

# Two dimensional symmetric and antisymmetric exponential functions and orthogonal polynomials

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**Abstract.** The symmetric and antisymmetric exponential functions of two variables, based on the permutation group  $S_2$ , are considered. Explicit formulas for the corresponding families of the orthogonal polynomials are derived and the properties of these orthogonal polynomials, such as their generating functions, continuous and discrete orthogonality, are presented. A connection of these polynomials with characteristic polynomials of some particular matrices is shown.

## 1. Introduction

The complex valued functions on the Euclidean space  $\mathbb{R}^n$ , symmetric or antisymmetric with respect to the permutation group  $S_n$ , are dealt with in quantum theory or theory of integrable systems. The functions considered in this paper form a family of special functions called the symmetric/antisymmetric exponential functions. They are introduced in general for any integer  $n$  in [9]. By bringing forward the construction of the associated orthogonal polynomials, we extend the study of their two dimensional case from [5]. After recalling the symmetric and antisymmetric exponential functions of two variables, we focus on families of the associated symmetric and antisymmetric orthogonal polynomials. A detailed study of the properties of the three variable exponential functions and the corresponding orthogonal polynomials is contained in the forthcoming paper [2].

Note that families of orthogonal polynomials of a similar construction are also explored elsewhere — see for example [1, 3, 7, 8, 14, 15, 19, 20, 21]. The symmetric and antisymmetric exponential functions are closely related to the symmetric [11], antisymmetric [12] and  $E$ -orbit functions [13] and to the corresponding orthogonal polynomials [4, 12, 16, 17, 18]. Through the inclusion of the alternating subgroup  $Alt_n \subset S_n$ , they are also related to alternating exponential functions [6, 10].

## 2. Two dimensional symmetric exponential functions and corresponding orthogonal polynomials

The two dimensional symmetric exponential functions  $E_{(k,l)}^+ : \mathbb{R}^2 \rightarrow \mathbb{C}$ , with  $k, l \in \mathbb{Z}$  and  $k \geq l$ , are defined by the formula

$$E_{(k,l)}^+(x, y) = \left| \frac{e^{2\pi i k x} e^{2\pi i k y}}{e^{2\pi i l x} e^{2\pi i l y}} \right|^+ = e^{2\pi i (kx+ly)} + e^{2\pi i (ky+lx)}. \quad (1)$$

Due to their periodicity and symmetry, the functions  $E_{(k,l)}^+$  can be considered on the closure of the fundamental domain  $F(S_2^{\text{aff}})$  only ([9]), where

$$F(S_2^{\text{aff}}) = \{(x, y) \in (0, 1) \times (0, 1) \mid x > y\}.$$

For detailed properties of these functions, we refer to the paper [5].

An explicit calculation of the corresponding orthogonal polynomials is based on the observation that multiplying two symmetric exponential functions gives a decomposition rule

$$E_{(k,l)}^+(x, y) E_{(m,n)}^+(x, y) = E_{(k+m, l+n)}^+(x, y) + E_{(k+n, l+m)}^+(x, y). \quad (2)$$

Introducing the following substitution for the two symmetric exponential functions,

$$X = E_{(1,0)}^+, \quad Y = E_{(1,1)}^+, \quad (3)$$

two-variable polynomials  $P_{(k,l)}^+(X, Y)$  labeled by the pairs  $(k, l)$ , with  $k \geq l$  and  $k, l \geq 0$ , are determined by the equation

$$P_{(k,l)}^+(X(x, y), Y(x, y)) = E_{(k,l)}^+(x, y).$$

Using the decomposition rule (2), we obtain the following table for the several lowest polynomials  $P_{(k,l)}^+(X, Y)$ :

$$\begin{array}{ll} P_{(0,0)}^+ = 2, & P_{(3,2)}^+ = \frac{1}{4}XY^2, \\ P_{(1,0)}^+ = X, & P_{(3,3)}^+ = \frac{1}{4}Y^3, \\ P_{(1,1)}^+ = Y, & P_{(4,0)}^+ = X^4 - 2X^2Y + \frac{1}{2}Y^2, \\ P_{(2,0)}^+ = X^2 - Y, & P_{(4,1)}^+ = \frac{1}{4}XY(2X^2 - 3Y), \\ P_{(2,1)}^+ = \frac{XY}{2}, & P_{(4,2)}^+ = \frac{1}{4}Y^2(X^2 - Y), \\ P_{(2,2)}^+ = \frac{Y^2}{2}, & P_{(4,3)}^+ = \frac{1}{8}XY^3, \\ P_{(3,0)}^+ = X^3 - \frac{3}{2}XY, & P_{(4,4)}^+ = \frac{1}{8}Y^4. \\ P_{(3,1)}^+ = \frac{1}{2}Y(X^2 - Y), & \end{array}$$

Below we determine directly the explicit formula and the generating function for the polynomials  $P_{(k,l)}^+$ . Note that in view of (2) we have, for any admissible  $n$  and  $k$ ,

$$\begin{aligned} P_{(n,k)}^+ P_{(0,0)}^+ &= P_{(n,k)}^+ + P_{(n,k)}^+ = 2P_{(n,k)}^+, \\ P_{(1,1)}^+ P_{(n-1,k-1)}^+ &= P_{(n,k)}^+ + P_{(n,k)}^+ = 2P_{(n,k)}^+, \end{aligned}$$

therefore  $P_{(n,k)}^+ = \frac{Y}{2} P_{(n-1,k-1)}^+$ . Iterating this relation, we see that

$$P_{(n,k)}^+ = \left(\frac{Y}{2}\right)^k P_{(n-k,0)}^+. \quad (4)$$

The recurrence relation (4) gives us

$$P_{(n+1,0)}^+ P_{(1,0)}^+ = P_{(n+1,1)}^+ + P_{(n+2,0)}^+ = \frac{Y}{2} P_{(n,0)}^+ + P_{(n+2,0)}^+,$$

and thus the polynomials  $P_{(k,l)}^+$  fulfill the following two-step recurrence relations

$$P_{(n+1,0)}^+ X = \frac{Y}{2} P_{(n,0)}^+ + P_{(n+2,0)}^+. \quad (5)$$

The equation (5) is a linear difference equation for  $P_{(n,0)}^+$  with constant coefficients and satisfying initial conditions  $P_{(0,0)}^+ = 2$ ,  $P_{(1,0)}^+ = X$ . The polynomial

$$P_{(n,0)}^+(X, Y) = \frac{1}{2^n} \left( \left( X - \sqrt{X^2 - 2Y} \right)^n + \left( X + \sqrt{X^2 - 2Y} \right)^n \right)$$

is an explicit solution of the difference equation (5). Now we can derive the following explicit formula for the polynomials  $P_{(n,k)}^+$ ,

$$P_{(n,k)}^+(X, Y) = \frac{Y^k}{2^n} \left( \left( X - \sqrt{X^2 - 2Y} \right)^{n-k} + \left( X + \sqrt{X^2 - 2Y} \right)^{n-k} \right).$$

Applying the formulae for the symmetric polynomials given in (4) and (5), we obtain the generating function  $G_+$  for  $P_{(n,k)}^+$

$$G_+(X, Y, r, s) = \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} P_{(n,k)}^+(X, Y) r^n s^k = \frac{4(Xr + Xs - rsY - 2)}{(r^2Y - 2rX + 2)(-s^2Y + 2sX - 2)}. \quad (6)$$

Note that the Jacobian  $J$  of the mapping (3) is equal to

$$J(x, y) = -8\pi^2 e^{2i\pi(x+y)} (e^{2i\pi x} - e^{2i\pi y}).$$

By changing the variables, the Jacobian  $J$  may be expressed as

$$J(X, Y) = -4\pi^2 Y \sqrt{X^2 - 2Y}.$$

Using the continuous orthogonality of the  $E_{(k,l)}^+$  functions [5, 9], the continuous orthogonality of the polynomials  $P_{(n,k)}^+$  follows,

$$\int_{F_2} P_{(n,k)}^+(X, Y) \overline{P_{(n',k')}^+(X, Y)} \frac{1}{-4\pi^2 Y \sqrt{X^2 - 2Y}} dX dY = G_{nk}^+ \delta_{nn'} \delta_{kk'}, \quad (7)$$

where the symbol  $G_{kl}^+$  is defined by

$$G_{nk}^+ = \begin{cases} 2, & n = k \\ 1, & \text{otherwise,} \end{cases}$$

and  $F_2$  is the transformed fundamental region  $F(S_2^{\text{aff}})$  via injective mapping (3):

$$F_2 = \{(X, Y) \in \mathbb{C}^2 \mid |Y| = 2, 2X = Y\bar{X}, |X| < 2\}.$$

Similarly, the discrete orthogonality relation from [5] yields the following formula for the discrete orthogonality of the polynomials  $P_{(n,k)}^+, P_{(n',k')}^+$  with  $n, k, n', k' \in \{0, 1, \dots, N-1\}$ ,

$$\begin{aligned} & \sum_{\substack{r,s=0 \\ r \geq s}}^{N-1} (G_{rs}^+)^{-1} P_{(n,k)}^+ \left( e^{2\pi i \frac{r}{N}} + e^{2\pi i \frac{s}{N}}, 2e^{2\pi i (\frac{r}{N} + \frac{s}{N})} \right) \overline{P_{(n',k')}^+ \left( e^{2\pi i \frac{r}{N}} + e^{2\pi i \frac{s}{N}}, 2e^{2\pi i (\frac{r}{N} + \frac{s}{N})} \right)} = \\ & = G_{nk}^+ N^2 \delta_{nn'} \delta_{kk'}. \end{aligned}$$

Note an interesting connection of the polynomials  $P_{(n,k)}^+$  with the characteristic polynomials of the following matrix. Let us consider  $n \times n$  matrix  $A^{(n)}$  with entries

$$A_{ij}^{(n)} = \min\{2i-1, 2j-1\}, \text{ i.e. } A^{(n)} = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ 1 & 3 & 3 & 3 & \dots \\ 1 & 3 & 5 & 5 & \dots \\ 1 & 3 & 5 & 7 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

By taking into account a suitable multiple of characteristic polynomials of  $A^{(n)}$  multiplied by scalar  $X^2$  in variable  $Y$ , a modified characteristic polynomial  $P_n(X, Y)$  may be expressed as

$$P_n(X, Y) = \frac{1}{2^{n-1}} \det(X^2 A^{(n)} - YI),$$

where  $I$  is the  $n \times n$  identity matrix. Computing several first  $P_n$ 's, we obtain

$$\begin{aligned} P_1(X, Y) &= X^2 - Y = P_{(2,0)}^+(X, Y), \\ P_2(X, Y) &= \frac{1}{2}(2X^4 - 4X^2Y + Y^2) = P_{(4,0)}^+(X, Y), \\ P_3(X, Y) &= \frac{1}{4}(4X^6 - 12X^4Y + 9X^2Y^2 - Y^3) = P_{(6,0)}^+(X, Y), \quad \text{etc.} \end{aligned}$$

From (2), by induction, we can derive the relation

$$P_n(X, Y) = P_{(2n,0)}^+(X, Y), \quad n \geq 0.$$

### 3. Two dimensional antisymmetric exponential functions and the corresponding orthogonal polynomials

The two dimensional antisymmetric exponential functions  $E_{(k,l)}^- : \mathbb{R}^2 \rightarrow \mathbb{C}$ , with  $k, l \in \mathbb{Z}$  and  $k > l$ , are defined similarly to the symmetric case by the formula

$$E_{(k,l)}^-(x, y) = \begin{vmatrix} e^{2\pi i kx} & e^{2\pi i ky} \\ e^{2\pi i lx} & e^{2\pi i ly} \end{vmatrix} = e^{2\pi i(kx+ly)} - e^{2\pi i(ky+lx)}.$$

Due to their periodicity and antisymmetry, the functions  $E_{(k,l)}^-$  can be considered on the fundamental domain  $F(S_2^{\text{aff}})$  only. For detailed properties of these functions, we refer to the paper [5].

The corresponding orthogonal polynomials are constructed in a similar manner as in the symmetric case. Note that the antisymmetric exponential functions fulfill the similar decomposition rule as the symmetric functions

$$E_{(k,l)}^-(x, y)E_{(m,n)}^-(x, y) = E_{(k+m, l+n)}^+(x, y) - E_{(k+n, l+m)}^+(x, y). \quad (8)$$

Two-variable polynomials  $P_{(k,l)}^-$  with the elementary variables (3) are labeled by the pairs  $(k, l)$ , with  $k > l$  and  $k, l \geq 0$ , and determined by the equation

$$P_{(k,l)}^-(X(x, y), Y(x, y)) = \frac{E_{(k,l)}^-(x, y)}{E_{(1,0)}^-(x, y)}.$$

Below we present a list of several lowest antisymmetric polynomials  $P_{(k,l)}^-$ :

$$\begin{aligned} P_{(1,0)}^- &= 1, & P_{(4,2)}^- &= \frac{XY^2}{4}, \\ P_{(2,0)}^- &= X, & P_{(4,3)}^- &= \frac{Y^3}{8}, \\ P_{(2,1)}^- &= \frac{Y}{2}, & P_{(5,0)}^- &= \frac{1}{4}(4X^4 - 6X^2Y + Y^2), \\ P_{(3,0)}^- &= X^2 - \frac{Y}{2}, & P_{(5,1)}^- &= \frac{1}{2}XY(X^2 - Y), \\ P_{(3,1)}^- &= \frac{XY}{2}, & P_{(5,2)}^- &= \frac{1}{8}Y^2(2X^2 - Y), \\ P_{(3,2)}^- &= \frac{Y^2}{4}, & P_{(5,3)}^- &= \frac{XY^3}{8}, \\ P_{(4,0)}^- &= X^3 - XY, & P_{(5,4)}^- &= \frac{Y^4}{16}, \\ P_{(4,1)}^- &= \frac{1}{4}Y(2X^2 - Y), \end{aligned}$$

Applying similar arguments as in Section 2, one can derive the explicit formula for the polynomials  $P_{(k,l)}^-$ ,

$$P_{(k,l)}^-(X, Y) = \frac{Y^l}{2^k \sqrt{X^2 - 2Y}} \left( \left( \sqrt{X^2 - 2Y} + X \right)^{k-l} - \left( X - \sqrt{X^2 - 2Y} \right)^{k-l} \right).$$

Their generating function, an analogy of (6), has the form

$$G_-(X, Y, r, s) = \sum_{k=0}^{+\infty} \sum_{l=0}^{+\infty} P_{(k,l)}^-(X, Y) r^k s^l = \frac{4(s-r)}{(r^2Y - 2rX + 2)(-s^2Y + 2sX - 2)}.$$

The antisymmetric polynomials  $P_{(k,l)}^-$  are continuously orthogonal with respect to the scalar product

$$\int_{F_2} P_{(k,l)}^-(X, Y) \overline{P_{(k',l')}^-(X, Y)} \frac{\sqrt{X^2 - 2Y}}{2\pi^2 Y^2} dX dY = \delta_{kk'} \delta_{ll'}$$

and their discrete orthogonality relation, for  $k, l, k', l' \in \{0, 1, \dots, N-1\}$ , is expressed by the following formula

$$\sum_{\substack{r,s=0 \\ r>s}}^{N-1} P_{(k,l)}^{-}\left(e^{2\pi i \frac{r}{N}} + e^{2\pi i \frac{s}{N}}, 2e^{2\pi i (\frac{r}{N} + \frac{s}{N})}\right) \overline{P_{(k',l')}^{-}\left(e^{2\pi i \frac{r}{N}} + e^{2\pi i \frac{s}{N}}, 2e^{2\pi i (\frac{r}{N} + \frac{s}{N})}\right)} \times \\ \left(-e^{-2i\pi(\frac{r}{N} + \frac{s}{N})} \left(e^{\frac{2i\pi r}{N}} - e^{\frac{2i\pi s}{N}}\right)^2\right) = N^2 \delta_{kk'} \delta_{ll'}.$$

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