+\partial_- + 2\kappa^2 - \mu^2}{\partial_+^2 + M_2^2} a_i \right] . \tag{D.1}$$

This looks different from the fluctuation Lagrangian found from the corresponding string action,

$$\mathcal{L}_1 = 2 \sum_{i=1,2} \left[\partial_+ a_i \partial_- a_i - \left(2\kappa^2 - \mu^2\right) a_i^2 \right] , \tag{D.2}$$

but the two Lagrangians are closely related as one can factorize the operator $\partial_+\partial_- + 2\kappa^2 - \mu^2$ in (D.1).

The Lagrangians $\mathcal{L}_1^{(2)}$ in (4.118) and $\mathcal{L}_1$ in (D.2) are, in fact, related by a nonlocal transformation. To

[1]The resulting determinant of the operator $\mathcal{O} = \partial_+^2 + M_2^2$ is equivalent to the (square of) massless operator determinant.

see this, it is useful to perform the following $O(2)$ rotations,

$$c_2 \to \tfrac{1}{\sqrt{2}}\left(c_2 - c_6\right) , \quad c_3 \to \tfrac{1}{\sqrt{2}}\left(c_3 + c_5\right) , \quad c_5 \to \tfrac{1}{\sqrt{2}}\left(c_2 + c_6\right) , \quad c_6 \to \tfrac{1}{\sqrt{2}}\left(c_3 - c_5\right) . \tag{D.3}$$

Then $\mathcal{L}_1^{(2)}$ splits into smaller subsectors. One contains a_1, c_2 and c_3,

$$\begin{aligned}\mathcal{L}_{a_1}^{(2)} &= 2\big[\partial_+ a_1 \partial_- a_1 - \left(2\kappa^2 - \mu^2\right) a_1^2\big] - 4M_1\left(\mu c_2 + \partial_- c_3\right) a_1 \\ &\quad - \sum_{j=2,3}\big[\partial_+ c_j \partial_- c_j + (2\kappa^2 - \mu^2) c_j^2\big] - 2\left(\mu \partial_+ c_3 + M_2 \partial_- c_3\right) c_2 .\end{aligned} \tag{D.4}$$

Another contains a_2, c_5 and c_6 with a similar Lagrangian. To decouple a_1 from c_2, c_3 we may apply the nonlocal transformation

$$\begin{aligned} a_1 &\to a_1 - \frac{\sqrt{2}\kappa\sqrt{\kappa^2-\mu^2}}{\partial_+\partial_- + 2\kappa^2 - \mu^2} c_2 - \frac{\sqrt{2}\kappa\sqrt{\kappa^2-\mu^2}\partial_-(\partial_-^2+\mu^2)}{\mu(\partial_+^2\partial_-^2 - \mu^4)} c_3 , \\ c_2 &\to c_2 + \frac{\partial_+\mu^2\left(\partial_+\partial_- + 2\kappa^2 - \mu^2\right) + \partial_-\left(\left(2\kappa^2-\mu^2\right)\partial_+\partial_- + \mu^4\right)}{\mu\left(\partial_+^2\partial_-^2 - \mu^4\right)} c_3 , \end{aligned} \tag{D.5}$$

leading to

$$\begin{aligned}\mathcal{L}_{a_1}^{(2)} &= 2\big[\partial_+ a_1 \partial_- a_1 - \left(2\kappa^2 - \mu^2\right) a_1^2\big] \\ &\quad + 2c_2 \frac{\partial_+^2\partial_-^2 - \mu^4}{\partial_+\partial_- + 2\kappa^2 - \mu^2} c_2 + c_3 \frac{\left(\partial_+\partial_- + 2\kappa^2 - \mu^2\right)\left(\partial_-^2 + \mu^2\right)\left(\partial_+^2 + \mu^2\right)}{\partial_+^2\partial_-^2 - \mu^4} c_3 .\end{aligned} \tag{D.6}$$

The physical part of this Lagrangian is the same as (D.2). The product of determinants resulting from integrating out c_2 and c_3 contains only trivial massless factors. The same is true in the a_2, c_5, c_6 sector.

Similar observations apply in the sector containing a_3 and a_4 described by the Lagrangian $\mathcal{L}_2^{(2)}$ in (4.118). Integrating out c_3, c_4 directly leads to

$$\tilde{\mathcal{L}}_2^{(2)} = 2\sum_{i=3,4} \partial_+ a_i \partial_- a_i + 4\left(\mu\partial_+ a_4 + M_2 \partial_- a_4\right) a_3 + 8M_1^2 a_3 \frac{\partial_-}{\partial_+} a_3 , \tag{D.7}$$

which looks different from the string theory counterpart,

$$\mathcal{L}_2 = 2\sum_{i=3,4} \partial_+ a_i \partial_- a_i + 4\left(\kappa^2 - \mu^2\right) a_3^2 + 4\mu\left(\partial_+ a_3 + \partial_- a_3\right) a_4 . \tag{D.8}$$

To find a transformation between $\mathcal{L}_2^{(2)}$ and (D.8) let us apply an $O(2)$ rotation

$$c_1 \to \tfrac{1}{\sqrt{2}}\left(c_1 + c_4\right)\,, \qquad c_4 \to \tfrac{1}{\sqrt{2}}\left(c_1 - c_4\right)\,, \tag{D.9}$$

and the following redefinition

$$c_1 \to c_1 + 2\sqrt{2}\tfrac{\kappa\sqrt{\kappa^2-\mu^2}}{\mu}\tfrac{1}{\partial_+}a_3\,, \qquad a_4 \to -a_4 - 2\tfrac{\kappa^2-\mu^2}{\mu}\tfrac{1}{\partial_+}a_3\,. \tag{D.10}$$

Then we get

$$\mathcal{L}^{(2)}_{a_3,a_4} = 2\sum_{i=3,4}\partial_+ a_i \partial_- a_i + 4\left(\kappa^2 - \mu^2\right)a_3^2 + 4\mu\left(\partial_+ a_3 + \partial_- a_3\right)a_4 - \sum_{j=1,4}\partial_+ c_j \partial_- c_j\,, \tag{D.11}$$

where the physical part is the same as in (D.8).

D.2 One-loop computation in $\delta A_+ = 0$ gauge

In this appendix we will perform the one-loop computation in the $\delta A_+ = 0$ gauge. This gauge was used in [73, 74, 75] to study S-matrices in the reduced theory by perturbing the deformed gWZW model around the BMN vacuum. A great advantage in the $\delta A_+ = 0$ gauge is that varying the fluctuation Lagrangian by δA_- yields an equation for $\eta^\perp$ (gauge equation) leading to the Lagrangian which involves only physical degrees of freedom once the gauge equation is solved. Although the solution of the gauge equation is expressed as a nonlocal function of the physical fields, the nonlocality disappears from the resulting Lagrangian in the case of the BMN vacuum. This makes it drastically easier to deal with the cubic and quartic terms. For the (S, J) folded string, however, we can not eliminate the nonlocality from the fluctuation Lagrangian, and the structure of the functional determinant contribution of the gauge equation for $\eta^\perp$ is complicated at higher orders. Hence it is still a challenging problem to carry out the two-loop computation in the $\delta A_+ = 0$ gauge.

Recall the classical solution in the reduced theory in (4.109) and (4.110). One can also set the coefficients of the fluctuation Lagrangian constant at one-loop level in the $\delta A_+ = 0$ gauge once we introduce fluctuation fields appropriately.

Consider fluctuations around a classical solution, g_0, $A_{0\pm}$ with vanishing fermions, as follows

$$
\begin{aligned}
&g = g_0 e^{\eta} = g_0(1 + \eta + \tfrac{1}{2}\eta^2 + \mathcal{O}(\eta^3))\,, \quad \eta \in \mathfrak{f}_0 \\
&A_+ = A_{+0} + \delta A_+\,, \qquad A_- = A_{-0} + \delta A_-\,,
\end{aligned}
\tag{D.12}
$$

and fermionic fluctuations Ψ_R, Ψ_L. The gauge condition we impose is $\delta A_+ = 0$. Then the quadratic fluctuations are described by the Lagrangian,

$$
\begin{aligned}
\mathcal{L}^{(2)} = \mathrm{STr}\Big[&\tfrac{1}{2}\mathscr{D}_+\eta\partial_-\eta - \mathscr{D}_+\eta\delta A_- + \tfrac{1}{2}\left[\eta, \mathscr{D}_+\eta\right] A_{-0} - \tfrac{\mu^2}{2}\left[\eta, g_0^{-1}Tg_0\right]\left[\eta, T\right] \\
&+\tfrac{1}{2}\Psi_R\left[T, \partial_-\Psi_R + \left[A_{0-}, \Psi_R\right]\right] + \tfrac{1}{2}\Psi_L\left[T, \partial_+\Psi_L + \left[A_{0+}, \Psi_L\right]\right] + \mu g_0^{-1}\Psi_L g_0\Psi_R\Big]\,,
\end{aligned}
\tag{D.13}
$$

where $\mathscr{D}_+ \; = \partial_+ \; + \left[g_0^{-1}\partial_+ g_0 + g_0^{-1}A_+ g_0, \; \right]$.

The Lagrangian (D.13) shows that the bosonic fluctuations and fermionic fluctuations are decoupled. First we will discuss the bosonic sector. By solving the gauge equation for the unphysical fields described above, we will derive the fluctuation Lagrangian containing only the physical fluctuations, and determine the functional determinant contributions to the partition function. Then we will derive the fluctuation Lagrangian and evaluate the one-loop contribution in the fermionic sector. Combining the results of these two sectors, we will show that the result obtained by using the PW identity is reproduced in the $\delta A_+ = 0$ gauge. Finally a brief comment on the two-loop computation in the $\delta A_+ = 0$ gauge will be given.

Bosonic fluctuations

Variation of the fluctuation Lagrangian by δA_- yields an equation for $\eta^{\perp}$ and its solution is a nonlocal function of $\eta^{\parallel}$. This implies that nonlocal terms appear in the resulting fluctuation Lagrangian, which is also observed in appendix D.1. In fact, the bosonic Lagrangians we will obtain here are the same as those found in appendix D.1.

Since (4.109) is the parameterization such that $g_0^{-1}\partial_+ g_0$ and $g_0^{-1}Tg_0$ are constants, any rotation for bosonic fluctuations is not needed. Then we will introduce the component fields of the bosonic fluctuations,

$$
\eta^{\parallel} = \begin{pmatrix} \eta^{\parallel}_A & 0 \\ 0 & \eta^{\parallel}_S \end{pmatrix},
$$

$$
\eta^{\parallel}_A = \begin{pmatrix} 0 & 0 & a_1 + ia_2 & a_3 + ia_4 \\ 0 & 0 & a_3 - ia_4 & -a_1 + ia_2 \\ a_1 - ia_2 & a_3 + ia_4 & 0 & 0 \\ a_3 - ia_4 & 0 & 0 & \end{pmatrix}, \tag{D.14}
$$

$$
\eta^{\parallel}_S = \begin{pmatrix} 0 & 0 & b_1 + ib_2 & b_3 + ib_4 \\ 0 & 0 & -b_3 + ib_4 & b_1 - ib_2 \\ -b_1 + ib_2 & b_3 + ib_4 & 0 & 0 \\ -b_3 + ib_4 & -b_1 - ib_2 & 0 & 0 \end{pmatrix},
$$

which correspond to physical fields in the reduced theory, and

$$
\eta^{\perp} = \begin{pmatrix} \eta^{\perp}_A & 0 \\ 0 & \eta^{\perp}_S \end{pmatrix},
$$

$$
\eta^{\perp}_A = \begin{pmatrix} ic_1 & c_2 + ic_3 & 0 & 0 \\ -c_2 + ic_3 & -ic_1 & 0 & 0 \\ 0 & 0 & ic_4 & c_5 + ic_6 \\ 0 & 0 & -c_5 + ic_6 & -ic_4 \end{pmatrix}, \tag{D.15}
$$

$$
\eta^{\perp}_S = \begin{pmatrix} id_1 & d_2 + id_3 & 0 & 0 \\ -d_2 + id_3 & -id_1 & 0 & 0 \\ 0 & 0 & id_4 & d_5 + id_6 \\ 0 & 0 & -d_5 + id_6 & -id_4 \end{pmatrix},
$$

which are unphysical fields in the reduced theory, and will be eliminated from the fluctuation Lagrangian by solving the equation $\frac{\delta \mathcal{L}^{(2)}}{\delta \delta A_-} = 0$.

As shown in other approaches in this thesis, the AdS_5 sector and the S^5 sector are decoupled at one-loop level. As the S^5 part of the classical solution is the vacuum, the Lagrangian for the quadratic fluctuations in the S^5 sector takes the simple form,

$$\mathcal{L}^{(2)}_{S^5} = \mathrm{STr}\left[\frac{1}{2}\partial_+\eta_A\partial_-\eta_A - \partial_+\eta_A\delta A_- - \frac{\mu^2}{2}\left([\eta_A, T]\right)^2\right]. \tag{D.16}$$

We can integrate out the unphysical fields by solving the equation, which is derived by varying this Lagrangian by δA_-,

$$\partial_+\eta_S^\perp = 0\,, \tag{D.17}$$

which eliminates the kinetic term for the unphysical fields. Recalling $\left[\eta^\perp, T\right] = 0$, then we find that the potential term does not contain the unphysical fields. Hence the resulting Lagrangian contains only the physical fluctuations $\eta_A^\perp$,

$$\mathcal{L}^{(2)}_{S^5} = \mathrm{STr}\left[\frac{1}{2}\partial_+\eta_A^\parallel\partial_-\eta_A^\parallel - \frac{\mu^2}{2}\left(\left[\eta_A^\parallel, T\right]\right)^2\right]. \tag{D.18}$$

In terms of the component fields introduced in (D.14), the Lagrangian is written as

$$\mathcal{L}^{(2)}_{S^5} = 2\sum_{i=1}^{4}\left(\partial_+ b_i\partial_- b_i - \mu^2 b_i^2\right)\,. \tag{D.19}$$

This shows that the four physical fields are massive fluctuations with characteristic frequencies,

$$\sqrt{n^2+\mu^2}\,, \tag{D.20}$$

which agrees with the string theory result [69].

Let us discuss the AdS_5 sector. First we shall integrate out the unphysical fields. Varying the fluctuation Lagrangian (D.13) by δA_- yields the following equation,

$$\left(\mathscr{D}_+\eta\right)^\perp = 0\,. \tag{D.21}$$

This equation can be solved once we rewrite it in terms of the component fields, (D.14) and (D.15),

$$\begin{aligned}
&\partial_+ c_1 - 2M_1 a_3 = 0\,, \quad \partial_+ c_2 - 2M_1 a_2 - M_2 c_3 = 0\,, \quad \partial_+ c_3 + 2M_1 a_1 + M_2 c_2 = 0\,,\\
&\partial_+ c_4 - 2M_1 a_3 = 0\,, \quad \partial_+ c_5 + 2M_1 a_2 - M_2 c_6 = 0\,, \quad \partial_+ c_6 + 2M_1 a_1 + M_2 c_5 = 0\,,
\end{aligned} \tag{D.22}$$

where M_1 and M_2 are constants,

$$M_1 = \frac{\kappa\sqrt{\kappa^2-\mu^2}}{\mu}\,, \quad M_2 = \frac{2\kappa^2-\mu^2}{\mu}\,. \tag{D.23}$$

The solution is

$$\begin{gathered} c_1 = c_4 = 2M_1 \tfrac{1}{\partial_+} a_3\,, \quad c_2 = \tfrac{2M_1}{M_2^2+\partial_+^2}\left(\partial_+ a_2 - M_2 a_1\right)\,, \quad c_3 = \tfrac{2M_1}{M_2^2+\partial_+^2}\left(-\partial_+ a_1 - M_2 a_2\right)\,, \\ c_5 = \tfrac{2M_1}{M_2^2+\partial_+^2}\left(-\partial_+ a_1 - M_2 a_2\right)\,, \quad c_6 = \tfrac{2M_1}{M_2^2+\partial_+^2}\left(-\partial_+ a_1 + M_2 a_2\right)\,. \end{gathered} \tag{D.24}$$

After solving the gauge equation (D.21) the Lagrangian takes the following form,

$$\begin{aligned} \mathcal{L}^{(2)}_{AdS_5} = \mathrm{STr}\Big[& \tfrac{1}{2}\left(\partial_+\eta^{\|} + [A_+, \eta^{\|}]\right)\left(\partial_-\eta^{\|} + [A_-, \eta^{\|}]\right) - \tfrac{\mu^2}{2}\left[\eta^{\|}, \left(g_0^{-1} T g_0\right)^{\perp}\right][\eta^{\|}, T] \\ & -\tfrac{1}{2}\left[\mathcal{A}_+^{\|}, \partial_-\eta^{\|} + [A_-, \eta^{\|}]\right]\eta^{\perp} + \tfrac{\mu^2}{2}\left[[\eta^{\|}, T]\,, \left(g_0^{-1} T g_0\right)^{\|}\right]\eta^{\perp}\Big]\,, \end{aligned} \tag{D.25}$$

where $\mathcal{A}_+ = g_0^{-1}\partial_+ g_0 + g_0^{-1} A_{+0} g_0$. For the choice of g_0 and A_{+0} in (4.109), $\mathcal{A}_+$ has nonzero components in both the $\mathfrak{m}$ and $\mathfrak{h}$ spaces, and $\mathcal{A}_+^{\perp}$ is equal to A_{+0} by definition.

Plugging (D.14) and (D.15) with (D.24) into the Lagrangian (D.25), then we get the fluctuation Lagrangian expressed in terms of the physical component fields only. The fluctuation fields decouple into two smaller subsectors. The first subsector, containing a_1 and a_2, is described by the Lagrangian,

$$\mathcal{L}_1 = 2\sum_{i=1,2}\left[\partial_+ a_i \partial_- a_i - (2\kappa^2-\mu^2)a_i^2 + 4M_1^2 a_i \frac{\partial_+\partial_- + (2\kappa^2-\mu^2)}{M_2^2+\partial_+^2} a_i\right]\,, \tag{D.26}$$

where M_1 and M_2 are defined in (D.23). Due to the existence of the nonlocal terms, this Lagrangian is different form the fluctuation Lagrangian found by perturbing the Nambu action [IWA, 97],

$$\mathcal{L}_1 = 2\sum_{i=1,2}\left[\partial_+ a_i \partial_- a_i - (2\kappa^2-\mu^2)a_i^2\right]\,. \tag{D.27}$$

However, the Lagrangian (D.26) yields the correct fluctuation frequencies, $\sqrt{n^2+2\kappa^2-\mu^2}$, which agree with the string theory result [69]. The reason for this result is as follows. The present Lagrangian (D.26) is rewritten as

$$\mathcal{L}_1 = -2\sum_{i=1,2} a_i \mathcal{O}_1 \mathcal{O}_2 a_i\,, \quad \mathcal{O}_1 = \partial_+\partial_- + (2\kappa^2-\mu^2)\,, \quad \mathcal{O}_2 = 1 - \frac{4M_1^2}{M_2^2+\partial_+^2}\,. \tag{D.28}$$

We have $\det(\mathcal{O}_1\mathcal{O}_2) = \det\mathcal{O}_1 \det\mathcal{O}_2$ and $\mathcal{O}_1$ is the operator found in the string theory result (D.27). In the present case $\det\mathcal{O}_2$ gives a trivial contribution, then the frequencies derived from the Lagrangian (D.26) are the same as the frequencies obtained by solving $\det\mathcal{O}_1 = 0$. Note also that the $\mu \to 0$ limit where the classical string is not stretching in S^5 is well-defined only in the Lagrangian (D.27).

The other subsector contains a_3 and a_4 and is described by the Lagrangian,

$$\mathcal{L}_2 = 2\left[\sum_{i=3,4}\partial_+ a_i\partial_- a_i + 4M_1^2 a_3\frac{\partial_-}{\partial_+}a_3 + 2a_3\left(M_2\partial_- + \mu\partial_+\right)a_4\right]. \tag{D.29}$$

Again this Lagrangian is different from the Lagrangian found in [IWA, 97],

$$\mathcal{L}_2 = 2\sum_{i=3,4}\partial_+ a_i\partial_- a_i - 4\left(\kappa^2-\mu^2\right)a_3^2 + 4\mu\left(\partial_+ a_3 + \partial_- a_3\right)a_4\,. \tag{D.30}$$

These two Lagrangian have the same foundational determinant and then give the same set of characteristic frequencies,

$$\sqrt{n^2 + 2\kappa^2 \pm 2\sqrt{\kappa^4 + n^2\mu^2}}\,, \tag{D.31}$$

At the level of the Lagrangian, the two Lagrangians (D.29) and (D.30) are related by a nonlocal transformation; by replacing $a_4 \to a_4 - 2\frac{\kappa^2-\mu^2}{\mu}\frac{1}{\partial_+}a_3$ in (D.29) we obtain the Lagrangian (D.30).

Let us evaluate the functional determinant in the bosonic sector. From the Lagrangian (D.26) and (D.29) we find that the functional determinant contribution of the physical fluctuations. Here we should also take into account the functional determinant contributions from the equations (D.17), (D.22), and the nonvanishing contributions arising from fixing the H gauge, $\delta A_+ = 0$. Then the total functional determinant in the bosonic sector is given by

$$\begin{aligned}&\left(\left[\det\left(\partial_+^2 + M_2^2\right)\right]^2\right)^{1/2}\left(\left[\det\left(\partial_+\partial_- + 2\kappa^2 - \mu^2\right)\right]^2\left[\det\left(\partial_+\partial_- + \mu^2\right)\right]^4\right.\\ &\quad\left.\times\left[\det\left(\partial_+^2\partial_-^2 + \left(\partial_+^2 + \partial_-^2\right)\mu^2 + 2\partial_+\partial_-\left(2\kappa^2 - \mu^2\right)\right)\right]\left[\det\left(\partial_+^2 + \mu^2\right)\right]^2\right)^{-1/2},\end{aligned} \tag{D.32}$$

where the physical contributions are correctly reproduced. The unphysical part in this functional determinant is $\left(\left[\det\left(\partial_+^2 + M_2^2\right)\right]^2 / \left[\det\left(\partial_+^2 + \mu^2\right)\right]^2\right)^{1/2}$. By the same argument as the treatment of the shift in (4.69), one can confirm that the shifts M_2^2, μ^2 are eliminated, then the unphysical part totally vanishes. Hence our result in the bosonic sector in the $\delta A_+ = 0$ gauge agrees with that found in the approach based on the PW identity in chapter 4.

Fermionic fluctuations

As the fermionic sector does not contain the fluctuations of the gauge fields δA_-, the computation is straightforward. We define component fields of the fermionic fluctuations in the following way,

$$\Psi_R = \begin{pmatrix} 0 & \mathfrak{X}_R \\ \mathfrak{Y}_R & 0 \end{pmatrix}, \quad \Psi_L = \begin{pmatrix} 0 & \mathfrak{X}_L \\ \mathfrak{Y}_L & 0 \end{pmatrix}, \tag{D.33}$$

where

$$\mathfrak{X}_R = \begin{pmatrix} 0 & 0 & \alpha_1 + i\alpha_2 & \alpha_3 + i\alpha_4 \\ 0 & 0 & -\alpha_3 + i\alpha_4 & \alpha_1 - i\alpha_2 \\ \alpha_5 + i\alpha_6 & \alpha_7 - i\alpha_8 & 0 & 0 \\ \alpha_7 + i\alpha_8 & -\alpha_5 + i\alpha_6 & 0 & 0 \end{pmatrix}, \tag{D.34}$$

$$\mathfrak{Y}_R = \begin{pmatrix} 0 & 0 & -\alpha_6 - i\alpha_5 & -\alpha_8 - i\alpha_7 \\ 0 & 0 & \alpha_8 - i\alpha_7 & -\alpha_6 + i\alpha_5 \\ \alpha_2 + i\alpha_1 & \alpha_4 - i\alpha_3 & 0 & 0 \\ \alpha_4 + i\alpha_3 & -\alpha_2 + i\alpha_1 & 0 & 0 \end{pmatrix}, \tag{D.35}$$

and

$$\mathfrak{X}_L = \begin{pmatrix} 0 & 0 & (\beta_1 + i\beta_2)v^* & (\beta_3 + i\beta_4)v^* \\ 0 & 0 & (\beta_3 - i\beta_4)v & (-\beta_1 + i\beta_2)v \\ (\beta_5 + i\beta_6)v^* & (-\beta_7 + i\beta_8)v^* & 0 & 0 \\ (\beta_7 + i\beta_8)v & (\beta_5 - i\beta_6)v & 0 & 0 \end{pmatrix}, \tag{D.36}$$

$$\mathfrak{Y}_L = \begin{pmatrix} 0 & 0 & (-\beta_6 - i\beta_5)v & (-\beta_8 - i\beta_7)v^* \\ 0 & 0 & (-\beta_8 + i\beta_7)v & (\beta_6 - i\beta_5)v^* \\ (\beta_2 + i\beta_1)v & (-\beta_4 + i\beta_3)v^* & 0 & 0 \\ (\beta_4 + i\beta_3)v & (\beta_2 - i\beta_1)v^* & 0 & 0 \end{pmatrix}, \tag{D.37}$$

where all component fields are real Grassmann. The extra factor $v_\tau = \exp(i\kappa^2\tau/\mu)$ is introduced in $\mathfrak{X}_L$ and $\mathfrak{Y}_L$ such that all coefficients in the Lagrangian become constant. The resulting Lagrangian is

$$\mathcal{L}_{\rm F} = 2\Bigg[\sum_{i=1}^{8}(\alpha_i\partial_-\alpha_i + \beta_i\partial_+\beta_i) - \mu\,(\alpha_1\alpha_2 + \alpha_3\alpha_4 + \alpha_5\alpha_6 - \alpha_7\alpha_8 - \beta_1\beta_2 - \beta_3\beta_4 - \beta_5\beta_6 + \beta_7\beta_8) \\ +2\kappa\,(\alpha_1\beta_4 + \alpha_2\beta_3 - \alpha_3\beta_2 - \alpha_4\beta_1 - \alpha_5\beta_8 + \alpha_6\beta_7 + \alpha_7\beta_6 - \alpha_8\beta_5)\Bigg]\,, \tag{D.38}$$

from which one can derive characteristic frequencies of the fermionic fluctuations. They agree with the string theory result [69] up to certain trivial shift and exactly agree with the result obtained by using the PW identity in (4.133). Then the functional determinants also match (cf. (4.134)),

$$\det\left(\partial_+^2\partial_-^2 + 2\partial_+\partial_-\kappa^2 + \frac{1}{4}\left(\partial_+^2 + \partial_-^2\right)\mu^2 + \frac{1}{16}\left(4\kappa^2 - \mu^2\right)^2\right)\,. \tag{D.39}$$

Directly from the agreement of (D.32) and (D.39) with the results in section 4.4.1, we find that the one-loop contribution to the partition function is given by (4.135).

Let us comment on the two-loop computation in this approach. The $\delta A_+ = 0$ gauge does work properly at one-loop order, but is not expected to be useful in the two-loop computation. The primary reason for this is the difficulty in dealing with the nonlocality which can not be eliminated in the case of the (S,J) folded string. The nonlocality enters when we solve the gauge equation derived by varying the fluctuation Lagrangian by δA_-. Up to quartic order, the gauge equation is written as

$$\left(\mathcal{D}_+\eta - \frac{1}{2}\left[\eta, \mathcal{D}_+\eta\right] - 2\Psi_R\Psi_R T + \frac{1}{6}\left[\eta,\, \left[\eta, \mathcal{D}_+\eta\right]\right]\right)^{\perp} = 0\,. \tag{D.40}$$

Since it is hard to solve it exactly, we shall solve this equation order by order. First we shall consider the following expansion,

$$\eta^{\perp} = \eta^{\perp}_{(1)} + \eta^{\perp}_{(2)} + \eta^{\perp}_{(3)} + \cdots\,, \tag{D.41}$$

where the index (i) represents the i-th order of the physical fluctuations $\eta^{\parallel}$. At the leading order the equation takes the form,

$$\partial_+\eta^{\perp}_{(1)} + \left[A_+, \eta^{\perp}_{(1)}\right] + \left[\mathcal{A}^{\parallel}_+, \eta^{\parallel}\right] = 0\,. \tag{D.42}$$

which was already solved by (D.24) and the solution contains nonlocal terms. Next we shall solve the equation for $\eta^{\perp}_{(2)}$ using the first order solution. The equation for $\eta^{\perp}_{(2)}$ is expressed by using $\eta^{\perp}_{(1)}$,

$$\partial_+\eta^{\perp}_{(2)} + \left[A_+, \eta^{\perp}_{(2)}\right] - \frac{1}{2}\left[\eta^{\parallel}, \partial_+\eta^{\parallel} + \left[A_+, \eta^{\parallel}\right] + \left[\mathcal{A}^{\parallel}_+, \eta^{\perp}_{(1)}\right]\right] - 2\Psi_R\Psi_R T = 0\,. \tag{D.43}$$

Once the leading order solution $\eta^{\perp}_{(1)}$ is substituted, the latter two terms in this equation contain only the physical field $\eta^{\parallel}$. Then the form of the equation (D.43) is the same as that of (D.42), and can be in principle solved for $\eta^{\perp}_{(2)}$ by rewriting the equation in terms of the component fields.

The obstacle in redoing the computation for higher orders is the nonlocality of the solution; the solution involves nonlocal terms such as $\frac{1}{\partial_+}\left(a_i \frac{\partial_+}{B^2+\partial_+^2} a_j\right)$, and the nonlocality will be more complicated if we discuss quartic terms in the fluctuation Lagrangian. As a result, the fluctuation Lagrangian containing higher order terms looks almost intractable in this approach.

D.3 Two-loop computation in vacuum case

In chapter 4 we studied two-loop corrections near the folded string with large spin S in AdS_3 taking the limit $\mu \to 0$ in which the angular momentum in S^5 vanishes. Here we shall check that the two-loop correction vanishes in the opposite limit of the trivial reduced theory solution corresponding to the BMN vacuum, i.e. in the case when

$$\kappa \to \mu\,, \qquad \ell \to 0\,. \tag{D.44}$$

In this case it is useful to define the "S" part of fluctuation fields with an additional rescaling by $\mathrm{w} = e^{i\mu\tau}$ as follows (cf. (4.114))

$$\eta_S^{\parallel} = \begin{pmatrix} 0 & 0 & b_1 + ib_2 & (b_3 + ib_4)\mathrm{w} \\ 0 & 0 & (-b_3 + ib_4)\mathrm{w}^* & b_1 - ib_2 \\ -b_1 + ib_2 & (b_3 + ib_4)\mathrm{w} & 0 & 0 \\ (-b_3 + ib_4)\mathrm{w}^* & -b_1 - ib_2 & 0 & 0 \end{pmatrix},$$
$$\eta_S^{\perp} = \begin{pmatrix} id_1 & (d_2 + id_3)\mathrm{w} & 0 & 0 \\ (-d_2 + id_3)\mathrm{w}^* & -id_1 & 0 & 0 \\ 0 & 0 & id_4 & (d_5 + id_6)\mathrm{w} \\ 0 & 0 & (-d_5 + id_6)\mathrm{w}^* & -id_4 \end{pmatrix}. \tag{D.45}$$

Also, the component fields of the fermionic fluctuations are to be defined as (cf. (4.128))

$$\mathfrak{X}_R = \begin{pmatrix} 0 & 0 & (\alpha_1 + i\alpha_2)t_{1+} & (\alpha_3 + i\alpha_4)t_{2+} \\ 0 & 0 & (-\alpha_3 + i\alpha_4)t_{2+}^* & (\alpha_1 - i\alpha_2)t_{1+}^* \\ (\alpha_5 + i\alpha_6)t_{1+} & (\alpha_7 - i\alpha_8)t_{2+} & 0 & 0 \\ (\alpha_7 + i\alpha_8)t_{2+}^* & (-\alpha_5 + i\alpha_6)t_{1+}^* & 0 & 0 \end{pmatrix}, \tag{D.46}$$

$$\mathfrak{Y}_R = \begin{pmatrix} 0 & 0 & (-\alpha_6 - i\alpha_5)t_{1+}^* & (-\alpha_8 - i\alpha_7)t_{2+} \\ 0 & 0 & (\alpha_8 - i\alpha_7)t_{2+}^* & (-\alpha_6 + i\alpha_5)t_{1+} \\ (\alpha_2 + i\alpha_1)t_{1+}^* & (\alpha_4 - i\alpha_3)t_{2+} & 0 & 0 \\ (\alpha_4 + i\alpha_3)t_{2+}^* & (-\alpha_2 + i\alpha_1)t_{1+} & 0 & 0 \end{pmatrix}, \tag{D.47}$$

$$\mathfrak{X}_L = \begin{pmatrix} 0 & 0 & (\beta_1 + i\beta_2)t_{2-}^* & (\beta_3 + i\beta_4)t_{1-}^* \\ 0 & 0 & (\beta_3 - i\beta_4)t_{1-} & (-\beta_1 + i\beta_2)t_{2-} \\ (\beta_5 + i\beta_6)t_{2-}^* & (-\beta_7 + i\beta_8)t_{1-}^* & 0 & 0 \\ (\beta_7 + i\beta_8)t_{1-} & (\beta_5 - i\beta_6)t_{2-} & 0 & 0 \end{pmatrix}, \tag{D.48}$$

$$\mathfrak{Y}_L = \begin{pmatrix} 0 & 0 & (-\beta_6 - i\beta_5)t_{2-} & (-\beta_8 - i\beta_7)t^*_{1-} \\ 0 & 0 & (-\beta_8 + i\beta_7)t_{1-} & (\beta_6 - i\beta_5)t^*_{2-} \\ (\beta_2 + i\beta_1)t_{2-} & (-\beta_4 + i\beta_3)t^*_{1-} & 0 & 0 \\ (\beta_4 + i\beta_3)t_{1-} & (\beta_2 - i\beta_1)t^*_{2-} & 0 & 0 \end{pmatrix}, \tag{D.49}$$

where

$$t_{1\pm} = e^{i\frac{\ell^2(\tau\pm\sigma)}{2\mu}}, \qquad t_{2\pm} = e^{i\frac{(\ell^2+2\mu^2)\tau\pm\ell^2\sigma}{2\mu}}. \tag{D.50}$$

Taking the limit (D.44) in the two-loop diagrams one finds cancellations between A and S sectors in each type of diagrams leading to the vanishing two-loop correction.

One can also check this cancellation directly, by expanding near the reduced theory counterpart of the BMN vacuum

$$g_0 = \mathbf{I}_{8\times 8}, \qquad A_\pm = 0 . \tag{D.51}$$

In this case the τ, σ-dependent rescalings of fluctuations are not needed and $2d$ Lorentz invariance of the perturbation theory is manifest. One then finds for the individual diagram contributions to the coefficient in the two-loop effective action[2]

$$\begin{aligned}
&\text{bosonic sunset :} && J_A = -J_S = -\frac{3}{2} I[1]I[1]\,, \\
&\text{bosonic double}-\text{bubble :} && J_A = -\ J_S = -\frac{1}{2} I[1]I[1]\,, \\
&\text{fermionic sunset :} && J_A = -J_S = -6I[0]I[1] + 3[1]I[1]\,, \\
&\text{fermionic double}-\text{bubble :} && J_A = -J_S = -6I[0]I[1] - 4I[1]I[1]\,, \\
&\text{tadpole :} && J_A = -J_S = 0\,.
\end{aligned} \tag{D.52}$$

We conclude again that the sum of the A and S sector contributions vanishes.

[2]As above, here A and S stand for contributions from the fluctuations corresponding to reduced theory counterparts of the AdS_5 and S^5 sectors.

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