The Pion Mass Difference in the Nambu–Jona-Lasinio Model

We present the results of a gauge- and chirally invariant calculation of the electromagnetic mass splitting of the pion in the chiral limit. The calculation is done in the two-flavor version of the original Nambu–Jona-Lasinio model, in the Hartree approximation. We elucidate the special role which the electromagnetic contributions to the gap equation play in satisfying Dashen's theorem. Specifically, the neutral pion is unshifted from its Goldstone limit of zero mass by electromagnetic interactions. Reversing the conventional procedure, we determine the quark vacuum condensate value $\langle \bar{q}q \rangle = -(260 \pm 1 \text{ MeV})^3$ from the observed pion mass difference and the radiatively corrected pion weak decay constant f_π. We discuss the role of the pion electromagnetic form factor $F_\pi(q^2)$ in our result for the mass shift. The space-like region of $F_\pi(q^2)$ is shown to reproduce the data reasonably well. Implications for the short- and medium-range behavior of the pion charge distribution are discussed. Finally, we compare our results with those of meson-theoretic models and other, more recent, quark models.

Key Words: *pion mass difference, Nambu–Jona-Lasinio model, Dashen's theorem, electromagnetic pion form factor*

I. INTRODUCTION

The mass squared difference between the charged and neutral pion is one of the accurately known experimental numbers for the pion system. This difference is measured[1] as

$$m^2_{\pi^\pm} - m^2_{\pi^0} = [35.55 \pm 0.02 \text{ MeV}]^2.$$

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The problem of trying to understand this mass difference theoretically is an old one in hadron physics.[2–5] It has already been addressed in a number of models based on meson degrees of freedom, with a great deal of phenomenological success. The explanation of the pion mass difference was one of the major triumphs of the (chiral) current algebra method[3] which was repeated by the (phenomenological) effective Lagrangian approach.[4] This success has left the lasting (and correct) impression that the underlying approximate chiral symmetry plays a very important role in the problem. It has also been known since the work of Ref. 2 that essentially all of the pion mass difference can be attributed to the electromagnetic self-energy, in stark contrast to the $K^0 - K^{\pm}$ and the $n - p$ mass differences,[1,2] for which even the sign of the mass difference is opposite to the Coulomb energy of the quarks and antiquarks involved.

The development of the quark model of hadrons revived interest in understanding this mass shift, but this time in terms of the underlying quark degrees of freedom.[6] The isospin-violating mass splittings of the heavy (charmed and heavier) pseudoscalar mesons have been studied in nonrelativistic potential quark models with considerable success.[7] These models are obviously inadequate for the light pseudoscalar mesons, such as the π and K, due to the nonrelativistic nature of the models and the absence of the chiral symmetry, which is an essential feature of the dynamics of the light pseudoscalar mesons, in those models.

Several new attempts at solving the pion mass difference problem have been recorded in recent years.[8–10] Most of these efforts are based on some form of the pion chiral dynamics, or equivalently current algebra, in the "long distance" regime, and perturbative QCD in the "short distance" regime. There has even been one recent calculation in the "bosonized" version of the extended NJL model,[11] which turns out to be completely equivalent to the purely meson-theoretic calculation of Ref. 4 and is therefore of little interest here. We will try to compare the results of these calculations with those of ours[12] where possible.

In recent years we have seen remarkable progress in the understanding of the mass spectrum of the pseudoscalar meson nonet[13] in a relativistic, chiral quark model with dynamic constituent quark mass generation, the so-called Nambu–Jona-Lasinio (NJL) model.[14]

This Comment is concerned with the mass splitting between the charged and neutral pions in the minimal two-flavor version of the NJL model. In fact our analysis will shed as much light on the pion mass difference problem itself as on the structure of the NJL model. We discuss the exact, closed form results of a gauge and chirally invariant pion mass difference calculation in the chiral limit (current quark masses, and therefore the non-electromagnetic pion mass, equal to zero) of the NJL model within the Hartree approximation, to $O(\alpha)$, where $\alpha = e^2/4\pi \approx 1/137$ is the fine structure constant. We leave the full discussion of the small non-chiral corrections for another occasion. As possible downsides of this calculation, we ought to emphasize that the NJL model does not describe the quark confinement, and secondly that the Hartree approximation and the chiral invariance force us to keep only one-$q\bar{q}$-pair intermediate states in our calculation. At the hadronic level this would mean keeping only intermediate states involving single pions. Thus there is plenty of room for future improvements. In order to make this Comment equally accessible to the non-specialist and the expert in chiral models, we have included general introductions to the NJL model and to the problem of isospin-violating mass differences. Then we discuss our calculation in detail: firstly the construction of the gauge invariant set of Feynman graphs, then their evaluation and finally our numerical results and their interpretation. As the final piece of new work we talk about the EM self-mass of the neutral pion in this model and show the important role the modified gap equation plays in keeping the model self-consistent in the presence of an electromagnetic field, in accord with Dashen's theorem.[15]

II. THE TWO-FLAVOR NJL MODEL IN THE HARTREE APPROXIMATION

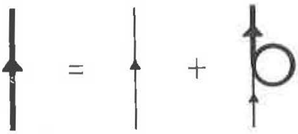
This section is concerned with the definition of the model and of the approximations used, as well as the proof of gauge invariance of the result. One of the most important features of this model is its chiral symmetry and its spontaneous breakdown induced by the dynamics, rather than by fiat. Our starting point is the two-flavor NJL Lagrangian,

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m^{o})\psi + G[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\tau\psi)^2] \qquad (2.1)$$

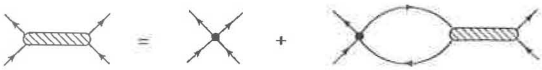
where m^o is the "current" quark mass, G is a dimensional coupling constant and τ are (Pauli) isospin matrices. The first and foremost property of this $\mathcal{L}$ is the equivalence of its internal flavor symmetries to those of the QCD Lagrangian. Specifically the symmetry group is $U_V(1) \otimes SU_L(2) \otimes SU_R(2)$ for $m^o = 0$, where the $U_V(1)$ expresses the baryon conservation and $SU_L(2) \otimes SU_R(2)$ represents the isospin symmetry of the left- and right-handed quarks, respectively. Note that the $U_A(1)$ symmetry is explicitly broken (in the maximal fashion) such that the "fourth" $T = 0$ pseudoscalar bound state completely disappears. Such "maximal" $U_A(1)$ symmetry breaking is nicely illustrated in the model defined by Eq. (2.10) of Ref. 13. In the limit when the coupling constants of the two terms appearing in that equation approach each other, the mass of the "fourth" pseudoscalar particle moves to infinity and completely decouples from the remaining pseudoscalars. The main point here is that there is no $U_A(1)$ problem with the $\mathcal{L}$ defined by Eq. (2.1) and that its removal has been accomplished in full accord with the QCD analysis of 't Hooft.[16] This fact clearly indicates the connection to QCD proper.

Note that the NJL Lagrangian of Eq. (2.1) contains a four-fermion contact interaction, very much like the old Fermi theory of weak interactions, which is not renormalizable in 3 + 1 dimensions. This means that in the one-loop approximation we will need at least one new arbitrary constant in order to make the amplitudes finite; this is equivalent to setting the mass scale of the theory. There are infinitely many ways of choosing this new constant, which is not apparent in the Lagrangian, and the way we will do it here is by cutting off the quark (Euclidean) four-momentum integrals at some cut-off value Λ. Since the Hartree procedure is a "one-loop" calculation, we end up with two free dimensional parameters that need fixing in the chiral limit: G and Λ. Neither of these quantities is an observable, so we are left with the task of finding two independent pion observables, and expressing them in terms of these two parameters. The standard choice for one of them has been the pion decay constant f_π. The other one is usually taken to be either the constituent quark mass or the quark condensate value (the two are simply related in the Hartree approximation). Neither is a true observable (the quarks are confined and we have no way of knowing their masses) and both of them are very model-dependent. Another candidate for the second observable is the charge radius of the pion. In the NJL model this quantity turns out to be a function of f_π *only*, so it is not suitable for our purposes. Although the NJL model results for f_π, m and the condensate density are slowly varying functions of the cut-off, thus implying a relative insensitivity of the predictions of the model under change of this free parameter, we will see that one can fix both of the free parameters with a great deal of accuracy using our analysis.

We next turn to the phenomenon of spontaneous chiral symmetry-breaking in the NJL model within the self-consistent Hartree approximation. This leads to a non-perturbative scheme defined by two integral equations of the Schwinger–Dyson type: (i) the one-body self-consistency (gap) equation (Fig. 1(a)), (ii) the two-body (Bethe–Salpeter) bound state equation (Fig. 1(b)). However, before embarking on such somewhat formal considerations



(a)



(b)

FIGURE 1 The Schwinger–Dyson equations that define the Hartree approximation to the NJL model: (a) The one-body (gap) equation, (b) The two-body (Bethe–Salpeter) equation. The bold-faced solid line denotes a "dressed" (constituent) quark, the thin solid line describes a "bare" (current) quark, but only in Fig. 1(a); in all subsequent diagrams the thin solid line stands for a dressed quark. The solid dot represents one of the two four-point interaction terms appearing in the NJL Lagrangian Eq. (2.1), and the shaded "balloon" in (b) denotes the bound-state (π,σ) propagator specified by Eq. (2.7).

that will be needed later, we want to point out that the salient features of chiral symmetry-breaking can be understood rather simply in the following way: Replace $\mathcal{L}$ in Eq. (2.1) by its approximate linearised equivalent

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m^{\circ} + 2Gn_s]\psi \tag{2.2}$$

where n_s is the quark condensate density

$$n_s = \langle o|\bar{\psi}(x)\ \psi(x)|o\rangle$$

in the *interacting* vacuum state $|o\rangle$. Notice that the structure of $|o\rangle$ does not have to be known at this stage. However, it has to be more complicated than the non-interacting vacuum since otherwise the condensate density would vanish. Notice also that in obtaining Eq. (2.2) the assumption $\langle o|\bar{\psi}(x)i\gamma_5\tau\psi(x)|o\rangle = 0$ has been introduced, i.e., that $|o\rangle$ has good isospin and parity.

The approximate Lagrangian of Eq. (2.2) now describes a free quark again, but with a modified, or constituent, mass

$$m = m^{\circ} - 2Gn_s. \tag{2.3}$$

A non-vanishing n_s thus signals the spontaneous dynamical breaking of chiral symmetry in the ground state, as opposed to the "mechanical," or explicit, breaking caused by inserting a non-zero current quark mass m° "by hand" into the Lagrangian. So from now on we will drop m° and concentrate on *dynamical* symmetry-breaking. Recalling[13] that the condensate density is determined by the trace of the quark propagator $S(x, x')$ over all its flavor, color and Dirac intrinsic variables, we rewrite Eq. (2.3) as

$$m = 2iG\ \mathrm{Tr}S(x, x) = 2iG\ \mathrm{Tr}\int \frac{d^4p}{(2\pi)^4} S(p). \tag{2.4}$$

A second basic approximation enters at this point, viz. that the propagator S is determined by the *same* Lagrangian that led to the relation (2.3) between m and n_s. This is obviously the self-consistency requirement that the mass determined by n_s is the same mass that enters S and thus determines n_s and m. The momentum space form of S is thus that of a free Dirac particle of mass m,

$$S(p) = (\not{p} - m + i\epsilon)^{-1} \tag{2.5}$$

and Eq. (2.4) becomes, for $m \neq 0$,

$$1 = 8iGN_cN_f \int^{\Lambda} \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2} \tag{2.6}$$

after cancelling a common factor of m. This relation, which determines m self-consistently and non-perturbatively in G, is originally due to Nambu and Jona-Lasinio[14] and is called the NJL gap equation because of its formal similarity with the BCS gap equation for a superconductor. At this point the anticipated divergence in the model has also emerged explicitly in the divergence of the integral on the right-hand side of Eq. (2.6), which we therefore will have to regularize in some way.

We now have the necessary background to approach the problem of dynamical symmetry-breaking more formally from the point of view of the Schwinger–Dyson equations shown pictorially in Fig. 1. The first Schwinger–Dyson equation (Fig. 1(a)) gives the integral equation for the quark propagator in momentum space, and is solved by the expression for $S(p)$ given in Eq. (2.5) above, where m is given by

$$-im = (-1)\, 2iG \operatorname{Tr} \int \frac{d^4p}{(2\pi)^4}\, iS(p)$$

in the one loop approximation. This obviously reproduces Eq. (2.4) again and thus leads us back to the transcendental equation (2.6) for the constituent quark mass.

The second Schwinger–Dyson equation (Fig. 1(b)) is an inhomogeneous Bethe–Salpeter equation describing the scattering of quarks and antiquarks. Due to the contact nature of our interaction, the solution of this integral equation is given by the geometric progression

$$-iM(k^2) = \frac{2iG}{1 - 2G\Pi(k^2)} \tag{2.7}$$

for a momentum transfer squared k^2 where

$$-i\Pi(k^2) = \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}[i\gamma_5 T_\pi^+ S(p + k)\, i\gamma_5 T_\pi S(p)] \tag{2.8}$$

is the lowest order irreducible pseudoscalar polarisation diagram. Here $T_\pi = (\tau_1 \pm i\tau_2)/\sqrt{2}$ or τ_3 are the projection operators onto the $\pi^\pm$ or π^o channels.

The position of the poles in the two-body propagator $M(k^2)$ determines the mass of the bound states in the theory, and the residue at these poles, the coupling of the bound states to the quarks. From Eq. (2.7) these residues are

$$g^2_{\pi qq} = \left[\frac{\partial \Pi}{\partial k^2}\right]^{-1}_{\text{pole}} \tag{2.9}$$

In our case there is only one pole in the isotriplet-pseudoscalar channel; this pole lies at $k^2 = 0$, in agreement with Goldstone's theorem[17] and corresponds to a degenerate isotriplet of massless pions in the NJL model. One sees this directly by computing the denominator in Eq. (2.7):

$$1 - 2G\Pi(k^2) = \left[1 - \frac{2iG}{m} \mathrm{Tr}S(x, x)\right] + [4iGN_cN_fI(k^2)]k^2 \tag{2.10}$$

where $iI(k^2)$ is defined by the expression

$$iI(k^2) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{[(p + k)^2 - m^2][p^2 - m^2]}$$

$$= iI(0) + \frac{1}{(4\pi)^2} \int_0^1 dx \ln\left[1 - \frac{k^2}{m^2} x(1 - x)\right] \tag{2.11}$$

The factor $I(0)$ contains the divergence that is present in the pion polarization loop at zero momentum transfer.

The first term on the right-hand side of Eq. (2.10) vanishes on account of the gap relation [Eqs. (2.4) or (2.6)] fixing m. Thus $M(k^2)$ has a pole at $k^2 = 0$, as anticipated. The coupling constant of the quarks to this mode is

$$g^2_{\pi qq} = -[2iN_cN_fI(0)]^{-1}. \tag{2.12}$$

This relation allows one to recast the scattering amplitude as

$$M(k^2) = \frac{g^2_{\pi qq}F_\pi^{-1}(k^2)}{k^2} \tag{2.13}$$

where we have defined

$$F_\pi(k^2) = I(k^2)/I(0). \tag{2.14}$$

We indicate later that $F_\pi(k^2)$ is identical to the electromagnetic form factor of a chiral (i.e., massless) pion within our scheme. Together, Eqs. (2.13) and (2.14) constitute the two essential ingredients for understanding the electromagnetic mass difference of pions. We discuss this problem next.

III. ISOSPIN-VIOLATING MASS DIFFERENCES

Formally the shift in the pole of $M(k^2)$ due to isospin-violating contributions determines the mass splitting of the pions. In QCD, however, there are only two known sources of isospin-violation:

(a) the current quark mass differences,
(b) the electro-weak interaction,

and the same is true in the NJL, or any other quark model. The first item on this list, which we call the "mechanical" mass difference, contributes 2% of the total mass shift and thus allows us to work in the chiral limit without making an obviously bad approx-

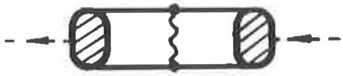


FIGURE 2 A schematic illustration of the non-relativistic Coulomb energy E_c, Eq. (3.1), of a two-body bound-state, e.g., the π. The wiggly line denotes the Coulomb gauge photon, the solid lines are the constituents and the dashed line is the bound-state. The shaded ovals, together with the constituent lines, represent the Schrödinger wave functions $\psi(r)$, which, in turn, determine the charge density $\rho_\pi(r)$.

imation. The reason for this mechanical mass difference being so small is that it has to be at least quadratic* in the current quark mass difference as a consequence of the isospin structure of the pions and the model-independent Gell-Mann, Oakes and Renner (GMOR) relation.[18] The coefficient of proportionality of the quadratic term is model dependent, and in our case turns out to be very close to unity. This fact, plus the commonly accepted values[19] of the current quark masses $m_u^\circ = 5.2$ MeV and $m_d^\circ = 1.8\ m_u^\circ$, lead immediately to our assertion about the size of the mechanical mass difference: to the lowest non-vanishing order one has $[\Delta m_\pi^2]_{\rm mech} \approx (m_u^\circ - m_d^\circ)^2 = (4.2\ \text{MeV})^2$.

The second item on the list can be divided into two subclasses: the weak and the electromagnetic contributions. The weak interaction's contribution is $O(G_F)$, where $G_F \approx 10^{-5}M_N^{-2}$ is the Fermi weak coupling constant, and hence it is negligible. We are thus left with only one important term: the electromagnetic interaction. However, before launching into the complexities of a fully gauge and Lorentz invariant calculation of the pionic electromagnetic (EM) self-energy, it is instructive to approach the problem on a more elementary level by looking at the Coulomb energy of two charged particles, in our case a quark and an anti-quark, that form a bound pair due to the strong interaction. The calculation is elementary (see Fig. 2), the exchanged photon being in the Coulomb gauge, and gives

$$E_c = \frac{e^2}{2}\int d^3\mathbf{r}\int d^3\mathbf{r}'\,\frac{\rho_\pi(\mathbf{r})\rho_\pi(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} = \frac{e^2}{2}\int \frac{d^3\mathbf{q}}{(2\pi)^3}\,\frac{F_\pi^2(-\mathbf{q}^2)}{\mathbf{q}^2} \qquad (3.1)$$

*This is not the case for strange and charmed meson mass differences.

where $\rho_\pi(\mathbf{r})$ is the charge density normalised to the total charge in units of the proton charge e, and $F_\pi(q^2)$ its Fourier transform. The Coulomb energy of a two-body bound state is clearly positive for two charges of equal sign (making up the $\pi^\pm$), and negative for two oppositely charged particles (the π^o). This simple consideration tells us without further ado that the charged versus neutral pion mass difference will be positive, as observed. This formula is not meant to describe the complete result for the non-relativistic EM self-energy of a bound state (for that see Friar[20]), but rather to indicate that the Coulomb energy has something to do with an integral over the EM form factor of the bound state. For comparison, the fully covariant field theoretic analog of Eq. (3.1) is the following expression[3]:

$$E_c = \frac{e^2}{2} \iint d^4x d^4y \, \langle \pi | T^*(j_\mu(x) j_\nu(y)) | \pi \rangle \, D_F^{\mu\nu}(x - y) \quad (3.2)$$

where $j_\mu(x)$ is the EM current operator, T^* denotes the *covariant* time ordered product that includes "sea-gull" terms if they are necessary for the EM gauge invariance, and $D_F^{\mu\nu}$ is the photon's Feynman propagator in configuration space.

An early attempt at a momentum space version of Eq. (3.2) was given by Barger and Kazes[5]; they came up with the following formula:

$$\Delta m_\pi^2 = m_{\pi^\pm}^2 - m_{\pi^o}^2 = 3e^2 i \int \frac{d^4q}{(2\pi)^4} \frac{F_\pi^2(q^2)}{q^2 + i\epsilon} \quad (3.3)$$

which for a monopole $F_\pi(q^2)$ (see below) leads to $\Delta m_\pi^2 = 3\alpha m_\rho^2/4\pi = (32\,\text{MeV})^2$, indeed close to experiment. Formula (3.3) is completely equivalent to the EM self-energy calculated from a scalar *point*-particle forward Compton amplitude multiplied by the square of the elastic *on-shell* form factor. However, it is clear that the intermediate-state pion in Fig. 3 is off-shell and that therefore the exactness of this formula is questionable. We will argue later that this expression is only correct for special kinds of pion EM interactions (vector meson dominance, or VMD, models), and that in the general case it violates the Ward–Takahashi identities.[21]

FIGURE 3 Two Feynman diagrams depicting the pion EM self-energy in the Barger–Kazes approach, Eq. (3.3). (a) The "sea-gull" diagram, (b) the pion-pole diagram. The wiggly line denotes a photon in an arbitrary gauge, the dashed line stands for a pion. The two pion–one-photon vertex is multiplied by $F_\pi(q^2)$, whereas the two-pion–two-photon vertex is multiplied by $F_\pi^2(q^2)$. The factor 1/2 in front of the graph in Fig. 3(a) is the appropriate symmetry number.

We thus suspect that the formula in Eq. (3.3) may be inapplicable to our problem and leave it for now. In the next section we obtain the exact gauge invariant quark–antiquark EM self-energy to $O(\alpha)$ in the $J^P = 0^-$, $T = 1$ channel.

IV. THE PION ELECTROMAGNETIC SELF-ENERGY IN THE NJL MODEL

We start out by minimally coupling the electromagnetic gauge field to the NJL Lagrangian by substituting

$$i\partial\!\!\!/ \rightarrow iD\!\!\!\!/ = i\partial\!\!\!/ - e_q A\!\!\!/ \tag{4.1}$$

in Eq. (2.1) where e_q is the quark charge, equal to $\frac{2}{3}e$ and $-\frac{1}{3}e$ for the u and d quarks, respectively. Note that the EM interaction explicitly breaks both $SU_L(2)$ and $SU_R(2)$ flavor symmetries. This means that not only do the pions acquire a mass, but they do so in an isospin asymmetric way. We compute the magnitude of this mass shift to $O(\alpha)$ by including the additional *gauge invariant* set of diagrams in the pion self-energy to the same order in α. As in Ref. 12 we call this additional contribution $\Pi_{\rm EM}^{T_3}(k^2)$ where $T_3 = \pm 1$ or 0 refers to the $\pi^\pm$ or the π^0. The $q\bar{q}$ scattering amplitude in the presence of EM interactions then reads

$$-iM_{T_3}(k^2) = \frac{2iG}{1 - 2G[\Pi(k^2) + \Pi_{\rm EM}^{T_3}(k^2)]} \tag{4.2}$$

in place of Eq. (2.7), with the non-electromagnetic part of the self-energy still being given by Eq. (2.8).

To construct the gauge-invariant pion EM self-energy in the NJL model we start from the Feynman graphs depicted in Figs. 4(a)–4(c). The first one describes the Coulomb interaction that we saw earlier in Fig. 2; the other two are the quark and antiquark EM self-energies, respectively. At first sight one might think that these three diagrams are the only ones of interest to $O(\alpha)$. This is not so. One finds that their sum is only gauge invariant for the neutral pion but not for the charged ones, so we are missing something. To find the missing diagrams we recall that one can have "vertex corrections" of the kind depicted in Fig. 4(d). Clearly, there are infinitely many such irreducible diagrams. Fortunately they can be summed to reproduce the exact $q\bar{q}$ scattering amplitude $-iM(k^2)$ of Eq. (2.7) again, and yield a single Feynman diagram (Fig. 5).

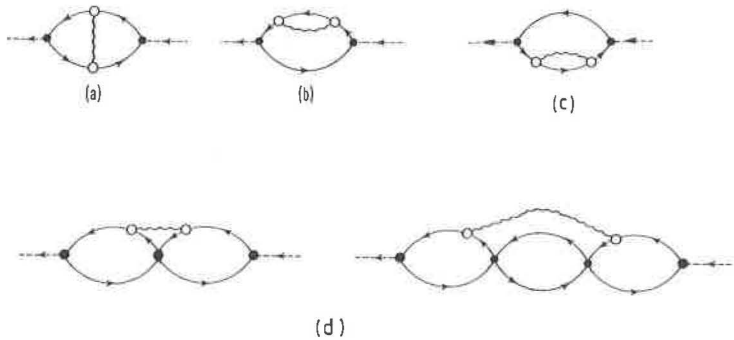


FIGURE 4 Feynman diagrams showing various parts of the pion's proper EM self-energy insertions in the NJL model. (a) The "Coulomb" graph, corresponding to the nonrelativistic graph shown in Fig. 2, (b) the quark and (c) antiquark EM self-energies, (d) two, of infinitely many, irreducible diagrams that contribute to the charged pion EM self-energy. The solid circles denote the effective πqq interaction with a coupling constant $g_{\pi qq}$ determined by Eq. (2.12), and the open circles denote "minimally coupled" (see Section IV) γqq EM vertices.

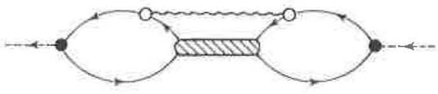


FIGURE 5 A single "dumbbell" diagram which sums up all of the graphs of the type shown in Fig. 4(d). Three other dumbbell graphs, with other possible momentum routings, enter the charged pion EM self-energy.

There are three other topologically equivalent diagrams with the charge flow in different directions.

The Feynman rules for calculating all these diagrams can be read off by installing Eq. (4.1) in the Lagrangian, Eq. (2.1). Working in an arbitrary gauge parametrized[21] by λ, both the gauge dependent and the gauge independent parts of these diagrams are expressed in closed form, as follows, in the chiral limit $k^2 = 0$:

$$\prod_{\text{EM}}^{T_3} (k^2 = 0) = i g_{\pi qq}^{-2} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 + i\epsilon} \Bigg\{ \text{Tr}(T_\pi^+ e_q T_\pi e_q) F_\pi(q^2)$$

$$\times \left[4 - \left(1 - \frac{1}{\lambda}\right)\right] + \text{Tr}(e_q^2 \{T_\pi, T_\pi^+\}) \left[\frac{1}{2}\left(1 - \frac{1}{\lambda}\right) F_\pi(q^2)\right.$$

$$\left. - 12 i g_{\pi qq}^2 J_1(q^2)\right] + e_\pi^2 F_\pi(q^2) \left[1 - \left(1 - \frac{1}{\lambda}\right)\right]\Bigg\} \quad (4.3)$$

with

$$J_1(q^2) = \int \frac{d^4 p}{(2\pi)^4} \frac{2p\cdot(p + q) - 4m^2}{(p^2 - m^2)^2[(p + q)^2 - m^2]}.$$

The last term in the expression (4.3) for the EM self-energy of the pion is the contribution from Fig. 5. We have implicitly assumed that all quark loops have been regulated via the Pauli–Villars prescription[21] which respects the Ward–Takahashi identities.

We are now in a position to check the gauge invariance of the pion's EM self-energy explicitly. The common charge factor multiplying the gauge dependent piece proportional to $(1 - \lambda^{-1})$ in Eq. (4.3) is

$$-\text{Tr}(T_\pi^+ e_q T_\pi e_q) + \frac{1}{2} \text{Tr}(e_q^2 \{T_\pi^+, T_\pi\}) - e_\pi^2$$

$$= -\left(\frac{5}{9} e^2 - e_\pi^2\right) + \frac{1}{2}\left(\frac{10}{9} e^2\right) - e_\pi^2 = 0 \quad (4.4)$$

by direct calculation. Thus the EM self-energy is fully gauge invariant. Note the regulatory role that the "dumbbell" diagram of Fig. 5 has played in maintaining the gauge invariance in all three pion channels. Without this term, which provides for the last factor to cancel out the pion's charge e_π^2 in Eq. (4.4), only the neutral pion channel would have been gauge invariant.

We return to the $q\bar{q}$ scattering amplitude in Eq. (4.2) and identify the pion masses $m_{T_3}^2$ in the presence of EM interactions from the poles of this expression that lie at the roots, in k^2, of

$$1 - 2G\left[\prod(k^2) + \prod{}_{\mathrm{EM}}^{T_3}(k^2)\right] = 0.$$

Since the shift of the $m_{T_3}^2$ away from the zero mass Goldstone mode limit is $O(\alpha)$, we expand the left-hand side of the above expression about $k^2 = 0$ to find that

$$m_{T_3}^2 \approx -g_{\pi qq}^2 \prod{}_{\mathrm{EM}}^{T_3}(0) \tag{4.5}$$

after using the relation (2.9) again. The minus sign in this expression in the $T_3 = 0$ channel (the π^0) is ominous since we already know that Π_{EM}^0 must be positive. We return to this point later on. For the pion mass squared difference itself, we obtain the following exact formula from Eqs. (4.5) and (4.3), expressed as an integral over the photon four-momentum q,

$$m_{\pi^\pm}^2 - m_{\pi^0}^2 = \Delta m_\pi^2 = 3e^2 i \int \frac{d^4q}{(2\pi)^4}\frac{F_\pi(q^2)}{q^2 + i\epsilon}. \tag{4.6}$$

All gauge dependent and/or isoscalar pieces have cancelled explicitly. Note that the elastic electromagnetic form factor $F_\pi(q^2)$ of the pion appears in this formula, but with only a single power,* as opposed to the Barger–Kazes formula, Eq. (3.3), in which it

*It is interesting that Bijnens and de Rafael (Ref. 10) have come up with an essentially equivalent formula in an entirely different model.

appears squared. Note also that the appearance of the elastic form factor is something of a surprise, because the diagrams involve the inelastic, or half-off-shell, EM vertex. In this case, due to the extreme simplicity of the NJL model and the chiral limit, we find that the on- and off-shell vertices entering our calculation are equivalent. The question remains: why only single power? The answer to this question is provided by the form that the exact $q\bar{q}$ scattering amplitude assumes in terms of the elastic form factor as given in Eq. (2.13); while the diagrams in Figs. 4(a) through 4(c) turn out to be proportional to the pion form factor, the diagram in Fig. 5 contains two factors of $F_\pi(q^2)$. The inverse dependence of $M(k^2)$ on $F_\pi(k^2)$ cancels one of these that appear in the numerator of the dumbbell diagram, and we are left with the linear dependence shown in Eq. (4.6). We add the comment without proof that all of the EM vertices satisfy the relevant Ward–Takahashi identities, thereby confirming the gauge invariance in a more general way.

V. THE ELECTROMAGNETIC PROPERTIES OF THE NJL PION

The essential ingredient in all these expressions for the pion mass splitting is the pion's electromagnetic form factor. In this section we calculate this form factor for a chiral pion in NJL, and compare the result both with experiment and with the empirical monopole form. A direct bonus of the NJL model is that one obtains a closed form expression for $F_\pi(q^2)$ from which the charge distribution of the NJL pion may readily be calculated as described below. Let us start with $F_\pi(q^2)$. This form factor is related to the sum of the two Feynman diagrams shown in Fig. 6. A direct calculation shows

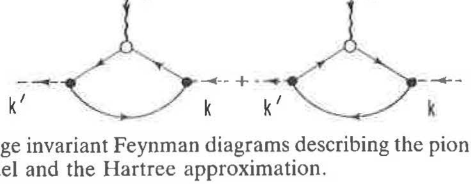


FIGURE 6 Gauge invariant Feynman diagrams describing the pion EM form factor in the NJL model and the Hartree approximation.

that $F_\pi(q^2)$ is exactly equal to the ratio $I(k^2)/I(0)$ in the chiral limit as already stated in Eq. (2.14). Performing the integral explicitly, one finds the closed form

$$F_\pi(-Q^2) = 1 + \frac{3m^2}{2\pi^2 f_\pi^2}\left[1 - \sqrt{1+\frac{4m^2}{Q^2}} \times \ln\left(\frac{Q}{2m} + \sqrt{1+\frac{Q^2}{4m^2}}\right)\right] \quad (5.1)$$

for space-like momentum transfers $q^2 = -Q^2 \leq 0$. One can check that $F_\pi(0) = 1$, as it should. In obtaining this answer, use has been made of the Goldberger–Treiman relation $m = f_\pi g_{\pi qq}$, which is automatically satisfied in the NJL model, to eliminate the quark–pion coupling constant in favor of the pion weak decay constant f_π.

The square of the expression (5.1) is plotted in Fig. 7 and compared with experiment and the empirical monopole form

$$F_\pi(q^2) = \left[1 - \frac{q^2}{m_V^2}\right]^{-1} \quad (5.2)$$

where $m_V \approx m_\rho = 770$ MeV is the rho meson mass. Note the closeness of the chiral, the non-chiral $F_\pi(q^2)^2$ and the monopole fit to the experimental data close to the origin. In particular, the calculated slope of the NJL $F_\pi(q^2)$ at the origin reproduces Tarrach's chiral limit result[22] for the pion charge radius squared, $\langle r_\pi^2\rangle = 3/(2\pi f_\pi)^2 = (0.59\text{ fm})^2$ that compares well with the measured value $(0.657(12)\text{ fm})^2$ of this quantity.[23b] Thus Eq. (5.1) essentially agrees with the monopole fit to the form factor and describes $F_\pi(q^2)$ very acceptably at low momentum transfer. At higher Q^2 values we see significant discrepancies, most notably the zero in the form factor at $-q^2 \approx 3(\text{GeV})^2$, signalled by the change to a positive slope in Fig. 7. All of this goes to show that the NJL model is a good approximation to nature only at low energies. The NJL form factor finally diverges logarithmically as Q^2 goes to infinity and the pion mass difference, as given by Eq. (4.6), diverges accordingly.

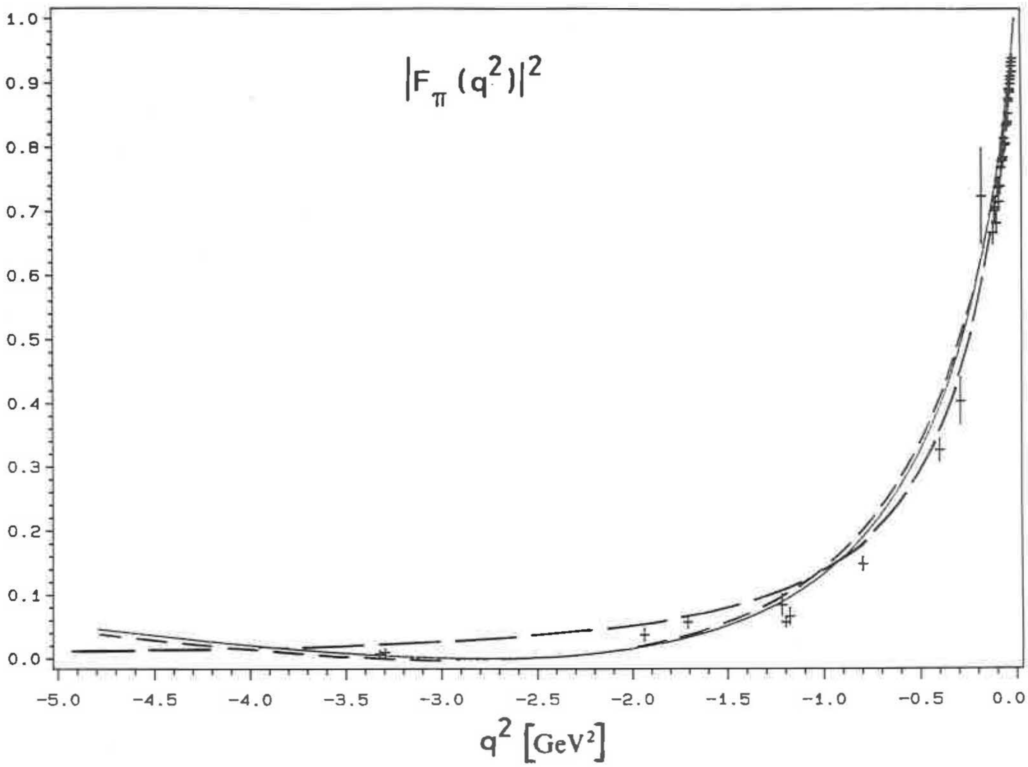


FIGURE 7 The square of the pion form factor in the space-like region $-5\ \mathrm{GeV}^2 < q^2 \leq 0$. Data are from Ref. 23(a). Note that $F_\pi(q^2)^2$, and not $F_\pi(q^2)$, enters the relevant cross-section from which it is determined experimentally. The solid curve is our result for a non-vanishing current quark mass ($m^\circ = 5$ MeV), while the short-dashed curve stands for our chiral-limit result, Eq. (5.1). Also shown is the empirical monopole fit (long-dashed curve), Eq. (5.2).

The same divergence problem can be seen from another point of view in configuration space, where the analogous quantities are the charge density and the Coulomb energy. The charge density can be calculated from the dispersive version of the form factor, which is readily obtained from our explicit form of $I(k^2)$ in Eq. (2.11), and methods familiar from Serber and Uehling's analysis[24] of the vacuum polarization diagram in QED. As opposed to that case, we find a closed form expression. The result is

$$\rho_\pi(r) = \left[1 - \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{dt}{t}\,\mathrm{Im}F_\pi(t)\right]\delta(\mathbf{r}) + \frac{3m^3}{4\pi^3 f_\pi^2}\,\frac{1}{r^2}\,K_1(2mr) \tag{5.3}$$

where K_1 is a modified cylindrical Bessel function. In view of the relation of $F_\pi(k^2)$ to $I(k^2)$ in Eq. (2.11), one can read off from the latter that the dispersive integral in Eq. (5.3) starts at the $q\bar{q}$ threshold $4m^2$, in direct conflict with confinement, as expected.

One can verify that this $\rho_\pi(r)$ is properly normalized to the total pionic charge consistent with $F_\pi(0) = 1$. In Fig. 8 we plot the radial charge distribution $4\pi r^2 \rho_\pi(r)$. For comparison we have also plotted the charge distribution due to the empirical monopole form for $F_\pi(-Q^2)$ given above in Eq. (5.2):

$$\rho_\pi(r) = \frac{m_V^2}{4\pi r}\, e^{-m_V r}. \tag{5.4}$$

The expression (5.4) shows the typical exponential decay for large r that is controlled by the vector meson mass m_V. In Fig. 8(a) we see that the result (5.4) is similar to Eq. (5.3) for r between 0.3 and 3 fm, despite the absence of explicit vector mesons in our Lagrangian. We see from Fig. 8(b) that the NJL model is unable to account for the interior region ($r \lesssim 0.3$ fm) of the pion, while reproducing the medium-range behavior reasonably well. This inadequacy comes from the small argument behavior of $K_1(2mr)$ which goes like $(2mr)^{-1}$ and thus causes the integral of the radial charge density to diverge like $\ln(mr)$ at $r \sim 0$. This again leads to an infinite EM mass difference.

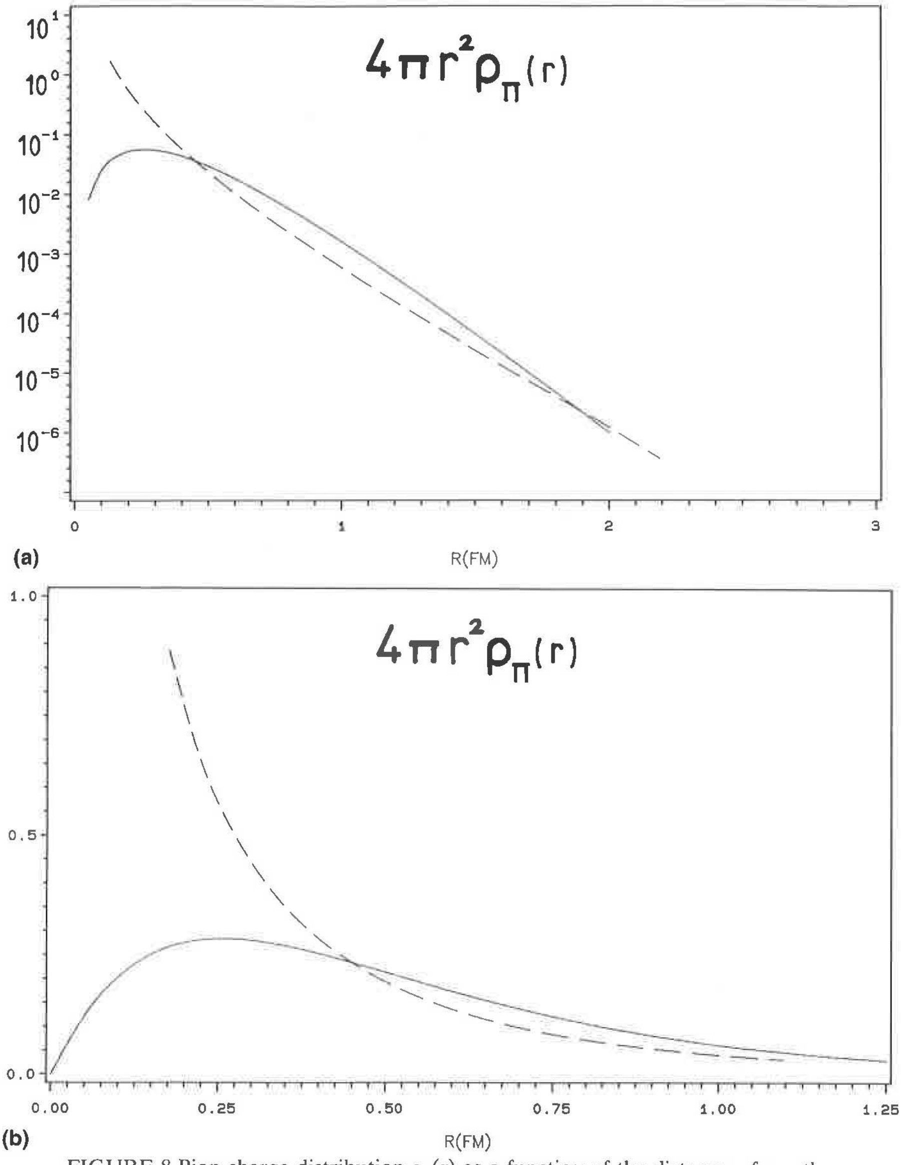


FIGURE 8 Pion charge distribution $\rho_\pi(r)$ as a function of the distance r from the center of the pion. The solid curve is the Fourier transformed monopole fit, Eq. (5.4), while the dashed curve represents our result, Eq. (5.3). Not shown is the singular behavior at the origin (see text). (a) Log-linear display of $4\pi r^2\rho_\pi(r)$ versus r; (b) linear display of $4\pi r^2\rho_\pi(r)$ versus r.

VI. EVALUATION OF THE PION MASS DIFFERENCE

The divergent natures of $F_\pi(-Q^2)$ at large space-like momenta, or of $\rho_\pi(r)$ at small r, that formally lead to an infinite EM mass difference for the pion, are of course symptoms of the same disease: the divergent nature of the NJL model. Fortunately we have forgotten one crucial point: the limitations set on the quark (Euclidean) four momenta in order to regularize the NJL model *also* constrain the *photon* momenta by momentum conservation; if the absolute values of the Euclidean quark momenta squared are limited by Λ^2, then the momentum squared carried by the photon cannot exceed $4\Lambda^2$. Equivalently, one must cut off the radial charge density at $r_o \approx (2\Lambda)^{-1}$, so that the inadequacy of the NJL form for $\rho_\pi(r)$ near the origin in Fig. 8(b) comes as no surprise. In other words, if we regularise the quark loops, the photon loops are automatically rendered finite without introducing yet another cut-off. Using this constraint, the evaluation of the integral expressing the mass difference is straightforward. After a Wick rotation of the photon four-momentum in Eq. (4.6) one finds

$$\Delta m_\pi^2 = \frac{3\alpha}{4\pi}\int_0^{4\Lambda^2} dQ^2 F_\pi(-Q^2).$$

Note that Δm_π^2 only depends on the behavior of the form factor in the space-like region. The remaining integration is elementary when the NJL form of $F_\pi(q^2)$, Eq. (5.1), is used, and

$$\Delta m_\pi^2 = \frac{3\alpha}{\pi} m^2 \left\{\left(1 + \frac{3}{2}\beta\right)\frac{\Lambda^2}{m^2} - \frac{1}{2}\beta\, sh^{-1}\frac{\Lambda}{m}\left[sh^{-1}\frac{\Lambda}{m} + 2\frac{\Lambda}{m}\sqrt{1 + \frac{\Lambda^2}{m^2}}\right]\right\} \tag{6.1}$$

where $\beta = 3m^2/(2\pi^2 f_\pi^2)$. This is our final result. We discuss its evaluation and how it impacts on NJL predictions of other physical quantities in a moment. However, before doing so, we would like to devote a few lines to the comparison with the result of Barger and Kazes. We have already asserted that the Barger–Kazes result violates both Ward–Takahashi identities, although the complete Compton amplitude is gauge invariant. But explicit calculations of Lee and Nieh[4] show that this is exactly the gauge invariant result one obtains from the simplest vector dominance model. Then how can it violate Ward identities? This apparent conflict is resolved as follows: in the VMD model the pion acquires an EM form factor due to the mixing between the photon and the neutral isovector vector meson, *not* because it (the pion) has an intrinsic structure. Hence the pion propagator remains the free one and the free Ward identities are satisfied due to the gauge invariant Lagrangian prescription of Kroll, Lee and Zumino,[25] and not due to the modification of the pion propagator. In the NJL model, on the other hand, the pion is a bound state with an intrinsic size and structure due its spatial extension. This structure substantially changes the propagation of the pion, and its Feynman propagator becomes significantly modified [compare with Eq. (2.13)] as compared with the free one. Here, the Ward identities are satisfied due to the inclusion of a complete set of gauge invariant Feynman diagrams. The first form of gauge invariance (VMD) leads to the Barger–Kazes mass formula, the second to our "linear in $F_\pi(q^2)$" mass formula.

Now let us evaluate Δm_π^2: in order to do so we need to know the two parameters of the NJL model: $G\Lambda^2$ and Λ. Up to now it has been customary to fix these by requiring that the quark condensate density $\langle\bar{\psi}\psi\rangle = \langle\bar{u}u\rangle + \langle\bar{d}d\rangle = 2\langle\bar{q}q\rangle$ and the pion weak decay constant f_π come out right. One typically has $\langle\bar{u}u\rangle = \langle\bar{d}d\rangle = -(250 \pm 50\ \text{MeV})^3$ and $f_\pi = 93$ MeV from experiment[1] (or, more accurately, $f_\pi = 92.4 \pm 0.2$ MeV, if $O(\alpha)$ EM corrections are considered[26]).

The details of the regularization of the quark loops enter at this point and we devote a few words to discussing the regularization problem. There are two quark loops to be regulated: the Hartree loop giving the condensate density or gap equation, and the pion self-energy loop. The divergence of the latter appears at zero momentum in the piece $iI(0)$, which determines $g_{\pi qq}^{-2}$. If we employ

Pauli–Villars regularization in order to stay consistent with our proof of gauge invariance, then one finds (see Ref. 13 for details)

$$\langle \bar{q}q \rangle = -\frac{3m^3}{4\pi^2}[(1 + 2x^2)\ln(1 + 2x^2) - 2(1 + x^2)\ln(1 + x^2)], \tag{6.2}$$

$$g_{\pi qq}^{-2} = \frac{f_\pi^2}{m^2} = \frac{3}{4\pi^2}[2\ln(1 + x^2) - \ln(1 + 2x^2)] \tag{6.3}$$

with $x = \Lambda_p/m$, where Λ_p is the Pauli–Villars regulating mass. We remark that it makes very little difference in practice if these expressions are replaced by their corresponding forms using a covariant cut-off, since the latter scheme with a cut-off $\Lambda^2 = (2\ln 2)\Lambda_p^2$ is essentially equivalent to the Pauli–Villars one with cut-off Λ_p^2. We exploit this feature to express the covariant cut-off appearing in the expression for Δm_π^2 in Eq. (6.1) in terms of x also, and then plot this together with $\langle \bar{q}q \rangle$ and m from Eqs. (6.2) and (6.3) as functions of x at fixed f_π.

These plots are shown in Fig. 9. Notice that the experimental range of quark condensate values is actually wider than that depicted in Fig. 9. In other words, the predicted range of the pion mass squared difference is at least (10–45 MeV)^2. This is a very wide range indeed and it clearly covers the experimental point at (35.55 MeV)^2. However, since we are dealing with an exact analytical expression for the mass difference in the Hartree approximation we realise that we can reverse the procedure and *determine* the quark condensate value from the experimental value of Δm_π^2 and f_π, instead of vice versa. We have thus found the "second pion observable" necessary for the complete determination of the free constants in the NJL model. These are now given by

$$\Lambda = 1088 \text{ MeV},$$

$$G\Lambda^2 = 3.82,$$

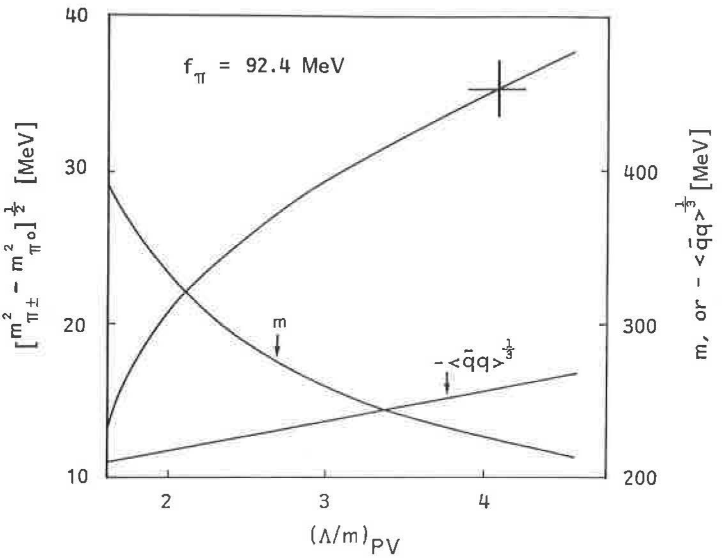


FIGURE 9 The pion EM mass difference, constituent quark mass m, and quark condensate $-\langle\bar{q}q\rangle^{1/3}$ in MeV, versus (Λ_p/m), as in Ref. 12. The pion weak decay constant is fixed at $f_\pi = 92.4$ MeV. Note that the vertical scale on the right-hand side refers to the quark mass and condensate, whereas the left-hand side scale refers to the pion mass difference. The experimental point $(\Delta m_\pi^2)^{1/2} = 35.55$ MeV is indicated by a cross.

so that

$$m = 225 \text{ MeV},$$

$$\langle\bar{q}q\rangle = -(260 \pm 1 \text{ MeV})^3.$$

We have used the EM corrected value $f_\pi = 92.4 \pm 0.2$ MeV mentioned previously and the measured value of $\Delta m_\pi^2 = (35.55 \pm 0.02 \text{ MeV})^2$ to make this determination. The "error" shown on the quark condensate reflects only the experimental errors in f_π and the pion mass difference. It is interesting to note that $\langle\bar{q}q\rangle$ falls nicely within the expected range for the quark condensate density. Furthermore, both the quark mass and the cut-off take on reasonable values for these parameters. Since the NJL pion form factor, which, as we have seen, gives a good description of the data, is central to these calculations, we draw the conclusion that the electromagnetic and weak decay properties of the pion provide an accurate method of determining quantities related to the breaking of chiral symmetry in the quark vacuum.

VII. DASHEN'S THEOREM

Let us note that it is not only $\pi^{\pm}$ which acquires a mass due to electromagnetism, but also the neutral pion. For the π^{o} all of this EM mass comes from the first three (effective sea-gull) diagrams in Fig. 4: the "Coulomb" graph (Fig. 4(a)) and the quark and antiquark EM self-energies (Figs. 4(b) and 4(c)). Hence it is not surprising that this EM mass squared shift of the π^{o} is negative because it is dominated by the negative Coulomb graph for all reasonable values of the cut-off. Evaluating Eq. (4.5) explicitly for the neutral pion one finds

$$m_{\pi^{o}}^{2} = -\left(\frac{10\alpha}{9\pi}\right)\beta\Lambda^{2}\left[1 - \frac{3}{2x^{2}}(sh^{-1}x)^{2}\right] \qquad (7.1)$$

where $x = \Lambda_{p}/m$ again. Numerically this π^{o} (mass)^{2} shift equals $-(43\text{ MeV})^{2}$ for the parameters established above. Since we started from a massless neutral pion, we seem to have obtained a tachyon—a clear sign of an instability in the theory.

This catastrophe is automatically and exactly remedied, in the chiral limit, by an additional gauge invariant Feynman diagram: the EM correction to the gap equation (Fig. 10). We have already seen in Section IV how the $q\bar{q}$ scattering amplitude $M(k^{2})$ is mod-

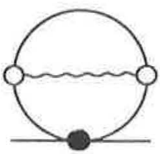


FIGURE 10 The constituent quark EM self-energy, to $O(\alpha)$, in the NJL model and the Hartree approximation. All lines and vertices are as defined in Fig. 1(b)–Fig. 6.

ified by the pion EM self-energy. As a consequence the mass of the neutral pion is shifted to an unphysical value *below* $k^2 = 0$ according to Eq. (7.1). So there is still a fundamental inconsistency in the calculation, or we have overlooked something again. What we in fact have overlooked is that the gap equation for the quark mass also contains EM corrections. This contribution to the quark self-energy is described by the (separately gauge invariant) diagram in Fig. 10. Calling this contribution Σ_{EM}, one finds by direct calculation that

$$\Sigma_{EM} = 2Gm\Pi^{o}_{EM}(0).$$

Inclusion of this graph into the gap equation *exactly* cancels the EM self-energy of the neutral pion! In detail,

$$m^2_{\pi^o} = -g^2_{\pi qq}[\Pi^{o}_{EM}(0) - (2Gm)^{-1}\Sigma_{EM}] \equiv 0.$$

This apparent miracle is a consequence of a deeper principle at work: the chiral symmetry. Roger Dashen has shown[15] on very general grounds that in the chiral limit of a chirally symmetric theory, the neutral pion must not acquire a mass due to EM self-interactions. We see that the corrected "gap" equation restores the self-consistency of the theory. The zero mass of the neutral pion comes about due to a cancellation of two separately gauge invariant sets of Feynman diagrams, as anticipated by Cabbibo and Maiani[27] in the linear sigma model. Note also that the magnitude of this additional term in Fig. 10 is $O(\alpha G\Lambda^2)$, i.e., a product of the EM and strong interactions, so that it cannot be treated perturbatively as an $O(\alpha)$ correction, but has to be included as a part of the self-consistent quark self-energy, as was done here. It is important to realise that Dashen's theorem is a consequence of the *exact* chiral symmetry; if the symmetry is no longer exact, due to, e.g., the finite current quark masses, even by a "small" amount, then the cancellation is not exact either, as can be explicitly shown in our model. This is not obvious from the general proof of the Dashen theorem: a simple-minded application of the same arguments to the non-chiral case seems to imply that the GMOR mass relation and the EM (mass)^2 are linearly additive. The cause of the breakdown of Dashen's theorem is, as anticipated by B. W.

Lee,[28] the presence of $O(\epsilon\alpha)$ terms, where $\epsilon = m^{o}/f_{\pi}$, which violate the theorem, but are not manifest in the above "derivation". These terms can be traced back to the violation of the basic assumption that, to lowest approximation, the axial current is conserved. In this sense our calculation is an example of chiral perturbation theory applied to our problem.

VIII. CONCLUSIONS AND OUTLOOK

We have presented here the results of a gauge invariant and chirally invariant calculation of the electromagnetic mass splitting of the pion in the chiral limit. The calculation was done in the two-flavor version of the original NJL model in the Hartree approximation. In agreement with Dashen's theorem, the (massless) neutral pion remains unshifted by electromagnetic interactions due to a subtle cancellation between two separately gauge invariant sets of Feynman diagrams, one of them being the electromagnetic modification of the gap equation. In order to obtain our main result for the pionic mass splitting, we also needed to calculate the electromagnetic form factor $F_{\pi}(q^2)$ of the pion. Only the space-like region of $F_{\pi}(q^2)$ contributes to the pionic mass shift; our result for $F_{\pi}(q^2)$ reproduces experimental data surprisingly well. Consequently we obtain the pion charge radius within 10%. The medium-range behavior of the pion charge distribution $\rho_{\pi}(r)$ is not inconsistent with that obtained in the ρ-dominance model (VMD).

There are several obvious tasks for the future:

(a) Include the (small) current quark mass effects into the calculation of the electromagnetic self-energy of the pion.
(b) Obtain a better description of the time-like $F_{\pi}(q^2)$.
(c) Extend the model calculation to three flavors in order to examine both pion and kaon mass splittings.
(d) Go beyond the Hartree approximation.
(e) Include confinement.

The first four problems, formidable as they might be, are perhaps of a more manageable nature; (e) is a fundamental problem. In either case they are likely to remain challenging for a long time to come.

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