Journal of Physics: Conference Series **68** (2007) 012034 doi:10.1088/1742-6596/68/1/012034

The Generalized Dark Radiation in Brane Cosmology

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Abstract. The effective Friedmann equation describing the evolution of a brane Universe in the cosmology of the Randall-Sundrum model includes a dark (or mirage or Weyl) radiation term. The brane evolution can be interpreted as the motion of the brane in an AdS-Schwarzschild bulk geometry. The energy density of the dark radiation is proportional to the black hole mass. I review how this result is generalized for an AdS bulk space with an arbitrary matter component. The mirage term retains its form, but the black hole mass is replaced by the covariantly defined integrated mass of the bulk matter. As this mass depends explicitly on the scale factor on the brane, the mirage term does not scale as pure radiation. For low energy densities the brane cosmological evolution is that of a four-dimensional Universe with two matter components: the matter localized on the brane and the mirage matter. There is conservation of energy between the two components. This behaviour indicates a duality between the bulk theory and a four-dimensional theory on the brane. I also discuss the cosmological evolution on a brane with induced gravity within a bulk with arbitrary matter content. The Friedmann equation now has two branches, distinguished by the two possible values of the parameter $\epsilon = \pm 1$. The branch with $\epsilon = 1$ is characterized by an effective cosmological constant and accelerated expansion for low energy densities. Another remarkable feature is that the contribution from the generalized dark radiation appears with a negative sign. As a result, the presence of the bulk corresponds to an effective negative energy density on the brane, without violation of the weak energy condition. In the branch with $\epsilon = 1$ the transition from a period of domination of the matter energy density by non-relativistic brane matter to domination by the generalized dark radiation can lead to a crossing of the phantom divide $w = -1$.

1. The generalized dark radiation

In the context of the Randall-Sundrum model [1], the Universe is identified with a four-dimensional hypersurface (a 3-brane) in a five-dimensional bulk with negative cosmological constant (AdS space). The action is

$$S = \int d^5x\sqrt{-g}\left(\Lambda + M^3 R + \mathcal{L}_B\right) + \int d^4x\sqrt{-\hat{g}}\ \left(-V + \mathcal{L}_b\right), \tag{1}$$

where R is the curvature scalar of the five-dimensional bulk metric g_{AB}, $-\Lambda$ the bulk cosmological constant ($\Lambda > 0$), V the brane tension, and $\hat{g}_{\alpha\beta}$ the induced metric on the brane. (We neglect higher curvature invariants in the bulk and induced gravity terms on the brane.) The Lagrangian density $\mathcal{L}_B$ describes the matter content (particles or fields) of the bulk, while the density $\mathcal{L}_b$ describes matter localized on the brane.

The geometry is non-trivial (warped) along the fourth spatial dimension, so that an effective localization of low-energy gravity takes place near the brane. (No such localization takes place for the bulk matter.) For low matter densities on the brane and a pure AdS bulk (no bulk matter), the cosmological evolution as seen by a brane observer reduces to the standard Friedmann-Robertson-Walker cosmology [2]. We parametrize the metric as

$$ds^2 = -m^2(\tau,\eta)d\tau^2 + a^2(\tau,\eta)d\Omega_k^2 + d\eta^2, \tag{2}$$

with $m(\tau,\eta=0)=1$. The brane is located at $\eta=0$, while we identify the half-space $\eta>0$ with the half-space $\eta<0$. The effective Friedmann equation at the location of the brane is

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{1}{6M_{\rm Pl}^2}\left[\tilde{\rho}\left(1+\frac{\tilde{\rho}}{2V}\right)+\rho_d\right] - \frac{k_c}{R^2} + \lambda, \tag{3}$$

with $R(\tau) = a(\tau,\eta=0)$ and $M_{\rm Pl}^2 = 12M^6/V$. The energy density $\tilde{\rho}$ corresponds to matter localized on the brane, and arises through the parametrization of the corresponding energy-momentum tensor as $\tilde{T}^A_B = \delta(\eta){\rm diag}(-\tilde{\rho},\tilde{p},\tilde{p},\tilde{p},0)$. The contribution $\sim\tilde{\rho}^2$ is typical of brane cosmologies and becomes negligible for $\tilde{\rho}\ll V$. The curvature term ($k_c=0,\pm1$) depends on the geometry of the maximally symmetric space with constant τ and η. The effective cosmological constant $\lambda = (V^2/12M^3-\Lambda)/12M^3$ can be set to zero through an appropriate fine-tuning of V and Λ. The conservation equation for the brane energy density is

$$\dot{\tilde{\rho}} + 3H(\tilde{\rho}+\tilde{p}) = 0. \tag{4}$$

The energy density $\rho_d = (12M^3/\pi^2V)(\mathcal{M}/R^4)$ depends on an arbitrary constant $\mathcal{M}$ and scales as conserved pure radiation. It is characterized as dark, or mirage, or Weyl radiation [2, 3, 4]. Its true nature becomes apparent in a Schwarzschild system of coordinates in which the metric in general can be written as [5]

$$ds^2 = -n^2(t,r)dt^2 + r^2d\Omega_k^2 + b^2(t,r)dr^2. \tag{5}$$

In this system the brane evolution described above corresponds to brane motion within a bulk with AdS-Schwarzschild geometry: $n^2 = 1/b^2 = \Lambda r^2/(12M^3)+k_c-\mathcal{M}/(6\pi^2M^3r^2)$. The constant $\mathcal{M}$ is identified with the mass of a black hole located at $r=0$. It can also be related to the value of the bulk Weyl tensor at the location of the brane $r=R(\tau)$ (see below).

This picture can be generalized for an arbitrary bulk energy-momentum tensor T^A_B using a covariant formalism [6]. The details of the calculation are given in ref. [7, 8]. The brane can be identified with a 4D hypersurface, whose spatial part (denoted by $\mathcal{D}$) is invariant under a six-dimensional group of isometries. The spatial curvature is determined by the value of k_c. The assumption of maximal symmetry of the spatial part, which is essential for our results, implies the existence of a preferred spacelike direction e^A, that represents the local axis of symmetry. For example, in the Gauss normal coordinate system (2) the preferred axis of symmetry is $\sim\partial_\eta$, while in the Schwarzschild system (5) it is $\sim\partial_r$. We consider an observer with a 5-velocity $\tilde{u}^A$ comoving with the brane. The spacelike unit vector field n^A is taken perpendicular to the brane trajectory ($n_A\tilde{u}^A=0$). We also consider a bulk observer with a 5-velocity u^A perpendicular to the preferred direction ($e_Au^A=0$).

Each spatial slice $\mathcal{D}$ is covariantly characterized by an average length scale function ℓ. The derivative of ℓ along the preferred direction (denoted by a prime) can be determined from the relation $D_Ae^A\equiv 3\ell'/\ell$, where D_A is the fully projected, perpendicular to u^A, covariant derivative [7, 8]. In the Schwarzchild system (5) we have $\ell=r$, while in the Gauss-normal system (2) $\ell=a(\tau,\eta)$. At the location of the brane, $\ell=R(\tau)$. The time evolution on the brane can be described in terms of the Hubble parameter, defined as $3H=\tilde{u}^\alpha{}_{;\alpha}$. It can be shown [7, 8] that

$$H^2 = \left(\frac{\dot{\ell}}{\ell}\right)^2 = \frac{1}{6M_{\rm Pl}^2}\tilde{\rho}\left(1+\frac{\tilde{\rho}}{2V}\right) + \frac{1}{6\pi^2M^3}\frac{\mathcal{M}(\ell,\tau)}{\ell^4} - \frac{k_c}{\ell^2} + \lambda, \tag{6}$$

where the dot indicates a derivative with respect to the proper time τ on the brane: $\dot{\ell} = \ell_{;\alpha}\tilde{u}^{\alpha}$.

The quantity $\mathcal{M}(\ell, \tau)$ is defined through the relation

$$\mathcal{M} = \int_{\ell_0}^{\ell} 2\pi^2 \rho \ell^3 d\ell + \mathcal{M}_0, \tag{7}$$

where $\rho \equiv T_{AB}u^A u^B$ is the bulk energy density as measured by the bulk observer. We can interpret $\mathcal{M}(\ell, \tau)$ as the generalized comoving mass of the bulk fluid within a spherical shell with radii ℓ_0 and ℓ. This interpretation is strictly correct only for $k = 1$. However, we shall refer to $\mathcal{M}$ as the integrated mass for all geometries of the spatial slices $\mathcal{D}$. The quantity $\bar{p} \equiv T_{AB}n^A n^B$ appearing in eq. (11) is the bulk pressure in the direction perpendicular to the brane, as measured by a brane observer. The dependence on the total integrated mass, irrespectively of the specific radial dependence of the energy density, is reminiscent of the implications of Birkhoff's theorem for the gravitational field generated by a matter distribution. Both results are consequences of the assumed rotational symmetry of the geometry, which is inherited by the matter distribution. If we employ the Schwarzchild system (5) we can set $\ell_0 = r_0 = 0$. Then, the integration constant $\mathcal{M}_0$ in eq. (7) can be interpreted as the mass of a black hole at $r = 0$.

A more intuitive physical interpretation of the contribution $\sim \mathcal{M}$ in eq. (6) can be given for a perfect bulk fluid. In this case $\mathcal{M}/(6\pi^2 M^3 \ell^4) = \rho/(12M^3) - \mathcal{E}/3$, where $\mathcal{E} \equiv C_{ACBD}\tilde{u}^A n^C \tilde{u}^B n^D$ is a scalar formed out of the bulk Weyl tensor [7, 8]. Both ρ and $\mathcal{E}$ are evaluated at the location of the brane. It is clear that the bulk affects the brane evolution through its energy density in the vicinity of the brane. The contribution ρ is related to the bulk matter, while $\mathcal{E}$ can be loosely interpreted as accounting for the effect of the local gravitational field. For $\rho = 0$ (AdS-Schwarzschild bulk) the whole effect arises through the gravitational field. This justifies the use of the term Weyl radiation for the contribution in the effective Friedmann equation.

We return to the general case of an arbitrary bulk energy-momentum tensor (general fluid). We can generalize eqs. (3), (4) by defining the effective energy density and pressure of the generalized dark radiation as

$$\rho_D = \frac{12M^3}{\pi^2 V}\frac{\mathcal{M}}{l^4}, \qquad p_D = \frac{\rho_D}{3} + \frac{8M^3}{V}\bar{p}. \tag{8}$$

We obtain [7, 8]

$$H^2 = \left(\frac{\dot{\ell}}{\ell}\right)^2 = \frac{1}{6M_{\rm Pl}^2}\left[\tilde{\rho}\left(1 + \frac{\tilde{\rho}}{2V}\right) + \rho_D\right] - \frac{k_c}{\ell^2} + \lambda, \tag{9}$$

$$\left[\dot{\tilde{\rho}} + 3H(\tilde{\rho} + \tilde{p})\right]\left(1 + \frac{\tilde{\rho}}{V}\right) = -\left[\dot{\rho}_D + 3H(\rho_D + p_D)\right]. \tag{10}$$

(We can set $\ell = R(\tau)$ by employing the coordinate system (2).) For $\tilde{\rho} \ll V$ the system behaves in a remarkably simple fashion. The brane cosmological evolution is that of a four-dimensional Universe with two matter components: the matter localized on the brane and the mirage matter. There is conservation of energy between the two components.

The effective equation of state of the mirage component is $p_D = p_D(\rho_D)$. This has a simple form for $\rho = 0$, in which case we recover the pure dark radiation with $p_D = \rho_D/3$. In more general cases, the determination of the equation of state requires the full knowledge of the brane and bulk dynamics. Similarly, the rate in which brane matter is transformed into mirage matter cannot be determined by our considerations. It requires explicit input about the interaction between the brane and the bulk matter. It is also noteworthy that in the general case the equation of state $p_D = p_D(\rho_D)$ depends explicitly on the scale ℓ. This is a consequence of the explicit breaking of scale invariance by the bulk distribution, which is reflected in the theory underlying the mirage component.

The evolution of H is determined by the Raychaudhuri equation

$$\dot{H} = -H^2 - \frac{1}{12M_{\rm Pl}^2}\left[(\tilde{\rho}+3\tilde{p}) + \frac{2\tilde{\rho}^2+3\tilde{\rho}\tilde{p}}{V}\right] - \frac{1}{6\pi^2 M^3}\frac{\mathcal{M}(\ell,\tau)}{\ell^4} - \frac{1}{M^3}\bar{p} + \lambda, \tag{11}$$

where $\dot{H} = H_{;\alpha}\tilde{u}^\alpha$. An interesting question concerns the possibility of having accelerated expansion on the brane as a result of the brane-bulk interaction. The Raychaudhuri equation (11) provides important intuition on this problem in a general framework. The acceleration parameter is proportional to $\dot{H} + H^2$. For this to be positive one or more of the following conditions must be satisfied: a) The effective cosmological constant λ is positive. b) The brane matter satisfies $\tilde{\rho} < V$ and $\tilde{p} < -\tilde{\rho}/3$. c) The brane matter satisfies $\tilde{\rho} > V$ and $\tilde{p} < -2\tilde{\rho}/3$. d) The comoving mass $\mathcal{M}$ is negative. e) The pressure $\bar{p}_{\|}$ of the bulk fluid perpendicularly to the brane, as measured by the brane observer, is negative. In general negative pressures are associated with field configurations. The possibility of a negative comoving mass seems problematic at first sight, as usually it implies the existence of naked singularities or instabilities. A counter example is a brane with negative tension in the two-brane model of [1].

2. Specific models

Let us consider now some particular examples that confirm the general picture we presented above. Our general strategy is to find an explicit solution of the Einstein equations in the bulk employing the Schwarzschild coordinates (5), and then introduce the brane as the bounday of the bulk space. This can be achieved only if there is an appropriate rate of energy exchange between the brane and the bulk. As a result, the conservation of the total energy in the brane and mirage components, as given by eq. (10), is guaranteed. In all the examples, the effective Friedmann equation includes the generalized dark radiation term, that has the form of eq. (8) with $\ell = R(\tau)$. The function $\mathcal{M}(R)$ can be determined explicitly as the integrated mass $\mathcal{M}(r)$ in the Schwarzschild frame. (An additional explicit dependence on time is possible for non-static bulk geometries, as we shall see below.)

• In the simplest example of ref. [9], the bulk is static in the Schwarzschild frame and the bulk matter distribution is very similar to that in the interior of a stellar object. The energy density has a profile $\rho(r)$ that can be determined through the solution of the Einstein equations. The integrated mass $\mathcal{M}(r)$ of this AdS-star has a non-trivial dependence on r. As a result, the generalized radiation term $\sim \mathcal{M}(R)/R^4$ does not scale as radiation.

• In ref. [10] an example of a non-static bulk is given. The bulk matter is pressureless, but has some initial outgoing velocity in the radial direction. The bulk metric is assumed to have the AdS-Tolman-Bondi form. In Schwarzschild coordinates it is given by

$$ds^2 = -dt^2 + b^2(t,r)dr^2 + S^2(t,r)d\Omega_k^2, \tag{12}$$

with $b(r,t)$ given by

$$b^2(t,r) = \frac{S_{,r}^2(t,r)}{k_c + f(r)}, \tag{13}$$

where the subscript denotes differentiation with respect to r, and $f(r)$ is an arbitrary function. The energy-momentum tensor of the bulk matter has the form $T^A_B = \mathrm{diag}\,(-\rho(t,r),\,0,\,0,\,0,\,0)$. The bulk fluid consists of successive shells marked by r, whose local density ρ is time-dependent. The function $S(t,r)$ describes the location of the shell marked by r at the time t. Notice that $S(r,t)$ is the actual radial coordinate, while r simply marks the successive shells. Thus, eq. (12) can be put in the form (5), if we express r as $r = r(S,t)$ and redefine t in order to eliminate the term $dtdS$ in the metric. It is more convenient, however, to match the metric (12) with (2) directly, through a transformation $t = t(\tau,\eta)$, $r = r(\tau,\eta)$ [10].

The Einstein equations reduce to

$$S_{,t}^2(t,r) = \frac{1}{6\pi^2 M^3}\frac{\mathcal{M}(r)}{S^2} - \frac{1}{12M^3}\Lambda S^2 + f(r) \tag{14}$$

$$\mathcal{M}_{,r}(r) = 2\pi^2 S^3 \rho\, S_{,r}. \tag{15}$$

The integrated mass $\mathcal{M}(r)$ of the bulk fluid incorporates the contributions of all shells between 0 and r. It can be obtained through the integration of eq. (15), in agreement with eq. (7) for $\ell = S$. Because of energy conservation it is independent of t, while ρ and S depend on both t and r. The function $f(r)$ determines the initial radial velocity of the bulk fluid, as can be seen from eq. (14).

It can be shown [10] that the effective Friedmann equation for the brane evolution has the form of eq. (9), with $\mathcal{M} = \mathcal{M}(r(\tau, \eta = 0))$ and $\ell = a(\tau, \eta = 0) = S(t(\tau, \eta = 0), r(\tau, \eta = 0))$. The mirage term does not scale as pure radiation, but has a complicated behaviour. The bulk pressure $\bar{p}$ perpendicularly to the brane, as measured by a brane observer, obeys $\bar{p} > 0$ [10], even though the pressure is zero for a bulk observer comoving with the fluid. This means that the equation of state of the mirage component has $p_D > \rho_D/3$.

• Another interesting case is discussed in refs. [11, 12]. The bulk metric is assumed to have the generalized AdS-Vaidya form

$$ds^2 = -n^2(u,r)\, du^2 + 2\epsilon\, du\, dr + r^2 d\Omega_k^2, \tag{16}$$

where

$$n^2(u,r) = \frac{1}{12M^3}\Lambda r^2 + k_c - \frac{1}{6\pi^2 M^3}\frac{\mathcal{M}(u,r)}{r^2}. \tag{17}$$

The parameter ϵ takes the values $\epsilon = \pm 1$. The energy-momentum tensor of the bulk matter that satisfies the Einstein equations is

$$T^u_{\ u} = T^r_{\ r} = -\frac{1}{2\pi^2}\frac{\mathcal{M}_{,r}}{r^3} \tag{18}$$

$$T^1_{\ 1} = T^2_{\ 2} = T^3_{\ 3} = -\frac{1}{6\pi^2}\frac{\mathcal{M}_{,rr}}{r^2} \tag{19}$$

$$T^u_{\ r} = \frac{1}{2\pi^2}\frac{\mathcal{M}_{,u}}{r^3}, \tag{20}$$

where the subscripts indicate derivatives with respect to r and u. The various energy conditions are satisfied if $\epsilon\mathcal{M}_{,u} \geq 0$, $\mathcal{M}_{,r} \geq 0$, $\mathcal{M}_{,rr} \leq 0$, $\mathcal{M}_{,r} \geq -r\mathcal{M}_{,rr}/3$.

The preferred axis of symmetry is $\sim \partial_r$, so that the average length scale function ℓ is $\ell = r$. It is then clear from eq. (18) that $\mathcal{M}$ is the integrated mass given by eq. (7). Another way to reach the same conclusion is to write the metric in Schwarzschild coordinates

$$ds^2 = -n^2(t,r)\, dt^2 + n^{-2}(t,r)\, dr^2 + r^2 d\Omega_k^2, \tag{21}$$

where

$$n^2(t,r) = n^2(u(t,r),r) = \frac{1}{12M^3}\Lambda r^2 + k_c - \frac{1}{6\pi^2 M^3}\frac{\mathcal{M}\left(u(t,r),r\right)}{r^2} \tag{22}$$

and $\partial u/\partial t = 1$, $\partial u/\partial r = \epsilon/n^2$. The non-zero components of the energy-momentum tensor that satisfies the Einstein equations for this metric are given by the same expressions as in eqs. (18)-(20), and the integrated mass is given by eq. (7).

It is not surprising, therefore, that the brane evolution is described by eq. (9) with $\mathcal{M}(\tau,\ell) = \mathcal{M}(u(\tau,\eta = 0),\ell)$. The effective pressure p_D, defined in eqs. (8), can be calculated to be

$$p_D = \frac{4M^3}{\pi^2 V}\left(\frac{\mathcal{M}}{\ell^4} - \frac{\partial\mathcal{M}/\partial\ell}{\ell^3}\right) - \frac{8M^3\epsilon}{V}\left(\dot{\tilde{\rho}} + 3H(\tilde{\rho} + \tilde{p})\right). \tag{23}$$

Journal of Physics: Conference Series **68** (2007) 012034 doi:10.1088/1742-6596/68/1/012034

For $\tilde{\rho} \ll V$ the second term in the r.h.s. can be neglected. The function $\mathcal{M}(u,r)$ is arbitrary. If it is assumed to have the form $\mathcal{M} = \mathcal{M}(u)$, the bulk energy-momentum tensor corresponds to a radiation field. The resulting cosmological solution describes a brane Universe that exchanges (emits or absorbs) relativistic matter with the bulk. For example, the form of $\mathcal{M}(u)$ can be matched to the rate of production of Kaluza-Klein gravitons during the collisions in a thermal bath of brane particles [4, 12]. In this case, the effective pressure of the mirage component becomes $p_D = \rho_D/3$. The system of equations (9), (10) for $\tilde{\rho} \ll V$ describes the evolution of a four-dimensional Universe with energy exchange between the brane matter component $\tilde{\rho}$ and the dark radiation component ρ_D.

Other choices for $\mathcal{M}(u,r)$ that satisfy the energy conditions are possible as well. If one assumes $\mathcal{M}(u,r) = \mathcal{M}(u)r$, the mirage component has $p_D = 0$ for $\tilde{\rho} \ll V$. This is the equation of state of non-relativistic matter. As a result, the mirage component can be characterized as mirage cold dark matter. However, the physical interpretation of a bulk geometry with $\mathcal{M}(u,r) = \mathcal{M}(u)r$ remains an open question.

• Our final example involves a bulk scalar field in a global monopole (hedgehog) configuration [10]. The field has four components ϕ^α, $\alpha = 1, 2, 3, 4$, and its action is invariant under a global $O(4)$ symmetry. The field configuration that corresponds to a global monopole is $\phi^\alpha = \phi(r)x^\alpha/r$. The asymptotic value of $\phi(r)$ for $r \to \infty$ is ϕ_0. The metric can be written in Schwarzschild coordinates, as in eq. (5), with $n = n(r)$, $b = b(r)$ and $k = 1$. For large r, the leading contribution of the monopole configuration to the energy-momentum tensor comes from the angular part of the kinetic term. The integrated mass can be calculated to be $\mathcal{M} = 3\pi^2\phi_0^2\ell^2/2$ for large ℓ. (The global monopole has a diverging mass in the limit $\ell \to \infty$.) As a result the effective energy density is $\rho_D = 18M^3\phi_0^2/(V\ell^2)$. In this case the mirage component scales $\sim \ell^{-2}$, similarly to the curvature term. This can be verified by calculating explicitly the effective pressure p_D, which turns out to be $p_D = -\rho_D/3$ for large ℓ.

3. The bulk-brane duality

We saw in the previous section that, for low energy densities, the cosmological evolution on the brane is typical of a four-dimensional Universe. In addition to the matter localized on the brane, a mirage matter component appears, which we characterized as generalized dark radiation. There is conservation of energy between the two components. The nature of the mirage component depends on the bulk matter. In particular, despite its characterization as generalized dark radiation, the mirage component can have a very general equation of state. If we define $p_D = w_D\rho_D$, there are configurations with

• $w_D > 1/3$: non-relativistic bulk matter in an AdS-Tolman-Bondi geometry;
• $w_D = 1/3$: AdS-Schwarzschild bulk geometry; AdS-Vaidya bulk with energy exchage between the brane and a radiation field in the bulk.
• $w_D = 0$: generalized AdS-Vaidya bulk;
• $w_D = -1/3$: global monopole in an AdS bulk;
• $w_D = -1$: constant field with a non-zero potential in the bulk (effective cosmological constant).

In the case of an AdS-Schwarzschild bulk the mirage component is pure radiation. Its appearance can be understood through the AdS/CFT correspondence [13, 14]. There is a duality that relates a supergravity theory, arising in the low energy limit of an appropriate compactification of a superstring theory, with a conformal field theory. In particular, the supergravity is defined on the product of a compact manifold and a five-dimensional manifold X_5 with a four-dimensional boundary M_4. The manifold X_5 asymptotically (near the boundary M_4) becomes an AdS$_5$ space. The conformal field theory is defined on M_4. It was suggested in ref. [15] that the cosmology in the Randall-Sundrum model can be understood through the AdS/CFT correspondence. The AdS bulk degrees of freedom correspond to a conformal field theory on the boundary of the bulk space. The dark radiation term is nothing but the energy density of the conformal degrees of freedom.

If there are non-zero bulk fields other than the gravitational field, the dual theory is not expected to be conformal. This is obvious if the bulk field profile introduces new energy scales, other than the fundamental Planck scale M and the cosmological constant Λ.As a result, it is not surprising that the effective equation of state of the generalized dark radiation can deviate significantly from that of pure radiation. The remarkable property is that the brane evolution at low energies can be described in four-dimensional terms for any bulk content. This implies that the dual description of a bulk gravity theory is quite general at low energies.

The breaking of conformal invariance is reflected in the trace of the energy-momentum tensor of the generalized dark radiation. From eq. (8) it is apparent that this is proportional to the pressure of the bulk fluid perpendicularly to the brane as measured by the brane observer. In cases in which the duality between a bulk theory with broken conformal invariance and a boundary theory is known, the trace of the energy-momentum tensor can also be expressed through the expectation value of an operator of the boundary conformal theory. The cosmological evolution on the brane at low energies can be derived either through an explicit solution of the five-dimensional Einstein equations or through the study of the dual theory in a cosmological context [16].

4. Induced gravity

A brane theory with an induced gravity term in the action, characterized by a length scale r_c, has several novel features. In the simplest example, the DGP model [17], the brane tension V and bulk cosmological constant Λ are neglected. Specific examples with an induced gravity term can be obtained in string theory, and are common in holographic descriptions [16, 18, 19, 20]. Despite possible problems at large distances [21], the DGP model predicts an interesting cosmological evolution [22]. The Friedmann equation has two branches, distinguished by the two possible values of the parameter $\epsilon = \pm 1$. The branch with $\epsilon = 1$ is characterized by an effective cosmological constant and accelerated expansion for low energy density of the brane matter [23]. The generalization of the model for non-zero V and Λ displays the same two branches [18]. For $\epsilon = -1$, the limit $r_c \to 0$ reproduces the cosmological evolution of the Randall-Sundrum model, in which the effective cosmological constant is zero. The branch with $\epsilon = 1$ displays late-time accelerated expansion, in complete analogy to the DGP model.

We are interested in describing the effects of the bulk matter on the brane cosmological evolution in the presence of an induced gravity term. We assume that the action takes the form

$$S = \int d^5x\sqrt{-g}\left(\Lambda + M^3 R + \mathcal{L}_B\right) + \int d^4x\sqrt{-\hat{g}}\,\left(-V + M^3 r_c R_4 + \mathcal{L}_b\right). \qquad (24)$$

This is a generalization of the action of eq. (1), to which a term proportional to the curvature scalar constructed out of the induced metric on the brane has been added. This term involves a new length scale r_c. In principle the value of r_c in units of M is calculable in the context of the fundamental theory underlying the brane Universe scenario. However, we adopt here a phenomenological approach and treat r_c as a free parameter.

The generalized Friedmann equation becomes [24]

$$\frac{r_c^2}{2}\left(H^2 + \frac{R_3}{6}\right) = 1 + \frac{r_c\left(V+\tilde{\rho}\right)}{12M^3} + \epsilon\left[1 + \frac{r_c\left(V+\tilde{\rho}\right)}{6M^3} + \frac{r_c^2\Lambda}{12M^3} - \frac{r_c^2\mathcal{M}}{6\pi^2 M^3\ell^4}\right]^{1/2}. \qquad (25)$$

The quantity $\mathcal{M}(\ell,\tau)$ is the comoving mass of the bulk fluid as defined in eq. (7). The above equation has two branches, characterized by the values of the parameter $\epsilon = \pm 1$. The branch with $\epsilon = 1$ is the normal branch. It leads to eq. (3) in the limit $r_c \to 0$, with Λ, V kept fixed and fine-tuned so that they satisfy $\lambda = \left(V^2/12M^3 - \Lambda\right)/12M^3 = 0$. The branch with $\epsilon = -1$ is the self-accelerating branch, in which an effective cosmological constant appears even when the fine-tuning $\lambda = 0$ is imposed.

5. Accelerated expansion and $w = -1$ crossing in the presence of induced gravity

The two values of ϵ in equation (25) correspond to two disconnected branches of solutions. The nature of the predicted expansion is clearer in the limit of low energy density. Let us consider first the Randall-Sundrum case [1], in which the bulk cosmological constant $-\Lambda$ and the brane tension V are related through $\Lambda = V^2/(12M^3)$. We define the energy scale $k = V/(12M^3) = \left[\Lambda/(12M^3)\right]^{1/2}$. The Friedmann equation (25) can be written as

$$\frac{r_c^2}{2}\left(H^2 + \frac{k_c}{\ell^2}\right) = 1 + kr_c + kr_c\frac{\tilde{\rho}}{V} + \epsilon\left[(1+kr_c)^2 + 2kr_c\frac{\tilde{\rho}}{V} - 2(kr_c)^2\frac{\tilde{\rho}_d}{V}\right]^{1/2}, \tag{26}$$

where $\tilde{\rho}_d = \mathcal{M}(\ell)/\left(k\pi^2\ell^4\right)$ is the effective energy density of the generalized dark radiation.

For $\tilde{\rho}, \tilde{\rho}_d \ll V$, keeping only the terms linear in $\tilde{\rho}$, $\tilde{\rho}_d$, we find

$$H^2 = \frac{2(1+\epsilon)(1+kr_c)}{r_c^2} + \frac{1+\epsilon+kr_c}{kr_c(1+kr_c)}\frac{\tilde{\rho}}{6\,(M^3/k)} - \frac{\epsilon}{1+kr_c}\frac{\tilde{\rho}_d}{6\,(M^3/k)} - \frac{k_c}{\ell^2}. \tag{27}$$

For $\epsilon = -1$ we have

$$H^2 = \frac{1}{6M_{\rm Pl}^2}\left(\tilde{\rho} + \tilde{\rho}_d\right) - \frac{k_c}{\ell^2}, \tag{28}$$

with $M_{\rm Pl}^2 = M^3(r_c + 1/k)$. The expansion is conventional, apart from the presence of the energy density of the generalized dark radiation.

For $\epsilon = 1$ and $kr_c \ll 1$ we obtain

$$H^2 = \frac{4}{r_c^2} + \frac{1}{6M_{\rm Pl}^2}\left(\tilde{\rho} - \frac{kr_c}{2}\tilde{\rho}_d\right) - \frac{k_c}{\ell^2}, \tag{29}$$

with $M_{\rm Pl}^2 = M^3 r_c/2$. For $\epsilon = 1$ and $kr_c \gg 1$ we have

$$H^2 = \frac{4k}{r_c} + \frac{1}{6M_{\rm Pl}^2}\left(\tilde{\rho} - \tilde{\rho}_d\right) - \frac{k_c}{\ell^2}, \tag{30}$$

with $M_{\rm Pl}^2 = M^3 r_c$. In both cases an effective cosmological constant appears, despite the fine tuning of the bulk cosmological constant and the brane tension. The other striking feature is the negative sign of the contribution proportional to the energy density of the dark radiation.

The above features also appear in the DGP model [17], characterized by $\Lambda = V = 0$. The Friedmann equation (25) now reads

$$\frac{r_c^2}{2}\left(H^2 + \frac{k_c}{\ell^2}\right) = 1 + r_c\frac{\tilde{\rho}}{12M^3} + \epsilon\left[1 + \frac{r_c}{6M^3}\left(\tilde{\rho} - \tilde{\rho}_d\right)\right]^{1/2}, \tag{31}$$

with $\tilde{\rho}_d = r_c\mathcal{M}(\ell)/\left(\pi^2\ell^4\right)$. For $\epsilon = 1$ and $\tilde{\rho} \ll M^3/r_c$ we have

$$H^2 = \frac{4}{r_c^2} + \frac{1}{6M_{\rm Pl}^2}\left(\tilde{\rho} - \frac{1}{2}\tilde{\rho}_d\right) - \frac{k_c}{\ell^2}, \tag{32}$$

with $M_{\rm Pl}^2 = M^3 r_c/2$.

It is obvious from the above that the brane cosmological expansion in the branch with $\epsilon = 1$ has novel properties arising from: a) an effective cosmological constant, and b) an effective negative energy density associated with the generalized dark radiation. The second feature is *not* a consequence of a violation of the weak energy condition, as the energy density is assumed *positive* both in the bulk and on the brane.

In the case of an AdS-Schwarzschild bulk we have $\tilde{\rho}_d \sim \ell^{-4}$. At late times the contribution from the dark radiation is subleading to the contribution from the brane matter $\tilde{\rho} \sim \ell^{-3}$. On the other hand, if there is a non-trivial matter configuration in the bulk so that $\tilde{\rho}_d \sim \ell^{-n}$ with $n < 3$, the cosmological constant and the effective negative energy density are the leading effects.

An example with $\tilde{\rho}_d \sim \ell^{-2}$ is given in reference [10]. The bulk is assumed to contain a scalar field in a global monopole (hedgehog) configuration. The field also interacts with the brane through a localized quadratic potential, so that the brane can be embedded in the bulk spacetime. For large ℓ, the dominant contribution to the integrated mass arises from the field kinetic term, so that $\mathcal{M} \sim \ell^2$ and $\tilde{\rho}_d \sim \ell^{-2}$. The contribution from the brane potential is $\sim \ell^{-4}$ and, therefore, negligible for large ℓ.

The evolution in the presence of a cosmological constant $\lambda_{\rm eff}$ and a matter contribution $\tilde{\rho} = \tilde{\rho}_0(\ell/\ell_0)^{-n}$ is determined by

$$\frac{1}{\tilde{\ell}}\frac{d\tilde{\ell}}{d\tilde{t}} = \left(1 + \delta\frac{c^2}{\tilde{\ell}^n}\right)^{1/2}, \tag{33}$$

where $\tilde{\ell} = \ell/\ell_0$, $\tilde{t} = (\lambda_{\rm eff}/6M^2)^{1/2}t$, $c^2 = \tilde{\rho}_0/\lambda_{\rm eff}$ and $\delta = \pm 1$. For $\delta = 1$ the solution is

$$\tilde{\ell}^{n/2} = c\sinh\left(\frac{n}{2}t + d_s\right), \tag{34}$$

with $d_s = \sinh^{-1}(1/c)$, while for $\delta = -1$ it is

$$\tilde{\ell}^{n/2} = c\cosh\left(\frac{n}{2}t + d_c\right), \tag{35}$$

with $d_c = \cosh^{-1}(1/c)$. The acceleration parameter $q = \ddot{\tilde{\ell}}\,\tilde{\ell}/(\dot{\tilde{\ell}})^2$ for $\delta = 1$ is

$$q = 1 - \frac{n}{2}\left[\cosh\left(\frac{n}{2}t + d_s\right)\right]^{-2}, \tag{36}$$

while for $\delta = -1$ it is

$$q = 1 + \frac{n}{2}\left[\sinh\left(\frac{n}{2}t + d_c\right)\right]^{-2}. \tag{37}$$

The Friedmann equation for the branch with $\epsilon = 1$ can be written in the form (33) when one of the energy densities $\tilde{\rho}$, $\tilde{\rho}_d$ is negligible. If the dominant matter contribution comes from the non-relativistic matter on the brane, we have $\delta = 1$, $n = 3$ and $q < 1$. If the dominant matter contribution comes from the generalized dark radiation with $n < 3$, we have $\delta = -1$ and $q > 1$.

For a cosmological fluid with an equation of state $p_{\rm eff} = w\rho_{\rm eff}$ and an effective state parameter w, the acceleration parameter is $q = -(1 + 3w)/2$. As a result, $q > 1$ implies $w < -1$, while $q < 1$ implies $w > -1$. It is apparent from our discussion above that the branch of equation (25) with $\epsilon = 1$ can lead to an evolution during which the line $w = -1$ is crossed. This happens if the cosmological constant is the leading contribution, and the brane dark matter dominates over the generalized dark radiation at early times, while the reverse takes place at later times.

Acknowledgments

This research is co-funded by the European Social Fund and National Resources - (EPEAEK II) PYTHAGORAS II (grant 70-03-7992).

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