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Special Issue
*Universe*: Feature Papers 2024 – Compact Objects
Edited by
Prof. Dr. Nicolas Chamel

MDPI

https://doi.org/10.3390/universe10110420 universe

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# Comparing a Gauge-Invariant Formulation and a "Conventional Complete Gauge-Fixing Approach" for $l = 0, 1$-Mode Perturbations on the Schwarzschild Background Spacetime

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**Abstract:** This article provides a comparison of the gauge-invariant formulation for $l = 0, 1$-mode perturbations on the Schwarzschild background spacetime, proposed by the same author in 2021, and a "conventional complete gauge-fixing approach" where the spherical harmonic functions $Y_{lm}$ as the scalar harmonics are used from the starting point. Although it is often stated that "gauge-invariant formulations in general-relativistic perturbations are equivalent to complete gauge-fixing approaches", we conclude that, as a result of this comparison, the derived solutions through the proposed gauge-invariant formulation and those through a "conventional complete gauge-fixing approach" are different. It is pointed out that there is a case where the boundary conditions and initial conditions are restricted in a conventional complete gauge-fixing approach.

**Keywords:** black hole; Schwarzschild spacetime; perturbation theory; gauge-invariance; gauge-fixing

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**Citation:** Nakamura, K. Comparing a Gauge-Invariant Formulation and a "Conventional Complete Gauge-Fixing Approach" for $l = 0, 1$-Mode Perturbations on the Schwarzschild Background Spacetime. *Universe* **2024**, *10*, 420. https://doi.org/10.3390/universe10110420

Academic Editor: Aharon Davidson

Received: 27 September 2024
Revised: 31 October 2024
Accepted: 31 October 2024
Published: 7 November 2024



## 1. Introduction

Through ground-based gravitational-wave detectors [1–4], many events of gravitational waves—mainly from black hole–black hole coalescences—have now been detected. We have reached a stage where there is no doubt about the existence of gravitational waves due to direct observations. One future direction of gravitational-wave astronomy will focus on achieving "precise science" through the statistics of numerous events. To further develop gravitational-wave science, projects focused on future ground-based gravitational-wave detectors [5,6] are also progressing towards creating more sensitive detectors. Some projects discussing space gravitational-wave antenna are also progressing [7–10]. Among the various targets for these detectors, the extreme mass ratio inspirals (EMRI), which produce gravitational waves from the motion of a stellar mass object around a supermassive black hole, are promising targets of the Laser Interferometer Space Antenna [7]. Since the mass ratio of this EMRI is very small, we can describe the gravitational waves from EMRIs through black hole perturbations [11]. Furthermore, the sophistication of higher-order black hole perturbation theories is required to support gravitational-wave physics as a precise science. The motivation of our previous series of papers, refs. [12–16], as well as this current paper, lies in enhancing the theoretical sophistication of black hole perturbation theories toward higher-order perturbations.

Realistic black holes have angular momentum, which requires us to apply the perturbation theory of a Kerr black hole for direct analyses related to the EMRI. However, we may say that further sophistication is possible, even in perturbation theories on the Schwarzschild background spacetime. Starting with the pioneering works by Regge and Wheeler [17] and Zerilli [18,19], there have been many studies on the perturbations in the Schwarzschild background spacetime [20–32]. In these works, perturbations are decomposed through the spherical harmonics $Y_{lm}$ because of the spherical symmetry of the background spacetime, and $l = 0, 1$ modes should be separately treated. These modes correspond to the monopole and dipole perturbations. These separate treatments make "gauge-invariant" treatments for $l = 0$ and $l = 1$ modes unclear.

Due to this situation, in our previous papers [12–16], we proposed a strategy of the gauge-invariant treatments of these $l = 0, 1$-mode perturbations. This strategy is outlined as Proposal 1 in Section 2 of this paper below. One of the key premises of our gauge-invariant perturbations is the distinction between the first-kind gauge and the second-kind gauge. The first-kind gauge is essentially the choice of the coordinate system on the single manifold, and we often use this first-kind gauge when we predict or interpret the measurement results of experiments and observations. On the other hand, the second-kind gauge is the choice of the point identifications between the points on the physical spacetime $\mathscr{M}_\epsilon$ and the background spacetime $\mathscr{M}$. This second-kind gauge has nothing to do with our physical spacetime $\mathscr{M}_\epsilon$. Although this difference is extensively explained in a previous paper [14], we also emphasize this difference in Section 2 of this paper. The proposal in the previous paper [14] is part of our development of the general formulations of a higher-order gauge-invariant perturbation theory on a generic background spacetime that is aimed toward unambiguous sophisticated nonlinear general-relativistic perturbation theories [33–38]. While we have applied this general framework to cosmological perturbations [39–41], we also applied it to black hole perturbations in our series of previous papers (i.e., refs. [12–16]), and in this paper. Even in the context of cosmological perturbation theories, the same problem as the above $l = 0, 1$-mode problem exists in gauge-invariant treatments of homogeneous modes of perturbations. In this sense, we can expect that the proposal in the previous paper [14] will be a clue to the same problem in gauge-invariant perturbation theory on the generic background spacetime.

In the previous paper [14], we also derived linearized Einstein equations in a gauge-invariant manner, following Proposal 1. Perturbations on the spherically symmetric background spacetime are classified into even- and odd-mode perturbations. In the same paper [14], we also provided a strategy to solve the odd-mode perturbations, including $l = 0, 1$ modes. Furthermore, we also derived the formal solutions for the $l = 0, 1$ odd-mode perturbations to the linearized Einstein equations, following Proposal 1. In another paper [15], we develop a strategy for solving the even-mode perturbations, including $l = 0, 1$ modes, and derived the formal solutions for the $l = 0, 1$ even-mode perturbations, also based on Proposal 1. In a further paper [16], we found the fact that the derived solutions in [15] realized the linearized version of two exact solutions: the Lemaître–Tolman–Bondi (LTB) solution [42] and the non-rotating C-metric [43,44]. This finding leads us to conclude that the solutions for even-mode perturbations derived in [15] are physically reasonable. This series of previous papers is the full version of our short paper, ref. [12]. Furthermore, brief discussions on the extension to the higher-order perturbations are given in [13].

On the other hand, it is well known that we cannot construct gauge-invariant variables for $l = 0, 1$ modes in the same manner as we do for $l \geq 2$ modes if we decompose the metric perturbations through the spherical harmonics as the scalar harmonics from the starting point. For this reason, we usually use gauge-fixing approaches. Furthermore, it is often said that "gauge-invariant formulations in general-relativistic perturbations are equivalent to complete gauge-fixing approaches". In this paper, we examine this statement through a comparison of our proposed gauge-invariant formulation, where we introduce singular harmonics at once, and then regularize them after the derivation of the mode-by-mode Einstein equation, with a "conventional complete gauge-fixing approach", where the spherical harmonic functions $Y_{lm}$ as the scalar harmonics are used from the starting point. As a result of this comparison, we conclude that our gauge-invariant formulation and the above "conventional complete gauge-fixing approach" are different, although these two formulations lead to similar solutions for $l = 0, 1$-mode perturbations. More specifically, there is a case where the boundary conditions and initial conditions are restricted in a "conventional complete gauge-fixing approach" where the decomposition of the metric perturbation by the spherical harmonics $Y_{lm}$ is used from the starting point.

The organization of this paper is as follows. In Section 2, we briefly review the premise of our series of papers [12–16], which are necessary for the ingredients of this paper. We also emphasize the difference between the concepts of the first- and second-kind gauges, and summarize the linearized Einstein equations for $l = 0, 1$ modes, and their formal solutions, in Section 2. In Section 3, we specify the rule of our comparison between our gauge-invariant formulation and a conventional gauge-fixing approach, and summarize the gauge-transformation rules for the metric perturbations of $l = 0, 1$ modes, because the above statement "gauge-invariant formulations in general-relativistic perturbations are equivalent to complete gauge-fixing approaches" includes some ambiguity. In Section 4, we discuss the linearized Einstein equations for $l = 1$ odd-modes and their solutions in the conventional gauge-fixing approach. In Section 5, we discuss the linearized Einstein equations for $l = 1$ even-modes and their solutions in the conventional gauge-fixing approach. In Section 6, we derive the solution to the linearized Einstein equations for $l = 0$ modes through the complete gauge-fixing, and discuss the comparison with the linearized LTB solution. Section 7 is devoted to a summary of this paper and discussions based on our results.

We use the notation used in the previous papers [12–16], and the unit $G = c = 1$, where $G$ is Newton's constant of gravitation and $c$ is the velocity of light.

## 2. Brief Review of a Gauge-Invariant Treatment of $l$ = 0, 1 Modes

In this section, we briefly review the premises of our series of papers [12–16] that are essential for the arguments of this paper. In Section 2.1, we introduce our framework of the gauge-invariant perturbation theory [33,34]. This framework is an crucial for our series of papers [12–16] and the current paper. Section 2.2 reviews the gauge-invariant perturbation theory on spherically symmetric spacetimes, which includes our proposal in refs. [12–16]. In Section 2.3, we summarize the linearized Einstein equations for $l = 1$ odd modes, and provide their "formal" solutions. Finally, in Section 2.4, we summarize the linearized Einstein equations for $l = 0, 1$ even modes with their "formal" solutions. The equations and their solutions in Sections 2.3 and 2.4 are derived based on our proposal in refs. [12–16]. These are necessary for the arguments presented in this paper.

### 2.1. General Framework of Gauge-Invariant Perturbation Theory

In any perturbation theory, we always consider two distinct spacetime manifolds. One is the physical spacetime $(\mathscr{M}_{\rm ph}, \bar{g}_{ab})$, which is identified with our nature itself, and we want to describe this spacetime $(\mathscr{M}_{\rm ph}, \bar{g}_{ab})$ by perturbations. The other is the background spacetime $(\mathscr{M}, g_{ab})$, which is prepared as a reference by hand. Furthermore, in any perturbation theory, we always write equations for the perturbation of the variable $Q$ as follows:

$$Q(\text{“}p\text{”}) = Q_0(p) + \delta Q(p). \tag{1}$$

In this equation, $Q(\text{“}p\text{”})$ in the left-hand side of Equation (1) is a variable on the physical spacetime $\mathscr{M}_\epsilon = \mathscr{M}_{\rm ph}$, whereas $Q_0(p)$ and $\delta Q(p)$ in the right-hand side of Equation (1) are variables on the background spacetime $\mathscr{M}$. Because we regard Equation (1) as a field equation, Equation (1) includes an implicit assumption of the existence of a point identification map $\mathscr{X}_\epsilon : \mathscr{M} \to \mathscr{M}_\epsilon : p \in \mathscr{M} \mapsto \text{“}p\text{”} \in \mathscr{M}_\epsilon$. This identification map is a "gauge choice" in general-relativistic perturbation theories. This is the notion of the "second-kind gauge" pointed out by Sachs [45–48]. Note that this second-kind gauge is a different notion from the degree of freedom of the coordinate transformation on a single manifold, which is called the "first-kind gauge" [14,39]. This distinction between the first- and the second-kind gauges was extensively explained in [14], and is also crucial for the understanding of the ingredients in this paper.

To compare the variable $Q$ on $\mathscr{M}_\epsilon$ with its background value $Q_0$ on $\mathscr{M}$, we use the pull-back $\mathscr{X}_\epsilon^*$ of the identification map $\mathscr{X}_\epsilon : \mathscr{M} \to \mathscr{M}_\epsilon$, and we evaluate the pulled-back variable $\mathscr{X}_\epsilon^* Q$ on the background spacetime $\mathscr{M}$. Furthermore, in perturbation theories, we expand the pull-back operation $\mathscr{X}_\epsilon^*$ to the variable $Q$ with respect to the infinitesimal parameter $\epsilon$ for the perturbation as follows:

$$\mathscr{X}_\epsilon^* Q = Q_0 + \epsilon {}^{(1)}_{\mathscr{X}}Q + O(\epsilon^2). \tag{2}$$

Equation (2) is evaluated on the background spacetime $\mathscr{M}$. When we have two different gauge choices, $\mathscr{X}_\epsilon$ and $\mathscr{Y}_\epsilon$, we can consider the "gauge-transformation", which is the change in the point-identification $\mathscr{X}_\epsilon \to \mathscr{Y}_\epsilon$. This gauge-transformation is given by the diffeomorphism $\Phi_\epsilon := (\mathscr{X}_\epsilon)^{-1} \circ \mathscr{Y}_\epsilon : \mathscr{M} \to \mathscr{M}$. Actually, the diffeomorphism $\Phi_\epsilon$ induces a pull-back from the representation $\mathscr{X}_\epsilon^* Q_\epsilon$ to the representation $\mathscr{Y}_\epsilon^* Q_\epsilon$, as $\mathscr{Y}_\epsilon^* Q_\epsilon(q) = \Phi_\epsilon^* \mathscr{X}_\epsilon^* Q_\epsilon(q)$ at any point $q \in \mathscr{M}$. From general arguments of the Taylor expansion [49–51], the pull-back $\Phi_\epsilon^*$ is expanded as

$$\mathscr{Y}_\epsilon^* Q_\epsilon(q) = \mathscr{X}_\epsilon^* Q_\epsilon(q) + \epsilon \pounds_{\xi_{(1)}} \mathscr{X}_\epsilon^* Q_\epsilon(q) + O(\epsilon^2), \tag{3}$$

where $\xi^a_{(1)}$ is the generator of $\Phi_\epsilon$. From Equations (2) and (3), the gauge-transformation for the first-order perturbation ${}^{(1)}Q$ is given by

$${}^{(1)}_{\mathscr{Y}}Q(q) - {}^{(1)}_{\mathscr{X}}Q(q) = \pounds_{\xi_{(1)}} Q_0(q) \tag{4}$$

at any point $q \in \mathscr{M}$. We also employ the "order by order gauge invariance" as a concept of gauge invariance [33,34]. We call the $k$th-order perturbation ${}^{(k)}_{\mathscr{X}}Q$ gauge-invariant if and only if

$${}^{(k)}_{\mathscr{X}}Q(q) = {}^{(k)}_{\mathscr{Y}}Q(q) \tag{5}$$

for any gauge choice $\mathscr{X}_\epsilon$ and $\mathscr{Y}_\epsilon$ at any point of $q \in \mathscr{M}$.

Based on the above setup, we proposed a formulation to construct gauge-invariant variables of higher-order perturbations [33,34]. First, we expand the metric on the physical spacetime $\mathscr{M}_\epsilon$, which has been pulled back to the background spacetime $\mathscr{M}$ through a gauge choice, $\mathscr{X}_\epsilon$, as follows:

$$\mathscr{X}_\epsilon^* \bar{g}_{ab} = g_{ab} + \epsilon\, {}_{\mathscr{X}}h_{ab} + O(\epsilon^2). \tag{6}$$

Although the expression (6) depends entirely on the gauge choice $\mathscr{X}_\epsilon$, henceforth, we do not explicitly express the index of the gauge choice $\mathscr{X}_\epsilon$ in the expression if there is no risk of confusion. A key premise of our formulation of higher-order gauge-invariant perturbation theory was the following conjecture [33,34] for the linear metric perturbation $h_{ab}$.

**Conjecture 1.** *If the gauge-transformation rule for a perturbative pulled-back tensor field $h_{ab}$ from the physical spacetime $\mathscr{M}_\epsilon$ to the background spacetime $\mathscr{M}$ is given by ${}_{\mathscr{Y}}h_{ab} - {}_{\mathscr{X}}h_{ab} = \pounds_{\xi_{(1)}} g_{ab}$ with the background metric $g_{ab}$, then there exists a tensor field $\mathscr{F}_{ab}$ and a vector field $Y^a$, such that $h_{ab}$ is decomposed as $h_{ab} =: \mathscr{F}_{ab} + \pounds_Y g_{ab}$, where $\mathscr{F}_{ab}$ and $Y^a$ are transformed as ${}_{\mathscr{Y}}\mathscr{F}_{ab} - {}_{\mathscr{X}}\mathscr{F}_{ab} = 0$ and ${}_{\mathscr{Y}}Y^a - {}_{\mathscr{X}}Y^a = \xi^a_{(1)}$ under the gauge-transformation, respectively.*

We call $\mathscr{F}_{ab}$ and $Y^a$ the "gauge-invariant" and "gauge-dependent" parts of $h_{ab}$, respectively.

The proof of Conjecture 1 is highly nontrivial [35–37], and it was found that the gauge-invariant variables are essentially non-local. Despite this non-triviality, once we accept Conjecture 1, we can decompose the linear perturbation of an arbitrary tensor field ${}^{(1)}_{\mathscr{X}}Q$, whose gauge-transformation is given by Equation (4), through the gauge-dependent part $Y_a$ of the metric perturbation in Conjecture 1, as

$${}^{(1)}_{\mathscr{X}}Q = {}^{(1)}\mathscr{Q} + \pounds_{{}_{\mathscr{X}}Y} Q_0, \tag{7}$$

where ${}^{(1)}\mathscr{Q}$ is the gauge-invariant part of the perturbation ${}^{(1)}_{\mathscr{X}}Q$. As examples, the linearized Einstein tensor ${}^{(1)}_{\mathscr{X}}G_a{}^b$ and the linear perturbation of the energy–momentum tensor ${}^{(1)}_{\mathscr{X}}T_a{}^b$ are also decomposed as follows:

$$ {}^{(1)}_{\mathscr{X}}G_a{}^b = {}^{(1)}\mathscr{G}_a{}^b[\mathscr{F}] + \pounds_{\mathscr{X}Y}G_a{}^b, \quad {}^{(1)}_{\mathscr{X}}T_a{}^b = {}^{(1)}\mathscr{T}_a{}^b + \pounds_{\mathscr{X}Y}T_a{}^b, \tag{8} $$

where $G_{ab}$ and $T_{ab}$ are the background values of the Einstein tensor and the energy–momentum tensor, respectively [34,41]. Through the background Einstein equation $G_a{}^b = 8\pi T_a{}^b$, the linearized Einstein equation ${}^{(1)}_{\mathscr{X}}G_{ab} = 8\pi{}^{(1)}_{\mathscr{X}}T_{ab}$ is automatically given in the gauge-invariant form

$$ {}^{(1)}\mathscr{G}_a{}^b[\mathscr{F}] = 8\pi{}^{(1)}\mathscr{T}_a{}^b[\mathscr{F},\phi], \tag{9} $$

even if the background Einstein equation is nontrivial. Here, "$\phi$" in Equation (9) symbolically represents the matter degree of freedom.

Finally, we comment on the coordinate transformation induced by the gauge-transformation $\Phi_\epsilon$ of the second kind [12,39]. To see this, we introduce the coordinate system $\{O_\alpha, \psi_\alpha\}$ on the background spacetime $\mathscr{M}$, where $O_\alpha$ are open sets on the background spacetime and $\psi_\alpha$ are diffeomorphisms from $O_\alpha$ to $\mathbb{R}^4$ ($4 = \dim \mathscr{M}$). The coordinate system $\{O_\alpha, \psi_\alpha\}$ is the set of collections of the pair of open sets $O_\alpha$ and diffeomorphism $\psi_\alpha : O_\alpha \mapsto \mathbb{R}^4$. If we employ a gauge choice $\mathscr{X}_\epsilon$ of the second kind, we have the correspondence of the physical spacetime $\mathscr{M}_\epsilon$ and the background spacetime $\mathscr{M}$. Together with the coordinate system $\psi_\alpha$ on $\mathscr{M}$, this correspondence $\mathscr{X}_\epsilon$ between $\mathscr{M}_\epsilon$ and $\mathscr{M}$ induces the coordinate system on $\mathscr{M}_\epsilon$. Actually, $\mathscr{X}_\epsilon(O_\alpha)$ for each $\alpha$ is an open set of $\mathscr{M}_\epsilon$. Then, $\psi_\alpha \circ \mathscr{X}_\epsilon^{-1}$ becomes a diffeomorphism from an open set $\mathscr{X}_\epsilon(O_\alpha) \subset \mathscr{M}_\epsilon$ to $\mathbb{R}^4$. This diffeomorphism $\psi_\alpha \circ \mathscr{X}_\epsilon^{-1}$ induces a coordinate system of an open set on $\mathscr{M}_\epsilon$. When we have two different gauge choices, $\mathscr{X}_\epsilon$ and $\mathscr{Y}_\epsilon$, of the second kind, $\psi_\alpha \circ \mathscr{X}_\epsilon^{-1} : \mathscr{M}_\epsilon \mapsto \mathbb{R}^4$ ($\{x^\mu\}$) and $\psi_\alpha \circ \mathscr{Y}_\epsilon^{-1} : \mathscr{M}_\epsilon \mapsto \mathbb{R}^4$ ($\{y^\mu\}$) become different coordinate systems on $\mathscr{M}_\epsilon$. We can also consider the coordinate transformation from the coordinate system $\psi_\alpha \circ \mathscr{X}_\epsilon^{-1}$ to another coordinate system $\psi_\alpha \circ \mathscr{Y}_\epsilon^{-1}$. Because the gauge-transformation $\mathscr{X}_\epsilon \to \mathscr{Y}_\epsilon$ is induced by the diffeomorphism $\Phi_\epsilon := (\mathscr{X}_\epsilon)^{-1} \circ \mathscr{Y}_\epsilon$, this diffeomorphism $\Phi_\epsilon$ induces the coordinate transformation as

$$ y^\mu(q) := x^\mu(p) = \left((\Phi_\epsilon^{-1})^* x^\mu\right)(q), \tag{10} $$

in the passive point of view [33,49], where $p$, $q \in \mathscr{M}$ are identified to the same point "$p$" $\in \mathscr{M}_\epsilon$ by the gauge choices $\mathscr{X}_\epsilon$ and $\mathscr{Y}_\epsilon$, respectively. If we represent this coordinate transformation in terms of the Taylor expansion (3), we have the coordinate transformation

$$ y^\mu(q) = x^\mu(q) - \epsilon\xi^\mu_{(1)}(q) + O(\epsilon^2). \tag{11} $$

We should emphasize that the coordinate transformation (11) is not the starting point of the gauge-transformation, but a result of the above framework.

On the other hand, we may consider the point replacement by the one-parameter diffeomorphism $s = \Psi_\lambda(r)$, where $s, r \in \mathscr{M}_\epsilon$ and $\lambda$ is an infinitesimal parameter satisfying $\Psi_{\lambda=0}(r) = r$. The pull-back $\Psi_\lambda^*$ of any tensor field $Q$ on $\mathscr{M}_\epsilon$ is given by

$$ \begin{aligned} Q(s) &= Q(\Psi_\lambda(r)) = (\Psi_\lambda^*)Q(r) \\ &= Q(r) + \lambda \pounds_\zeta Q\big|_{\lambda=0}(r) + O(\lambda^2), \end{aligned} \tag{12} $$

where $\zeta^a$ is the generator of the pull-back $\Psi_\lambda^*$. Equation (12) is just the formula of the Taylor expansion on the manifold $\mathscr{M}_\epsilon$ without using any coordinate system. At the same time, Equation (12) is the definition of the Lie derivative $\pounds_\zeta$ and its generator $\zeta^a$. In the literature, this formula is derived from the coordinate transformation

$$y^\mu(s) := x^\mu(r) + \lambda\zeta^\mu(r) + O(\lambda^2), \quad (13)$$

as explained in [14]. The Formula (12) of the Taylor expansion is defined without any coordinate system; the second term in the right-hand side of Equation (12) represents the actual difference of the tensor fields $Q(s)$ and $Q(r)$ in the different point $s \neq r$ and its physical meaning.

If we consider a formal metric decomposition $\bar{g}_{ab} = {}^{(0)}g_{ab} + \lambda h_{ab} + O(\lambda^2)$ within $\mathscr{M}_\epsilon$ as an example of the tensor field $Q$ in Equation (12), Formula (12) is given by

$${}^{(0)}g_{ab}(s) + \lambda h_{ab}(s) = {}^{(0)}g_{ab}(r) + \lambda\left(h_{ab}(r) + \pounds_\zeta {}^{(0)}g_{ab}\Big|_r\right) + O(\lambda^2). \quad (14)$$

Since the formula of the Taylor expansion (12) is derived from the infinitesimal coordinate transformation rule (13) in the literature, the term $\pounds_\zeta {}^{(0)}g_{ab}\Big|_r$ in the right-hand side of Equation (14) is often called "the degree of freedom of the coordinate transformation" and "unphysical degree of freedom". However, since the formula (14) is just a re-expression of the Taylor expansion (12) defined without any coordinate system, the term $\pounds_\zeta {}^{(0)}g_{ab}\Big|_r$ in Equation (14) should have its physical meaning. If we insist that the term $\pounds_\zeta {}^{(0)}g_{ab}\Big|_r$ that appears due to the point replacement on the single manifold is "unphysical", this directly leads the statement that "the famous arguments of the Killing vector and the symmetry of the spacetime are physically meaningless". Therefore, we have to emphasize that we "cannot" regard the second term of the right-hand side of Equation (12) as an "unphysical degree of freedom".

In our series of papers [12–16] and in this paper, we classify the term $\pounds_\zeta {}^{(0)}g_{ab}\Big|_r$ in Equation (14) as one of "gauge-degree of freedom of the first kind". This classification is based on the fact that this term can be eliminated the infinitesimal coordinate transformation (13) which operates within the single manifold $\mathscr{M}_\epsilon$. Furthermore, Equation (14) does not imply that $h_{ab}(s) = h_{ab}(r) + \pounds_\zeta {}^{(0)}g_{ab}\Big|_r$; rather, it indicates that ${}^{(0)}g_{ab}(s) = {}^{(0)}g_{ab}(r) + \lambda\pounds_\zeta {}^{(0)}g_{ab}\Big|_r + O(\lambda^2)$. Moreover, we also have to emphasize that the infinitesimal coordinate transformation (13) is essentially different from the infinitesimal coordinate transformation (11). The former represents the replacement of a point within the single manifold $\mathscr{M}_\epsilon$, while the latter refers to a change in the coordinate label at the same point in the background spacetime $\mathscr{M}$.

### 2.2. Linear Perturbations on Spherically Symmetric Background Spacetime

Here, we consider the 2+2 formulation of the perturbation of a spherically symmetric background spacetime, which originally proposed by Gerlach and Sengupta [24–27]. Spherically symmetric spacetimes are characterized by the direct product $\mathscr{M} = \mathscr{M}_1 \times S^2$, and their metric is

$$g_{ab} = y_{ab} + r^2\gamma_{ab}, \quad (15)$$

$$y_{ab} = y_{AB}(dx^A)_a(dx^B)_b, \quad \gamma_{ab} = \gamma_{pq}(dx^p)_a(dx^q)_b, \quad (16)$$

where $x^A = (t, r)$, $x^p = (\theta, \phi)$, and $\gamma_{pq}$ is the metric on the unit sphere. In the Schwarzschild spacetime, the metric (15) is given by

$$y_{ab} = -f(dt)_a(dt)_b + f^{-1}(dr)_a(dr)_b, \quad f = 1 - \frac{2M}{r}, \quad (17)$$

$$\gamma_{ab} = (d\theta)_a(d\theta)_b + \sin^2\theta(d\phi)_a(d\phi)_b = \theta_a\theta_b + \phi_a\phi_b, \quad (18)$$

$$\theta_a = (d\theta)_a, \quad \phi_a = \sin\theta(d\phi)_a. \quad (19)$$

On this background spacetime $(\mathscr{M}, g_{ab})$, the components of the metric perturbation is given by

$$h_{ab} = h_{AB}(dx^A)_a(dx^B)_b + 2h_{Ap}(dx^A)_{(a}(dx^p)_{b)} + h_{pq}(dx^p)_a(dx^q)_b. \tag{20}$$

Here, we note that the components $h_{AB}$, $h_{Ap}$, and $h_{pq}$ are regarded as components of scalar, vector, and tensor on $S^2$, respectively. In [14], we showed the linear independence of the set of harmonic functions

$$\left\{S_\delta, \hat{D}_p S_\delta, \epsilon_{pq}\hat{D}^q S_\delta, \frac{1}{2}\gamma_{pq}S_\delta, \left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\right)S_\delta, 2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S_\delta\right\}, \tag{21}$$

where $\hat{D}_p$ is the covariant derivative associated with the metric $\gamma_{pq}$ on $S^2$, $\hat{D}^p = \gamma^{pq}\hat{D}_q$, and $\epsilon_{pq} = \epsilon_{[pq]} = 2\theta_{[p}\phi_{q]}$ is the totally antisymmetric tensor on $S^2$. In the set of harmonic function (21), the scalar harmonic function $S_\delta$ is given by

$$S_\delta = \left\{ \begin{array}{ccc} Y_{lm} & \text{for} & l \geq 2; \\ k_{(\hat{\Delta}+2)m} & \text{for} & l = 1; \\ k_{(\hat{\Delta})} & \text{for} & l = 0, \end{array} \right. \tag{22}$$

where functions $k_{(\hat{\Delta})}$ and $k_{(\hat{\Delta}+2)m}$ are the kernel modes of the derivative operator $\hat{\Delta}$ and $[\hat{\Delta}+2]$, respectively, and we employ the explicit form of these functions as

$$k_{(\hat{\Delta})} = 1 + \delta \ln\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{1/2}, \quad \delta \in \mathbb{R}, \tag{23}$$

$$k_{(\hat{\Delta}+2,m=0)} = \cos\theta + \delta\left(\frac{1}{2}\cos\theta \ln\frac{1+\cos\theta}{1-\cos\theta} - 1\right), \quad \delta \in \mathbb{R}, \tag{24}$$

$$k_{(\hat{\Delta}+2,m=\pm 1)} = \left[\sin\theta + \delta\left(+\frac{1}{2}\sin\theta\ln\frac{1+\cos\theta}{1-\cos\theta} + \cot\theta\right)\right]e^{\pm i\phi}. \tag{25}$$

Then, we consider the mode decomposition of the components $\{h_{AB}, h_{Ap}, h_{pq}\}$ as follows:

$$h_{AB} = \sum_{l,m} \tilde{h}_{AB} S_\delta, \tag{26}$$

$$h_{Ap} = r\sum_{l,m}\left[\tilde{h}_{(e1)A}\hat{D}_p S_\delta + \tilde{h}_{(o1)A}\epsilon_{pq}\hat{D}^q S_\delta\right], \tag{27}$$

$$h_{pq} = r^2\sum_{l,m}\left[\frac{1}{2}\gamma_{pq}\tilde{h}_{(e0)}S_\delta + \tilde{h}_{(e2)}\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}^r\hat{D}_r\right)S_\delta + 2\tilde{h}_{(o2)}\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S_\delta\right]. \tag{28}$$

Since the linear independence of each element of the set of harmonic function (21) is guaranteed, the one-to-one correspondence between the components $\{h_{AB}, h_{Ap}, h_{pq}\}$ and the mode coefficients $\{\tilde{h}_{AB}, \tilde{h}_{(e1)A}, \tilde{h}_{(o1)A}, \tilde{h}_{(e0)}, \tilde{h}_{(e2)}, \tilde{h}_{(o2)}\}$ with the decomposition formulae (26)–(28) is guaranteed, including the $l = 0, 1$ mode if $\delta \neq 0$. Then, the mode-by-mode analysis including $l = 0, 1$ is possible when $\delta \neq 0$. However, the mode functions (23)–(25) are singular if $\delta \neq 0$. When $\delta = 0$, we have $k_{(\hat{\Delta})} \propto Y_{00}$ and $k_{(\hat{\Delta}+2)m} \propto Y_{1m}$, but the linear dependence of the set of harmonics (21) is lost in this case. Because of this situation, we proposed the following strategy.

**Proposal 1.** *We decompose the metric perturbation $h_{ab}$ on the background spacetime with the metrics (15)–(18) through Equations (26)–(28), with the harmonic function $S_\delta$ given by Equation (22). Then, Equations (26)–(28) become invertible, including $l = 0, 1$ modes. After deriving the mode-by-mode field equations, such as linearized Einstein equations by using the harmonic functions $S_\delta$, we choose $\delta = 0$ as the regular boundary condition for solutions when we solve these field equations.*

As shown in [14], once we accept Proposal 1, Conjecture 1 becomes the following statement.

**Theorem 1.** *If the gauge-transformation rule for a perturbative pulled-back tensor field $h_{ab}$ to the background spacetime $\mathscr{M}$ is given by ${}_{\mathscr{Y}}h_{ab} - {}_{\mathscr{X}}h_{ab} = £_{\xi_{(1)}}g_{ab}$ with the background metric $g_{ab}$ with spherically symmetry, then there exist a tensor field $\mathscr{F}_{ab}$ and a vector field $Y^a$, such that $h_{ab}$ is decomposed as $h_{ab} =: \mathscr{F}_{ab} + £_Y g_{ab}$, where $\mathscr{F}_{ab}$ and $Y^a$ are transformed into ${}_{\mathscr{Y}}\mathscr{F}_{ab} - {}_{\mathscr{X}}\mathscr{F}_{ab} = 0$ and ${}_{\mathscr{Y}}Y^a - {}_{\mathscr{X}}Y^a = \xi^a_{(1)}$ under the gauge transformation, respectively.*

Actually, the gauge-dependent variable $Y_a$ is given by

$$Y_a := \sum_{l,m} \tilde{Y}_A S_\delta (dx^A)_a + \sum_{l,m}\left(\tilde{Y}_{(e1)}\hat{D}_p S_\delta + \tilde{Y}_{(o1)}\epsilon_{pq}\hat{D}^q S_\delta\right)(dx^p)_a, \tag{29}$$

where

$$\tilde{Y}_A := r\tilde{h}_{(e1)A} - \frac{r^2}{2}\bar{D}_A\tilde{h}_{(e2)}, \tag{30}$$

$$\tilde{Y}_{(e1)} := \frac{r^2}{2}\tilde{h}_{(e2)}, \tag{31}$$

$$\tilde{Y}_{(o1)} := -r^2\tilde{h}_{(o2)}. \tag{32}$$

Furthermore, including $l = 0, 1$ modes, the components of the gauge-invariant part $\mathscr{F}_{ab}$ of the metric perturbation $h_{ab}$ is given by

$$\mathscr{F}_{AB} = \sum_{l,m} \tilde{F}_{AB} S_\delta, \tag{33}$$

$$\mathscr{F}_{Ap} = r\sum_{l,m} \tilde{F}_A \epsilon_{pq}\hat{D}^q S_\delta, \quad \hat{D}^p\mathscr{F}_{Ap} = 0, \tag{34}$$

$$\mathscr{F}_{pq} = \frac{1}{2}\gamma_{pq} r^2 \sum_{l,m}\tilde{F} S_\delta, \tag{35}$$

where $\tilde{F}_{AB}$, $\tilde{F}_A$, and $\tilde{F}$ are given by

$$\tilde{F}_{AB} := \tilde{h}_{AB} - 2\bar{D}_{(A}\tilde{Y}_{B)}, \tag{36}$$

$$\tilde{F}_A := \tilde{h}_{(o1)A} + r\bar{D}_A\tilde{h}_{(o2)}, \tag{37}$$

$$\tilde{F} := \tilde{h}_{(e0)} - \frac{4}{r}\tilde{Y}_A\bar{D}^A r + \tilde{h}_{(e2)} l(l+1). \tag{38}$$

Thus, we have constructed gauge-invariant metric perturbations on the Schwarzschild background spacetime including $l = 0, 1$ modes.

Furthermore, from Equations (33)–(38) for the gauge-invariant variables $\mathscr{F}_{AB}$, $\mathscr{F}_{Ap}$, and $\mathscr{F}_{pq}$, and gauge-dependent variables $Y_a$, defined by Equations (29)–(32), the original components $\{h_{AB}, h_{Ap}, h_{pq}\}$ of the metric perturbation (26)–(28) are given by

$$h_{ab} = \mathscr{F}_{ab} + £_Y g_{ab}. \tag{39}$$

This is the assertion of Theorem 1.

The aim of our proposal is to add a greater degree of freedom of the metric perturbations so that the decomposition (39) is guaranteed, even in the case of $l = 0, 1$ modes. Therefore, it is impossible to reach the final expression (39) if we treat the metric perturbation by using $S_{\delta=0} = Y_{lm}$ in the decomposition formulae (26)–(28) from the starting point.

To see this more specifically, we show an explicit example of the $l = 1$ odd-mode perturbation. If we apply our proposal to the decomposition formulae (26)–(28), $l = 1$ odd-mode perturbation is given by

$$h_{ab} = 2r\tilde{h}^{(l=1)}_{(o1)A}\epsilon_{pq}\hat{D}^q S_\delta (dx^A)_{(a}(dx^p)_{b)} + 2r^2\tilde{h}^{(l=1)}_{(o2)}\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S_\delta (dx^p)_{(a}(dx^q)_{b)}. \tag{40}$$

If we use $S_{\delta=0} = Y_{lm}$ from the starting point, $2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S_\delta = 0$. So, the mode coefficient $\tilde{h}^{(l=1)}_{(o2)}$ does not appear. When $\delta \neq 0$, the gauge-transformation rule of the variables $\tilde{h}^{(l=1)}_{(o1)A}$ and $\tilde{h}^{(l=1)}_{(o2)}$ are given by

$$\mathscr{Y}\tilde{h}^{(l=1)}_{(o1)A} - \mathscr{X}\tilde{h}^{(l=1)}_{(o1)A} = r\bar{D}_A\left(\frac{1}{r}\zeta_{(o)}\right), \quad \mathscr{Y}\tilde{h}^{(l=1)}_{(o2)} - \mathscr{X}\tilde{h}^{(l=1)}_{(o2)} = -\frac{1}{r}\zeta_{(o)}, \tag{41}$$

where the generator of the gauge-transformation is given by $\zeta_A = 0$ and $\zeta_p = r\zeta_{(o)}\epsilon_{pq}\hat{D}^q S_\delta$. If we use $S_{\delta=0} = Y_{lm}$ from the starting point, the gauge-transformation rule for the variable $\tilde{h}^{(l=1)}_{(o2)}$ does not appear. However, in our case, this gauge-transformation appears due to the fact $\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S_\delta \neq 0$. Inspecting these gauge-transformation rules, we rewrite Equation (40) as follows:

$$\begin{aligned} h_{ab} &= 2r\left[\left(\tilde{h}^{(l=1)}_{(o1)A} + r\bar{D}_A\tilde{h}_{(o2)}\right) - r\bar{D}_A\tilde{h}_{(o2)}\right]\epsilon_{pq}\hat{D}^q S_\delta (dx^A)_{(a}(dx^p)_{b)} \\ &\quad + 2r^2\tilde{h}^{(l=1)}_{(o2)}\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S_\delta (dx^p)_{(a}(dx^q)_{b)} \\ &= 2r\left[\tilde{F}_A - r\bar{D}_A\left(\frac{1}{r^2}\tilde{Y}_{(o1)}\right)\right]\epsilon_{pq}\hat{D}^q S_\delta (dx^A)_{(a}(dx^p)_{b)} \\ &\quad + 2r^2\left(\frac{1}{r^2}\tilde{Y}_{(o1)}\right)\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S_\delta (dx^p)_{(a}(dx^q)_{b)}. \end{aligned} \tag{42}$$

In Equation (42), $\tilde{F}_A$ is the gauge-invariant variable for the metric perturbation, defined by Equation (37), and $\tilde{Y}_{(o1)}$ is the gauge-dependent part of the metric perturbation, defined by Equation (32). The terms of $\tilde{Y}_{(o1)}$ in Equation (42) can be written in the form of the Lie derivative of the background metric, as in Equation (39). When $\delta = 0$, Equation (42) is given by

$$h_{ab} = 2r\left[\tilde{F}_A - r\bar{D}_A\left(\frac{1}{r^2}\tilde{Y}_{(o1)}\right)\right]\epsilon_{pq}\hat{D}^q S_\delta (dx^A)_{(a}(dx^p)_{b)}. \tag{43}$$

This is also given in the form (39).

When we construct the gauge-invariant variable $\tilde{F}_A$ in Equations (42) and (43), the existence of the coefficient $\tilde{h}^{(l=1)}_{(o2)} = \tilde{Y}_{(o1)}/r^2$ is essential, which does not appear in the treatment using the spherical harmonics $Y_{lm}$ from the starting point. Actually, in this conventional case, we cannot define the gauge-invariant variable for $l = 1$ odd-mode perturbation in the same manner as $l \geq 2$-mode case. Proposal 1 makes the degree of freedom of the metric perturbations increase. We have to emphasize that we can reach the decomposition (39) owing to this additional degree of freedom.

To discuss the linearized Einstein Equation (9), and the linear perturbation of the continuity equation,

$$\nabla_a {}^{(1)}\mathscr{T}_b{}^a = 0 \tag{44}$$

of the gauge-invariant energy–momentum tensor ${}^{(1)}\mathscr{T}_b{}^a := g^{ac}\,{}^{(1)}\mathscr{T}_{bc}$ on a vacuum background spacetime, we consider the mode-decomposition of the gauge-invariant part ${}^{(1)}\mathscr{T}_{bc}$ of the linear perturbation of the energy–momentum tensor through the set (21) of the harmonics as follows:

$$
\begin{aligned}
{}^{(1)}\mathscr{T}_{ab} \;=\;& \sum_{l,m} \tilde{T}_{AB} S_\delta (dx^A)_a (dx^B)_b + r \sum_{l,m} \left\{ \tilde{T}_{(e1)A} \hat{D}_p S_\delta + \tilde{T}_{(o1)A} \epsilon_{pr} \hat{D}^r S_\delta \right\} 2 (dx^A)_{(a} (dx^p)_{b)} \\
& + r^2 \sum_{l,m} \left\{ \tilde{T}_{(e0)} \frac{1}{2} \gamma_{pq} S_\delta + \tilde{T}_{(e2)} \left( \hat{D}_p \hat{D}_q S_\delta - \frac{1}{2} \gamma_{pq} \hat{D}_r \hat{D}^r S_\delta \right) \right. \\
& \qquad \left. + \tilde{T}_{(o2)} 2 \epsilon_{r(p} \hat{D}_{q)} \hat{D}^r S_\delta \right\} (dx^p)_a (dx^q)_b .
\end{aligned} \tag{45}
$$

In [14], we derived the linearized Einstein equations, discussed the odd-mode perturbation $\tilde{F}_{Ap}$ in Equation (34), and derived the $l = 1$ odd-mode solutions to these equations. The Einstein equation for even-mode $\tilde{F}_{AB}$ and $\tilde{F}$ in Equations (33) and (35), also derived in [14], and $l = 0, 1$ even-mode solutions, are derived in [15]. Since these solutions include the Kerr parameter perturbation and the Schwarzschild mass parameter perturbation of the linear order in the vacuum case, these are physically reasonable. Then, we conclude that our proposal is also physically reasonable. Furthermore, we also checked the fact that our derived solutions include the linearized LTB solution and non-rotating C-metric with the Schwarzschild background in [16].

The purpose of this paper is to compare our proposed gauge-invariant treatments for $l = 0, 1$-mode perturbations on the Schwarzschild background spacetime with conventional "complete gauge-fixing treatments" in which we use the spherical harmonics $Y_{lm}$ from the starting point. For this purpose, the linearized Einstein equations for the $l = 0, 1$-modes and their solutions based on our proposal are necessary. Therefore, we review them below.

### 2.3. $l = 1$ *Odd-Mode Linearized Einstein Equations and Solutions*

As derived in [14], the $l = 1$ odd-mode part in the linearized Einstein equations are simplified as the constraint equation

$$
\bar{D}_D (r \tilde{F}^D) = 0, \tag{46}
$$

and the evolution equation

$$
\begin{aligned}
& - \left[ \bar{D}^D \bar{D}_D - \frac{2}{r^2} \right] (r \tilde{F}_A) - \frac{2}{r^2} (\bar{D}^D r)(\bar{D}_A r)(r \tilde{F}_D) + \frac{2}{r} (\bar{D}^D r) \bar{D}_A (r \tilde{F}_D) \\
=\;& 16 \pi r \tilde{T}_{(o1)A},
\end{aligned} \tag{47}
$$

where we choose $\tilde{T}_{(o2)} = 0$ by hand. Furthermore, we have the continuity equation

$$
\bar{D}^C \tilde{T}_{(o1)C} + \frac{3}{r} (\bar{D}^D r) \tilde{T}_{(o1)D} = 0 \tag{48}
$$

for the $l = 1$ odd-mode matter perturbation, which is derived from the divergence of the first-order perturbation of the energy–momentum tensor.

We also note that the constraint (46) comes from the traceless part of the $(q, p)$-component of the Einstein Equation (9), which is the coefficient of the mode function $2 \epsilon_{r(p} \hat{D}_{q)} \hat{D}^p S_\delta$. If $\delta = 0$, i.e., the scalar harmonics $S_\delta$ are the spherical harmonics $Y_{lm}$, $2 \epsilon_{r(p} \hat{D}_{q)} \hat{D}^p S_\delta = 0$ for $l = 1$. Therefore, Constraint (46) does not appear when we use the spherical harmonics $Y_{lm}$ from the starting point.

The explicit strategy to solve these odd-mode perturbations and $l = 0, 1$ mode solutions was discussed in [14]. As a result, we obtained the $l = 1$ odd-mode "formal" solution as follows:

$$
2 \mathscr{F}_{Ap} (dx^A)_{(a} (dx^p)_{b)} = 6 M r^2 \left[ \int dr \frac{a_1(t, r)}{r^4} \right] \sin^2\theta (dt)_{(a} (d\phi)_{b)} + \pounds_V g_{ab}, \tag{49}
$$

$$
V_a = \Big( \beta_1 t + \beta_0 + W_{(o)}(t, r) \Big) r^2 \sin^2\theta (d\phi)_a, \tag{50}
$$

where

$$\begin{aligned} a_1(t,r) &= -\frac{16\pi}{3M} r^3 f \int dt \tilde{T}_{(o1)r} + a_{10} \\ &= -\frac{16\pi}{3M} \int dr r^3 \frac{1}{f} \tilde{T}_{(o1)t} + a_{10}. \end{aligned} \tag{51}$$

The constant $a_{10}$ in Equation (51) corresponds to the Kerr parameter. Furthermore, the function $W_{(o)}(t,r)$ in Equation (50) is related to the solution to the $l = 1$-mode Regge–Wheeler equation

$$\partial_t^2 Z_{(o)} - f\partial_r(f\partial_r Z_{(o)}) + \frac{1}{r^2} f[l(l+1) - 3(1-f)] Z_{(o)} = 16\pi f^2 \tilde{T}_{(o1)r}, \tag{52}$$

where $Z_{(o)} = rf\partial_r W_{(o)}$. We have to solve Equation (52) to obtain the function $W_{(o)}(t,r)$. In this sense, the solution (50) should be regarded as the "formal" one.

### 2.4. $l = 0,1$ *Even-Mode Linearized Einstein Equations and Solutions*

The $l = 0,1$ even-mode part of the linearized Einstein Equation (9) is summarized as follows:

$$\tilde{F}_D{}^D = 0, \tag{53}$$

$$\bar{D}^D \tilde{\mathbb{F}}_{AD} - \frac{1}{2} \bar{D}_A \tilde{F} = 16\pi r \tilde{T}_{(e1)A}, \tag{54}$$

where the variable $\tilde{\mathbb{F}}_{AB}$ is the traceless part of the variable $\tilde{F}_{AB}$, defined by

$$\tilde{\mathbb{F}}_{AB} := \tilde{F}_{AB} - \frac{1}{2} y_{AB} \tilde{F}_C{}^C \tag{55}$$

and we choose the component of the energy–momentum tensor by hand, so that $\tilde{T}_{(e2)} = 0$. Owing to the same choice $\tilde{T}_{(e2)} = 0$, we also have the evolution equations

$$\left( \bar{D}_D \bar{D}^D + \frac{2}{r} (\bar{D}^D r) \bar{D}_D - \frac{(l-1)(l+2)}{r^2} \right) \tilde{F} - \frac{4}{r^2} (\bar{D}_C r)(\bar{D}_D r) \tilde{\mathbb{F}}^{CD} = 16\pi S_{(F)}, \tag{56}$$

$$S_{(F)} := \tilde{T}_C{}^C + 4(\bar{D}_D r) \tilde{T}^D_{(e1)}, \tag{57}$$

$$\begin{aligned} &\left[ -\bar{D}_D \bar{D}^D - \frac{2}{r} (\bar{D}_D r) \bar{D}^D + \frac{4}{r} (\bar{D}^D \bar{D}_D r) + \frac{l(l+1)}{r^2} \right] \tilde{\mathbb{F}}_{AB} \\ &+ \frac{4}{r} (\bar{D}^D r) \bar{D}_{(A} \tilde{\mathbb{F}}_{B)D} - \frac{2}{r} (\bar{D}_{(A} r) \bar{D}_{B)} \tilde{F} \\ &= 16\pi S_{(\mathbb{F})AB}, \end{aligned} \tag{58}$$

$$\begin{aligned} S_{(\mathbb{F})AB} :=\ & \tilde{T}_{AB} - \frac{1}{2} y_{AB} \tilde{T}_C{}^C - 2\left( \bar{D}_{(A}(r\tilde{T}_{(e1)B)}) - \frac{1}{2} y_{AB} \bar{D}^D (r \tilde{T}_{(e1)D}) \right) \\ & + 2 y_{AB} (\bar{D}^C r) \tilde{T}_{(e1)C}, \end{aligned} \tag{59}$$

for the variable $\tilde{F}$ and the traceless variable $\tilde{\mathbb{F}}_{AB}$ with $l = 0,1$. We also have to take into account the even-mode part of the continuity equation as follows:

$$\bar{D}^C \tilde{T}_C{}^B + \frac{2}{r} (\bar{D}^D r) \tilde{T}_D{}^B - \frac{1}{r} l(l+1) \tilde{T}^B_{(e1)} - \frac{1}{r} (\bar{D}^B r) \tilde{T}_{(e0)} = 0, \tag{60}$$

$$\bar{D}^C \tilde{T}_{(e1)C} + \frac{3}{r} (\bar{D}^C r) \tilde{T}_{(e1)C} + \frac{1}{2r} \tilde{T}_{(e0)} = 0, \tag{61}$$

for $l = 0,1$.

We also note that the constraint (53) comes from the traceless part of the $(q,p)$-component of the Einstein tensor (9), which is the coefficient of the mode function $\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}^r\hat{D}_r\right)S_\delta$. If $\delta = 0$, i.e., the scalar harmonics $S_\delta$ is the spherical harmonics $Y_{lm}$, $\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}^r\hat{D}_r\right)S_\delta = 0$ for $l = 1$. Therefore, the constraint (53) does not appear when we use the spherical harmonics $Y_{lm}$ from the starting point.

#### 2.4.1. $l = 1$ Even-Mode Solution

In [15], we derived the $l = 1$ solution to Equations (53), (54), (56), (58), (60) and (61) with $l = 1$. For the $m = 0$ mode, in [15], we derived the following "formal" solution to the linearized Einstein equation:

$$\begin{aligned}\mathscr{F}_{ab} &= \pounds_V g_{ab} - \frac{16\pi r^2}{3(1-f)}\left[f^2\left\{\frac{1+f}{2}\tilde{T}_{rr} + rf\partial_r\tilde{T}_{rr} - \tilde{T}_{(e0)} - 4\tilde{T}_{(e1)r}\right\}(dt)_a(dt)_b\right.\\ &\quad + \frac{2r}{f}\left\{\partial_t\tilde{T}_{tt} - \frac{3f(1-f)}{2r}\tilde{T}_{tr}\right\}(dt)_{(a}(dr)_{b)}\\ &\quad + \frac{r}{f}\left\{\partial_r\tilde{T}_{tt} - \frac{3(1-3f)}{2rf}\tilde{T}_{tt}\right\}(dr)_a(dr)_b\\ &\quad \left. + r^2\tilde{T}_{tt}\gamma_{ab}\right]\cos\theta,\end{aligned} \tag{62}$$

where the vector field $V_a$ is given by

$$V_a := -r\partial_t\Phi_{(e)}\cos\theta(dt)_a + \left(\Phi_{(e)} - r\partial_r\Phi_{(e)}\right)\cos\theta(dr)_a - r\Phi_{(e)}\sin\theta(d\theta)_a. \tag{63}$$

Here, the variable $\Phi_{(e)}$ is a solution to the $l = 1$ Zerilli equation,

$$\begin{aligned}&-\partial_t^2\Phi_{(e)} + f\partial_r\left[f\partial_r\Phi_{(e)}\right] - \frac{f(1-f)}{r^2}\Phi_{(e)}\\ =& \frac{4\pi r}{3(1-f)}\left[3(1-3f)\tilde{T}_{tt} + (1+f)f^2\tilde{T}_{rr} - 2rf\partial_r\tilde{T}_{tt}\right.\\ &\left. + 2rf^3\partial_r\tilde{T}_{rr} - 2f^2\tilde{T}_{(e0)} - 8f^2\tilde{T}_{(e1)r}\right].\end{aligned} \tag{64}$$

If we obtain the solution to the $l = 1$ Zerilli Equation (64), we can write the explicit form of the solution (62) through the generator $V_a$, defined by Equation (63). In this sense, we have to regard the solution (62) as a "formal" one.

#### 2.4.2. $l = 0$ Even-Mode Solution

On the other hand, for the $l = 0$-mode, we may choose $\tilde{T}_{(e1)A} = 0$ in Equations (53), (54), (56)–(61) with $l = 0$. Then, we derived the $l = 0$-mode solution

$$\begin{aligned}\mathscr{F}_{ab} &= \frac{2}{r}\left(M_1 + 4\pi\int dr\frac{r^2}{f}T_{tt}\right)\left((dt)_a(dt)_a + \frac{1}{f^2}(dr)_a(dr)_a\right)\\ &\quad + 2\left[4\pi r\int dt\left(\frac{1}{f}\tilde{T}_{tt} + f\tilde{T}_{rr}\right)\right](dt)_{(a}(dr)_{b)} + \pounds_V g_{ab},\end{aligned} \tag{65}$$

where

$$V_a = \left(\frac{f}{4}\Upsilon + \frac{rf}{4}\partial_r\Upsilon - \frac{r\Xi(r)}{(1-3f)} + f\int dr\frac{2\Xi(r)}{f(1-3f)^2}\right)(dt)_a + \frac{1}{4f}r\partial_t\Upsilon(dr)_a. \tag{66}$$

Here, the variable $\tilde{F} =: \partial_t \Upsilon$ must satisfy the equation

$$\begin{aligned} & -\frac{1}{f}\partial_t^2\Upsilon + \partial_r(f\partial_r\Upsilon) + \frac{3(1-f)}{r^2}\Upsilon - \frac{8}{r^3}\int dt m_1(t,r) - \frac{4}{1-3f}\partial_r\Xi(r) \\ = \quad & 16\pi\int dt\left(-\frac{1}{f}\tilde{T}_{tt} + f\tilde{T}_{rr}\right), \end{aligned} \tag{67}$$

where $m_1(t,r)$ is given by

$$m_1(t,r) = 4\pi\int dr\frac{r^2}{f}\tilde{T}_{tt} + M_1 = 4\pi\int dt r^2 f\tilde{T}_{tr} + M_1, \tag{68}$$

where $M_1$ is the constant corresponding to the Schwarzschild mass perturbation, and $\Xi(r)$ is an arbitrary function of $r$.

## 3. Rule of Comparison and Gauge-Transformation Rules in a Conventional Gauge-Fixing

The main purpose of this paper is to check the following statement: "Gauge-invariant formulations of perturbations are equivalent to complete gauge-fixing approaches". However, this statement is too ambiguous to check if we have the background knowledge explained in Section 2. For example, there is no explanation of the terminology "gauge" in this statement. Therefore, we have to clarify this statement as follows.

**Description 1 (Our rule for comparison).** *First of all, we assume the terminology "gauge" in the statement "Gauge-invariant formulations of perturbations are equivalent to complete gauge-fixing approaches" is the second-kind gauge, which is explained in Section 2.1. Therefore, the gauge-transformation rule for this statement is given by (4). Furthermore, the degree of freedom that changes under this gauge-transformation rule is regarded as "unphysical". In the context of this statement, "gauge-fixing" is a specification of some perturbative variables through the degree of freedom of the generator* $\xi^a_{(1)}$*. Furthermore, "complete gauge-fixing" a specification of some perturbative variables through the "entire" degree of freedom of the generator* $\xi^a_{(1)}$*.*

These are all rules of our game to compare our gauge-invariant formulation and a "conventional complete gauge-fixing approach" in which we use the spherical harmonics $Y_{lm}$ from the starting point.

### 3.1. Metric Perturbations

Based on the conceptual premise outlined above, we examine the metric perturbation $h_{ab}$ on the background Schwarzschild spacetime $(\mathscr{M}, g_{ab})$, whose metric $g_{ab}$ is given by Equations (15)–(19). We consider the components of the metric perturbation $h_{ab}$ as Equation (20), and the decomposition of these components $\{h_{AB}, h_{Ap}, h_{pq}\}$ as Equations (26)–(28). Our focus is on the case where $S_\delta = Y_{lm} =: S$, where $Y_{lm}$ is the conventional spherical harmonics.

Since we choose $S = Y_{lm}$, we have $\hat{D}_p S = \epsilon_{pq}\hat{D}^q S = 0$, $\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}^r\hat{D}_r\right)S = 2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S = 0$ for $l = 0$ modes. For $l = 1$ modes, we have $\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}^r\hat{D}_r\right)S = 2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S = 0$. Due to these facts, we cannot construct gauge-invariant variables for $l = 0, 1$ modes in a similar manner to the derivation of Equations (36)–(38) for $l \geq 2$ modes. Then, we cannot use the gauge-invariant formulation reviewed in Section 2. Instead, we have to fix the second-kind gauge-degree of freedom to exclude the "unphysical" degree of freedom.

### 3.2. Conventional Gauge-Transformation Rules for $l = 0, 1$ Metric Perturbations

Here, we consider the gauge-transformation rule,

$${}_{\mathscr{Y}}h_{ab} - {}_{\mathscr{X}}h_{ab} = \pounds_\xi g_{ab} = 2\nabla_{(a}\xi_{b)}. \tag{69}$$

From this gauge-transformation rule, we can derive the gauge-transformation rule for the components $\{h_{AB}, h_{Ap}, h_{qp}\}$, defined by Equation (20), is given by

$$\mathscr{Y}h_{AB} - \mathscr{X}h_{AB} = \pounds_{\xi}g_{AB} = \nabla_A\xi_B + \nabla_B\xi_A = \bar{D}_A\xi_B + \bar{D}_B\xi_A, \tag{70}$$

$$\mathscr{Y}h_{Ap} - \mathscr{X}h_{Ap} = \pounds_{\xi}g_{Ap} = \nabla_A\xi_p + \nabla_p\xi_A = \bar{D}_A\xi_p + \hat{D}_p\xi_A - \frac{2}{r}(\bar{D}_A r)\xi_p, \tag{71}$$

$$\mathscr{Y}h_{pq} - \mathscr{X}h_{pq} = \pounds_{\xi}g_{pq} = \nabla_p\xi_q + \nabla_q\xi_p = \hat{D}_p\xi_q + \hat{D}_q\xi_p + 2r(\bar{D}^A r)\gamma_{pq}\xi_A \tag{72}$$

from the formulae summarized in Appendix B of [14]. Here, we consider the Fourier transformation of $\xi_a =: \xi_A(dx^A)_a + \xi_p(dx^p)_a$ as

$$\xi_A = \sum_{l,m}\zeta_A S, \quad \xi_p = r\sum_{l,m}\left(\zeta_{(e)}\hat{D}_p S + \zeta_{(o)}\epsilon_{pq}\hat{D}^p S\right). \tag{73}$$

In terms of these Fourier transformed variables, Equations (70)–(72) are given by

$$\mathscr{Y}h_{AB} - \mathscr{X}h_{AB} = \bar{D}_A\xi_B + \bar{D}_B\xi_A = \sum_{l,m}2\bar{D}_{(A}\zeta_{B)}S, \tag{74}$$

$$\begin{aligned}\mathscr{Y}h_{Ap} - \mathscr{X}h_{Ap} &= \bar{D}_A\xi_p + \hat{D}_p\xi_A - \frac{2}{r}(\bar{D}_A r)\xi_p \\ &= r\sum_{l,m}\left[\left(\frac{1}{r}\zeta_A + \bar{D}_A\zeta_{(e)} - \frac{1}{r}(\bar{D}_A r)\zeta_{(e)}\right)\hat{D}_p S \right.\\ &\qquad \left. + \left(\bar{D}_A\zeta_{(o)} - \frac{1}{r}(\bar{D}_A r)\zeta_{(o)}\right)\epsilon_{pq}\hat{D}^p S\right],\end{aligned} \tag{75}$$

$$\begin{aligned}\mathscr{Y}h_{pq} - \mathscr{X}h_{pq} &= \hat{D}_p\xi_q + \hat{D}_q\xi_p + 2r(\bar{D}^A r)\gamma_{pq}\xi_A \\ &= r^2\sum_{l,m}\left[\frac{4}{r}\left(-\frac{l(l+1)}{2}\zeta_{(e)} + (\bar{D}^A r)\zeta_A\right)\frac{1}{2}\gamma_{pq}S \right.\\ &\qquad + \frac{2}{r}\zeta_{(e)}\left(\hat{D}_p\hat{D}_q S - \frac{1}{2}\gamma_{pq}\hat{D}^r\hat{D}_r S\right) \\ &\qquad \left. - \frac{1}{r}\zeta_{(o)}2\epsilon_{r(q}\hat{D}_{p)}\hat{D}^r S\right].\end{aligned} \tag{76}$$

Here, we used the property of the mode function $S$ is an eigen function of the Laplacian $\hat{D}_r\hat{D}^r := \gamma^{rs}\hat{D}_r\hat{D}_s$ as

$$\hat{D}_r\hat{D}^r S = -l(l+1)S, \quad S = Y_{lm}. \tag{77}$$

### 3.2.1. $l \geq 2$ Perturbations

For $l \geq 2$ modes, the set of mode functions

$$\left\{S, \hat{D}_p S, \epsilon_{pq}\hat{D}^q S, \frac{1}{2}\gamma_{pq}S, \left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}_r\hat{D}^r\right)S, 2\epsilon_{r(p}\hat{D}^r\hat{D}_{q)}S\right\} \tag{78}$$

is a linear-independent set of the tensor field of the rank 0, 1, and 2, even if $S = Y_{lm}$. Then, we may compare Equations (26)–(28) and Equations (74)–(76). As a result of this comparison, we obtain the gauge-transformation rules for the mode functions $\{\tilde{h}_{AB}, \tilde{h}_{(e1)A}, \tilde{h}_{(o1)A}, \tilde{h}_{(e0)}, \tilde{h}_{(e2)}, \tilde{h}_{(o2)}\}$ as

$$
\begin{aligned}
{}_{\mathscr{Y}}\tilde{h}_{AB} - {}_{\mathscr{X}}\tilde{h}_{AB} &= 2\bar{D}_{(A}\zeta_{B)}, &(79)\\
{}_{\mathscr{Y}}\tilde{h}_{(e1)A} - {}_{\mathscr{X}}\tilde{h}_{(e1)A} &= \frac{1}{r}\zeta_A + \bar{D}_A\zeta_{(e)} - \frac{1}{r}(\bar{D}_A r)\zeta_{(e)}, &(80)\\
{}_{\mathscr{Y}}\tilde{h}_{(o1)A} - {}_{\mathscr{X}}\tilde{h}_{(o1)A} &= \bar{D}_A\zeta_{(o)} - \frac{1}{r}(\bar{D}_A r)\zeta_{(o)}, &(81)\\
{}_{\mathscr{Y}}\tilde{h}_{(e0)} - {}_{\mathscr{X}}\tilde{h}_{(e0)} &= \frac{4}{r}\left(-\frac{l(l+1)}{2}\zeta_{(e)} + (\bar{D}^A r)\zeta_A\right), &(82)\\
{}_{\mathscr{Y}}\tilde{h}_{(e2)} - {}_{\mathscr{X}}\tilde{h}_{(e2)} &= \frac{2}{r}\zeta_{(e)}, &(83)\\
{}_{\mathscr{Y}}\tilde{h}_{(o2)} - {}_{\mathscr{X}}\tilde{h}_{(o2)} &= -\frac{1}{r}\zeta_{(o)}. &(84)
\end{aligned}
$$

These gauge-transformation rules yields the well-known fact we can construct gauge-invariant variables and gauge-dependent variables, and that Conjecture 1 is valid for $l \geq 2$ modes.

#### 3.2.2. $l = 1$ Perturbations

As shown in Appendix A in Ref. [14], for $l = 1$ modes,

$$\hat{D}_p Y_{1m} \neq 0 \neq \epsilon_{pq}\hat{D}^q Y_{1m} \tag{85}$$

but

$$\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\hat{D}_r\hat{D}^r\right)Y_{1m} = 0 = 2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r Y_{1m}. \tag{86}$$

In this case, the Fourier transformation (26)–(28) of the metric perturbation with $S_\delta = S = Y_{lm}$ is given by

$$
\begin{aligned}
h_{AB} &= \sum_m \tilde{h}_{AB} S, &(87)\\
h_{Ap} &= \sum_m r\tilde{h}_{(e1)A}\hat{D}_p S + \sum_m r\tilde{h}_{(o1)A}\epsilon_{pq}\hat{D}^q S, &(88)\\
h_{pq} &= \sum_m \frac{r^2}{2}\gamma_{pq}\tilde{h}_{(e0)} S. &(89)
\end{aligned}
$$

Furthermore, the gauge-transformation rules (74)–(76) for each $m$ mode are given by

$$
\begin{aligned}
{}_{\mathscr{Y}}\tilde{h}_{AB} - {}_{\mathscr{X}}\tilde{h}_{AB} &= 2\bar{D}_{(A}\zeta_{B)}, &(90)\\
{}_{\mathscr{Y}}\tilde{h}_{(e1)A} - {}_{\mathscr{X}}\tilde{h}_{(e1)A} &= \frac{1}{r}\zeta_A + \bar{D}_A\zeta_{(e)} - \frac{1}{r}(\bar{D}_A r)\zeta_{(e)}, &(91)\\
{}_{\mathscr{Y}}\tilde{h}_{(o1)A} - {}_{\mathscr{X}}\tilde{h}_{(o1)A} &= \bar{D}_A\zeta_{(o)} - \frac{1}{r}(\bar{D}_A r)\zeta_{(o)}, &(92)\\
{}_{\mathscr{Y}}\tilde{h}_{(e0)} - {}_{\mathscr{X}}\tilde{h}_{(e0)} &= -\frac{4}{r}\zeta_{(e)} + \frac{4}{r}(\bar{D}^A r)\zeta_A. &(93)
\end{aligned}
$$

Comparing the gauge-transformation rules (79)–(84) for $l \geq 2$ with the gauge-transformation rules (90)–(93), it is easy to find that the gauge-transformation rules (83) and (84) for $l \geq 2$ do not appear in the $l = 1$-mode gauge-transformation. This is due to Equation (86), and the reason why we cannot construct gauge-invariant variables for $l = 1$ mode perturbations in a similar method to the $l \geq 2$ case.

3.2.3. $l = 0$ Perturbations

The spherical harmonic function $S = Y_{lm}$ is constant when $l = 0$. In this case, the Fourier transformation (26)–(28) of the metric perturbation with $S_\delta = S = Y_{lm}$ is given by

$$h_{AB} = \tilde{h}_{AB} S, \tag{94}$$
$$h_{Ap} = 0, \tag{95}$$
$$h_{pq} = \frac{r^2}{2}\gamma_{pq}\tilde{h}_{(e0)} S. \tag{96}$$

The gauge-transformation rules (74)–(76) are given by

$$_{\mathscr{Y}}\tilde{h}_{AB} - {}_{\mathscr{X}}\tilde{h}_{AB} = 2\bar{D}_{(A}\zeta_{B)} = \bar{D}_A\zeta_B + \bar{D}_B\zeta_A, \tag{97}$$
$$_{\mathscr{Y}}\tilde{h}_{(e0)} - {}_{\mathscr{X}}\tilde{h}_{(e0)} = \frac{4}{r}(\bar{D}^A r)\zeta_A. \tag{98}$$

It is easy to see that many gauge-transformation rules in (79)–(84) are missing. This is due to the following facts:

$$\hat{D}_p S = \epsilon_{pq}\hat{D}^q S = 0, \quad \left(\hat{D}_q\hat{D}_p - \frac{1}{2}\gamma_{pq}\hat{D}_r\hat{D}^r\right) S = 2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S = 0. \tag{99}$$

These are the reasons why we cannot construct gauge-invariant variables for $l = 0$-mode perturbations in a similar method to the $l \geq 2$ case.

## 4. $l$ = 1 Odd-Mode Perturbation in the Conventional Approach

As discussed in Section 3.2.2, $l = 1$ mode perturbations are given by Equations (87)–(89). In particular, among these expressions of the $l = 1$-mode perturbation, odd-mode perturbation is given by

$$h_{AB} = 0, \quad h_{Ap} = r\tilde{h}_{(o1)A}\epsilon_{pq}\hat{D}^q S, \quad h_{pq} = 0. \tag{100}$$

As the property of the $l = 1$ spherical harmonics, we obtain Condition (86). For odd-mode perturbation we have

$$2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r Y_{1m} = 0. \tag{101}$$

Here, we apply the notation that is introduced by Equations (A14) and (A15) in Appendix A, and we obtain

$$\bar{h}_{AB} = 0, \quad \bar{h}_A{}^D = 0, \quad \bar{h}^{pD} := \gamma^{pq} y^{DE} r\tilde{h}_{(o1)E}\epsilon_{qr}\hat{D}^r S, \quad \bar{h}_p{}^q = 0, \quad \bar{h}^{pq} = 0, \tag{102}$$

for $l = 1$ odd-mode perturbations. Through this notation of the metric perturbation, the linear perturbations (A16)–(A18) of the Einstein tensor are given by

$$^{(1)}G_A{}^B = 0, \tag{103}$$

$$\begin{aligned} ^{(1)}G_A{}^q = \frac{1}{2r}\Bigg[&\bar{D}_A\bar{D}^C\tilde{h}_{(o1)C} - \bar{D}_C\bar{D}^C\tilde{h}_{(o1)A} - \frac{2}{r}(\bar{D}^C r)\bar{D}_C\tilde{h}_{(o1)A} \\ &+\frac{3}{r}(\bar{D}^C r)\bar{D}_A\tilde{h}_{(o1)C} - \frac{1}{r}(\bar{D}_A r)\bar{D}^C\tilde{h}_{(o1)C} + \frac{1}{2r^2}(\bar{D}_C r)(\bar{D}^C r)\tilde{h}_{(o1)A} \\ &-\frac{2}{r^2}(\bar{D}_A r)(\bar{D}^C r)\tilde{h}_{(o1)C} + \frac{3}{2r^2}\tilde{h}_{(o1)A}\Bigg]\epsilon^{qr}\hat{D}_r S, \end{aligned} \tag{104}$$

$$^{(1)}G_p{}^q = -\frac{1}{2r^2}\bar{D}^C\left(r\tilde{h}_{(o1)C}\right)\gamma^{qr}2\epsilon_{s(p}\hat{D}_{r)}\hat{D}^s S = 0. \tag{105}$$

The final equality in Equation (105) is due to the property of the spherical harmonics $Y_{lm}$ (101). Furthermore, Expression (104) is gauge-invariant under the gauge-transformation rule (92). On the other hand, the $\bar{D}^{C}\left(r\tilde{h}_{(o1)C}\right)$ in Equation (105) is not gauge-invariant, as shown below. However, the gauge-invariance of the linear-order Einstein tensor ${}^{(1)}G_{a}{}^{b}$ is guaranteed by the identity $2\epsilon_{s(p}\hat{D}_{r)}\hat{D}^{s}S = 0$ for the $l = 1$ odd-mode perturbations.

For the $l = 1$ odd-mode perturbations, the components of the linearized energy–momentum tensor are summarized as

$$ {}^{(1)}\mathcal{T}_{A}{}^{B} = 0, \quad {}^{(1)}\mathcal{T}_{A}{}^{q} = \frac{1}{r}\tilde{T}_{(o1)A}\epsilon^{qr}\hat{D}_{r}S, \quad {}^{(1)}\mathcal{T}_{p}{}^{q} = 0. \tag{106} $$

Then, the linearized Einstein equations for the odd-mode perturbations are given by

$$ \begin{aligned} & \bar{D}_{A}\bar{D}^{C}\tilde{h}_{(o1)C} - \bar{D}_{C}\bar{D}^{C}\tilde{h}_{(o1)A} \\ & -\frac{2}{r}(\bar{D}^{C}r)\bar{D}_{C}\tilde{h}_{(o1)A} + \frac{3}{r}(\bar{D}^{C}r)\bar{D}_{A}\tilde{h}_{(o1)C} - \frac{1}{r}(\bar{D}_{A}r)\bar{D}^{C}\tilde{h}_{(o1)C} \\ & +\frac{1}{2r^{2}}(\bar{D}_{C}r)(\bar{D}^{C}r)\tilde{h}_{(o1)A} - \frac{2}{r^{2}}(\bar{D}_{A}r)(\bar{D}^{C}r)\tilde{h}_{(o1)C} + \frac{3}{2r^{2}}\tilde{h}_{(o1)A} \\ = \; & 16\pi\tilde{T}_{(o1)A}, \end{aligned} \tag{107} $$

Although the equation

$$ \bar{D}^{C}\left(r\tilde{h}_{(o1)C}\right) = 0 \tag{108} $$

does not appear from Equation (105) due to the fact that $2\epsilon_{s(p}\hat{D}_{r)}\hat{D}^{s}Y_{1m} = 0$, we may use Equation (108) as a gauge condition. As noted in Section 3.2.2, the gauge-transformation rule for the variable $\tilde{h}_{(o1)C}$ is given by Equation (92). From the gauge-transformation rule (92), we consider the gauge-transformation rule for the left-hand side of Equation (108) as

$$ \bar{D}^{A}\left(r{}_{\mathscr{Y}}\tilde{h}_{(o1)A}\right) - \bar{D}^{A}\left(r{}_{\mathscr{X}}\tilde{h}_{(o1)A}\right) \;=\; r\bar{D}^{A}\bar{D}_{A}\zeta_{(o)} - (\bar{D}^{A}\bar{D}_{A}r)\zeta_{(o)}. \tag{109} $$

Using the background Einstein equation (Equation (B67) of Appendix B in ref. [14]), we consider the equation

$$ r\bar{D}^{C}\bar{D}_{C}\zeta_{(o)} - \frac{1}{r}\left(1 - (\bar{D}^{C}r)(\bar{D}_{C}r)\right)\zeta_{(o)} = -\bar{D}^{C}\left(r{}_{\mathscr{X}}\tilde{h}_{(o1)C}\right) \tag{110} $$

More explicitly, Equation (110) is given by

$$ -\partial_{t}^{2}\zeta_{(o)} + f\partial_{r}\left(f\partial_{r}\zeta_{(o)}\right) - \frac{f}{r^{2}}(1-f)\zeta_{(o)} = -\frac{f}{r}\bar{D}^{C}\left(r{}_{\mathscr{X}}\tilde{h}_{(o1)C}\right) \tag{111} $$

If we choose $\zeta_{(o)}$ as a special solution to Equation (111), or equivalently Equation (110), Equation (108) is regarded as a gauge condition in the $\mathscr{Y}$-gauge, i.e.,

$$ \bar{D}^{C}\left(r{}_{\mathscr{Y}}\tilde{h}_{(o1)C}\right) \;=\; 0. \tag{112} $$

We have to emphasize that this is not a complete gauge-fixing, since there is room for a choice of $\zeta_{(o)}$ that satisfies the homogeneous equation of Equation (110),

$$ r\bar{D}^{C}\bar{D}_{C}\zeta_{(o)} - \frac{1}{r}\left(1 - (\bar{D}^{C}r)(\bar{D}_{C}r)\right)\zeta_{(o)} = 0, \tag{113} $$

i.e.,

$$ -\partial_{t}^{2}\zeta_{(o)} + f\partial_{r}\left(f\partial_{r}\zeta_{(o)}\right) - \frac{f}{r^{2}}(1-f)\zeta_{(o)} = 0. \tag{114} $$

Under the gauge choice (112), Equation (107) is given by

$$\begin{aligned} & \left[\bar{D}_C\bar{D}^C - \frac{2}{r^2}\right]\left(r\tilde{h}_{(o1)A}\right) - \frac{2}{r^2}(\bar{D}_A r)(\bar{D}^C r)\left(r\tilde{h}_{(o1)C}\right) + \frac{2}{r}(\bar{D}^C r)\bar{D}_A\left(r\tilde{h}_{(o1)C}\right) \\ = \;& 16\pi r\tilde{T}_{(o1)A}. \end{aligned} \tag{115}$$

Although the variable $r\tilde{h}_{(o1)C}$ is a gauge-dependent variable that is different from the gauge-invariant variable $r\tilde{F}_A$, we can obtain Equation (115) if we replace the variable $r\tilde{F}_A$ in Equation (47) with the variable $r\tilde{h}_{(o1)C}$.

We also note that the gauge condition (112) coincides with Equation (46). However, we must replace the gauge-invariant variable $r\tilde{F}^D$ with the gauge-dependent variable $r\tilde{h}_{(o1)C}$ in Equation (46) to confirm this coincidence. Furthermore, we also take into account the continuity Equation (48) of the $l = 1$ odd-mode perturbation of the energy–momentum tensor. Thus, the equations to be solved for the $l = 1$ odd-mode perturbation are the gauge condition (108), evolution Equation (115), and continuity Equation (48). These equations coincide with Equations (46)–(48), except for the fact that the variable to be obtained is not the gauge-invariant variable $r\tilde{F}^D$, but the gauge-dependent variable $r\tilde{h}_{(o1)C}$. Then, through the same logic in ref. [14], we obtain the solution to the gauge condition (112), evolution Equation (115), and continuity Equation (48) as follows:

$$\begin{aligned} 2r\tilde{h}_{(o1)C}\sin^2\theta(dx^C)_{(a}(d\phi)_{b)} = 6Mr^2\left[\int dr\frac{a_1(t,r)}{r^4}\right]\sin^2\theta(dt)_{(a}(d\phi)_{b)} \\ +\pounds_V g_{ab},, \end{aligned} \tag{116}$$

$$V_a = (\beta_1 t + \beta_0 + W_{(o)}(t,r))r^2\sin^2\theta(d\phi)_a. \tag{117}$$

Here, we concentrate only on the $m = 0$ solution, $\beta_1$ and $\beta_0$ are constant, and $Z_{(o)} = rf\partial_r W_{(o)}(t,r)$ is an arbitrary function that satisfies the equation

$$\partial_t^2 Z_{(o)} - f\partial_r(f\partial_r Z_{(o)}) + \frac{1}{r^2}f[2-3(1-f)]Z_{(o)} = 16\pi f^2\tilde{T}_{(o1)r}. \tag{118}$$

where $a_1(t,r)$ in Equation (116) is given by Equation (6.44) in ref. [14], i.e.,

$$\begin{aligned} a_1(t,r) &= -\frac{16\pi}{3M}r^3 f\int dt\tilde{T}_{(o1)r} + a_{10} \\ &= -\frac{16\pi}{3M}\int dr r^3\frac{1}{f}\tilde{T}_{(o1)t} + a_{10}, \end{aligned} \tag{119}$$

where $a_{10}$ is the perturbative Kerr parameter.

It is important to note that the formal solution described by Equations (116)–(119) has the same form as the formal solution described by Equations (49)–(52). However, we have to emphasize that the variable $\mathscr{F}_{Ap}$ in Equation (49) is gauge-invariant, as noted by Equation (43), and the vector field $V_a$ in Equation (49) is also gauge-invariant. On the other hand, the variable $\tilde{h}_{(o1)C}$ is still gauge-dependent. Actually, there are remaining gauge-degrees of freedom of the generator $\zeta_{(o)}$ that satisfy Equation (114) as noted above. This is clear from the fact that the gauge-transformation rule (92) with Equation (114) gives a still non-trivial transformation rule. This remaining gauge-degree of freedom is so-called "residual gauge". For this reason, there is a possibility that $V_a$ in Equation (116) includes this residual gauge.

To determine whether $V_a$ in Equation (116) includes the "residual gauge", we need to examine Equation (114). According to our rule for comparing our gauge-invariant formulation with a conventional gauge-fixed approach, we consider the terms in $V_a$ that satisfy Equation (114) as representing the second-kind gauge-degrees of freedom. These degrees of freedom are considered "unphysical degrees of freedom". To clarify this "unphysical degree of freedom", we introduce the indicator function $\mathfrak{R}_{(o)}[*]$ as

$$\mathfrak{R}_{(o)}[\zeta_{(o)}] := -\partial_t^2\zeta_{(o)} + f\partial_r\left(f\partial_r\zeta_{(o)}\right) - \frac{f}{r^2}(1-f)\zeta_{(o)}. \quad (120)$$

If $\mathfrak{R}_{(o)}[\zeta_{(o)}] = 0$, we should regard $\zeta_{(o)}$ as the second-kind gauge, and we regard $\zeta_{(o)}$ as an "unphysical degree of freedom".

We can easily confirm that

$$\mathfrak{R}_{(o)}[(\beta_1 t + \beta_0)r] = 0. \quad (121)$$

Then, according to our above rule of comparison, we have to conclude that the rigidly rotating term $(\beta_1 t + \beta_0)r$ in $V_a$ in Equation (117) should be regarded as the second-kind gauge-degree of freedom, namely the "unphysical degree of freedom".

Next, we consider the term $W_{(o)}(t,r)r$ in Equation (117). In this case, the direct calculation of the indicator $\mathfrak{R}_{(o)}[W_{(o)}(t,r)r]$ is useless. However, it is useful to consider the $r$-derivative of $\mathfrak{R}_{(o)}[W_{(o)}(t,r)r]$, and we can show that

$$rf\partial_r\left(\frac{1}{r}\mathfrak{R}_{(o)}[W_{(o)}(t,r)r]\right) = -\partial_t^2 Z_{(o)} + f\partial_r(f\partial_r Z_{(o)}) - \frac{f}{r^2}(2-3(1-f))Z_{(o)}, \quad (122)$$

where we used $Z_{(o)} = rf\partial_r W_{(o)}(t,r)$. Since the left-hand side of Equation (122) coincides with the right-hand side of Equation (118), through Equation (118), we obtain

$$rf\partial_r\left(\frac{1}{r}\mathfrak{R}_{(o)}[W_{(o)}(t,r)r]\right) = -16\pi f^2 \tilde{T}_{(o1)r}. \quad (123)$$

or, equivalently,

$$\mathfrak{R}_{(o)}[W_{(o)}(t,r)r] = -16\pi r\int dr\frac{f}{r}\tilde{T}_{(o1)r}. \quad (124)$$

Then, if we consider the case where $\tilde{T}_{(o1)r} \neq 0$, we have nonvanishing $\mathfrak{R}_{(o)}[W_{(o)}(t,r)r]$. This means that the $W_{(o)}(t,r)$ term does not belong to the second-kind gauge, and we have to regard the degree of freedom $W_{(o)}(t,r)$ in Equation (117) is a "physical one".

On the other hand, even in the case where $\tilde{T}_{(o1)r} = 0$, Equation (122) is valid. If $W_{(o)}(t,r)$ belongs to the second-kind gauge, i.e., $\mathfrak{R}_{(o)}[W_{(o)}(t,r)r] = 0$, Equation (122) yields

$$-\partial_t^2 Z_{(o)} + f\partial_r(f\partial_r Z_{(o)}) - \frac{f}{r^2}(2-3(1-f))Z_{(o)} = 0. \quad (125)$$

However, even if $Z_{(o)}$ is a solution to Equation (125), we cannot directly yield $\mathfrak{R}_{(o)}[W_{(o)}(t,r)r] = 0$. Therefore, considering the following two sets of function $W_{(o)}(t,r)r$ as

$$\mathfrak{G}_{(o)} := \left\{rW_{(o)}(t,r)\middle|\mathfrak{R}_{(o)}[W_{(o)}(t,r)r] = 0\right\}, \quad (126)$$

$$\mathfrak{H}_{(o)} := \left\{rW_{(o)}(t,r)\middle|Z_{(o)} = rf\partial_r W_{(o)}(t,r), -\partial_t^2 Z_{(o)} + f\partial_r(f\partial_r Z_{(o)}) - \frac{f}{r^2}(2-3(1-f))Z_{(o)} = 0.\right\}, \quad (127)$$

we obtain the relation

$$\mathfrak{G}_{(o)} \subset \mathfrak{H}_{(o)}. \quad (128)$$

According to our rule of comparison, this indicates that a part of the solution to Equation (125) should be regarded as a gauge-degree of freedom of the second kind, which is the "unphysical degree of freedom".

Furthermore, to obtain the explicit solution $W_{(o)}(t,r)$, we have to solve Equation (118) with appropriate boundary conditions. Equation (118) is an inhomogeneous second-order linear differential equation for $Z_{(o)}$, and its boundary conditions are adjusted by the homogeneous solutions to Equation (118), i.e., the element of the set of functions $\mathfrak{H}_{(o)}$. However, a part of this homogeneous solution to Equation (118) should be regarded as an unphysical degree of freedom in the "complete gauge-fixing approach", as mentioned above. Therefore, in the conventional "complete gauge-fixing approach", the boundary conditions for Equation (118) is restricted. In this sense, a conventional "complete gauge-fixing approach" includes a stronger restriction than our proposed gauge-invariant formulation.

## 5. $l = 1$ Even-Mode Perturbation in the Conventional Approach

As discussed in Section 3.2.2, $l = 1$ mode perturbations are given by Equations (87)–(89). In particular, among these expressions of the $l = 1$-mode perturbations, even-mode perturbation is given by

$$h_{AB} = \tilde{h}_{AB}S, \quad h_{Ap} = r\tilde{h}_{(e1)A}\hat{D}_pS, \quad h_{pq} = \frac{r^2}{2}\gamma_{pq}\tilde{h}_{(e0)}S. \tag{129}$$

As the property of the $l = 1$ spherical harmonics, we obtain the condition (86). For even-mode perturbation, we have

$$\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}_r\hat{D}^r\right)Y_{1m} = 0. \tag{130}$$

The gauge-transformation rules for the $l = 1$ even-mode perturbations are given by Equations (90), (91), and (93). Inspecting the gauge-transformation rules (91) and (93), we define the variable $\tilde{H}$ by

$$\tilde{H} \quad := \quad \tilde{h}_{(e0)} - 4(\bar{D}^Ar)\tilde{h}_{(e1)A}. \tag{131}$$

The gauge-transformation rule of $\tilde{H}$ is given by

$${}_{\mathscr{Y}}\tilde{H} - {}_{\mathscr{X}}\tilde{H} \quad = \quad -4(\bar{D}^Ar)\bar{D}_A\zeta_{(e)} - \frac{4}{r}\left(1 - (\bar{D}^Ar)(\bar{D}_Ar)\right)\zeta_{(e)}. \tag{132}$$

Furthermore, inspecting gauge-transformation rules (90) and (91), we also define the variable $\tilde{H}_{AB}$ by

$$\tilde{H}_{AB} \quad := \quad \tilde{h}_{AB} - \bar{D}_A\left(r\tilde{h}_{(e1)B}\right) - \bar{D}_B\left(r\tilde{h}_{(e1)A}\right). \tag{133}$$

The gauge-transformation of $\tilde{H}_{AB}$ is given by

$${}_{\mathscr{Y}}\tilde{H}_{AB} - {}_{\mathscr{X}}\tilde{H}_{AB} \quad = \quad -2r\bar{D}_A\bar{D}_B\zeta_{(e)} + 2(\bar{D}_A\bar{D}_Br)\zeta_{(e)}. \tag{134}$$

Definitions (131) and (133) of the variables $\tilde{H}$ and $\tilde{H}_{AB}$, respectively, are analogous to the gauge-invariant variable $\tilde{F}$ and $\tilde{F}_{AB}$ defined by Equations (36) and (38) for $l \geq 2$ modes, respectively. However, the variables $\tilde{H}$ and $\tilde{H}_{AB}$ are not gauge-invariant. The employment of the variables $\tilde{H}$ and $\tilde{H}_{AB}$ corresponds to the gauge-fixing of the gauge-degree of freedom of the generator $\zeta_A$ and the specification of the component $\tilde{h}_{(e1)A}$ of the metric perturbation. On the other hand, the gauge-transformation rules (90), (91), and (93) include the gauge degree of freedom of the generator $\zeta_{(e)}$. This gauge-degree of freedom appears in the gauge-transformation rules (132) and (134).

The linearized Einstein tensor for $l = 1$ even modes in terms of $\tilde{H}_{AB}$ and $\tilde{H}$ are given as

$$y_{CB}{}^{(1)}G_{A}{}^{C}/S = \left[-\frac{1}{2}\bar{D}_C\bar{D}^C + \frac{3}{r^2} - \frac{1}{r}(\bar{D}^C r)\bar{D}_C\right]\tilde{H}_{AB} + \bar{D}_{(A}\bar{D}^C\tilde{H}_{B)C} - \frac{1}{2}\bar{D}_A\bar{D}_B\tilde{H}_C{}^C$$
$$+\frac{2}{r}(\bar{D}^C r)\left[\bar{D}_{(A}\tilde{H}_{B)C} - \frac{1}{r}(\bar{D}_C r)\tilde{H}_{AB}\right] - \frac{1}{2}\bar{D}_A\bar{D}_B\tilde{H} - \frac{1}{r}(\bar{D}_{(A} r)\bar{D}_{B)}\tilde{H}$$
$$+\frac{1}{2}y_{AB}\left[\left[\bar{D}_C\bar{D}^C + \frac{3}{r}(\bar{D}^C r)\bar{D}_C\right]\tilde{H} + \left\{\bar{D}_D\bar{D}^D - \frac{5}{r^2}\right\}\tilde{H}_C{}^C\right.$$
$$+\frac{1}{r}(\bar{D}^C r)\left\{2\bar{D}_C\tilde{H}_D{}^D + \frac{3}{r}(\bar{D}_C r)\tilde{H}_D{}^D\right\} - \bar{D}^C\bar{D}^D\tilde{H}_{CD}$$
$$\left. -\frac{2}{r}(\bar{D}^D r)\left\{2\bar{D}^C\tilde{H}_{CD} + \frac{1}{r}(\bar{D}^C r)\tilde{H}_{CD}\right\}\right], \tag{135}$$

$${}^{(1)}G_{A}{}^{q} = \left[\frac{1}{2r^2}\bar{D}^C\tilde{H}_{AC} - \frac{1}{2r^2}\bar{D}_A\tilde{H}_C{}^C + \frac{1}{2r^3}(\bar{D}_A r)\tilde{H}_C{}^C - \frac{1}{4r^2}\bar{D}_A\tilde{H}\right]\hat{D}^q S, \tag{136}$$

$$\gamma_{qr}{}^{(1)}G_{p}{}^{r} = \left[+\frac{1}{4}\bar{D}_C\bar{D}^C\tilde{H} + \frac{1}{2r}(\bar{D}^C r)\bar{D}_C\tilde{H} + \frac{1}{2}\left\{+\bar{D}_C\bar{D}^C + \frac{1}{r}(\bar{D}^C r)\bar{D}_C - \frac{1}{r^2}\right\}\tilde{H}_D{}^D\right.$$
$$\left. -\frac{1}{2}\left\{\bar{D}_C + \frac{2}{r}(\bar{D}_C r)\right\}\bar{D}_D\tilde{H}^{CD}\right]\gamma_{pq}S$$
$$-\frac{1}{2r^2}\tilde{H}_C{}^C\left(\hat{D}_p\hat{D}_q S - \frac{1}{2}\gamma_{pq}\hat{D}^r\hat{D}_r S\right). \tag{137}$$

Since there is the identity (130) for $l = 1$-mode perturbations, the last term of Equation (137) does not appear for $l = 1$ even modes. Furthermore, the expressions (135) and (136) are gauge-invariant under the gauge-transformation rules (132) and (134). We also note that the component $\gamma_{qr}{}^{(1)}G_p{}^r$ is gauge-invariant, except for the last term in Equation (137). Actually, the variable $\tilde{H}_C{}^C$ is gauge-dependent, as shown below. However, the gauge-invariance of $\gamma_{qr}{}^{(1)}G_p{}^r$ is guaranteed by the identity (130).

On the other hand, the $l = 1$ even components of them are given by

$${}^{(1)}\mathcal{T}_{A}{}^{B} = S\tilde{T}_A{}^B, \tag{138}$$

$$\gamma_{qr}{}^{(1)}\mathcal{T}_{A}{}^{r} = \frac{1}{r}\tilde{T}_{(e1)A}\hat{D}_q S, \tag{139}$$

$$\gamma_{qr}{}^{(1)}\mathcal{T}_{p}{}^{r} = \tilde{T}_{(e0)}\frac{1}{2}\gamma_{pq}S + \tilde{T}_{(e2)}\left(\hat{D}_p\hat{D}_q S - \frac{1}{2}\gamma_{pq}\hat{D}_r\hat{D}^r S\right) \tag{140}$$

Here, again, we note that Equation (130) satisfies as a mathematical identity for $l = 1$-mode perturbations. Therefore, the last term of Equation (140) does not appear for $l = 1$ even modes.

If Equation (130) is not satisfied, we have the equation

$$\left[-\frac{1}{2r^2}\tilde{H}_C{}^C\right]\left(\hat{D}_p\hat{D}_q S - \frac{1}{2}\gamma_{pq}\hat{D}^r\hat{D}_r S\right) = 8\pi\tilde{T}_{(e2)}\left(\hat{D}_p\hat{D}_q S - \frac{1}{2}\gamma_{pq}\hat{D}_r\hat{D}^r S\right) \tag{141}$$

as one of the components of the linearized Einstein equation. However, Equation (130) implies that this equation is identically satisfied. Therefore, this equation does not restrict $\tilde{H}_C{}^C$ or $\tilde{T}_{(e2)}$, and there is no other restriction of them. Here, we note that the tensor field

$\tilde{H}_{AB}$ is not gauge-invariant, as shown in Equation (134). Since we may freely choose $\tilde{H}_C{}^C$ and $\tilde{T}_{(e2)}$, we choose $\tilde{T}_{(e2)} = 0$ by hand, and choose

$$H_C{}^C = 0 \tag{142}$$

as a gauge condition.

Actually, from the gauge-transformation rule (134), we obtain

$$y^{AB}{}_{\mathscr{Y}}\tilde{H}_{AB} - y^{AB}{}_{\mathscr{X}}\tilde{H}_{AB} = -2r\left[-\frac{1}{f}\partial_t^2\zeta_{(e)} + \partial_r(f\partial_r\zeta_{(e)}) - \frac{1-f}{r^2}\zeta_{(e)}\right]. \tag{143}$$

Then, if we choose $\zeta_{(e)}$ as a special solution to the equation

$$\frac{1}{2r}y^{AB}{}_{\mathscr{X}}\tilde{H}_{AB} = -\frac{1}{f}\partial_t^2\zeta_{(e)} + \partial_r(f\partial_r\zeta_{(e)}) - \frac{1-f}{r^2}\zeta_{(e)}, \tag{144}$$

we may regard ${}_{\mathscr{Y}}\tilde{H}_C^C = 0$ in the $\mathscr{Y}$-gauge. We have to note that this gauge-fixing (143) is not complete gauge-fixing. There is a remaining degree of freedom in the choice of $\zeta_{(e)}$ as the homogeneous solution $\zeta_{(e)h}$ to the wave equation

$$-\frac{1}{f}\partial_t^2\zeta_{(e)h} + \partial_r(f\partial_r\zeta_{(e)h}) - \frac{1-f}{r^2}\zeta_{(e)h} = 0. \tag{145}$$

Due to the gauge condition (142), the components (135)–(137) of the linearized Einstein tensor and components (138)–(140) of the linear-order energy–momentum tensor yield the linearized Einstein equations. The $(A,p)$-components (equivalently, $(p,B)$-components) of the linearized Einstein equations are given by

$$\bar{D}^C\tilde{H}_{AC} - \frac{1}{2}\bar{D}_A\tilde{H} = 16\pi r\tilde{T}_{(e1)A}. \tag{146}$$

Equation (146) coincides with Equation (54), because the tensor $\tilde{H}_{AC}$ is traceless due to the gauge condition (142).

From the trace part of the $(p,q)$-component of the linearized Einstein equation, Equation (146) yields the component (61) of the continuity equation of the linearized energy–momentum tensor.

The trace part of $(A,B)$-component of the linearized Einstein equation with Equation (146) gives

$$\bar{D}_C\bar{D}^C\tilde{H} + \frac{2}{r}(\bar{D}_C r)\bar{D}^C\tilde{H} - \frac{4}{r^2}(\bar{D}^C r)(\bar{D}^D r)\tilde{H}_{CD} = 16\pi\left(\tilde{T}_C{}^C + 4(\bar{D}^D r)\tilde{T}_{(e1)D}\right). \tag{147}$$

This coincides with the $l = 1$ version of Equation (56) with the source term (57), because $\tilde{H}_{CD}$ is a traceless tensor. However, we have to emphasize that $\tilde{H}$ nor $\tilde{H}_{CD}$ are not gauge-invariant, as shown above.

Finally, the traceless part of $(A,B)$-component of the linearized Einstein equation with Equations (146) and (147) yields

$$\begin{aligned}
&\left[-\bar{D}_C\bar{D}^C - \frac{2}{r}(\bar{D}^C r)\bar{D}_C + \frac{4}{r}(\bar{D}^C\bar{D}_C r) + \frac{2}{r^2}\right]\tilde{H}_{AB} + \frac{4}{r}(\bar{D}^C r)\bar{D}_{(A}\tilde{H}_{B)C} \\
&-\frac{2}{r}(\bar{D}_{(A}r)\bar{D}_{B)}\tilde{H} \\
= \;& 16\pi\left[\tilde{T}_{AB} - \frac{1}{2}y_{AB}\tilde{T}_C{}^C - 2\left(\bar{D}_{(A}(r\tilde{T}_{(e1)B)}) - \frac{1}{2}y_{AB}\bar{D}^C(r\tilde{T}_{(e1)C})\right)\right. \\
&\left. +2y_{AB}(\bar{D}^D r)\tilde{T}_{(e1)D}\right].
\end{aligned} \tag{148}$$

This coincides with the $l = 1$ version of Equation (58) with the source term (59), because the variable $H_{AB}$ is traceless due to the gauge condition (142).

In addition to these linearized Einstein equations, the following linearized perturbations of the continuity equation should be satisfied:

$$\bar{D}^C \tilde{T}_C{}^B + \frac{2}{r}(\bar{D}^D r)\tilde{T}_D{}^B - \frac{2}{r}\tilde{T}^B_{(e1)} - \frac{1}{r}(\bar{D}^B r)\tilde{T}_{(e0)} = 0, \quad (149)$$

$$\bar{D}^C \tilde{T}_{(e1)C} + \frac{3}{r}(\bar{D}^C r)\tilde{T}_{(e1)C} + \frac{1}{2r}\tilde{T}_{(e0)} = 0. \quad (150)$$

Together with the gauge condition (142), Equations (146)–(150) coincide with Equations (53)–(61) with $l = 1$ in our gauge-invariant formulation developed in refs. [12–16]. In ref. [15], we derived the formal solution to Equations (53)–(61) with $l = 1$ as Equation (62) with Equations (63) and (64), and $m = 0$. Due to the coincidence of the set of equations, this formal solution should be the $m = 0$ formal solution to Equation (142), and Equations (146)–(150). Then, as the $m = 0$ solution, we obtain

$$\begin{aligned}
\mathscr{H}_{ab} &:= H_{AB}\cos\theta (dx^A)_{(a}(dx^B)_{b)} + \frac{r^2}{2} H \cos\theta \gamma_{ab} \\
&= \pounds_V g_{ab} - \frac{16\pi r^2}{3(1-f)}\left[ f^2\left\{ \frac{1+f}{2}\tilde{T}_{rr} + r f \partial_r \tilde{T}_{rr} - \tilde{T}_{(e0)} - 4\tilde{T}_{(e1)r} \right\}(dt)_a(dt)_b \right. \\
&\qquad + \frac{2r}{f}\left\{ \partial_t \tilde{T}_{tt} - \frac{3f(1-f)}{2r}\tilde{T}_{tr} \right\}(dt)_{(a}(dr)_{b)} \\
&\qquad + \frac{r}{f}\left\{ \partial_r \tilde{T}_{tt} - \frac{3(1-3f)}{2rf}\tilde{T}_{tt} \right\}(dr)_a(dr)_b) \\
&\qquad \left. + r^2 \tilde{T}_{tt}\gamma_{ab} \right] \cos\theta,
\end{aligned} \quad (151)$$

where $\gamma_{ab} = \gamma_{pq}(dx^p)_a(dx^q)_b$, $V_a = g_{ab}V^b$ is the vector field defined by

$$V_a := -r\partial_t \Phi_{(e)} \cos\theta (dt)_a + (\Phi_{(e)} - r\partial_r \Phi_{(e)}) \cos(dr)_a - r\Phi_{(e)} \sin\theta (d\theta)_a \quad (152)$$

and $\Phi_{(e)}$ is a solution to the equation

$$-\frac{1}{f}\partial_t^2 \Phi_{(e)} + \partial_r\left[ f\partial_r \Phi_{(e)} \right] - \frac{1-f}{r^2}\Phi_{(e)} = \frac{16\pi r}{3(1-f)} S_{(\Phi_{(e)})}, \quad (153)$$

$$S_{(\Phi_{(e)})} = \frac{1}{2} r\partial_t T_{tr} - \frac{1}{2} r\partial_r \tilde{T}_{tt} + \frac{1-4f}{2f}\tilde{T}_{tt} - \frac{f}{2}\tilde{T}_{rr} - f\tilde{T}_{(e1)r}. \quad (154)$$

As in the case of the $l = 1$ odd-mode perturbation, the tensor field $\mathscr{F}_{ab}$ in Equation (62) is gauge-invariant. Then, $V_a$ in Equation (62), which defined by Equation (63), is also gauge-invariant. Therefore, we regarded $V_a$ in Equation (62) as a first-kind gauge. On the other hand, the tensor field $\mathscr{H}_{ab}$ in Equation (151) is gauge-dependent in the sense of the second kind due to the gauge-transformation rules (132) and (134) for the components $\tilde{H}$ and $\tilde{H}_{AB}$, respectively. Actually, there is a remaining gauge-degree of freedom of the generator $\zeta_{(e)}$ which satisfies Equation (145), as noted above. This remaining gauge-degree of freedom is so-called "residual gauge". For this reason, there is a possibility that $V_a$ in Equation (151) includes this "residual gauge", while $V_a$ in Equation (62) is gauge-invariant.

Since we already fixed the gauge-degree of freedom $\zeta_{(e)}$ so that Equation (142) is satisfied, the remaining gauge-degree of freedom $\zeta_{(e)h}$ must satisfy the Equation (145). Therefore, to clarify whether $V_a$ in Equation (151) includes the "residual gauge", we have to confirm Equation (145). Within our rules to compare our gauge-invariant formulation with a conventional gauge-fixed approach, we regard the term in $V_a$ in Equation (151) that satisfies Equation (145) as a second-kind gauge-degree of freedom, and we regard this degree of freedom as an "unphysical degree of freedom". As in the case of the $l = 1$ odd-mode perturbations, we introduce the indicator function $\mathfrak{R}_{(e)}[*]$ as

$$\mathfrak{R}_{(e)}[\zeta_{(e)h}] := -\frac{1}{f}\partial_t^2\zeta_{(e)h} + \partial_r(f\partial_r\zeta_{(e)h}) - \frac{1-f}{r^2}\zeta_{(e)h}. \tag{155}$$

If $\mathfrak{R}_{(e)}[\zeta_{(e)}] = 0$, we should regard $\zeta_{(e)}$ as a the second-kind gauge-degree of freedom, and we regard $\zeta_{(e)}$ as an "unphysical degree of freedom".

To clarify the second-kind gauge-degree of freedom, we consider the gauge-transformation rule of the tensor field $\mathscr{H}_{ab}$ through the gauge-transformation rules (132) and (134) as

$$\begin{aligned}
&{}_{\mathscr{Y}}\mathscr{H}_{ab} - {}_{\mathscr{X}}\mathscr{H}_{ab} \\
:=\;& ({}_{\mathscr{Y}}H_{AB} - {}_{\mathscr{X}}H_{AB})\cos\theta(dx^A)_{(a}(dx^B)_{b)} + \frac{r^2}{2}({}_{\mathscr{Y}}H - {}_{\mathscr{X}}H)\cos\theta\gamma_{ab} \\
=\;& -2r\left[\partial_t^2\zeta_{(e)h} - \frac{f(1-f)}{2r}\partial_r\zeta_{(e)h} + \frac{f(1-f)}{2r^2}\zeta_{(e)h}\right]\cos\theta(dt)_a(dt)_b \\
& -4r\left[\partial_t\partial_r\zeta_{(e)h} - \frac{1-f}{2rf}\partial_t\zeta_{(e)h}\right]\cos\theta(dt)_{(a}(dr)_{b)} \\
& -2r\left[\partial_r^2\zeta_{(e)h} + \frac{1-f}{2rf}\partial_r\zeta_{(e)h} - \frac{1-f}{2r^2f}\zeta_{(e)h}\right]\cos\theta(dr)_a(dr)_b \\
& -2r^2\left(f\partial_r\zeta_{(e)h} + \frac{1}{r}(1-f)\zeta_{(e)h}\right)\cos\theta\gamma_{ab} && (156)\\
=:\;& \pounds_W g_{ab}, && (157)
\end{aligned}$$

where we defined

$$W_a := -r\partial_t\zeta_{(e)h}\cos\theta(dt)_a + \left(\zeta_{(e)h} - r\partial_r\zeta_{(e)h}\right)\cos\theta(dr)_a - r\zeta_{(e)h}\sin\theta(d\theta)_a. \tag{158}$$

Comparing Equation (157) with the generator (158) and Equation (151) with the generator (152), there is the possibility that the Lie derivative term $\pounds_V g_{ab}$ in the solution (151) is a "residual gauge degree of freedom" with the identification

$$\Phi_{(e)} = \zeta_{(e)h}. \tag{159}$$

For this reason, we check the indicator (155). From Equation (153), and we obtain

$$\mathfrak{R}_{(e)}[\Phi_{(e)}] := \frac{16\pi r}{3(1-f)}S_{(\Phi_{(e)})}. \tag{160}$$

From this result, if $S_{(\Phi_{(e)})} = 0$, the variable $\Phi_{(e)}$ should be regarded as the second-kind gauge-degree of freedom, and is the "unphysical degree of freedom". On the other hand, in the non-vacuum case $S_{\Phi_{(e)}} \neq 0$, the indicator (160) yields $\mathfrak{R}_{(e)}[\Phi_{(e)}] \neq 0$. This means that $\Phi_{(e)}$ is not the second-kind gauge-degree of freedom, but is the "physical degree of freedom" in the non-vacuum case $S_{\Phi_{(e)}} \neq 0$. Due to this existence of the source term, the identification (159) is impossible in the case of the non-vacuum situation. Rather, this term is gauge-invariant in the sense of the second kind, and we regard this term as a gauge-degree of freedom of the first kind in the non-vacuum case.

Although we have $\mathfrak{R}_{(e)}[\Phi_{(e)}] = 0$ in the case where $S_{\Phi_{(e)}} = 0$, and we should regard $\Phi_{(e)}$ is a "gauge degree of freedom of the second kind", Equation (160) indicates that

$$\mathfrak{R}_{(e)}[\Phi_{(e)}] = -\frac{1}{f}\partial_t^2\Phi_{(e)} + \partial_r(f\partial_r\Phi_{(e)}) - \frac{1-f}{r^2}\Phi_{(e)} = 0. \tag{161}$$

This coincides with the left-hand side of Equation (153). Therefore, as in the case of the $l = 1$ odd-mode perturbations, we consider the following two sets of function $\Phi_{(e)}$ as

$$\mathfrak{G}_{(e)} := \left\{ \Phi_{(e)} \middle| \mathfrak{R}_{(e)}[\Phi_{(e)}] = 0 \right\}, \tag{162}$$

$$\mathfrak{H}_{(e)} := \left\{ \Phi_{(e)} \middle| -\frac{1}{f}\partial_t^2\Phi_{(e)} + \partial_r(f\partial_r\Phi_{(e)}) - \frac{1-f}{r^2}\Phi_{(e)} = 0 \right\}. \tag{163}$$

From Equation (161), we obtain the relation

$$\mathfrak{G}_{(e)} = \mathfrak{H}_{(e)}. \tag{164}$$

This indicates that any homogeneous solution (without source term) to Equation (153) should be regarded as a gauge-degree of freedom of the second kind, which is the "unphysical degree of freedom".

On the other hand, to obtain the explicit solution $\Phi_{(e)}$ in the case $S_{\Phi_{(e)}} \neq 0$, which is regarded as a "physical degree of freedom", we have to solve Equation (153) with appropriate boundary conditions. Equation (153) is an inhomogeneous second-order linear differential equation for $\Phi_{(e)}$, and its boundary conditions are adjusted by the homogeneous solutions to Equation (153), i.e., the element of the set of functions $\mathfrak{H}_{(e)}$. However, any homogeneous solution to Equation (153) should be regarded as an "unphysical degree of freedom" in the "complete gauge-fixing approach", as mentioned above. Therefore, in the conventional "complete gauge-fixing approach", we have to impose the boundary conditions for Equation (118) using a homogeneous solution, which is regarded as an "unphysical degree of freedom", to obtain a "physical solution" to Equation (118) with nonvanishing source term $S_{\Phi_{(e)}} \neq 0$. This situation is a dilemma. In this sense, as in the case of $l = 1$ odd-mode perturbations, a conventional "complete gauge-fixing approach" includes a stronger restriction than our proposed gauge-invariant formulation.

## 6. *l* = 0 Mode Perturbation in the Conventional Approach

### *6.1. Einstein Equations for $l = 0$ Mode Perturbations*

Here, we consider the $l = 0$-mode perturbations, which are described by Equations (94)–(96). By substituting Equations (94)–(96) into Equations (A16)–(A18) in Appendix A, we obtain the $l = 0$ mode perturbations of the linearized Einstein tensor. We use the background Einstein equations (Equation (B67) and (B68) of Appendix B in [14]). Then, we obtain the non-trivial components of the $l = 0$-mode Einstein equation ${}^{(1)}G_a{}^b = 8\pi{}^{(1)}T_a{}^b$ are given by

$$\begin{aligned}
& -\frac{1}{2}\bar{D}_C\bar{D}^C\tilde{h}_A{}^B + \frac{1}{2}\bar{D}^B\bar{D}^C\tilde{h}_{AC} + \frac{1}{2}\bar{D}_A\bar{D}_C\tilde{h}^{BC} - \frac{1}{2}\bar{D}_A\bar{D}^B\tilde{h}_C{}^C \\
& -\frac{1}{r}(\bar{D}^Cr)\bar{D}_C\tilde{h}_A{}^B + \frac{1}{r}(\bar{D}^Cr)\bar{D}^B\tilde{h}_{AC} + \frac{1}{r}(\bar{D}_Cr)\bar{D}_A\tilde{h}^{BC} - \frac{2}{r^2}(\bar{D}^Cr)(\bar{D}_Cr)\tilde{h}_A{}^B \\
& -\frac{1}{2r}(\bar{D}^Br)\bar{D}_A\tilde{h}_{(e0)} - \frac{1}{2r}(\bar{D}_Ar)\bar{D}^B\tilde{h}_{(e0)} - \frac{1}{2}\bar{D}_A\bar{D}^B\tilde{h}_{(e0)} + \frac{2}{r^2}\tilde{h}_A{}^B \\
& +y_A{}^B\Big(-\frac{1}{2}\bar{D}_C\bar{D}_D\tilde{h}^{CD} + \frac{1}{2}\bar{D}_C\bar{D}^C\tilde{h}_D{}^D - \frac{2}{r}(\bar{D}_Dr)\bar{D}_C\tilde{h}^{CD} + \frac{1}{r}(\bar{D}^Cr)\bar{D}_C\tilde{h}_D{}^D \\
& \qquad -\frac{1}{r^2}(\bar{D}_Cr)(\bar{D}_Dr)\tilde{h}^{CD} + \frac{3}{2r^2}(\bar{D}^Dr)(\bar{D}_Dr)\tilde{h}_C{}^C - \frac{3}{2r^2}\tilde{h}_C{}^C \\
& \qquad +\frac{1}{2r^2}\tilde{h}_{(e0)} + \frac{3}{2r}(\bar{D}^Cr)\bar{D}_C\tilde{h}_{(e0)} + \frac{1}{2}\bar{D}_C\bar{D}^C\tilde{h}_{(e0)}\Big) \\
= \; & 8\pi\tilde{T}_A{}^B,
\end{aligned} \tag{165}$$

$$\begin{aligned}
& \bar{D}_C\bar{D}^C\tilde{h}_D{}^D - \bar{D}_C\bar{D}_D\tilde{h}^{CD} - \frac{2}{r}(\bar{D}_Cr)\bar{D}_D\tilde{h}^{CD} + \frac{1}{r}(\bar{D}^Cr)\bar{D}_C\tilde{h}_D{}^D \\
& +\frac{1}{2}\bar{D}_C\bar{D}^C\tilde{h}_{(e0)} + \frac{1}{r}(\bar{D}^Cr)\bar{D}_C\tilde{h}_{(e0)} \\
= \; & 8\pi\tilde{T}_{(e0)}.
\end{aligned} \tag{166}$$

Here, we decomposition of the component $\tilde{h}_{AB}$ as

$$\tilde{h}_{AB} =: \tilde{\mathbb{H}}_{AB} + \frac{1}{2}y_{AB}\tilde{h}_C{}^C. \tag{167}$$

In terms of the variables defined by Equation (167), the linearization of the Einstein equations for $l = 0$ mode are summarized as follows. The trace of ${}^{(1)}G_A{}^B = 8\pi{}^{(1)}T_A{}^B$ for $l = 0$ mode is given by

$$\begin{aligned} &-\frac{1}{r^2}\tilde{h}_C{}^C - \frac{2}{r}(\bar{D}_C r)\left(+\bar{D}_D\tilde{\mathbb{H}}^{DC} + \frac{1}{r}(\bar{D}_D r)\tilde{\mathbb{H}}^{CD}\right) \\ &+\frac{1}{2}\bar{D}_C\bar{D}^C\tilde{h}_{(e0)} + \frac{2}{r}(\bar{D}_C r)\bar{D}^C\tilde{h}_{(e0)} + \frac{1}{r^2}\tilde{h}_{(e0)} = 8\pi\tilde{T}_C{}^C. \end{aligned} \tag{168}$$

On the other hand, the traceless part of ${}^{(1)}G_A{}^B = 8\pi{}^{(1)}T_A{}^B$ for $l = 0$ mode is given by

$$\begin{aligned} &-\frac{1}{2}\left[\bar{D}_C\bar{D}^C + \frac{2}{r}(\bar{D}^C r)\bar{D}_C + \frac{4}{r^2}(\bar{D}^C r)(\bar{D}_C r) - \frac{4}{r^2}\right]\tilde{\mathbb{H}}_{AB} \\ &+\bar{D}_{(A}\bar{D}^C\tilde{\mathbb{H}}_{B)C} - \frac{1}{2}y_{AB}\bar{D}_C\bar{D}_D\tilde{\mathbb{H}}^{CD} + \frac{2}{r}(\bar{D}^C r)\left(\bar{D}_{(A}\tilde{\mathbb{H}}_{B)C} - \frac{1}{2}y_{AB}\bar{D}^D\tilde{\mathbb{H}}_{DC}\right) \\ &+\frac{1}{r}\left((\bar{D}_{(A} r)\bar{D}_{B)}\tilde{h}_E{}^E - \frac{1}{2}y_{AB}(\bar{D}^C r)\bar{D}_C\tilde{h}_D{}^D\right) - \frac{1}{2}\left[\bar{D}_A\bar{D}_B\tilde{h}_{(e0)} - \frac{1}{2}y_{AB}\bar{D}_C\bar{D}^C\tilde{h}_{(e0)}\right] \\ &-\frac{1}{r}\left[(\bar{D}_{(A} r)\bar{D}_{B)}\tilde{h}_{(e0)} - \frac{1}{2}y_{AB}(\bar{D}^C r)\bar{D}_C\tilde{h}_{(e0)}\right] = 8\pi\left[\tilde{T}_{AB} - \frac{1}{2}y_{AB}\tilde{T}_C{}^C\right]. \end{aligned} \tag{169}$$

Furthermore, ${}^{(1)}G_p{}^q = 8\pi{}^{(1)}T_p{}^q$ for $l = 0$ mode is given by

$$\begin{aligned} &\frac{1}{2}\bar{D}_C\bar{D}^C\tilde{h}_{(e0)} + \frac{1}{r}(\bar{D}^C r)\bar{D}_C\tilde{h}_{(e0)} - \bar{D}_C\bar{D}_D\tilde{\mathbb{H}}^{CD} - \frac{2}{r}(\bar{D}_C r)\bar{D}_D\tilde{\mathbb{H}}^{CD} + \frac{1}{2}\bar{D}_C\bar{D}^C\tilde{h}_D{}^D \\ = \;& 8\pi\tilde{T}_{(e0)}. \end{aligned} \tag{170}$$

### *6.2. Gauge-Fixing for $l = 0$ Mode Perturbations*

We consider the gauge-transformation rules (97) and (98) of the mode coefficient $\tilde{h}_{AB}$ and $\tilde{h}_{(e0)}$. If these gauge-transformation rules are those of the second kind, we should exclude these gauge-degrees of freedom through some gauge-fixing procedure, because the degree of freedom of the second-kind gauge is the unphysical degree of freedom.

In the static chart, the metric $y_{AB}$ is given by Equation (17). Through the static chart (17), the gauge-transformation rule (98) is given by

$${}_{\mathscr{Y}}\tilde{h}_{(e0)} - {}_{\mathscr{X}}\tilde{h}_{(e0)} = \frac{4}{r}f\zeta_r, \tag{171}$$

and we may choose $\zeta_r$ so that

$${}_{\mathscr{Y}}\tilde{h}_{(e0)} \;=\; {}_{\mathscr{X}}\tilde{h}_{(e0)} + \frac{4}{r}f\zeta_r = 0, \tag{172}$$

i.e.,

$$\zeta_r = -\frac{r}{4f}\,{}_{\mathscr{X}}\tilde{h}_{(e0)}. \tag{173}$$

Through this gauge-fixing, we may regard $\tilde{h}_{(e0)} = 0$ in the $\mathscr{Y}$-gauge.

As the gauge-fixing for $\tilde{h}_{AB}$, we fix the gauge so that $\tilde{h}_{AB}$ is traceless. This gauge-fixing so that $\tilde{h}_{AB}$ is traceless makes it easy to compare with our gauge-invariant expression in refs. [12–16]. From the trace of the gauge-transformation rule Equation (97), we obtain

$$y^{AB}{}_{\mathcal{Y}}\tilde{h}_{AB} - y^{AB}{}_{\mathcal{X}}\tilde{h}_{AB} \quad = \quad {}_{\mathcal{Y}}\tilde{h}_{C}{}^{C} - {}_{\mathcal{X}}\tilde{h}_{C}{}^{C} = 2\bar{D}^{C}\zeta_{C}. \tag{174}$$

In the static chart (17) of the Schwarzschild spacetime, the gauge-transformation rule (174) is given by

$${}_{\mathcal{Y}}\tilde{h}_{C}{}^{C} - {}_{\mathcal{X}}\tilde{h}_{C}{}^{C} \quad = \quad -2f^{-1}\partial_{t}\zeta_{t} + 2\partial_{r}(f\zeta_{r}). \tag{175}$$

From this gauge-transformation rule and the gauge-fixing (173), we may fix the gauge degree of freedom $\zeta_t$ so that

$$\begin{aligned} 0 \quad &= \quad {}_{\mathcal{Y}}\tilde{h}_{C}{}^{C} \\ &= \quad {}_{\mathcal{X}}\tilde{h}_{C}{}^{C} - 2f^{-1}\partial_{t}\zeta_{t} + 2\partial_{r}\left(-\frac{r}{4}{}_{\mathcal{X}}\tilde{h}_{(e0)}\right). \end{aligned} \tag{176}$$

Through this gauge-fixing, we may regard $\tilde{h}_{C}{}^{C} = 0$. However, we have to emphasize that the degree of freedom of the choice of the generator $\zeta_t$ remains the degree of freedom of an arbitrary function of $r$.

Thus, through the gauge-fixing (173) and (176), we may regard

$$\tilde{h}_{(e0)} = \tilde{h}_{C}{}^{C} = 0 \tag{177}$$

Under the gauge-fixing condition (177), linearization of the Einstein equations for $l = 0$ mode are summarized as

$$-(\bar{D}_{C}r)\bar{D}_{D}\tilde{\mathbb{H}}^{DC} - \frac{1}{r}(\bar{D}_{C}r)(\bar{D}_{D}r)\tilde{\mathbb{H}}^{CD} = 4\pi r\tilde{T}_{C}{}^{C}, \tag{178}$$

$$\begin{aligned} &-\frac{1}{2}\left[\bar{D}_{C}\bar{D}^{C} + \frac{2}{r}(\bar{D}^{C}r)\bar{D}_{C} + \frac{4}{r^{2}}(\bar{D}^{C}r)(\bar{D}_{C}r) - \frac{4}{r^{2}}\right]\tilde{\mathbb{H}}_{AB} + \bar{D}_{(A}\bar{D}^{C}\tilde{\mathbb{H}}_{B)C} \\ &-\frac{1}{2}y_{AB}\bar{D}_{C}\bar{D}_{D}\tilde{\mathbb{H}}^{CD} + \frac{2}{r}(\bar{D}^{C}r)\left(+\bar{D}_{(A}\tilde{\mathbb{H}}_{B)C} - \frac{1}{2}y_{AB}\bar{D}^{D}\tilde{\mathbb{H}}_{DC}\right) \\ = \quad &8\pi\left[\tilde{T}_{AB} - \frac{1}{2}y_{AB}\tilde{T}_{C}{}^{C}\right], \end{aligned} \tag{179}$$

and

$$-\bar{D}_{C}\bar{D}_{D}\tilde{\mathbb{H}}^{CD} - \frac{2}{r}(\bar{D}_{C}r)\bar{D}_{D}\tilde{\mathbb{H}}^{CD} = 8\pi\tilde{T}_{(e0)}. \tag{180}$$

### 6.3. Component Expression of the Linearized $l = 0$ Gauge-Fixed Field Equations

Here, we consider the component representations of Equations (178)–(180). To do this, we consider the components of the traceless tensor $\tilde{\mathbb{H}}_{AB}$ as

$$\tilde{\mathbb{H}}_{AB} =: X_{(e)}\left[(dt)_{A}(dt)_{B} + f^{-2}(dr)_{A}(dr)_{B}\right] + 2Y_{(e)}(dt)_{(A}(dr)_{B)}. \tag{181}$$

Here, we use the background Einstein equation (B65) of Appendix B in [14], i.e.,

$$\partial_{r}f = \frac{1-f}{r}, \tag{182}$$

and

$$\partial_{r}^{2}f = \partial_{r}\left(\frac{1-f}{r}\right) = -\frac{2(1-f)}{r^{2}} \tag{183}$$

Through Equations (182) and (183), Equation (178) is given by

$$rf\partial_t Y_{(e)} - rf\partial_r X_{(e)} - fX_{(e)} = -4\pi r^2\left(\tilde{T}_{tt} - f^2\tilde{T}_{rr}\right). \tag{184}$$

The $(A,B) = (t,t)$ and $(A,B) = (r,r)$ components of Equation (179) yield the same equation as

$$rf\partial_t Y_{(e)} = 4\pi r^2\left(\tilde{T}_{tt} + f^2\tilde{T}_{rr}\right). \tag{185}$$

The $(A,B) = (t,r)$ component of Equation (179) is given by

$$\partial_t X_{(e)} = 8\pi rf\tilde{T}_{tr}. \tag{186}$$

In terms of the components (181), Equation (180) is given by

$$-\partial_t^2 X_{(e)} - f^2\partial_r^2 X_{(e)} - \frac{2}{r}f^2\partial_r X_{(e)} + 2f^2\partial_t\partial_r Y_{(e)} + \frac{1}{r}f(1+f)\partial_t Y_{(e)} = 8\pi f^2\tilde{T}_{(e0)}. \tag{187}$$

On the other hand, the even-mode perturbation of the divergence of the energy–momentum tensor is given by Equations (60) and (61). However, Equation (61) does not appear, due to $\hat{D}_p S = \epsilon_{pq}\hat{D}^q S = 0$. Then, the non-trivial $l = 0$ components of the divergence of the energy–momentum tensor are given by

$$\bar{D}^C\tilde{T}_C{}^B + \frac{2}{r}(\bar{D}^D r)\tilde{T}_D{}^B - \frac{1}{r}(\bar{D}^B r)\tilde{T}_{(e0)} = 0. \tag{188}$$

The $B = t$ component of Equation (188) is given by

$$\partial_t\tilde{T}_{tt} - f^2\partial_r\tilde{T}_{rt} - \frac{1}{r}f(1+f)\tilde{T}_{rt} = 0, \tag{189}$$

and the $B = r$ component of Equation (188) is given by

$$-f\partial_t\tilde{T}_{tr} + f^3\partial_r\tilde{T}_{rr} + \frac{1}{2r}(1-f)\tilde{T}_{tt} + \frac{1}{2r}f^2(3+f)\tilde{T}_{rr} - \frac{1}{r}f^2\tilde{T}_{(e0)} = 0. \tag{190}$$

Substituting Equation (185) into Equation (184), we obtain

$$\partial_r\left(rX_{(e)}\right) = r\partial_r X_{(e)} + X_{(e)} = 8\pi\frac{r^2}{f}\tilde{T}_{tt}. \tag{191}$$

On the other hand, from the substitution of Equations (185), (186) and (191) into Equation (187), we obtain

$$0 = -f\partial_t\tilde{T}_{tr} + f^3\partial_r\tilde{T}_{rr} + \frac{1}{2r}(1-f)\tilde{T}_{tt} + \frac{1}{2r}f^2(3+f)\tilde{T}_{rr} - \frac{1}{r}f^2\tilde{T}_{(e0)}.$$

This coincides with Equation (190), which indicates that Equation (187) does not give any new information other than Equation (190).

Here, we consider the integrability of Equation (186) and (191) as follows:

$$\begin{aligned}\partial_r(\partial_t(rX_{(e)})) - \partial_t(\partial_r(rX_{(e)})) &= \partial_r\left(8\pi r^2 f\tilde{T}_{tr}\right) - \partial_t\left(8\pi\frac{r^2}{f}\tilde{T}_{tt}\right)\\ &= -8\pi r^2\left[+\partial_t\tilde{T}_{tt} - f^2\partial_r\tilde{T}_{tr} - \frac{1}{r}f(1+f)\tilde{T}_{tr}\right]\\ &= 0,\end{aligned} \tag{192}$$

where we used Equation (189) in the last equality. This means that the $t$-component (189) of the continuity equation guarantees the integrability of Equations (186) and (191). Then, from Equation (191), we may write the solution to Equations (186) and (191) under the integrability condition (189) as

$$X_{(e)} = \frac{1}{r}\left[2M_1 + 8\pi\int dr\frac{r^2}{f}\tilde{T}_{tt}\right], \tag{193}$$

where $M_1$ is the constant of integration. This $M_1$ corresponds to the perturbation of the Schwarzschild mass parameter.

On the other hand, in the linearized Einstein equation, there is no equation for $\partial_r Y_{(e)}$ which guarantees the integrability condition for Equation (185). However, we may write the solution to the Equation (185) as

$$Y_{(e)} = \frac{4\pi r}{f}\int dt\left(\tilde{T}_{tt} + f^2\tilde{T}_{rr}\right) + Y_{(e)0}(r). \tag{194}$$

where $Y_{(e)0}(r)$ is an arbitrary function of $r$. There is no equation that determines the arbitrary function $Y_{(e)0}(r)$ of $r$ within the linearized Einstein equations.

From Definition (181) of the metric perturbation $\tilde{\mathbb{H}}_{AB}$ and the solutions (193) and (194), we obtain

$$\begin{aligned}\tilde{\mathbb{H}}_{AB} &= \frac{2}{r}\left(M_1 + 4\pi\int dr\frac{r^2}{f}\tilde{T}_{tt}\right)\left((dt)_A(dt)_B + f^{-2}(dr)_A(dr)_B\right)\\ &\quad +2\left[4\pi r\int dt\left(\frac{1}{f}\tilde{T}_{tt} + f\tilde{T}_{rr}\right) + Y_{(e)0}(r)\right](dt)_{(A}(dr)_{B)},\end{aligned} \tag{195}$$

where the components of the energy–momentum tensor satisfy the continuity Equations (189) and (190).

To interpret the term of $Y_{(e)0}(r)$ in Equation (195), we consider the term $\pounds_V g_{ab}$ with the generator $V_a$, whose components are given by

$$V_a = V_t(r)(dt)_a. \tag{196}$$

The nonvanishing components of $\pounds_V g_{ab}$ are given by

$$\pounds_V g_{tr} = f\partial_r\left(\frac{1}{f}V_t\right). \tag{197}$$

Choosing $V_t$ so that

$$f\partial_r\left(\frac{1}{f}V_t\right) = Y_{(e)0}(r), \quad V_t = f\int dr\frac{1}{f}Y_{(e)0}(r), \tag{198}$$

we obtain

$$\pounds_V g_{tr} = Y_{(e)0}(r). \tag{199}$$

Thus, the solution (195) is given by

$$\begin{aligned}\tilde{\mathbb{H}}_{ab} &= \frac{2}{r}\left(M_1 + 4\pi\int dr\frac{r^2}{f}\tilde{T}_{tt}\right)\left((dt)_a(dt)_b + f^{-2}(dr)_a(dr)_b\right)\\ &\quad +8\pi r\int dt\left(\frac{1}{f}\tilde{T}_{tt} + f\tilde{T}_{rr}\right)(dt)_{(a}(dr)_{b)} + \pounds_V g_{ab},\end{aligned} \tag{200}$$

where

$$V_a = \left( f \int dr \frac{1}{f} Y_{(e)0}(r) \right) (dt)_a. \tag{201}$$

As noted just after Equation (176), there is still the remaining degree of freedom of the gauge, whose generator is given by

$$\zeta_a = \zeta_t(r)(dt)_a, \tag{202}$$

where $\zeta_t(r)$ is an arbitrary function of $r$. Therefore, we may regard the degree of freedom of $V_a$ given in Equation (201) as possibly being eliminated as the second-kind gauge-degree of freedom (202). According to our rule of the comparison between our gauge-invariant formulation and a "conventional complete gauge-fixing approach", we have to regard that the degree of freedom of $V_a$ given in Equation (201) is "unphysical".

Thus, as the "physical solution" for $l = 0$-mode linearized Einstein equation based of a "conventional complete gauge-fixing approach", we obtain

$$\begin{aligned} h_{ab} &= \frac{2}{r}\left( M_1 + 4\pi \int dr \frac{r^2}{f} \tilde{T}_{tt} \right) \left( (dt)_a(dt)_b + f^{-2}(dr)_a(dr)_b \right) \\ &\quad + 8\pi r \int dt \left( \frac{1}{f} \tilde{T}_{tt} + f \tilde{T}_{rr} \right) (dt)_{(a}(dr)_{b)}. \end{aligned} \tag{203}$$

This solution (203) coincides with the solution obtained in refs. [12,15] except for the terms of the Lie derivative of the background metric $g_{ab}$. Therefore, we conclude that the $l = 0$ solution except for the terms of the Lie derivative of the background metric $g_{ab}$ obtained in refs. [12,15] can be also obtained as (203) through a complete gauge-fixing approach.

The difference between our solution in refs. [12,15] is in the term of the Lie derivative of the background metric. As shown above, according to our rule of comparison between our gauge-invariant formulation and a "conventional complete gauge-fixing approach", all terms of the Lie derivative of the background metric should be regarded as the second-kind gauge-degree of freedom, and these are "unphysical degree of freedom" in the above solution (203). On the other hand, in our gauge-invariant formulation developed in refs. [12–16], $l = 0, 1$-mode metric perturbations are given in a gauge-invariant form, and the terms of the Lie derivative of the background metric is included in these gauge-invariant variables. Since the second-kind gauge-degree of freedom is completely excluded in our gauge-invariant formulation, we cannot regard these terms of the Lie derivative of the background metric as an "unphysical degree of freedom". Therefore, we regard the terms of the Lie derivative of the background metric in these gauge-invariant variables as first-kind gauges that have some physical meaning. This point is the essential difference of the solutions in refs. [12–16] and the above solutions based on a "conventional complete gauge-fixing approach". This difference leads to the confusion of the interpretation of the perturbative Tolman Bondi solution, as shown in the next subsection.

### *6.4. Comparing with Lemaître–Tolman–Bondi Solution*

#### 6.4.1. Perturbative Expression of the LTB Solution on Schwarzschild Background Spacetime

Here, we consider the Lemaître–Tolman–Bondi (LTB) solution [42], which is an exact solution to the Einstein equation with the matter field

$$T_{ab} = \rho u_a u_b, \quad u_a = -(d\tau)_a, \tag{204}$$

and the metric

$$g_{ab} = -(d\tau)_a(d\tau)_b + \frac{(\partial_R r)^2}{1 + f(R)}(dR)_a(dR)_b + r^2\gamma_{ab}, \quad r = r(\tau, R). \tag{205}$$

This solution is a spherically symmetric solution to the Einstein equation. The function $r = r(\tau, R)$ satisfies the differential equation

$$(\partial_\tau r)^2 = \frac{F(R)}{r} + f(R). \tag{206}$$

Note that $F(R)$ is an arbitrary function of $R$ representing the dust matter's initial distribution. $f(R)$ is also an arbitrary function of $R$ that represents the initial distribution of the energy of the dust field in the Newtonian sense. The solution to Equation (206) is given in three cases.

**Description 2.**

*(i)* $f(R) > 0$*:*

$$r = \frac{F(R)}{2f(R)}(\cosh\eta - 1), \quad \tau_0(R) - \tau = \frac{F(R)}{2f(R)^{3/2}}(\sinh\eta - \eta), \tag{207}$$

*(ii)* $f(R) < 0$*:*

$$r = \frac{F(R)}{-2f(R)}(1 - \cos\eta), \quad \tau_0(R) - \tau = \frac{F(R)}{2(-f(R))^{3/2}}(\eta - \sin\eta), \tag{208}$$

*(iii)* $f(R) = 0$*:*

$$r = \left(\frac{9F(R)}{4}\right)^{1/3} [\tau_0(R) - \tau]^{2/3}. \tag{209}$$

The energy density $\rho$ is given by

$$8\pi\rho = \frac{\partial_R F}{(\partial_R r) r^2}. \tag{210}$$

The LTB solution includes the three arbitrary functions: $f(R)$, $F(R)$, and $\tau_0(R)$.

Here, we consider the vacuum case $\rho = 0$. In this case, from Equation (210), we have

$$\partial_R F = 0, \quad F = 2M, \tag{211}$$

where $M$ is the Schwarzschild metric with the mass parameter $M$. Furthermore, we consider the case $f(R) = 0$. Here, we chose $\tau_0 = R$, i.e., $\partial_R \tau_0 = 1$. In this case, Equation (209) yields

$$(dR)_a = (d\tau)_a + \left(\frac{2M}{r}\right)^{-1/2} (dr)_a. \tag{212}$$

In this case, the metric (205) is given by

$$\begin{aligned} g_{ab} &= -(d\tau)_a (d\tau)_b + (\partial_R r)^2 (dR)_a (dR)_b + r^2 \gamma_{ab} \\ &= -f(dt)_a (dt)_b + f^{-1} (dr)_a (dr)_b + r^2 \gamma_{ab}, \quad f = 1 - \frac{2M}{r}, \end{aligned} \tag{213}$$

where $(dt)_a$ is defined by

$$(dt)_a := (d\tau)_a - f^{-1}(1-f)^{1/2} (dr)_a. \tag{214}$$

We also note that the degree of freedom of the choice of the coordinate $R$ is completely fixed through the choice $\tau_0 = R$, though there remains a degree of freedom of the choice of $R = R(\tilde{R})$ in the exact solution (205).

Now, we consider the perturbation of the Schwarzschild spacetime, which is derived by the exact LTB solution (205) so that

$$F(R) = 2[M + \epsilon m_1(R)] + O(\epsilon^2), \tag{215}$$
$$f(R) = 0 + \epsilon f_1(R) + O(\epsilon^2), \tag{216}$$
$$\tau_0(R) = R + \epsilon \tau_1(R) + O(\epsilon^2). \tag{217}$$

Through these perturbations (215)–(217), we consider the perturbative expansion of the function $r$, which is determined by Equation (206):

$$r(\tau, R) = r_s(\tau, R) + \epsilon r_1(\tau, R) + O(\epsilon^2). \tag{218}$$

Here, the function $r_s(\tau, R)$ is given by Equation (209), i.e.,

$$r_s(\tau, R) = r(\tau, R) = \left(\frac{9M}{2}\right)^{1/3} [R - \tau]^{2/3}. \tag{219}$$

In Equations (217) and (219), we chose the background value of the function $\tau_0(R)$ to be $R$.

Through this perturbative expansion, we evaluate $O(\epsilon^1)$ perturbation of Equation (206) through Equation (212) as

$$(1 - f)^{1/2}(\partial_\tau r_1) + \frac{m_1(R)}{r} - \frac{M}{r^2} r_1 + \frac{1}{2} f_1(R) = 0, \tag{220}$$

where we used Equation (212) and the replacement $r_s \to r$. The solution to Equation (220) is given by

$$\begin{aligned} r_1 &= \left(\frac{M}{6}\right)^{1/3} \frac{m_1(R)}{M} [R - \tau]^{2/3} - \frac{3}{20}\left(\frac{6}{M}\right)^{1/3} f_1(R)[R - \tau]^{+4/3} \\ &\quad + B(R)[R - \tau]^{-1/3}. \end{aligned} \tag{221}$$

From the comparison with Equation (219), $B(R)$ is the perturbation of the $\tau_1(R)$ as $\tau_0(R) = R + \tau_1(R)$ in the exact solution (207)–(209). Furthermore, the solution (221) can be also derived from the exact solution (207)–(209). From Equation (210), the perturbative dust energy density is given by

$$8\pi\rho = \frac{2\partial_R m_1(R)}{(\partial_R r) r^2}. \tag{222}$$

Through the perturbative solution (221), the metric (205) is given by

$$\begin{aligned} g_{ab} &= -(d\tau)_a(d\tau)_b + (\partial_R r)^2 (dR)_a (dR)_b + r^2 \gamma_{ab} \\ &\quad + \epsilon[(2(\partial_R r_1) - f_1(\partial_R r))(\partial_R r)(dR)_a(dR)_b + 2 r r_1 \gamma_{ab}] + O(\epsilon^2) \\ &=: g_{ab}^{(0)} + \epsilon\, {}_{\mathscr{X}}h_{ab} + O(\epsilon^2). \end{aligned} \tag{223}$$

As shown in Equation (213), the background metric $g_{ab}^{(0)}$ is given by the Schwarzschild metric in the static chart. On the other hand, the linear order perturbation ${}_{\mathscr{X}}h_{ab}$ (in the gauge $\mathscr{X}_\epsilon$) is given by

$${}_{\mathscr{X}}h_{ab} := (2(\partial_R r_1) - f_1(R)(\partial_R r))(\partial_R r)(dR)_a(dR)_b + 2 r r_1 \gamma_{ab}. \tag{224}$$

Here, we fixed the second-kind gauge so that

$$\mathscr{X}_\epsilon : (\tau, R, \theta, \phi) \in \mathscr{M}_{ph} \mapsto (\tau, R, \theta, \phi) \in \mathscr{M}. \tag{225}$$

Through this second-kind gauge choice and the choice of the background radial coordinate $R$ in Equation (213), the degree of freedom of the choice of the radial coordinate $R = R(\tilde{R})$ is also completely fixed, even in the linearized version of the exact solution (224), though there remains a degree of freedom of the choice of $R = R(\tilde{R})$ in the exact solution (205).

Of course, if we employ the different gauge choice $\mathscr{Y}_\epsilon$ from the above gauge-choice $\mathscr{X}_\epsilon$, we obtain the different expression of the metric perturbation ${}_{\mathscr{Y}}h_{ab}$. Actually, we may choose $\mathscr{Y}_\epsilon$ as the identification of

$$\mathscr{Y}_\epsilon : (\tau + \epsilon\xi^\tau(\tau, R), R + \epsilon\xi^R(\tau, R), \theta, \phi) \in \mathscr{M}_{ph} \mapsto (\tau, R, \theta, \phi) \in \mathscr{M}. \tag{226}$$

In this identification, the metric on $\mathscr{M}_{ph}$ pulled back to $\mathscr{M}$ is given by

$$\begin{aligned} {}_{\mathscr{Y}_\epsilon}g_{ab} &= {}_{\mathscr{X}_\epsilon}g_{ab} + \epsilon \pounds_\xi g^{(0)}_{ab} + O(\epsilon^2) \\ &= g^{(0)}_{ab} + \epsilon\left( {}_{\mathscr{X}}h_{ab} + \pounds_\xi g^{(0)}_{ab} \right) + O(\epsilon^2), \end{aligned} \tag{227}$$

where $\xi^a = \xi^\tau(\partial_\tau)^a + \xi^R(\partial_R)^a$ is the generator of second-kind gauge-transformation $\mathscr{X}_\epsilon \to \mathscr{Y}_\epsilon$.

#### 6.4.2. Expression of the Perturbative LTB Solution in Static Chart

Here, we consider the expression of the linear perturbation ${}_{\mathscr{X}}h_{ab}$ given by Equation (224). From Equations (212) and (214), with $F = 2M$, we obtain

$$(dR)_a = (dt)_a + f^{-1}(1-f)^{-1/2}(dr)_a, \quad f = 1 - \frac{2M}{r}, \tag{228}$$

$$(d\tau)_a = (dt)_a + f^{-1}(1-f)^{1/2}(dr)_a. \tag{229}$$

First, we consider the perturbation of the energy–momentum tensor of the matter field. In the case of the LTB solution, the matter field is characterized by the dust field whose energy–momentum tensor (204) is given by

$$T_{ab} = \rho u_a u_b, \quad u_a = -(d\tau)_a, \quad u^a = (\partial_\tau)^a. \tag{230}$$

In our case, the linearized Einstein equation gives Equation (222), i.e.,

$$8\pi\rho = \frac{2\partial_R m_1(R)}{(\partial_R r)r^2} = \frac{\partial_R m_1(R)}{4\pi r^2}(1-f)^{-1/2}. \tag{231}$$

On the other hand, substituting Equation (229) into Equation (230), we obtain

$$\begin{aligned} T_{ab} &= \rho(d\tau)_a(d\tau)_b \\ &= \rho\left((dt)_a + f^{-1}(1-f)^{1/2}(dr)_a\right)\left((dt)_b + f^{-1}(1-f)^{1/2}(dr)_b\right) \\ &= \rho(dt)_a(dt)_b + \rho\frac{(1-f)^{1/2}}{f}2(dt)_{(a}(dr)_{b)} + \rho\frac{1-f}{f^2}(dr)_a(dr)_b. \end{aligned} \tag{232}$$

Then, we obtain the components of the energy–momentum tensor for the static coordinate $(t, r)$ as

$$\tilde{T}_{tt} = \rho, \quad \tilde{T}_{tr} = \frac{(1-f)^{1/2}}{f}\rho, \quad \tilde{T}_{rr} = \frac{1-f}{f^2}\rho, \quad \tilde{T}_{(e0)} = 0. \tag{233}$$

From this component, we can confirm the continuity Equations (189) and (190).

As derived in ref. [16], the linearized Tolman–Bondi solution (224) with the Schwarzschild background is given by

$$\mathscr{X}h_{ab} = \frac{2m_1(R)}{r}\left[(dt)_a(dt)_b + \frac{1}{f^2}(dr)_a(dr)_b\right] + \frac{2-f}{f(1-f)^{1/2}}\frac{m_1(R)}{r}2(dt)_{(a}(dr)_{b)} + \pounds_{V_{(LTB)}}g_{ab}, \quad (234)$$

where $V_{(LTB)a}$ is given by

$$V_{(LTB)a} := \left[(1-f)^{1/2}r_1 + \frac{1}{2}f\int dt f_1(R)\right](dt)_a + \frac{r_1}{f}(dr)_a. \quad (235)$$

As shown in ref. [16], the $l = 0$ solution (203) with the components (233) of the linearized energy–momentum tensor realize the first line of Equation (234). In this sense, the solutions (203) of the $l = 0$-mode perturbations are justified by the LTB solutions.

However, we emphasize that the linearized LTB solution (234) does have the term $\pounds_{V_{LTB}}g_{ab}$. As noted by Equations (226) and (227), we can always eliminate the terms of the Lie derivative of the background metric as the second-kind gauge-degree of freedom. Therefore, according to our rule of the comparison, the term $\pounds_{V_{(LTB)}}g_{ab}$ should be regarded as the second-kind gauge-degree of freedom in the "conventional gauge-fixing approach" discussed in this paper. In this case, we have to regard that the perturbation $f_1(R)$ of the initial distribution $f(R)$ of the energy of dust field in Equation (206) is an "unphysical degree of freedom", in spite of the fact that the behavior of the LTB solution crucially depends on the signature of the function $f(R)$, as shown in Equations (207)–(209).

## 7. Summary and Discussion

In this paper, we have discussed comparison of our gauge-invariant formulation for $l = 0, 1$ perturbations on the Schwarzschild background spacetimes proposed in [12–16] and a "conventional complete gauge-fixing approach". It is well-known that we cannot construct gauge-invariant variables for $l = 0, 1$-mode perturbations through a same manner as for $l \geq 2$ mode perturbations if we use the decomposition formulae (26)–(28) with the spherical harmonic functions $Y_{lm}$ as the scalar harmonics $S_\delta$. In our gauge-invariant formulation for $l = 0, 1$ perturbations on the Schwarzschild background spacetime, we proposed the introduction of the singular harmonic function at once. Due to this, we can construct gauge-invariant variables for $l = 0, 1$-mode perturbations through a similar manner to the $l \geq 2$ modes of perturbations. After deriving the mode-by-mode perturbative Einstein equations in terms of the gauge-invariant variables, we impose the regularity on the introduced singular harmonics when we solve the derived Einstein equations. This approach enables us to obtain formal solutions to the $l = 0, 1$-mode linearized Einstein equations without the specification of the components of the linear perturbation of the energy–momentum tensor [12,14,15]. Our proposal also allow us to develop higher-order perturbations of the Schwarzschild spacetime [13]. Furthermore, we verified that our derived solutions realized the linearized version of the LTB solution and non-rotating C-metric [15]. In this sense, we conclude that our proposal is physically reasonable. On the other hand, it is often said that "gauge-invariant formulations in general-relativistic perturbations are equivalent to complete gauge-fixing approaches". For this reason, we examine this statement through the comparison of our gauge-invariant formulation and a "conventional complete gauge-fixing approach", in which we use the spherical harmonic functions $Y_{lm}$ as the scalar harmonics $S_\delta$ from the starting point.

After reviewing the concept of "gauges" in general relativistic perturbation theories, our proposed gauge-invariant formulation for the $l = 0, 1$-mode perturbations, and our derived $l = 0, 1$-mode solutions, we considered $l = 1$ odd-mode perturbations, $l = 1$ even-mode perturbations, and $l = 0$ even-mode perturbations separately. As a result, it is shown that we can derive similar solutions even through the "conventional complete gauge-fixing approach". However, it is important to note that the derived solutions are slightly different from those derived based on our gauge-invariant formulation, especially from a conceptual point of view.

In the case of $l = 1$ odd-mode perturbations, we derived the formal solution to the linearized Einstein equation through our proposed gauge-invariant formulation. This formal solution includes the term of the Lie derivative of the background metric. In our gauge-invariant formulation, we describe the solutions only through gauge-invariant variables. Therefore, we should regard the term of the Lie derivative of the background metric as gauge-invariant. On the other hand, in the conventional gauge-fixing approach where we use the spherical harmonics $Y_{lm}$ from the starting point, we cannot construct gauge-invariant variable for $l = 1$ odd-mode perturbations in the same way we can for $l \geq 2$-mode perturbations. For this reason, we have to treat gauge-dependent variables for perturbations. Nevertheless, the linearized Einstein equations and the continuity equations of the linearized energy–momentum tensor for $l = 1$ odd-mode perturbations in terms of these gauge-dependent variables have the completely same form as those is derived through our gauge-invariant formulation. Consequently, the formal solutions derived through our gauge-invariant formulation must be the formal solutions to these linearized Einstein equations, and the continuity equation of the linearized energy–momentum tensor in terms of gauge-dependent variables. As mentioned earlier, we treat gauge-dependent variables in the conventional gauge-fixing approach. Therefore, the above Lie derivative terms of the background metric may include the gauge-degree of freedom of the second kind, which should be regarded as an "unphysical degree of freedom". Examining the residual gauge-degree of freedom, we conclude that the above Lie derivative terms of the background metric include the second-kind gauge-degree of freedom. However, in our formal solution, there is a variable that should be obtained by solving the $l = 1$ Regge–Wheeler equation. We conclude that the solution to this $l = 1$ Regge–Wheeler equation is not the gauge-degree of freedom of the second kind, but a physical degree of freedom in the non-vacuum case. Furthermore, we have to impose appropriate boundary conditions to solve this $l = 1$ Regge–Wheeler equation. Since the $l = 1$ Regge–Wheeler equation is an inhomogeneous linear second-order partial differential equation, the boundary conditions for an inhomogeneous linear second-order partial differential equation are adjusted by the homogeneous solutions to this linear second-order partial differential equation. According to the check of the residual gauge-degree of freedom, we have to conclude that a part of homogeneous solutions to this equation is the gauge-degree of freedom of the second kind. We must exclude this part from our consideration because this gauge degree of freedom is "unphysical". This exclusion is the restriction of the boundary conditions of the linearized Einstein equations.

In the case of $l = 1$ even-mode perturbations, the situation is worse than the $l = 1$ odd-mode case. As in the $l = 1$ odd-mode case, we obtained the formal solution to the linearized Einstein equation through our proposed gauge-invariant formulation. This formal solution includes the term of the Lie derivative of the background metric, which is gauge-invariant within our proposed gauge-invariant formulation. On the other hand, in the conventional gauge-fixing approach where we use the spherical harmonics $Y_{lm}$ from the starting point, we cannot construct gauge-invariant variable for $l = 1$ even-mode perturbations in the same manner as we can for $l \geq 2$-mode perturbations. Consequently, we have to treat gauge-dependent variables for perturbations. As in the $l = 1$ odd-mode case, the linearized Einstein equations and the continuity equations of the linearized energy–momentum tensor for $l = 1$ even-mode perturbations, which are expressed by these gauge-dependent variables, retain the same form derived through our gauge-invariant formulation. Therefore, the same formal solutions derived through our gauge-invariant formulation should be the formal solutions to these linearized Einstein equations and the continuity equation of the linearized energy–momentum tensor in terms of gauge-dependent variables. As previously mentioned, we treat gauge-dependent variables in the conventional gauge-fixing approach as in the $l = 1$ odd-mode case. Therefore, the above Lie derivative terms of the background metric may include the second-kind gauge-degree of freedom, which should be regarded as an "unphysical degree of freedom". However, in our formal solution, there is a variable that must be obtained by solving the $l = 1$ Zerilli equation. We conclude that the solution to this $l = 1$ Zerilli equation is not the gauge-degree of freedom of the second kind, but the physical degree of freedom in the non-vacuum case. Furthermore, we have to impose appropriate boundary conditions to solve this $l = 1$ Zerilli equation in the non-vacuum case, since the $l = 1$ Zerilli equation is an inhomogeneous linear second-order partial differential equation. The homogeneous solutions to this linear second-order partial differential equation adjust the boundary conditions for an inhomogeneous linear second-order partial differential equation. According to the check of the residual gauge-degree of freedom, we conclude that all homogeneous solutions to this equation are the gauge degree of freedom of the second kind. We have to eliminate these homogeneous solutions from our consideration because these are regarded as "unphysical". This situation leads us to the dilemma of needing to impose boundary conditions using these "unphysical degrees of freedom" in order to obtain the "physical solution" through the $l = 1$ Zerilli equation.

In the case of $l = 0$ even-mode perturbations, we obtain the complete gauge-fixed solution. This solution does not include any terms involving the Lie derivative of the background metric, as these terms are regarded as the gauge degree of freedom of the second kind. In contrast, the solution derived from our proposed gauge-invariant formulation includes the terms of the Lie derivatives of the background metric, which are regarded as the first-kind gauge in our gauge-invariant formulation. In our formulation, these are "physical". This difference creates a problem when comparing our derived solution with the linearized LTB solution. In the linearized LTB solution, the initial energy distribution $f_1(R)$ of the dust field in the Newtonian sense is incorporated within the terms of the Lie derivative of the background metric. When we interpret this linearized exact solution by the solution using our gauge-invariant formulation, this initial energy distribution is treated as a physical degree of freedom. Conversely, if we use a conventional complete gauge-fixing approach to understand the same linearized exact solution, we must consider the initial energy distribution of the dust field as the gauge-degree of freedom of the second kind, thus labeling it as "unphysical". Since the behavior of the exact LTB solution significantly depends on this initial energy distribution of the dust field, it is unreasonable to classify this initial degree of freedom as unphysical.

In summary, we have to conclude that there is a case where the boundary conditions and initial conditions are restricted in the conventional complete gauge-fixing approach, where we use the decomposition of the metric perturbation by the spherical harmonics $Y_{lm}$ from the starting point. On the other hand, such a situation does not occur in our proposed gauge-invariant formulation. As a theory of physics, this point should be regarded as the incompleteness of the conventional complete gauge-fixing approach where we use the decomposition of the metric perturbation by the spherical harmonics $Y_{lm}$ from the starting point. This is the main result of this paper.

Let us discuss differences between our proposed gauge-invariant formulation and a conventional complete gauge-fixed approach that begins with the decomposition of the metric perturbation using the spherical harmonics $Y_{lm}$. The purpose of introducing singular harmonic functions in our proposed gauge-invariant formulation is to enhance the degree of freedom in order to clarify the distinction between the gauge-degree of freedom of the second kind and the physical degree of freedom. As emphasized in Section 3, if we set $S = Y_{lm}$, we find $\hat{D}_p S = \epsilon_{pq}\hat{D}^q S = 0$, $\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}^r\hat{D}_r\right)S = 2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S = 0$ for $l = 0$ modes. We also have $\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}^r\hat{D}_r\right)S = 2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S = 0$ for $l = 1$ modes. This is the crucial reason why we cannot construct gauge-invariant variables for $l = 0, 1$-mode metric perturbation in the same way as we do for $l \geq 2$ modes. Due to these vanishing vector- or tensor-harmonics, the corresponding mode coefficients do not appear, and we cannot construct gauge-invariant variables for $l = 0, 1$-modes in the same manner as for $l \geq 2$ modes. In our series of papers [12–16], we regarded that this is due to the lack of the degree of freedom. Consequently, in our proposed gauge-invariant formulation, we introduced singular harmonic functions $k_{\hat{\Delta}}$ and $k_{(\hat{\Delta}+2)m}$ for $l = 0$ and $l = 1$ modes, respectively. With the introduction of these singular harmonic functions, we now have $\hat{D}_p S \neq 0 \neq \epsilon_{pq}\hat{D}^q S$ and $\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}^r\hat{D}_r\right)S \neq 0 \neq 2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S$ for $l = 0$ modes, and $\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}^r\hat{D}_r\right)S \neq 0 \neq 2\epsilon_{r(p}\hat{D}_{q)}\hat{D}^r S$ for $l = 1$ modes. Thanks to these non-vanishing vector- or tensor-harmonics, the associated mode coefficients emerge, allowing us to construct gauge-invariant variables for $l = 0, 1$-modes in a similar manner to the case of $l \geq 2$ modes. This leads to the development of the gauge-invariant perturbation theory for all modes [12,14–16], and the development of the higher-order gauge-invariant perturbations [13].

As noted in [14], the decomposition using the spherical harmonics $Y_{lm}$ from the starting point corresponds to the imposition of the regular boundary conditions on $S^2$ for the metric perturbations from the starting point. In this sense, our introduction of the singular harmonic functions corresponds to a change in the boundary conditions on $S^2$. As shown in [14], this change of boundary conditions on $S^2$ allows us to clearly distinguish the gauge-degree of freedom of the second kind and the physical degree of freedom. Specifically, we can easily construct gauge-invariant variables for $l = 0, 1$-mode perturbations. This indicates that the imposition of the boundary conditions on $S^2$ and the construction of the gauge-invariant variables does not commute within the calculation process. This is the appearance of the non-locality of $l = 0, 1$-mode perturbations as pointed out in ref. [37]. Consequently, we have identified a conceptual difference between the $l = 0, 1$-mode solutions discussed in this paper. In the conventional complete gauge-fixing approach, which we use the decomposition by the harmonic function $Y_{lm}$ from the starting point, the degree of freedom of the metric perturbations is insufficient to distinguish between the gauge-degree of freedom of the second kind, i.e., "unphysical modes" and "physical modes". Despite this situation of the lack of degree of freedom, if one proceeds with the "complete gauge-fixing" as a means of elimination of unphysical modes, this approach results in constraints on the boundary conditions and initial conditions as demonstrated in this paper.

On the other hand, in our proposed gauge-invariant formulation, we encounter no conceptual difficulties, unlike issues related to the restriction of the boundary conditions and initial conditions pointed out in this paper. This is due to the sufficient degree of freedom of the metric perturbations. Incidentally, due to this sufficient degree of freedom of the metric perturbations through the introduction of the singular harmonic functions, our proposed gauge-invariant variables are equivalent to variables of the complete gauge-fixing within our proposed formulation in which the degree of freedom of the metric perturbations is sufficiently extended. Furthermore, we can develop higher-order perturbation theory without gauge ambiguities if we apply our proposal and there are wide applications of the higher-order gauge-invariant perturbations on the Schwarzschild background spacetime, as briefly discussed in ref. [13]. We leave these further developments of the application of our formulation to specific problems related to perturbations in the Schwarzschild background spacetime as future works.

**Funding:** This research received no external funding.

**Data Availability Statement:** No new data were created.

**Acknowledgments:** The author acknowledges Masashi Kimura for pointing out essential typos in the previous version of this manuscript.

**Conflicts of Interest:** The author declares no conflicts of interest.

## Appendix A. Linearized Einstein Tensor

In this Appendix A, we derive components of the linearized Einstein tensor. Although similar formulae are derived in Appendix C in [14] in the gauge-invariant form, in this paper, we have to derive the components of the Einstein tensor with the spherically background spacetime without any gauge-fixing. In this case, the formulation of the "gauge-ready formulation" proposed in ref. [52] in the context of cosmological perturbation theories is an appropriate formulation. Therefore, in this Appendix A, we derive the components of the linear-order Einstein tensor without any gauge-fixing following the philosophy of the "gauge-ready formulation" 'in ref. [52]. The derived formulae in the philosophy of "gauge ready formulation" is useful in the ingredient of this manuscript.

Here, we consider the metric perturbation as

$$_{\mathscr{X}}\bar{g}_{ab} = g_{ab} + \epsilon\,{}_{\mathscr{X}}h_{ab} + O(\epsilon^2). \tag{A1}$$

The connection $C^{c}{}_{ab}$ between the covariant derivative $\bar{\nabla}_a$ associated with the metric $\bar{g}_{ab}$ and the covariant derivative $\nabla_a$ associated with the metric $g_{ab}$ is given by

$$C^{c}{}_{ab} = \frac{1}{2}\bar{g}^{cd}(\nabla_a \bar{g}_{db} + \nabla_b \bar{g}_{da} - \nabla_d \bar{g}_{ab}), \tag{A2}$$

where $\bar{g}^{ab}$ is the inverse of $\bar{g}_{ab}$. Here, we expand the connection $C^{c}{}_{ab}$ with respect to $\epsilon$ as

$$C^{c}{}_{ab} = \epsilon\,{}^{(1)}C^{c}{}_{ab} + O(\epsilon^2). \tag{A3}$$

Then, we have

$${}^{(1)}C^{c}{}_{ab} = \frac{1}{2}g^{cd}(\nabla_a h_{db} + \nabla_b h_{da} - \nabla_d h_{ab}). \tag{A4}$$

The relation between the Riemann curvature $\bar{R}_{abc}{}^{d}$ associated with the metric $\bar{g}_{ab}$ and the curvature $R_{abc}{}^{d}$ associated with the background metric $g_{ab}$ is given by

$$\bar{R}_{abc}{}^{d} = R_{abc}{}^{d} - 2\nabla_{[a}C^{d}{}_{b]c} + 2C^{e}{}_{c[a}C^{d}{}_{b]e}, \tag{A5}$$

Then, we have

$$\bar{R}_{abc}{}^{d} = R_{abc}{}^{d} - 2\epsilon\nabla_{[a}{}^{(1)}C^{d}{}_{b]c} + O(\epsilon^2). \tag{A6}$$

The perturbative expansion of the Ricci tensor $\bar{R}_{ac}$ is given by

$$\bar{R}_{ac} = R_{ac} - 2\epsilon\nabla_{[a}{}^{(1)}C^{b}{}_{b]c} + O(\epsilon^2). \tag{A7}$$

The perturbative expansion of the curvature $\bar{R}_{a}{}^{c}$ is given by

$$\begin{aligned} \bar{R}_{a}{}^{d} &= \bar{g}^{cd}\bar{R}_{ac} \\ &= \bar{g}^{cd}\bar{R}_{ac} + \epsilon\left(-2g^{cd}\nabla_{[a}{}^{(1)}C^{b}{}_{b]c} - h^{cd}R_{ac}\right) + O(\epsilon^2). \end{aligned} \tag{A8}$$

The perturbation of the scalar curvature is given by

$$\begin{aligned} \bar{R} &= \bar{g}^{ac}\bar{R}_{ac} \\ &= R + \epsilon\left(-h^{ac}R_{ac} - 2g^{ac}\nabla_{[a}{}^{(1)}C^{b}{}_{b]c}\right) + O(\epsilon^2). \end{aligned} \tag{A9}$$

Then, the perturbative expansion of the Einstein tensor $\bar{G}_{a}{}^{d}$ is given by

$$\begin{aligned} \bar{G}_{a}{}^{d} &= \bar{R}_{a}{}^{d} - \frac{1}{2}\delta_{a}{}^{d}\bar{R} \\ &= G_{a}{}^{d} + \epsilon\left(-2g^{cd}\nabla_{[a}{}^{(1)}C^{b}{}_{b]c} + \delta_{a}{}^{d}g^{ec}\nabla_{[e}{}^{(1)}C^{b}{}_{b]c} - h^{cd}R_{ac} + \frac{1}{2}\delta_{a}{}^{d}h^{ec}R_{ec}\right) + O(\epsilon^2) \\ &=: G_{a}{}^{d} + \epsilon\,{}^{(1)}G_{a}{}^{d} + O(\epsilon^2). \end{aligned} \tag{A10}$$

Namely, the 1st-order perturbation of the Einstein tensor is given by

$$ {}^{(1)}G_a{}^d = -2g^{cd}\nabla_{[a}{}^{(1)}C^b{}_{b]c} + \delta_a{}^d g^{ec}\nabla_{[e}{}^{(1)}C^b{}_{b]c} - h^{cd}R_{ac} + \frac{1}{2}\delta_a{}^d h^{ec}R_{ec}. \tag{A11} $$

Substituting Equation (A4) into Equation (A11), we obtain

$$ \begin{aligned} {}^{(1)}G_a{}^d = & -\frac{1}{2}\nabla_b\nabla^b h_a{}^d + R_{abc}{}^d h^{bc} \\ & +\frac{1}{2}\left(g_{ac}\nabla^d - 2\delta_{[a}{}^d\nabla_{c]}\right)\nabla_b h^{cb} - \frac{1}{2}\left(\nabla_a\nabla^d - \delta_a{}^d\nabla_c\nabla^c\right)h_b{}^b \\ & -\frac{1}{2}R_{ac}h^{dc} + \frac{1}{2}R^{df}h_{af} + \frac{1}{2}\delta_a{}^d h^{ec}R_{ec}. \end{aligned} \tag{A12} $$

In the case of the vacuum background spacetime $R_{ab} = 0$, which we consider in this paper, we obtain

$$ \begin{aligned} {}^{(1)}G_a{}^d = & -\frac{1}{2}\nabla_b\nabla^b h_a{}^d + R_{abc}{}^d h^{bc} \\ & +\frac{1}{2}\left(g_{ac}\nabla^d - 2\delta_{[a}{}^d\nabla_{c]}\right)\nabla_b h^{cb} - \frac{1}{2}\left(\nabla_a\nabla^d - \delta_a{}^d\nabla_c\nabla^c\right)h_b{}^b. \end{aligned} \tag{A13} $$

To derive the component of ${}^{(1)}G_a{}^d$, we denote the components of the metric perturbation and the derivative operator as

$$ \begin{aligned} & \bar{h}_{AB} := h_{AB}, \quad \bar{h}_A{}^D := y^{DE}h_{AE}, \quad \bar{h}^{pD} := \gamma^{pq}y^{DE}h_{qE}, \\ & \bar{h}_p{}^q := \gamma^{qr}h_{pr}, \quad \bar{h}^{pq} := \gamma^{pr}\gamma^{qs}h_{rs}, \end{aligned} \tag{A14} $$

and

$$ \bar{D}^C = y^{CE}\bar{D}_E, \quad \hat{D}^p = \gamma^{pq}\hat{D}_q. \tag{A15} $$

Now, the components of ${}^{(1)}G_a{}^d$ in terms of the variable defined by Equation (A14) as follows:

$$ \begin{aligned} {}^{(1)}G_A{}^B = & -\frac{1}{2}\bar{D}_C\bar{D}^C\bar{h}_A{}^B - \frac{1}{2r^2}\hat{D}_p\hat{D}^p\bar{h}_A{}^B - \frac{2}{r^2}(\bar{D}^C r)(\bar{D}_C r)\bar{h}_A{}^B + \frac{2}{r^2}\bar{h}_A{}^B \\ & +\frac{1}{2}\bar{D}^B\bar{D}^C\bar{h}_{AC} + \frac{1}{2}\bar{D}_A\bar{D}_C\bar{h}^{BC} - \frac{1}{r}(\bar{D}^C r)\bar{D}_C\bar{h}_A{}^B - \frac{1}{2}\bar{D}_A\bar{D}^B\bar{h}_C{}^C \\ & +\frac{1}{r}(\bar{D}^C r)\bar{D}^B\bar{h}_{AC} + \frac{1}{r}(\bar{D}_C r)\bar{D}_A\bar{h}^{BC} \\ & +\frac{1}{2r^2}\bar{D}^B\hat{D}^p\bar{h}_{Ap} + \frac{1}{2r^2}\bar{D}_A\hat{D}_p\bar{h}^{Bp} \\ & -\frac{1}{2r^2}\bar{D}_A\bar{D}^B\bar{h}_r{}^r + \frac{1}{2r^3}(\bar{D}_A r)\bar{D}^B\bar{h}_r{}^r + \frac{1}{2r^3}(\bar{D}^B r)\bar{D}_A\bar{h}_r{}^r - \frac{1}{r^4}(\bar{D}_A r)(\bar{D}^B r)\bar{h}_r{}^r \\ & +y_A{}^B\left(-\frac{1}{2}\bar{D}_C\bar{D}_D\bar{h}^{CD} + \frac{1}{2}\bar{D}_C\bar{D}^C\bar{h}_D{}^D + \frac{1}{2r^2}\hat{D}_p\hat{D}^p\bar{h}_C{}^C - \frac{2}{r}(\bar{D}_D r)\bar{D}_C\bar{h}^{CD}\right. \\ & \qquad +\frac{1}{r}(\bar{D}^C r)\bar{D}_C\bar{h}_D{}^D - \frac{1}{r^2}(\bar{D}_C r)(\bar{D}_D r)\bar{h}^{CD} + \frac{3}{2r^2}(\bar{D}^D r)(\bar{D}_D r)\bar{h}_C{}^C \\ & \qquad -\frac{1}{r^2}\bar{D}_C\hat{D}_p\bar{h}^{pC} - \frac{1}{r^3}(\bar{D}_C r)\hat{D}_p\bar{h}^{Cp} \\ & \qquad +\frac{1}{2r^2}\bar{D}_C\bar{D}^C\bar{h}_r{}^r + \frac{1}{2r^4}\hat{D}_p\hat{D}^p\bar{h}_r{}^r - \frac{1}{2r^3}(\bar{D}^C r)\bar{D}_C\bar{h}_r{}^r - \frac{1}{2r^4}\hat{D}_p\hat{D}_s\bar{h}^{ps} \\ & \qquad \left. +\frac{1}{2r^4}(\bar{D}_C r)(\bar{D}^C r)\bar{h}_r{}^r - \frac{3}{2r^2}\bar{h}_C{}^C\right). \end{aligned} \tag{A16} $$

$$
\begin{aligned}
{}^{(1)}G_{A}{}^{q} \quad = \quad & \frac{1}{2r^2}\hat{D}^q\bar{D}^C\bar{h}_{AC} - \frac{1}{2r^2}\bar{D}_A\hat{D}^q\bar{h}_C{}^C + \frac{1}{2r^3}(\bar{D}_A r)\hat{D}^q\bar{h}_C{}^C \\
& -\frac{1}{2r^2}\bar{D}_C\bar{D}^C\bar{h}_A{}^q - \frac{1}{2r^4}\hat{D}_r\hat{D}^r\bar{h}_A{}^q + \frac{1}{2r^4}\bar{h}_A{}^q + \frac{1}{2r^4}\hat{D}^q\hat{D}^p\bar{h}_{Ap} \\
& +\frac{1}{2r^2}\bar{D}_A\bar{D}_D\bar{h}^{qD} - \frac{1}{r^3}(\bar{D}_A r)\bar{D}_D\bar{h}^{qD} - \frac{1}{r^4}(\bar{D}_A r)(\bar{D}_D r)\bar{h}^{qD} + \frac{1}{r^3}(\bar{D}_D r)\bar{D}_A\bar{h}^{qD} \\
& +\frac{1}{2r^4}\bar{D}_A\hat{D}_s\bar{h}^{qs} - \frac{1}{r^5}(\bar{D}_A r)\hat{D}_s\bar{h}^{qs} - \frac{1}{2r^4}\bar{D}_A\hat{D}^q\bar{h}_r{}^r + \frac{1}{r^5}(\bar{D}_A r)\hat{D}^q\bar{h}_r{}^r,
\end{aligned} \tag{A17}
$$

$$
\begin{aligned}
{}^{(1)}G_{p}{}^{q} \quad = \quad & -\frac{1}{2r^2}\bar{D}_C\bar{D}^C\bar{h}_p{}^q - \frac{1}{2r^4}\hat{D}_s\hat{D}^s\bar{h}_p{}^q + \frac{1}{r^3}(\bar{D}_C r)\bar{D}^C\bar{h}_p{}^q - \frac{2}{r^4}(\bar{D}_C r)(\bar{D}^C r)\bar{h}_p{}^q + \frac{2}{r^4}\bar{h}_p{}^q \\
& -\frac{1}{2r^2}\hat{D}_p\hat{D}^q\bar{h}_C{}^C + \frac{1}{2r^2}\hat{D}_p\bar{D}_C\bar{h}^{qC} + \frac{1}{2r^2}\hat{D}^q\bar{D}^C\bar{h}_{pC} \\
& +\frac{1}{2r^4}\hat{D}_p\hat{D}_r\bar{h}^{qr} + \frac{1}{2r^4}\hat{D}^q\hat{D}^r\bar{h}_{pr} - \frac{1}{2r^4}\hat{D}_p\hat{D}^q\bar{h}_r{}^r \\
& +\gamma_p{}^q\Bigg(-\frac{1}{2}\bar{D}_C\bar{D}_D\bar{h}^{CD} - \frac{1}{r}(\bar{D}_C r)\bar{D}_D\bar{h}^{CD} + \frac{1}{2}\bar{D}_C\bar{D}^C\bar{h}_D{}^D + \frac{1}{2r}(\bar{D}^C r)\bar{D}_C\bar{h}_D{}^D \\
& \qquad +\frac{1}{2r^2}\hat{D}_s\hat{D}^s\bar{h}_C{}^C - \frac{1}{r^2}\bar{D}_C\hat{D}_r\bar{h}^{Cr} \\
& \qquad -\frac{1}{2r^4}\hat{D}_r\hat{D}_s\bar{h}^{sr} - \frac{3}{2r^4}\bar{h}_r{}^r + \frac{1}{2r^2}\bar{D}_C\bar{D}^C\bar{h}_r{}^r - \frac{1}{r^3}(\bar{D}^C r)\bar{D}_C\bar{h}_r{}^r \\
& \qquad +\frac{2}{r^4}(\bar{D}_C r)(\bar{D}^C r)\bar{h}_r{}^r + \frac{1}{2r^4}\hat{D}_s\hat{D}^s\bar{h}_r{}^r\Bigg).
\end{aligned} \tag{A18}
$$

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